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Clasificación homotópica de álgebras de camino de Leavitt simples puramente infinitas

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Buenos Aires, 2019
Fecha de defensa: 15/03/2019

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En esta tesis investigamos en qué medida las teorías de homología escisivas, invariantes por homotopía y matricialmente estables nos ayudan a distinguir dos álgebras de camino de Leavitt $L(E)$ y $L(F)$ de grafos E y F sobre un anillo conmutativo ℓ. Este trabajo está dividido en dos partes.

En la primera (Capítulo 2) consideramos álgebras de camino de Leavitt de grafos generales sobre anillos conmutativos arbitrarios. La K-teoría algebraica bivariante kk es la teoría de homología universal con respecto a las propiedades mencionadas; probamos un teorema de estructura para álgebras de camino de Leavitt unitales en kk. Mostramos que bajo leves hipótesis en el anillo ℓ, para un grafo E con finitos vértices y matriz de incidencia reducida A_E, la estructura de $L(E)$ depende solamente en las clases de isomorfía del conúcleo de la matriz $I - A_E$ y el de su transpuesta, que son respectivamente los grupos $KH^1(L(E)) = kk_{-1}(L(E), \ell)$ y $KH_0(L(E)) = kk(\ell, L(E))$. Por tanto, si $L(E)$ y $L(F)$ son álgebras de Leavitt unitales tales que $KH_0(L(E)) \cong KH_0(L(F))$ y $KH^1(L(E)) \cong KH^1(L(F))$ entonces ninguna teoría de homología con las tres propiedades mencionadas puede distinguirlas. Además probamos que, para álgebras de camino de Leavitt, kk tiene varias propiedades similares a las que la K-teoría bivariante de Kasparov tiene para C^*-algebras de grafo, incluyendo análogos a los Teoremas de coeficientes universales y de Künneth de Rosenberg y Schochet.

En la segunda parte (Capítulo 3) abordamos el problema de clasificación de álgebras de camino de Leavitt simples puramente infinitas de grafos finitos sobre un cuerpo ℓ. Es un problema abierto determinar cuándo el par $(K_0(L(E)), [1_{L(E)}])$, que consiste del grupo de Grothendieck junto con la clase $[1_{L(E)}]$ de la identidad, es un invariante completo para la clasificación, a menos de isomorfismos, de álgebras de camino de Leavitt de grafos finitos que son simples puramente infinitas. Nosotros mostramos que $(K_0(L(E)), [1_{L(E)}])$ es un invariante completo para el problema de clasificación de dichas álgebras a menos de equivalencia homotópica polinomial. Para esto, desarrollamos aún más el estudio de la K-teoría algebraica bivariante de álgebras de Leavitt inciada en la parte previa y obtenemos otros resultados de interés.

Palabras claves : K-teoría algebraica bivariante, clasificación homotópica, álgebras de camino de Leavitt, álgebras simples puramente infinitas, teorema de coeficientes universales.

Homotopy classification of purely infinite simple Leavitt path algebras

In this thesis we investigate to what extent homotopy invariant, excisive and matrix stable homology theories help one distinguish between the Leavitt path algebras $L(E)$ and $L(F)$ of graphs E and F over a commutative ground ring ℓ. This work is divided in two parts.

In the first one (Chapter 2) we consider Leavitt path algebras of general graphs over general ground rings. Bivariant algebraic K-theory kk is the universal homology theory with the properties above; we prove a structure theorem for unital Leavitt path algebras in kk. We show that under very mild assumptions on ℓ, for a graph E with finitely many vertices and reduced incidence matrix A_E, the structure of $L(E)$ depends only on the isomorphism classes of the cokernels of the matrix $I-A_E$ and of its transpose, which are respectively the kk groups $KH^1(L(E)) = kk_{-1}(L(E),\ell)$ and $KH_0(L(E)) = kk(\ell, L(E))$. Hence if $L(E)$ and $L(F)$ are unital Leavitt path algebras such that $KH_0(L(E)) \cong KH_0(L(F))$ and $KH^1(L(E)) \cong KH^1(L(F))$ then no homology theory with the above properties can distinguish them. We also prove that for Leavitt path algebras, kk has several properties similar to those that Kasparov's bivariant K-theory has for C^*-graph algebras, including analogues of the Universal coefficient and Künneth theorems of Rosenberg and Schochet.

In the second part (Chapter 3) we address the classification problem for purely infinite simple Leavitt path algebras of finite graphs over a field ℓ. There is an open question which asks whether the pair $(K_0(L(E)), [1_{L(E)}])$, consisting of the Grothendieck group together with the class $[1_{L(E)}]$ of the identity, is a complete invariant for the classification, up to algebra isomorphism, of those Leavitt path algebras of finite graphs which are purely infinite simple. We show that $(K_0(L(E)), [1_{L(E)}])$ is a complete invariant for the classification of such algebras up to polynomial homotopy equivalence. To prove this we further develop the study of bivariant algebraic K-theory of Leavitt path algebras started in the previous part and obtain several other results of independent interest.

Keywords: Bivariant algebraic K-theory, homotopy classification, Leavitt path algebra, purely infinite simple algebra, universal coefficient theorem.

Acknowledgement / Agradecimientos

I want to thanks the referees for their meticulous reading of this Thesis. Also, I want to thanks Gene for all his support and enthusiasm.

A Willie por su paciencia y buena predisposición.

A mis viejos por toda la ayuda que me brindaron todos estos años.

A todos mis amigos que me acompañaron durante todos estos años.

A Belén, aunque sigo sin saber porqué.

Contents

Introducción

Un grafo dirigido E consiste de un conjunto E^0 de vértices y un conjunto de aristas E^1 junto con funciones de rango y fuente $r, s : E^1 \to E^0$. Esta tesis se ocupa de las álgebras de camino de Leavitt $L(E)$ de un grafo dirigido E sobre un anillo conmutativo ℓ ([1]). Cuando $\ell = \mathbb{C}$, $L(E)$ es un álgebra normada; su completación es la C^*-álgebra de grafo $C^*(E)$. Un grafo E se dice finito o numerable si E^0 y E^1 son ambos finitos o numerables. Un resultado de Cuntz y Rørdam ([25, Teorema 6.5]) dice que las álgebras de grafo simples puramente infinitas asociadas a grafos finitos, i.e. las álgebras de Cuntz-Krieger simples puramente infinitas, se clasifican a menos de isomorfismos (estables) mediante el grupo de Grothendieck K_0. Es un problema abierto determinar si un resultado similar vale para álgebras de Leavitt [3]. En esta tesis probamos que K_0 clasifica álgebras de camino de Leavitt a menos de (M_2-) equivalencia homotópica. En el siguiente teorema y en el resto de la tesis, usamos la siguiente notación. Notamos $\iota_2 : R \to M_2R$ a la inclusión de un álgebra en la esquina superior izquierda del álgebra de matrices, $\phi \approx \psi$ para indicar que dos morfismos de álgebras ϕ y ψ son (polinomialmente) homotópicos y $\phi \approx_{M_2} \psi$ para indicar que $\iota_2\phi \approx \iota_2\psi$. También escribimos $[1_R]$ para la clase de la identidad de un álgebra unital en el grupo K_0. El resultado principal de esta tesis es el siguiente.

Theorem 0.1. *Sean E y F grafos finitos y ℓ un cuerpo. Supongamos que $L(E)$ y $L(F)$ son simples puramente infinitas. Sea $\xi : K_0(L(E)) \to K_0(L(F))$ un isomorfismo de grupos. Entonces*

- *Existen morfismos de álgebras no nulos $\phi : L(E) \leftrightarrow L(F) : \psi$ tales que $K_0(\phi) = \xi$, $K_0(\psi) = \xi^{-1}$, $\psi\phi \approx_{M_2} \mathrm{id}_{L(E)}$ y $\phi\psi \approx_{M_2} \mathrm{id}_{L(F)}$.*
- *Si además $\xi([1_{L(E)}]) = [1_{L(F)}]$ entonces ϕ y ψ pueden ser tomados como morfismos unitales tales que $\psi\phi \approx \mathrm{id}_{L(E)}$ y $\phi\psi \approx \mathrm{id}_{L(F)}$.*

El resultado de clasificación de álgebras de Cuntz-Krieger usa la K-teoría bivariante de Kasparov, $C^* \to KK$, que es universal entre los funtores de la categoría de C^*-álgebras separables en categorías trianguladas que son escisivos, estables por operadores compactos e invariantes por homotopías continuas. Motivados por esto, analizamos el problema de clasificación para álgebras de camino de Leavitt en términos de la K-teoría bivariante algebraica $j : \mathrm{Alg}_\ell \to kk$ introducida en [15] y [24], que es universal entre las teorías de homología de la categoría de ℓ-álgebras que son escisivas, matricialmente estables e invariantes por homotopía polinomial. Acá una *teoría de homología* de la categoría Alg_ℓ de álgebras es simplemente un funtor $X : \mathrm{Alg}_\ell \to \mathcal{T}$ con valores en una categoría triangulada $\mathcal{T}$. Si S es un conjunto y $A \in \mathrm{Alg}_\ell$, escribimos $M_S A$ por el álgebra de matrices $M : S \times S \to A$ de soporte finito. Decimos que una teoría de homología X es S*-estable* si para todo $s \in S$ y $A \in \mathrm{Alg}_\ell$, la inclusión $\iota_s : A \to M_S A$, $\iota_s(a) = \epsilon_{s,s} \otimes a$ induce un isomorfismo $X(\iota_s)$. Notamos E^0 y E^1 a los conjuntos de vértices y aristas del grafo E. Decimos que X es E-estable si es $E^0 \sqcup E^1 \sqcup \mathbb{N}$-estable. Así, si E^0 y E^1 son ambos numerables, la E-estabilidad es lo mismo que la estabilidad respecto de $M_\infty = M_\mathbb{N}$. Para dos álgebras $A, B \in \mathrm{Alg}_\ell$, la afirmación $j(A) \cong j(B)$ es equivalente a la afirmación que $X(A) \cong X(B)$ para toda teoría de homología escisiva, invariante por homotopía y E-estable X. Sea $\Omega : kk \to kk$ la inversa del funtor de suspensión; si $A, B \in \mathrm{Alg}_\ell$, notamos

$$kk_n(A, B) = \hom_{kk}(j(A), \Omega^n j(B)), \quad kk(A, B) = kk_0(A, B).$$

Por [15, Teorema 8.2.1], fijando en la primera variable el anillo base, recobramos la K-teoría homotópica de Weibel KH [32]:

$$kk_n(\ell, B) = KH_n(B).$$

Si ℓ es $\mathbb{Z}$ o un cuerpo, entonces $KH_*(L(E)) = K_*(L(E))$ es la K-teoría algebraica de Quillen.

Sea

$$KH^n(B) := kk_{-n}(B, \ell).$$

Recordemos que un vértice $v \in E^0$ es *regular* si emite un número no nulo y finito de aristas y que es *singular* en caso contrario. Escribimos $\mathrm{reg}(E)$ y $\mathrm{sing}(E)$ a los conjuntos de vértices regulares y singulares. Sea $A_E \in \mathbb{Z}^{\mathrm{reg}(E)\times E^0}$ la matriz cuya entrada (v, w) es el número de aristas que van desde v hasta w y sea $I \in \mathbb{Z}^{E^0\times\mathrm{reg}(E)}$ la matriz que resulta de remover a la matriz identidad las columnas que corresponden a los vértices singulares. Se sigue de [4] que si $KH_0(\ell) = \mathbb{Z}$, $KH_{-1}(\ell)=0$ y E es finito, entonces para la matriz de incidencia reducida A_E tenemos que

$$KH_0(L(E)) = \mathrm{Coker}(I - A_E^t). \tag{1}$$

Aquí mostramos (ver Sección 2.3) que, abusando de notación, y escribiendo I por I^t,

$$KH^1(L(E)) = \mathrm{Coker}(I - A_E). \tag{2}$$

Para $n \geq 0$, notamos L_n al álgebra de caminos de Leavitt del grafo con un vértice y n bucles. Así $L_0 = \ell$ y $L_1 = \ell[t, t^{-1}]$. Probamos el siguiente teorema de estructura.

Theorem 0.2. *Supongamos que $KH_0(\ell) = \mathbb{Z}$ y $KH_{-1}(\ell)=0$. Sea E un grafo tal que E^0 es finito. Sean $d_1, \dots, d_n$, $d_i \backslash d_{i+1}$ los factores invariantes del grupo de torsión $\tau(E) = \mathrm{tors}(K_0(L(E)))$, $s = \#\,\mathrm{sing}(E)$ y $r = \mathrm{rk}(KH^1(L(E)))$. Sea $j : \mathrm{Alg}_\ell \to kk$ la teoría de homología escisiva, invariante por homotopía polinomial y E-estable universal. Entonces*

$$j(L(E)) \cong j(L_0^s \oplus L_1^r \oplus \bigoplus_{i=1}^{n} L_{d_i+1})$$

En particular, toda álgebra de caminos de Leavitt unital con KH_0 trivial es cero en kk. Por ejemplo, tanto L_2 como su empalme de Cuntz L_{2^-} ([25]) son cero en kk. Además tenemos el siguiente corolario; desde aquí, en toda afirmación que involucre la imagen por j de finitas álgebras de caminos de Leavitt de grafos $E_1, \ldots, E_n$, se entiende que j se refiere a la $j \sqcup_{i=1}^n E_i$-estable.

Corollary 0.3. *Sea ℓ como en el Teorema 0.2. Sean E y F grafos con finitos vértices. Las siguientes afirmaciones son equivalentes.*

i) $j(L(E)) \cong j(L(F))$.

ii) $KH_0(L(E)) \cong KH_0(L(F))$ *y* $KH^1(L(E)) \cong KH^1(L(F))$.

iii) $KH_0(L(E)) \cong KH_0(L(F))$ *y* $\#\operatorname{sing}(E) = \#\operatorname{sing}(F)$.

Proof. No es difícil de ver , usando (1) y (2) (ver Lema 2.16) que los grupos $KH_0(L(E))$ y $KH^1(L(E))$ tienen subgrupos de torsión isomorfos y que

$$\#\operatorname{sing}(E) = \operatorname{rk} KH_0(L(E)) - \operatorname{rk} KH^1(L(E)). \tag{3}$$

El corolario se sigue inmediatamente de esto y del teorema anterior. □

Para poner el resultado previo en perspectiva, recordemos que E. Ruiz y M. Tomforde mostraron en [29] que si ℓ es un cuerpo, $L(E)$ y $L(F)$ son simples y tanto E como F tienen emisores infinitos, entonces la condición iii) del corolario es equivalente a que $L(E)$ y $L(F)$ sean Morita equivalentes. Nuestros resultados se aplican de forma mucho más general, pero son en principio más débiles, visto que álgebras kk-isomorfas pueden no ser Morita equivalentes. Por ejemplo L_2 no es Morita equivalente al anillo 0. Observar que la identidad (3) nos ayuda a reemplazar las condiciones sobre $\#\operatorname{sing}$ por condiciones puramente K-teóricas u homológicas sobre KH^1.

En el siguiente teorema y en adelante, escribimos $[A, R]$ y $[A, R]_{M_2}$ por el conjunto de clases de homotopía y M_2-homotopía de morfismos $A \to R$. Si además, A y R son unitales, escribimos $[A, R]_1$ por el conjunto de clases de homotopía de morfismos unitales $A \to R$. Recordemos que un anillo R es *K_n-regular* si el morfismo canónico $K_n(R) \to K_n(R[t_1, \ldots, t_m])$ es un isomorfismo para todo m. Por ejemplo, toda álgebra de Leavitt es K_n-regular para todo $n \in \mathbb{Z}$, por el Ejemplo 2.8. Además del Teorema de estructura 0.2, podemos calcular los grupos de K-teoría algebraica bivariante de álgebras de caminos de Leavitt en algunos casos particulares como muestra el siguiente teorema.

Theorem 0.4. *Sea ℓ un cuerpo. Sean E un grafo finito tal que $L(E)$ es simple y R un álgebra unital simple puramente infinita. Supongamos que R es K_1-regular. Entonces el morfismo canónico*

$$[L(E), R]_{M_2} \setminus \{0\} \to kk(L(E), R)$$

es un isomorfismo de monoides.

El teorema previo es el resultado técnico principal y es la clave para la demostración del Teorema 0.1. Gracias a la observación 3.32, podemos ver al Teorema 0.4 como una generalización del teorema de Ara, Goodearl y Pardo [5] que dice que si R es como en el teorema y $\mathcal{V}(R)$ es el monoide de clases de equivalencia Murray-von Neumann sobre matrices idempotentes en $M_\infty R$, entonces $K_0(R) = \mathcal{V}(R) \setminus \{0\}$. De hecho, el último resultado es usado en la demostración del Teorema 0.4.

En similitud con el caso de álgebras de operadores, probamos (Corolario 2.23) que si ℓ y E son como en el Teorema 0.2 y $R \in \mathrm{Alg}_\ell$, entonces existe una sucesión exacta corta

$$\begin{aligned}0 \to \mathrm{Ext}^1_{\mathbb{Z}}(KH_0(L(E)), KH_{n+1}(R)) \to kk_n(L(E), R) &\xrightarrow{[KH_0, \gamma^* KH_1]} \\ \mathrm{Hom}(KH_0(L(E)), KH_n(R)) \oplus \mathrm{Hom}(\mathrm{Ker}(I - A_E^t), KH_{n+1}(R)) &\to 0. \quad (4)\end{aligned}$$

Observar que, para la K-teoría topológica, $K_1^{\mathrm{top}}(C^*(E)) = \mathrm{Ker}(I - A_E^t)$, por tanto, sustituyendo KH y kk por K^{top} y KK en (4) obtenemos el UCT de [28, Teorema 1.17]. Más aún, en la Proposición 2.26 probamos un análogo al teorema Künneth de [28, Teorema 1.18].

Hasta aquí en esta introducción sólo discutimos resultados para E con finitos vértices y ℓ tal que $KH_0(\ell) = \mathbb{Z}$ y $KH_{-1}\ell = 0$. Sin hipótesis en ℓ mostramos que si E y F tienen finitos vértices y $\theta \in kk(L(E), L(F))$ entonces

$$\theta \text{ es un isomorfismo} \iff KH_0(\theta) \text{ y } KH_1(\theta) \text{ son isomorfos.} \quad (5)$$

Sin embargo, no es cierto que las álgebras de camino de Leavitt unitales con KH_0 and KH_1 isomorfos son kk-isomorfas, incluso cuando ℓ es un cuerpo (ver Observación 2.12). Por tanto, en vista del Corolario 0.3, el par (KH_0, KH^1) es un mejor invariante para álgebras de camino de Leavitt que el par (KH_0, KH_1).

Sean ℓ y E arbitrarios y sea $R \in \mathrm{Alg}_\ell$. Si I es un conjunto, escribimos

$$R^{(I)} = \bigoplus_{i \in I} R$$

para notar al álgebra de funciones $I \to R$ de soporte finito. Sea $X : \mathrm{Alg}_\ell \to \mathcal{T}$ una teoría de homología escisiva, invariante por homotopía polinomial y E-estable. Supongamos además que las sumas directas de cardinal a lo sumo $\#E^0$ existen en $\mathcal{T}$ y que para cualquier familia de álgebras $\{R_i : i \in I\}$ el morfismo natural

$$\bigoplus_{i \in I} X(R_i) \to X(\bigoplus_{i \in I} R_i)$$

es un isomorfismo si $\#I \leq \#E^0$. Probamos en el Teorema 2.7 que existe un triángulo distinguido en $\mathcal{T}$ de la siguiente forma

$$X(R)^{(\mathrm{reg}(E))} \xrightarrow{I - A_E^t} X(R)^{(E^0)} \longrightarrow X(L(E) \otimes R). \quad (6)$$

Esto se aplica, en particular, cuando tomamos $X = KH$, generalizando [4, Teorema 8.4]. Por tanto, obtenemos una sucesión exacta larga

$$KH_{n+1}(L(E)\otimes R)^{(E^0)} \to KH_n(R)^{(\mathrm{reg}(E))} \xrightarrow{I-A_E^t} KH_n(R)^{(E^0)} \to KH_n(L(E)\otimes R). \tag{7}$$

Cuando R es regular supercoherente podemos sustituir KH por K en (7), generalizando [4, Teorema 7.6] (ver Ejemplo 2.8). No se sabe si existen sumas directas infinitas en kk; sin embargo las sumas directas finitas existen, y j conmuta con ellas. Entonces cuando E^0 es finito y ℓ es arbitrario, podemos poner $X = j$ en el resultado previo y obtener un triángulo distinguido

$$j(R)^{\mathrm{reg}(E)} \xrightarrow{I-A_E^t} j(R)^{E^0} \longrightarrow j(L(E)\otimes R). \tag{8}$$

Este triángulo es el punto de partida que usamos para establecer todos los resultados sobre álgebras de Leavitt en esta tesis.

Desde ahora asumimos, a menos que se especifique lo contrario, que ℓ es un cuerpo.

También probamos otros resultados que pensamos que tienen interés en sí mismo. Por ejemplo tenemos el siguiente teorema, probado en el Corolario 3.19.

Theorem 0.5. *Sean E un grafo tal que $L(E)$ es simple y sea R un álgebra unital simple puramente infinita.*

i) Si E es numerable entonces $L(E)$ es isomorfo a una subálgebra de $M_\infty R$.

ii) Si E es finito y $[1_R] = 0$ en $K_0(R)$, entonces $L(E)$ es isomorfo a una subálgebra unital de R.

iii) Si E es finito, entonces $L(E)$ es isomorfo a una subálgebra de R.

En el siguiente teorema y en adelante, usaremos la noción de anillo regular supercoherente de [11]. Por ejemplo, $L(E)$ es regular supercoherente para todo grafo finito E ([1, Lema 6.4.16]).

Theorem 0.6. *Sea E un grafo finito tal que $L(E)$ es simple y R un álgebra unital simple puramente infinita, regular supercoherente. Entonces $[L(E), L_2]_1 = [L(E), L_2]_{M_2} \setminus \{0\}$, $[L(E), R\otimes L_2]_1 = [L(E), R\otimes L_2]_{M_2} \setminus \{0\}$, y ambos conjuntos tienen exactamente un elemento cada uno.*

En particular, el Teorema 0.6 implica que si $d : L_2 \to L_2 \otimes L_2$, $d(x) = 1 \otimes x$ y $\phi : L_2 \to L_2 \otimes L_2$ es un homomorfismo no nulo, entonces $\phi \approx_{M_2} d$ y que si ϕ es unital entonces $\phi \approx d$.

Por (2) y [16, Teorema 5.3], cuando E es finito y regular $KH^1(L(E))$ es isomorfo al grupo de extensiones de la C^*-álgebra de E por el álgebra de operadores compactos. Veremos que $KH^1(L(E))$ está también vinculada a las extensiones

$$0 \to M_\infty \to \mathcal{E} \to L(E) \to 0.$$

Uno puede formar un monoide abeliano de clases de homotopía de tales extensiones (ver Sección 1.1); escribimos $\mathcal{E}xt(L(E))$ para su completación a grupo. Cuando E^0 es finito y E^1 es numerable, existe un morfismo natural

$$\mathcal{E}xt(L(E), R) \to kk_{-1}(L(E), R). \tag{9}$$

Nosotros probamos en la Proposición 2.15 que, bajo la suposición del Teorema 0.2, si además E no tiene fuentes y $R = \ell$, entonces el morfismo (9) es sobreyectivo. Más aún, tenemos lo siguiente

Theorem 0.7. *Sea E un grafo finito tal que $L(E)$ is simple. Sea R un álgebra de división o un álgebra unital simple puramente infinita y K_0-regular. Entonces el morfismo natural (9) es un isomorfismo*

$$\mathcal{E}xt(L(E), R) \xrightarrow{\sim} kk_{-1}(L(E), R).$$

Si además $K_0(L(E))$ es de torsión, entonces para todo R como en el Teorema 0.7 (en particular, para $R = \ell$ y toda álgebra de caminos de Leavitt unital simple puramente infinita R), tenemos

$$\mathcal{E}xt(L(E), R) = \operatorname{Ext}^1_{\mathbb{Z}}(K_0(L(E)), K_0(R)). \tag{10}$$

Esta tesis está organizada de la siguiente manera: En el Capítulo 1 daremos los preliminares. En la primera sección recordaremos las nociones de homotopía algebraica, probaremos algunos lemas elementales sobre ella, y los usaremos para definir, para cada par de algebras A y R con R unital, un grupo $\mathcal{E}xt(A, R)$ de clases de homotopía de extensiones de A por $M_\infty R$. Luego recordarmeos algunas propiedades básicas sobre kk y cuasi-morfismos y probaremos algunos lemas técnicos. En la última sección recordaremos las definiciones de álgebras de camino de Cohn y de Leavitt.

En el capítulo 2 trabajaremos con álgebras de camino de Leavitt sobre un anillo commutativo ℓ. Todos los resultados de este capítulo provienen de [13]. La Sección 2.1 está dedicada a la caracterización de la imagen por $j : \mathrm{Alg}_\ell \to kk$ del álgebras de camino de Cohn $C(E)$ de grafo E. El álgebra $C(E)$ recibe un morfismo canónico $\phi : \ell^{(E^0)} \to C(E)$. Probamos, en el Teorema 2.1, que la teoría de homología esciva, invariante por homotopía polinomial y E-estable universal j manda al morfismo ϕ a un isomorfismo.

$$j(\ell^{(E^0)}) \cong j(C(E)). \tag{11}$$

La demostración usa cuasi-morfismos, en el espíritu de la demostración de Cuntz de la periodicidad de Bott en K-teoría para C^*-álgebras. En la Sección 2.2 usamos los resultados de la Sección 2.1 para obtener el triángulo (8). Con este triángulo obtenemos los resultados de las Secciones 2.3 and 2.4. En la primera, probamos el Teorema 0.2 (Teorema 2.17), que clasifica las álgebras de camino de Leavitt en kk. En la segunda, introducimos una filtración descendente $\{kk(L(E), R)^i : 0 \leq i \leq 2\}$ en $kk(L(E), R)$ para toda álgebra R y toda álgebra de camino de Leavitt unital $L(E)$ y calculamos los cocientes $kk(L(E), R)^i / kk(L(E), R)^{i+1}$ (Teorema 2.21).

Usamos esto para probar el teorema de coeficienes universales (4) en el Corolario 2.23 y el Teorema de Künneth en la Proposición 2.26.

En el capítulo 3 trabajamos con álgebras de caminos de Leavitt simples de grafos finitos sobre un cuerpo ℓ. Todos los resultados de este capítulo provienen de [14]. En la Sección 3.1 recordamos los resultados de Ara, Goodearl and Pardo sobre la K-teoría de álgebras simples puramente infinitas. También probamos (Corolario 3.10) que si R es K_1-regular, simple puramente infinita y unital, entonces $K_1(R)$ es isomorfo al grupo $\pi_0(U(R))$ de componentes conexas polinomiales del grupo de elementos inversibles de R. En la Sección 3.2 probamos el Teorema 0.4 (Teorema 3.12) que usamos en la Sección 3.3 para establecer el Teorema 0.1 (Teorema 3.33), que es el resultado principal de esta tesis. Las Secciones 3.4 y 3.5 están dedicadas a probar los Teoremas 0.7 (Teorema 3.36) y 0.6 (Teorema 3.42).

Introduction

A directed graph E consists of a set E^0 of vertices and a set E^1 of edges together with source and range functions $r, s : E^1 \to E^0$. This thesis is concerned with the Leavitt path algebra $L(E)$ of a directed graph E over a commutative ring ℓ ([1]). When $\ell = \mathbb{C}$, $L(E)$ is a normed algebra; its completion is the graph C^*-algebra $C^*(E)$. A graph E is called finite or countable if both E^0 and E^1 are finite or countable. A result of Cuntz and Rørdam ([25, Theorem 6.5]) says that the purely infinite simple graph algebras associated to finite graphs, i.e. the purely infinite simple Cuntz-Krieger algebras, are classified up to (stable) isomorphism by the Grothendieck group K_0. It is an open question whether a similar result holds for Leavitt path algebras [3]. Here we prove that K_0 classifies simple Leavitt path algebras up to (M_2-) homotopy equivalence. In the following theorem and elsewhere, we use the following notations. We write $\iota_2 : R \to M_2R$ for the inclusion of an algebra into the upper left hand corner of the matrix algebra, $\phi \approx \psi$ to indicate that two algebra homomorphisms ϕ and ψ are (polynomially) homotopic and $\phi \approx_{M_2} \psi$ to mean that $\iota_2\phi \approx \iota_2\psi$. We also put $[1_R]$ for the K_0-class of the identity of a unital algebra R. The main theorem of this thesis is the following.

Theorem 0.1. *Let E and F be finite graphs and ℓ a field. Assume that $L(E)$ and $L(F)$ are purely infinite simple. Let $\xi : K_0(L(E)) \to K_0(L(F))$ be an isomorphism of groups. Then*

- *There exist nonzero algebra homomorphisms $\phi : L(E) \leftrightarrow L(F) : \psi$ such that $K_0(\phi) = \xi$, $K_0(\psi) = \xi^{-1}$, $\psi\phi \approx_{M_2} \mathrm{id}_{L(E)}$ and $\phi\psi \approx_{M_2} \mathrm{id}_{L(F)}$.*
- *If moreover $\xi([1_{L(E)}]) = [1_{L(F)}]$ then ϕ and ψ can be chosen to be unital homomorphisms such that $\psi\phi \approx \mathrm{id}_{L(E)}$ and $\phi\psi \approx \mathrm{id}_{L(F)}$.*

The classification result of Cuntz-Krieger algebras uses Kasparov's bivariant K-theory, $C^* \to KK$, which is universal among functors from the category of separable C^*-algebras to triangulated categories that are excisive, stable under compact operators and invariant under continuous homotopies. Motivated by this, we analyze the classification problem for Leavitt path algebras in terms of the algebraic bivariant K-theory $j : \mathrm{Alg}_\ell \to kk$ introduced in [15] and [24], which is universal among those homology theories of the category of ℓ-algebras which are excisive, matrix stable and invariant under polynomial homotopies. Here a *homology theory* of the category Alg_ℓ of algebras is simply a functor $X : \mathrm{Alg}_\ell \to \mathcal{T}$ with values in some triangulated category $\mathcal{T}$. If S is a set and $A \in \mathrm{Alg}_\ell$, we write M_SA for the algebra of those matrices $M : S \times S \to A$ which are finitely supported. We call a homology theory X S*-stable* if for $s \in S$ and $A \in \mathrm{Alg}_\ell$, the inclusion $\iota_s : A \to M_S A$, $\iota_s(a) = \epsilon_{s,s} \otimes a$ induces an isomorphism $X(\iota_s)$. Write E^0 and E^1 for the sets of vertices and edges of the graph E. We call X E*-stable* if it is $E^0 \sqcup E^1 \sqcup \mathbb{N}$-stable. Thus if E^0 and E^1 are both countable, E-stability is the same as stability with respect to $M_\infty = M_{\mathbb{N}}$. For two algebras $A, B \in \mathrm{Alg}_\ell$, the statement $j(A) \cong j(B)$ is equivalent the statement that $X(A) \cong X(B)$ for any excisive, homotopy invariant and E-stable homology theory X. Let $\Omega : kk \to kk$ be the inverse suspension; if $A, B \in \mathrm{Alg}_\ell$, put

$$kk_n(A, B) = \hom_{kk}(j(A), \Omega^n j(B)), \quad kk(A, B) = kk_0(A, B).$$

By [15, Theorem 8.2.1], setting the first variable equal to the ground ring we recover Weibel's homotopy algebraic K-theory KH [32]:

$$kk_n(\ell, B) = KH_n(B).$$

If ℓ is either $\mathbb{Z}$ or a field, then $KH_*(L(E)) = K_*(L(E))$ is Quillen's K-theory. Set

$$KH^n(B) := kk_{-n}(B, \ell).$$

Recall that a vertex $v \in E^0$ is *regular* if it emits a nonzero finite number of edges and that it is *singular* otherwise. Write $\mathrm{reg}(E)$ and $\mathrm{sing}(E)$ for the sets of regular and of singular edges. Let $A_E \in \mathbb{Z}^{\mathrm{reg}(E)\times E^0}$ be the matrix whose (v, w) entry is the number of edges from v to w and let $I \in \mathbb{Z}^{E^0\times \mathrm{reg}(E)}$ be the matrix that results from the identity matrix upon removing the columns corresponding to singular vertices. It follows from [4] that if $KH_0(\ell) = \mathbb{Z}$, $KH_{-1}(\ell)$=0 and E^0 is finite, then for the reduced incidence matrix A_E we have

$$KH_0(L(E)) = \mathrm{Coker}(I - A_E^t). \tag{12}$$

We show here (see Section 2.3) that, abusing notation, and writing I for I^t,

$$KH^1(L(E)) = \mathrm{Coker}(I - A_E). \tag{13}$$

For $n \geq 0$, let L_n be the Leavitt path algebra of the graph with one vertex and n loops. Thus $L_0 = \ell$ and $L_1 = \ell[t, t^{-1}]$. We prove the following structure theorem.

Theorem 0.2. *Assume that $KH_0(\ell) = \mathbb{Z}$ and $KH_{-1}(\ell)$=0. Let E be a graph such that E^0 is finite. Let $d_1, \dots, d_n$, $d_i \backslash d_{i+1}$ be the invariant factors of the torsion group $\tau(E) =$ $\mathrm{tors}(K_0(L(E)))$, $s = \#\,\mathrm{sing}(E)$ and $r = \mathrm{rk}(KH^1(L(E)))$. Let $j : \mathrm{Alg}_\ell \to kk$ be the universal excisive, homotopy invariant, E-stable homology theory. Then*

$$j(L(E)) \cong j(L_0^s \oplus L_1^r \oplus \bigoplus_{i=1}^{n} L_{d_i+1})$$

In particular, any unital Leavitt path algebra with trivial KH_0 is zero in kk. For example both L_2 and its Cuntz splice L_{2^-} ([25]) are zero in kk. We also have the following corollary; here, and in any other statement which involves the image under j of the Leavitt path algebras of finitely many graphs $E_1, \ldots, E_n$, j is understood to refer to the $\sqcup_{i=1}^n E_i$-stable j.

Corollary 0.3. *Let ℓ be as in Theorem 0.2. The following are equivalent for graphs E and F with finitely many vertices.*

i) $j(L(E)) \cong j(L(F))$.

ii) $KH_0(L(E)) \cong KH_0(L(F))$ *and* $KH^1(L(E)) \cong KH^1(L(F))$.

iii) $KH_0(L(E)) \cong KH_0(L(F))$ *and* $\#\operatorname{sing}(E) = \#\operatorname{sing}(F)$.

Proof. It is not hard to check, using (12) and (13) (see Lemma 2.16) that the groups $KH_0(L(E))$ and $KH^1(L(E))$ have isomorphic torsion subgroups and that

$$\#\operatorname{sing}(E) = \operatorname{rk} KH_0(L(E)) - \operatorname{rk} KH^1(L(E)). \tag{14}$$

The corollary is immediate from this and the theorem above. □

To put the above result in perspective, let us recall that E. Ruiz and M. Tomforde have shown in [29] that if ℓ is a field, $L(E)$ and $L(F)$ are simple and both E and F have infinite emitters, then condition iii) of the corollary holds if and only if $L(E)$ and $L(F)$ are Morita equivalent. Our result applies far more generally, but it is in principle weaker, since kk-isomorphic algebras need not be Morita equivalent. For example L_2 is not Morita equivalent to the 0 ring. Observe also that the identity (14) helps us replace the graphic condition about $\#\operatorname{sing}$ by the purely K-theoretic or homological condition about KH^1.

In the next theorem and elsewhere, we write $[A, R]$ and $[A, R]_{M_2}$ for the set of homotopy classes and M_2-homotopy classes of homomorphisms $A \to R$. If moreover, A and R are unital, we write $[A, R]_1$ for the set of homotopy classes of unital homomorphisms $A \to R$. Recall that a ring R is *K_n-regular* if the canonical map $K_n(R) \to K_n(R[t_1, \ldots, t_m])$ is an isomorphism for every m. For example, every Leavitt path algebra is K_n-regular for all $n \in \mathbb{Z}$, by Example 2.8. Besides the structure Theorem 0.2, we can also compute the bivariant K-theory groups for Leavitt path algebras in some specific cases as the following theorem shows.

Theorem 0.4. *Let ℓ be a field. Let E be a finite graph such that $L(E)$ is simple and R a purely infinite simple unital algebra. Assume that R is K_1-regular. Then the canonical map*

$$[L(E), R]_{M_2} \setminus \{0\} \to kk(L(E), R)$$

is an isomorphism of monoids.

The above theorem is the main technical result and it is key for the proof of Theorem 0.1. Thanks to Remark 3.32, we may view Theorem 0.4 as a generalization of the theorem of Ara, Goodearl and Pardo [5] which says that if R is as in the theorem and $\mathcal{V}(R)$ is the monoid of Murray-von Neumann equivalence classes of idempotent matrices in $M_\infty R$, then $K_0(R) = \mathcal{V}(R) \setminus \{0\}$. In fact, the latter result is used in the proof of Theorem 0.4.

As a similarity with the operator algebra case, we prove (Corollary 2.23) that if ℓ and E are as in Theorem 0.2 and $R \in \mathrm{Alg}_\ell$, then there is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(KH_0(L(E)), KH_{n+1}(R)) \to kk_n(L(E), R) \xrightarrow{[KH_0,\gamma^* KH_1]} \mathrm{Hom}(KH_0(L(E)), KH_n(R)) \oplus \mathrm{Hom}(\mathrm{Ker}(I - A_E^t), KH_{n+1}(R)) \to 0. \quad (15)$$

Observe that, for operator algebraic K-theory, $K_1^{\mathrm{top}}(C^*(E)) = \mathrm{Ker}(I - A_E^t)$, so substituting K^{top} and KK for KH and kk in (15) one obtains the usual UCT of [28, Theorem 1.17]. Moreover, in Proposition 2.26 we also prove an analogue of the Künneth theorem of [28, Theorem 1.18].

Up to here in this introduction we have only discussed results that hold for E with finitely many vertices and for ℓ such that $KH_0(\ell) = \mathbb{Z}$ and $KH_{-1}\ell = 0$. With no hypothesis on ℓ we show that if E and F have finitely many vertices and $\theta \in kk(L(E), L(F))$ then

$$\theta \text{ is an isomorphism} \iff KH_0(\theta) \text{ and } KH_1(\theta) \text{ are isomorphisms.} \quad (16)$$

It is however not true that unital Leavitt path algebras with isomorphic KH_0 and KH_1 are kk-isomorphic, even when ℓ is a field (see Remark 2.12). Thus in view of Corollary 0.3, the pair (KH_0, KH^1) is a better invariant of Leavitt path algebras than the pair (KH_0, KH_1).

Next let ℓ and E be arbitrary and let $R \in \mathrm{Alg}_\ell$. If I is a set, write

$$R^{(I)} = \bigoplus_{i \in I} R$$

for the algebra of finitely supported functions $I \to R$. Let $X : \mathrm{Alg}_\ell \to \mathcal{T}$ be an excisive, homotopy invariant, E-stable homology theory. Further assume that direct sums of at most $\#E^0$ summands exist in $\mathcal{T}$ and that for any family of algebras $\{R_i : i \in I\}$ the natural map

$$\bigoplus_{i \in I} X(R_i) \to X(\bigoplus_{i \in I} R_i)$$

is an isomorphism if $\#I \leq \#E^0$. We prove in Theorem 2.7 that there is a distinguished triangle in $\mathcal{T}$ of the following form

$$X(R)^{(\mathrm{reg}(E))} \xrightarrow{I - A_E^t} X(R)^{(E^0)} \longrightarrow X(L(E) \otimes R). \quad (17)$$

This applies, in particular, when we take $X = KH$, generalizing [4, Theorem 8.4]. Thus we get a long exact sequence

$$KH_{n+1}(L(E) \otimes R)^{(E^0)} \to KH_n(R)^{(\mathrm{reg}(E))} \xrightarrow{I - A_E^t} KH_n(R)^{(E^0)} \to KH_n(L(E) \otimes R) \quad (18)$$

When R is regular supercoherent we may substitute K for KH in (18), generalizing [4, Theorem 7.6] (see Example 2.8). Infinite direct sums are not known to exist in kk; however finite direct sums do exist, and j does commute with them. Hence when E^0 is finite and ℓ is arbitrary, we may take $X = j$ above to obtain a distinguished triangle

$$j(R)^{\mathrm{reg}(E)} \xrightarrow{I-A_E^t} j(R)^{E^0} \longrightarrow j(L(E)\otimes R). \tag{19}$$

This triangle is the starting point that we use to establish all the results on unital Leavitt path algebras in this thesis.

From now on we assume, unless otherwise stated, that ℓ is a field.

We also prove other results which we think are of independent interest. For example we have the following embedding theorem, proved in Corollary 3.19.

Theorem 0.5. *Let E be a graph such that $L(E)$ is simple and let R be a unital purely infinite simple algebra.*

i) If E is countable then $L(E)$ embeds as a subalgebra of $M_\infty R$.

ii) If E is finite and $[1_R] = 0$ in $K_0(R)$, then $L(E)$ embeds as a unital subalgebra of R.

iii) If E is finite, then $L(E)$ embeds as a subalgebra of R.

In the next theorem and elsewhere we use the notion of regular supercoherent ring from [11]. For example, $L(E)$ is regular supercoherent for every finite graph E ([1, Lemma 6.4.16]).

Theorem 0.6. *Let E be finite graph such that $L(E)$ is simple and R a purely infinite simple, regular supercoherent unital algebra. Then $[L(E), L_2]_1 = [L(E), L_2]_{M_2}\setminus\{0\}$, $[L(E), R\otimes L_2]_1 = [L(E), R\otimes L_2]_{M_2}\setminus\{0\}$, and both sets have exactly one element each.*

In particular, Theorem 0.6 implies that if $d : L_2 \to L_2\otimes L_2$, $d(x) = 1\otimes x$ and $\phi : L_2 \to L_2\otimes L_2$ is a nonzero homomorphism, then $\phi \approx_{M_2} d$ and that if ϕ is unital then $\phi \approx d$.

By (13) and [16, Theorem 5.3], when E is finite and regular $KH^1(L(E))$ is isomorphic to the group of extensions of the C^*-algebra of E by the algebra of compact operators. We shall see presently that $KH^1(L(E))$ is also related to algebra extensions

$$0 \to M_\infty \to \mathcal{E} \to L(E) \to 0.$$

One can form an abelian monoid of homotopy classes of such extensions (see Section 1.1); we write $\mathcal{E}xt(L(E))$ for its group completion. When E^0 is finite and E^1 is countable, there is a natural map

$$\mathcal{E}xt(L(E), R) \to kk_{-1}(L(E), R). \tag{20}$$

We show in Proposition 2.15 that, under the assumptions of Theorem 0.2, if in addition E has no sources and $R = \ell$, then the map (20) is onto. Moreover, we have the following.

Theorem 0.7. *Let E be a finite graph such that $L(E)$ is simple. Let R be either a division algebra or a K_0-regular purely infinite simple unital algebra. Then the natural map (20) is an isomorphism*

$$\mathcal{E}xt(L(E), R) \xrightarrow{\sim} kk_{-1}(L(E), R).$$

If furthermore $K_0(L(E))$ is torsion, then for every R as in Theorem 0.7 (in particular, for $R = \ell$ and for every purely infinite simple unital Leavitt path algebra R), we have

$$\mathcal{E}xt(L(E), R) = \operatorname{Ext}^1_{\mathbb{Z}}(K_0(L(E)), K_0(R)). \tag{21}$$

This thesis is organized as follows:

Preliminaries are given in Chapter 1. In the first section we recall some basic notions about algebraic homotopy, prove some elementary lemmas about it, and use them to define, for every pair of algebras A and R with R unital, a group $\mathcal{E}xt(A, R)$ of virtual homotopy classes of extensions of A by $M_\infty R$. Then we recall some basic properties of kk and quasi-homomorphisms and prove a few technical lemmas. In the last section we recall the definitions of Cohn and Leavitt path algebras.

In chapter 2 we deal with Leavitt path algebras over a commutative ground ring ℓ. All the results of this Chapter are extracted from [13]. Section 2.1 is devoted to the characterization of the image under $j : \mathrm{Alg}_\ell \to kk$ of the Cohn path algebra $C(E)$ of a graph E. The algebra $C(E)$ receives a canonical homomorphism $\phi : \ell^{(E^0)} \to C(E)$. We prove in Theorem 2.1 that the universal excisive, homotopy invariant, E-stable homology theory j maps ϕ to an isomorphism

$$j(\ell^{(E^0)}) \cong j(C(E)). \tag{22}$$

The proof uses quasi-homomorphisms, much in the spirit of Cuntz' proof of Bott periodicity for C^*-algebra K-theory. In Section 2.2 we use the result of Section 2.1 to obtain the distinguished triangle (19). With this triangle we get the results of Sections 2.3 and 2.4. In the first one, we prove Theorem 0.2 (Theorem 2.17) classifying Leavitt path algebras in kk. In the second one, we introduce a descending filtration $\{kk(L(E), R)^i : 0 \le i \le 2\}$ on $kk(L(E), R)$ for every algebra R and every unital Leavitt path algebra $L(E)$ and compute the slices $kk(L(E), R)^i / kk(L(E), R)^{i+1}$ (Theorem 2.21). We use this to prove the universal coefficient theorem (15) in Corollary 2.23 and the Künneth theorem in Proposition 2.26.

In Chapter 3 we deal with simple Leavitt path algebras of finite graphs over a field ℓ. All the results of this Chapter are extracted from [14]. In Section 3.1 we recall the results of Ara, Goodearl and Pardo on K-theory of purely infinite simple algebras. We also prove (Corollary 3.10) that if R is a K_1-regular, purely infinite simple and unital algebra, then $K_1(R)$ is isomorphic to the group $\pi_0(U(R))$ of polynomially connected components of the group of invertible elements of R. In Section 3.2 we prove the main technical Theorem 0.4 (Theorem 3.12) and we use it in Section 3.3 devoted to the proof of Theorem 0.1 (Theorem 3.33) which is the main result of this thesis. Sections 3.4 and 3.5 are devoted to prove Theorems 0.7 (Theorem 3.36) and 0.6 (Theorem 3.42).

Chapter 1

Preliminaries

Resumen del capítulo

En este capítulo damos una breve introducción a los conceptos básicos que utilizaremos a lo largo de la tesis. El mismo se divide en tres secciones.

Por un lado, en la primer sección, recordamos el concepto homotopía polinomial e introducimos la noción de M_2-homotopía. Para cada álgebra A y para cada ideal $B \triangleleft R$ de una C_2-álgebra R, le damos una estructura de monoide abeliano al conjunto de clases de M_2-homotopía $[A, B]_{M_2}$ de morfismos de A en B. Por otro lado, recordamos las construcciones de cono $\Gamma(R)$ y suspensión $\Sigma(R)$ de Wagoner para un álgebra R y mostramos que si R es unital y A es un álgebra arbitraria, los morfismos de álgebras $A \to \Gamma(R)$ clasifican las extensiones

$$0 \to M_\infty(R) \to \mathcal{E} \to A \to 0.$$

Con esto en mano, unificamos los conceptos antes vistos y definimos el grupo $\mathcal{E}xt(A, R)$ como la completación a grupo del monoide abeliano $[A, \Sigma(R)]^+_{M_2}$.

En la segunda sección recordamos los conceptos de teoría de homología, escisión, invarianza homotópica polinomial, estabilidad matricial y cuasi-morfismos. Revemos las propiedades de la teoría de homología universal $j : \mathrm{Alg}_\ell \to kk$ de Cortiñas-Thom y demostramos algunos lemas técnicos que necesitaremos a lo largo de la tesis.

En la última sección establecemos la notación básica sobre álgebras de caminos de Cohn y Leavitt.

1.1 Homotopy and extensions

Let ℓ be a commutative ring. Let Alg_ℓ be the category of associative, not necessarily unital algebras over ℓ. If $B \in \mathrm{Alg}_\ell$, we write $\mathrm{ev}_i : B[t] \to B$, $\mathrm{ev}_i(f) = f(i)$, $i = 0, 1$ for the evaluation map. Let $\phi_0, \phi_1 : A \to B$ be two algebra homomorphisms; an *elementary homotopy* from ϕ_0 to ϕ_1 is an algebra homomorphism $H : A \to B[t]$ such that $\mathrm{ev}_0 H = \phi_0$ and $\mathrm{ev}_1 H = \phi_1$. We say that two algebra homomorphisms $\phi, \psi : A \to B$ are *homotopic*, and write $\phi \approx \psi$, if for some $n \geq 1$ there is a finite sequence $\phi = \phi_0, \ldots, \phi_n = \psi$ such that for each $0 \leq i \leq n-1$ there is an elementary homotopy from ϕ_i to ϕ_{i+1}. We write

$$[A, B] = \hom_{\mathrm{Alg}_\ell}(A, B)/\approx$$

for the set of homotopy classes of homomorphisms $A \to B$.

Lemma 1.1. *Let A be a ring. Then the maps $\iota_2, \iota_2' : A \to M_2A$, $\iota_2(a) = \epsilon_{1,1} \otimes a$, $\iota_2'(a) = \epsilon_{2,2} \otimes a$ are homotopic.*

Proof. Let $R = \tilde{A}$ be the unitalization. Consider the element

$$U(t) = \begin{bmatrix} (1-t^2) & (t^3-2t) \\ t & (1-t^2) \end{bmatrix} \in \mathrm{GL}_2\, R[t].$$

Let $\mathrm{ad}(U(t)) : M_2(R[t]) \to M_2(R[t])$ be the conjugation map. Then $H = \mathrm{ad}(U(t))\iota_2 : A \to M_2A[t]$, satisfies $\mathrm{ev}_0 H = \iota_2$, $\mathrm{ev}_1 H = \iota_2'$. □

Let A and R be algebras, $\phi, \psi \in \hom_{\mathrm{Alg}_\ell}(A, R)$ and $\iota_2 : R \to M_2R$, as in Lemma 1.1. We say that ϕ and ψ are *M_2-homotopic*, and write $\phi \approx_{M_2} \psi$, if $\iota_2\phi \approx \iota_2\psi$. Put

$$[A, R]_{M_2} = \hom_{\mathrm{Alg}_\ell}(A, R)/\approx_{M_2}.$$

Let C be an algebra, $A, B \subset C$ subalgebras and $\mathrm{inc}_A : A \to C$, $\mathrm{inc}_B : B \to C$ the inclusion maps. Let $x, y \in C$ such that $yAx \subset B$ and $axya' = aa'$ for all $a, a' \in A$. Then

$$\mathrm{ad}(y, x) : A \to B, \quad \mathrm{ad}(y, x)(a) = yax \tag{1.1}$$

is a homomorphism of algebras, and we have the following.

Lemma 1.2. *Let A, B, C and x, y be as above. Then $\mathrm{inc}_B \mathrm{ad}(y, x) \approx_{M_2} \mathrm{inc}_A$. If moreover $A = B$ and $yA, Ax \subset A$, then $\mathrm{ad}(y, x) \approx_{M_2} \mathrm{id}_A$.*

Proof. Consider the diagonal matrices $\bar{y} = \mathrm{diag}(y, 1)$, $\bar{x} = \mathrm{diag}(x, 1) \in M_2\tilde{C}$. One checks that $a\bar{x}\bar{y}a' = aa'$ for all $a, a' \in M_2A$. Hence $\phi := \mathrm{ad}(\bar{y}, \bar{x}) : M_2A \to M_2C$ is a homomorphism. Moreover we have $\phi\iota_2 = \iota_2 \mathrm{inc}_B \mathrm{ad}(y, x)$ and $\phi\iota_2' = \iota_2' \mathrm{inc}_A$. Thus applying Lemma 1.1 twice, we get

$$\iota_2 \mathrm{inc}_B \mathrm{ad}(y, x) \approx \iota_2' \mathrm{inc}_A \approx \iota_2 \mathrm{inc}_A.$$

This proves the first assertion. Under the hypothesis of the second assertion, ϕ maps $M_2A \to M_2A$, and we have $\phi\iota_2 = \iota_2 \mathrm{ad}(y, x)$ and $\phi\iota_2' = \iota_2'$. The proof is immediate from this using Lemma 1.1. □

A *C_2-algebra* is a unital algebra R together with a unital algebra homomorphism from the Cohn algebra C_2 to R (see [10]). Equivalently, R is a unital algebra together with elements $x_1, x_2, y_1, y_2 \in R$ satisfying $y_i x_j = \delta_{i,j}$.

If R is a C_2-algebra the map

$$\boxplus : R \oplus R \to R, \quad a \boxplus b = x_1 a y_1 + x_2 b y_2 \tag{1.2}$$

is an algebra homomorphism. An *infinite C_2-algebra* is a C_2-algebra together with an endomorphism $\phi : R \to R$ such that for all $a \in R$ we have

$$a \boxplus \phi(a) = \phi(a).$$

In the following lemma and elsewhere, if M is an abelian monoid, we write M^+ for the group completion.

The main reason of why we are interested in C_2-algebras is the following.

Lemma 1.3. *Let A be an algebra, $R = (R, x_1, x_2, y_1, y_2)$ a C_2-algebra, and $B \triangleleft R$ an ideal. Then* (1.2) *induces an operation in $[A, B]_{M_2}$ which makes it into an abelian monoid whose neutral element is the zero homomorphism. If furthermore R is an infinite C_2-algebra, then $[A, R]^+_{M_2} = 0$.*

Proof. By Lemma 1.2, the homomorphisms $B \to B$, $b \mapsto x_i b y_i$ $(i = 0, 1)$ are M_2-homotopic to the identity. Hence to prove the first assertion, it suffices to show that (1.2) associative and commutative up to M_2-homotopy. This is straightforward from Lemma 1.2, since all diagrams involved commute up to a map of the form (1.1). The second assertion is clear. $\square$

Example 1.4. *Any purely infinite simple unital algebra (see Section 3.1 for a definition of purely infinite simple algebra) is a C_2-algebra, by* [5, Proposition 1.5].

For a set S and an algebra A, we write $M_S A$ for the algebra of those matrices $M : S \times S \to A$ which are finitely supported. We write $M_S = M_S \ell$ and $\epsilon_{s,t} \in M_S$ for the matrix whose only nonzero entry is a 1 at the (s, t)-spot $(s, t \in S)$. We also consider the algebra

$$\Gamma_S(R) := \{A : S \times S \to R \mid \#\operatorname{supp} A_{i,*}, \#\operatorname{supp} A_{*,i} < \infty\}$$

of those matrices which have finitely many nonzero coefficients in each row and column. If $\#S = n < \infty$, then $\Gamma_S = M_S = M_n$ is the usual matrix algebra. We use special notation for the case $S = \mathbb{N}$; we write M_∞ for $M_\mathbb{N}$ and Γ for $\Gamma_\mathbb{N}$. Observe that $M_\infty R$ is an ideal of $\Gamma(R)$. Put

$$\Sigma(R) = \Gamma(R)/M_\infty R. \tag{1.3}$$

The algebras $\Gamma(R)$ and $\Sigma(R)$ are Wagoner's *cone* and *suspension* algebras [31]. A *$*$-algebra* is an algebra R equipped with an involutive algebra homomorphism $R \to R^{op}$. For example ℓ is a $*$-algebra with trivial involution. If R is a $*$-algebra, the conjugate matricial transpose makes both $\Gamma_S(R)$ and $M_S R$ into $*$-algebras.

Example 1.5. *If R is a unital algebra, its cone $\Gamma(R)$ is an infinite C_2-algebra ([31]) and $\Sigma(R)$ is a C_2-algebra. For every algebra R, $\Gamma(R) \lhd \Gamma(\tilde{R})$ and $\Sigma(R) \lhd \Sigma(\tilde{R})$. By definition, we have an exact sequence*

$$0 \to M_\infty R \to \Gamma(R) \to \Sigma(R) \to 0. \tag{1.4}$$

Lemma 1.6. *Let R be a unital algebra and let $\mathcal{E}$ be an algebra containing $M_\infty R$ as an ideal. Then there exists a unique algebra homomorphism $\psi = \psi_{\mathcal{E}} : \mathcal{E} \to \Gamma(R)$ which restricts to the identity on $M_\infty R$.*

Proof. If $a \in \mathcal{E}$ then for each $i, j \in \mathbb{N}$ there is a unique element $a_{i,j} \in R$ such that $(\epsilon_{i,i} \otimes 1)a(\epsilon_{j,j} \otimes 1) = \epsilon_{i,j} \otimes a_{i,j}$. One checks that $\psi : \mathcal{E} \to \Gamma(R)$, $\psi(a) = (a_{i,j})$ satisfies the requirements of the lemma. □

It follows from Lemma 1.6 that if R is unital then every exact sequence of algebras

$$0 \to M_\infty R \to \mathcal{E} \to A \to 0 \tag{1.5}$$

induces a homomorphism $\psi : A \to \Sigma(R)$ and that (1.5) is isomorphic to the pullback along ψ of (1.4). Hence we may regard $[A, \Sigma(R)]_{M_2}$ as the abelian monoid of homotopy classes of all sequences (1.5). Put

$$\mathcal{E}xt(A, R) = [A, \Sigma(R)]^+_{M_2}, \quad \mathcal{E}xt(A) = \mathcal{E}xt(A, \ell). \tag{1.6}$$

Observe that, by Lemma 1.3, any sequence (1.5) which is split by an algebra homomorphism $A \to \mathcal{E}$ maps to zero in $\mathcal{E}xt(A, R)$.

1.2 Homology theories and algebraic bivariant K-theory

Let $\mathcal{T}$ be a triangulated category and Ω the inverse suspension functor of $\mathcal{T}$. A *homology theory* with values in $\mathcal{T}$ is a functor $X : \mathrm{Alg}_\ell \to \mathcal{T}$. An *extension* of algebras is a short exact sequence of algebra homomorphisms

$$(E) : 0 \to A \to B \to C \to 0 \tag{1.7}$$

which is ℓ-linearly split. We write $\mathcal{E}$ for the class of all extensions. An *excisive homology theory* for ℓ-algebras with values in $\mathcal{T}$ consists of a functor $X : \mathrm{Alg}_\ell \to \mathcal{T}$, together with a collection $\{\partial_E : E \in \mathcal{E}\}$ of maps $\partial_E^X = \partial_E \in \hom_{\mathcal{T}}(\Omega X(C), X(A))$. The maps ∂_E are to satisfy the following requirements.

i) For all $E \in \mathcal{E}$ as above,

$$\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

is a distinguished triangle in $\mathcal{T}$.

ii) If

$$\begin{tikzcd}
(E): & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\
(E'): & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'
\end{tikzcd}$$

is a map of extensions, then the following diagram commutes

$$\begin{tikzcd}
\Omega X(C) \arrow[r, "\partial_E"] \arrow[d, "\Omega X(\gamma)"'] & X(A) \arrow[d, "X(\alpha)"] \\
\Omega X(C') \arrow[r, "\partial_{E'}"'] & X(A)
\end{tikzcd}$$

Observe that if $X : \mathrm{Alg}_\ell \to \mathcal{T}$ is excisive and $A, B \in \mathrm{Alg}_\ell$, then the canonical map $X(A) \oplus X(B) \to X(A \oplus B)$ is an isomorphism. Let I be a set. We say that a homology theory $X : \mathrm{Alg}_\ell \to \mathcal{T}$ is *I-additive* if first of all direct sums of cardinality $\leq \#I$ exist in $\mathcal{T}$ and second of all the map

$$\bigoplus_{j\in J} X(A_j) \to X(\bigoplus_{j\in J} A_j)$$

is an isomorphism for any family of algebras $\{A_j : j \in J\} \subset \mathrm{Alg}_\ell$ with $\#J \leq \#I$.

We say that the functor $X : \mathrm{Alg}_\ell \to \mathcal{T}$ is *homotopy invariant* if for every $A \in \mathrm{Alg}_\ell$, X maps the inclusion $A \subset A[t]$ to an isomorphism.

Let S be a set, $s \in S$ and let

$$\iota_s : A \to M_S A, \quad \iota_s(a) = \epsilon_{s,s} \otimes a \qquad (A \in \mathrm{Alg}_\ell). \tag{1.8}$$

Call X *M_S-stable* if for every $A \in \mathrm{Alg}_\ell$, it maps $\iota_s : A \to M_S A$ to an isomorphism. This definition is independent of the element $s \in S$, by the argument of [12, Lemma 2.2.4]. One can further show, using [12, Proposition 2.2.6] and [24, Example 5.2.6] that if S is infinite and X is M_S-stable, and T is a set such that $\#T \leq \#S$, then X is M_T-stable.

Definition 1.7. *Let $A, B \in \mathrm{Alg}_\ell$. A* quasi-homomorphism *from A to B is a pair of homomorphisms $\phi, \psi : A \to D \in \mathrm{Alg}_\ell$, where D contains B as an ideal, such that*

$$\phi(a) - \psi(a) \in B \qquad (a \in A).$$

We use the notation

$$(\phi, \psi) : A \to D \rhd B.$$

Two algebra homomorphisms $\phi, \psi : A \to B$ are said to be *orthogonal*, in symbols $\phi \perp \psi$, if $\phi(x)\psi(y) = 0 = \psi(x)\phi(y)$ $(x, y \in A)$. If $\phi \perp \psi$ then $\phi + \psi$ is an algebra homomorphism.

Proposition 1.8. *([17, Proposition 3.3]) Let $X : \mathrm{Alg}_\ell \to \tau$ be an excisive homology theory and let $(\phi, \psi) : A \to D \rhd B$ be a quasi-homomorphism. Then, there is an induced map*

$$X(\phi, \psi) : X(A) \to X(B)$$

which satisfies the following naturality conditions:

1. $X(\phi, 0) = X(\phi)$.

2. $X(\phi, \psi) = -X(\psi, \phi)$.

3. *If (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are quasi-homomorphisms $A \to D \rhd B$ with $\phi_1 \perp \phi_2$ and $\psi_1 \perp \psi_2$, then $(\phi_1 + \phi_2, \psi_1 + \psi_2)$ is a quasi-homomorphism and*

$$X(\phi_1 + \phi_2, \psi_1 + \psi_2) = X(\phi_1, \psi_1) + X(\phi_2, \psi_2).$$

4. $X(\phi, \phi) = 0$.

5. *If $\alpha : C \to A$ is an ℓ-algebra homomorphism, then*

$$X(\phi\alpha, \psi\alpha) = X(\phi, \psi)X(\alpha).$$

6. *If $\eta : D \to D'$ is an ℓ-algebra homomorphism which maps B into an ideal $B' \lhd D'$, then*

$$X(\eta\phi, \eta\psi) = X(\eta|_B)X(\phi, \psi).$$

7. *Let $H = (H^+, H^-) : A \to D[t] \rhd B[t]$ with $ev_0 \circ H = (\phi^+, \phi^-)$ and $ev_1 \circ H = (\psi^+, \psi^-)$. If, in addition, X is homotopy invariant then*

$$X(\phi^+, \phi^-) = X(\psi^+, \psi^-).$$

8. *Let (ψ, ϱ) be another quasi-homomorphism $A \to D \rhd B$. Then (ϕ, ϱ) is a quasi-homomorphism and*

$$X(\phi, \varrho) = X(\phi, \psi) + X(\psi, \varrho).$$

The excisive homology theories form a category, where a homomorphism between the theories $X : \mathrm{Alg}_\ell \to \mathcal{T}$ and $Y : \mathrm{Alg}_\ell \to \mathcal{U}$ is a triangulated functor $G : \mathcal{T} \to \mathcal{U}$ such that

$$\begin{tikzcd} \mathrm{Alg}_\ell \arrow[r, "X"] \arrow[dr, "Y"'] & \mathcal{T} \arrow[d, "G"] \\ & \mathcal{U} \end{tikzcd}$$

commutes, and such that for every extension (1.7) in $\mathcal{E}$, the natural isomorphism $\phi : G(\Omega X(C)) \to \Omega Y(C)$ makes the following into a commutative diagram

$$\begin{tikzcd} G(\Omega X(C)) \arrow[r, "G(\partial_E^X)"] \arrow[d, "\phi"'] & Y(A) \\ \Omega Y(C). \arrow[ur, "\partial_E^Y"'] & \end{tikzcd}$$

In [15] a functor $j : \mathrm{Alg}_\ell \to kk$ was defined which is an initial object in the full subcategory of those excisive homology theories which are homotopy invariant and M_∞-stable. It was shown in [24] that, for any fixed infinite set S, by a slight variation of the construction of [15] one obtains an initial object in the full subcategory of those excisive and homotopy invariant homology theories which are M_S-stable. Starting in the next section we shall fix S and use j and kk for the universal excisive, homotopy invariant and S-stable homology theory and its target triangulated category. Moreover, we shall often omit j from our notation, and say, for example, that an algebra homomorphism is an isomorphism in kk or that a diagram in Alg_ℓ commutes in kk or that a sequence of algebra maps

$$A \to B \to C$$

is a triangle in kk to mean that j applied to the corresponding morphism, diagram or sequence is an isomorphism, a commutative diagram or a distinguished triangle. Also, since as explained above, in kk the corner inclusion $\iota_s : A \to M_S A$ is independent of s, we shall simply write ι for $j(\iota_s)$.

The loop functor Ω in kk and its inverse have a concrete description as follows. Let $\Omega_1 = t(t-1)\ell[t]$, $\Omega_{-1} = (t-1)\ell[t, t^{-1}]$. For $A \in \mathrm{Alg}_\ell$ we have

$$\Omega^{\pm 1} j(A) = j(\Omega_{\pm 1} \otimes A). \tag{1.9}$$

For $A, B \in \mathrm{Alg}_\ell$ and $n \in \mathbb{Z}$, set

$$kk_n(A, B) = \hom_{kk}(j(A), \Omega^n j(B)), \qquad kk(A, B) = kk_0(A, B). \tag{1.10}$$

The groups $kk_*(A, B)$ are the *bivariant K-theory* groups of the pair (A, B). Setting $A = \ell$ in (1.10) we recover the homotopy algebraic K-groups of Weibel [32]; there is a natural isomorphism ([15, Theorem 8.2.1], [24, Theorem 5.2.20])

$$kk_*(\ell, B) \xrightarrow{\sim} KH_*(B) \qquad (B \in \mathrm{Alg}_\ell). \tag{1.11}$$

Example 1.9. *Let T be an infinite set and $j : \mathrm{Alg}_\ell \to kk$ the universal homotopy invariant, excisive and M_T-stable homology theory. If $R \in \mathrm{Alg}_\ell$, then the functor $j((-) \otimes R) : \mathrm{Alg}_\ell \to kk$ is again a homotopy invariant, M_T-stable, excisive homology theory. Hence it gives rise to a triangulated functor $kk \to kk$. In particular, triangles in kk are preserved by tensor products. Moreover, the tensor product induces a "cup product"*

$$\cup : kk(A, B) \otimes kk(R, S) \to kk(A \otimes R, B \otimes S), \quad \xi \cup \eta = (B \otimes \eta) \circ (\xi \otimes R).$$

Remark 1.10. *Even though KH is I-additive for every set I, the universal functor $j : \mathrm{Alg}_\ell \to kk$ is not known to be infinitely additive.*

Lemma 1.11. *Let $\{A_i : i \in I\} \subset \mathrm{Alg}_\ell$ be a family of algebras, $A = \bigoplus_{i\in I} A_i$, T a set, $\mathfrak{j} : I \to T$ a function and $v \in T$. Then the homomorphism*

$$\iota_{\mathfrak{j}} : A \to M_T A, \quad \iota_{\mathfrak{j}}(\sum_i a_i) = \sum_{i\in I} \epsilon_{\mathfrak{j}(i),\mathfrak{j}(i)} \otimes a_i$$

is homotopic to ι_v.

Proof. Because $(M_T A)[x] = \bigoplus_{i\in I}(M_T A_i[x])$, we may assume that I has a single element, in which case the lemma follows using a rotational homotopy, as in the proof of Lemma 1.1. □

Lemma 1.12. *Let $\{S_i : i \in I\}$ be a family of sets, $\sigma_i : S_i \to S_i$ an injective map, $(\sigma_i)_* : M_{S_i} \to M_{S_i}$, $(\sigma_i)_*(\epsilon_{s,t}) = \epsilon_{\sigma_i(s),\sigma_i(t)}$ the induced homomorphism, $D = \bigoplus_{i\in I} M_{S_i}$, and $\sigma_* = \bigoplus_{i\in I}(\sigma_i)_* : D \to D$. If $X : \mathrm{Alg}_\ell \to \mathcal{T}$ is M_2-invariant, then $X(\sigma_*)$ is the identity map.*

Proof. The map σ_i induces an ℓ-module homomorphism $\ell^{(S_i)} \to \ell^{(S_i)}$ whose matrix $[\sigma_i]$ is an element of the ring Γ_{S_i} of Section 1.1. Let $[\sigma_i]^*$ be the transpose matrix; we have $[\sigma_i]^*[\sigma_i] = 1$, and $(\sigma_i)_*(a) = [\sigma_i]a[\sigma_i^*]$ ($a \in M_{S_i}$). Hence for $[\sigma] = \bigoplus_{i\in I}[\sigma_i] \in R = \bigoplus_{i\in I}\Gamma_{S_i}$, we have $\sigma_*(a) = [\sigma]a[\sigma]^*$. Since $D \triangleleft R$, $X(\sigma_*)$ is the identity by [12, Proposition 2.2.6]. □

Proposition 1.13. *Let $\{S_i : i \in I\}$ be a family of sets, $v_i \in S_i$ and $S = \sqcup_{i\in I} S_i$. Let $f = \bigoplus_{i\in I} \iota_{v_i} : \ell^{(I)} \to \oplus_{i\in I} M_{S_i}$. Let T be an infinite set with $\#T \geq \#S$. Let $j : \mathrm{Alg}_\ell \to kk$ be the universal excisive, homotopy invariant and M_T-stable homology theory. Then $j(f)$ is an isomorphism.*

Proof. Put $D = \bigoplus_{i\in I} M_{S_i}$. Let $\mathrm{inc} : D \to M_S\ell^{(I)}$ be the inclusion. By Lemma 1.11, the composite $\mathrm{inc}\, f$ equals the canonical inclusion ι in kk. Next let $g = (M_S f)\,\mathrm{inc} : D \to M_S D$. We have $g(\epsilon_{\alpha,\beta}) = \epsilon_{\alpha,\beta} \otimes \epsilon_{v_i,v_i}$ ($\alpha, \beta \in S_i$). For each $i \in I$ extend the coordinate permutation map $S_i \times \{v_i\} \to \{v_i\} \times S_i$, to a bijection $\sigma_i : S \times S_i \to S \times S_i$, and let $(\sigma_i)_*$ be the induced automorphism of $M_S M_{S_i} \cong M_{S\times S_i}$. Consider the automorphism $\sigma_* = \bigoplus_{i\in I}(\sigma_i)_* : M_S D \to M_S D$; by Lemmas 1.11 and 1.12, $\iota = j(\sigma_* g) = j(g)$. From what we have just seen and Example 1.9, in kk the following diagram commutes and its horizontal arrows are isomorphisms.

$$\begin{tikzcd}
\ell^{(I)} \arrow[r, "\iota"] \arrow[d, "f"'] & M_S\ell^{(I)} \arrow[r, "M_S\iota"] \arrow[d, "M_S f"'] & M_S M_S \ell^{(I)} \arrow[d, "M_S M_S f"] \\
D \arrow[r, "\iota"'] \arrow[ur, "\mathrm{inc}"] & M_S D \arrow[r, "M_S\iota"'] \arrow[ur, "M_S\,\mathrm{inc}"] & M_S M_S D
\end{tikzcd}$$

It follows that $M_S f$ and f are isomorphisms in kk. □

1.3 Cohn and Leavitt path algebras

A directed *graph* is a quadruple $E = (E^0, E^1, r, s)$ where E^0 and E^1 are the sets of vertices and edges, and r and s are the *range* and *source* functions $E^1 \to E^0$. We call E *finite* if both E^0 and E^1 are finite. A vertex $v \in E^0$ is a *sink* if $s^{-1}(v) = \emptyset$ and is an *infinite emitter* if $s^{-1}(v)$ is infinite. A vertex v is *singular* if it is either a sink or an infinite emitter; we call v *regular* if it is not singular. A vertex $v \in E^0$ is a *source* if $r^{-1}(v) = \emptyset$. We write $\mathrm{sink}(E)$, $\mathrm{inf}(E)$ and $\mathrm{sour}(E)$ for the sets of sinks, infinite emitters, and sources, and $\mathrm{sing}(E)$ and $\mathrm{reg}(E)$ for those of singular and of regular vertices.

A finite *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case $|\mu| := n$ is the *length* of μ. We view the vertices of E as paths of length 0. Write $\mathcal{P}(E)$ for the set of all finite paths in E. The range and source functions r, s extend to $\mathcal{P}(E) \to E^0$ in the obvious way. An edge f is an *exit* for a path $\mu = e_1 \dots e_n$ if there exist i such that $s(f) = s(e_i)$ and $f \neq e_i$. A path $\mu = e_1 \dots e_n$ with $n \geq 1$ is a *closed path* at v if $s(e_1) = r(e_n) = v$. A closed path $\mu = e_1 \dots e_n$ at v is a *cycle* at v if $s(e_j) \neq s(e_i)$ for $i \neq j$.

The *Cohn path algebra* $C(E)$ of a graph E is the quotient of the free associative ℓ-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$, subject to the relations:

(V) $v \cdot w = \delta_{v,w} v$.

(E1) $s(e) \cdot e = e = e \cdot r(e)$.

(E2) $r(e) \cdot e^* = e^* = e^* \cdot s(e)$.

(CK1) $e^* \cdot f = \delta_{e,f}\, r(e)$.

The algebra $C(E)$ is in fact a $*$-algebra; it is equipped with an involution $* : C(E) \to C(E)^{op}$ which fixes vertices and maps $e \mapsto e^*$ ($e \in Q^1$). Condition V says that the vertices of E are orthogonal idempotents in $C(E)$. Hence the subspace generated by E^0 is a subalgebra of $C(E)$, isomorphic to the algebra $\ell^{(E^0)}$ finitely supported functions $E^0 \to \ell$. For $v \in E^0$, let $\chi_v \in \ell^{(E^0)}$ be the characteristic function of $\{v\}$. We have a monomorphism

$$\varphi : \ell^{(E^0)} \to C(E), \quad \varphi(\chi_v) = v. \tag{1.12}$$

Observe that if E^0 is finite, then $\ell^{(E^0)} = \ell^{E^0}$ is the algebra of all functions $E^0 \to \ell$.

Associate an element $m_v \in C(E)$ to each $v \in E^0 \setminus \mathrm{inf}(E)$ as follows

$$m_v = \begin{cases} \sum_{e \in s^{-1}(v)} ee^* & \text{if } v \in \mathrm{reg}(E) \\ 0 & \text{if } v \in \mathrm{sink}(E). \end{cases}$$

Observe that m_v satisfies the following identities:

$$m_v = m_v^*, \;\; m_v^2 = m_v, \;\; m_v w = \delta_{w,v} m_v, \;\; m_v e = \delta_{v,s(e)} e \quad (e \in E^1). \tag{1.13}$$

Let $C^m(E)$ be the $*$-algebra obtained from $C(E)$ by formally adjoining an element m_v for each $v \in \inf(E)$ subject to the identities (1.13). We have a canonical $*$-homomorphism

$$\mathrm{can} : C(E) \to C^m(E). \tag{1.14}$$

Let $\mathcal{P} = \mathcal{P}(E)$. For $v \in E^0$, set

$$\mathcal{P}_v = \{\mu \in \mathcal{P}(E) \mid r(\mu) = v\}, \quad \mathcal{P}^v = \{\mu \in \mathcal{P} \mid s(\mu) = v\}. \tag{1.15}$$

Let $\Gamma_{\mathcal{P}}$ be the ring introduced in Section 1.1. Using the notation (1.15) in the summation indexes, define a $*$-homomorphism

$$\begin{gathered}\rho : C^m(E) \to \Gamma_{\mathcal{P}}, \\ \rho(v) = \sum_{\alpha\in\mathcal{P}^v} \epsilon_{\alpha,\alpha}, \quad \rho(e) = \sum_{\alpha\in\mathcal{P}^{r(e)}} \epsilon_{e\alpha,\alpha}, \quad (v \in E^0, e \in E^1) \\ \rho(m_w) = \sum_{\alpha\in\mathcal{P}^w, |\alpha|\geq 1} \epsilon_{\alpha,\alpha} \quad (w \in \inf(E)).\end{gathered} \tag{1.16}$$

Lemma 1.14. *The maps* (1.14) *and* (1.16) *are monomorphisms.*

Proof. It is well-known that the set

$$\mathcal{B}_1 = \{\alpha\beta^* \mid \alpha, \beta \in \mathcal{P}, r(\alpha) = r(\beta)\}$$

is a basis of $C(E)$ ([1, Proposition 1.5.6]). Set

$$\mathcal{B}_2 = \{\alpha m_v \beta^* \mid \alpha, \beta \in \mathcal{P}_v, v \in \inf(E)\}.$$

It follows from (1.13) that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ generates $C^m(E)$ as an ℓ-module. It is clear that ρ is injective on $\mathcal{B}$; hence it suffices to show that the set $\rho(\mathcal{B}) \subset \Gamma_{\mathcal{P}}$ is ℓ-linearly independent. Let $\mathcal{F} \subset \mathcal{B}$ be a finite set and $c : \mathcal{F} \to \ell \setminus \{0\}$ a function such that

$$\sum_{x\in\mathcal{F}} c_x x = 0.$$

Let $Q = \{(\alpha, \beta) \in \mathcal{P}^2 \mid r(\alpha) = r(\beta)\}$; give Q a partial order by setting $(\alpha, \beta) \geq (\alpha', \beta')$ if and only if there exists $\theta \in \mathcal{P}_{r(\alpha)}$ such that $\alpha' = \alpha\theta$, $\beta' = \beta\theta$. Let $f : \mathcal{B} \to Q$, $f(\alpha\beta^*) = (\alpha, \beta)$, $f(\alpha m_v \beta^*) = (\alpha, \beta)$. Assume that $\mathcal{F} \neq \emptyset$. Then $f(\mathcal{F})$ has a maximal element (α, β). If $\alpha\beta^* \in \mathcal{F}$, then $\rho(\alpha\beta^*)$ is the only matrix in $\rho(\mathcal{F})$ whose (α, β) entry is nonzero. Thus $c_{\alpha\beta^*} = 0$, a contradiction. Hence $v = r(\alpha) \in \inf(E)$, $\alpha\beta^* \notin \mathcal{F}$ and $\alpha m_v \beta^* \in \mathcal{F}$. Then $f(\mathcal{F} \setminus \{\alpha m_v \beta^*\})$ contains only finitely many elements of the form $(\alpha e, \beta e)$ with $e \in s^{-1}(v)$. However $\rho(\alpha m_v \beta^*)_{\alpha e, \beta e} = 1$ for every $e \in s^{-1}(v)$. Thus $c_{\alpha m_v \beta^*} = 0$, which again is a contradiction. Hence $\mathcal{F}$ must be empty; this concludes the proof. □

Remark 1.15. *By Lemma 1.14 we may identify $C^m(E)$ with its image in $\Gamma_{\mathcal{P}}$. Under this identification, the formula*

$$m_v = \sum_{e\in s^{-1}(v)} ee^*$$

holds for every $v \in E^0$.

Set

$$C^m(E) \ni q_v = v - m_v \quad (v \in E^0). \tag{1.17}$$

Consider the following ideals of $C^m(E)$

$$\mathcal{K}(E) = \langle q_v \mid v \in \operatorname{reg}(E)\rangle \subset \hat{\mathcal{K}}(E) = \langle q_v \mid v \in E^0\rangle. \tag{1.18}$$

For $v \in E^0$ let $q_v \in C(E)$ be the element (1.17). The *Leavitt path algebra $L(E)$* is the quotient of $C(E)$ modulo the relation

$$(CK2) \quad q_v = 0 \quad (v \in \operatorname{reg}(E)).$$

In other words, for the ideal $K(E) \triangleleft C(E)$ of (1.18), we have a short exact sequence

$$0 \to \mathcal{K}(E) \to C(E) \to L(E) \to 0. \tag{1.19}$$

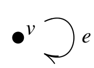
It follows from [1, Proposition 1.5.11] that the sequence (1.19) is ℓ-linearly split, and is thus an algebra extension in the sense of Section 1.2.

Example 1.16. *Let A_n be the graph*

$$\bullet^{v_1} \xrightarrow{f_1} \bullet^{v_2} \xrightarrow{f_2} \bullet^{v_3} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{f_{n-1}} \bullet^{v_n}.$$

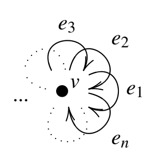
Then, there is an isomorphism $L(A_n) \cong M_n$ (see [1, Section 1.3]).

Example 1.17. *Let R_1 be the graph*



Then, there is an isomorphism $L(R_1) \cong \ell[x, x^{-1}]$ (see [1, Section 1.3]).

Example 1.18. *Let R_n with $n \geq 2$ be the graph*



Then, $L(R_n) = L(1, n)$ the Leavitt algebra of type $(1, n)$ introduced in [21] (see [1, Section 1.3]).

The *adjacency matrix* $A'_E \in \mathbb{Z}^{((E^0\setminus\inf(E))\times E^0)}$ is the matrix whose entries are given by

$$(A'_E)_{v,w} = \#\{e \in E^1 \ : \ s(e) = v \text{ and } r(e) = w\}.$$

The *reduced adjacency matrix* is the matrix $A_E \in \mathbb{Z}^{(\operatorname{reg}(E))\times E^0)}$ which results from A_E upon removing the rows corresponding to sinks. We also consider the matrix

$$I \in \mathbb{Z}^{(E^0\times \operatorname{reg}(E))}, \quad I_{v,w} = \delta_{v,w}.$$

Chapter 2

Algebraic bivariant K-theory and Leavitt path algebras

Resumen del capítulo

En este capítulo trabajaremos con álgebras de camino de Leavitt sobre un anillo commutativo ℓ.

La primera sección está dedicada a la caracterización de la imagen por $j : \mathrm{Alg}_\ell \to kk$ del álgebras de camino de Cohn $C(E)$ de grafo E. El álgebra $C(E)$ recibe un morfismo canónico $\phi : \ell^{(E^0)} \to C(E)$. Probamos, en el Teorema 2.1, que la teoría de homología escisiva, invariante por homotopía polinomial y E-estable universal j manda al morfismo ϕ a un isomorfismo.

$$j(\ell^{(E^0)}) \cong j(C(E)). \tag{2.1}$$

La demostración usa los cuasi-morfismos mencionados en el capítulo anterior, en el espíritu de la demostración de Cuntz de la periodicidad de Bott en K-teoría para C^*-álgebras. La demostración de este hecho está divida en cuatro subsecciones con tres lemas intercalados.

En la segunda sección usamos los resultados de la sección anterior junto con la extensión (1.19) para obtener el triángulo (8). También mostramos un resultado interesante que se desprende del triángulo (8) (ver Proposición 2.11) que dice que un morfismo $\theta \in kk(L(E), L(F))$ entre álgebras de camino de Leavitt unitales induce un isomorfismo en KH_i para $i = 0, 1$ entonces es un isomorfismo en kk.

En la Sección 2.3, bajo mínimas hipótesis en ℓ, demostramos el teorema de estructura (Teorema 2.17) que dice que para toda álgebra de caminos de Leavitt unital existe una única descomposición

$$j(L(E)) \cong j(L_0^s \oplus L_1^r \oplus \bigoplus_{i=1}^{n} L_{d_i+1})$$

donde L_n es el álgebra de Leavitt de tipo $(1, n)$, $d_1, \ldots, d_n$ son los coeficientes de estructura de $KH_0(L(E))$ y s y r están relacionados con los rangos de los grupos $KH_0(L(E))$ y $KH^1(L(E))$.

Por último, relacionamos este resultado con los resultados obtenidos por Ruiz y Tomforde (ver [29]) para grafos E con finitos vértices e infinitas aristas.

En la última sección introducimos una filtración descendiente $\{kk(L(E),R)^i : 0 \le i \le 2\}$ en $kk(L(E),R)$ para toda álgebra R y toda álgebra de camino de Leavitt unital $L(E)$ y calculamos los cocientes $kk(L(E),R)^i/kk(L(E),R)^{i+1}$ (Teorema 2.21). Con esto en mano, demostramos:

1) Existe una sucesión exacta corta (Teorema de coeficientes universales)

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(KH_0(L(E)), KH_{n+1}(R)) \to kk_n(L(E),R) \xrightarrow{[KH_0,\gamma^* KH_1]} \operatorname{Hom}(KH_0(L(E)), KH_n(R)) \oplus \operatorname{Hom}(\operatorname{Ker}(I - A^t_E), KH_{n+1}(R)) \to 0.$$

2) Existe una sucesión exacta corta (Fórmula de Künneth)

$$0 \to KH^1(L(E)) \otimes KH_{n+1}(R) \oplus \operatorname{Ker}(I - A_E) \otimes KH_n(R) \to kk(L(E),R) \to \operatorname{Tor}^1_{\mathbb{Z}}(KH^1(L(E)), KH_n(R)) \to 0.$$

2.1 The Cohn path algebra in kk

We shall say that a homology theory is *E-stable* if it is stable with respect to a set of cardinality $\#(E^0 \sqcup E^1 \sqcup \mathbb{N})$.

Recall (see equation 1.12) that we have a monomorphism

$$\varphi : \ell^{(E^0)} \to C(E), \quad \varphi(\chi_v) = v.$$

The main result of this section is the following theorem.

Theorem 2.1. *Let φ be the algebra homomorphism (1.12) and let $j : \mathrm{Alg}_\ell \to kk$ be the universal excisive, homotopy invariant and E-stable homology theory. Then $j(\varphi)$ is an isomorphism.*

Corollary 2.2. *Let E and F be graphs and $j : \mathrm{Alg}_\ell \to kk$ the universal excisive, homotopy invariant and $E \sqcup F$-stable homology theory. Assume that $KH_0(\ell) \cong \mathbb{Z}$. Then $C(E)$ and $C(F)$ are isomorphic in kk if and only if $\#E^0 = \#F^0$.*

Proof. By Theorem 2.1, $C(E)$ and $C(F)$ are isomorphic in kk if and only if $\ell^{(E^0)}$ and $\ell^{(F^0)}$ are. If $\#E^0 = \#F^0$ then $\ell^{(E^0)}$ and $\ell^{(F^0)}$ are isomorphic in Alg_ℓ, and therefore also in kk. Assume conversely that $\ell^{(E^0)}$ and $\ell^{(F^0)}$ are isomorphic in kk. Then in view of (1.11) and of the hypothesis that $KH_0(\ell) \cong \mathbb{Z}$, we have $\#E^0 = \#F^0$. ◻

The proof of Theorem 2.1 is organized in four parts, with three lemmas interspersed.

2.1.1 Proof of Theorem 2.1, part I

Recall from Section 1.3 that we have elements $q_v \in C^m(E)$ and ideals of $C^m(E)$

$$\mathcal{K}(E) = \langle q_v \mid v \in \operatorname{reg}(E)\rangle \subset \hat{\mathcal{K}}(E) = \langle q_v \mid v \in E^0\rangle. \tag{2.2}$$

One checks, using [1, Proposition 1.5.11] that the maps

$$M_{\mathcal{P}_v} \to \hat{\mathcal{K}}(E), \quad \epsilon_{\alpha,\beta} \mapsto \alpha q_v \beta^* \quad (v \in E^0)$$

assemble to an isomorphism

$$\bigoplus_{v\in E^0} M_{\mathcal{P}_v} \xrightarrow{\sim} \hat{\mathcal{K}}(E). \tag{2.3}$$

Observe that (2.3) restricts to an isomorphism

$$\bigoplus_{v\in \operatorname{reg}(E)} M_{\mathcal{P}_v} \xrightarrow{\sim} \mathcal{K}(E). \tag{2.4}$$

Let $\hat{\iota} : \ell^{(E^0)} \to \hat{\mathcal{K}}(E)$ be the homomorphism that sends the canonical basis element χ_v to q_v and let $\xi : C(E) \to C^m(E)$ be the $*$-homomorphism determined by

$$\xi(v) = m_v, \quad \xi(e) = em_{r(e)}.$$

One checks that (can, ξ) is a quasi-homomorphism $C(E) \to C^m(E) \triangleright \hat{\mathcal{K}}(E)$. From the equality $\mathrm{can}\,\varphi = \xi\varphi + \hat{\iota}$ and items (1), (3), (4) and (5) of Proposition 1.8, it follows that

$$j(\mathrm{can}, \xi) j(\varphi) = j(\mathrm{can}\,\varphi, \xi\varphi) = j(\xi\varphi + \hat{\iota}, \xi\varphi) = j(\xi\varphi, \xi\varphi) + j(\hat{\iota}, 0) = j(\hat{\iota}).$$

By Proposition 1.13, $\hat{\iota}$ is an isomorphism in kk. Hence

$$j(\hat{\iota})^{-1} j(\mathrm{can}, \xi) j(\varphi) = 1_{j(\ell^{(E^0)})}.$$

It remains to show that

$$j(\varphi) j(\hat{\iota})^{-1} j(\mathrm{can}, \xi) = 1_{C(E)}. \tag{2.5}$$

Let $\mathcal{P} = \mathcal{P}(E)$; consider the algebra $M_{\mathcal{P}}$ of finite matrices indexed by $\mathcal{P}$. Let $\hat{\varphi} : \hat{\mathcal{K}}(E) \to M_{\mathcal{P}} C(E)$ be the homomorphism that sends $\alpha q_v \beta^*$ to $\epsilon_{\alpha,\beta} \otimes v$, where $\epsilon_{\alpha,\beta}$ is the matrix unit. We shall need a twisted version $\hat{\iota}_\tau$ of $\hat{\iota}$; this is the $*$-homomorphism

$$\hat{\iota}_\tau : C(E) \to M_{\mathcal{P}} C(E), \quad \hat{\iota}_\tau(v) = \epsilon_{v,v}, \quad \hat{\iota}_\tau(e) = \epsilon_{s(e),r(e)} \otimes e \qquad (v \in E^0, e \in E^1). \tag{2.6}$$

We have a commutative diagram

$$\begin{tikzcd} \ell^{(E^0)} \arrow[r, "\hat{\iota}"] \arrow[d, "\varphi"'] & \hat{\mathcal{K}}(E) \arrow[d, "\hat{\varphi}"] \\ C(E) \arrow[r, "\hat{\iota}_\tau"'] & M_{\mathcal{P}} C(E) \end{tikzcd} \tag{2.7}$$

Lemma 2.3. *Let $\alpha \in \mathcal{P}$ and let $\iota_\alpha : C(E) \to M_{\mathcal{P}}C(E)$ as in (1.8). Then ι_α and the map $\hat{\iota}_\tau$ of (2.6) induce the same isomorphism in kk.*

Proof. Because j is E-stable, it is $M_{\mathcal{P}}$-stable, whence ι_α is an isomorphism and does not depend on α. Hence we may and do assume that $\alpha = w \in E^0$. Because j is homotopy invariant, it is enough to find a polynomial homotopy between ι_w and $\hat{\iota}_\tau$. For each $v \in E^0 \setminus \{w\}$ set

$$\begin{aligned} A_v =& [(1-t^2)\epsilon_{w,w} + (t^3 - 2t)\epsilon_{w,v} + t\epsilon_{v,w} + (1-t^2)\epsilon_{v,v}] \otimes v, \\ B_v =& [(1-t^2)\epsilon_{w,w} + (2t - t^3)\epsilon_{w,v} - t\epsilon_{v,w} + (1-t^2)\epsilon_{v,v}] \otimes v, \quad A_w = \epsilon_{w,w} \otimes w = B_w. \end{aligned}$$

The desired homotopy is the homomorphism $H : C(E) \to M_{\mathcal{P}}C(E)[t]$ defined by

$$H(v) = A_v(\epsilon_{v,v} \otimes v)B_v, \quad H(e) = A_{s(e)}(\epsilon_{s(e),r(e)} \otimes e)B_{r(e)}, \quad H(e^*) = A_{r(e)}(\epsilon_{r(e),s(e)} \otimes e^*)B_{s(e)}.$$

□

2.1.2 Proof of Theorem 2.1, part II

Let

$$M_{\mathcal{P}}C(E) \supset \mathfrak{A} = \operatorname{span}\{\epsilon_{\gamma,\delta} \otimes \alpha\beta^* \mid s(\alpha) = r(\gamma), s(\beta) = r(\delta), r(\alpha) = r(\beta)\}.$$

One checks that $\mathfrak{A}$ is a subalgebra containing the images of both $\hat{\iota}_\tau$ and $\hat{\varphi}$. From the commutative diagram 2.7 we obtain, by corestriction, another commutative diagram

$$\begin{tikzcd} \ell^{E^0} \arrow[r, "\hat{\iota}"] \arrow[d, "\varphi"'] & \hat{\mathcal{K}}(E) \arrow[d, "\hat{\varphi}"] \\ C(E) \arrow[r] & \mathfrak{A} \end{tikzcd} \tag{2.8}$$

By Lemma 2.3, the bottom arrow of (2.8) is a monomorphism in *kk*. We shall abuse notation and write $\hat{\iota}_\tau$ for the latter map.

Let $\widetilde{C}^m(E)$ be the unitalization; put $R = \Gamma_{\mathcal{P}}\widetilde{C}^m(E)$. Consider the homomorphism $\rho' = \rho \otimes 1 : C(E) \to R$. One checks that the subalgebra $\mathfrak{A} \subset R$ is closed under both left and right multiplication by elements in the image of ρ'. We can thus form the semi-direct product $C^m(E) \ltimes \mathfrak{A} = C^m(E) \ltimes_{\rho'} \mathfrak{A}$. As an ℓ-module, $C^m(E) \ltimes \mathfrak{A}$ is just $C^m(E) \oplus \mathfrak{A}$. Multiplication is defined by the rule

$$(r, x) \cdot (s, y) = (rs, \rho'(r)x + y\rho'(s) + xy).$$

Let J be the ideal in $C^m(E) \ltimes \mathfrak{A}$ generated by the elements $(\alpha q_v \beta^*, -\epsilon_{\alpha,\beta} \otimes v)$ with $v = r(\alpha) = r(\beta)$. One checks that

$$J = \operatorname{span}\{(\alpha q_v \beta^*, -\epsilon_{\alpha,\beta} \otimes v) \ : \ v = r(\alpha) = r(\beta)\}.$$

Set

$$D = (C^m(E) \ltimes \mathfrak{A})/J.$$

Lemma 2.4. *The composite of the inclusion and projection maps* $\mathfrak{A} = 0 \rtimes \mathfrak{A} \subset C^m(E) \ltimes \mathfrak{A} \to D$ *is injective.*

Proof. It follows from (2.3) that there is an injective homomorphism

$$\jmath : \hat{\mathcal{K}}(E) \to \mathfrak{A}, \quad \jmath(\alpha q_v \beta^*) = \epsilon_{\alpha,\beta} \otimes v \quad (r(\alpha) = r(\beta) = v).$$

Let $\mathrm{inc} : \hat{\mathcal{K}}(E) \to C^m(E)$ be the inclusion. Observe that J is the image of the map $\mathrm{inc} \rtimes (-\jmath) : \hat{\mathcal{K}}(E) \to C^m(E) \rtimes \mathfrak{A}$. In particular, the projection $\pi : C^m(E) \ltimes \mathfrak{A} \to C^m(E)$ is injective on J. It follows that $J \cap (0 \rtimes \mathfrak{A}) = 0$; this finishes the proof. □

2.1.3 Proof of Theorem 2.1, part III

By Lemma 2.4, we may regard $\mathfrak{A}$ as an ideal of D. Let $\Upsilon : C^m(E) \to D$ be the composite of the inclusion $C^m(E) \subset C^m(E) \rtimes \mathfrak{A}$ and the projection $C^m(E) \rtimes \mathfrak{A} \to D$. We may embed diagram (2.8) into a commutative diagram

$$\begin{tikzcd}
\ell^{(E^0)} \arrow[r, "\hat{\iota}"] \arrow[d, "\varphi"'] & \hat{\mathcal{K}}(E) \arrow[r] \arrow[d, "\hat{\varphi}"] & C(E) \arrow[d, "\Upsilon"] \\
C(E) \arrow[r, "\hat{\iota}_\tau"] & \mathfrak{A} \arrow[r] & D
\end{tikzcd} \tag{2.9}$$

Let $\psi_0 = \Upsilon\,\mathrm{can}$, $\psi_1 = \Upsilon\xi$. Note that $\psi_1 \perp \hat{\iota}_\tau$, so $\psi_{1/2} = \psi_1 + \hat{\iota}_\tau$ is an algebra homomorphism. We have quasi-homomorphisms

$$(\psi_0, \psi_1), (\psi_0, \psi_{1/2}), (\psi_{1/2}, \psi_1) : C(E) \to D \rhd \mathfrak{A}.$$

Lemma 2.5. *The quasi-homorphism* $(\psi_0, \psi_{1/2})$ *induces the zero map in kk.*

Proof. Let $H^+ : C(E) \to D[t]$ be the algebra homorphism determined by setting

$$H^+(v) = (v, 0), \quad H^+(e) = (em_{r(e)}, 0) + (1 - t^2)(0, \epsilon_{s(e),r(e)} \otimes e) + t(0, \epsilon_{e,r(e)} \otimes r(e))$$
$$H^+(e^*) = (m_{r(e)}e^*, 0) + (1 - t^2)(0, \epsilon_{r(e),s(e)} \otimes e) + (2t - t^3)(0, \epsilon_{r(e),e} \otimes r(e))$$

for $v \in E^0$ and $e \in E^1$. It is a matter of calculation to show that H^+ a homotopy between ψ_0 and $\psi_{1/2}$, and that $(H^+, \psi_{1/2}) : C(E) \to D[t] \rhd \mathfrak{A}[t]$ is a homotopy between $(\psi_0, \psi_{1/2})$ and $(\psi_{1/2}, \psi_{1/2})$. Hence by item (7) of Proposition 1.8, we obtain

$$j(\psi_0, \psi_{1/2}) = j(\psi_{1/2}, \psi_{1/2}) = 0$$

as wanted. □

2.1.4 Proof of Theorem 2.1, conclusion

Using the commutativity of diagram (2.9) and items (6), (8) and (1) of Proposition 1.8 and Lemma 2.5 we have

$$j(\hat{\varphi})j(\mathrm{can},\xi) = j(\psi_0,\psi_1) = j(\psi_0,\psi_{1/2}) + j(\psi_{1/2},\psi_1) = j(\hat{\iota}_\tau).$$

On the other hand

$$j(\hat{\varphi})j(\mathrm{can},\xi) = j(\hat{\iota}_\tau)j(\varphi)j(\hat{\iota})^{-1}j(\mathrm{can},\xi).$$

Hence

$$j(\hat{\iota}_\tau) = j(\hat{\iota}_\tau)j(\varphi)j(\hat{\iota})^{-1}j(1,\xi)$$

Since $j(\hat{\iota}_\tau)$ is a monomorphism, this implies that

$$1_{j(C(E))} = j(\varphi)j(\hat{\iota})^{-1}j(1,\xi).$$

This finishes the proof. □

2.2 A triangle for $L(E)$

Recall that we have an ℓ-linearly split short exact sequence 1.19

$$0 \to \mathcal{K}(E) \to C(E) \to L(E) \to 0.$$

and is thus an algebra extension in the sense of Section 1.2.

Proposition 2.6. *Let $j : \mathrm{Alg}_\ell \to kk$ be as in Theorem 2.1.*

i) There is a distinguished triangle in kk

$$\ell^{(\mathrm{reg}(E))} \xrightarrow{f} \ell^{(E^0)} \longrightarrow L(E). \tag{2.10}$$

ii) Let $\xi_v : \ell \to \ell^{(\mathrm{reg}(E))}$ be the inclusion in the v-summand and let $c_v \in \mathbb{Z}^{(E^0)\times\{v\}}$ be the v-column of the matrix $I - A_E^t$ ($v \in \mathrm{reg}(E)$). Under the isomorphism (1.11), the composite $fj(\chi_v)$ corresponds to the map

$$1 \otimes c_v : KH_0(\ell) \to KH_0(\ell) \otimes \mathbb{Z}^{(E^0)}$$

Proof. Consider the map $q : \ell^{(\mathrm{reg}(E))} \to \mathcal{K}(E)$ $q(\chi_v) = q_v$. In view of (2.4), $j(q)$ is an isomorphism by Proposition 1.13. By Theorem 2.1, the map $j(\phi)$ is an isomorphism. Hence the kk-triangle induced by (1.19) is isomorphic to the triangle (2.10) where for the inclusion $\mathrm{inc} : \mathcal{K}(E) \subset C(E)$, we have $f = j(\phi)^{-1}j(\mathrm{inc})j(q)$. This proves i). To prove ii), fix $v \in \mathrm{reg}(E)$ and consider the elements q_v, m_v and $ee^* \in C(E)$ ($e \in E^1$, $s(e) = v$). As the latter elements are idempotent, we regard them as homomorphisms $\ell \to C(E)$. In particular, $q_v = \mathrm{inc}\, q\chi_v$. Because $q_v \perp m_v$ and $v = q_v + m_v$, $j(q_v) = j(v) - j(m_v)$. On the other hand, by (CK1), $j(m_v) = \sum_{s(e)=v} j(r(e))$. Summing up, $q_v = j(v) - \sum_{s(e)=v} j(r(e))$; this proves ii). □

Theorem 2.7. *Let $X : \mathrm{Alg}_\ell \to \mathcal{T}$ be an excisive, homotopy invariant, E-stable and E^0-additive homology theory and let $R \in \mathrm{Alg}_\ell$. Then (2.10) induces a triangle in $\mathcal{T}$*

$$X(R)^{(\mathrm{reg}(E))} \xrightarrow{I-A_E^t} X(R)^{(E^0)} \longrightarrow X(L(E)\otimes R)\ .$$

Proof. Tensoring the triangle (2.10) by R yields another triangle in kk, by Example 1.9. By the universal property of j, applying X to the latter triangle gives a distinguished triangle in $\mathcal{T}$. Now apply Proposition 2.6(ii) and the E^0-additivity hypothesis on X to finish the proof. □

Example 2.8. *Theorem 2.7 applies to $X = KH$ and arbitrary E, generalizing* [4, Theorem 8.4] *from the row-finite to the general case. Recall a ring A is K_n*-regular *if for every $m \geq 1$, the inclusion $A \to A[t_1, \dots, t_m]$ induces an isomorphism $K_n(A) \to K_n(A[t_1, \dots, t_m])$. We call A K*-regular *if it is K_n-regular for all n. By* [32, Proposition 1.5]*, the canonical map $K(A) \to KH(A)$ is a weak equivalence when A is K-regular. For example, when $\ell = \mathbb{Z}$ and R is any regular supercoherent ring, then $L(E)\otimes R$ is K-regular (by the argument of* [4, page 23]*), so we may replace KH by K to obtain the following triangle in the homotopy category of spectra which generalizes* [4, Theorem 7.6]

$$K(R)^{(\mathrm{reg}(E))} \xrightarrow{I-A_E^t} K(R)^{(E^0)} \longrightarrow K(L(E)\otimes R)\ .$$

In particular this applies when $R = \ell$ is a field. When E^0 is finite and ℓ is arbitrary, Theorem 2.7 also applies to the universal homology theory $j : \mathrm{Alg}_\ell \to kk$ of Theorem 2.1. In particular, if $\#E^0 < \infty$ we have a triangle in kk

$$\ell^{\mathrm{reg}(E)} \xrightarrow{I-A_E^t} \ell^{E^0} \longrightarrow L(E). \tag{2.11}$$

In particular $L(E)$ belongs to the bootstrap category of [15, Section 8.3] *whenever E^0 is finite, or equivalently, when $L(E)$ is unital* [1, Lemma 1.2.12].

Remark 2.9. *When E is finite, we can also fit $L(E)$ into a kk-triangle associated to a matrix with entries in $\{0,1\}$. Let $B'_E \in \{0,1\}^{(E^1\sqcup \mathrm{sink}(E))\times(E^1\sqcup\mathrm{sink}(E))}$,*

$$(B'_E)_{x,y} = \begin{cases} \delta_{r(x),s(y)} & x, y \in E^1 \\ \delta_{r(x),y} & x \in E^1, y \in \mathrm{sink}(E) \\ 0 & x \in \mathrm{sink}(E) \end{cases}$$

The matrix $B'_E = A'_{E'}$ is the incidence matrix of the maximal out-split graph E' of [1, Definition 6.3.23]*. Since by* [1, Proposition 6.3.25]*, $L(E) \cong L(E')$ in Alg_ℓ, (2.11) gives a triangle*

$$\ell^{E^1} \xrightarrow{I-B_E^t} \ell^{E^1\sqcup\mathrm{sink}(E)} \longrightarrow L(E).$$

Here $I, B_E^t \in (E^1 \sqcup \mathrm{sink}(E)) \times E^1$ are obtained from the identity matrix and from $(B'_E)^t$ by removing the columns corresponding to sinks.

Remark 2.10. *In* [15]*, a functor* $j' : \mathrm{Alg}_\ell \to kk'$ *was constructed that is universal among those homotopy invariant and* M_∞*-stable homology theories which are excisive with respect to all, not just the linearly split short exact sequences of algebras* (1.7)*. The suspension functor in* kk' *is induced by Wagoner's suspension* (1.3)*; we have* $\Omega^{-1}j = j\Sigma$*. The universal property of* j *implies that there is a triangulated functor* $F : kk \to kk'$ *such that* $j' = Fj$*, and it follows from* [15, Theorem 8.2.1] *that* $F : KH_n(R) = kk_n(\ell, R) \to kk'_n(\ell, R)$ *is an isomorphism for all* $n \in \mathbb{Z}$ *and* $R \in \mathrm{Alg}_\ell$*. Note that when* E^0 *is finite and* E^1 *is countable, Theorem 2.7 applies to* $X = j'$*. It follows that* $F_n : kk(L(E), R) \to kk'_n(L(E), R)$ *is an isomorphism for all* $n \in \mathbb{Z}$ *and* $R \in \mathrm{Alg}_\ell$*. In particular, if* R *is unital,* E^1 *is countable and* E^0 *is finite, then for the* $\mathcal{E}xt$*-group we have a natural map*

$$\mathcal{E}xt(L(E), R) \to kk_{-1}(L(E), R).$$

Convention 2.12. *From now on, every statement about the image under* j *of the Cohn or Leavitt path algebras of finitely many graphs* $E_1, \ldots, E_n$ *will refer to the* $\sqcup_{i=1}^n E_i$*-stable, homotopy invariant, excisive homology theory* $j : \mathrm{Alg}_\ell \to kk$*.*

One easy application of Theorem 2.7 is the following proposition:

Proposition 2.11. *Let* E *and* F *be graphs and* $\theta \in kk(L(E), L(F))$*. Assume that* E^0 *and* F^0 *are finite and that* $KH_i(\theta)$ *is an isomorphism for* $i = 0, 1$*. Then* θ *is an isomorphism. In particular* $KH_n(\theta)$ *is an isomorphism for all* $n \in \mathbb{Z}$*.*

Proof. The map θ induces a natural transformation $\theta_A : kk(A, L(E)) \to kk(A, L(F))$ ($A \in \mathrm{Alg}_\ell$). Our hypothesis that $KH_i(\theta)$ is an isomorphism for $i = 0, 1$ says that $\theta_{\Omega^{-i}j(\ell)}$ is an isomorphism. Since F^0 is finite by assumption, this implies that also $\theta_{\Omega^{-i}j(\ell^{F^0})}$ and $\theta_{\Omega^{-i}j(\ell^{\mathrm{reg}(F)})}$ are isomorphisms. Hence applying $\theta : kk(-, L(E)) \to kk(-, L(F))$ to the triangle

$$\ell^{\mathrm{reg}(F)} \xrightarrow{I - A_F^t} \ell^{F^0} \longrightarrow L(F)$$

and using the five lemma, we obtain that $\theta_{L(E)}$ is an isomorphism. In particular there is an element $\mu \in kk(L(F), L(E))$ such that $\mu\theta = 1_{L(F)}$. Our hypothesis implies that $KH_i(\mu)$ must be an isomorphism for $i = 0, 1$. Hence reversing the role of E and F in the previous argument shows that μ has a left inverse. It follows that θ is an isomorphism. □

Remark 2.12. *The conclusion of Proposition 2.11 does not follow if we only assume that there are group isomorphisms* $\theta_i : KH_i(L(E)) \xrightarrow{\sim} KH_i(L(F))$ $(i = 0, 1)$*. For example, over* $\ell = \mathbb{Q}$*,* $K_0(L_0) = K_0(L_1) = \mathbb{Z}$ *and* $K_1(L_0) = \mathbb{Q}^* \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{(\mathbb{N})} \cong K_1(L_1)$*. However they are not isomorphic in* kk*, since they have different periodic cyclic homology:* $HP_1(L_0) = 0$ *and* $HP_1(L_1) = \mathbb{Q}$*.*

2.3 A structure theorem for Leavitt path algebras in *kk*

Standing assumptions 2.13. *From here on, we shall assume that the commutative base ring ℓ satisfies the following conditions.*

i) $KH_{-1}(\ell) = 0$.

ii) The natural map $\mathbb{Z} = K_0(\mathbb{Z}) = KH_0(\mathbb{Z}) \to KH_0(\ell)$ *is an isomorphism.*

Moreover, all graphs considered henceforth are assumed to have finitely many vertices. In particular, all Leavitt path algebras will be unital.

Remark 2.13. *Any regular supercoherent ground ring ℓ satisfies standing assumption i), and moreover any Leavitt path algebra over ℓ is K-regular. Hence all statements of this section are valid for regular supercoherent ℓ satisfying standing assumption ii), with K_0 substituted for KH_0. In particular, this applies when $\ell = \mathbb{Z}$ or any field.*

Definition 2.14. *Let $L(E)$ the Leavitt path algebra associated to the graph E. Put*

$$KH^1(L(E)) = kk_{-1}(L(E), \ell).$$

It follows from (2.11) and the standing assumptions that, abusing notation, and writing I for I^t,

$$KH^1(L(E)) \cong \operatorname{Coker}(I - A_E : \mathbb{Z}^{E^0} \to \mathbb{Z}^{\operatorname{reg}(E)}). \tag{2.14}$$

Proposition 2.15. *(Compare* [16, Theorem 5.3]*.) Let E be a graph with finitely many vertices, such that E^1 is countable and* $\operatorname{sour}(E) = \emptyset$. *Then the natural map of Remark 2.10 is a surjection*

$$\mathcal{E}xt(L(E)) \twoheadrightarrow KH^1(L(E)). \tag{2.15}$$

Proof. Our hypothesis on E imply that, with the notation of(1.15), we have $\#\mathcal{P}_v = \#\mathbb{N}$ for all $v \in E^0$. Hence by (2.4), $\mathcal{K}(E) \cong M_\infty \ell^{\operatorname{reg}(E)}$, and (1.19) is an extension of $L(E)$ by $M_\infty \ell^{\operatorname{reg}(E)}$. Let $\psi : L(E) \to \Sigma(\ell)^{\operatorname{reg}(E)}$ be its classifying map and for $v \in \operatorname{reg}(E)$ let $\pi_v : \Sigma(\ell)^{\operatorname{reg}(E)} \to \Sigma(\ell)$ be the projection, and put $\psi_v = \pi_v \psi$. With the notation of Remark 2.10 we have a triangle in kk'

$$j(\ell^{E^0}) \to j(L(E)) \xrightarrow{\psi} j(\Sigma(\ell)^{\operatorname{reg}(E)}) \to j(\Sigma(\ell)^{E^0}).$$

Applying $kk'(-, \Sigma(\ell))$ to it and using Remark 2.10 we see that $KH^1(L(E))$ is generated by the kk-classes of the ψ_v; since these are in the image of (2.15), it follows that the latter map is surjective. □

Lemma 2.16.

i) The groups $KH^1(L(E))$ and $KH_0(L(E))$ have isomorphic torsion subgroups.

ii) $\#\operatorname{sing}(E) = \operatorname{rk}(KH_0(L(E)) - \operatorname{rk}(KH^1(L(E))$.

Proof. Let $D = \operatorname{diag}(d_1, \dots, d_n, 0, \dots, 0) \in \mathbb{Z}^{E^0 \times \operatorname{reg}(E)}$, $d_i \geq 2$, $d_i \backslash d_{i+1}$ be the Smith normal form of $I - A_E^t$. Then D^t is the Smith normal form of $I - A_E$, whence

$$\operatorname{tors} KH_0(L(E)) = \bigoplus_{i=1}^{n} \mathbb{Z}/d_i = \operatorname{tors} KH^1(L(E)). \tag{2.16}$$

Similarly,

$$\begin{aligned}\operatorname{rk} KH_0(L(E)) - \operatorname{rk} KH^1(L(E)) =& (\# E^0 - \operatorname{rk}(I - A_E)) - (\# \operatorname{reg}(E) - \operatorname{rk}(I - A_E))) \\ =& \# \operatorname{sing}(E).\end{aligned}$$

□

We shall write

$$\tau(E) = \operatorname{tors} KH_0(L(E)).$$

For $0 \leq n \leq \infty$, let $\mathcal{R}_n$ be the graph with exactly one vertex and n loops and let $L_n = L(\mathcal{R}_n)$. Thus $L_0 = \ell$, $L_1 = \ell[t, t^{-1}]$ is the algebra of Laurent polynomials and for $2 \leq n < \infty$, $L_n = L(1, n)$ is the Leavitt algebra of [21]. By (2.11), $j(L_\infty) \cong j(L_0)$ and we have a distinguished triangle in kk

$$j(\ell) \xrightarrow{n-1} j(\ell) \longrightarrow j(L_n) \qquad (n \geq 1). \tag{2.17}$$

Theorem 2.17. *Let E be a graph with finitely many vertices. Assume that ℓ satisfies the standing assumptions 2.13. Let $d_1, \dots, d_n$, $d_i \backslash d_{i+1}$ be the invariant factors of the finite abelian group $\tau(E)$, $s = \# \operatorname{sing}(E)$ and $r = \operatorname{rk}(KH^1(L(E)))$. Let $j : \operatorname{Alg}_\ell \to kk$ be the universal excisive, homotopy invariant, E-stable homology theory. Then*

$$j(L(E)) \cong j(L_0^s \oplus L_1^r \oplus \bigoplus_{i=1}^{n} L_{d_i+1}).$$

Proof. Let $D = \operatorname{diag}(d_1, \dots, d_n, 0, \dots, 0) \in \mathbb{Z}^{E^0 \times \operatorname{reg}(E)}$. Then there are $P \in \operatorname{GL}_{\# E^0} \mathbb{Z}$, $Q \in \operatorname{GL}_{\# \operatorname{reg}(E)} \mathbb{Z}$ such that $P(I - A_E^t)Q = D$ where $D := diag(d_1, \dots, d_r, 0, \dots, 0)$. Hence we have the following commutative square in kk with vertical isomorphisms

$$\begin{tikzcd} j(\ell^{\operatorname{reg}(E)}) \arrow[r, "I-A_E^t"] \arrow[d, "Q^{-1}"] & j(\ell^{E^0}) \arrow[d, "P"] \\ j(\ell^{\operatorname{reg}(E)}) \arrow[r, "D"'] & j(\ell^{E^0}) \end{tikzcd}$$

Hence both rows have isomorphic cones. By (2.11), the cone of the top row is $L(E)$; by (2.17) and Lemma 2.16 that of the bottom row is $L_0^s \oplus L_1^r \oplus \bigoplus_{i=1}^{n} L_{d_i+1}$. □

Corollary 2.18. *The following are equivalent for graphs E and F with finitely many vertices.*

i) $j(L(E)) \cong j(L(F))$.

ii) $KH_0(L(E)) \cong KH_0(L(F))$ *and* $KH^1(L(E)) \cong KH^1(L(F))$.

iii) $KH_0(L(E)) \cong KH_0(L(F))$ *and* $\#\operatorname{sing}(E) = \#\operatorname{sing}(F)$.

Proof. Immediate from Lemma 2.16 and Theorem 2.17. □

Remark 2.19. *Let E and F be as in Corollary 2.18. Assume in addition that ℓ is a field, that $L(E)$ and $L(F)$ are simple and that* $\inf(E) \neq \emptyset \neq \inf(F)$. *In* [29, Theorem 7.4]*, E. Ruiz and M. Tomforde show that under these assumptions condition iii) of Corollary 2.18 is equivalent to the existence of a Morita equivalence between $L(E)$ and $L(F)$. It follows that for such E and F, the algebras $L(E)$ and $L(F)$ are isomorphic in kk if and only if they are Morita equivalent. Ruiz and Tomforde show also that under the additional assumption that the group of invertible elements $U(\ell)$ has no free quotients, the condition that $\#\operatorname{sing}(E) = \#\operatorname{sing}(F)$ in iii) can be replaced by the condition that $K_1(L(E)) \cong K_1(L(F))$. The additional assumption guarantees that* $\operatorname{rk}(K_1(L(E))) = \operatorname{rk}(\operatorname{Ker}(1 - A_E^t)) = \operatorname{rk}(KH^1(L(E))$ *whenever $\#E^0 < \infty$, so that* $\#\operatorname{sing}(E) = \operatorname{rk}(K_0(L(E)) - \operatorname{rk}(K_1(L(E)))$.

2.4 A canonical filtration in $kk(L(E),R)$

Let ℓ be a ground ring satisfying the Standing assumptions 2.13, let E be a graph with finitely many vertices, $L(E)$ its Leavitt path algebra over ℓ, and $n \in \mathbb{Z}$. It follows from (2.10) that we have an exact sequence

$$0 \to KH_n(\ell) \otimes KH_0(L(E)) \longrightarrow KH_n(L(E)) \to \operatorname{Ker}((I - A_E^t) \otimes KH_{n-1}(\ell)) \to 0. \tag{2.18}$$

Lemma 2.20. *The map $KH_n(\ell) \otimes KH_0(L(E)) \longrightarrow KH_n(L(E))$ of (2.18) is the cup product map of Example 1.9.*

Proof. Because by assumption 2.13 (ii), $KH_0(\ell) = \mathbb{Z}$, for any finite set X, the cup product of Example 1.9 gives an isomorphism

$$\cup : KH_n(\ell) \otimes KH_0(\ell^X) \xrightarrow{\sim} KH_n(\ell^X). \tag{2.19}$$

Hence by (2.10) we have a commutative diagram with exact rows

$$\begin{tikzcd}
KH_n(\ell^{\operatorname{reg}(E)}) \arrow[r, "I-A_E^t"] & KH_n(\ell^{E^0}) \arrow[r] & KH_n(L(E)) \\
KH_n(\ell) \otimes KH_0(\ell^{\operatorname{reg}(E)}) \arrow[r, "I-A_E^t"'] \arrow[u, "\cup"] & KH_n(\ell) \otimes KH_0(\ell^{E^0}) \arrow[r] \arrow[u, "\cup"] & KH_n(\ell) \otimes KH_0(L(E)). \arrow[u, "\cup"]
\end{tikzcd}$$

□

Let R be an algebra and $n \in \mathbb{Z}$. Consider the map

$$KH_n : kk(L(E), R) \to \operatorname{Hom}_{\mathbb{Z}}(KH_n(L(E)), KH_n(R)). \tag{2.20}$$

Define a descending filtration $\{kk(L(E), R)^i \mid 0 \le i \le 2\}$ on $kk(L(E), R)$ as follows. Let

$$kk(L(E), R)^0 = kk(L(E), R), \quad kk(L(E), R)^1 = \operatorname{Ker} KH_0, \tag{2.21}$$

$$kk(L(E), R)^2 = (\operatorname{Ker} KH_1) \cap kk(L(E), R)^1. \tag{2.22}$$

It follows from the definition of $kk(L(E), R)^0$ and $kk(L(E), R)^1$ that KH_0 induces a canonical homomorphism

$$kk(L(E), R)^0 / kk(L(E), R)^1 \to \hom(KH_0(L(E)), KH_0(R)). \tag{2.23}$$

Let $\xi \in kk(L(E), R)^1$; by Lemma 2.20, $KH_1(\xi)$ vanishes on the image of $KH_1(\ell)^{(E^0)}$, whence it induces a map $\operatorname{Ker}(I - A_E^t) \to KH_1(R)$. Thus we have a map

$$kk(L(E), R)^1 / kk(L(E), R)^2 \to \hom(\operatorname{Ker}(I - A_E^t), KH_1(R)). \tag{2.24}$$

Let $\xi \in kk(L(E), R)^2$; embed ξ into a distinguished triangle

$$C_\xi \to L(E) \xrightarrow{\xi} R. \tag{2.25}$$

We have an extension of abelian groups

$$(\mathcal{E}(\xi)) \quad 0 \to KH_1(R) \to K_0(C_\xi) \to KH_0(L(E)) \to 0. \tag{2.26}$$

Let

$$kk(L(E), R)^2 \to \operatorname{Ext}^1_{\mathbb{Z}}(KH_0(L(E)), KH_1(R)), \quad \xi \mapsto [\mathcal{E}(\xi)]. \tag{2.27}$$

Theorem 2.21. *Let E be a graph with finitely many vertices, ℓ a ring satisfying the Standing assumptions 2.13, $L(E)$ the Leavitt path algebra over ℓ and R an ℓ-algebra. Then the maps (2.23), (2.24) and (2.27) are isomorphisms.*

Proof. Observe that if B is an algebra and X a finite set, then the isomorphism (1.11) induces an isomorphism $kk_n(\ell^X, B) \xrightarrow{\sim} \hom(\mathbb{Z}^X, KH_n(B))$. Using this and applying $kk(-, R)$ to the triangle (2.11) we obtain an exact sequence

$$\begin{aligned} &\operatorname{Hom}(\mathbb{Z}^{E^0}, KH_1(R)) \to \operatorname{Hom}(\mathbb{Z}^{\operatorname{reg}(E)}, KH_1(R))) \to kk(L(E), R) \\ &\qquad \to \operatorname{Hom}(\mathbb{Z}^{E^0}, KH_0(R)) \to \operatorname{Hom}(\mathbb{Z}^{\operatorname{reg}(E)}, KH_0(R)). \end{aligned} \tag{2.28}$$

Since

$$0 \to \operatorname{Ker}(I - A_E^t) \to \mathbb{Z}^{\operatorname{reg}(E)} \to \mathbb{Z}^{E^0} \to KH_0(L(E)) \to 0 \tag{2.29}$$

is a free $\mathbb{Z}$-module resolution, the kernel of the last map in (2.28) is

$\mathrm{Hom}(KH_0(L(E)), KH_0(R))$, and it is straightforward to check that the induced surjection

$$kk(L(E), R) \twoheadrightarrow \mathrm{Hom}(KH_0(L(E)), KH_0(R))$$

is precisely the map KH_0 of (2.20). Hence the cokernel of the first map in (2.28) is $kk(L(E), R)^1$, and again because (2.29) is a free resolution, we have a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(KH_0(L(E), KH_1(R)) \to kk(L(E), R)^1 \to \mathrm{Hom}(\mathrm{Ker}(I - A^t_E), KH_1(R)) \to 0. \quad (2.30)$$

It is again straightforward to check that the surjective map from $kk(L(E), R)^1$ in (2.30) is (2.24). Hence by (2.30) we have an isomorphism

$$kk(L(E), R)^2 \xrightarrow{\sim} \mathrm{Ext}^1_{\mathbb{Z}}(KH_0(L(E), KH_1(R)) \quad (2.31)$$

It remains to show that the above isomorphism agrees with (2.27).

Let $\xi \in kk(L(E), R)^2$ and let $\partial : j(L(E)) \to \Omega^{-1} j(\ell)^{\mathrm{reg}(E)}$ be the boundary map in (2.11). Because $KH_0(\xi) = 0$, there is an element $\hat{\xi} \in kk_1(\ell^{\mathrm{reg}(E)}, R)$ such that $\xi = \hat{\xi}\partial$. Hence because kk is triangulated, there exists $\theta \in kk(\ell^{E^0}, C_\xi)$ such that we have a map of distinguished triangles

$$\begin{tikzcd}
j(\ell)^{\mathrm{reg}(E)} \arrow[r] \arrow[d, "\Omega j\hat{\xi}"] & j(\ell)^{E^0} \arrow[r] \arrow[d, "\theta"] & j(L(E)) \arrow[r, "\partial"] \arrow[d, equal] & \Omega^{-1} j(\ell)^{\mathrm{reg}(E)} \arrow[d, "\hat{\xi}"] \\
\Omega j(R) \arrow[r] & C_\xi \arrow[r] & j(L(E)) \arrow[r, "\xi"'] & j(R)
\end{tikzcd}$$

Applying the functor $kk(\ell, -)$ and using that $KH_1(\xi) = 0$, we obtain a map of extensions

$$\begin{tikzcd}
\mathrm{Ker}(I - A^t_E) \arrow[r] \arrow[d] & \mathbb{Z}^{\mathrm{reg}(E)} \arrow[r] \arrow[d, "\hat{\xi}"] & \mathbb{Z}^{E^0} \arrow[r] \arrow[d, "\theta"] & K_0(L(E)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_1(R) \arrow[r] & K_0(C_\xi) \arrow[r] & K_0(L(E)) \arrow[r] & 0
\end{tikzcd} \quad (2.32)$$

By definition, (2.31) maps ξ to the class $[\hat{\xi}]$ of $\hat{\xi}$ modulo the image of $\mathrm{Hom}(\mathbb{Z}^{E^0}, KH_1(R))$. It is clear from (2.32) that $[\hat{\xi}] = [C_\xi]$. □

Corollary 2.22. *Let $\xi \in kk(L(E), R)$ and let C_ξ be as in (2.25). Then $\xi = 0$ if and only if $KH_0(\xi) = KH_1(\xi) = 0$ and the extension (2.26) is split.*

In the next corollary we shall use the fact that, since $\mathrm{Ker}(I - A^t_E)$ is a free abelian group, the canonical surjection $KH_1(L(E)) \to \mathrm{Ker}(I - A^t_E)$ admits a section

$$\gamma : \mathrm{Ker}(I - A^t_E) \to KH_1(L(E)). \quad (2.33)$$

The map γ induces a natural transformation

$$\gamma^* : \mathrm{Hom}(KH_1(L(E)), -) \to \mathrm{Hom}(\mathrm{Ker}(I - A^t_E), -).$$

Corollary 2.23. *(UCT) For every $n \in \mathbb{Z}$ we have an exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(KH_0(L(E)), KH_{n+1}(R)) \to kk_n(L(E), R) \xrightarrow{[KH_0, \gamma^* KH_1]} \operatorname{Hom}(KH_0(L(E)), KH_n(R)) \oplus \operatorname{Hom}(\operatorname{Ker}(I - A^t_E), KH_{n+1}(R)) \to 0.$$

Proof. In view of (1.9) we may assume that $n = 0$. By Theorem 2.21 the map $KH_0 : kk(L(E), R) \to \hom(KH_0(L(E)), KH_0(R))$ is a surjection; by definition, its kernel is $kk(L(E), R)^1$, and $\gamma^* KH_1$ induces the map (2.24), which is surjective by Theorem 2.21. Hence $[KH_0, \gamma^* KH_1]$ is surjective, and its kernel is by definition $kk(L(E), R)^2$, which, again by Theorem 2.21, is $\operatorname{Ext}^1_{\mathbb{Z}}(KH_0(L(E)), KH_1(R))$. $\square$

Lemma 2.24. *Let E be a graph and R an algebra. Assume that $\#E^0 < \infty$. Then the composition map induces an isomorphism*

$$KH^1(L(E)) \otimes KH_1(R) \xrightarrow{\sim} kk(L(E), R)^1$$

Proof. By our Standing assumptions, $KH_{-1}\ell = 0$; it follows from this that

$KH^1(L(E)) = kk_{-1}(L(E), \ell)^1$ and that the composition map lands in $kk(L(E), R)^1$. In particular, writing $^\vee$ for the dual group, we have $KH^1(L(E))/kk_{-1}(L(E), \ell)^2 = \operatorname{Ker}(I - A^t_E)^\vee$; since the latter is free, tensoring with $KH_1(R)$ we obtain the top exact sequence of the commutative diagram below; the bottom row is exact by Theorem 2.21.

$$\begin{tikzcd}
\operatorname{Ext}^1_{\mathbb{Z}}(\tau(E), \mathbb{Z}) \otimes KH_1(R) \arrow[r, tail] \arrow[d] & KH^1(LE) \otimes KH_1(R) \arrow[r, two heads] \arrow[d] & \operatorname{Ker}(I - A^t_E)^\vee \otimes KH_1(R) \arrow[d] \\
\operatorname{Ext}^1_{\mathbb{Z}}(\tau(E), KH_1(R)) \arrow[r, tail] & kk(L(E), R)^1 \arrow[r, two heads] & \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Ker}(I - A^t_E), KH_1(R)).
\end{tikzcd}$$

One checks, using the fact that for a free, finitely generated group L, $L^\vee \otimes (-) \cong \operatorname{Hom}_{\mathbb{Z}}(L, -)$, that the vertical arrows on the right and left are isomorphisms; it follows that the vertical arrow at the middle is an isomorphism as well. $\square$

Lemma 2.25. *Let E and R be as in Lemma 2.24. There is an exact sequence*

$$\operatorname{Ker}(I - A_E) \otimes KH_0(R) \hookrightarrow \operatorname{Hom}(KH_0(L(E)), KH_0(R)) \twoheadrightarrow \operatorname{Tor}^1_{\mathbb{Z}}(KH^1(L(E)), KH_0(R)).$$

Proof. It follows from (2.11) that we have a free $\mathbb{Z}$-module resolution

$$0 \to \operatorname{Ker}(I - A_E) \to (\mathbb{Z}^{E^0})^\vee \to (\mathbb{Z}^{\operatorname{reg}(E)})^\vee \to KH^1(L(E)) \to 0.$$

Now tensor by $KH_0(R)$ and observe that

$$\operatorname{Ker}((I - A_E) \otimes \operatorname{id}_{KH_0(R)}) = \hom(KH_0(L(E)), KH_0(R)).$$

$\square$

Proposition 2.26. *(Künneth theorem) Let $L(E)$ and R be as in Theorem 2.21 and $n \in \mathbb{Z}$. Then there is an exact sequence*

$$\begin{aligned}0 \to KH^1(L(E)) \otimes KH_{n+1}(R) \oplus \operatorname{Ker}(I - A_E) \otimes KH_n(R) \to kk(L(E), R) \\ \to \operatorname{Tor}^1_{\mathbb{Z}}(KH^1(L(E)), KH_n(R)) \to 0.\end{aligned}$$

Proof. It suffices to prove the proposition for $n = 0$. By Theorem 2.21 we have a canonical surjection $\pi : kk(L(E), R) \to \operatorname{Hom}(KH_0(L(E)), KH_0(R))$. By Lemma 2.25 we have an inclusion

$$\operatorname{inc} : \operatorname{Ker}(I - A_E) \otimes KH_0(R) \subset \operatorname{Hom}(KH_0(L(E)), KH_0(R)). \tag{2.34}$$

Let $Q = \pi^{-1}(\operatorname{Ker}(I - A_E) \otimes KH_0(R))$; by Lemmas 2.24 and 2.25 we have exact sequences

$$\begin{gathered}0 \to Q \to kk(L(E), R) \to \operatorname{Tor}^1_{\mathbb{Z}}(KH^1(L(E)), KH_0(R)) \to 0 \\ 0 \to KH^1(L(E)) \otimes KH_1(R) \to Q \to \operatorname{Ker}(I - A_E) \otimes KH_0(R) \to 0.\end{gathered} \tag{2.35}$$

We have to show that the second sequence above splits. Let $\theta : \operatorname{Ker}(I - A_E) \to KH^0(L(E))$ be a section of the canonical projection. One checks that for inc as in (2.34), the composite

$$\theta' : \operatorname{Ker}(I - A_E) \otimes KH_0(R) \xrightarrow{\theta \otimes \mathrm{id}} KH^0(L(E)) \otimes KH_0(R) \xrightarrow{\circ} kk(L(E), R)$$

satisfies $\pi\theta' = \operatorname{inc}$. It follows that the sequence (2.35) splits, completing the proof. □

Remark 2.27. *The key property of the algebra $B = L(E)$ that we have used in this section is that for some $m, n \in \mathbb{N}$ and $M \in \mathbb{Z}^{m\times n}$ we have a distinguished triangle in kk*

$$j(\ell)^n \xrightarrow{M} j(\ell)^m \to j(B).$$

All the results and proofs in this section apply to any algebra B with the above property, substituting M for $I - A^t_E$, and assuming of course that ℓ satisfies the Standing assumptions 2.13. However one can show, using the Smith normal form of M, that any such B is kk-isomorphic to the sum of Leavitt path algebra and a number of copies of the suspension $\Omega_{-1}(\ell)$.

Chapter 3

Homotopy classification of unital purely infinite simple Leavitt path algebras

Resumen del capítulo

En el capítulo 3 trabajamos con álgebras de caminos de Leavitt simples de grafos finitos sobre un cuerpo ℓ.

En la Sección 3.1 recordamos los resultados de Ara, Goodearl and Pardo sobre la K-teoría de álgebras simples puramente infinitas. También probamos (Corolario 3.10) que si R es K_1-regular, simple puramente infinita y unital, entonces $K_1(R)$ es isomorfo al grupo $\pi_0(U(R))$ de componentes conexas polinomiales del grupo de elementos inversibles de R.

En la Sección 3.2 probamos que para toda álgebra de caminos de Leavitt simple de grafo finito y para toda álgebra unital simple puramente infinita y K_1-regular R el morfismo de monoides

$$j : [L(E), R]_{M_2} \setminus \{0\} \to kk(L(E), R)$$

es un isomorfismo.

En la tercera Sección demostramos el Teorema principal de esta tesis, que es el siguiente:

Theorem 3.1. *Sean E y F grafos finitos y ℓ un cuerpo. Supongamos que $L(E)$ y $L(F)$ son simples puramente infinitas. Sea $\xi : K_0(L(E)) \to K_0(L(F))$ un isomorfismo de grupos. Entonces*

- *Existen morfismos de álgebras no nulos $\phi : L(E) \leftrightarrow L(F) : \psi$ tales que $K_0(\phi) = \xi$, $K_0(\psi) = \xi^{-1}$, $\psi\phi \approx_{M_2} \mathrm{id}_{L(E)}$ y $\phi\psi \approx_{M_2} \mathrm{id}_{L(F)}$.*
- *Si además $\xi([1_{L(E)}]) = [1_{L(F)}]$ entonces ϕ y ψ pueden ser tomados como morfismos unitales tales que $\psi\phi \approx \mathrm{id}_{L(E)}$ y $\phi\psi \approx \mathrm{id}_{L(F)}$.*

Este resultado se sigue fácilmente de todos los resultados obtenidos en esta tesis hasta el momento.

En la Sección 3.4 probamos que si R es un álgebra unital de división o un álgebra unital simple puramente infinita K_0-regular entonces existe un isomorfismo $\mathcal{E}xt(L(E), R) \cong kk_{-1}(L(E), R)$ para toda álgebra de caminos de Leavitt simple de grafo finito.

En la Sección 3.5 probamos que para toda álgebra de caminos de Leavitt simple de grafo finito y toda álgebra unital, simple puramente infinita y regular supercoherente, cualquier par de morfismos $L(E) \to L_2$ y cualquier par de morfismos $L(E) \to L_2 \otimes R$ son M_2-homotópicos y, más aún, si los morfismos son unitales entonces son homotópicos.

3.1 Purely infinite algebras and K-theory

Let R be a ring; write $\mathrm{Idem}(R)$ for the set of idempotent elements. Let $p, q \in \mathrm{Idem}(R)$. We write $p \sim q$ if p and q are *Murray-von Neumann equivalent* [5]; that is, if there exist elements $x \in pRq$ and $y \in qRp$ such that $xy = p$ and $yx = q$. We call such pair (x, y) an *MvN equivalence* from p to q and write $(x, y) : p \sim q$.

Put $\mathrm{Idem}_n(R) = \mathrm{Idem}(M_n(R))$, $1 \leq n \leq \infty$. If R is unital, we write

$$\mathcal{V}_n(R) = \mathrm{Idem}_n(R)/\sim \qquad (1 \leq n < \infty), \quad \mathcal{V}(R) = \mathrm{Idem}_\infty(R)/\sim .$$

Remark 3.2. *One may also define $\mathcal{V}(R)$ as the set of isomorphism classes of finitely generated projective right modules. The equivalence between the two definitions follows from* [27, Theorem 1.2.3] *and* [8, Propositions 4.2.5 and 4.3.1]. *One checks that if $f : R \to S$ is a homomorphism and $f(1) = p$, then under the identification, the map $\mathcal{V}(R) \to \mathcal{V}(S)$ induced by $M_\infty R \to M_\infty S$ corresponds to the scalar extension functor $\otimes_R pS$.*

If $p, q \in \mathrm{Idem}(R)$ and $pq = qp = 0$ we say that p and q are *orthogonal* and write $p \perp q$ to indicate this. An idempotent p in a ring R is *infinite* if there exist orthogonal idempotents $q, r \in R$ such that $p = q + r$, $p \sim q$ and $r \neq 0$. A ring R is said to be *purely infinite simple* if for every nonzero element $x \in R$ there exist $s, t \in R$ such that sxt is an infinite idempotent. If R is unital this is equivalent to asking that R not be a division ring and that for every $x \in R$ there are $a, b \in R$ such that $axb = 1$.

The graphs E such that $L(E)$ is purely infinite simple are completely characterized by [1, Theorem 3.1.10]. We wish to express this result using the notion of cofinality; we recall the definition from [1, Definitions 2.9.4]. Let $\mathfrak{X}_E$ be the set whose elements are the infinite paths of E and also the finite paths which end at a singular vertex of E. The graph E is called *cofinal* if for every vertex $v \in E^0$ and every $\gamma \in \mathfrak{X}_E$ there exists a path from v to some vertex w in γ.

Theorem 3.3. [1, Lemma 2.9.6 and Theorem 3.1.10] *$L(E)$ is purely infinite simple if and only if E is cofinal, has at least one cycle and every cycle of E has an exit.*

The following theorem describing K_0 and K_1 of purely infinite simple unital rings is due to Ara, Goodearl and Pardo. If R is a unital ring, write $U(R)$ for the group of invertible elements of R.

Theorem 3.4. [5, Corollary 2.3 and Theorem 2.4] *If R is a purely infinite simple unital ring, then*

$$K_0(R) = \mathcal{V}(R)\backslash\{[0]\}$$

$$K_1(R) = U(R)^{ab}.$$

Proposition 3.5. *Let R be a purely infinite simple unital ring. Then the map $\iota : \mathcal{V}_1(R) \to \mathcal{V}(R)$ is an isomorphism. Moreover, for every $n \geq 1$ and every element $(q_1, \ldots, q_n) \in \mathrm{Idem}_\infty(R)^n$ there exists $(p_1, \ldots, p_n) \in \mathrm{Idem}_1(R)^n$, such that $p_i \sim q_i$ in $\mathrm{Idem}_\infty(R)$ and such that $p_i \perp p_j$ for $i \neq j$.*

Proof. This is straightforward from [5, Proposition 1.5 and Lemma 1.1] □

Combining Proposition 3.5 and Theorem 3.4 we obtain the following.

Corollary 3.6. *Let R be a purely infinite simple unital ring. Then*

$$K_0(R) \cong \mathcal{V}_1(R)\backslash\{[0]\}.$$

Corollary 3.7. *Let R be a purely infinite simple unital ring and let $e, f \in R$ be nonzero idempotents. Then the following are equivalent*

(1) $e \sim f$.

(2) $[e] = [f]$ *in* $K_0(R)$.

If furthermore $e, f \in \mathrm{Idem}_1(R)\backslash\{0, 1\}$ then the above conditions are also equivalent to the following.

(3) There exists $u \in U(R)$ such that $f = ueu^{-1}$.

(4) There exists a commutator $u \in [U(R), U(R)]$ such that $f = ueu^{-1}$.

Proof. The equivalence of (1) and (2) follows from Corollary 3.6. By [8, Proposition 4.2.5], (3) is equivalent to having simultaneously $e \sim f$ and $1 - e \sim 1 - f$. Hence to prove that (1) implies (3) it only remains to show that $1 - e \sim 1 - f$. But

$$[e] + [1 - e] = [1] = [f] + [1 - f]$$

in $K_0(R)$ and $[e] = [f]$, implies $[1 - e] = [1 - f]$ in $K_0(R)$ and therefore in $\mathcal{V}_1(R)$. Hence $1 - e \sim 1 - f$. Next we show that (3) implies (4). Because R is simple and $f \neq 1$, $1 - f$ is a full idempotent. Hence $(1 - f)L(E)(1 - f)$ is purely infinite simple (by [5, Corollary 1.7]) and the inclusion induces an isomorphism $K_1((1 - f)R(1 - f)) \xrightarrow{\sim} K_1(R)$. By Theorem 3.4, this implies that the induced map $U((1 - f)R(1 - f))^{ab} \to U(R)^{ab}$ is an isomorphism. Since the latter map sends $[\xi] \mapsto [\xi + f]$, there is an element $\omega \in U((1 - f)R(1 - f))$ such that $[\omega + f] = [u^{-1}]$. Then $(\omega + f)u \in [U(R), U(R)]$ and $(\omega + f)ueu^{-1}(\omega^{-1} + f) = f$. To prove that (4) implies (1) take $x = eu^{-1}f$ and $y = fue$; we have $xy = e$ and $yx = f$. □

Let $G : \mathrm{Alg}_\ell \to \mathfrak{Grp}$ be a functor from algebras to groups and let $A \in \mathrm{Alg}_\ell$. The *connected component* of $G(A)$ is the subgroup

$$G(A) \supset G(A)^0 = \{g \mid (\exists u(t) \in G(A[t])) \quad u(0) = 1, u(1) = g\}.$$

Observe that $G(A)^0$ is a normal subgroup. We write

$$\pi_0 G(A) = G(A)/G(A)^0.$$

The *Karoubi-Villamayor* K_1-group ([20]) is

$$KV_1(A) = \pi_0(\mathrm{GL}(A)).$$

Observe that every elementary matrix is in $\mathrm{GL}(A)^0$. It follows that we have a surjective homomorphism

$$K_1(A) \twoheadrightarrow KV_1(A). \tag{3.1}$$

By [32, Proposition 1.5], the map (3.1) is an isomorphism whenever A is K_1-regular.

Lemma 3.8. *Let R be a unital ring.*

i) If $p \in \mathrm{Idem}(R)$ and $u \in U(pRp)^0$, then $u + 1 - p \in U(R)^0$.

ii) Let $x_1, \dots, x_n, y_1, \dots, y_n \in R$ such that $y_i x_j y_i = \delta_{i,j} y_i$, $x_i y_j x_i = \delta_{i,j} x_i$. Set $p_i = x_i y_i$, $q_i = y_i x_i$, $P = \bigoplus_{i=1}^n p_i R$, $Q = \bigoplus_{i=1}^n q_i R$. Then the map

$$c_{y,x} := \mathrm{End}_R(P) = \bigoplus_{i,j} p_j R p_i \to \bigoplus_{i,j} q_j R q_i = \mathrm{End}_R(Q),$$
$$a \mapsto \mathrm{diag}(y_1, \dots, y_n) a \, \mathrm{diag}(x_1, \dots, x_n)$$

is an isomorphism which sends $U(\mathrm{End}_R(P))^0$ isomorphically onto $U(\mathrm{End}_R(Q))^0$.

Proof. Straightforward. □

Proposition 3.9. *Let R be a unital purely infinite simple ring. Then the canonical map $\pi_0(U(R)) \to \pi_0(\mathrm{GL}(R)) = KV_1(R)$ is an isomorphism.*

Proof. We know from Theorem 3.4 and (3.1) that $U(R) \to KV_1(R)$ is surjective. The kernel of this map is $U(R) \cap \mathrm{GL}(R)^0$; it is clear that it contains $U(R)^0$. We have to show that

$$U(R) \cap \mathrm{GL}(R)^0 \subset U(R)^0. \tag{3.2}$$

We claim that the argument of the proof that $[\mathrm{GL}(R), \mathrm{GL}(R)] \cap U(R) \subset [U(R), U(R)]$ in [5, Theorem 2.3] can be adapted to prove (3.2). The proof in *loc.cit.* has two parts. The first part shows that if $0 \neq p \in \mathrm{Idem}(R)$ and $u \in [\mathrm{GL}(R), \mathrm{GL}(R)] \cap U(R)$ satisfies

$$u = p + (1 - p)u(1 - p) \tag{3.3}$$

then $u \in [U(R), U(R)]$. Using the same argument and taking Lemma 3.8 into account, one shows that if (3.3) is in $\mathrm{GL}(R)^0$, then it must be in $U(R)^0$. In the second part of the proof of [5, Theorem 2.3] it is observed that for adequately chosen idempotents e and $f \in T = eRe$ and elements $x_1, y_1, \dots, x_n, y_n \in R$, the assignment $a \mapsto \mathrm{diag}(y_1, \dots, y_n)a\,\mathrm{diag}(x_1, \dots, x_n)$ induces an isomorphism between R and the subring

$$M_n(T) \supset S = \{(a_{i,j}) : a_{i,n} \in Tf, a_{n,i} \in fT \text{ for all } 1 \le i \le n\}.$$

Let $\mathcal{E} \subset U(R)$ be the image under the isomorphism $U(S) \xrightarrow{\sim} U(R)$ of the subgroup generated by the set of those elementary matrices $1 + a\epsilon_{i,j}$ $i \neq j$ which are elements of S. The authors then proceed, using the argument of the proof of [22, Theorem 2.2], to show that any $u \in U(R)$ is congruent modulo $\mathcal{E}$ to one of the form of (3.3). In view of Lemma 3.8 and of the fact that elementary matrices above are in $U(S)^0$, this shows that any $u \in U(R)$ is congruent modulo $U(R)^0$ to one of the form (3.3). This finishes the proof. □

Corollary 3.10. *If R is unital, purely infinite simple and K_1-regular then $K_1(R) = \pi_0(U(R))$.*

Let A be an algebra. Identify $\mathrm{Hom}_{\mathrm{Alg}_\ell}(\ell, A) = \mathrm{Idem}_1(A)$ via the bijection $\phi \mapsto \phi(1)$. We say that two idempotents $p, q \in \mathrm{Idem}_1(A)$ are *homotopic*, and write $p \approx q$, if the corresponding homomorphisms $\ell \to A$ are homotopic.

Lemma 3.11. *Let A be an algebra and $p \in \mathrm{Idem}_1(A)$. Then $p \approx 0$ if and only if $p = 0$. If A is unital, then $p \approx 1$ if and only if $p = 1$.*

Proof. The if part of both assertions is clear. One checks that if $x \in \{0, 1\}$ and $p(t) \in \mathrm{Idem}_1(A[t])$ satisfies $p(0) = x$, then $p = x$. The only if part of both assertions follows from this. □

3.2 *kk*-maps as homotopy maps

The main goal of this section is to prove the following theorem.

Theorem 3.12. *Let E be a finite graph such that $L(E)$ is simple and R a purely infinite simple unital algebra. Assume that R is K_1-regular. Then the canonical map*

$$[L(E), R]_{M_2} \setminus \{0\} \to kk(L(E), R)$$

is an isomorphism of groups.

3.2.1 Non-purely infinite case

Let E be a finite graph such that $L(E)$ is simple. If $L(E)$ is not purely infinite, then it follows from [1, Lemma 2.9.5] and source elimination [1, Definition 6.3.26] that $L(E) \cong M_n$ for some $1 \leq n < \infty$. Hence, it is sufficient to show that $j : [M_n, R]_{M_2} \setminus \{0\} \to kk(M_n, R)$ is an isomorphism.

Recall (see 1.1) that a *C_2-algebra* is a unital algebra R together with a unital homomorphism from the Cohn algebra C_2 to R. Thus a C_2-algebra is a unital algebra together with elements $x_1, x_2, y_1, y_2 \in R$ such that $y_i x_j = \delta_{i,j}$. For example, if R is a purely infinite simple unital algebra then R is a C_2-algebra (see [5, Proposition 1.5]). Put

$$\boxplus : R \oplus R \to R, \quad a \boxplus b = x_1 a y_1 + x_2 b y_2. \tag{3.4}$$

Lemma 3.13. *Let R_1 and R_2 be C_2-algebras and let $A_1 \triangleleft R_1$ and $A_2 \triangleleft R_2$ ideals. Let $\boxplus_i : A_i \oplus A_i \to A_i$ be the sum operation (3.4). Then the maps*

$$\boxplus_1 \otimes \mathrm{id}_{A_2}, \mathrm{id}_{A_1} \otimes \boxplus_2 : A_1 \otimes A_2 \oplus A_1 \otimes A_2 \to A_1 \otimes A_2$$

are M_2-homotopic.

Proof. Straightforward from Lemma 1.2. □

Let C be an algebra, $A, B \subset C$ subalgebras and $x, y \in C$ satisfying $xAy \subset B$ and $ayxa' = aa'$ $(a, a' \in A)$; then the following map is an algebra homomorphism

$$\mathrm{ad}(x, y) : A \to B, \quad \mathrm{ad}(x, y)(a) = xay. \tag{3.5}$$

If C is unital and $y = x^{-1}$, then $\mathrm{ad}(x, y) = \mathrm{ad}(x)$ is the usual conjugation map.

Lemma 3.14. *Let A and R be algebras, with A finitely generated. Then:*

i) The canonical map

$$[A, M_\infty R] \to [A, M_\infty R]_{M_2}$$

is bijective.

ii) If furthermore R is a C_2-algebra then the canonical map

$$[A, R]_{M_2} \to [A, M_\infty R]_{M_2}$$

is an isomorphism of monoids.

Proof.

i) Because A is finitely generated,

$$[A, M_\infty R] = \operatorname*{colim}_n [A, M_{2^n} R] = \operatorname*{colim}_n [A, M_{2^n} R]_{M_2} = [A, M_\infty R]_{M_2}.$$

ii) Because R is an C_2-algebra, the map $[A,R]_{M_2} \to [A, M_\infty R]_{M_2}$ is a monoid homomorphism by Lemma 3.13. We have to prove that it is bijective. Observe that M_2R is again a C_2-algebra. Hence in view of the proof of part i), it suffices to show that $[A,R]_{M_2} \to [A, M_2R]_{M_2}$ is bijective. Let $x = \epsilon_{1,1} \otimes x_1 + \epsilon_{1,2} \otimes x_2$ and $y = \epsilon_{1,1} \otimes y_1 + \epsilon_{2,1} \otimes y_2$. By Lemma 1.2, the following diagram is M_2-homotopy commutative

$$\begin{tikzcd} M_2R \arrow[r, "\mathrm{ad}(x,y)"] \arrow[dr, equal] & \iota_2(R) \arrow[d, "\mathrm{inc}"] \\ & M_2R. \end{tikzcd}$$

It follows that the map of ii) is surjective. Injectivity follows similarly. □

Lemma 3.15. *Let $\phi, \psi : A \to R$ be algebra homomorphisms with R unital. Assume that there are $n \geq 1$ and $u \in \mathrm{GL}_n(R)$ such that $\mathrm{ad}(u)\iota_n\phi = \iota_n\psi$. Then there are elements $x, y \in R$ such that $\mathrm{ad}(x,y)\phi = \psi$. If moreover A, ϕ and ψ are unital, then we may choose x invertible and $y = x^{-1}$.*

Proof. Put $v = u^{-1}$. It follows from the identity $\mathrm{ad}(u)\iota_n\phi = \iota_n\psi$ that for every $a \in A$, $u_{1,1}\phi(a)v_{1,1} = \psi(a)$ and $u_{i,1}\phi(a) = \phi(a)u_{1,i} = 0$ if $i \neq 1$. Hence $x = u_{1,1}$ and $y = v_{1,1}$ satisfy $\mathrm{ad}(x,y)\phi = \psi$ and if ϕ and ψ are unital, then $xy = yx = 1$. □

Proposition 3.16. *Let R be a unital, purely infinite simple, K_0-regular algebra and $n \geq 1$. Then the natural monoid maps*

$$[M_n, R]_{M_2} \to [M_n, M_\infty R] \setminus \{0\} \to kk(M_n, R) \cong kk(\ell, R) \cong K_0(R)$$

are bijective. Moreover, for nonzero algebra homomorphisms $M_n \to M_\infty R$ as well as for unital algebra homomorphisms $M_n \to R$, being homotopic is the same as being conjugate.

Proof. Because as explained above, any purely infinite simple unital algebra is a C_2-algebra, the map $[M_n, R]_{M_2} \to [M_n, M_\infty R]$ is an isomorphism of monoids by Lemma 3.14. Since $(\iota_n)^* : kk(M_n, R) \to kk(\ell, R) = K_0R$ is an isomorphism, to prove that the map $[M_n, M_\infty R] \setminus \{0\} \xrightarrow{\sim} kk(M_n, R)$ is surjective, it suffices, by Corollary 3.6, to show that the image of its composite with ι_n^* contains the class of every nonzero idempotent in R. Let $p \in \mathrm{Idem}_1 R \setminus \{0\}$; by Proposition 3.5 we may choose $q \in \mathrm{Idem}_1 R$, $q \sim p$, and an embedding $\theta : M_n \to R$ sending $\epsilon_{1,1} \to q$. Hence the map of the proposition is surjective. If two homomorphisms $\phi, \psi \in \mathrm{Hom}_{\mathrm{Alg}_\ell}(M_n, M_\infty R)$ induce the same K_0-element then they are conjugate by the argument of the proof of [19, Lemma 15.23(b)], and therefore homotopic by Lemma 3.14 and Lemma 1.2 . From what we have just proved and Lemma 3.15, it follows that if two unital homomorphisms $M_n \to R$ are homotopic then they are conjugate. This finishes the proof. □

Remark 3.17. *Since K_n-regularity implies K_{n-1}-regularity* [30], *Proposition 3.16 implies Theorem 3.12 in the case when $L(E)$ is simple and not pure infinite.*

3.2.2 Lifting K-theory maps to algebra maps:K_0

Recall that a vertex $v \in E^0$ is *singular* if it is either a sink or an infinite emitter, and that it is *regular* otherwise. We write $\operatorname{reg}(E)$, $\operatorname{sink}(E)$, $\operatorname{sour}(E)$ and $\inf(E)$ for the sets of regular vertices, sinks, sources, and infinite emitters, and put $\operatorname{sing}(E) = \operatorname{sink}(E) \cup \inf(E)$.

Let R and S be unital algebras and $\xi : K_0(R) \to K_0(S)$. We call ξ *unital* if $\xi([1_R]) = [1_S]$.

Theorem 3.18. *Let E be a graph, R a purely infinite simple unital algebra, and $\xi : K_0(L(E)) \to K_0(R)$ a group homomorphism. Set $\iota : R \to M_\infty(R)$, $\iota(a) = \epsilon_{1,1} \otimes a$.*

i) If E is countable, then there exists a nonzero algebra homomorphism $\psi : L(E) \to M_\infty R$ such that $K_0(\psi) = K_0(\iota)\xi$.

ii) If E is finite, then there exists a nonzero algebra homomorphism $\psi : L(E) \to R$ such that $K_0(\psi) = \xi$.

iii) If E^0 is finite, E^1 countable and ξ unital, then there is a unital homomorphism $\phi : L(E) \to R$ such that $K_0(\phi) = \xi$.

Proof. Assume first that E is countable and row-finite. By Theorem 3.4 there are orthogonal idempotents $\{p_e : e \in E^1\} \cup \{p_v : v \in \operatorname{sing}(E)\} \subset \operatorname{Idem}_\infty(R) \setminus \{0\}$ such that $[p_v] = \xi[v]$ and $[p_e] = \xi[ee^*]$ in $K_0(R)$ ($v \in \operatorname{sink}(E)$, $e \in E^1$). If $e \in E^1$ and $r(e) \in \operatorname{reg}(E)$ then

$$[p_e] = [\sum_{f \in E^1, s(f)=r(e)} p_f].$$

Hence for $\sigma_f = \sum_{f \in E^1, s(f)=r(e)} p_f$ there are elements $x_e, y_e \in M_\infty(R)$ implementing an MvN equivalence $p_e \sim \sigma_e$. Similarly if $e \in E^1$ and $r(e) = v \in \operatorname{sink}(E)$, then there is an MvN equivalence $(x_e, y_e) : p_e \sim p_v$ with $x_e, y_e \in M_\infty R$. One checks that the prescriptions

$$\psi(e) = x_e, \psi(e^*) = y_e \quad (e \in E^1), \quad \psi(v) = p_v \quad (v \in \operatorname{sink}(E))$$

define a nonzero algebra homomorphism $\psi : L(E) \to M_\infty R$. Let $\tau : M_\infty M_\infty \to M_\infty M_\infty$, $\tau(x \otimes y) = y \otimes x$; one checks that $\tau \otimes Id_R$ induces the identity of $K_0(M_\infty R)$. By construction $K_0(\psi)$ agrees with $K_0(\tau \otimes 1)K_0(\iota)\xi = K_0(\iota)\xi$ on the classes of those vertices which are sinks and on those of elements of the form ee^* ($e \in E^1$). Since the latter generate $K_0(L(E))$ (by [1, Theorem 3.2.5]), we have $K_0(\psi) = K_0(\iota)\xi$.

For general countable E, let E_δ be a desingularization and $f : L(E) \to L(E_\delta)$ the canonical homomorphism [2, Section 5]; then $K_0(f)$ is an isomorphism. Hence by what we have just proved, there exists an algebra homomorphism $\psi' : L(E_\delta) \to M_\infty(R)$ such that $K_0(\psi') = K_0(\iota)\xi K_0(f)^{-1}$. Then $\phi = \psi' f$ satisfies $K_0(\psi) = K_0(\iota)\xi$. This proves i). Next assume that E^1 is countable, that E^0 is finite and that $\xi([1_{L(E)}] = [1_R]$. Let $\psi : L(E) \to M_\infty(R)$ be a homomorphism such that $K_0(\iota)\xi = K_0(\psi)$. Set $p = \psi(1)$; then $\psi(L(E)) \subset pM_\infty(R)p$ and there is an MvN equivalence $(x, y) : p \sim \epsilon_{1,1}$. It follows that there is a unique unital homomorphism $\phi : L(E) \to R$ such that $\iota\phi = \operatorname{ad}(y, x)\psi$. By Lemma 1.2, ϕ satisfies the requirements of iii).

Finally assume that E is finite. By Corollary 3.6 and Proposition 3.5 there are orthogonal idempotents $\{p_e : e \in E^1\} \cup \{p_v : v \in \mathrm{sink}(E)\} \subset \mathrm{Idem}_1(R) \setminus \{0\}$ such that $[p_v] = \xi[v]$ and $[p_e] = \xi[ee^*]$ ($v \in \mathrm{sink}(E)$, $e \in E^1$). If $e \in E^1$ and $r(e) \notin \mathrm{sink}(E)$ then by Corollary 3.6, for σ_e as in the proof of Theorem 3.18 there are elements $x_e \in p_e R \sigma_e$ and $y_e \in \sigma_e R p_e$ such that $p_e = x_e y_e$ and $\sigma_e = y_e x_e$. Similarly, if $e \in E^1$ and $r(e) = v \in \mathrm{sink}(E)$, then there are $x_e \in p_e R p_v$ and $y_e \in p_v R p_e$ such that $y_e x_e = p_v$ and $x_e y_e = p_e$. One checks that the prescriptions

$$\psi(e) = x_e, \psi(e^*) = y_e \quad (e \in E^1), \quad \psi(v) = p_v \quad (v \in \mathrm{sink}(E))$$

define a nonzero algebra homomorphism $\psi : L(E) \to R$ such that $K_0(\psi) = \xi$. □

Corollary 3.19. *Let R be a unital purely infinite algebra and E a graph such that $L(E)$ is simple.*

i) If E is countable, then $L(E)$ embeds as a subalgebra of $M_\infty R$.

ii) If E^1 is countable, E^0 is finite and $[1_R] = 0$ in $K_0(R)$, then $L(E)$ embeds as a unital subalgebra of R.

iii) If E is finite then $L(E)$ embeds as a subalgebra of R.

Proof. Apply Theorem 3.18 to the trivial homomorphism $\xi = 0$. □

Remark 3.20. *It follows from Corollary 3.19 that any purely infinite algebra R such that $[1_R] = 0$ contains L_2 as a unital subalgebra. Hence by* [9, Theorem 4.1]*, if E is countable (resp. finite), then $L(E)$ embeds as a subalgebra (resp. a unital subalgebra) of R, independently of whether $L(E)$ is simple or not.*

Corollary 3.21. *Let E be a countable graph with finite E^0. Assume that $K_0(L(E))$ is finite and let $d_1, \dots, d_n$, $d_i \backslash d_{i+1}$ be its invariant factors. Let $j : \mathrm{Alg}_\ell \to kk$ be the canonical functor* ([15])*. Then there is an algebra homomorphism $\psi : L(E) \to M_\infty(\bigoplus_{i=1}^n L_{d_i+1})$ such that $j(\psi)$ is an isomorphism in kk. If moreover $L(E)$ is purely infinite simple then there is an algebra homomorphism $\phi : \bigoplus_{i=1}^n L_{d_i+1} \to M_\infty L(E)$ such that $\iota^{-1} j(\phi)$ and $\iota^{-1} j(\psi)$ are inverse isomorphisms in kk. If E is finite then the same holds with $L(E)$ substituted for $M_\infty(L(E))$.*

Proof. Assume that E is countable with finite E^0. By part ii) of Theorem 3.18, for each $1 \le i \le n$, there is a homomorphism $\psi_i : L(E) \to M_\infty L_{d_i+1}$ such that $K_0(\psi_i)$ is the projection from $K_0(L(E)) = \bigoplus_{j=1}^n \mathbb{Z}/d_j$ onto the copy of $\mathbb{Z}/d_i$. The map

$$\psi = (\psi_1, \dots, \psi_n) : L(E) \to M_\infty(\bigoplus_{i=1}^b L_{d_i+1})$$

then induces an isomorphism in K_0. In view of Lemma 2.20 and of the fact that, since $K_0(L(E))$ is finite, $\mathrm{Ker}(I - A_E^t) = 0$, this implies that $K_1(\psi)$ is an isomorphism too. Hence

$j(\psi)$ is an isomorphism by Proposition 2.11. Assume furthermore that $L(E)$ is purely infinite simple. Consider the graph

$$F = \sqcup_{i=1}^{n} \mathcal{R}_{d_i+1}.$$

Then $L(F) = \bigoplus_{j=1}^{n} L_{d_{j+1}}$. The homomorphism ϕ of the corollary is obtained by applying Theorem 3.18 to $\xi = K_0(\psi)^{-1}\iota : K_0(L(F)) \to K_0(L(E))$. This proves the first assertion of the corollary; the second, for finite E, is proved similarly, using part iii) of Theorem 3.18. □

Let E be a finite graph; if $X \subset L(E)$, write $\operatorname{span}(X)$ for the subspace generated by X. In the following proposition and elsewhere we consider the following "diagonal" subalgebra of $L(E)$

$$DL(E) = \operatorname{span}(\operatorname{sink}(E) \cup \{ee^* : e \in E^1\}) \subset L(E).$$

Proposition 3.22 below will be needed in the next section.

Proposition 3.22. *Let E and R be as in part iii) of Theorem 3.18. Assume that $L(E)$ is simple and let $\phi, \psi : L(E) \to R$ be nonzero algebra homomorphisms such that $K_0(\phi) = K_0(\psi)$. Then there exists an algebra homomorphism $\psi' : L(E) \to R$ such that $j(\psi) = j(\psi')$ in kk and $\psi'_{|DL(E)} = \phi_{|DL(E)}$.*

Proof. First assume that $\phi(1) = \psi(1) = p$. For each $e \in E^1$ and each $v \in \operatorname{sink}(E)$ choose MvN equivalences $(x_e, y_e) : \phi(ee^*) \sim \psi(ee^*)$ and $(x_v, y_v) : \phi(v) \sim \psi(v)$. Define $x = \sum_{e\in E^1} x_e + \sum_{v\in\operatorname{sink}(E)} x_v$ and $y = \sum_{e\in E^1} y_e + \sum_{v\in\operatorname{sink}(E)} y_v$. Then $x, y \in pRp$ and $xy = p = yx$. Hence $\psi' : L(E) \to R$, $\psi'(a) = x\psi(a)y$ satisfies $\psi'_{|DL(E)} = \phi_{|DL(E)}$. Moreover $j(\psi) = j(\psi')$ by Lemma 1.2. Next assume that $\phi(1) \neq \psi(1)$ and that none of them is equal to 1. Then by Corollary 3.7, there is an element $u \in U(R)$ such that $u\phi(1)u^{-1} = \psi(1)$. Hence we can replace ψ by $a \mapsto u\psi(a)u^{-1}$ and we are in the above case. Finally, if $\phi(1) \neq \psi(1)$ and one of them, say $\psi(1)$, is 1, we can replace ϕ by a unital homomorphism by Theorem 3.18 and we are again in the first case. □

3.2.3 Lifting K-theory maps to algebra maps: K_0 and K_1

Let E be a finite graph; below we will give a right inverse of the surjective map

$$\partial : K_1(L(E)) \twoheadrightarrow \operatorname{Ker}(I - A_E^t). \tag{3.6}$$

Observe that the analogue of the map (3.6) in the C^*-algebra setting is an isomorphism; an explicit formula for its inverse was given by Rørdam in [25, page 33] in the case when E is regular. We shall show that in the purely algebraic case considered here, the same formula gives a right inverse of (3.6), even for singular E.

Let $I - B_E^t$ be as in Remark 2.9. Let

$$s^* : \mathbb{Z}^{E^0} \to \mathbb{Z}^{(E^1)\sqcup\operatorname{sink}(E)},\ s^*(\chi_v) = \begin{cases} \sum_{s(e)=v} \chi_e & v \in \operatorname{reg}(E) \\ \chi_v & v \in \operatorname{sink}(E) \end{cases}$$

By [4, formula 4.1], we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^{E^1} \arrow[r, "I-B_E^t"] & \mathbb{Z}^{E^1\sqcup \mathrm{sink}(E)} \\
\mathbb{Z}^{\mathrm{reg}(E)} \arrow[u, "s^*"] \arrow[r, "I-A_E^t"'] & \mathbb{Z}^{E^0} \arrow[u, "s^*"']
\end{tikzcd}$$

In particular, s^* maps $\mathrm{Ker}(I - A_E^t) \to \mathrm{Ker}(I - B_E^t)$. Furthermore it is an isomorphism by the dual of [4, Lemma 4.3]. Let $x = (x_v) \in \mathrm{Ker}(I - A_E^t) \subseteq \mathbb{Z}^{\mathrm{reg}(E)}$. Set $y = s^*(x) \in \mathrm{Ker}(I - B_E^t)$. Let

$$S = \{(e, j) : y_e \neq 0, 1 \leq j \leq |y_e|\} \tag{3.7}$$

Consider the diagonal matrix $V = V(x) \in M_S(L(E))$,

$$V_{(e,j),(e,j)} = \begin{cases} e & \text{if } y_e > 0 \\ e^* & \text{if } y_e < 0 \end{cases}$$

Let $p = 1 - VV^*$, $q = 1 - V^*V$. Observe that $p, q \in M_S(DL(E))$. Moreover, for $\Lambda = E^1 \sqcup \mathrm{sink}(E)$, $DL(E) \cong \ell^\Lambda$ and we may regard $p = (p_\alpha)$ and $q = (q_\alpha)$ as Λ-tuples of diagonal matrices in M_S whose entries are in $\{0, 1\}$. One checks, using that $y \in \mathrm{Ker}(I - B_E^t)$, that for each $\alpha \in \Lambda$, p_α and q_α have the same number of nonzero coefficients. Hence we may choose for each α a matrix $W_\alpha \in p_\alpha M_S q_\alpha$ with coefficients in $\{0, 1\}$ such that $W_\alpha W_\alpha^t = p_\alpha$ and $W_\alpha^t W_\alpha = q_\alpha$. Further, we may even require that

$$(W_\alpha)_{(e,i),(f,j)} = 1 \Rightarrow (p_\alpha)_{(e,i),(e,i)} = (q_\alpha)_{(f,j),(f,j)} = 1. \tag{3.8}$$

We shall use (3.8) in the proof of Lemma 3.24 below. Let $W = W(x) \in M_S(DL(E))$ be the matrix corresponding to (W_α); then

$$WW^* = 1 - VV^*, \quad W^*W = 1 - V^*V \text{ and } W^*V = V^*W = 0. \tag{3.9}$$

Put

$$U(x) = V(x) + W(x). \tag{3.10}$$

It follows from (3.9) that $U(x)U(x)^* = U(x)^*U(x) = 1$.

Proposition 3.23. *Let $x \in \mathrm{Ker}(I-A_E^t)$, $[U(x)] \in K_1(L(E))$ the class of the element (3.10) and $\partial : K_1(L(E)) \to \mathrm{Ker}(I - A_E^t)$ as in (3.6). Then $\partial(U(x)) = -x$.*

Proof. We keep the notation of the paragraph preceding the proposition. Let $C(E)$ be the Cohn path algebra; consider the subalgebra

$$DC(E) = \mathrm{span}(\{q_v : v \in \mathrm{reg}(E)\} \cup \mathrm{sink}(E) \cup \{ee^* : e \in E^1\} \subset C(E).$$

Consider the diagonal matrix $\hat{V}$ defined by the same prescription as V but regarded now as an element of $M_S(C(E))$. Let $\hat{W} \in M_S(DC(E))$ be the image of W under the map induced by the obvious inclusion $DL(E) \subset DC(E)$; put $\hat{U} = \hat{V} + \hat{W}$. Consider the matrix

$$h = \begin{bmatrix} 2\hat{U} - \hat{U}\hat{U}^*\hat{U} & \hat{U}\hat{U}^* - 1 \\ 1 - \hat{U}^*\hat{U} & \hat{U}^* \end{bmatrix} \in M_{S\sqcup S}(C(E)).$$

By [12, Section 2.4] (see also [26, Definition 9.1.3]), h is invertible and

$$\partial([U]) = [h1_S h^{-1}] - [1_S].$$

Here 1_S is the $S \times S$ identity matrix, located in the upper left corner.

One checks that $\hat{U} = \hat{U}\hat{U}^*\hat{U}$, and that

$$\partial([U]) = [1 - \hat{U}^*(x_i)\hat{U}] - [1 - \hat{U}\hat{U}^*] \in K_0(\bigoplus_{v\in\operatorname{reg}(E)} \ell q_v) \cong \mathbb{Z}^{\operatorname{reg}(E)}. \tag{3.11}$$

One checks, using (3.11) and the fact that $x \in \operatorname{Ker}(I - A_E^t)$, that

$$\partial([U]) = -[\sum_{v\in E^0} x_v q_v].$$

This finishes the proof. □

In principle, the assignment $\operatorname{Ker}(I - A_E^t) \to K_1(L(E))$, $[x] \mapsto [U(x)]$ is just a set theoretic map. A group homomorphism with similar properties is obtained as follows. Choose a basis $\mathfrak{B} = \{x_i\}$ of the free abelian group $\operatorname{Ker}(I - A_E^t)$; let

$$\gamma = \gamma_{\mathfrak{B}} : \operatorname{Ker}(I - A_E^t) \to K_1(L(E)), \quad \gamma(\sum_i n_i x_i) = \sum_i n_i[U(x_i)]. \tag{3.12}$$

Let E be a finite graph such that $L(E)$ is purely infinite simple. Then $\operatorname{sink}(E) = \emptyset$, by [1, Theorem 3.1.10 (iii) and (iii')]. Let $\phi : L(E) \to R$ be a unital algebra homomorphism with R purely infinite simple. Set

$$R_\phi = \{x \in R : \phi(ee^*)x = x\phi(ee^*), \quad \text{for all } e \in E^1\}. \tag{3.13}$$

Note that

$$R_\phi = \oplus_{e\in E^1}\phi(ee^*)R\phi(ee^*).$$

Because $L(E)$ is simple, $\phi(\alpha) \neq 0$ ($\alpha \in E^1$), whence each of the inclusions $\phi(\alpha\alpha^*)R\phi(\alpha\alpha^*) \subset R$ induces an isomorphism in K_1. Hence the direct sum $R_\phi \subset R^{E^1}$ of those inclusions induces an isomorphism

$$K_1(R_\phi) = \bigoplus_{e\in E^1} K_1(\phi(ee^*)R\phi(ee^*)) \xrightarrow{\sim} K_1(R)^{E^1}. \tag{3.14}$$

Let $\iota : K_1(R_\phi) \to K_1(R)$ be the map induced by the inclusion $R_\phi \subset R$. Consider the bilinear map

$$\langle \cdot, \cdot \rangle : \mathbb{Z}^{E^1} \times K_1(R_\phi) \to K_1(R), \quad \langle x, y \rangle = \sum_i x_i \iota(y_i). \tag{3.15}$$

Observe that $\langle \cdot, \cdot \rangle$ is a perfect pairing; indeed the adjoint homomorphism $K_1(R_\phi) \to K_1(R)^{E^1}$ is the isomorphism (3.14).

Lemma 3.24. *(cf.*[25, Lemma 3.5].*) Let E be a finite graph such that $L(E)$ is purely infinite simple, R a purely infinite simple unital algebra, and ϕ and $\psi : L(E) \to R$ unital homomorphisms. Assume that ϕ and ψ agree on $DL(E)$. Let*

$$u = \sum_{\alpha \in E^1} \psi(\alpha)\phi(\alpha^*) \in R_\phi = R_\psi.$$

Then

$$K_1(\psi)(\gamma(x)) = \langle x, [u] \rangle + K_1(\phi)(\gamma(x)) \textit{ for all } x \in \operatorname{Ker}(I - A_E^t). \tag{3.16}$$

Proof. Observe that $\psi(e)\phi(e^*) \in U(\phi(e)R\phi(e^*))$ $(e \in E^1)$, whence $u \in U(R_\phi)$. Let $\{\chi_e : e \in I\}$ be the canonical basis of $\mathbb{Z}^I$. One checks that

$$\langle \chi_e, [u] \rangle = \psi(e)\phi(e^*) + 1 - \phi(ee^*). \tag{3.17}$$

To prove the lemma, we may assume that x is an element of the basis $\mathfrak{B}$ of $\operatorname{Ker}(I - A_E^t)$ used in (3.12) to define γ . Then taking (3.17) into account and adopting the notations and conventions used in the definition of $U(x)$, one computes that the right hand side of equation (3.16) is

$$\sum_{y_e > 0} y_e[\psi(e)\phi(e^*) + 1 - \phi(ee^*)] + [\phi(U(x))] - \sum_{y_e < 0} y_e[\phi(e)\psi(e^*) + 1 - \phi(ee^*)]. \tag{3.18}$$

Let S be as in (3.7). Consider the diagonal matrices $P, Q \in M_S L(E)$ with diagonal entries as follows

$$P_{(e,j),(e,j)} = \begin{cases} \psi(e)\phi(e^*) + 1 - \phi(ee^*) & \text{if } y_e > 0 \\ 1 & \text{if } y_e < 0 \end{cases}$$

$$Q_{(e,j),(e,j)} = \begin{cases} 1 & \text{if } y_e > 0 \\ \phi(e)\psi(e^*) + 1 - \phi(ee^*) & \text{if } y_e < 0 \end{cases}$$

Observe that (3.18) is $[P\phi(U(x))Q]$. Hence it suffices to show that $K_1(\psi)(U(x)) = [P\phi(U(x))Q]$; we shall show that in fact $\psi(U(x)) = P\phi(U(x))Q$. Recall that $U(x) = V(x) + W(x)$. It is immediate from the definition of $V(x)$ that $\psi(V(x)) = P\phi(V(x))Q$. Hence since W has coefficients in $DL(E)$, it only remains to show that $\phi(W(x)) = P\phi(W(x))Q$. A tedious but straightforward calculation, using (3.8) shows that

$$\phi(W(x)_\alpha)_{(e,i),(f,j)} = (P\phi(W(x)_\alpha)Q)_{(e,i),(f,j)} \qquad \forall (e,i), (f,j) \in S, \quad \alpha \in \Lambda.$$

This completes the proof. □

Remark 3.25. *Recall that if $L(E)$ is unital, we have an exact sequence*

$$0 \to K_0(L(E)) \otimes K_1(\ell) \to K_1(L(E)) \to \mathrm{Ker}(I - A_E^t) \to 0.$$

It follows from Lemma 2.20 that if $R \in \mathrm{Alg}_\ell$ is K_1-regular and $\xi \in kk(L(E), R)$, then $K_1(\xi)$ restricts to the composite of $K_0(\xi) \otimes \mathrm{id}$ with the cup product $K_0(R) \otimes K_1(\ell) \to K_1(R)$.

Theorem 3.26. *Let E be a finite graph and S an algebra. Assume that $L(E)$ is simple and that S is unital, purely infinite simple and K_1-regular. Let $\xi_0 : K_0(L(E)) \to K_0(S)$ and $\xi_1 : \mathrm{Ker}(I - A_E^t) \to K_1(S)$ be group homomorphisms. Then there exists a nonzero algebra homomorphism $\phi : L(E) \to S$ such that $K_0(\phi) = \xi_0$ and such that $K_1(\phi)\gamma = \xi_1$. If moreover ξ_0 is unital then we can choose ϕ to be a unital homomorphism $L(E) \to S$.*

Proof. By Theorem 3.18, there exists a nonzero algebra homomorphism $\phi_0 : L(E) \to S$ such that $K_0(\phi_0) = \xi_0$, and if ξ_0 is unital then we may choose ϕ_0 unital. If $L(E)$ is not purely infinite, then by Remark 3.17, $L(E) \cong M_n$ for some $1 \leq n < \infty$. Hence $\mathrm{Ker}(I - A_E^t) = 0$ and $K_1(L(E)) = K_0(L(E)) \otimes U(\ell)$. Assume that $L(E)$ is purely infinite simple. Let $R = \phi_0(1) S \phi_0(1)$ and let $\bar{\phi}_0 : L(E) \to R$ be the corestriction of ϕ_0 and $\mathrm{inc} : R \to S$ the inclusion. Since $\mathrm{Ker}(I - A_E^t)$ is a direct summand of $\mathbb{Z}^{\mathrm{reg}(E)}$ and $\langle \cdot, \cdot \rangle$ is a perfect pairing, there exists $\theta \in K_1(R_{\bar{\phi}_0})$ such that

$$\langle -, \theta \rangle = K_1(\mathrm{inc})^{-1} \xi_1 - K_1(\bar{\phi}_0)\gamma.$$

Because $R_{\bar{\phi}_0}$ is a direct sum of purely infinite simple algebras, by Theorem 3.4 there exists $g \in U(R_{\bar{\phi}_0})$ such that $[g] = \theta$. Define $\bar{\phi} : L(E) \to R$ by setting $\bar{\phi}_{|E^0} = (\bar{\phi}_0)_{|E^0}$, $\bar{\phi}(e) = g\bar{\phi}_0(e)$, $\bar{\phi}(e^*) = \bar{\phi}_0(e^*)g^{-1}$. Observe that $\bar{\phi}$ and $\bar{\phi}_0$ agree on $DL(E)$; in particular, $\bar{\phi}$ is unital. Hence by Lemma 3.24, we have

$$K_1(\bar{\phi})\gamma = K_1(\bar{\phi}_0)\gamma + \langle -, [u] \rangle.$$

But it follows from the formula defining u in Lemma 3.24 and the definition of $\bar{\phi}$ that $u = g$. Hence

$$K_1(\bar{\phi})\gamma = K_1(\mathrm{inc})^{-1}\xi_1.$$

Set $\phi = \mathrm{inc}\,\bar{\phi}$; then $K_1(\phi)\gamma = \xi_1$. Further, $K_0(\phi) = K_0(\phi_0) = \xi_0$ because ϕ and ϕ_0 agree on E^0. It is clear by construction that if ϕ_0 is unital homomorphism, then ϕ is also unital. □

3.2.4 Lifting *kk*-maps to algebra maps

Let $\phi, \psi : A \to B$ be algebra homomorphisms. Put

$$C_{\phi,\psi} = \{(a, f) \in A \oplus B[t] : f(0) = \phi(a), f(1) = \psi(a)\}.$$

Let $\pi : C_{\phi,\psi} \to A$, $\pi(a, f) = a$; we have an algebra extension

$$\Omega B \to C_{\phi,\psi} \xrightarrow{\pi} A. \tag{3.19}$$

Lemma 3.27. *Let $j : \mathrm{Alg}_\ell \to kk$ be the canonical functor. The sequence (3.19) induces the following distinguished triangle in kk*

$$j(\Omega B) \longrightarrow j(C_{\phi,\psi}) \xrightarrow{j(\pi)} j(A) \xrightarrow{j(\phi)-j(\psi)} j(B).$$

Proof. By definition of $C_{\phi,\psi}$, we have a map of extensions

$$\begin{tikzcd} \Omega B \arrow[r] \arrow[d, equal] & C_{\phi,\psi} \arrow[r, "\pi"] \arrow[d] & A \arrow[d, "(\phi,\psi)"] \\ \Omega B \arrow[r] & B[t] \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"] & B \oplus B \end{tikzcd} \tag{3.20}$$

Let $\Delta : B \to B \oplus B$, $\Delta(b) = (b, b)$. One checks that the *kk*-triangle associated to the bottom row of (3.20) is isomorphic to

$$j(\Omega B) \xrightarrow{0} j(B) \xrightarrow{j(\Delta)} j(B \oplus B) \xrightarrow{[\mathrm{id},-\mathrm{id}]} B.$$

Let $\xi : j(A) \to j(B)$ be the boundary map in the triangle induced by (3.19). It follows from (3.20) that there is a commutative diagram

$$\begin{tikzcd} j(A) \arrow[r, "\xi"] \arrow[d, "(j(\phi),j(\psi))"] & j(B) \arrow[d, equal] \\ j(B) \oplus j(B) \arrow[r, "{[\mathrm{id},-\mathrm{id}]}"'] & j(B). \end{tikzcd}$$

Hence $\xi = j(\phi) - j(\psi)$. □

Let R be a unital, purely infinite simple algebra, let E be a finite graph such that $L(E)$ is simple and let $\phi, \psi : L(E) \to R$ be nonzero algebra homomorphisms which agree on $DL(E)$. Let R_ϕ be as in (3.13). Put $p = \phi(1)$ and let

$$B = pRp.$$

By corestriction, we may consider ϕ and ψ as homomorphisms $L(E) \to B$. Let

$$C = \{f \in B[t] \mid (\exists a \in L(E))\ \ \phi(a) = f(0),\ \ \psi(a) = f(1)\}.$$

Since $L(E)$ is simple, the map

$$C_{\phi,\psi} \to C, \quad (a, f) \mapsto f$$

is an isomorphism. We shall identify $C = C_{\phi,\psi}$. Assume that R is K_1-regular. Then B is K_1-regular also, whence $K_0(\Omega B) = KV_1(B) = K_1(B)$. Hence the extension (3.19) induces an exact sequence

$$K_1(B) \xrightarrow{\partial'} K_0(C) \xrightarrow{\pi} K_0(L(E)) \xrightarrow{\phi-\psi} K_0(B) \tag{3.21}$$

The following two lemmas adapt [25, Lemmas 3.2 and 3.3] to the purely algebraic case.

Lemma 3.28. *Let u be as in Lemma 3.24, ∂' as in (3.21) and $\langle\cdot,\cdot\rangle$ as in (3.15). Let $\sigma \in K_0(C)^{E^1}$, $\sigma_e = [\phi(ee^*)]$. Then for every $x \in \mathbb{Z}^{E^1}$ we have*

$$\langle x, [u]\rangle = -\langle (I - A_E^t)x, \sigma\rangle$$

Proof. Let $u_e = u\phi(ee^*) + 1 - \phi(ee^*)$ ($e \in E^1$). By Whitehead's lemma there is $U_e(t) \in \mathrm{GL}(B[t])$ such that $U_e(0) = 1$ and $U_e(1) = \mathrm{diag}(u_e, u_e^{-1})$. Set $V_e(t) = U_e(t)\,\mathrm{diag}(\phi(e), 0)$, $W_e(t) = \mathrm{diag}(\phi(e^*), 0)U_e(t)^{-1}$. Now proceed as in the proof of [25, Lemma 3.2], substituting $U_e(t)^{-1}$ and $W_e(t)$ for $U_e(t)^*$ and $V_e(t)^*$. □

Lemma 3.29. *Let $\lambda : R_\phi \to R_\phi$, $\lambda(a) = \sum_{e\in E^1} \phi(e)a\phi(e^*)$. If $j(\phi) = j(\psi) \in kk(L(E), B)$ then there is $v \in U(R_\phi)$ such that $[u] = [v^{-1}\lambda(v)] \in K_1(R_\phi)$.*

Proof. The proof is the same as that of [25, Lemma 3.3]. □

Let S be an algebra, E a finite graph, and $\phi, \psi : L(E) \to S$ algebra homomorphisms. We say that ϕ and ψ are 1-*step* ad-*homotopic* if either

a) there is an MvN equivalence $(u, u') : \psi(1) \sim \phi(1)$ such that $\mathrm{ad}(u, u')\phi = \psi$,

   or

b) ϕ and ψ agree on $DL(E)$ and for $B = \phi(1)S\phi(1)$ there is $U(t) \in U(B_\phi[t])$ such that $U(0) = 1$ and $\phi_{i+1}(e) = U(1)\phi(e)$, $\psi(e^*) = \phi_i(e^*)U(1)^{-1}$.

We say that ϕ and ψ are *n-step* ad-*homotopic* if there is a sequence of algebra homomorphisms $\phi_i : L(E) \to S$, $1 \le i \le n$, such that $\phi_1 = \phi$, $\phi_n = \psi$, and ϕ_i and ϕ_{i+1} are 1-step ad-homotopic for $1 \le i \le n-1$. Two unital homomorphisms ϕ and ψ are *n-step unitally* ad-*homotopic* if they are n-ad-homotopic and the ϕ_i can be chosen to be unital for all $1 \le i \le n$. Call ϕ and ψ (unitally) ad-*homotopic* if they are n-step (unitally) ad-homotopic for some n.

Remark 3.30. *Observe that if in a) above ϕ and ψ are unital, then $u \in U(S)$, so that ϕ and ψ are conjugate in the usual, unital sense. Note also that in the situation b) above, ϕ and ψ are homotopic. It follows that a unital homomorphism $\phi : L(E) \to L(E)$ is unitally* ad-*homotopic to the identity if and only if it is homotopic to* ad(u) *for some $u \in U(L(E))$.*

Theorem 3.31. *Let E be a finite graph and R a unital algebra. Assume that $L(E)$ and R are purely infinite simple and that R is K_1-regular. Then the canonical map*

$$j : [L(E), R]_{M_2} \setminus \{0\} \to kk(L(E), R) \tag{3.22}$$

is an isomorphism of groups. In particular, $[L(E), R]_{M_2} \setminus \{0\}$ is the group completion of $[L(E), R]_{M_2}$. Moreover, we have the following:

i) If $\xi \in kk(L(E), R)$, then there is a nonzero algebra homomorphism $\phi : L(E) \to R$ such that $j(\phi) = \xi$. Moreover, ϕ may be chosen to be unital if ξ is.

ii) Two nonzero (unital) algebra homomorphisms $\phi, \psi : L(E) \to R$ satisfy $j(\phi) = j(\psi)$ if and only if they are M_2-homotopic if and only if they are (unitally) ad*-homotopic if and only if they are* 3*-step (unitally)* ad*-homotopic.*

Proof. The map $[L(E), R]_{M_2} \to kk(L(E), R)$ is a monoid homomorphism by the same argument as in Proposition 3.16.

Let $\xi \in kk(L(E), R)$ and let $\gamma : \operatorname{Ker}(I - A_E^t) \to K_1(L(E))$ be as in (3.12). By Theorem 3.26 there exists a nonzero algebra homomorphism $\psi : L(E) \to R$ such that $K_0(\xi) = K_0(\psi)$ and $K_1(\xi)\gamma = K_1(\psi)\gamma$. Let $B = \psi(1)R\psi(1)$, $\mathrm{inc} : B \to R$ the inclusion and $\bar{\psi} : L(E) \to B$ the corestriction of ψ. Then $j(\mathrm{inc})$ is an isomorphism, and for $\eta = j(\mathrm{inc})^{-1}\xi$ we have $\eta - j(\bar{\psi}) \in kk(L(E), B)^2 \cong \operatorname{Ext}^1_{\mathbb{Z}}(K_0(L(E)), K_1(B))$, by Theorem 2.21. To prove that the map of the theorem is surjective, it suffices to show that there exists $u \in U(R_\psi)$ such that for $\phi : L(E) \to B$, $\phi(e) = u\psi(e)$, $\phi(e^*) = \psi(e^*)u^{-1}$, we have $\eta - j(\bar{\psi}) = j(\phi) - j(\bar{\psi})$. The argument of the proof of [25, Theorem 3.1] shows this. Next we show that (3.22) is injective, and that the different notions of homotopy agree. It follows from Lemma 3.14, Lemma 1.2 and the definition of ad-homotopy that ad-homomotopic homomorphisms $L(E) \to R$ are M_2-homotopic, and from the universal property of kk that j sends homotopic maps to equal maps. Conversely, let $\phi, \psi : L(E) \to R$ be algebra homomorphisms such that $j(\phi) = j(\psi)$. Then $K_0(\phi) = K_0(\psi)$, whence there exist for each $e \in E^1$ elements $u_e \in \phi(ee^*)R\psi(ee^*)$ and $u'_e \in \psi(ee^*)R\phi(ee^*)$ such that $u_e u'_e = \phi(ee^*)$ and $u'_e u_e = \psi(ee^*)$. Thus $u = \sum_{e\in E^1} u_e \in \phi(1)R\psi(1)$, $u' = \sum_{e\in E^1} u'_e \in \psi(1)R\phi(1)$, and $\psi' = \operatorname{ad}(u, u')\psi$ agrees with ϕ on $DL(E)$. Hence upon spending a 1-step ad-homotopy from ψ to ψ' if necessary, we may assume that ϕ and ψ agree on $DL(E)$. Let $B = \phi(1)R\phi(1)$ and let $u \in B_\phi$ be as in Lemma 3.24; we have

$$\psi(e) = u\phi(e), \quad \psi(e^*) = \phi(e^*)u^{-1}. \tag{3.23}$$

Observe that, because R is purely infinite and K_1-regular, the same is true of B. By Lemma 3.29 and K_1-regularity of B, there are $v \in U(B_\phi)$ and $U(t) \in U(B[t])$ such that $U(0) = 1$ and $U(1) = v^{-1}\lambda(v)u^{-1}$. Hence, upon using a second 1-step ad-homotopy, we may assume that $u = v^{-1}\lambda(v)$. A calculation shows that $\psi = \operatorname{ad}(v)\phi$. Thus a third 1-step ad-homotopy concludes the proof of the nonunital part of the theorem. If ξ is unital, then by Theorem 3.26 there is a unital algebra homomorphism $\psi : L(E) \to R$ such that $K_0(\xi) = K_0(\psi)$ and $K_1(\xi)\gamma = K_1(\psi)\gamma$. The argument used above to prove the surjectivity of (3.22) subsituting ξ for η, shows that there is a unital algebra homomorphism $\phi : L(E) \to R$ such that $j(\phi) = \xi$. Finally the same argument used above for nonunital homomorphisms shows that two unital homomorphisms $L(E) \to R$ go to the same element in kk if and only if they are unitally 3-step ad-homotopic. □

Remark 3.32. *By Lemma 3.14, we have that if R and $L(E)$ are as in Theorem 3.31, then $[L(E), M_\infty R]$ is an abelian monoid, with operation induced by the map (3.4), and the canonical homomorphism $[L(E), M_\infty R] \setminus \{0\} \to kk(L(E), R)$ is an isomorphism of groups.*

3.3 Homotopy classification

The main result of this thesis is the following:

Theorem 3.33. *Let E and F be finite graphs such that $L(E)$ and $L(F)$ are purely infinite simple. Let $\xi_0 : K_0(L(E)) \xrightarrow{\sim} K_0(L(F))$ be an isomorphism. Then*

- *There exist nonzero algebra homomorphisms $\phi : L(E) \leftrightarrow L(F) : \psi$ such that $K_0(\phi) = \xi$, $K_0(\psi) = \xi^{-1}$, $\psi\phi \approx_{M_2} \mathrm{id}_{L(E)}$ and $\phi\psi \approx_{M_2} \mathrm{id}_{L(F)}$.*
- *If moreover $\xi([1_{L(E)}]) = [1_{L(F)}]$ then ϕ and ψ can be chosen to be unital homomorphisms such that $\psi\phi \approx \mathrm{id}_{L(E)}$ and $\phi\psi \approx \mathrm{id}_{L(F)}$.*

Proof. Because $\mathrm{Ker}(I - A_E^t)$ and $\mathrm{Ker}(I - A_F^t)$ are isomorphic to the quotients of $K_0(L(E))$ and $K_0(L(F))$ modulo torsion, the assumed isomorphism ξ_0 induces an isomorphism $\xi_1 : \mathrm{Ker}(I - A_E^t) \xrightarrow{\sim} \mathrm{Ker}(I - A_F^t)$. By Corollary 2.23 , there exists $\xi \in kk(L(E), L(F))$ such that for the injective homomorphism $\gamma_F : \mathrm{Ker}(I - A_F^t) \to K_1(L(F))$ of (3.12), we have $K_0(\xi) = \xi_0$ and $K_1(\xi)\gamma_E = \gamma_F \xi_1$. Hence $\xi \in kk(L(E), L(F))$ is an isomorphism by Proposition 2.11. By Theorem 3.31 there are algebra homomorphisms $\phi : L(E) \to L(F)$ and $\psi : L(F) \to L(E)$ such that $j(\phi) = \xi$ and $j(\psi) = \xi^{-1}$, which may be chosen unital if ξ_0 is. Again by Theorem 3.31, $\phi\psi$ and $\psi\phi$ are M_2-homotopic to the respective identity maps. If moreover ϕ and ψ are unital, then by Theorem 3.31, $\phi\psi$ and $\psi\phi$ are unitally ad-homotopic to identity maps. Hence by Remark 3.30 there are $u \in U(L(E))$ and $v \in U(L(F)$ such that $\mathrm{ad}(v)\phi\psi$ and $\psi\phi\,\mathrm{ad}(u)$ are homotopic to identity maps. Hence ψ is a homotopy equivalence. Upon replacing ϕ by the homotopy inverse of ψ, we get the last statement of the theorem. □

Recall from [8, Chapter III, Section 6.2] that a *scaled ordered group* is an ordered group together with a choice of order unit. If R is a unital algebra, then $K_0(R)$ has a natural structure of scaled ordered group whose positive cone is the image of $\mathcal{V}(R)$ and whose order unit is $[1_R]$.

We say that two unital algebras R and S are *unitally homotopy equivalent* if there are unital homomorphisms $\phi : R \to S$ and $\psi : S \to R$ such that $\psi\phi$ and $\phi\psi$ are homotopic to the respective identity maps.

Corollary 3.34. *Let E and F be finite graphs such that $L(E)$ and $L(F)$ are simple. Assume that $K_0(L(E))$ and $K_0(L(F))$ are isomorphic as scaled ordered groups. Then either*

i) there is $1 \leq n$ such that $L(E) \cong L(F) \cong M_n$

or

ii) $L(E)$ and $L(F)$ are purely infinite and unitally homotopy equivalent.

Proof. By Remark 3.17 if $L(E)$ is simple but not purely infinite, then there is $n \geq 1$ such that $L(E) \cong M_n$. In this case $K_0(L(E)) \cong \mathbb{Z}$ with the usual order and $[1_{L(E)}]$ corresponds to n. On the other hand if R is a purely infinite simple unital algebra, then every element of $K_0(R)$ is nonnegative, by Theorem 3.4. The proof is concluded using Theorem 3.33 and observing that the identity is the only automorphism of $\mathbb{Z}$ as an ordered group. □

3.4 More on algebra extensions

Let R be an algebra. For $x \in R^{\mathbb{N}}$, let $\operatorname{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\}$. For a matrix $a \in R^{\mathbb{N}\times\mathbb{N}}$ and $i \in \mathbb{N}$, put $a_{i,*}$ and $a_{*,i}$ for the i^{th} row and column, and set

$$\mathfrak{I}(a) = \{a_{i,j} : i, j \in \mathbb{N}\} \subset R,$$
$$N(a) = \sup\{\#\operatorname{supp}(a_{i,*}), \#\operatorname{supp}(a_{*,i}) : i \in \mathbb{N}\}.$$

In Section 1.1 we defined the Wagoner's cone and suspension

$$\Gamma R = \{a \in R^{\mathbb{N}\times\mathbb{N}} : \text{each row and column of } a \text{ is finitely supported}\}$$
$$\Sigma R = \Gamma R/M_\infty R.$$

We also have the Karoubi's *cone* and *suspension* defined as

$$\Gamma' R = \{a \in \Gamma R : \#\mathfrak{I}(a) < \infty \text{ and } N(a) < \infty\},$$
$$\Sigma' R = \Gamma' R/M_\infty R.$$

Proposition 3.35. *Let R be either a division algebra or a purely infinite simple unital algebra. Then ΣR and $\Sigma' R$ are purely infinite simple.*

Proof. It suffices to show that if $M \in \Gamma R \setminus M_\infty R$ then there are $A, B \in \Gamma' R$ such that $AMB = 1$. The conditions defining Γ' and Γ imply that there are infinite, strictly increasing sequences $Y = \{y_1, y_2, \dots\}, N = \{N_1 = 1, N_2, \dots\} \subset \mathbb{N}$ such that for each j, $\emptyset \neq \operatorname{supp}(m_{*,y_j}) \subset [N_j + 1, N_{j+1}]$. Let B_1 be the matrix whose n^{th} column is the canonical basis element ϵ_{y_n}. The support of the j^{th}-column of the matrix MB_1 is contained in $[N_j + 1, N_{j+1}]$. Choose, for each j, an element $x_j \in [N_{j+1}, N_{j+1}]$ such that $(MB_1)_{x_j,j} \neq 0$. Let A_1 be the matrix whose j^{th} row is the basis element ϵ_{x_j}. The matrix A_1MB_1 is diagonal, and all its diagonal entries are nonzero. Hence by our hypothesis on R there are diagonal matrices A_2 and B_2 such that $A_2A_1MB_1B_2 = 1$. □

Recall from Lemma 1.6 and the paragraph following it that when R is unital, every extension of an algebra A by $M_\infty R$ is classified by a homomorphism $A \to \Sigma R$. By Lemma 1.3, the sets $[A, \Sigma R]_{M_2}$ and $[A, \Sigma' R]_{M_2}$ are abelian monoids with the sum induced by (3.4). Put

$$\mathcal{E}xt(A, R) = [A, \Sigma R]_{M_2}, \quad \mathcal{E}xt(A, R)_f = [A, \Sigma' R]_{M_2}.$$

By definition, there is a canonical map $\mathcal{E}xt(A, R)_f \to \mathcal{E}xt(A, R)$; by Remark 2.10 there is also a natural map $\mathcal{E}xt(A, R) \to kk_{-1}(A, R)$.

Theorem 3.36. *Let R be either a division algebra or a K_0-regular purely infinite simple unital algebra and E a finite graph such that $L(E)$ is simple. Then the canonical maps*

$$\mathcal{E}xt(L(E), R)_f \to \mathcal{E}xt(L(E), R) \to kk_{-1}(L(E), R)$$

are isomorphisms. Moreover every nonzero element of each of these groups represents the M_2-homotopy class of a nontrivial extension of $L(E)$ by $M_\infty(R)$.

Proof. Since ℓ is a field, Σ and Σ' are models for the suspension functor. By Proposition 3.35, ΣR and $\Sigma' R$ are purely infinite simple. Now apply Theorem 3.31 to prove the first assertion. The second assertion follows from Theorem 3.31 and Lemma 1.6. □

Corollary 3.37. *[cf.* [16, Theorem 5.3]*] For E as in the theorem above, we have*

$$\mathcal{E}xt(L(E), \ell) = \operatorname{Coker}(I - A_E).$$

Proof. Immediate from Theorem 3.36 and the the fact that $KH^1(L(E)) = \operatorname{Coker}(I - A_E)$ Formula 2.14. □

Corollary 3.38. *Let E and R be as in Theorem 3.36. Then there is an exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(K_0(L(E)), K_0(R)) \to \mathcal{E}xt(L(E), R) \to \\ \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Ker}(I - A^t_E), K_0(R)) \oplus \operatorname{Hom}_{\mathbb{Z}}(K_0(L(E)), K_{-1}R) \to 0.$$

Proof. Immediate from Theorem 3.36 and Corollary 2.23. □

Example 3.39. *If R is either ℓ or a purely infinite simple unital Leavitt path algebra, then $K_{-1}R = 0$, so the exact sequence of Corollary 3.38 becomes*

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(K_0(L(E)), K_0(R)) \to \mathcal{E}xt(L(E), R) \to \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Ker}(I - A^t_E), K_0(R)) \to 0.$$

If furthermore $K_0(L(E))$ is torsion, then $\operatorname{Ker}(I - A^t_E) = 0$, and we get a canonical isomorphism

$$\mathcal{E}xt(L(E), R) = \operatorname{Ext}^1_{\mathbb{Z}}(K_0(L(E)), K_0(R)).$$

3.5 Maps into tensor products with L_2

Lemma 3.40. *Let E be a graph and let $\phi : L(E) \to R$ be an algebra homomorphism. Then $\phi \approx 0 \iff \phi = 0$.*

Proof. It suffices to show that if $H : L(E) \to R[t]$ satisfies $\operatorname{ev}_0 H = 0$, and $v \in E^0$, then $H(v) = 0$. This follows from Lemma 3.11. □

A unital algebra R is *regular supercoherent* if for every $n \geq 0$, $R[t_1, \dots, t_n]$ is regular coherent in the sense of [11].

Lemma 3.41. *Let E be graph and R a regular supercoherent algebra. Then $L(E) \otimes R$ is K-regular. In particular, $L(E) \otimes L(F)$ is K-regular for every finite graph F.*

Proof. By definition, $R_n = R[t_1, \dots, t_n]$ is regular supercoherent for every $n \geq 0$. Hence by Example 2.8 the canonical map $K_*(R_n \otimes L(E)) \xrightarrow{\sim} KH_*(R_n \otimes L(E)) = KH_*(R_0 \otimes L(E))$ is an isomorphism for every n, whence also $K_*(R_0 \otimes L(E)) \to K_*(R_n \otimes L(E))$ is an isomorphism for all n. The second assertion follows from the first, using [1, Lemma 6.4.16]. □

Let R, S be unital algebras. Put

$$[R, S] \supset [R, S]_1 = \{[f] : f \text{ unital }\}.$$

Theorem 3.42. *Let E be finite graph such that $L(E)$ is simple and R a purely infinite simple regular supercoherent algebra. Then $[L(E), L_2]_1 = [L(E), L_2]_{M_2} \setminus \{0\}$, $[L(E), R \otimes L_2]_1 = [L(E), R \otimes L_2]_{M_2}$, and both sets have exactly one element each.*

Proof. By Remark 3.17, Proposition 3.16 and Theorem 3.12, $[L(E), L_2]_{M_2} \setminus \{0\}$ has exactly one element, since $j(L_2) = 0$ in kk; by Corollary 3.19 this element is the class of a unital homomorphism. Next let $\phi, \psi : L(E) \to L_2$ be unital homomorphisms. If $L(E)$ is not purely infinite, then by Proposition 3.16, ϕ and ψ are conjugate, and therefore homotopic, since by Corollary 3.10, $\pi_0(U(L_2)) = K_1(L_2) = 0$. If $L(E)$ is purely infinite, then by part iii) of Theorem 3.31, ϕ and ψ are 3-step unitally ad-homotopic. Hence by Remark 3.30 and the argument we have just used, $\phi \approx \psi$. Thus the assertions about homomorphisms $L(E) \to L_2$ are proved. It is well-known that the tensor product of a unital simple algebra with a unital central simple algebra is again simple. By [6, Theorem 4.2], L_2 is central, so $R \otimes L_2$ is simple. Moreover, $R \otimes L_2$ is purely infinite by [7, Theorem 7.9]. Hence using that $j(R \otimes L_2) = 0$ in kk and applying Lemmas 3.40 and 3.41, Proposition 3.16 and Theorem 3.31, we obtain

$$[L(E), R \otimes L_2]_{M_2} \setminus \{0\} = kk(L(E), R \otimes L_2) = 0.$$

By Corollary 3.19 there is a unital homomorphism $\phi : L(E) \to L(F) \otimes L_2$. If ψ is another, then $\phi \approx \psi$ by Lemma 3.15 and the argument above. □

Example 3.43. *Let R be as in Theorem 3.42, let $d : L_2 \to R \otimes L_2$, $a \mapsto 1 \otimes a$ and let $\phi : L_2 \to R \otimes L_2$ be any homomorphism. Setting $L(E) = L_2$ in Theorem 3.42 we get that if ϕ is nonzero then it is M_2-homotopic to d and that if ϕ is unital then it is homotopic to d.*

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