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## **Estudio del operador biclique aplicado a distintas clases de grafos**

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# Estudio del operador biclique aplicado a distintas clases de grafos

Una biclique en un grafo es un subgrafo inducido bipartito completo maximal. El estudio de las bicliques ha recibido mucha atención en los últimos tiempos. El grafo biclique de  $G$ ,  $KB(G)$ , es el grafo de intersección de las bicliques de  $G$ . Este fue definido y caracterizado recientemente. Sin embargo, aún sigue abierta la pregunta sobre la existencia de un algoritmo eficiente que resuelva el problema de reconocimiento de grafos biclique. En esta tesis estudiamos el problema de reconocimiento de grafos biclique de algunas clases de grafos. Se pretende con esto, acercarse al problema de reconocimiento de grafos biclique en general, encontrando clases donde el problema de decidir si un grafo es grafo biclique sea polinomial o se pueda probar que es NP-completo. Entre otras, en este trabajo estudiamos el operador biclique aplicado a los grafos bipartitos cordales, split y bipartitos de permutación. También estudiamos el problema de reconocimiento de la clase biclique inversa de los grafos completos, es decir, dado un grafo, decidir si su grafo biclique es completo. Cabe mencionar que, dado que la cantidad de bicliques de un grafo puede ser exponencial, no siempre es eficiente construir el grafo biclique para responder esta pregunta.

**Palabras claves:** Bicliques, cordal, grafo biclique, grafo clique, permutación, split.

# Study of the biclique operator applied to different graph classes

A biclique in a graph is a maximal bipartite complete induced subgraph. The study of bicliques has received a lot of attention in the last years. The biclique graph of  $G$ ,  $KB(G)$ , is the intersection graph of the bicliques of  $G$ . It was defined and characterized recently. Nevertheless, the time complexity of the recognition problem for biclique graphs is still open. In this work we study the problem of recognizing biclique graphs in several classes of graphs. By finding classes where the problem of deciding if a graph is a biclique graph of the class is polynomially solvable and other properties of biclique graphs, we intend to approach to the solution of the general recognition problem. For example, in this work we study the biclique operator applied to chordal, split and bipartite permutation graphs. We also study the problem of recognizing the biclique inverse class of complete graphs, that is, given a graph, the problem of deciding if its biclique graph is complete. It is worth to mention that given that the quantity of bicliques of a graph can be exponential, it is not always efficient to built a biclique graph to answer this question.

**Keywords:** Bicliques, biclique graphs, chordal, clique graph, permutation, split.

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# 1 Introducción

Dada una familia de subconjuntos  $H$ , el grafo de intersección de  $H$  es el grafo cuyo conjunto de vértices es  $H$ , y existe una arista entre los subconjuntos  $E$  y  $F$  de  $H$  si y sólo si  $E$  y  $F$  se intersecan. Un grafo  $G$  se dice que es de intersección cuando existe una familia de conjuntos  $H$  tal que  $G$  es isomorfo al grafo de intersección de  $H$ . Los grafos de intersección han recibido mucha atención en el estudio de algoritmos y propiedades en grafos y sus aplicaciones. Ejemplos de grafos de intersección son los grafos de intervalos, donde  $H$  es una familia de intervalos, los grafos cordales, donde  $H$  es una familia de subárboles de un árbol, los grafos arco-circulares, donde  $H$  es una familia de arcos alrededor de un círculo, los grafos de permutación, donde  $H$  es una familia de segmentos que unen dos líneas rectas paralelas, los grafos clique, donde  $H$  es la familia de cliques de un grafo y los grafos de línea, donde  $H$  es la familia de aristas de un grafo [FG65, Hub74, Kle76, RS71, RW65, Whi32]. Las clases de grafos mencionadas tienen aplicaciones en genética, biología molecular, control del tránsito (diseño de semáforos), almacenamiento de mercadería, ordenamiento de vagones de un ferrocarril, estadística y diseño de compiladores [CDG02, Hsu85, Sta67, Sto68, Tuc74].

Los grafos de intervalos fueron introducidos por Hajós en 1957 [Haj57] y Benzer en 1959 [Ben59]. Estos grafos tienen una gran cantidad de aplicaciones y hay varios problemas NP-Hard para el caso general que resultan ser polinomiales para esta clase de grafos. Por ejemplo, los problemas de colooreo, clique máximo, conjunto independiente máximo y mínimo cubrimiento por cliques [Gol80, Gav74]. Existen varias caracterizaciones para esta clase de grafos [LB62, GH64, BL76, KM89, HMPV00].

Los grafos clique han sido introducidos por Hamelink [Ham68]. Existe un teorema de reconocimiento de grafos clique dado por Roberts y Spencer [RS71], pero éste no condujo a un algoritmo eficiente de reconocimiento. Después de más de 40 años, se probó que el problema de reconocimiento de grafos clique es NP-completo [AFdFG13], es decir, no existe un algoritmo de tiempo polinomial para resolver este problema (si  $P \neq NP$ ). Por otro lado, existen numerosos trabajos sobre el operador clique aplicado a diversas clases de grafos, es decir, la clase  $K(\mathcal{A})$ , donde  $\mathcal{A}$  es una clase de grafos. La mayoría de las clases cuyos grafos clique se han caracterizado hasta el momento se pueden clasificar en tres tipos: clase fija ( $K(\mathcal{A}) = \mathcal{A}$ ), clase cerrada ( $K(\mathcal{A}) \subset \mathcal{A}$ ) y la clase  $\mathcal{A}$ , donde  $\mathcal{A} \cap K(\mathcal{A}) \neq \emptyset$  y  $K(K(\mathcal{A})) \subset \mathcal{A}$  [Szw03].

Por ejemplo, se probó que  $K(\text{clique-Helly}) = \text{clique-Helly}$ ,  $K(\text{cordal}) = \text{dualmente cordal}$  y  $K(\text{intervalos}) = \text{intervalos propios}$ . Se ha estudiado el operador clique para varias otras clases de grafos, como los grafos sin diamantes, split, desmantelables, arco circulares y los grafos self-clique ( $(K(G) \cong G)$  [Gut96, GO95, GZ, LNL02, PS99, BDLS03, DL01, LSS10]).

También, existen resultados sobre las clases clique inversas: dada una clase  $\mathcal{A}$  se aborda el problema de reconocimiento de  $K^{-1}(\mathcal{A})$ , es decir, el problema de decidir si dado un grafo cualquiera, su grafo clique pertenece a la clase  $\mathcal{A}$ . Por ejemplo, se ha estudiado este problema para el caso donde  $\mathcal{A}$  son los grafos bipartitos, cordales, clique Helly, intervalos y  $e$ -comparabilidad [Pro98, PS00a, PS00b, PS01, Szw03].

Por otro lado, se han realizado trabajos acerca del operador clique iterado, es decir, el grafo que resulta de aplicar el operador clique sucesivamente. En particular se ha analizado la convergencia de un grafo bajo el operador clique: un grafo  $G$  es clique-convergente si existen  $n, m \in \mathbb{N}$ , con  $n < m$  tales que  $K^n(G)$  y  $K^m(G)$  son isomorfos; un grafo  $G$  es clique-divergente en caso de que tales  $n$  y  $m$  no existan.

Decidir si un grafo es clique-divergente o clique-convergente es un problema muy estudiado [HS72, Esc73, LNL97, LNL99, LNL00, Piz03, Piz04, BP86, Hed86, PRS86, LNL02, LNLP02, LNLP03, FANLP04, LdMM<sup>+</sup>04, LNLP04, LNLP06, FPVF08]. Hay resultados recientes que indican que este problema es muy difícil, como muestra el reciente trabajo de Cedillo y Pizaña presentado en el LAWCG'2018 en Rio de Janeiro [CP18].

Un concepto natural relacionado con las cliques es el de bicliques. Una biclique en un grafo  $G$  es un subgrafo inducido bipartito completo maximal. El estudio de las bicliques ha recibido mucha atención en los últimos tiempos en diferentes contextos [GG78, Tuz84, BBD06, CM07, CF06, Mül96, Mül97, Pri97, Pri00, Pee03, GM09].

En los últimos años, muchos problemas clásicos de la literatura estudiados sobre cliques, fueron considerados en el contexto de bicliques. En este sentido, se estudiaron los grafos biclique Helly y su versión hereditaria (la familia de bicliques satisface la propiedad de Helly), biclique-coloreo de grafos, conjunto independiente de bicliques y transversal de bicliques, entre otros. En particular fue definido y caracterizado el grafo biclique, biclique iterado y el grafo  $e$ -biclique [Gro06, GS07, GHS12, GST14, DGG<sup>+</sup>17, GM17]

Existen diversos problemas relacionados con las bicliques en la teoría de autómatas, teoría de lenguajes, compresión de grafos, órdenes parciales, inteligencia artificial, teoría de la información, redes sociales, medicina, genética y biología [AVJ98, Hae01, KRRT99, NK08, ABP<sup>+</sup>10, BZC<sup>+</sup>03, LSL06]. Por otro lado, las bicliques son mencionadas en el estudio de diversas familias de grafos, por ejemplo, los grafos bipartitos cordales, los retractos y los bigrafos de intervalo, entre otros [GG78, BFH93].

El grafo biclique de  $G$ ,  $KB(G)$ , es el grafo de intersección de las bicliques de  $G$ . Este fue definido y caracterizado recientemente [GS10]. Sin embargo, aún sigue abierta la pregunta sobre la existencia de un algoritmo eficiente que resuelva el problema de reconocimiento de grafos biclique.

Pocos resultados existen sobre el estudio del grafo biclique aplicado a alguna clase [GG18]. En [Gro06], Groshaus prueba que  $KB(\mathcal{BHD}) = \mathcal{CHBDI}$ , donde  $\mathcal{BHD}$  es la clase de los grafos bipartitos biclique-Helly sin vértices dominados y  $\mathcal{CHBDI}$  es la clase de grafos sin vértices dominados tal que la familia de cliques puede ser independientemente Helly-bicubierta.

El grafo biclique puede pensarse como un operador que transforma una clase de grafos en otra. Por lo tanto puede estudiarse el grafo biclique iterado, es decir, el grafo que resulta de aplicar el operador biclique sucesivamente. Se han caracterizado los grafos bicliques-divergentes y bicliques-convergentes y se presentó un algoritmo lineal de reconocimiento de cada comportamiento [GM09, GM13, GGM16]. Por otro lado, el grafo e-biclique es el grafo de intersección de las bicliques, considerando intersección en aristas. Se caracterizó a los grafos e-biclique y se los relacionó con el grafo línea y el grafo biclique [GHS12].

Cabe mencionar que el problema del grafo clique posee algunas similitudes y otras diferencias considerables con el problema del grafo biclique. Por ejemplo, la caracterización de grafos clique está basada en la propiedad de Helly y un resultado análogo, aunque considerablemente más complicado, se pudo obtener para grafos biclique [GS10]. Si miramos el problema del operador iterado, encontramos que este no está resuelto en el caso del grafo clique, es decir, en el caso general no existen caracterizaciones que definan si un grafo diverge o no bajo el operador clique. Aún restringido a algunas clases, son muy escasos los resultados en la literatura al momento. Sin embargo, sorpresivamente, cuando se estudió el problema análogo para el operador biclique, se consiguieron caracterizaciones muy simples de los grafos convergentes y divergentes, que dieron lugar a un algoritmo lineal para decidir el comportamiento de cualquier grafo.

En general, el objetivo de esta tesis es comprender mejor al grafo biclique, encontrando propiedades que encaminen y ayuden a la resolución del problema de reconocimiento en el caso general, ya que probar si el problema de reconocimiento es NP-completo no parece ser una tarea sencilla (recordemos que este es aún un problema abierto).

En esta tesis estudiamos el operador biclique aplicado a varias clases de grafos. Es de interés, por un lado, conocer clases de grafos biclique, es decir, familias de grafos que sean grafos biclique. Se pretende con esto acercarse al problema de reconocimiento de grafos biclique en general, encontrando clases donde el problema de decidir si un grafo es grafo biclique o no sea polinomial o se pueda probar que es NP-completo.

Por otro lado, análogamente al estudio existente sobre el grafo clique, estudiamos el operador biclique aplicado a diversas clases de grafos, es decir,  $KB(\mathcal{A})$ , para algunas clases de grafos  $\mathcal{A}$ , como los grafos bipartitos cordales, split, bipartitos de permutación y otras. También estudiamos el problema de reconocimiento de la clase biclique inversa  $KB^{-1}(\mathcal{A})$ , es decir, dado un grafo, decidir si su grafo biclique pertenece a cierta clase de grafos  $\mathcal{A}$ . Este problema fue analizado para la clase de los grafos completos. Cabe mencionar que, dado que la cantidad de bicliques de un grafo puede ser exponencial, no siempre es eficiente construir el grafo biclique para responder esta pregunta.

También, en esta tesis presentamos varias propiedades de la clase de grafos biclique en general y restringido a algunas clases. Entre otros resultados, probamos que todo grafo es subgrafo inducido de algún grafo biclique.

Notamos que algunas partes de este texto están en inglés, debido a que fueron escritas como artículos. Los Capítulos 3, 6 y 7 están parcialmente o completamente en inglés.

La tesis está organizada de la siguiente forma. En el Capítulo 2 presentamos los conceptos preliminares necesarios. A continuación, se encuentran cinco capítulos que tratan problemas del tipo  $KB(\mathcal{A})$  y luego un capítulo acerca del problema  $KB^{-1}(\mathcal{A})$ .

En el Capítulo 3 estudiamos los grafos biclique de algunas clases como completos, árboles y otras. Probamos que las clases de los grafos biclique (en general),  $KB(\text{Bipartito } C_4\text{-free})$ ,  $KB(\text{Split})$  y la clase de los grafos clique no tienen ningún subgrafo inducido prohibido. Es decir, todo grafo  $H$  es subgrafo inducido de un grafo de estas clases. En particular, construimos en tiempo polinomial un grafo  $G$  (de cada una de estas clases) tal que  $H$  es subgrafo inducido de  $KB(G)$  (o  $K(G)$ ).

También en este capítulo presentamos caracterizaciones (algunas parciales) de los grafos biclique de algunas clases como grafos completos, árboles y grafos con cintura por lo menos 5.

En el Capítulo 4 analizamos la clase de los grafos bipartitos cordales. Por un lado caracterizamos la clase de grafos biclique de los grafos bipartitos cordales libres de dominó. Por otro lado, proporcionamos una propiedad para los grafos biclique de los bipartitos cordales que no son necesariamente cordales.

Luego, estudiamos la clase de grafos split. En primer lugar, en el Capítulo 5, estudiamos propiedades del grafo biclique de un grafo split. En particular estudiamos la conectividad de los grafos biclique de grafos split y probamos que, para esa subclase de grafos biclique, vale la Conjetura de Groshaus-Montero [GM17].

**Conjetura 1.1** ([GM17] (Conjetura 6.3. pg. 17)). *Todo grafo biclique es hamiltoniano.*

Luego, en el Capítulo 6, abordamos el problema de reconocimiento de grafos biclique de los grafos split, es decir, decidir si dado un grafo cualquiera  $G$ , existe un grafo split  $H$  tal que  $G = KB(H)$ . Resolvimos este problema para una subclase de los grafos split, proporcionando un algoritmo polinomial de reconocimiento. Explicamos también nuestra intuición sobre el hecho de que el problema general para grafos split es NP-hard.

En el Capítulo 7 estudiamos los grafos biclique de los bigrafos de intervalo (BI). Probamos que  $KB(BI)$  es subconjunto propio de la clase de co-comparabilidad libres de  $K_{1,4}$ . También estudiamos la sub-clase de los bigrafos de intervalos propios (BIP), que es la misma que la de los grafos bipartitos de permutación. Probamos que  $KB(BIP)$  es igual a la clase de los cuadrados de los grafos línea de BIP, o sea,  $KB(BIP) = (L(BIP))^2$ . Sin embargo todavía no tenemos un algoritmo polinomial para el reconocimiento. Para una subclase de BIP pudimos encontrar un algoritmo polinomial de reconocimiento para los grafos biclique y lo presentamos.

Sobre el problema  $KB^{-1}(\mathcal{A})$  tenemos el Capítulo 8 que trata de la clase biclique inversa de los completos,  $KB^{-1}(\text{completos})$ . En este capítulo estudiamos esta clase y obtuvimos un resultado análogo al del operador clique, probando que decidir si un grafo pertenece a  $KB^{-1}(\text{completos})$  es co-NP-completo.

En el Capítulo 9 damos las conclusiones del trabajo, con algunas ideas de trabajo futuro.

## 2 Preliminares

### 2.1. Nociones básicas de teoría de grafos

Un *grafo* es un par ordenado  $G = (V(G), E(G))$ , donde  $V(G)$  es un conjunto finito no vacío y  $E(G)$  es un conjunto de pares no ordenados  $vw$  con  $v, w \in V(G)$  y  $v \neq w$ . El conjunto  $V(G)$  es el *conjunto de vértices* de  $G$  y sus elementos son los *vértices* de  $G$ , mientras que el conjunto  $E(G)$  es el *conjunto de aristas* de  $G$  y sus elementos son las *aristas* de  $G$ . La cantidad de vértices de  $G$  es llamada el *orden* de  $G$ . Como es usual en la literatura, denotamos  $n = |V(G)|$  y  $m = |E(G)|$ , a menos que se indique lo contrario. El único grafo de orden 1 es llamado grafo *trivial*. Observe que en nuestra definición de grafo,  $vv \notin E(G)$  para  $v \in V(G)$ ,  $vw \in E(G)$  si y sólo si  $wv \in E(G)$ , y  $|\{vw \in E(G) \mid v, w \in V(G)\}| \leq 1$ . En la literatura, se hace referencia a estas condiciones como grafos *sin loops*, *no dirigidos*, y *simples*, respectivamente. Un vértice  $v$  es *adyacente* a un vértice  $w$  cuando  $vw$  es una arista del grafo. También decimos que  $v$  es *vecino* de  $w$  cuando  $v$  es adyacente a  $w$ . Si  $a$  y  $b$  son vértices adyacentes decimos que  $a$  y  $b$  son *adyacentes por* la arista  $ab$ . La *vecindad* de  $v$  es el conjunto  $N_G(v)$  de todos los vecinos de  $v$ . El cardinal de  $N_G(v)$  es el *grado* de  $v$  y es denotado por  $d_G(v)$ . Los valores mínimo y máximo de entre todos los grados de todos los vértices son respectivamente denotados por  $\delta(G)$  y  $\Delta(G)$ . Si  $\delta(G) = \Delta(G)$  se tiene que  $G$  es un grafo *regular*. La *vecindad cerrada* de  $v$  es el conjunto  $N_G[v] = N_G(v) \cup \{v\}$ . Si  $N_G[v] = V(G)$  decimos que  $v$  es un *vértice universal* mientras que si  $N_G(v) = \emptyset$  decimos que  $v$  es un *vértice aislado*. Omitiremos el subíndice en  $N$  y en  $d$  cuando no haya ambigüedad sobre  $G$ . Un *camino* en un grafo  $G$  es una secuencia  $v_1, \dots, v_k$  de vértices tal que  $v_i$  es adyacente a  $v_{i+1}$ , para todo  $1 \leq i < k$ . Se dice que tal camino es un camino *entre*  $v_1$  y  $v_k$ , su longitud es  $k - 1$  y si todos los vértices son distintos lo denotamos  $P_{k-1}$ .

Dado un camino en un grafo  $G$ , si todos los vértices del camino son diferentes, salvo el primero y el último que son iguales y el camino tiene al menos longitud 1, decimos que el camino es un *ciclo*. Por simplicidad diremos que  $v_0, \dots, v_{k-1}$  es un *ciclo* para  $v_0 \neq v_{k-1}$  cuando  $v_0, \dots, v_{k-1}, v_0$  sea un ciclo. Al referirnos a un ciclo de longitud  $k$ ,  $v_0, \dots, v_{k-1}$ , utilizaremos “+” para representar la suma módulo  $k$ ; por ejemplo como el ciclo  $v_0, v_1, v_2, v_3, v_4, v_5, v_6$  tiene longitud 7, tenemos  $v_{5+3} = v_1$ . Un grafo  $H$  es un *subgrafo* del grafo  $G$  si  $V(H) \subseteq V(G)$  y  $E(H) \subseteq E(G)$ . Si además  $E(H) = \{vw \in E(G) \mid v, w \in V(H)\}$ ,  $H$  es un subgrafo *inducido* de  $G$ . Para cada  $W \subseteq V(G)$ , el subgrafo de  $G$  *inducido por*  $W$  es el único subgrafo inducido de  $G$  cuyo conjunto de vértices es  $W$ . Denotamos por  $G[W]$  al subgrafo de  $G$  inducido por  $W$ . Un subgrafo de  $G$  cuyo conjunto de vértices es  $V(G)$  es llamado un *subgrafo generador*. Para cada  $W \subset V(G)$ , denotamos por  $G \setminus W$

el subgrafo de  $G$  inducido por  $V(G) \setminus W$ . De la misma forma, para cada  $F \subseteq E(G)$ , denotamos por  $G \setminus F$  el subgrafo generador de  $G$  cuyo conjunto de aristas es  $E(G) \setminus F$ . Un grafo es *conexo* si posee un camino entre cualquier par de vértices. Un grafo es *disconexo* si no es conexo. Una *componente conexa*, o simplemente una *componente*, es un subgrafo conexo maximal. Un grafo es un *árbol* si es conexo y sin ciclos. Un árbol es un *caterpillar* si eliminando todos sus vértices de grado uno se obtiene un camino. La *distancia* de dos vértices  $v$  y  $w$  que pertenecen a una misma componente conexa en un grafo  $G$ , denotada por  $d_G(v, w)$ , es el mínimo entre las longitudes de todos los caminos entre  $v$  y  $w$ . La distancia entre  $v$  y  $w$  es infinita cuando  $v$  y  $w$  pertenecen a diferentes componentes conexas de un grafo. Como antes, omitiremos el subíndice cuando no haya ambigüedad sobre  $G$ . La *cintura* de un grafo es la longitud del ciclo más corto contenido en dicho grafo. Si el grafo no posee ciclos, su cintura se define como infinita. Para todo  $k \in \mathbb{N}_0$ , la  $k$ -ésima potencia de  $G$ , denotada por  $G^k$ , es el grafo que tiene los mismos vértices que  $G$  y tal que dos vértices son adyacentes en  $G^k$  si y sólo si su distancia en  $G$  es menor o igual a  $k$ . Dos vértices  $v$  y  $w$  son *mellizos verdaderos*, o simplemente *mellizos*, cuando  $N[v] = N[w]$ . Nos referiremos a  $v$  y  $w$  como *mellizos falsos* cuando  $N(v) = N(w)$ . Dos grafos  $G$  y  $H$  son *isomorfos* si existe un mapeo uno-a-uno  $f$  entre  $V(G)$  y  $V(H)$  tal que  $vw \in E(G)$  si y sólo si  $f(v)f(w) \in E(H)$ . El mapeo  $f$  es llamado un *isomorfismo* entre  $G$  y  $H$ . El *complemento* de un grafo  $G$ , denotado por  $\overline{G}$ , es el grafo que tiene los mismos vértices que  $G$  y es tal que dos vértices son adyacentes en  $\overline{G}$  si y sólo si ellos no son adyacentes en  $G$ . Llamamos *conectividad por vértices* de un grafo  $G = (V, E)$ , y la denotamos  $\kappa(G)$ , a la cantidad mínima de vértices de  $V$  que deben eliminarse para que el grafo sea disconexo o se reduzca a un único vértice. Llamamos *conectividad por aristas* de un grafo  $G = (V, E)$ , y la denotamos  $\lambda(G)$ , a la cantidad mínima de aristas de  $E$  que deben eliminarse para que el grafo sea disconexo. Un subconjunto  $L$  de  $E(G)$  se llama *corte-arista restringido* si  $G \setminus L$  es disconexo y cada componente contiene al menos dos vértices. La *conectividad-arista restringida*  $\lambda_2(G)$  es el cardinal mínimo de todos los corte-arista restringidos. Un grafo es *completo* si todos sus vértices son adyacentes de a pares. El grafo completo de tamaño  $n$  es denotado por  $K_n$ . Una *clique* de un grafo es un subgrafo completo maximal. Un *conjunto independiente* es un conjunto de vértices no adyacentes de a pares. Un grafo es *bipartito* si existe una partición  $V_1, V_2$  de  $V(G)$  tal que  $V_1$  y  $V_2$  son conjuntos independientes no vacíos. Un grafo bipartito es *bipartito completo* si todo vértice de una partición es adyacente a todo vértice de la otra partición. Un grafo bipartito completo de  $n, m$  vértices en cada partición es denotado por  $K_{n,m}$ . Llamamos al grafo bipartito completo de la forma  $K_{1,m}$  una *estrella*. El vértice  $v$  de la partición que tiene un único vértice se llama *centro* de la *estrella*. Un *grafo split*  $H = (K \cup S, E)$  es un grafo tal que su conjunto de vértices puede particionarse en una clique,  $K$ , y en un conjunto independiente,  $S$ . Llamamos a los vértices de  $S$  satélites del grafo. La excentricidad  $\epsilon(v)$  de un vértice  $v$  es la distancia más grande entre  $v$  y cualquier otro vértice. El diámetro  $d$  de un grafo es la máxima excentricidad de cualquier vértice en el grafo.

# 3 Sobre $KB$ de algunas clases

En este capítulo presentamos un resultado no solo relacionado con la clase de grafos biclique de algunas clases sino que lo extendemos para la clase de grafos clique.

En particular, probamos en la Sección 3.1 que cualquier grafo  $H$  es subgrafo inducido de algún grafo biclique de un grafo split, es subgrafo inducido de un grafo biclique de un grafo bipartito sin  $C_4$  y es subgrafo inducido de un grafo clique.

Ya se sabía que la clase de grafos biclique y la clase de grafos clique no son clases hereditarias. Este resultado extiende ese concepto, ya que tiene como consecuencia que ninguna de esas clases (grafos biclique de split, biclique de bipartito sin  $C_4$  y grafos clique) tiene subgrafos prohibidos.

De este resultado se obtiene otra consecuencia para el caso de grafos cuadrados: todo grafo  $H$  es subgrafo inducido del cuadrado de un grafo bipartito sin  $C_4$ . Más aún, todo grafo  $H$  es subgrafo inducido del grafo de intersección de estrellas de un grafo bipartito sin  $C_4$ .

Las demostraciones son constructivas, es decir, construimos en cada caso el grafo cuyo grafo biclique/clique contiene a  $H$  como subgrafo inducido.

Cabe resaltar que estos grafos contruidos tienen tamaño polinomial en el tamaño de  $H$ .

Para algunas clases clásicas de grafos, como completos, árboles, caminos, ciclos y grafos con cintura por lo menos 5, el grafo biclique es fácilmente caracterizado. Presentamos estas caracterizaciones usando grafo línea y grafo cuadrado en la Sección 3.2.

## 3.1. $KB(\text{bipartite})$ , $KB(\text{split})$ and $K(G)$

Given a graph  $H$ , we construct a graph  $G$  in each class ( $C_4$ -free bipartite, split, general) such that  $H$  is isomorphic to an induce subgraph of  $KB(G)$  or  $K(G)$ .

Let  $H$  be a graph with vertices  $v_1 \cdots v_n$  and edges  $e_{ij} = v_i v_j$ , with  $i \leq j$ .

We construct the graph  $H_1^*$  as follows:  $V(H_1^*) = V(H) \cup E(H)$ .  $E(H_1^*)$  is defined as follows: If  $v_i$  is adjacent to  $v_j$  in  $H$ , then  $v_i$  is adjacent to  $v_j$  in  $H_1^*$ . Also, for every  $e_{ij} \in V(H_1^*)$ ,  $e_{ij}$  is adjacent to  $v_i$ ,  $v_j$ ,  $e_{ik}$  for  $k > i$ ,  $e_{ki}$  for  $k < i$ ,  $e_{lj}$  for  $l < j$  and  $e_{jl}$  for every  $j < l$ , with  $1 \leq k, l \leq n$ .

We construct the graph  $H_2^*$  as follows: Let  $T(H)$  the family of all triangles of  $H$ . Denote the triangle  $\{v_i, v_j, v_k\} \in T(H)$  by  $t_{ijk}$ , with  $i < j < k$ .  $V(H_2^*) = V(H) \cup T(H)$  and  $E(H_2^*)$  is defined as follows: If  $v_i$  is adjacent to  $v_j$  in  $H$ , then  $v_i$  is adjacent to  $v_j$  in  $H_2^*$ . Every vertex  $t_{ijk}$  is adjacent to  $v_i, v_j, v_k$ . Also,  $t_{ijk}$  is adjacent to  $t_{ijm}, t_{ijk}$  is adjacent to  $t_{mjk}$  and  $t_{ijk}$  is adjacent to  $t_{imk}$ , that is, two triangles of  $H$  are adjacent in  $H_2^*$  when they share an edge (two vertices).

**Teorema 3.1.** *Let  $H$  be a graph. Then  $H$  is an induced subgraph of some biclique graph of a split graph. Also it is an induced subgraph of a clique graph of some graph. Moreover,  $H$  is an induced subgraph of  $H_1^*$  and  $H_2^*$ ,  $H_1^*$  is a biclique graph of a  $C_4$ -free bipartite graph and  $H_2^*$  is a clique graph.*

*Proof.* We construct a graph  $G$  for each case and prove that  $H$  is an induced subgraph of  $KB(G)$  or of  $K(G)$ .

Let  $|V(H)| = n$ . The graph  $G$  is defined as follows:  $V(G) = V(H) \cup E(H) \cup M$  where  $M = \{m_1 \dots m_n\}$ . Also,  $v_i$  and  $m_i$  are adjacent and  $v_i$  is adjacent to every  $e_{ij}, e_{ji} \in V(G)$  (recall that  $e_{ij}$  and  $e_{ji}$  are edges of  $H$ ). We define the rest of the edges of  $G$  for the other cases:

1. For the case  $G$  is a  $C_4$ -free bipartite graph, we add no other edges. The graph  $G$  constructed is  $C_4$ -free bipartite.
2. Case  $G$  is a split graph:  $E(H)$  is the set of satellites. Add edges so that the set  $M \cup V(H)$  induces the complete part. Clearly  $G$  is a split graph.
3. Case  $H$  is a subgraph of  $K(G)$ :  $m_i$  is adjacent to every  $e_{ij} \in V(G)$ ,  $i < j$  and  $e_{ji} \in V(G)$ ,  $j < i$  ( $v_i, m_i$  are twins.) Also  $e_{ij}$  is adjacent to  $e_{ik}, e_{ki} \in N[v_i]$ , that is,  $N[v_i]$  is a complete graph.

We remark that the same result follows without vertices of  $M$ .

We prove all cases together, referring to biclique/cliques for each case.

For the bipartite case and the clique case, consider  $N_i = N[v_i]$ . For the split case consider  $N_i = \{v_i, m_i\} \cup S(v_i)$ . In any case,  $N_i$  induces a biclique/clique in  $G$ .

Also, observe that for  $i < j$ , if  $N_i \cap N_j \neq \emptyset$  then  $N_i \cap N_j = e_{ij}$ . Consider the subgraph  $H'$  of  $KB(G)/K(G)$  induced by vertices corresponding to biclique/cliques  $N_1 \dots N_n$ . Observe that vertices  $N_i, N_j, i < j$  of  $KB(G)/K(G)$  are adjacent if and only if  $N_i \cap N_j \neq \emptyset$ , that is, if and only if  $e_{ij}$  belongs to  $N_i \cap N_j$ . It follows that vertices  $N_i, N_j$  of  $KB(G)/K(G)$  are adjacent if and only if  $v_i, v_j$  are adjacent in  $H$ . Associating each  $v_i \in H$  to vertex  $N_i \in V(H')$ , we obtain that  $H$  is isomorphic to an induced subgraph of  $KB(G)/K(G)$ .

We prove that when  $G$  is a  $C_4$ -free bipartite graph,  $KB(G)$  is isomorphic to  $H_1^*$ . Observe that for every edge  $e_{ij}$  of  $H$  there is a unique biclique in  $G$  that contains vertex  $e_{ij}$  that is not a vertex of  $H'$ . That vertex corresponds to the biclique  $B_{e_{ij}} = \{e_{ij}\}\{v_i, v_j\}$ . Moreover, those are the unique vertices of  $KB(G) \setminus H'$ . Each vertex  $B_{e_{ij}}$  in  $KB(G)$  is adjacent to  $N_i, N_j$ . Observe that  $N_i, N_j$  are adjacent in  $KB(G)$ . Also,  $B_{e_{ij}}$  is adjacent to every vertex

$B_{e_{ik}}, B_{e_{ki}}, B_{e_{jl}}$  and  $B_{e_{lj}}$  of  $KB(G)$ . Associating vertex  $e_{ij}$  of  $H_1^*$  to vertex  $B_{e_{ij}}$ , it follows that  $KB(G)$  is isomorphic to  $H_1^*$ .

Next we prove that  $K(G)$  is isomorphic to  $H_2^*$ , where  $G$  is constructed for the clique graph case (graph  $G$  of item 3).

Observe that for every triangle  $v_i, v_j, v_k$  in  $H$ , there exists a different clique in  $G$  that is not a vertex in  $H'$ . That vertex corresponds to a clique of the form  $t'_{ijk} = \{e_{ij}, e_{jk}, e_{ik}\}$ . Moreover, those are the unique vertices of  $K(G) \setminus H'$ . Finally, vertex  $t'_{ijk}$  is adjacent to vertices  $N_i, N_j, N_k$  in  $K(G)$  and  $N_i, N_j, N_k$  induce a triangle in  $K(G)$ . If  $t'_{ijk}$  is adjacent to  $t'_{lms}$  then  $e_{ab}$  is contained in both triangles in  $G$ , that is,  $e_{ab} \in \{e_{ij}, e_{jk}, e_{ik}\}$ . That means that triangle  $N_i, N_j, N_k$  and triangle  $N_l, N_m, N_s$  share an edge. Concluding, associating vertices  $t_{ijk}$  of  $H_2^*$  to vertices  $t'_{ijk}$  of  $K(G)$ , it follows that  $t_{ijk}$  and  $t_{lms}$  are adjacent if and only if both triangles share an edge.

We conclude that  $KB(G) \cong H_1^*/K(G) \cong H_2^*$ .  $\square$

**Corolario 3.2.** *Every  $H$  is an induced subgraph of a square of a graph  $G$ , where  $G$  is a  $C_4$ -free bipartite graph. Moreover, every  $H$  is the induced subgraph of a star graph of a graph  $G$ , where  $G \in C_4$ -free bipartite.*

### 3.2. Characterizations of the biclique graph of other classes

For some classical classes as complete graphs, trees, paths, cycles, and graphs with girth at least 5, the biclique graph can be easily characterized.

If  $H$  is a complete graph, the bicliques of  $H$  are its edges, so  $KB(H) = L(H)$ .

Let  $\mathcal{G}_k$  be the class of graphs with girth at least  $k$ . Note that  $\mathcal{G}_{k+1} \subset \mathcal{G}_k$ , for every  $k \geq 3$ . Let  $leaves(H)$  be the set of vertices of degree 1 of  $H$ . If  $H \in \mathcal{G}_5$ , then the bicliques of  $H$  are all stars and it follows that  $KB(H) = (H - leaves(H))^2$ .

As trees belong to the class  $\mathcal{G}_5$ , to decide if a graph  $H$  is the biclique graph of a tree is the same as to decide if  $H$  is the square of a tree.

See Table 3.1 for a brief list of similar results, along with the time complexity of the problem of recognizing if a graph  $G$  belongs to  $K(\mathcal{A})$ , for certain class  $\mathcal{A}$ .

To decide if a graph  $G \in L(\text{complete})$  can be done in polynomial time [Leh74, LTVM15, Rou73].

To decide if  $G \in (\mathcal{G}_k)^2$  there is a polynomial time algorithm for  $k \geq 6$  and is NP-complete for  $k \leq 4$  [FLLT12]. For the case of trees, it can be done in linear time ( $\mathcal{O}(n + m)$ ) [LS95].

class $\mathcal{A}$	$KB(G)$ , $G \in \mathcal{A}$	class $KB(\mathcal{A})$	complexity
complete	$L(G)$	$L(\text{complete})$	P
tree	$(G - \text{leaves}(G))^2$	$(\text{tree})^2$	P (linear)
path ( $G = P_n$ )	$\emptyset$ , for $n = 1$ $K_1$ , for $n = 2$ $(P_{n-2})^2$ , for $n > 2$	$(\text{path})^2$	P
cycle ( $G = C_n$ )	$K_1$ , for $n = 4$ $(C_n)^2$ , for $n \neq 4$	$(\text{cycle})^2 - K_4 + K_1$	P
$\mathcal{G}_k$ , for $k \geq 5$	$(G - \text{leaves}(G))^2$	$(\mathcal{G}_k)^2$ , for $k \geq 5$	P, for $k \geq 6$ open, for $k = 5$
$\mathcal{IBG}$	—	$\subset K_{1,4}$ -free co-comparability (see Chapter 7)	open
$\mathcal{PIB} (= \mathcal{BPG})$	$(L(S(G))^2$	$(L(\mathcal{PIB}))^2$ (see Chapter 7)	open
$\mathcal{PIB}\text{-ASG}$	$(L(S(G))^2$	1- $\mathcal{PIG}$ (see Chapter 7)	P
$\mathcal{BHD}$	—	$\mathcal{CHBDI}$ [Gro06]	open

Cuadro 3.1: Biclique graph of some classes. At column “ $KB(G)$ ,  $G \in \mathcal{A}$ ” we can find a brief description of the graph  $KB(G)$  for each class; at column “class  $KB(\mathcal{A})$ ” appears the class that is equal to  $KB(\mathcal{A})'$  except for the case of  $\mathcal{IBG}$ , where we show its super-class. At column “complexity” we present the complexity (if known) of recognizing  $KB(\mathcal{A})$ .

Note also that it remains open the complexity of deciding if a graph belongs to  $(\mathcal{G}_5)^2$  [FLLT12]. As the class  $KB(\mathcal{G}_5) = (\mathcal{G}_5)^2$ , it is also unknown the complexity of the recognition of the class  $KB(\mathcal{G}_5)$ .

Let  $\mathcal{BHD}$  be the class of bipartite biclique-Helly graphs with no dominated vertices. Let  $\mathcal{CHBDI}$  be the class of graphs such that the family of cliques is independent Helly-bicovered and has no dominated vertices. Groshaus proved that  $KB(\mathcal{BHD}) = \mathcal{CHBDI}$  [Gro06].

# 4 Grafo Bipartito Cordal

En esta sección buscamos acercarnos a la caracterización de la clase de grafos biclique de los grafos bipartitos cordales.

Buscamos un resultado análogo al encontrado para el operador clique:  $K(\text{cordal}) = \text{dualmente cordal}$  [ABV98, Gut96, SB94]. Obtuimos resultados acerca de los grafos bicliques de los grafos bipartitos cordales y una subclase de ellos. A diferencia del caso clique, para la subclase de los bipartitos cordales estudiada no es cierto que todo grafo dualmente cordal es grafo biclique de un bipartito cordal dominó-free. Probamos que si  $G$  es bipartito cordal, entonces  $KB(G)$  es cordal o sus ciclos de longitud mínima (sin considerar los triángulos) están en una rueda (i.e. existe un vértice adyacente a todos los vértices del ciclo); y si  $G$  es un grafo bipartito cordal dominó-free, luego  $KB(G)$  es dualmente cordal.

## 4.1. Nociones de grafos utilizadas en este capítulo

Un grafo bipartito es *bipartito cordal* si y sólo si no tiene ciclos inducidos distintos de  $C_4$ .

Un grafo es *dualmente cordal* si y sólo si es el grafo clique de algún grafo cordal.

Un grafo  $G$  es *dominó* si es isomorfo al grafo  $D$  cuyo conjunto de vértices es  $V(D) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  y cuyo conjunto de aristas es  $E(D) = \{v_1v_2, v_1v_4, v_2v_3, v_2v_5, v_3v_6, v_4v_5, v_5v_6\}$ . El grafo de la izquierda de la figura 4.1 es un grafo dominó.

Un grafo  $G$  es *rueda* si todos sus vértices menos uno,  $v$ , inducen un ciclo y  $v$  es un vértice universal.

## 4.2. Clase $KB(\text{bipartito cordal})$

**Teorema 4.1.** *Sea  $G$  bipartito cordal. Entonces  $G$  es dominó-free si y sólo si  $G^2$  es cordal.*

*Demostración.* Para probar el “sólo si” supongamos que  $G^2$  no es cordal. Consideremos un ciclo inducido de  $G^2$  de longitud mínima:  $C = c_0, c_1, \dots, c_{n-1} \in G^2$  con  $n \geq 4$ . Dado que en  $G^2$  la distancia entre vértices adyacentes en el ciclo de  $G^2$  es 1, es decir,  $d_{G^2}(c_i, c_{i+1}) = 1$ ,  $\forall 0 \leq i \leq n - 1$ , la distancia de estos vértices en  $G$  debe ser menor o igual que 2:  $d_G(c_i, c_{i+1}) \leq 2$ ,  $\forall 0 \leq i \leq n - 1$ . Luego,  $\forall 0 \leq i \leq n - 1$ ,  $c_i$  es adyacente en  $G$  a  $c_{i+1}$  o sino existe  $d_i$  tal que  $c_i d_i, d_i c_{i+1} \in E(G)$ .

Observemos que todo  $P_3$  inducido en  $G^2$  contiene al menos una arista que no está en  $G$ . Por lo tanto, si consideramos el ciclo formado por los vértices de  $C$  y los  $d_i$  mencionados, tenemos un ciclo de longitud mayor que 4 y por lo tanto existen dos vértices no consecutivos en ese ciclo que son adyacentes (cuerda).

Observamos que ninguno de esos dos vértices pertenecen a  $C$ , ya que de otra forma si  $d_i$  es adyacente en  $G$  a  $c_k$ , entonces en  $G^2$  los vértices  $c_i$  y  $c_{i+1}$  son adyacentes a  $c_k$  y por lo tanto  $C$  tiene tamaño 3 o tiene una cuerda, lo que es una contradicción.

Sean  $d_k$  y  $d_i$  dos vértices adyacentes a distancia mínima en  $C$ , es decir, tal que el camino  $c_{k+1}, c_{k+2}, c_{k+3}, \dots, c_i$  o el camino  $c_k, c_{k-1}, \dots, c_{i+1}$  sea mínimo de entre todos los caminos entre vértices del tipo  $d$  adyacentes. Sin pérdida de generalidad, vamos a suponer que el camino más corto es  $c_{k+1}, c_{k+2}, c_{k+3}, \dots, c_i$ . Observemos que como  $G$  es bipartito,  $d_k \neq d_{i+1}$  y  $d_k \neq d_{i-1}$ , sino se formaría un triángulo.

Supongamos que  $c_{k+1}$  no es adyacente a  $c_i$  en  $G^2$ . Entonces, en  $G^2$ ,  $d_k, c_{k+1}, c_{k+2}, \dots, c_i$  forma un ciclo que tiene longitud estrictamente menor que  $C$ , lo cual contradice la hipótesis de que  $C$  es mínimo. Por lo tanto,  $c_{k+1}$  es adyacente a  $c_i$  en  $G^2$ .

Observemos también que  $c_{k+1}$  es adyacente a  $c_i$  en  $G^2$  ya que por como elegimos  $d_k$  y  $d_i$ , sabemos que no hay cuerdas en el camino  $d_k, c_{k+1}, \dots, c_i, d_i$  y por lo tanto su longitud es 4.

Queremos probar que  $c_k$  es adyacente a  $c_{i+1}$  en  $G$ . Supongamos que no. Consideremos ahora el ciclo  $d_k, c_k, c_{k-1}, c_{k-2}, \dots, c_{i+2}, c_{i+1}$  en  $G^2$  (observemos que  $c_{i+1}$  es adyacente a  $d_k$ ). Este ciclo no puede ser inducido en  $G^2$  porque asumimos que  $C$  tiene longitud mínima. Entonces, deben existir cuerdas en  $G^2$  entre  $d_k$  y todos los vértices  $c_{k-1}, c_{k-2}, \dots, c_{i+2}$ , siendo éstas aristas de  $G^2$  que no están en  $G$  (ya que sino en  $G^2$  habría aristas entre los vértices de  $C$ ). Análogamente, considerando el ciclo  $c_k, c_{k-1}, c_{k-2}, \dots, c_{i+2}, c_{i+1}, d_i$  y usando el mismo argumento anterior, concluimos que  $d_i$  es adyacente a todos los vértices  $c_{k-1}, c_{k-2}, \dots, c_{i+2}, \dots, c_{i+2}$  en  $G^2$ . Recordemos que estas aristas no existen en  $G$ .

Por último, consideremos las aristas en  $G^2$  del camino  $(c_k, c_{k-1}, c_{k-2} \dots, c_{i+2}, c_{i+1})$ . Si ninguna arista pertenece a  $G$ , entonces por cada arista existe un vértice diferente en  $G$  adyacente a sus extremos. Por lo tanto, en  $G$  tenemos un ciclo impar formado por esos vértices,  $c_k, c_{k-1}, c_{k-2} \dots, c_{i+2}, c_{i+1}$  y  $d_i, d_k$ , lo cual es un absurdo.

Supongamos que  $c_k$  es adyacente a  $c_{k-1}$  en  $G$ . Como  $d_i$  es adyacente a  $c_k$  y a  $c_{k-1}$  en  $G^2$ , existe un vértice  $x$  adyacente a  $d_i$  y a  $c_k$  en  $G$  y un vértice  $y$  adyacente a  $d_i$  y a  $c_{k-1}$  en  $G$ . Los vértices  $d_i, x, y, c_k$  y  $c_{k-1}$  forman un  $C_5$  o tres de ellos forman un triángulo.

El caso  $c_{i+1}$  adyacente a  $c_{i+2}$  en  $G$  es análogo considerando el vértice  $d_k$ .

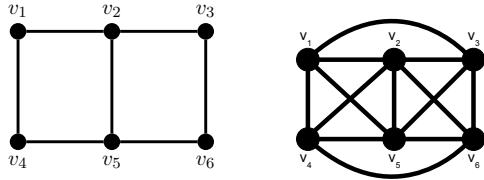


Figura 4.1: Dominó en  $G$  y el subgrafo que induce en  $G^2$

Por último, si ninguno de los casos anteriores ocurre, entonces como mencionamos antes, existe una arista en el camino  $c_{k-1}, c_{k-2} \dots c_{i+2}$  que está en  $G$  y utilizando el mismo argumento anterior, llegamos a la conclusión de que tenemos en  $G$  un  $C_5$  o un triángulo, lo que es un absurdo.

Concluimos que  $c_k$  es adyacente a  $c_{i+1}$  en  $G^2$  y por lo tanto, como vimos antes, también en  $G$ .

Finalmente, los vértices  $d_k, c_{k+1}, c_i, d_i, c_{i+1}, c_k$  inducen un dominó, lo cual contradice la hipótesis.

Concluimos entonces que  $G^2$  no tiene ciclos inducidos de longitud al menos 4.

Para probar el “si”, supongamos que  $G$  no es dominó-free. Sea  $D$  un dominó inducido en  $G$  donde

$V(D) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  y  $E(D) = \{v_1v_2, v_1v_4, v_2v_3, v_2v_5, v_3v_6, v_4v_5, v_5v_6\}$  (en la figura 4.1 se muestra este subgrafo inducido).

Luego en  $G^2$ ,  $v_1, v_2, v_3, v_4, v_5, v_6$  inducen un grafo donde todos los vértices son adyacentes entre sí salvo  $v_1$  y  $v_6$  y  $v_3$  y  $v_4$  como se muestra en la figura 4.1. Las adyacencias se producen porque los vértices están a distancia menor o igual a 2 en  $G$ . Las no adyacencias se producen porque los vértices están a distancia mayor a 2 en  $G$ . Por ejemplo,  $v_1$  y  $v_6$  no están a distancia menor o igual a 2 en  $G$  ya que no son adyacentes y para estar a distancia 2 debería existir  $v$  adyacente a  $v_1$  y  $v_6$ , lo que no puede ser ya que  $v_1, v_4, v_5, v_6, v$  formaría un ciclo de longitud 5 lo que no es posible en un grafo bipartito. Luego  $v_1, v_3, v_6, v_4$  inducen un  $C_4$  en  $G^2$ , lo que es una contradicción dado que  $G^2$  es cordal.  $\square$

**Teorema 4.2** (Tesis Groshaus, ver Corollary 6.2, [ GS08 ]). *Sea  $G$  un grafo bipartito. Luego  $G$  es  $C_6$ -free y sólo si  $KB(G) = K(G^2)$ .*

**Teorema 4.3.** *Sea  $G$  un grafo bipartito cordal dominó-free. Luego  $KB(G)$  es dualmente cordal.*

*Demostración.* Sea  $G$  bipartito cordal dominó-free. Luego, por Teorema 4.1,  $G^2$  es cordal y por definición,  $K(G^2)$  dualmente cordal. Por otra parte,  $G$  no contiene  $C_6$  ni triángulos como subgrafos inducidos, por el Teorema 4.2,  $KB(G) = K(G^2)$ . Concluimos que  $KB(G)$  es dualmente cordal.  $\square$

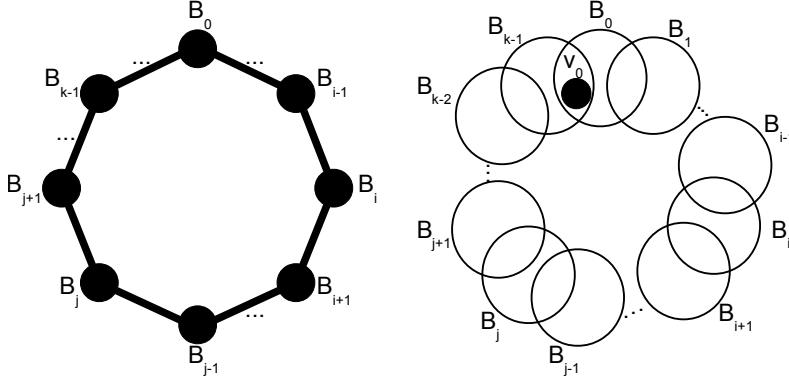


Figura 4.2:  $C_k$  en  $KB(G)$  y en  $G$

**Teorema 4.4.** Si  $G$  es bipartito cordal, entonces  $KB(G)$  es cordal o sus ciclos de longitud mínima (sin considerar los triángulos) están en una rueda (es decir, existe un vértice adyacente a todos los vértices del ciclo)

*Demostración.* Si  $KB(G)$  no es cordal,  $KB(G)$  tiene ciclos inducidos distintos de  $C_3$ . Sea  $C_k = \{B_0, \dots, B_{k-1}\}$  un ciclo inducido de longitud mínima de  $KB(G)$  sin considerar los  $C_3$ . Dado que  $B_0, B_{k-1} \in E(KB(G))$ ,  $B_0 \cap B_{k-1} \neq \emptyset$  en  $G$  y por tanto existe  $v_0 \in B_0 \cap B_{k-1}$ , como se muestra en la Figura 4.2

Partiendo de  $v_0$  definimos recursivamente el ciclo  $C$  de  $G$  de la siguiente forma:

- $v_0 \in V(C)$
  - Para  $1 \leq i \leq k-1$ ,
- si existe  $v_i \in B_{i-1} \cap B_i$  tal que  $v_i$  y  $v_{i-1}$  están en distinto conjunto de la partición de  $B_{i-1}$ ,  $v_i \in V(C)$  y  $v_{i-1}v_i \in E(C)$ .

Si no, existe  $v_i \in B_{i-1} \cap B_i$  tal que  $v_{i-1}$  y  $v_i$  están en el mismo conjunto de la partición de  $B_{i-1}$  y existe  $w_{i-1} \in B_{i-1}$  tal que  $w_{i-1}$  está en distinto conjunto de la partición que  $v_{i-1}$  en  $B_{i-1}$ ; definimos  $w_{i-1}, v_i \in V(C)$  y  $v_{i-1}w_{i-1}, w_{i-1}v_i \in E(C)$ . Observar que  $w_{i-1} \notin B_i$ . Una situación posible se muestra en la figura 4.3

- Si  $v_{k-1}$  y  $v_0$  están en distinta partición de  $B_{k-1}$ ,  $v_{k-1}v_0 \in E(C)$ . Si no, si  $w_0 \in C$  y  $w_0 \in B_{k-1}$ , quitamos  $v_0$  de  $C$  (y por tanto la arista  $v_0w_0$ ) y agregamos a  $C$  la arista  $v_{k-1}w_0$ . Por último renombramos  $w_0$  con  $v_0$ . Si no (i.e.  $v_{k-1}$  y  $v_0$  están en la misma partición de  $B_{k-1}$  y  $w_0 \notin B_{k-1}$ ), existe  $w_{k-1} \in B_{k-1}$  en distinta partición que  $v_{k-1}$  en  $B_{k-1}$ , definimos  $v_{k-1}w_{k-1}, w_{k-1}v_0 \in E(C)$ .

Las tres posibles formas de cerrar el ciclo se muestran en la figura 4.4

Luego,  $C$  es un ciclo (posiblemente no inducido) en  $G$ , cuya longitud  $l$  es tal que  $k \leq l \leq 2k$  ya que por cada biclique contiene al menos una arista y a los sumo dos.

1. Si  $k = 4$ , i.e.,  $C_4 = \{B_0, B_1, B_2, B_3\} \subset KB(G)$  consideraremos dos casos

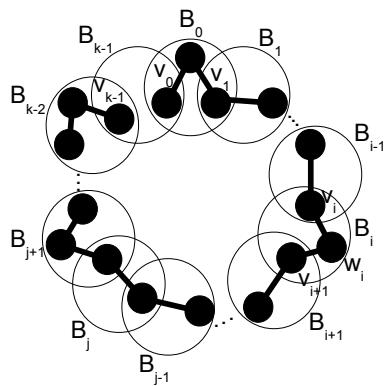


Figura 4.3: Ciclo en  $G$  sin cerrar

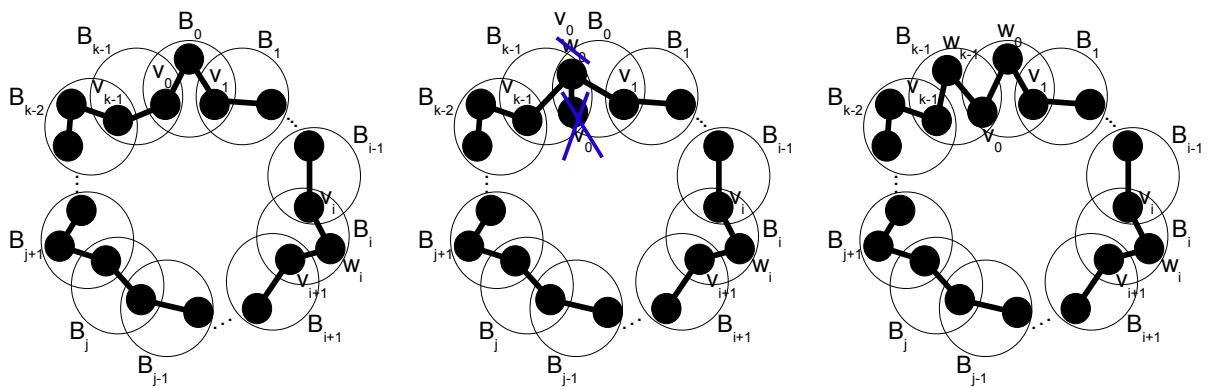


Figura 4.4: Posibles formas de cerrar el ciclo en  $G$

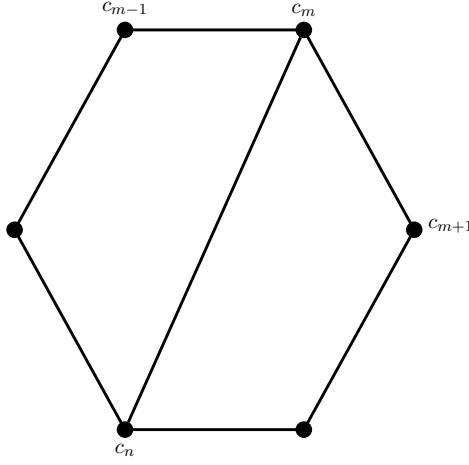


Figura 4.5: Ciclo de longitud 6 formado por 4 bicliques de  $G$

- $|C| = 4$ , i.e.,  $C$  tiene una única arista por biclique y por tanto  $C = \{v_0, v_1, v_2, v_3\}$  y  $C$  es un bipartito completo en  $G$ . Sea  $B$  una biclique de  $G$  tal que  $C \subset B$ , luego  $B \cap B_i \neq \emptyset$  para todo  $0 \leq i \leq 3$  ya que  $v_i \in B_i$  para todo  $0 \leq i \leq 3$ . Además  $B \neq B_i$  para todo  $0 \leq i \leq 3$  ya que  $B_i \cap B_{i+2} = \emptyset$ . Luego,  $BB_i \in E(KG(B))$  para todo  $0 \leq i \leq 3$ .
- $|C| = t > 4$ , luego  $t = 6 \vee 8$ , ya que  $|C|$  no es impar porque  $G$  es bipartito.

- a) Si  $t = 6$ ,  $C = \{c_0, c_1, \dots, c_5\}$ , y dado que  $C \subset G$  y  $G$  es bipartito cordal, existen  $0 \leq m \leq 2$ ,  $n = m + 3$  tal que  $c_m c_n \in E(G)$ , lo que se muestra en la figura 4.5

Además  $\{c_m\} \cup \{c_{m-1}, c_n, c_{m+1}\}$  es un bipartito completo inducido en  $G$ , ya que  $c_{m-1} c_{m+1}, c_{m-1} c_n, c_{m+1} c_n \notin E(G)$  puesto que sino  $G$  tendría  $C_3$ . Sea  $B \subset G$  una biclique que contiene a  $\{c_m\} \cup \{c_{m-1}, c_n, c_{m+1}\}$ , luego  $(\{c_m\} \cup \{c_{m-1}, c_n, c_{m+1}\}) \cap c_l c_{l+1} \neq \emptyset$  para todo  $0 \leq l \leq 5$ , es decir la biclique  $B$  interseca todas las aristas de  $C$ , y dado que toda biclique  $B_i$  contiene algún vértice de  $C$  tenemos  $B \cap B_i \neq \emptyset$  para todo  $0 \leq i \leq 3$ .

- b) Si  $t = 8$ ,  $C = \{v_0, w_0, v_1, w_1, v_2, w_2, v_3, w_3\}$ . Dado que  $G$  es bipartito cordal, existe  $w_r$  con  $0 \leq r \leq 3$  tal que  $w_r v_{r+2} \in E(G)$  o  $w_r v_{r+3} \in E(G)$ . En la figura 4.6 se muestra esta situación para  $r = 3$ :

Luego  $\{w_r\} \cup \{v_r, v_{r+2}, v_{r+1}\}$  o  $\{w_r\} \cup \{v_r, v_{r+3}, v_{r+1}\}$  es un bipartito completo en  $G$  ya que sino  $G$  tendría triángulos. Sea  $B \subset G$  la biclique que contiene a este bipartito completo. Luego  $B \cap B_l \neq \emptyset$  para  $l = r-1, r, r+1$  ya que  $v_r \in B_{r-1}, v_r \in B_r, v_{r+1} \in B_{r+1}$ . Además  $B \cap B_{r+2} \neq \emptyset$  en cualquiera de los dos casos ya que  $v_{r+2}, v_{r+3} \in B_{r+2}$ . Luego  $B \cap B_i \neq \emptyset$  para todo  $0 \leq i \leq 3$ .

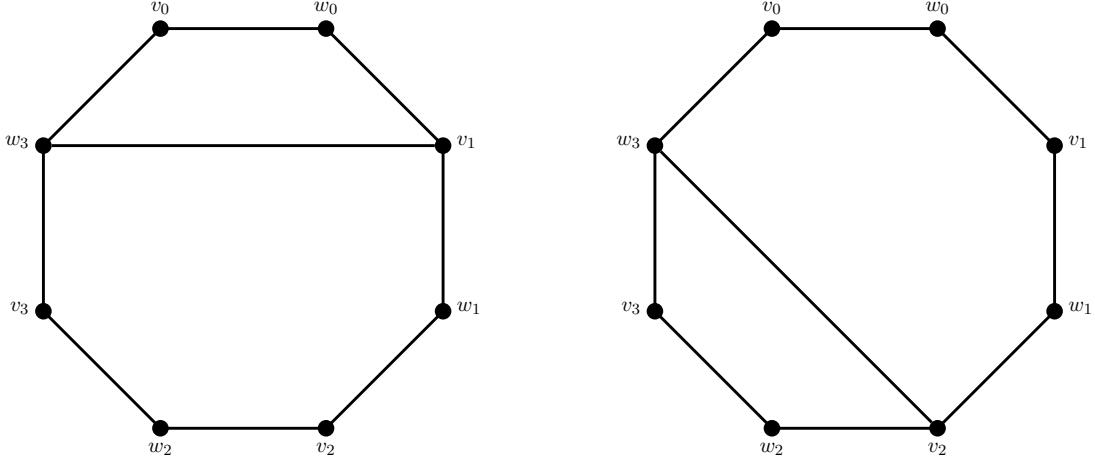


Figura 4.6: Ciclo de longitud 8 formado por 4 bicliques de  $G$

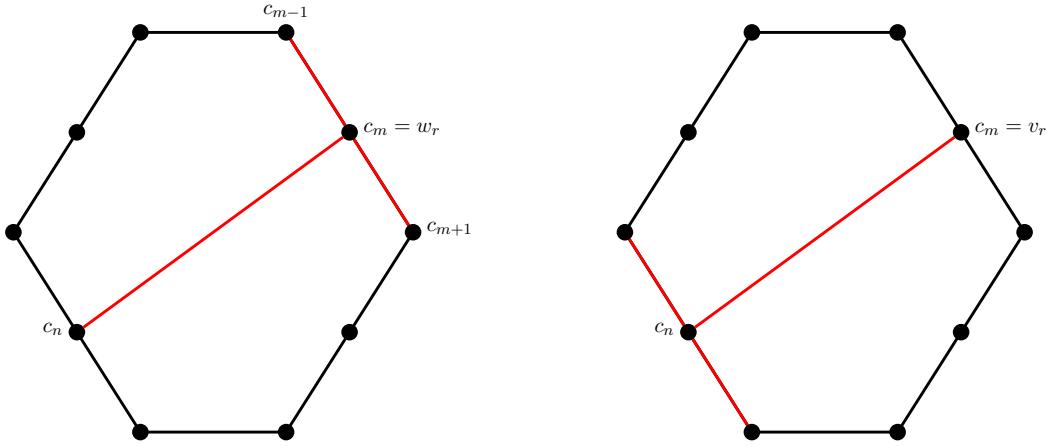


Figura 4.7: Ciclo de longitud 10 en  $G$

2. Si  $k > 4$ , sea  $C = \{c_1, c_2, \dots, c_t\}$ , donde  $c_1 = v_0$ ,  $k \leq t \leq 2k$  y  $t$  es par ya que  $G$  es bipartito. Dado que  $G$  es bipartito cordal, existen  $1 \leq m < n \leq t$  tales que  $c_m c_n \in E(G) \setminus E(C)$ . Si  $c_m = w_r$  para algún  $0 \leq r \leq k-1$  consideramos  $B' = \{c_m\} \cup \{c_{m-1}, c_n, c_{m+1}\}$ , sino, i.e  $c_m = v_r$  para algún  $0 \leq r \leq k-1$ , consideramos  $B' = \{c_n\} \cup \{c_{n-1}, c_m, c_{n+1}\}$ . Luego  $B'$  es un bipartito completo en  $G$  ya que si no  $G$  tendría triángulos. Luego, existe una biclique  $B \subset G$  que contiene a  $B'$ . Estas dos posibilidades se muestran en la figura 4.7 para  $t = 10$ :

Si  $B' = \{c_m\} \cup \{c_{m-1}, c_n, c_{m+1}\}$ , tenemos  $c_m = w_r, c_{m-1} = v_r, c_{m+1} = v_{r+1}$  y por tanto  $B \cap B_j \neq \emptyset$  para  $j = r-1, r, r+1$ . Además, existe  $0 \leq l \leq k-1, l \neq r-1, r, r+1$ , tal que  $c_n \in B_l$  ya que sino  $c_m$  y  $c_n$  estarían en un triángulo. Luego  $B \cap B_l \neq \emptyset$ . Concluimos que  $B \cap B_i \neq \emptyset$  para todo  $0 \leq i \leq k-1$  ya que sino se formarían ciclos más grandes que tres y más chicos que  $k$  en  $KB(G)$ . Entonces, el vértice

correspondiente a  $B$  en  $KB(G)$  es adyacente a todo vértice del ciclo.

Si  $B' = \{c_n\} \cup \{c_{n-1}, c_m, c_{m+1}\}$ , podemos realizar un análisis similar al anterior considerando que  $c_m = v_r$  y por tanto  $B \cap B_j \neq \emptyset$  para  $j = r - 1, r$ , y existe  $v_l \in \{c_{n-1}, c_n, c_{m+1}\}$  con  $l \neq r - 1, r$  y  $l - 1 \neq r$ , por tanto  $B \cap B_j \neq \emptyset$  para  $j = l - 1, l$ .

□

Notemos que solo conseguimos probar la existencia del vértice universal al ciclo cuando consideramos el ciclo de menor tamaño. Dejamos como problema abierto estudiar mas propiedades de esta clase.

# 5 Grafo Split

## 5.1. Preliminares

En esta sección introducimos algunas definiciones y repetimos otras que consideramos importantes para la comprensión de este capítulo. Un *grafo split*  $H = (K \cup S, E)$  es un grafo tal que su conjunto de vértices puede particionarse en una clique,  $K$ , y en un conjunto independiente,  $S$ . Llamamos a los vértices de  $S$  satélites del grafo y abreviamos grafo split con SG. Dado un SG  $H = (K \cup S, E)$  y un vértice  $v \in K$ , denotamos por  $S_H(v)$  el conjunto de satélites adyacentes a  $v$ , o sea,  $S_H(v) = N_H(v) \cap S$ . Los vértices de  $K$  pueden particionarse en partes tales que los vértices con el mismo conjunto de satélites están en la misma partición. Es decir, sea  $\mathcal{X}_H = \{X_1, \dots, X_\ell\}$  una partición de  $K$  tal que  $x, y \in X_i$  para alguna  $i$  si y solo si  $S_H(x) = S_H(y)$ . Denotamos por  $X_H(v)$  la parte a la que  $v$  pertenece, es decir, si  $v \in X_i$  luego  $X_H(v) = X_i$ .

Denotamos por  $S_H(X_i)$  el conjunto de satélites de los vértices de  $X_i$ , es decir,  $S_H(X_i) = S_H(x)$ , para  $x \in X_i$ . Si  $X_z$  es una parte tal que  $S_H(X_z) = \emptyset$ , luego  $X_z$  se llama la *parte cero*.

El conjunto parcialmente ordenado (poset) asociado a  $H$  es la relación binaria  $(\mathcal{X}_H, \preceq_H)$  tal que  $X_i \preceq_H X_j$  si y solo si  $S_H(X_i) \subseteq S_H(X_j)$ . Si  $X_i \preceq_H X_j$  o  $X_j \preceq_H X_i$  decimos que  $X_i, X_j$  son *comparables* y lo denotamos por  $X_i \leftrightarrow X_j$ . Por otro lado, si  $X_i, X_j$  son partes incomparables, lo denotamos por  $X_i \not\leftrightarrow_H X_j$ . El subíndice de  $S_H$ ,  $\mathcal{X}_H$ ,  $\preceq_H$  y  $\not\leftrightarrow_H$  lo omitimos cuando es claro cuál es el grafo  $H$ .

Note que  $(\mathcal{X}_H, \preceq_H)$  es una relación reflexiva, antisimétrica y transitiva, y por tanto es efectivamente un poset.

Denotamos  $C(n, k)$  al número combinatorio de numerador  $n$  y denominador  $k$ .

## 5.2. Bicliques de los grafos split

En esta sección estudiamos las bicliques de los grafos split. Sea  $H = (K \cup S, E)$  un SG con poset asociado  $(\mathcal{X}, \preceq)$ . Cada biclique de  $H$  tiene uno o dos vértices de  $K$ . Una biclique que tiene un único vértice  $v$  de  $K$  es una *biclique estrella* formada por  $v$  como su centro y su conjunto de satélites,  $S(v)$ . Tal biclique existe si y solo si para cada parte  $X_i \in \mathcal{X}$ , con  $X_i \neq X(v)$ ,  $S(X_i) \cap S(X(v)) \neq \emptyset$ . Además  $|S(v)| \geq 2$  ya que sino podría considerarse un completo  $K$  mayor.

Llamamos *biclique estrella tipo uno* a una biclique estrella  $B$  con centro en  $v$  y tal que  $\exists u, w \in K$  tal que  $S(u) \cap S(w) \cap S(v) = \emptyset$ .

Llamamos *biclique estrella tipo dos* a una biclique estrella  $B$  con centro en  $v$  y tal que  $\forall u, w \in K, S(u) \cap S(w) \cap S(v) \neq \emptyset$ .

Dada una biclique  $B$  de  $H$  que no es una biclique estrella, llamamos a la arista de  $K$  la *arista base* de  $B$ . Algunas bicliques tienen también vértices de  $S$ . Cada arista de  $K$  es la arista base de una o dos bicliques de  $H$ .

Si dos bicliques tienen un vértice de  $K$  en común decimos que esas bicliques son  *$K$ -intersectantes*. Si dos bicliques tienen solamente satélites en común decimos que esas bicliques son  *$S$ -intersectantes*.

Las bicliques de  $H$  que no son bicliques estrellas pueden clasificarse en tres tipos diferentes. Considerando la biclique  $B$  de  $H$  y su arista base  $uv$ .

Si  $X(u) = X(v)$ , luego llamamos a  $B = uv$  una *biclique arista*, y la denotamos  $B_{uv}$  (o  $B_{vu}$ ) de  $X(u)$ . Decimos que  $B$  es una biclique arista de  $X_i$  si  $u, v \in X_i$ .

Si  $X(u) \neq X(v)$  y  $X(v) \preceq X(u)$ , luego  $B$  contiene también los satélites de  $S(X(u)) \setminus S(X(v))$ , la llamamos una *s-biclique simple*, y nos referimos a ella como  $B_{uv}$  desde  $X(v)$  a  $X(u)$ . Es decir, la dirección de la biclique es desde  $X(v)$  a  $X(u)$ .

Si  $X(u) \leftrightarrow X(v)$ , existen dos bicliques con la misma arista base. Una biclique contiene satélites de  $S(X(u)) \setminus S(X(v))$  y la denotamos por  $B_{uv}$ . En este caso decimos que la biclique  $B_{uv}$  es una biclique con dirección desde  $X(v)$  a  $X(u)$ . La otra biclique contiene los satélites de  $S(X(v)) \setminus S(X(u))$  y nos referimos a ella como  $B_{vu}$  con dirección desde  $X(u)$  a  $X(v)$ . Llamamos a estas bicliques *s-bicliques dobles*.

Observe que si  $B_{uv}$  es una biclique arista simple desde  $X(v)$  a  $X(u)$ , no existe una biclique arista  $B_{vu}$ , ni simple ni doble.

Si la dirección de la s-biclique (simple o doble) no es conocida (o no importa), decimos que es una s-biclique entre  $X(u)$  y  $X(v)$ .

En la Figura 5.1 se muestra un SG donde por claridad no hemos dibujado las aristas correspondientes a la clique  $K$ , que es un  $K_6$  formado por los vértices  $\{k_1, k_2, k_3, k_4, k_5, k_6\}$ . El conjunto de satélites es  $\{s_1, s_2, s_3, s_4\}$ .

En este ejemplo tenemos que  $\{k_3\} \cup \{s_1, s_2, s_3\}$  es una biclique estrella con centro en  $K_3$  tipo uno,  $B = k_2k_6$  es una biclique arista que denotamos  $B_{k_2k_6}$  o  $B_{k_6k_2}$ . La biclique  $\{k_4\} \cup \{k_5, s_3, s_4\}$  es una s-biclique simple y nos referimos a ella como  $B_{k_4k_5}$  desde  $X(k_5)$  a  $X(k_4)$ . La arista  $k_5k_6$  es arista base de dos bicliques:  $\{k_5\} \cup \{k_6, s_2\}$  y  $\{k_6\} \cup \{k_1, s_1\}$ , denotamos  $B_{k_5k_6}$  a la primera (y decimos que es una biclique con dirección desde  $X(k_6)$  a  $X(k_5)$ ) y  $B_{k_6k_5}$  a la segunda.

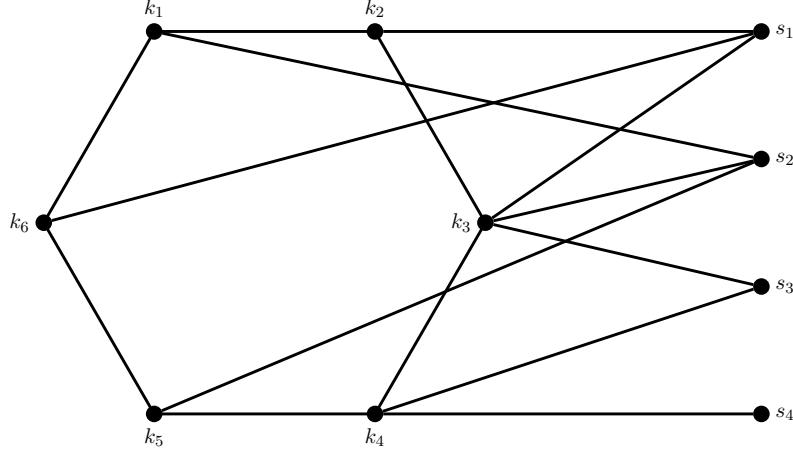


Figura 5.1: Grafo Split con biclique estrella

### 5.3. Propiedades de los grafos split

En esta sección consideraremos que  $H$  es un SG tal que  $H = (K \cup S, E)$  con  $|K| = n$  y  $G$  es su grafo biclique, es decir,  $G = KB(H)$ .

**Proposición 5.1.**  $C(n, 2) \leq |V(G)| \leq 2C(n, 2) + n$

*Demostración.* Dos aristas distintas de  $K$  no pueden estar en la misma biclique y por tanto existen al menos  $C(n, 2)$  bicliques en  $H$ . Por otra parte, la máxima cantidad posible de bicliques estrella es una por cada vértice de  $K$ , es decir  $n$ , y la máxima cantidad posible de bicliques arista es dos por cada arista  $(u, v) \in K$ :  $B_{uv}$  y  $B_{vu}$ , es decir, las bicliques arista son a lo sumo  $2C(n, 2)$ . Luego la cantidad máxima de bicliques es  $2C(n, 2) + n$ .  $\square$

Es claro que existen grafos split tales que su clique  $K$  tiene  $n$  vértices y su grafo biclique tiene exactamente  $c(n, 2)$  vértices. La Figura 5.2. muestra un grafo donde su clique tiene cuatro vértices y su grafo biclique tiene exactamente  $2C(4, 2) + 4$  (tiene una biclique estrella por cada vértice de la clique y dos bicliques con la misma arista base por cada arista de la clique, lo que prueba que los límites establecidos en la proposición son lo más ajustados posibles. Nuevamente no dibujamos aristas correspondientes a la clique  $K$  inducida por  $k_1, k_2, k_3, k_4$ .

**Proposición 5.2.** *El diámetro de  $G$  es menor o igual que 2.*

*Demostración.* Sean  $B_1, B_2 \in V(G)$ . Si  $B_1B_2 \notin E(G)$ , existen  $u \in B_1 \cap K, v \in B_2 \cap K, u \neq v$ . Claramente,  $u \notin B_2, v \notin B_1$ . Luego  $B_{uv}$  o  $B_{vu}$  es una biclique que contiene a la arista  $uv$  y la distancia entre  $B_1$  y  $B_2$  en  $G$  es 2.  $\square$

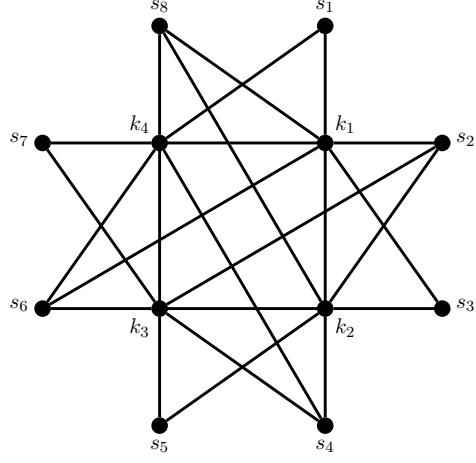


Figura 5.2: Grafo Split con  $2C(n, 2) + n$  bicliques

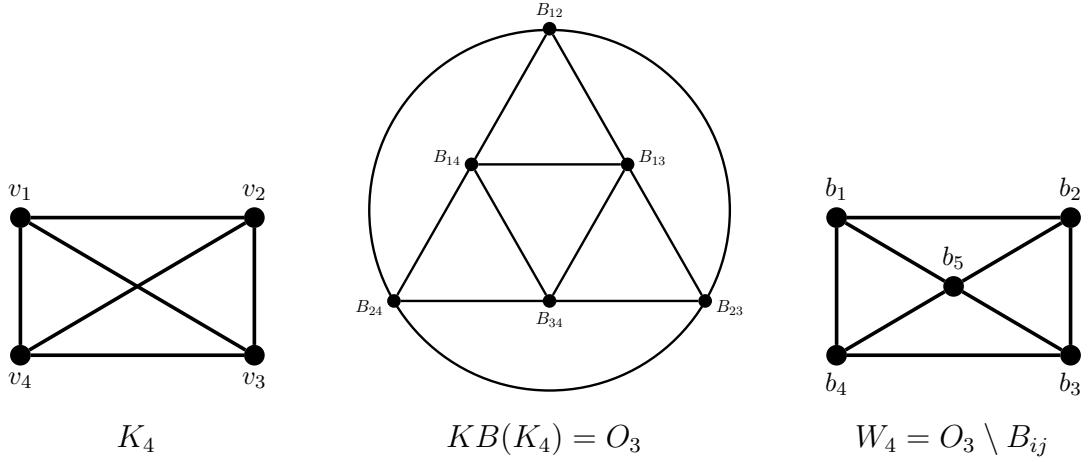


Figura 5.3:  $K_4$ ,  $O_3$ ,  $W_4$

Veamos con un ejemplo que la propiedad de pertenecer a la clase  $KB(Split)$  no es hereditaria.

Más aún, existe  $G \in KB(Split)$  tal que  $\forall B \in V(G), G \setminus \{B\} \notin KB(Split)$ . Por ejemplo,  $O_3 \in KB(Split)$  ya que  $KB(K_4) = O_3$  y  $W_4 = O_3 \setminus B_{ij} \notin KB(Split)$  (Figura 5.3).  $W_4 \notin KB(Split)$  debido a que por la Proposición 5.1 la cantidad  $c$  de bicliques varía entre  $C(n, 2)$  y  $2C(n, 2) + n$  y en este caso, al ser  $c = 5$  tendríamos  $n = 3$  y que las cinco bicliques se intersectarían dos a dos.

En los próximos lemas, estudiamos los grados de los vértices de grafos en esta clase.

**Lema 5.3.** *Si  $B$  es una biclique estrella tipo uno de  $H$ , entonces existen al menos  $2(n - 1)$  bicliques de  $H$  distintas de  $B$  que intersectan a  $B$ .*

*Demostración.* Sea  $B$  una biclique estrella tipo uno con centro en  $v$  y sea  $u, w$  los vértices de  $K$  tales que  $S(v) \cap S(u) \cap S(w) = \emptyset$ .

Sean  $S_1 = S(u) \cap S(v)$  y  $S_2 = S(v) \setminus S_1$ . Luego  $S_1 \neq \emptyset$  ya que sino  $B$  no sería una biclique estrella puesto que podría extenderse a  $u$  y  $S_2 \neq \emptyset$  ya que sino  $S(v) \subset S_1$  y dado que  $S(v) \cap S(w) \neq \emptyset$  (nuevamente para que no pueda extenderse la biclique estrella a  $w$ ) tendríamos  $S(v) \cap S(u) \cap S(w) \neq \emptyset$ . Sean  $C_1 = \{k \in K \text{ tal que } S(k) \cap S(v) \subset S_1\}$ ,  $C_2 = \{k \in K \text{ tal que } S(k) \cap S(v) \subset S_2\}$ ,  $C_3 = K \setminus C_1 \setminus C_2$ , i.e.  $C_3 = \{k \in K \text{ tal que } S(k) \cap S_1 \neq \emptyset \wedge S(k) \cap S_2 \neq \emptyset\}$  (recordemos que no puede existir ningún  $k \in K$  que no sea adyacente a ninguno  $s \in S(v)$  ya que sino la biclique podría extenderse a  $k$  y no ser biclique estrella) y sea  $c_j = |C_j|$  para  $1 \leq j \leq 3$ . Luego  $c_1, c_2, c_3 \geq 1$  ya que  $u \in C_1, w \in C_2, v \in C_3$ .

Luego, las bicliques intersectantes a  $B$  son:

- $2c_1c_2$  bicliques cuya arista base  $xy$  es tal que  $x \in C_1 \wedge y \in C_2$ , ya que para cada  $x \in C_1$  y cada  $y \in C_2$  existen dos bicliques con la arista base  $xy$ : una que contiene los satélites de  $S_1$  y la otra los satélites de  $S_2$ . Todas estas bicliques son *S-intersectantes* con  $B$ .
- al menos  $c_1c_3$  bicliques cuya arista base  $xy$  es tal que  $x \in C_1, y \in C_3$ , ya que estas bicliques se extenderían al menos a un satélite de  $S_2$ .
- al menos  $c_2c_3$  bicliques cuya arista base  $xy$  es tal que  $x \in C_2, y \in C_3$ , ya que estas bicliques se extenderían al menos a un satélite de  $S_1$ .

Luego la cantidad de bicliques de  $H$  que intersecan a  $B$  es al menos  $2c_1c_2 + c_1c_3 + c_2c_3$ .

Dado que:

- $c_1 + c_2 + c_3 = n$
- $c_1, c_2, c_3 \geq 1$
- $2c_1(c_2 - 1) - 2(c_2 - 1) \geq 0$  (lo que se deduce de (b) ya que  $c_1 \geq 1$  implica  $c_1(2(c_2 - 1)) \geq 2(c_2 - 1)$ )

Tenemos que la cantidad  $c$  de bicliques que intersecan a  $B$  satisface:

$$\begin{aligned} c &\geq 2c_1c_2 + c_1c_3 + c_2c_3 = 2c_1c_2 + (c_1 + c_2)c_3 \stackrel{(a)}{=} \\ &= 2c_1c_2 + (c_1 + c_2)(n - c_1 - c_2) \stackrel{(b)}{\geq} \\ &\geq 2c_1c_2 + 2(n - c_1 - c_2) = \\ &= 2c_1c_2 - 2c_1 - 2c_2 + 2n = \\ &= 2c_1(c_2 - 1) - 2(c_2 - 1) - 2 + 2n \stackrel{(c)}{\geq} \\ &\geq -2 + 2n = 2(n - 1) \end{aligned}$$

Luego las bicliques intersectantes a  $B$  son al menos  $2(n - 1)$  □

**Lemma 5.4.** Si  $B$  es una biclique estrella tipo dos de  $H$ , entonces existen al menos  $2(n - 1)$  bicliques de  $H$  distintas de  $B$  que intersectan a  $B$ .

*Demostración.* Las bicliques intersectantes a  $B$  son:

- $n - 1$  bicliques estrella: cada vértice  $u \in K \setminus \{v\}$ , sería el centro de una biclique estrella que intersecta a  $B$ . En efecto, la biclique estrella con centro en  $u \in K \setminus \{v\}$  no puede extenderse a  $w \in K \setminus \{u\}$  ya que  $\forall u, w \in K, S(u) \cap S(w) \neq \emptyset$ . Estas bicliques intersecan a  $B$  en  $S(u)$  que es no vacío.
- $n - 1$  bicliques que tienen dos vértices de  $K$ : existen  $n - 1$  bicliques  $K$ -intersectantes a  $B$  cuyos vértices de  $K$  son  $v$  y  $u \in K \setminus \{v\}$ .

Luego las bicliques intersectantes a  $B$  son  $2(n - 1)$  □

**Teorema 5.5.**  $\delta(G) \geq 2(n - 2)$

*Demostración.* Si  $B$  es una biclique de  $H$  que tiene dos vértices  $u, v$  de  $K$ , existirán al menos  $2(n - 2)$  bicliques  $K$ -intersectantes a  $B$  que contienen dos vértices de  $K$ :  $n - 2$  que contienen a  $u$  y un vértice  $x \in K \setminus \{u, v\}$  y  $n - 2$  que contienen a  $v$  y un vértice  $y \in K \setminus \{u, v\}$ .

Si  $B$  es una biclique estrella de  $H$  con centro en  $v$  y conjunto de satélites  $S(v)$ , luego las únicas posibilidad son:  $\exists u, w \in K$  tal que  $N(u) \cap N(w) \cap S(v) = \emptyset$  o  $\forall u, w \in K, N(u) \cap N(w) \cap S(v) \neq \emptyset$ , luego, por lemas 5.3 y 5.4 tendrá al menos  $2(n - 1)$  bicliques intersectantes.

Concluimos que el grado de todo vértice de  $G = KB(H)$  es mayor o igual a  $2(n - 2)$  □

Utilizando los resultados anteriores, vamos a estudiar la arista-conectividad y la conectividad de los grafos biclique de los grafos split.

**Lemma 5.6** ([CS69]). Si  $G'$  es un grafo  $m$ -arista conexo, luego su grafo líneas  $L(G')$  es  $(2m - 2)$ -arista conexo.

**Teorema 5.7.**  $\lambda(G) \geq 2(n - 2)$ .

*Demostración.* Sea  $G'$  un grafo desconexo que se obtiene de  $G$  quitando  $\lambda(G)$  aristas de  $G$  y sean  $C_1$  y  $C_2$  componentes conexas distintas de  $G'$ .

Sea  $G_1$  el subgrafo inducido en  $G$  por las bicliques que contienen dos vértices de  $K$  y  $G_2$  el inducido por las bicliques estrella. Luego:

- Si  $V(G_1) \cap V(C_1) \neq \emptyset$  y  $V(G_1) \cap V(C_2) \neq \emptyset$ , al remover las aristas debemos necesariamente haber desconectado  $G_1$ . Dado que  $\lambda(K_n) = n - 1$ , por Lema 5.6  $\lambda(L(K_n)) = 2(n-1)-2 = 2(n-2)$  y como las bicliques que contienen dos vértices de  $K$  poseen una arista del grafo completo  $K$ , tenemos  $\lambda(G_1) \geq \lambda(L(K_n)) = 2(n-2)$ . Por lo tanto  $\lambda(G) \geq \lambda(G_1) \geq 2(n-2)$ .
- Si no,  $V(G_1) \subset V(C_1)$  o  $V(G_1) \subset V(C_2)$ . Consideramos  $V(G_1) \subset V(C_1)$ . Luego, tenemos que si  $B_1 \in V(C_2)$  entonces  $B_1$  es una biclique estrella y consideramos dos casos:

**Caso 1.** Si  $B_1$  es una es de tipo uno, como se prueba en la demostración del lema 5.3 las bicliques de  $G_1$  intersectantes a  $B_1$  son  $2(n-1)$  y por tanto  $\lambda(G) \geq 2(n-1)$ .

**Caso 2.** Si  $B_1$  es de tipo dos. Si  $|V(C_2)| = 1$ , es decir  $C_2$  solamente contiene a la biclique  $B_1$ , como probamos en la demostración del lema 5.4 las bicliques de  $G_1$  intersectantes a  $B_1$  son  $n-1$  y las bicliques estrellas intersectantes a  $B_1$  (que en este caso serán todas vértices de  $C_1$ ) son  $n-1$ . Luego necesitamos quitar al menos  $2(n-1)$  aristas para desconectar  $G$ . Si  $|V(C_2)| > 1$ , es decir  $C_2$  contiene otra biclique  $B_2$  además de  $B_1$ , como probamos en la demostración del lema 5.4 las bicliques de  $G_1$  intersectantes a  $B_1$  son  $n-1$  y como probamos en los lemas 5.3 y 5.4 las bicliques de  $G_1$  intersectantes a  $B_2$  son al menos  $n-1$ . Luego necesariamente deben haberse quitado al menos  $2(n-1)$  aristas para desconectar  $G$ .

□

**Lemma 5.8** ([HRV04](Teorema 3, pg. 8)). *Sea  $G'$  un grafo tal que  $V(G') \geq 4$  y  $G'$  no es una estrella, luego  $\kappa(L(G')) = \lambda_2(G')$ .*

**Lemma 5.9.**  $\kappa(L(K_n)) = 2(n-2)$ , si  $n \geq 4$ .

*Demostración.* Si  $n \geq 4$ , sea  $S$  un corte-arista restringido de  $K_n$  de cardinal mínimo y  $V_1$  y  $V_2$  las particiones en las que queda dividido luego de la remoción de  $S$ . Consideremos  $k_1, k_2 \in V_1$  y  $k_3, k_4 \in V_2$ . Dado que  $V_1$  y  $V_2$  están desconectados deben haber sido removidas las  $|V_2|$  aristas  $k_1k_i$  y las  $|V_2|$  aristas  $k_2k_i$  para todo  $k_i \in V_2$ , las  $|V_1| - 2$  aristas  $k_3, k_j$  y  $|V_1| - 2$  aristas  $k_4, k_j$  para todo  $k_j \in V_1, j \neq 1, 2$  (no consideramos las aristas  $k_3k_1, k_3k_2, k_4k_1, k_4k_2$  porque ya fueron contadas). Considerando que  $|V_1| + |V_2| = n$ , tenemos que  $S \geq |V_2| + |V_2| + |V_1| - 2 + |V_1| - 2 = 2(n-2)$ . Además dado que definiendo  $V_1 = \{k_1, k_2\}$  y  $V_2 = V(G) \setminus V_1$ , tenemos que la remoción de las aristas entre ambos conjuntos genera un corte-arista de tamaño  $2(n-2)$ . Concluimos que, si  $n \geq 4$ ,  $\lambda_2(K_n) = 2(n-2)$  y por propiedad 5.8,  $\kappa(L(K_n)) = 2(n-2)$ . □

**Teorema 5.10.**  $\kappa(G) \geq 2(n-2) - 1$ .

*Demostración.* Si  $n = 1, 2$  el resultado es trivial.

Si  $n = 3$ ,  $G$  es un grafo completo ya que todas sus bicliques se intersectan (el resultado para bicliques que contienen dos vértices de  $K$  es obvio y si existiese una biclique estrella

debería intersectarse a toda biclique estrella y a toda biclique que contiene dos vértices de  $K$ ).

Si  $n \geq 4$  sea  $G_1$  el subgrafo inducido en  $G$  por las bicliques que contienen dos vértices de  $K$  y  $G_2$  el inducido por las bicliques estrellas.

Por Lema 5.9, la conectividad por vértices de  $G_1$  será de al menos  $2(n - 2)$ .

Dado que toda biclique estrella interseca a una biclique que tiene dos vértices en  $K$ , la única forma de obtener un grafo desconexo removiendo vértices de  $G$  es remover  $R \subset V(G_1)$  y  $S \subset V(G_2)$  de manera que para toda biclique estrella  $B \in V(G_2) \setminus S$ , las únicas bicliques con dos vértices de  $K$  que la interseque sean bicliques de  $R$ .

Sea  $B_1 \in V(G_2) \setminus S$  consideramos dos casos teniendo en cuenta su tipo de biclique estrella:

**Caso 1.** Si  $B_1$  es de tipo uno. Como se prueba en la demostración del Lema 5.3 las bicliques de  $G_1$  intersectantes a  $B_1$  son  $2(n - 1)$ . Luego  $|R| = 2(n - 1)$

**Caso 2.** Si  $B_1$  es de tipo dos. Si  $|V(G_2) \setminus S| = 1$ , es decir  $V(G_2) \setminus S$  solamente contiene a la biclique  $B_1$ , como probamos en la demostración del lema 5.4 las bicliques de  $G_1$  intersectantes a  $B_1$  son  $n - 1$  y las bicliques estrellas intersectantes a  $B_1$  (que en este caso serán todas bicliques de  $S$ ) son  $n - 1$ . Luego hemos quitado al menos  $|R| + |S| = 2(n - 1)$  vértices para desconectar  $G$ .

Si  $|V(G_2) \setminus S| > 1$ , es decir  $V(G_2) \setminus S$  contiene otra biclique  $B_2$  además de  $B_1$ , si  $B_2$  es de tipo uno las bicliques de  $G_1$  intersectantes a  $B_2$  son  $2(n - 1)$ , si no como probamos en la demostración del Lema 5.4 las bicliques de  $G_1$  intersectantes a  $B_1$  son al menos  $n - 1$  y las intersectantes a  $B_2$  son al menos  $n - 1$ .

Si  $u$  es el centro de  $B_1$ ,  $v$  el centro de  $B_2$  y  $B_{uv}$  y  $B_{vu}$  son s-bicliques dobles, las bicliques de  $G_1$  intersectantes a  $B_1$  y  $B_2$  son  $2(n - 1)$  (las  $n - 1$  bicliques de  $G_1$  de la forma  $B_{uw}$  y las  $(n-1)$  bicliques de la forma  $B_{vw}$ ).

Pero si  $B_{uv}$  es una s-biclique simple, los mismos argumentos del parrafo anterior nos aseguran la intersección con una biclique menos, por lo tanto necesitamos quitar al menos  $2(n - 1) - 1$  bicliques de  $G_1$ .

□

Para concluir este capítulo, vamos a probar un resultado para la clase de grafos biclique de split que responde positivamente a la Conjetura Groshaus-Montero (Conjetura 1.1) sobre la relación de grafos biclique y grafos Hamiltonianos.

**Teorema 5.11.**  *$G$  es hamiltoniano.*

*Demostración.* Por simplicidad consideramos  $K = \{1, 2, \dots, n\}$  y que si entre los vértices  $i$  y  $j$ , para  $1 \leq i < j \leq n$ , existe una única biclique la denotamos  $b_{ij}$  (sin importar si la biclique es una biclique arista o una s-biclique) y si existen dos bicliques llamamos a una biclique  $b_{ij}$  y a la otra  $b'_{ij}$  (sin importar la dirección de la s-biclique). Construimos

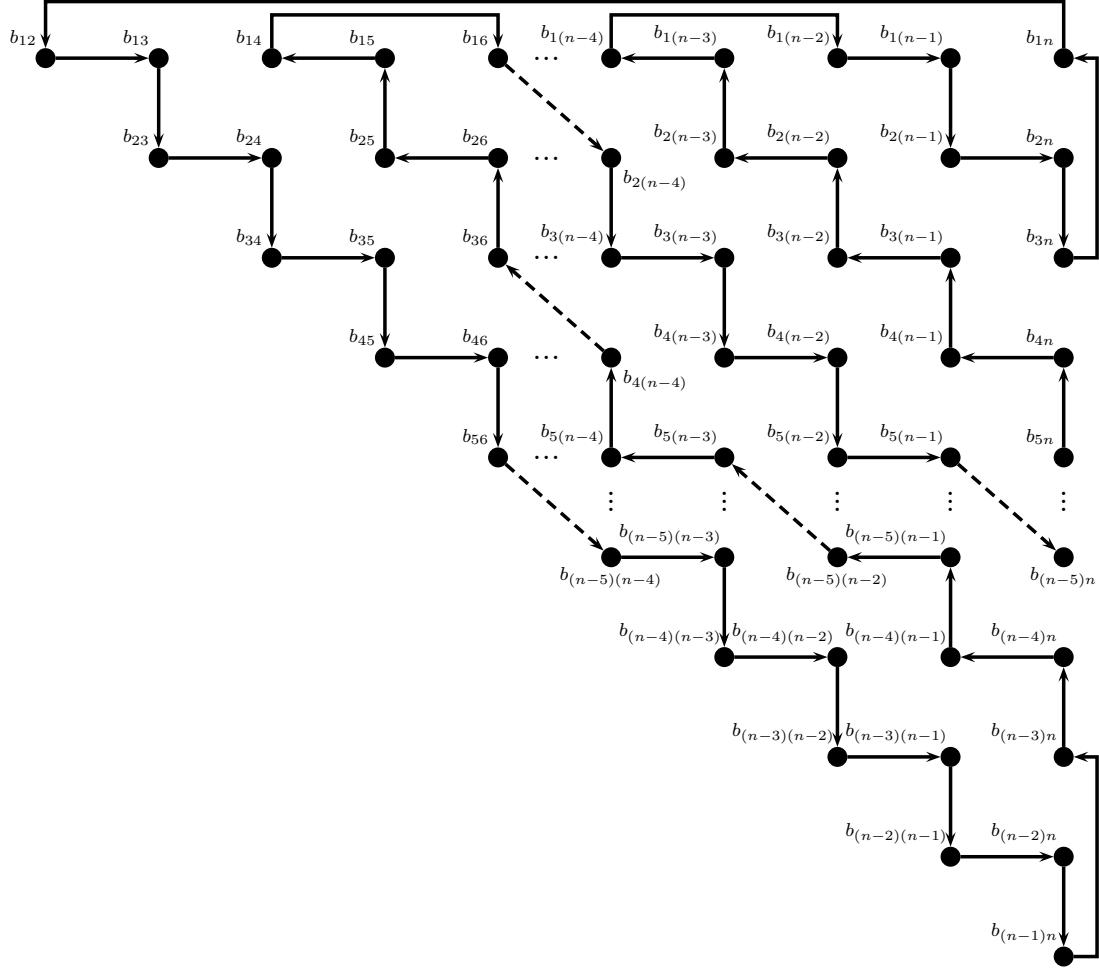


Figura 5.4: Ciclo hamiltoniano en  $G$  si  $n \equiv 0 \pmod{4}$

un ciclo hamiltoniano tomando primero una biclique por cada dos vértices de  $K$ . Luego anexamos al ciclo, si existen, el resto de las bicliques que contienen dos vértices de  $K$  y por último las bicliques estrella.

Si consideramos una única biclique por cada dos vértices de  $K$ , encontrar un ciclo hamiltoniano  $C$  para ese conjunto de vértices es equivalente a encontrarlo para el grafo líneas del grafo completo  $K_n$ . Construimos el ciclo  $C$  como se muestra en las figuras 5.4, 5.5, 5.6, 5.7.

Luego anexamos al ciclo, si existen, el resto de las bicliques que contienen dos vértices de  $K$ :

Sea  $s(b_{ij})$  el nodo siguiente en el ciclo  $C$  a  $b_{ij}$ . Para cada  $b'_{ij}$ , modificamos  $C$  de la siguiente manera:

- eliminamos la arista  $b_{ij}s(b_{ij})$

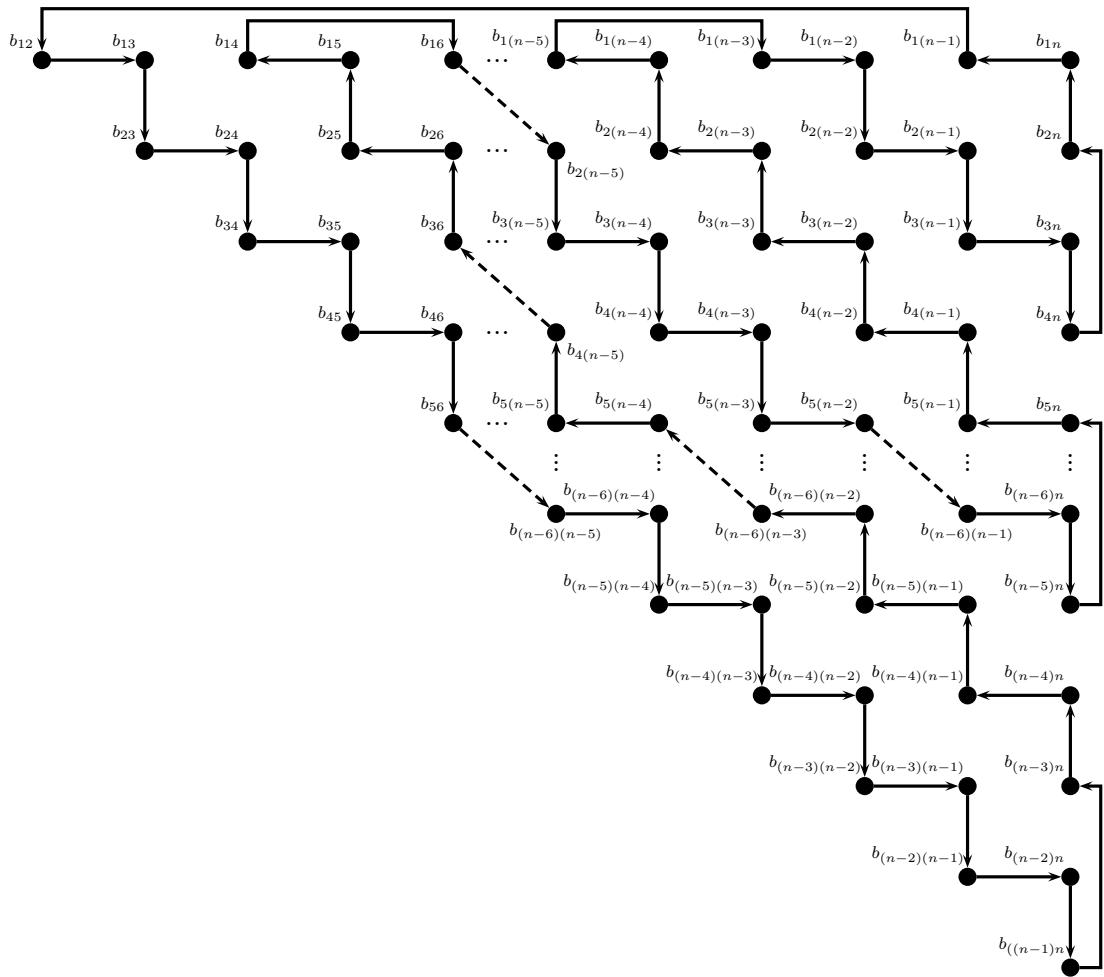


Figura 5.5: Ciclo hamiltoniano en  $G$  si  $n \equiv 1 \pmod{4}$

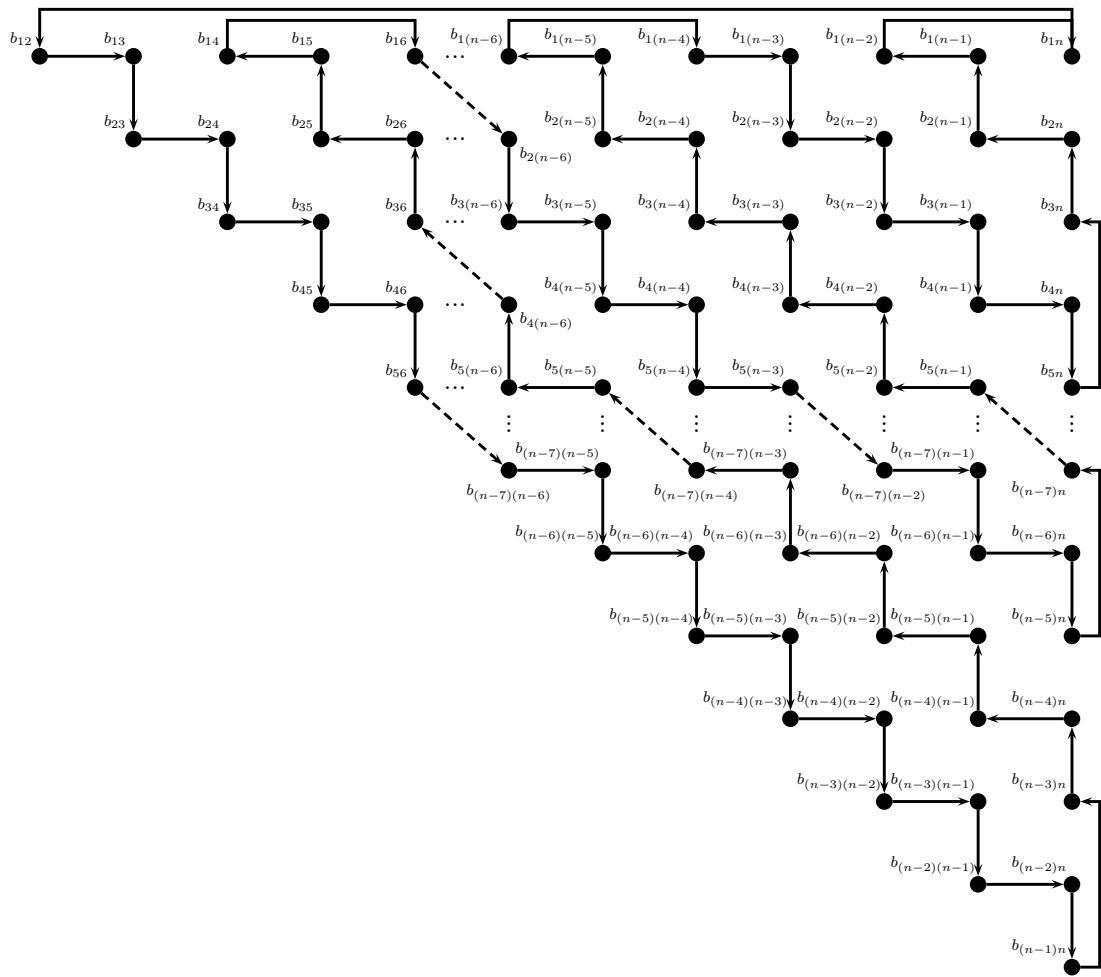


Figura 5.6: Ciclo hamiltoniano en  $G$  si  $n \equiv 2 \pmod{4}$

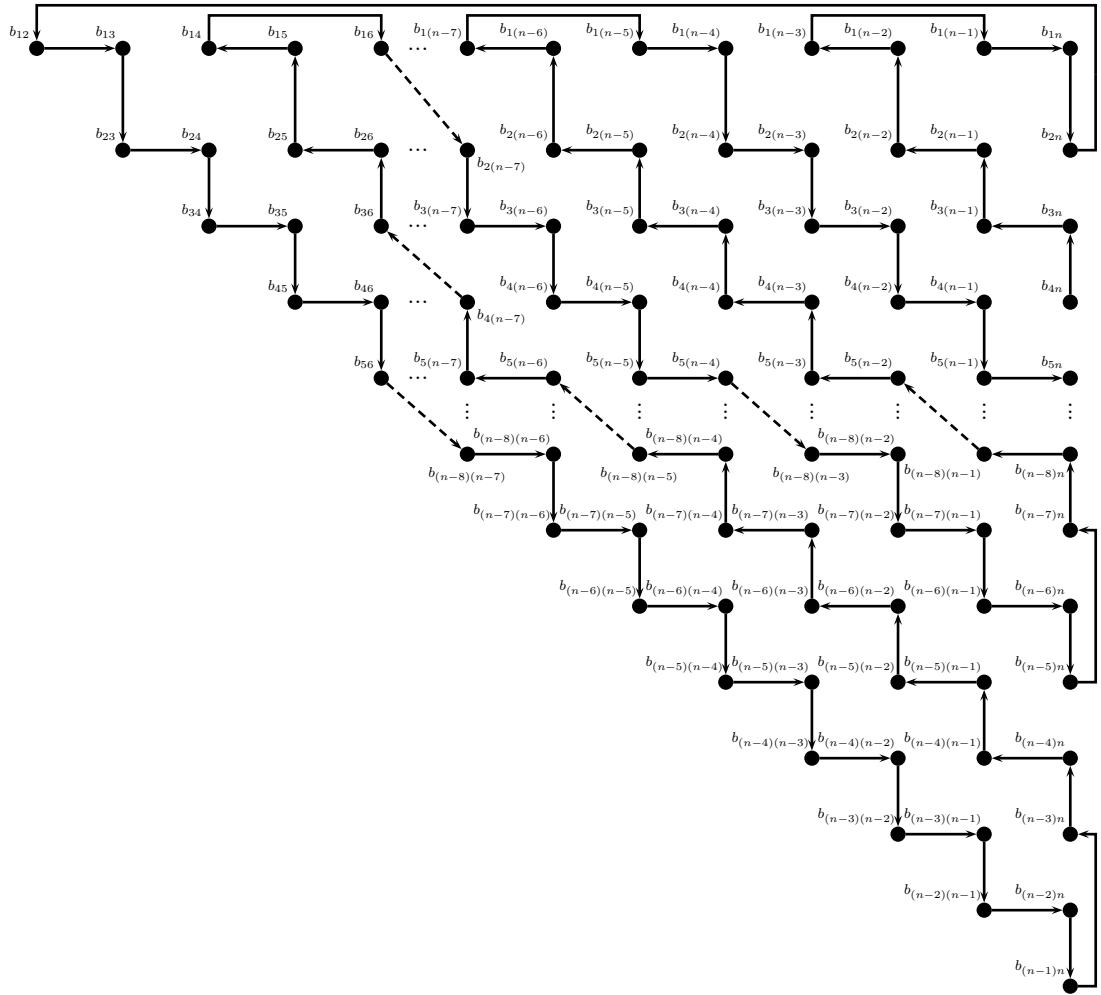


Figura 5.7: Ciclo hamiltoniano en  $G$  si  $n \equiv 3 \pmod{4}$

- agregamos las aristas  $b_{ij}b'_{ij}$  y  $b'_{ij}s(b_{ij})$ .

Así obtenemos un ciclo hamiltoniano  $C'$  que incluye todas las bicliques que contienen dos vértices de  $K$ .

Por último extendemos el ciclo hamiltoniano para incluir las bicliques estrella:

Denotamos  $b_i$  a la biclique estrella de centro  $i$ . Sea  $s'(b_{ij})$  el nodo siguiente en el ciclo  $C'$  a  $b_{ij}$ . Para cada biclique estrella  $b_i$  tal que  $1 \leq i \leq n - 2$ , modificamos  $C'$  de la siguiente manera:

- eliminamos la arista  $b_{i(i+1)}s'(b_{i(i+1)})$
- agregamos las aristas  $b_{i(i+1)}b_i$  y  $b_is'(b_{i(i+1)})$ .

Observemos que el vértice  $i$  pertenece a  $s'(b_{ij})$  por construcción y entonces existen las aristas  $b_{i(i+1)}b_i$  y  $b_is'(b_{i(i+1)})$ .

Si  $b_{n-1}$  es una biclique estrella eliminamos la arista  $b_{(n-3)(n-1)}s'(b_{(n-3)(n-1)})$  y agregamos las aristas  $b_{(n-3)(n-1)}b_{n-1}$  y  $b_{n-1}s'(b_{(n-3)(n-1)})$ . Si  $b_n$  es una biclique estrella eliminamos la arista  $b_{(n-2)n}s'(b_{(n-2)n})$  y agregamos las aristas  $b_{(n-2)n}b_n$  y  $b_ns'(b_{(n-2)n})$ .

El nuevo ciclo  $C''$  es hamiltoniano. Concluimos que  $KB(H)$  es Hamiltoniano.

□

# 6 Reconocimiento $KB(Split)$

En este capítulo estudiamos el problema de reconocimiento de grafos biclique de grafos split. En este trabajo resolvemos este problema para una subclase de grafos split, que denominamos *split separables anidados*. Esta clase contiene, por ejemplo, a los grafos threshold y tiene intersección con la clase de grafos split de comparabilidad.

Un grafo es *split separable* (SSG) cuando es un grafo split y para todo vértice  $v \in K$ , existe un vértice  $u \in K$  tal que  $S(v) \cap S(u) = \emptyset$ .

Un grafo *split separable anidado* (NSSG) es un grafo split separable tal que para todo par de vértices  $u, v \in K$ , si  $S(u) \cap S(v) \neq \emptyset$ , entonces  $S(u) \subseteq S(v)$  o  $S(v) \subseteq S(u)$ . La definición de esta clase aparece cuando construimos el candidato a preimagen del grafo que queremos determinar si es biclique de un grafo NSSG. Los grafos en esta clase proporcionan condiciones suficientes para resolver las condiciones de las vecindades de los vértices del completo  $K$  de una posible preimagen, proporcionado por el algoritmo.

Un grafo *threshold* es un grafo split tal que para todo par de vértices  $u, v \in K$ ,  $S(u) \subseteq S(v)$  o  $S(v) \subseteq S(u)$ .

En este capítulo presentamos un algoritmo de reconocimiento para determinar si un grafo es grafo biclique de un grafo split separable anidado, es decir, si dado un grafo  $G$ , existe un grafo NSSG  $H$  tal que  $G = KB(H)$ . El algoritmo, además de dar la respuesta, en caso afirmativo, devuelve el grafo  $H$ .

También caracterizamos la clase de los grafos split anidados.

Para decidir si un grafo  $G$  es un grafo biclique de un NSSG construimos un candidato  $H$  tal que si  $G$  es  $KB(F)$  para algún NSSG  $F$  entonces  $G$  es isomorfo a  $KB(H)$  ( $G \cong KB(H)$ ). Para esto, primero estudiamos las propiedades de las bicliques en de un NSSG  $H$ . Además, estudiamos el grafo  $KB(H)$ . En particular, estudiamos propiedades específicas de los vértices y las cliques de  $G$  tal que caracterizan los vértices que representan cada tipo de biclique de  $H$  (arista-bicliques y s-bicliques).

Para resolver nuestro problema en cuestión, dado un grafo  $G$  utilizamos estas propiedades para descubrir qué vértices son candidatos a ser los vértices correspondientes a cada tipo de biclique de una posible preimagen de  $G$ , donde una preimagen es un grafo NSSG  $H$  tal que  $KB(H) \cong G$ .

La idea del algoritmo es la siguiente: primero, descubrir cuáles son los únicos vértices de  $G$  candidatos a representar las aristas-bicliques de una preimagen (Sección 6.2.3). Este proceso nos da la parte completa  $K$  de una posible preimagen (Observación 6.9). Además,

buscamos en  $G$  posibles candidatos a una familia particular de cliques que existe en todo NSSG, llamadas cliques-estrellas (star cliques). Estas cliques particulares son usadas para encontrar los vértices de  $G$  candidatos a representar las s-bicliques de un candidato a preimagen (Observación 6.10). Combinando todo lo anterior, construimos conjuntos de satélites de un posible candidato a preimagen  $H$  de tal manera que, si  $G$  es un grafo biclique de algún NSSG, entonces  $G \cong KB(H)$ .

En la Sección 6.1 presentamos algunas definiciones. La Sección 6.2 contiene propiedades sobre bicliques de un grafo NSSG y su grafo biclique. En la Sección 6.3 presentamos un algoritmo polinomial para reconocer grafos biclique de NSSGs. También explicamos cada paso del algoritmo y probamos su correctitud.

Nota: El resto de este capítulo es parte de un paper (en preparación) y por este motivo está en inglés. Para facilitar la lectura y evitar ambigüedades de traducción decidimos repetir en inglés algunas definiciones específicas utilizadas en esta sección y además agregamos nuevas definiciones.

## 6.1. Preliminaries

A *split graph*  $H$  is a graph such that the vertex set can be partitioned into a clique  $K$  and an independent set  $S$ . The clique part  $K$  is called the *complete part* and the independent set  $S$  is called the *satellite part* and it is denoted as  $H = (K \cup S, E)$ . The vertices of the satellite part are called *satellites* of the graph.

Given a vertex  $v \in K$ , let  $S(v)$  be the set of satellites adjacent to  $v$ , that is,  $S(v) = N(v) \cap S$ .

A *separable split graph* (SSG)  $H$  is a split graph where for every vertex  $v \in K$  there is a vertex  $u \in K$  such that  $S(v) \cap S(u) = \emptyset$ .

A *nested separable split graph* (NSSG)  $H$  is a SSG such that for every pair of vertices  $u, v \in K$ , if  $S(u) \cap S(v) \neq \emptyset$ , then  $S(u) \subseteq S(v)$  or  $S(v) \subseteq S(u)$ .

Two graphs  $G$  and  $G'$  are *isomorphic* if there is a one-to-one mapping  $f$  between  $V(G)$  and  $V(G')$  such that  $vw \in E(G)$  if and only if  $f(v)f(w) \in E(G')$ .

## 6.2. Bicliques of a split graph

In this section we study the bicliques of split graphs. Let  $H = (K \cup S, E)$  be a split graph with associated poset  $(\mathcal{X}, \preceq)$ . Each biclique of a split graph  $H$  has one or two vertices of  $K$ . A biclique with only one vertex  $v$  of  $K$  is a *star biclique* formed by  $v$  as its center and its set of satellites,  $S(v)$ . Such biclique exists if and only if for every part  $X_i \in \mathcal{X}$ , with  $X_i \neq X(v)$ ,  $S(X_i) \cap S(X(v)) \neq \emptyset$ . Observe that SSGs are exactly the split graphs such that every biclique has two vertices of  $K$ .

Given a biclique  $B$  of  $H$ , call the edge of  $K$  the *base edge* of  $B$ . Some bicliques have also vertices of  $S$ . Each edge of  $K$  is the base edge of one or two bicliques of  $H$ .

If two bicliques have a vertex of  $K$  in common it is said that these bicliques are  *$K$ -intersecting*. If two bicliques have only satellites in common it is said that these bicliques are  *$S$ -intersecting*.

The bicliques of  $H$  that are not star bicliques can be classified in three different types. Consider the biclique  $B$  of  $H$  and its base edge  $uv$ .

If  $X(u) = X(v)$ , then  $B = uv$  and it is called an *edge-biclique*, and is denoted by  $B_{uv}$  (or  $B_{vu}$ ) of  $X(u)$ . Say that  $B$  is an edge-biclique of  $X_i$  if  $u, v \in X_i$ .

If  $X(v) \preceq X(u)$ , then  $B$  contains also the satellites of  $S(X(u)) \setminus S(X(v))$  and it is called a *single s-biclique*, denoted by  $B_{uv}$  from  $X(v)$  to  $X(u)$  (not  $B_{vu}$ ).

If  $X(u) \leftrightarrow X(v)$ , there are two bicliques with the same base edge. One biclique contains satellites of  $S(X(u)) \setminus S(X(v))$  and it is denoted by  $B_{uv}$ . In this case we say that the biclique  $B_{uv}$  is a biclique from  $X(v)$  to  $X(u)$ . The second biclique contains the satellites of  $S(X(v)) \setminus S(X(u))$  and it is denoted by  $B_{vu}$  from  $X(u)$  to  $X(v)$ . They are called *double s-bicliques*.

If the direction of the s-biclique from  $X_i$  to  $X_j$  (single or double) is not known (or does not matter), we say that it is an s-biclique of  $X_i$  and  $X_j$ .

A *star clique*  $C^v$  of  $G \cong KB(H)$  is a complete subgraph of  $G$  (a clique when  $|K| \geq 4$ ) with all the bicliques of  $H$  that contain vertex  $v \in K$ . That is, the base edges of the bicliques in  $C^v$  form a star subgraph in  $K$ . An edge of  $G$  is called an *old edge* if it is an edge of some star clique. The other edges are called *new edges*.

### 6.2.1. Bicliques of an NSSG and its biclique graph

We present some observations and lemmas about the bicliques and the parts of an NSSG (or a SSG) in order to understand their properties. These observations are used for the construction of the algorithm presented in Section 6.3.

We start by giving an idea of why we restrict to this class.

As mentioned before, SSG are exactly those split graphs such that every biclique has a base edge. If  $G$  is a biclique graph of a SSG  $F$ , as we will see later, the set of minimum degree vertices of  $G$  is associated with bicliques with base edge in the same part and this follows only because  $F$  does not contain star bicliques. If such star bicliques exist in a preimage of  $G$ , the degree of the corresponding vertex in  $G$  depends on the number of parts of the preimage and the size of each one. Since when we are looking for the preimage we do not have that information, the set of minimum degree vertices of  $G$  would not be useful. On the other hand, since we need to find the correspondence between vertices of  $G$  and the bicliques of a preimage, the fact that every biclique is associated to an edge of the complete part of the preimage gives a tool to construct the preimage.

The reason why we restricted to NSSGs is that, when constructing the preimage, we need to find the order of the parts and the direction of the bicliques, that is, knowing the base edge we need to know from which part to which part the biclique is. If we do not restrict to NSSG, with only the information about the intersection of bicliques, the set of satellites of each part (and the direction of the bicliques) can not be guessed (at least not following the procedure we propose).

In the following, let  $H$  be a split graph with associated poset  $(\mathcal{X}, \preceq)$ . It will be remarked at the beginning of each observation if it holds for split graphs in general, SSGs or just for NSSGs.

**Observation 6.1 (SSG).** Let  $B_{wx}$  and  $B_{yz}$  be two bicliques of  $H$ .  $B_{wx}$  and  $B_{yz}$  intersect iff  $\{w, x\} \cap \{y, z\} \neq \emptyset$  or  $(S(X(w)) \setminus S(X(x))) \cap (S(X(y)) \setminus S(X(z))) \neq \emptyset$ .

The next observations show some differences between bicliques  $B_{uv}$  and  $B_{vu}$  in order to distinguish them in  $G$ .

**Observation 6.2 (NSSG).** Let  $B_1$  and  $B_2$  be two double  $s$ -bicliques with the same base edge  $uv$ . Suppose  $X_k$  is a part different from  $X(u)$  and  $X(v)$  such that  $X_k \leftrightarrow X(v)$ . Then, if there is a biclique  $B$  of parts  $X_k$  and  $X(u)$  such that  $B$  is adjacent to  $B_1$  by new edge and not adjacent to  $B_2$ , then  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$ .

**Observation 6.3 (NSSG).** Let  $X_i$ ,  $X_j$  and  $X_k$  be three parts of  $\mathcal{X}$  such that  $X_i \preceq X_j$ ,  $X_i \leftrightarrow X_k$  and  $X_j \leftrightarrow X_k$ . Let  $B$  be a biclique from  $X_i$  to  $X_j$ . Then  $B$  intersects every biclique from  $X_k$  to  $X_j$  and  $B$  does not  $S$ -intersect any of the bicliques from  $X_j$  to  $X_k$ , neither bicliques from  $X_k$  to  $X_i$  nor from  $X_i$  to  $X_k$ . See Figure 6.1

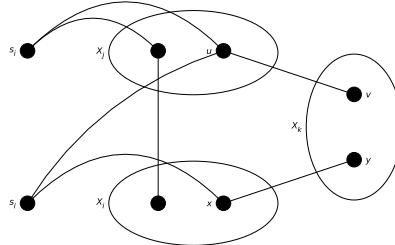


Figura 6.1: Bicliques  $B_{uv}$  and  $B_{xy}$  with satellites  $s_j \in S(X_j) \setminus S(X_i) \setminus S(X_k)$  and  $s_i \in S(X_i) \setminus S(X_k)$ , and a biclique from  $X_i$  to  $X_j$ .

Let  $T$  be the set of parts of  $\mathcal{X}$  that are comparable with every other part and let  $P$  be the set of parts that are not in  $T$ , that is, the parts that are incomparable with some other part. Note that if  $H$  contains a zero part  $X_z$ , then  $X_z \in T$ .

**Observation 6.4 (NSSG).** Let  $T$  and  $P$  be defined as above. Let the parts  $X_i \in T$ ,  $X_k \in P$ . Then  $X_i \preceq X_k$  if and only if  $X_i$  is the zero part. In that case,  $X_i \preceq X_j$  for all  $X_j$ .

**Observation 6.5 (split).** If  $P = \emptyset$  then  $H$  is a threshold graph and  $\preceq$  is a total order. In that case, the biclique graph of  $H$  is the same as the biclique graph of  $H'$  where  $H'$  is the threshold graph with the same parts of  $H$ , with the reverse order, and then  $G$  is also the biclique graph of  $H'$ .

**Observation 6.6 (SSG).**  $T \neq \emptyset$  if and only if  $H$  contains a zero part  $X_z$ .

**Lemma 6.7 (SSG).** If  $H$  contains a pair of parts  $X_i \leftrightarrow X_j$ , and  $T = \emptyset$ , then there always exists a part  $X_k$  such that  $X_k \leftrightarrow X_i$  and  $X_k \leftrightarrow X_j$ .

*Proof.* Suppose  $T = \emptyset$  and there are two parts  $X_i$  and  $X_j$  such that  $X_i \preceq X_j$ . By Observation 6.6, there is no zero part. As  $H$  is a SSG there is a part  $X_k$  such that  $S(X_k) \cap S(X_j) = \emptyset$ , and then  $X_k \leftrightarrow X_j$ . As  $S(X_i) \subseteq S(X_j)$ ,  $S(X_k) \cap S(X_i) = \emptyset$  and it follows that  $X_k \leftrightarrow X_i$ .  $\square$

### 6.2.2. Biclique graph of an NSSG

Next we go through properties of the biclique graph of an NSSG. They are used during the construction of the algorithm of the next section. Some of the following observations are meant to characterize star cliques in a biclique graph in order to find them in a given graph  $G$ . Also, they point out properties of  $K$ -intersecting bicliques and  $S$ -intersecting bicliques.

Let  $G \cong KB(H)$ , where  $H = (K \cup S, E)$  is a split graph. Note that the vertices of  $G$  are the bicliques of  $H$ .

Throughout this text, when we talk about bicliques we refer to bicliques of  $H$  and when we say cliques, we refer to cliques of  $G$ .

Two intersecting bicliques in  $H$  are adjacent vertices in  $G$ , then we use the expressions “bicliques intersect”, “bicliques are adjacent” and “bicliques are neighbours” all with the same meaning.

**Observation 6.8 (split).** An edge-biclique is incident only to old edges of  $G$ . The old edges are incident in  $G$  to bicliques that  $K$ -intersect in  $H$ . The new edges are incident to bicliques that  $S$ -intersect in  $H$ .

By Observation 6.1, if two edge-bicliques are adjacent in  $G$  then they are of the same part. Let  $\mathcal{E}$  be the set of edge-bicliques of  $H$ .

**Observation 6.9 (split).** Each connected component  $CC_i$  of  $G[\mathcal{E}]$  corresponds to a part  $X_i$  of  $\mathcal{X}$  in  $H$ . The vertices (edge-bicliques of  $H$ ) of the connected component  $CC_i$  of  $G[\mathcal{E}]$  are the edges of the graph  $H[X_i]$  (induced subgraph of  $H$  by  $X_i$ ) which is a complete graph and  $|CC_i| = \binom{|X_i|}{2}$ .

**Observation 6.10 (SSG).** Let  $B_{uv}$  and  $B_{vw}$  be two edge-bicliques of  $H$  adjacent in  $G$ . The intersection of the closed neighbourhoods in  $G$  of  $B_{uv}$  and  $B_{vw}$  is formed by the star clique  $C^v$  and the edge-biclique  $B_{uw}$ . That is,  $N_G[B_{uv}] \cap N_G[B_{vw}] = C^v \cup \{B_{uw}\}$ . Observe that the biclique  $B_{uw}$  is adjacent only to  $B_{uv}$  and  $B_{vw}$  when restricted to  $N_G[B_{uv}] \cap N_G[B_{vw}]$ . See Figure 6.2, where the base edges of the bicliques of  $H$  are showed.

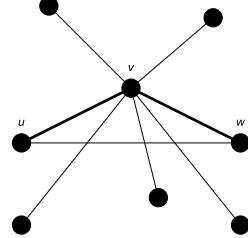


Figura 6.2: Two incident edge-bicliques in  $H$  (thick edges) and the common incident (base edges of) bicliques.

**Observation 6.11 (SSG).** Every vertex of  $G$  belongs to exactly 2 star cliques. The intersection of two star cliques  $C^u$  and  $C^v$  are 1 or 2 vertices, depending if the vertices  $u$  and  $v$  are at comparable parts or not.

The following observation is used to identify single and double bicliques, which gives the parts of the candidate preimage of  $G$ .

**Observation 6.12 (SSG).** Let  $B_1$  and  $B_2$  be two edge-bicliques of different parts  $X_i$  and  $X_j$ . If  $X_i \leftrightarrow X_j$ , then  $B_1$  and  $B_2$  have exactly 4 (four) neighbours bicliques in common. These neighbours are single s-bicliques. If  $X_i \leftrightarrow X_j$ , then they have exactly 8 (eight) neighbours bicliques in common. These neighbours are double s-bicliques.

**Observation 6.13 (split).** Each s-biclique (single or double) of  $X_i$  and  $X_j$  is adjacent only to edge-bicliques of parts  $X_i$  and  $X_j$ .

The following observation is used to find the candidate order of the parts of a candidate preimage of  $G$ .

**Observation 6.14 (split).** Let  $X_i, X_j, X_k$  be three comparable different parts. For every pair of bicliques  $B_1$ , of parts  $X_i$  and  $X_j$ , and  $B_2$ , of parts  $X_i$  and  $X_k$ , such that  $B_1$  and  $B_2$  are not  $K$ -intersecting,  $B_1$  is not adjacent by new edge to  $B_2$  if and only if  $X_i$  is “in the middle”, that is,  $X_j \preceq X_i \preceq X_k$  or  $X_k \preceq X_i \preceq X_j$ . See Figure 6.3.

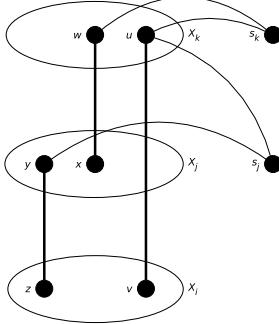


Figura 6.3: Bicliques  $B_{uv}$ ,  $B_{wx}$  and  $B_{yz}$  with satellites  $s_j \in S(X_j) \setminus S(X_i)$  and  $s_k \in S(X_k) \setminus S(X_j)$ .

### 6.2.3. Degree of the vertices of the biclique graph of a SSG

In this section we study the degrees of vertices of  $G \cong KB(H)$ . The goal is to characterize vertices that correspond to edge bicliques, based on their degree.

Let  $n = |K|$ ,  $N = |V(G)|$  and  $n_i = |X_i|$ . Given some part  $X_i$ , let  $N_{\leftrightarrow}^i$  be a set of indexes  $j$  such that  $X_i \leftrightarrow X_j$ . That is,  $N_{\leftrightarrow}^i = \{j \mid X_i \leftrightarrow X_j\}$ .

Let  $B_{uv}$  be a vertex of  $G$  and  $\alpha_{uv}$  be the number of bicliques intersecting  $B_{uv}$  only in satellites. We study the degree of each biclique.

**Case  $B_{uv}$  is an edge-biclique of  $X_i$**  : The number of edges in  $K$  incident to  $uv$  is  $2(n - 2)$ . Also, we need to consider the number of edges that are base edges of double s-bicliques and incident to  $uv$ , as there are two bicliques for each one of these edges. This number is  $2 \sum_{k \in N_{\leftrightarrow}^i} n_k$ . Then  $d_G(B_{uv}) = 2(n - 2) + 2 \sum_{k \in N_{\leftrightarrow}^i} n_k$ .

**Case  $B_{uv}$  is a single s-biclique with  $u \in X_i$  and  $v \in X_j$**  : The number of edges that are base edges of double s-bicliques and incident to  $uv$  is  $\sum_{k \in N_{\leftrightarrow}^i} n_k + \sum_{k \in N_{\leftrightarrow}^j} n_k$  as  $u$  and  $v$  are at different parts. Then,  $d_G(B_{uv}) = 2(n - 2) + \sum_{k \in N_{\leftrightarrow}^i} n_k + \sum_{k \in N_{\leftrightarrow}^j} n_k + \alpha_{uv}$ .

**Case  $B_{uv}$  is a double s-biclique with  $u \in X_i$  and  $v \in X_j$**  : This case is similar to the case above, except for the fact that we need to add 1 (counting the other double s-biclique  $B_{vu}$ ) and that  $X_i \leftrightarrow X_j$ , so we have to subtract 1 (because of  $v$ ) when summing up  $\sum_{k \in N_{\leftrightarrow}^i} n_k$  and 1 (because of  $u$ ) when summing up  $\sum_{k \in N_{\leftrightarrow}^j} n_k$ . Then,  $d_G(B_{uv}) = 2(n - 2) + \sum_{k \in N_{\leftrightarrow}^i} n_k + \sum_{k \in N_{\leftrightarrow}^j} n_k + \alpha_{uv} - 1$ .

We restrict our study to NSSGs such that each part contains at least 3 vertices. That restriction is used in Theorem 6.15 which is used to find the edge-bicliques.

We use the result about degrees in  $KB(H)$  to dismantle the graph by deleting edge bicliques of the candidate preimage of  $G$  and find its parts. First we need the following definition.

Given any split graph  $H = (K \cup S, E)$ , define the *2-biclique graph of  $H$*  denoted by  $KB_2(H)$ , as the intersection graph of the bicliques of  $H$  with 2 vertices of  $K$ . Note that if  $H$  is a SSG, then  $KB_2(H) \cong KB(H)$ .

**Theorem 6.15.** *Let  $G \cong KB_2(H)$ , for some split graph  $H$ , such that every part contains at least 3 vertices, and let  $M$  be the set of vertices of  $G$  of minimum degree. Then  $G \setminus (M \cup N_G(M))$  is the 2-biclique graph of a split graph. Moreover, let  $\mathcal{X}'$  be the set of parts of  $H$  with edge-bicliques in  $M$  and let  $U = \bigcup_{X_i \in \mathcal{X}'} X_i$ . Then,  $G \setminus (M \cup N_G(M)) = KB_2(H \setminus U)$ .*

*Proof.* The vertices of  $M$  correspond to edge-bicliques of one or more parts of  $H$ , since each part contains at least 3 vertices and then  $\alpha_{uv} \geq 4$ . Let  $H' = H \setminus U$ . Note that  $H'$  is a split graph. Observe that the bicliques of  $H'$  with two vertices of  $K' = K \setminus U$ , are also bicliques of  $H$ . So  $KB_2(H') \subseteq KB_2(H)$ .

The bicliques of  $H$  that are not bicliques of  $H'$  are bicliques with one or two vertices of  $K$  and at least one of them is in  $U$ . Let  $B$  be a biclique of  $H$  with an edge at  $K$ . If  $B \cap U = \emptyset$  then  $B$  is a biclique of  $H'$ . If the vertices of  $B \cap K$  are all in  $U$  and they are of the same part, then  $B \in M$ . If the vertices of  $B \cap K$  are of different parts and at least one of them is in  $U$ , them  $B \in N_G(M)$ .

So  $G \setminus (M \cup N_G(M)) = KB_2(H')$ . □

### 6.3. Recognition of a biclique graph of an NSSG

In this section we study the problem of recognizing biclique graphs of NSSGs. We restrict the problem to NSSGs such that each part contains at least 3 vertices (Bnss). The recognition problem consists in deciding, given a graph  $G$ , whether there exists an NSSG  $H$  with at least 3 vertices in each part such that  $G \cong KB(H)$ . In this case, we call  $H$  a preimage graph of  $G$ . From now on, when we refer to an NSSG we consider an NSSG with at least 3 vertices in each part.

The algorithm we present to solve the Bnss problem consists in constructing a candidate preimage NSSG  $H$  and check whether  $G \cong KB(H)$ . Therefore, if the algorithm answers YES, it is because it actually finds a preimage, that is, it does not give a false positive answer. On the other hand, we need to prove that if  $G \cong KB(F)$  for some NSSG  $F$ , the algorithm constructs an NSSG  $H$  such that  $G \cong KB(H)$ .

The algorithm is composed of 6 steps which will be explained in detail in the next subsection.

First the algorithm looks for the partition  $\mathcal{X}$  of the candidate  $H$ . For that, it finds the vertices of  $G$  that are candidates for edge bicliques and the connected components of the subgraph of  $G$  induced by these vertices.

In the next steps the algorithm finds the vertices of  $G$  candidate to being s-bicliques, double s-bicliques and constructs the candidate preimage  $H$ .

Each star clique is associated with a vertex of  $K$ , so each vertex of  $G$  is associated with two vertices of  $K$  (the star cliques of Observation 6.11). To define a bijection between vertices of  $G$  and bicliques of  $H$  (for the isomorphism) these two vertices of  $K$  must be ordered in the biclique ( $B_{uv}$  or  $B_{vu}$ ). So, given a s-biclique (vertex of  $G$ ) with base-edge  $uv$  it is necessary to know if that s-biclique is  $B_{uv}$  or  $B_{vu}$ . Assuming that this information is known, the set  $K$ , the family of sets of satellites and the graph  $H$  are constructed.

Outline of the Algorithm:

Given a graph  $G = (V, E)$ , the steps of the algorithm are:

1. Find the candidates edge-bicliques  $\mathcal{E}$ ;
2. Find the connected components of  $G[\mathcal{E}]$ ,  $CC_1, \dots, CC_\ell$  and generate the candidate partition  $\mathcal{X}$  and discover the size  $n_i$  of each part.
3. Find the candidates star cliques  $\mathcal{C}$  intersection of the neighbourhood of adjacent edge-bicliques) and mark the old edges and the new edges
4. Mark the candidate single s-bicliques and the double s-bicliques;
5. Find the associated poset  $(X, \preceq)$  and the “directions” of the double s-bicliques;
6. Generate the sets  $S(X_1), \dots, S(X_\ell)$ , construct a candidate NSSG  $H$  and check if  $G \cong KB(H)$ .

### 6.3.1. Detailing each step of the algorithm

In this section, we describe each step of the algorithm in detail.

#### Step (1) - Find $\mathcal{E}$ .

Based on the idea of Theorem 6.15, we use the following polynomial algorithm. Consider the set  $M$  of vertices of minimum degree and set them as edge bicliques. Next consider the graph  $G'$ , where  $G' = G \setminus (M \cup N_G(M))$  and repeat until  $G' = \emptyset$ . Let  $\mathcal{E}$  be the set of edge bicliques found with this algorithm.

Observe that the degrees of the edge bicliques of two parts  $X_i, X_j$  can be the same, and then two or more parts are found together at the same set  $M$ , so the candidate parts of  $K$  should be separated later.

### **Step (2) - Find $CC_1, \dots, CC_\ell$ and generate $\mathcal{X}$ .**

In this step the algorithm computes the connected components,  $CC_1, \dots, CC_\ell$ , of  $G[\mathcal{E}]$  in order to find the partition  $\mathcal{X}$  of the candidate preimage  $H$ .

Recall that each connected component of the graph  $G[\mathcal{E}]$  corresponds to the edge-bicliques of each part of the partition  $\mathcal{X} = \{X_1, \dots, X_\ell\}$  of  $H$  (Observation 6.9). So,  $|CC_i| = \binom{|X_i|}{2}$  and there is a bijection associating vertices of  $\mathcal{E}$  with edges of  $X_i$ .

For each connected component  $CC_i$  found by the algorithm, check if there exists a positive integer number  $n_i \geq 3$  such that  $|CC_i| = \binom{n_i}{2}$ . If some connected component  $CC_i$  is such that there is no integer  $n_i$ , answer NO, that is,  $G$  is not the biclique graph of a SSG. Finally, for each  $CC_i$  associate a part  $X_i$  of the complete subgraph  $K$  in the candidate graph  $H$ . Then  $K = \bigcup_{X_i \in \mathcal{X}} X_i$ .

### **Step (3) - Finding star cliques, old and new edges.**

To find<sup>1</sup> the star cliques we use the idea of Observation 6.10.

For each connected component  $CC_i$ , for each pair of intersecting edge-bicliques  $B_1$  and  $B_2$  of  $X_i$  (adjacent vertices of  $CC_i$ ), compute their closed neighbourhood intersection. Check if there is a unique vertex in this intersection that is adjacent only to  $B_1$  and  $B_2$  and check if the rest of the intersection is a clique. If so, that clique is a candidate star clique  $C_j$ . Otherwise, the algorithm answers NO. Associate that star clique  $C_j$  with a vertex  $v_j$  of  $X_i$  of  $H$ .

Every vertex  $v$  of  $G$  is in the intersection of two star cliques  $C_i, C_j$  (by Observation 6.11). Then, we define the following function from  $V(G)$  to edges of the the complete  $K$  of the candidate preimage  $H$ :  $\phi(v) = v_i v_j$  if  $v$  of  $G$  is in the intersection of two star cliques  $C_i, C_j$ .

Recall that if  $G$  is a biclique graph of an NSSG, this step can be done since we assume that each part has at least 3 vertices.

Next, set as old edges the edges of the star cliques and set as new edges the rest of them (Observation 6.8).

### **Step (4) - Single and double s-bicliques.**

In this step the s-bicliques are found. Also, single and double bicliques are identified by analyzing the intersection of the neighborhoods of edge-bicliques in  $G$  according to Observations 6.12 and 6.13. In this step also the two parts associated to the s-biclique are given.

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<sup>1</sup>This step can be done while computing the connected components, at Step (2).

To find out which s-bicliques are single or double and their direction, the order  $\preceq$  of the parts need to be established. First, the candidate binary relation  $\leftrightarrow$  is defined. That is, considering parts  $X_i$  and  $X_j$  and a pair of edge-bicliques, one of each part, and the size of the intersection of its neighborhood, if it contains 8 bicliques (Observation 6.12) then it sets  $X_i \leftrightarrow X_j$ . If it contains exactly 4 bicliques, then it sets  $X_i \leftrightarrow X_j$ . In any other case it answers NO.

Also, a biclique  $B$  is set as single s-biclique if its parts are comparable and are set as double s-bicliques if its parts are incomparable. Note that, if  $\phi(B) = v_i v_j$ , the parts associated to  $B$  are the parts where  $v_i$  and  $v_j$  belong to.

### Step (5) - Find $(X, \preceq)$ and the “directions” of the double s-bicliques.

First the order  $\preceq$  of the comparable parts are determined.

If  $P = \emptyset$ , by Observation 6.14, we determine the consecutive pairs. That is, given 3 parts, we discover which one is candidate of being in the middle, so the other two are not consecutive. Considering a new part at a time, we find a “chain” of parts: for each part we know which are the parts at both sides. Choose one of the two possible directions, that is, choose a part  $X$  of the extremes of the chain to be the first one in the chain, that is  $X \preceq X_i$  for every part  $X_i$  and the rest of the order is induced by it.

Next, suppose  $P \neq \emptyset$ . Note that if  $P \neq \emptyset$  then  $|P| \geq 2$ .

Suppose  $T = \emptyset$ , then for each pair of comparable parts  $X_i, X_j$ , let  $X_k$  be a part incomparable with both of them. Let  $B$  be a candidate s-biclique of  $X_i$  and  $X_j$  and let  $B'$  be a candidate s-biclique of  $X_k$  and  $X_i$  or  $X_j$  such that  $B$  and  $B'$  are adjacent by a new edge. If  $B'$  is of  $X_j$  then set  $X_i \preceq X_j$ , otherwise, set  $X_j \preceq X_i$ .

Now, suppose  $P \neq \emptyset$  and  $T \neq \emptyset$ . The algorithm looks for the candidate for the zero part  $X_z$ .

If  $|T| = 1$ , set as the zero part  $X_z$  the unique part of  $T$ .

If  $|T| = 2$ , let  $T = \{X_i, X_j\}$ . Let  $X_k \in P$ . Note that  $X_k \leftrightarrow X_i$  and  $X_k \leftrightarrow X_j$  and  $X_k$  must be in the middle of them, as one among  $X_i$  and  $X_j$  is the zero part. Consider another part  $X_{k'} \in P$ ,  $X_{k'} \leftrightarrow X_k$  and consider  $B_1$  a biclique of parts  $X_i$  and  $X_k$  and a biclique  $B_2$  of parts  $X_i$  and  $X_{k'}$  that is not adjacent to  $B_1$  by old edge. If  $B_1$  and  $B_2$  are adjacent by new edge then set  $X_j$  as the zero part. Then set  $X_k, X_{k'} \preceq X_i$  and  $X_j \preceq X_k, X_{k'}$ . If  $B_1$  and  $B_2$  are not adjacent by new edge, consider  $B_3, B_4$  bicliques of parts  $X_j$  and  $X_k$  and  $X_j$  and  $X_{k'}$  respectively, that are not adjacent by old edge. If  $B_3, B_4$  are adjacent by new edge then set  $X_i$  as the zero part and then set  $X_k, X_{k'} \preceq X_j$  and  $X_i \preceq X_k, X_{k'}$ . Otherwise, if there is no new edge between  $B_3$  and  $B_4$ , then, if  $|P| = 2$ , then choose any part of  $T$  and set it as the zero part  $X_z$ . Otherwise, if  $|P| > 2$ , there is  $X_l \in P$  comparable with one of  $X_k, X_{k'}$ . Suppose  $X_l \leftrightarrow X_k$ . Then,  $X_k \preceq X_l$ . Consider a biclique  $B_5$  of  $X_l, X_i$  that is not adjacent by old edge to  $B_2$  nor to  $B_4$ . If  $B_5$  is adjacent by new edge to  $B_2$  then  $X_i$  is not the zero part,  $X_j$  is set as the zero part,  $X_k, X_{k'}, X_l \preceq X_i$  and  $X_j \preceq X_k, X_{k'}, X_l$ .

If  $B_5, B_2$  are not adjacent by new edge, then  $X_i$  is set as the zero part, and then set  $X_k, X_{k'}, X_l \preceq X_j$  and  $X_i \preceq X_k, X_{k'}, X_l$ .

If  $|T| > 2$ , choose one part of  $P$ ,  $X_k$ . Choose two parts of  $T$ ,  $X_i$  and  $X_j$ , such that  $X_k$  is in the middle (use Observation 6.14 to find out which one is in the middle). Note that if  $X_k$  is not in the middle, then by Observation 6.4  $X_i$  and  $X_j$  are not candidate for the zero part ( $X_z$ ).

If  $X_k$  is in the middle, again by Observation 6.4 one of  $X_i$  or  $X_j$  are candidate for the zero part  $X_z$ . Choose another part  $X_l \in T$ , and repeat the same process with parts  $X_i, X_l$  and  $X_k$ . If  $X_k$  is again in the middle, then  $X_i$  is the candidate for the zero part, else  $X_j$  is the candidate for the zero part.

Let  $X_z$  be the candidate for the zero part. Then, set  $X_z \preceq X_a$ , for every  $X_a \in \mathcal{X}$ , and  $X_b \preceq X_c$  for every  $X_b \in P$  and every  $X_c \in T$ , with  $X_c \neq X_z$ .

Now the order for the other pairs is decided.

Let  $X_i \leftrightarrow X_j$  be two parts that are not ordered yet. Clearly,  $X_i$  and  $X_j$  are different from  $X_z$ . Use Observation 6.14 to find out which one is in the middle, among  $X_i, X_j$  and  $X_z$ . As  $X_z$  can not be in the middle, if  $X_i$  is in the middle set  $X_i \preceq X_j$ , if  $X_j$  is in the middle set  $X_j \preceq X_i$ . The directions of the single s-bicliques are completed.

Next, the directions of the double s-bicliques are found (Observation 6.2).

If there is a pair of double s-bicliques for which the direction is known,  $B_{uv}$  and  $B_{vu}$ , for any other double s-biclique  $B$  with base edge  $wz$ ,  $z \in X(u)$ ,  $w \in X(v)$ , if  $B$  is adjacent by new edge to  $B_{uv}$  set  $B = B_{zw}$ , and if  $B$  is adjacent by new edge to  $B_{vu}$ , set  $B = B_{wz}$ . We call this procedure propagation and it is used every time a double s-biclique direction is determined.

Consider the pair of double s-bicliques  $B_1, B_2$  with base edge  $uv$  for which we need to determine the direction.

First consider the case there are only two parts in  $\mathcal{X}$ ,  $X(u), X(v)$ . In this case, set  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$ .

Now suppose there are more than 2 parts.

If the candidate for the zero part  $X_z$  exists ( $T \neq \emptyset$ ), consider a biclique  $B$  from  $X_z$  to  $X(u)$ . If  $B$  is adjacent by new edge to  $B_1$  and not adjacent to  $B_2$  set  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$ . If  $B$  is not adjacent by new edge to  $B_1$  and adjacent by new edge to  $B_2$  set  $B_1 = B_{vu}$  and  $B_2 = B_{uv}$ . Otherwise, if  $B$  is adjacent to both  $B_1$  and  $B_2$  or to none, answer NO, since  $G$  is not a biclique graph of an NSSG.

Next, suppose there are at least 3 parts and  $T = \emptyset$  (there is no candidate for the zero part).

If there is a part  $X_i$  incomparable with  $X(u)$  and  $X(v)$ , then consider any biclique  $B$  of  $X_i$  and  $X(u)$ , with base edge  $wz$ ,  $u \neq z \in X(u)$ ,  $w \in X_i$  adjacent by new edge to  $B_1$  or  $B_2$ . If  $B$  is adjacent by new edge only to  $B_1$  then set  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$  and

$B = B_{zw}$ . If  $B$  is adjacent by new edge only to  $B_2$  then  $B_2 = B_{uv}$  and  $B_1 = B_{vu}$  and  $B = B_{zw}$ . If  $B$  is adjacent to both or no such a  $B$  exists, answer NO.

Next, suppose such a part  $X_i$  does not exist.

Suppose there is a part  $X_j$  such that  $X_j \preceq X(u)$  or  $X_j \preceq X(v)$ . Suppose  $X_j \preceq X(u)$ . Recall that since  $X_j$  is not the zero part, then  $X_j \leftrightarrow X(v)$ . Consider a biclique  $B$  from  $X_j$  to  $X(u)$ , not adjacent by old edge to  $B_1$  or with  $B_2$ . If it is adjacent by new edge only to  $B_1$  then set  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$ . If  $B$  is adjacent by new edge only to  $B_2$  then  $B_2 = B_{uv}$  and  $B_1 = B_{vu}$ . If  $B$  is not adjacent by new edge to  $B_1$  nor  $B_2$  or  $B$  is adjacent by new edge to both  $B_1$  and  $B_2$ , answer NO. The case  $X_j \preceq X(v)$  is analogous.

Otherwise, if there is no part  $X_j$  such that  $X_j \preceq X(u)$  or  $X_j \preceq X(v)$ , suppose there is a part  $X_k$  such that  $X(u) \preceq X_k$  or  $X(v) \preceq X_k$ .

Suppose  $X(v) \preceq X_k$ . Then, if  $X_k \leftrightarrow X(u)$ , consider a biclique  $B$  with base edge  $wz$ ,  $u \neq w \in X(u)$ ,  $z \in X_k$  adjacent to at least one of  $B_1, B_2$ . If  $B$  is adjacent only to  $B_1$  by new edge then set  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$  and  $B = B_{wz}$ . If  $B$  is adjacent only to  $B_2$  by new edge then set  $B_2 = B_{uv}$ ,  $B_1 = B_{vu}$  and  $B = B_{wz}$ . If such a  $B$  does not exists or  $B$  is adjacent by new edge to both, then answer NO. The case  $X(u) \preceq X_k$  is analogous.

If none of the cases above holds, answer NO.

When this procedure ends, the direction of every double s-biclique is given.

#### Step (6) - Generate $H$ and check if $G \cong KB(H)$ .

To finish the construction of  $H$ , it is necessary to find, for each  $i$ , a set of vertices  $S(X_i)$  satisfying the order, that is, a candidate family of sets of satellites.

Let  $\prec$  be the *transitive and reflexive reduction* of the order  $\preceq$ , that is,  $X_i \prec X_j$  if and only if  $X_i \neq X_j$ ,  $X_i \preceq X_j$  and there is no  $X_k$  such that  $X_k \neq X_i$ ,  $X_k \neq X_j$  and  $X_i \preceq X_k \preceq X_j$ . Note that  $\prec$  can be computed in polynomial time [AGU72].

As the zero part  $X_z$  has no satellites,  $S(X_z) = \emptyset$ . Note that the relation  $\prec$  of an NSSG induces a forest where the vertices are the parts that are not the zero part, and each component is a rooted tree. The rooted tree has the local maximum as root and if  $X_i \prec X_j$  then  $X_j$  is the father of  $X_i$ . So, in each component, every pair of parts of the same level are incomparable.

Given the relation  $\prec$  of the parts we need to construct the sets of satellites of each part.

The algorithm for constructing the set of satellites of the candidate graph  $H$  is the following.

Start with the leafs. For each part  $X_i$  that is a leaf, add a different vertex to  $S(X_i)$ , that is, set  $S(X_i) = \{s_i\}$ . Next, for each part  $X_j$  such that every part  $X_i$ , with  $X_i \prec X_j$ , is already visited, consider the different cases:

**Case 1:** There exist only two parts  $X_i, X_{i'} \prec X_j$ . Consider a biclique from  $X_i$  to  $X_j$  and a biclique from  $X_{i'}$  to  $X_j$ , not adjacent by old edge. If there is a new edge between those bicliques, set  $S(X_j) = S(X_i) \cup S(X_{i'}) \cup \{v\}$ , for a new vertex  $v$ . If there is no new edge between them set  $S(X_j) = S(X_i) \cup S(X_{i'})$ .

**Case 2:** There exist at least 3 parts  $X_i, X_{i'}, X_{i''} \prec X_j$ . Then, set  $S(X_j) = \bigcup_{X_i \prec X_j} S(X_i)$ .

Repeat until every part is visited.

Finally, having the family of sets of satellites  $S(X_1), \dots, S(X_\ell)$  of the candidate graph  $H$ , the algorithm decides if  $G \cong KB(H)$ , for  $H$  defined as follows:

Let  $H = (K \cup S, E)$  where  $K = \bigcup_{i=1}^\ell X_i$  and  $S = \bigcup_{i=1}^\ell S(X_i)$ . For the edge set, let  $E_K = \binom{K}{2}$  and  $E_{X_i} = \{\{v, s\} \mid v \in X_i, s \in S(X_i)\}$ , for  $1 \leq i \leq \ell$ . Let  $E = E_K \cup \bigcup_{i=1}^\ell E_{X_i}$ .

Next, for checking whether  $G \cong KB(H)$ , define the function  $\phi^* : V(G) \rightarrow V(KB(H))$ , as  $\phi^*(v) = v_{ij}$ , if  $\phi(v) = v_iv_j$  and  $v$  corresponds to a biclique from  $X(v_j)$  to  $X(v_i)$ , that is,  $v$  corresponds to biclique  $B_{v_iv_j}$ .

It only remains to check if the function given above induces an isomorphism between  $G$  and  $KB(H)$ . If that function is an isomorphism, then, the algorithm answers YES. Otherwise, the algorithm answers NO.

### 6.3.2. Proof of the algorithm

Using the observations of the previous section, we prove that indeed, the algorithm given in the previous section is a polynomial algorithm that can be used for recognizing biclique graphs of NSSGs.

We remark that we are aware that the time complexity can be improved considerably by using the correct data structure and saving information during the process and also by changing the way some steps are done. We recall that, as mentioned in the introduction, to find the best time complexity of an algorithm for the biclique graph recognition problem is not a goal of this work, but to prove its polynomiality.

The following lemmas are used to find the order of the parts of the candidate  $H$  built by the algorithm.

**Lemma 6.16 (NSSG).** *Let  $X_k \leftrightarrow X_{k'}$  be parts of  $P$  and  $X_i$  be a part of  $T$ . Then if there exist two bicliques  $B_1, B_2$  of  $X_i$  and  $X_k$  and of  $X_i$  and  $X_{k'}$ , respectively, such that they  $S$ -intersect, then  $X_k, X_{k'} \preceq X_i$ . Otherwise, either  $X_i$  is the zero part or for any other part  $X_l \in P$ ,  $X_l$  is comparable with exactly one of  $X_k$  or  $X_{k'}$  and  $X_l$  incomparable with the other. In the latter case, every biclique  $B_3$  of parts  $X_i$  and  $X_l$  that does not  $K$ -intersect  $B_1$  nor  $B_2$ ,  $S$ -intersects  $B_1$  or  $B_2$ . Moreover, any two bicliques of any other two incomparable parts of  $P$  and  $X_i$   $S$ -intersect.*

*Proof.* If  $S(X_i) \neq \emptyset$ , then  $X_k, X_{k'} \preceq X_i$  and  $B_1, B_2$  contain satellites of  $X_i$  that are not satellites of  $X_k$  or  $X_{k'}$  respectively. If  $S(X_{k'}) \cup S(X_k) \neq S(X_i)$  then  $B_1$  and  $B_2$  intersect in a satellite of  $X_i$ . If  $B_1, B_2$  do not  $S$ -intersect, then  $S(X_{k'}) \cup S(X_k) = S(X_i)$ . Consider any  $X_l \in P$ ,  $X_l \leftrightarrow X_k$ . Let  $B_3$  be a biclique from  $X_l$  to  $X_i$ . If  $B_3$  and  $B_1$  do not  $S$ -intersect, as proved before,  $S(X_l) \cup S(X_k) = S(X_i)$ . But then, since  $H$  is an NSSG,  $S(X_l) \subseteq S(X_{k'})$  and  $S(X_l) \cap S(X_k) \neq \emptyset$  which is a contradiction since  $H$  is an NSSG. We conclude that  $B_3$   $S$ -intersects  $B_1$  and  $B_2$ . The same argument follows for  $B_2$  and bicliques of  $X_i$  and any other two incomparable parts of  $P$ .  $\square$

**Lemma 6.17 (NSSG).** *Let  $B_1, B_2$  be two bicliques of an NSSG from  $X_i$  to  $X_j$ , and  $X_{i'}$  to  $X_j$ , respectively, such that  $X_i, X_{i'} \preceq X_j$ . Then they have no satellites in common if and only if  $X_i, X_{i'}$  are incomparable and  $S(X_j) = S(X_i) \cup S(X_{i'})$ . In that case, there is no other  $X \preceq X_j$ ,  $X \leftrightarrow X_i$  and  $X \leftrightarrow X_{i'}$ .*

*Proof.* Let  $B_1, B_2$  be two bicliques of an NSSG  $H$  from  $X_i$  to  $X_j$ , and from  $X_{i'}$  to  $X_j$ , respectively, such that  $X_i, X_{i'} \preceq X_j$ . Suppose there is a vertex  $v \in S(X_j)$ ,  $v \notin S(X_i), S(X_{i'})$ . Then,  $v$  belongs to  $B_1$  and  $B_2$ . Conversely, if  $S(X_j) = S(X_i) \cup S(X_{i'})$ , then  $B_1$  contains only vertices of  $S(X_{i'})$  and  $B_2$  contains only vertices of  $S(X_i)$ . If  $X_i \leftrightarrow X_{i'}$  then  $B_1$  and  $B_2$  do not intersect in satellites (as  $H$  is an NSSG). Finally, suppose there is  $X \preceq X_j$ . Then,  $X$  intersects  $X_i$  or  $X_{i'}$  (or both). Since  $H$  is an NSSG,  $X$  is not incomparable with both  $X_i, X_{i'}$ .  $\square$

**Theorem 6.18.** *Let  $G$  be a graph with  $N$  vertices and  $M$  edges. Then  $G$  is a biclique graph of an NSSG  $F$  with at least 3 vertices in each part, if and only if the algorithm answers YES. Also, the algorithm runs in polynomial time.*

*Proof.* If the algorithm answers YES it is because  $G \cong KB(H)$ , where  $H$  is the NSSG constructed by the algorithm. Then,  $G$  is a biclique graph of an NSSG.

Conversely, if  $G \cong KB(F)$  where  $F$  is an NSSG, we need to prove that the algorithm answers YES.

We prove that the algorithm goes through every step until the last one, where it answers YES.

**Step (1).** First we prove that the set  $\mathcal{E}$  returned by the algorithm is the set of edge-bicliques of  $F$ . By Theorem 6.15, if  $G$  is a biclique graph of some SSG  $F$ , then the graph obtained by removing from  $G$  vertices of smallest degree and its neighbors is the biclique graph of  $F'$ , where  $F'$  is the graph obtaining from  $F$  after removing vertices of one or more parts of  $F$ . Therefore, the vertices of  $G$  that represent edge-bicliques in  $F'$  are also vertices representing edge-bicliques of  $F$ . By induction, we conclude that the vertices added to  $\mathcal{E}$  correspond to edge-bicliques of one or more parts of  $F$ , which completes the proof. This step can be done in polynomial time.

**Step (2).** As mentioned before, by Observation 6.9, the connected components of  $G[\mathcal{E}]$  are the edge bicliques of each part of  $F$ . Then, in this step the algorithm finds indeed the parts of  $F$ . Again, finding the connected components can be done in polynomial time.

**Step (3).** By Observation 6.10, as  $G$  is a biclique graph of an NSSG, the intersection of the neighborhood of intersecting edge bicliques contains a star clique and one vertex adjacent only to both edge bicliques. Since every star clique belongs to the intersection of some pair of edge bicliques, every star clique of  $G$  is found by the algorithm. Every edge-biclique  $B_{v_i v_j}$  of  $F$  is in the intersection of two star cliques. Each star clique contains every edge biclique of  $F$  that contains  $v_i$  or  $v_j$  respectively. That is, each star biclique is identified with a vertex of  $F$ .

Now, consider the star cliques  $C_1, \dots, C_l$  found by the algorithm. If we rename the vertices of  $F$  in a way that  $C_i$  is the star clique associated to vertex  $v_i$ , (that is, name a vertex of  $F$  by  $v_i$  if  $C_i$  contains all the bicliques incident to it), then the function  $\phi$  of step 3 gives, for every vertex of  $G$ , the edge of  $K$  that is the base-edge of the biclique of  $F$  that it represents. Recall that if we restrict to edge-bicliques and single s-bicliques,  $\phi$  associates vertices of  $G$  to the corresponding biclique. For the double s-bicliques, we still need to find out which of the two vertices with the same base edge is each double s-biclique.

Finally, the algorithm sets the edges of star cliques as old edges. Since the star cliques found by the algorithm correspond to star cliques of  $F$ , every old edge of  $G$  is set as an old edge by the algorithm and if the algorithm sets an edge as an old edge it is because it belongs to a star clique and then, it is an old edge of  $G$ . We conclude that the algorithm finds the old and new edges of  $G$ . Recall that an old edge corresponds to intersection of bicliques in vertices of  $K$  and a new edge corresponds to intersection of bicliques only in satellites in  $F$ . This is done while finding the star cliques.

This step can be done in polynomial time.

**Step (4).** By Observation 6.12 , the intersection in  $G$  of the neighborhood of two edge bicliques of different parts contains exactly 8 bicliques if and only if they belong to incomparable parts. In that case, those 8 bicliques are the double s-bicliques that have an extreme in each of the incomparable parts.

Also, since  $G$  is a biclique graph of  $F$ , when the intersection in  $G$  of the neighborhood of two edge bicliques of different parts does not contain 8 bicliques, it contains exactly 4 bicliques and the parts of the edge bicliques are comparable.

As the intersection of the neighborhood of two edge bicliques of different parts can be done in polynomial time, this step can be done in polynomial time.

**Step (5).** Up to now, the algorithm have found every pair comparable and incomparable parts of  $F$ , the edge bicliques, the single and double s-bicliques and the new and old edges represents the intersection of the bicliques in satellites of  $F$  or vertices of the complete part of  $F$ , respectively. The algorithm continues looking for the order of the comparable parts. We need to prove that if two parts  $X_i \leftrightarrow X_j$  of  $F$  are such that  $S(X_i) \subseteq S(X_j)$  in  $F$ , then the algorithm sets  $X_i \preceq X_j$ .

Consider the sets  $P$  and  $T$  described in the algorithm. If  $F$  is a threshold graph then  $P = \emptyset$  and the algorithm uses Observation 6.14 to find the consecutive pairs. Then the algorithm constructs  $H$  choosing an order for the consecutive pairs such that if  $S(X_i) \subseteq S(X_j)$  in  $F$ , then  $X_i \preceq X_j$  for every  $X_i, X_j$  or if  $S(X_i) \supseteq S(X_j)$  in  $F$ , then  $X_j \preceq X_i$  in  $H$  for every  $X_i, X_j$ . By Observation 6.5, in both cases  $KB(F)$  and  $KB(H)$  are isomorphic and then the algorithm answers YES.

Otherwise,  $P \neq \emptyset$  and the algorithm continues. If  $P \neq \emptyset$  and  $T = \emptyset$ , by Observation 6.6 and Lemma 6.7, there is no zero part and for every pair of parts  $X_i \leftrightarrow X_j$  there always exists a part  $X_k$  such that  $X_k \leftrightarrow X_i$  and  $X_k \leftrightarrow X_j$ . Then, by Observation 6.3, checking the adjacency at new edges of the vertices of  $G$  associated with the candidate s-bicliques of parts  $X_i, X_j$  and  $X_k$ , the order of  $X_i, X_j$  can be decided: Consider a biclique  $B_1$  of parts  $X_i$  and  $X_j$ ,  $S(X_i) \subseteq S(X_j)$  in  $F$  and a part  $X_k$  incomparable with both parts. Then if a biclique  $B_2$  of parts  $X_k$  and any of  $X_j$  or  $X_i$  have a new edge with  $B_1$ , since  $S(X_i) \subseteq S(X_j)$  in  $F$ , the satellites in common of  $B_1$  and  $B_2$  are in  $S(X_j)$  and they can not belong to  $S(X_i)$ . It means that  $B_2$  is a biclique from  $X_k$  to  $X_j$  and the algorithm concludes that  $X_j \geq X_i$ . Recall that we already proved that the new edges set by the algorithm correspond exactly to intersections in satellites of the corresponding bicliques of  $F$ . This can be done in  $O(n^3)$

Finally, suppose  $T \neq \emptyset$ . The algorithm looks for the zero part  $X_z$ , that exists by Observation 6.6.

Since  $X_z \in T$ , if  $|T| = 1$  then  $X_z$  is found.

If  $|T| = 2$ , suppose  $T = \{X_i, X_j\}$  and consider two parts in  $P$ ,  $X_k \leftrightarrow X_{k'}$ . By Observations 6.4 and 6.14 and Lemma 6.7,  $X_i$  or  $X_j$  is the zero part. Suppose, w.l.o.g that  $X_i$  is the zero part. So, for every part  $X \in P$ ,  $X_i \preceq X \preceq X_j$ . Also consider bicliques  $B$  and  $B'$  from  $X_k$  and  $X_{k'}$  respectively, to some of  $X_i$  or  $X_j$ . Suppose,  $X_j$ . If  $B$  and  $B'$  are adjacent by a new edge it means they share a satellite in  $F$ . Then, since  $X_k \leftrightarrow X_{k'}$ , the satellite is in  $X_j$  and the algorithm concludes that  $X_i$  is the zero part, as desired. If  $B$  and  $B'$  are not adjacent by new edge, it means that  $S(X_j) = S(X_k) \cup S(X_{k'})$ . Therefore the algorithm considers bicliques of  $X_i$  and  $X_k, X_{k'}$ . Since  $X_i$  is the zero part, the bicliques are not adjacent by new edge.

If there is no other part in  $P$ , then observe that the biclique graph of  $F$  and the biclique graph of  $F'$ , where  $F'$  is the graph obtained by changing the set of satellites of parts  $X_j$  and  $X_i$ , that is, setting  $S(X_i) = S(X_k) \cup S(X_{k'})$  and  $S(X_j) = \emptyset$ , are isomorphic. The algorithm then chooses one of  $X_i$  or  $X_j$  to be the zero part, constructing the graph  $H$  isomorphic to  $F$  or  $F'$ .

Otherwise, consider another part  $X_l \in P$ . Then, by Lemma 6.16, any  $X_l$  in  $P$  must be incomparable with exactly one of  $X_k, X_{k'}$  and comparable with the other. Moreover,  $S(X_l)$  is included in  $S(X_k)$  or  $S(X_{k'})$  since  $S(X_l) \subseteq S(X_k) \cup S(X_{k'})$ . Then, any biclique of  $X_k$  and  $X_i$  has all the satellites of  $X_k$  and then it intersects in satellites the biclique of  $X_j, X_{k'}$  that contains all the satellites of  $X_k$ . Then, there

is a new edge between those bicliques and the algorithm concludes that  $X_i$  is the zero part, as desired.

Finally, if  $|T| > 2$  and  $|P| \neq 0$ , let  $X_k \in P$ . There are parts  $X_i$  and  $X_j$  such that  $X_k$  is in the middle of  $X_i$  and  $X_j$ , as  $|T| > 1$ . Suppose again that  $X_i$  is the zero part. Any biclique  $B_1$  of  $X_i$  and  $X_k$  is a biclique with all the satellites of  $X_k$  and any biclique  $B_2$  of  $X_k$ ,  $X_j$  is a biclique from  $X_k$  to  $X_j$  with all the satellites of  $X_j$  that are not satellites of  $X_k$ . Let  $X_l \in T$  (chosen by the algorithm) different from  $X_i$  and  $X_j$ . Since  $X_k \in P$ ,  $X_l \in T$ ,  $X_l$  is not the zero part and by Observation 6.4,  $S(X_k) \subseteq S(X_l)$ , the algorithm concludes that  $X_k$  is in the middle between  $X_i$  and  $X_l$  the algorithm sets  $X_i$  as the zero part, as desired.

We conclude that if  $F$  contains a zero part,  $X_z$ , then the algorithm finds it.

Next, the algorithm gives an order to any comparable pair  $X_i, X_j$  (not ordered yet). As  $X_i$  and  $X_j$  are not the zero part  $X_z$ , among  $X_i, X_j$  and  $X_z$ , one of  $X_i$  or  $X_j$  is in the middle and the order is given.

In addition, the algorithm considers the double s-bicliques.

First lets prove that the propagation procedure is correct. Let  $B_{uv}$  and  $B_{vu}$  be a pair of double s-bicliques for which the direction is known. Let  $B$  be any double s-biclique with base edge  $wz$  of the same parts ( $z \in X(u)$ ,  $w \in X(v)$ ) that do not  $K$ -intersect  $B_{uv}$  nor  $B_{vu}$ . As the satellites of  $B$  are the satellites of  $S(u)$  or  $S(v)$ ,  $B$  is adjacent by new edge to  $B_{uv}$  or  $B_{vu}$ , not both. So the direction of  $B$  is determined and the propagation procedure works.

Now let  $B_1, B_2$  be two s-bicliques with the same base edge  $uv$  and  $X(u) \leftrightarrow X(v)$ . Suppose  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$ .

Suppose  $X(u)$  and  $X(v)$  are the unique parts of  $F$ . Recall that any double s-biclique adjacent by new edge to  $B_1$  has satellites in common with it, then they are from the same part to the same part. If the algorithm sets  $B_1 = B_{vu}$  and  $B_2 = B_{uv}$ , by the propagation procedure, every biclique with the same satellites of  $B_1$  is set from  $X(v)$  to  $X(u)$ . It follows that both possible constructions of  $H$ , where  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$  or  $B_1 = B_{vu}$  and  $B_2 = B_{uv}$  are graphs such that its biclique graphs are isomorphic and then  $KB(F)$  is isomorphic to  $KB(H)$  for any construction of  $H$ .

Now consider the case  $F$  contains a zero part,  $X_z$ . Since the algorithm has already identified it, it considers the biclique  $B$  from  $X_z$  to  $X(u)$ . Then  $B$  contains all the satellites of  $X(u)$  and can only be adjacent by a new edge to  $B_1$ , that is, to the biclique from  $X(v)$  to  $X(u)$ . Then, the algorithm answers that  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$  as desired.

Now suppose there are at least 3 parts and  $T = \emptyset$ . Then  $F$  does not contain a zero part. Suppose there exists a part  $X_i$  of  $F$  that is incomparable with  $X(u)$  and  $X(v)$ .

If a biclique  $B$  of  $X_i$ ,  $X(u)$ , with base edge  $wz$ ,  $w \neq z \in X(u)$ ,  $w \in X_i$  is adjacent by new edge to  $B_1$ , then  $B$  and  $B_1$  have a satellite in common. Since  $F$  is an NSSG,  $S(X_i)$  and  $S(X(v))$  have no intersection. Therefore, the satellite in common is in  $S(X(u))$  which implies that  $B$  is from  $X_i$  to  $X(u)$  and it is not adjacent to  $B_2$  by new edge. Then, the algorithm will conclude that  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$  as desired.

If  $F$  does not have a part incomparable with both  $X(v)$ ,  $X(u)$ , suppose there is a part  $X_i$  such that  $X_i \preceq X(u)$  or  $X_i \preceq X(v)$ . If  $X_i \preceq X(u)$  and  $B$  is a biclique from  $X_i$  to  $X(u)$ , it is clear that  $B$  is not adjacent to  $B_2$ , and it is adjacent by new edge to  $B_1$ . It follows that the algorithm sets  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$  as desired. Analogously for the case  $X_i \preceq X(u)$ .

If such a  $X_i$  does not exists, then there is a part  $X \preceq X_i$ , for  $X$  being  $X(u)$  or  $X(v)$ . Suppose  $X(v) \preceq X_i$ . Then, the algorithm considers a biclique  $B$  with base edge  $kl$ ,  $u \neq k \in X(u)$ ,  $l \in X_i$  adjacent to at least one of  $B_1$ ,  $B_2$ . Recall that such a biclique always exists since any biclique from  $X_i$  to  $X(u)$  intersects  $B_1$  if  $X(u) \leftrightarrow X_i$  and does not intersect  $B_2$ . Then the algorithm sets  $B_1 = B_{uv}$  and  $B_2 = B_{vu}$  as desired. Analogously for the case  $X(u) \preceq X_i$ .

We affirm that some of the cases above holds, since, otherwise, every  $X_i$  is such that  $X(u), X(v) \preceq X_i$ , and every  $X_j$  is such that  $X_j \leftrightarrow X_i$  (otherwise,  $X_j \leftrightarrow X(u)$ ,  $X_j \leftrightarrow X(v)$ ). But then,  $X_i$  is in  $T$  which is a contradiction.

This step can be done in polynomial time.

**Step (6).** Since the partial order of the parts found by the algorithm and the partial order of the parts of  $F$  are the same (or the inverse in some case), the set of satellites can always be constructed. Recall that the sets of satellites of  $H$  are not the same as the sets of satellites of  $F$  but they verify the same relation of inclusion. Also, observe that if  $X_i, X_j \preceq X_k$ , the bicliques from  $X_i, X_j$  to  $X_k$  can have or not satellites in common, that is, if  $B_1$  is a biclique from  $X_i$  to  $X_k$  and  $B_2$  is a biclique from  $X_j$  to  $X_k$ , then  $B_1$  and  $B_2$  have not satellites in common if and only if  $S(X_k) = S(X_i) \cup S(X_j)$ .

Let  $S(X_1), \dots, S(X_k)$  be the set of satellites constructed by the algorithm. We need to prove that two bicliques of  $H$  intersect in satellites if and only if their corresponding bicliques in  $F$  intersect in satellites. First that correspondance needs to be defined. The function  $\phi^* : V(G) \rightarrow V(KB(H))$  is defined as  $\phi^*(v) = v_{ij}$ , if  $\phi(v) = v_iv_j$  and  $v$  corresponds to a biclique from  $X(v_j)$  to  $X(v_i)$  that is,  $v$  corresponds to biclique  $B_{v_iv_j}$ . As the bicliques of  $F$  are the vertices of  $G$  then  $\phi^*$  is the correspondance between bicliques of  $F$  and bicliques of  $H$ .

It is enough to prove it for bicliques from parts  $X_i, X_{i'}$  to  $X_j$  such that  $X_i, X_{i'} \prec X_j$ . Let  $B_1, B_2$  be two bicliques from  $X_i, X_{i'}$  to  $X_j$ , respectively. Suppose that the bicliques of  $F$ , corresponding  $B_1$  and  $B_2$  ( $B'_1 = \phi^{*-1}(B_1)$  and  $B'_2 = \phi^{*-1}(B_2)$ ) have a satellite in common. If there is no other part  $X \prec X_j$ , since there is a new edge between  $B_1$  and  $B_2$  in  $G$ , there is a new edge between every pair of bicliques of

the same parts that are not incident. Then, the algorithm adds a new vertex  $v$  to  $S(X_j)$  and then  $v$  belong to  $B_1$  and  $B_2$ . If there is another part  $X \prec X_j$ , then the algorithm sets  $S(X_j) = \bigcup_{X \prec X_j} S(X)$ . But then, any pair of bicliques from any part  $X \prec X_j$  to  $X_j$  share a satellite.

For the converse, we prove that if  $B_1$  and  $B_2$  are bicliques from part  $X_i \prec X_j$  to  $X_j$  and from  $X_{i'} \preceq X_j$  to  $X_j$ , respectively, such that  $B'_1$  and  $B'_2$  do not share a satellite in  $F$  then  $B_1$  and  $B_2$  do not share a satellite in  $H$ . Recall that if  $B_1$  and  $B_2$  do not share a satellite, then every other biclique of the same parts do not share satellites. But if they do not share a satellite, by Lemma 6.17 there is no other  $X \prec X_j$  in  $F$ . The algorithm follows Case 1 and sets  $S(X_j) = S(X_i) \cup S(X_{i'})$  so no biclique from  $X_{i'}$ , or  $X_i$  to  $X_j$  in  $H$  has satellites in common.

It follows that  $\phi^*(v)$  is an isomorphism and the algorithm answers YES.

This step can be done in polynomial time.

□

## 6.4. Characterization of nested split graphs

We defined nested separable split graphs as a subclass of split graphs for which we solved the recognition problem of its biclique graphs. The definition is based on properties needed in some of the steps of the algorithm. In this section we study the class of nested split graphs, giving a characterization by forbidden induced subgraphs. The class of separable split graphs is not a hereditary class so the NSSG class can not be characterized by forbidden induced subgraphs.

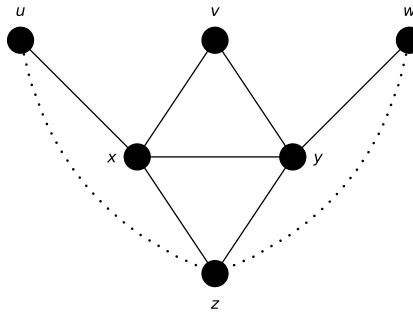


Figura 6.4: Set  $\mathcal{F}$  of forbidden induced subgraphs for nested split graphs. The dotted edges can be present or not, generating 3 graphs.

**Theorem 6.19.** *A split graph  $H = (K \cup S, E)$  with at least 3 vertices in the complete part  $K$  is a nested split graph if and only if it is  $\mathcal{F}$ -free, where  $\mathcal{F}$  is the set of graphs of Figure 6.4.*

*Proof.* Let  $H$  be a split graph. Let  $(X, \preceq)$  be as defined at Section 5.1.

Suppose  $H$  is not a nested split graph. So, there exist two parts  $X_i$  and  $X_j$  such that  $X_i \nleftrightarrow X_j$  and  $S(X_i) \cap S(X_j) \neq \emptyset$ . Let  $x \in X_i$  and  $y \in X_j$ . As  $X_i \nleftrightarrow X_j$  there are satellites  $u$  and  $w$  such that  $u$  is adjacent to  $x$  and not to  $y$  and  $w$  is adjacent to  $y$  and not to  $x$ . As  $S(X_i) \cap S(X_j) \neq \emptyset$ , there is a satellite  $v$  such that  $v$  is adjacent to both  $x$  and  $y$ . Let  $z \in K$  such that  $z$  is not adjacent to  $v$ . Note that such  $z$  exists as no satellite is adjacent to every vertex of  $K$ .

It follows that the graph  $H[\{u, v, w, x, y, z\}]$  is isomorphic to one of the graphs of  $\mathcal{F}$ , depending on the adjacencies of vertices of  $z$  and  $u, w$ .

Conversely, suppose  $H$  contains an induced subgraph isomorphic to some graph of  $\mathcal{F}$  and call the vertices of that subgraph by its names in Figure 6.4. That is,  $u, v, w, x, y$  and  $z$ . Observe that  $x$  and  $y$  are not satellites (as they are not simplicial vertices), so  $x, y \in K$ . Also, note that  $u$  and  $w$  are satellites (as  $u$  is not adjacent to  $y$  and  $w$  is not adjacent to  $x$ ). Vertex  $v$  (or  $z$ , depending the existence of other edges) is also a satellite. Consider the parts  $X(x)$  and  $X(y)$ . It is clear that  $X(x) \nleftrightarrow X(y)$  and  $S(X(x)) \cap S(X(y)) \neq \emptyset$  and then  $H$  is not a nested split graph.  $\square$

## 6.5. Final remarks

If  $G$  is the biclique graph of some split graph  $H$ , we give the relations that the family of satellites of  $H$  must verify in order to preserve the intersection of the bicliques (edges of  $G$ ). Any class of split graphs that allows us to construct the family of subsets of satellites from that relations can be considered. The nested property arised as a sufficient condition to solve that problem. We leave as an open problem to find other classes where the same strategy can be used . On the other hand, finding those sets of satellites implies finding the order of the parts preserving the intersections of the bicliques which, in some cases, seems to be hard. That is why we conjecture that the problem of deciding if a graph is the biclique graph of a split graph is an NP-hard problem.

On the other hand, we remind that for guessing those restrictions, we need to find the parts of the preimage and for that we use the fact that it does not contain star bicliques. We leave as an open problem to find other subclasses of split graphs where, using another strategy, the family of parts can be identified by a property of the vertices of its biclique graph.

# 7 Bigrafos de intervalos

En este capítulo estudiamos el problema de reconocimiento de grafos biclique de bigrafos de intervalos ( $\mathcal{IBG}$ ). Esto es, dado un grafo  $G$ , el problema de decidir si  $G$  es grafo biclique de un bigrafo de intervalos  $H$ .

La clase de bigrafos de intervalos fue definida por Harary, Kabell y McMorris como *grafos de bi-intervalos* [HKM82]. Müller presentó un algoritmo polinomial de reconocimiento de esta clase [Mül97]. En este capítulo, también estudiamos la clase de grafos biclique de bigrafos de intervalos propios ( $\mathcal{PIB}$ ).

En particular, probamos que  $KB(\mathcal{IBG}) \subset$  grafos de co-comparabilidad libres de  $K_{1,4}$ . Por otro lado, presentamos una caracterización de la clase  $KB(\mathcal{PIB})$  utilizando el concepto de grafo linea y grafo cuadrado. Luego, Presentamos una subclase de  $\mathcal{PIB}$ , proporcionando un algoritmo de reconocimiento de los grafos biclique de esta clase.

Recordemos que el problema de reconocer grafos biclique de grafos bipartitos es un problema aún abierto ( $KB(G)$ ,  $G$  bipartito)

En la Sección 7.1 damos algunas definiciones. El estudio de las bicliques de  $\mathcal{IBG}$ s y algunas propiedades de la clase  $KB(\mathcal{IBG})$  son presentadas en la Sección 7.2. En la Sección 7.3 y 7.4 presentamos algunas propiedades de las bicliques de un  $\mathcal{PIB}$  y la clase  $KB(\mathcal{PIB})$ .

Aclaración: Durante el transcurso de esta tesis, probamos inicialmente el resultado:  $KB(\mathcal{PIB}) \subset$  grafos de co-comparabilidad libres de  $K_{1,4}$ . Luego, colaborando con el trabajo de tesis de maestría de Edmilson Pereira da Cruz [dC18], se probó que ese resultado vale para  $KB(\mathcal{IBG})$ , lo cual extiende nuestro resultado inicial. Decidimos colocar en esta tesis directamente el resultado más general. Este resultado fue enviado [CGGP18].

Nota: En la sección que sigue, para coherencia y facilidad del lector, repetimos en inglés las definiciones específicas de este capítulo que esta escrito enteramente en inglés.

## 7.1. Definitions

A *line graph* of a graph  $H$  is the intersection graph of its edges and it is denoted by  $L(H)$ .

Let  $G^2$  denote the *square of graph*  $G$ , that is,  $G^2$  contains the vertices and edges of  $G$  plus edges connecting vertices at distance 2 in  $G$ .

A graph  $G$  is a *co-comparability graph* if there is a partial order over its vertex set such that two vertices are comparable if and only if there is no edge connecting them [GH64].

An *interval graph* is a graph such that its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if their corresponding intervals intersect.

A *proper interval graph* is an interval graph that admits a representation such that all the intervals are inclusion-free.

A bipartite graph  $G = (A \cup B, E)$  — with bipartition  $(A, B)$  — is an *interval bigraph* ( $\mathcal{IBG}$ ) if the vertices of  $G$  can be represented by a family of intervals on the real line  $I_v$ ,  $v \in A \cup B$ , so that, for  $u \in A$  and  $w \in B$ ,  $u$  and  $w$  are adjacent in  $G$  if and only if  $I_u$  and  $I_w$  intersect. Call that family of intervals a *bipartite interval model* of  $G$ , with two sets of intervals:  $I_A = \{I_v \mid v \in A\}$  and  $I_B = \{I_v \mid v \in B\}$ .

Given a model of an interval bigraph  $G$ , for each vertex  $v \in V(G)$  let  $s(v)$  be the *beginning* (leftmost) point of the interval  $I_v$  and let  $t(v)$  be the *ending* (rightmost) point of  $I_v$ . Using the usual order of the real numbers, note that  $s(v) \leq t(v)$  for every vertex  $v \in V(G)$ .

For any interval bigraph there is always a bipartite interval model of  $G$  such that the endpoints of the intervals are all different, that is, for every pair of vertices  $u, v$ , the endpoints  $s(u), t(u), s(v)$  and  $t(v)$  are all different.

Observe that intervals  $I_u$  and  $I_v$  intersect if and only if  $s(u) < t(v)$  and  $s(v) < t(u)$ .

Let  $G = (A \cup B, E)$  be an interval bigraph. Define  $<_G$  such that  $u <_G v$  if and only if  $u$  and  $v$  are of the same part and  $s(u) < s(v)$ .

Let  $a = uv$  and  $b = wx$  be two edges of  $G$ , with  $u, w \in A$  and  $v, x \in B$ . We say that  $a$  and  $b$  *cross* if  $u <_G w$  and  $x <_G v$ .

A *proper bipartite interval model* is a model such that every interval is not contained in another interval of the same set. An interval bigraph that admits a proper interval model is called *proper interval bigraph* ( $\mathcal{PIB}$ ).

A graph  $G$  is a *permutation graph* if there are two permutations  $\pi_1$  and  $\pi_2$  of  $V(G)$  such that there is an edge  $uv$  if and only if the vertices  $u$  and  $v$  appear in different relative order in  $\pi_1$  and in  $\pi_2$ . A graph  $G$  is a *bipartite permutation graph* ( $\mathcal{BPG}$ ) if it is bipartite and a permutation graph.

The functions  $f_G, l_G : V(G) \rightarrow V(G)$  (*first* and *last*, respectively) are defined, for an interval bigraph  $G$  with order  $<_G$ , as follows:  $f_G(v)$  is the first vertex adjacent to  $v$  using order  $<_G$ ; and  $l_G(v)$  is the last vertex adjacent to  $v$  using order  $<_G$ .

## 7.2. Biclique graph of an Interval bigraph

In this section we study the bicliques of an interval bigraph  $G = (A \cup B, E)$  with order  $<_G$ . Given a set  $S$  of vertices of  $G$ , let  $f_A(S)$  and  $l_A(S)$  be, respectively, *first* and *last* vertices of  $S$  in  $A$ , with respect to  $<_G$ , and  $f_B(S)$  and  $l_B(S)$  be, respectively, *first* and *last* vertices of  $S$  in  $B$ , with respect to  $<_G$ .

Let  $P$  and  $Q$  be two bicliques of an interval bigraph  $G = (A \cup B, E)$  with order  $<_G$ . Define the partial order  $\prec$  as follows:  $P \prec Q$  if  $l_A(P) <_G f_A(Q)$  and  $l_B(P) <_G f_B(Q)$ . We say that  $P$  and  $Q$  are *comparable* if  $P \prec Q$  or  $Q \prec P$ .

The following lemma concludes that, given two non-intersecting bicliques, the relative order of the first vertices of its parts in  $A$  is the same as the relative order of the first vertices of its parts in  $B$ .

**Lemma 7.1.** *Let  $P$  and  $Q$  be two non-intersecting bicliques of an interval bigraph  $G = (A \cup B, E)$  with order  $<_G$ .  $f_A(P) <_G f_A(Q)$  if and only if  $f_B(P) <_G f_B(Q)$ .*

*Proof.* Suppose that  $P \cap Q = \emptyset$  and  $f_A(P) <_G f_A(Q)$ . Note that  $s(f_A(P)) < s(f_A(Q))$ .

Also note that  $s(f_B(P)) < t(f_A(P))$  and  $s(f_A(P)) < t(f_B(P))$ , as the intervals intersect.

Let  $y \in Q \cap A$ . So  $s(f_A(Q)) \leq s(y)$  (by definition of  $f_A(Q)$ ) and then  $s(f_A(P)) < s(y) < t(y)$ . Then, for every  $y \in Q \cap A$ ,  $s(f_B(P)) < t(y)$ , otherwise  $I_y \subset I_{f_A(P)}$ . Note that if  $I_u$  and  $I_v$  are intervals of the same part and  $I_u \subset I_v$ , then  $I_v$  intersects every interval that intersects  $I_u$ , that is, the neighbourhood of  $u$  is contained in the neighbourhood of  $v$ . This implies that every biclique that contains  $u$  also contains  $v$ . So, if  $I_y \subset I_{f_A(P)}$  then  $f_A(P)$  belongs to  $Q$ , which is false.

As  $f_B(P)$  does not belong to  $Q$ , there exists  $y \in Q \cap A$  such that  $t(f_B(P)) < s(y)$ . Also, we have that  $s(y) < t(f_B(Q))$ , as  $y$  belongs to  $Q$ . Then  $t(f_B(P)) < t(f_B(Q))$ .

As  $f_B(Q)$  does not belong to  $P$ ,  $I_{f_B(P)} \not\subset I_{f_B(Q)}$  and  $s(f_B(P)) < t(f_B(P)) < t(f_B(Q))$ , then  $s(f_B(P)) < s(f_B(Q))$  and thus  $f_B(P) <_G f_B(Q)$ .

To prove the converse just change the roles of  $A$  and  $B$ . □

The next lemma shows that the partial order  $\prec$  over the family of bicliques of  $G$  is such that two non-intersecting bicliques of  $G$  are comparable.

**Lemma 7.2.** *Let  $P$  and  $Q$  be two bicliques of an interval bigraph  $G = (A \cup B, E)$  with orders  $<_G$  and  $\prec$ .  $P$  and  $Q$  do not intersect if and only if  $P$  and  $Q$  are comparable.*

*Proof.* Suppose  $P$  and  $Q$  are comparable. Without loss of generality, suppose that  $P \prec Q$ , that is,  $l_A(P) <_G f_A(Q)$  and  $l_B(P) <_G f_B(Q)$ . It follows that there is no vertex that belongs to both  $P$  and  $Q$  and then  $P \cap Q = \emptyset$ .

Conversely suppose  $P \cap Q = \emptyset$ . Without loss of generality, suppose that  $f_A(P) <_G f_A(Q)$ .

Suppose also, in order to get a contradiction, that  $f_A(Q) \leq_G l_A(P)$ . As  $P \cap Q = \emptyset$ ,  $f_A(Q) \neq l_A(P)$ , so  $f_A(Q) <_G l_A(P)$ . The vertex  $v = f_A(Q) \notin P$ , then there is a vertex  $w \in P \cap B$  such that  $v$  and  $w$  are not adjacent. Then either  $t(v) < s(w)$  or  $t(w) < s(v)$ . If  $t(w) < s(v)$ , as  $v <_G l_A(P)$ ,  $s(v) < s(l_A(P))$  and then  $t(w) < s(v) < s(l_A(P))$ , which is not possible, since  $w$  is adjacent to  $l_A(P)$ . Hence,  $t(v) < s(w)$ . As  $f_A(P)$  is adjacent to  $w$  and  $s(f_A(P)) < s(v) < t(v) < s(w)$ , it follows that  $t(f_A(P)) > s(w) > t(v)$ . So  $I_v \subset I_{f_A(P)}$  and then  $f_A(P)$  is in biclique  $Q$ . Consequently,  $l_A(P) <_G f_A(Q)$ .

By Lemma 7.1, and using the same argument, we get that  $l_B(P) <_G f_B(Q)$ . Therefore,  $P$  and  $Q$  are comparable.  $\square$

Using the result of the previous lemma, we can state that the biclique graph of an interval bigraph is a co-comparability graph.

**Theorem 7.3.**  $KB(\mathcal{IBG}) \subset$  co-comparability graphs.

*Proof.* Let  $G = (A \cup B, E)$  be an interval bigraph with order  $<_G$ . The order  $\prec$  is an order on  $V(KB(G))$  (bicliques of  $G$ ). By Lemma 7.2, two vertices of  $KB(G)$  are adjacent if and only if they are not comparable. Therefore,  $KB(G)$  is a co-comparability graph.

As  $P_3$  is a co-comparability graph but it is not a biclique graph of any graph [GS10],  $KB(\mathcal{IBG}) \subset$  co-comparability graphs.  $\square$

Now we present other properties of the bicliques of an interval bigraph  $G$ . Such properties are related to some special edges of the bicliques of  $G$ , connecting the first vertex of one part to the last vertex in the other part (by order  $<_G$ )

**Lemma 7.4.** *Let  $P$  and  $Q$  be two bicliques of an interval bigraph  $G = (A \cup B, E)$  with order  $<_G$ . The following are all true:*

1.  $f_A(P) <_G f_A(Q) \implies l_B(P) <_G l_B(Q)$ ;
2.  $f_B(P) <_G f_B(Q) \implies l_A(P) <_G l_A(Q)$ ;
3.  $l_A(P) <_G l_A(Q) \implies f_B(P) \leq_G f_B(Q)$ ;
4.  $l_B(P) <_G l_B(Q) \implies f_A(P) \leq_G f_A(Q)$ .

*Proof.* Let  $P$  and  $Q$  be two bicliques of an interval bigraph  $G = (A \cup B, E)$  with order  $<_G$ .

1. As  $f_A(P) <_G f_A(Q)$ , the vertex  $f_A(P)$  does not belong to  $Q$ , so  $t(f_A(P)) < s(l_B(Q))$ . But  $s(l_B(P)) < t(f_A(P))$ , since  $f_A(P)$  and  $l_B(P)$  are adjacent. So  $s(l_B(P)) < t(f_A(P)) < s(l_B(Q))$  and then  $s(l_B(P)) < s(l_B(Q))$ . Consequently  $l_B(P) <_G l_B(Q)$ .
2. Similar to 1. Just change the roles of  $A$  and  $B$ .
3. By 2. above, if  $f_B(Q) <_G f_B(P)$  then  $l_A(Q) <_G l_A(P)$ , so if  $l_A(P) <_G l_A(Q)$  then  $f_B(P) \leq_G f_B(Q)$ .

4. Similar to 3. Just change the roles of  $A$  and  $B$ .

□

Note that Lemma 7.4 states that the edges  $f_A(P)l_B(P)$  and  $f_A(Q)l_B(Q)$  do not cross and can have a vertex in common. The same for edges  $l_A(P)f_B(P)$  and  $l_A(Q)f_B(Q)$ .

Using the previous lemma we can prove that  $KB(\mathcal{IBG}) \subseteq K_{1,4}$ -free graphs.

**Theorem 7.5.** *The biclique graph of an interval bigraph is  $K_{1,4}$ -free.*

*Proof.* Let  $G = (A \cup B, E)$  be an interval bigraph with order  $<_G$ . Let  $S_1, S_2, S_3$ , and  $S_4$  be four non-intersecting bicliques of  $G$  such that  $S_1 \prec S_2 \prec S_3 \prec S_4$ . Suppose  $S_5$  is a different biclique. We will see that  $S_5$  does not intersect all the others four bicliques.

Suppose  $S_5$  intersects  $S_1$ , and suppose, without loss of generality, that  $S_1 \cap S_5 \cap A \neq \emptyset$ . So,  $f_A(S_5) \leq_G l_A(S_1)$ , and then  $f_A(S_5) <_G f_A(S_2)$ . See Figure 7.1.

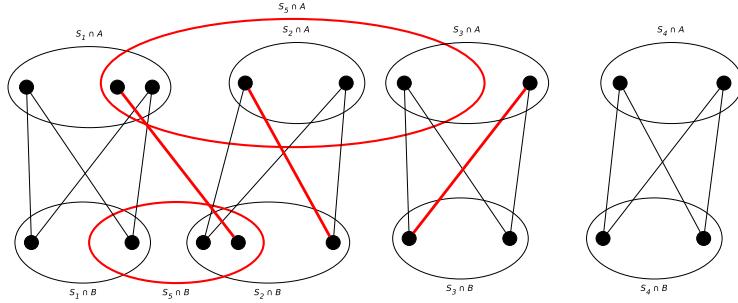


Figura 7.1: Biclique  $S_5$  does not intersect biclique  $S_4$ .

By Lemma 7.4,  $l_B(S_5) <_G l_B(S_2)$  and then  $S_4 \cap S_5 \cap B = \emptyset$ . So, if  $S_5$  does intersect  $S_4$ , the vertices of that intersection are in part  $A$ .

Since  $f_B(S_5) \leq_G l_B(S_5)$  and  $l_B(S_2) <_G f_B(S_3)$ ,  $f_B(S_5) <_G f_B(S_3)$ .

By Lemma 7.4,  $l_A(S_5) <_G l_A(S_3)$ . It follows that  $S_5$  does not intersect  $S_4$ .

We conclude that there is no induced subgraph of  $KB(G)$  isomorphic to  $K_{1,4}$ . □

As a corollary of Theorems 7.3 and 7.5 we can state the following theorem, that is one of the main results of this work.

**Theorem 7.6.**  *$KB(\mathcal{IBG}) \subset K_{1,4}$ -free co-comparability graphs.*

### 7.3. The bicliques in a proper interval bigraph

In this section we present some properties of the bicliques of a proper interval bigraph that are used in the next section.

We start with this natural observation.

**Observation 7.7.** *Let  $G = (A \cup B, E)$  be a proper interval bigraph with a proper interval model. Let  $u$  and  $v$  be two vertices of the same part ( $A$  or  $B$ ). Then,  $s(x) < s(y)$  if and only if  $t(x) < t(y)$ .*

We present the next lemma, using the same ideas of Lemma 7.4, for the class of proper interval bigraphs. When  $G$  is a proper interval bigraph, assume that the order  $<_G$  is induced by a proper interval model.

**Lemma 7.8.** *Let  $G = (A \cup B, E)$  be a proper interval bigraph with order  $<_G$ . Let  $P$  and  $Q$  be two bicliques of  $G$ . The following are true:*

1.  $l_A(P) <_G l_A(Q) \implies f_B(P) <_G f_B(Q);$
2.  $l_B(P) <_G l_B(Q) \implies f_A(P) <_G f_A(Q).$

*Proof.*

1. Suppose  $l_A(P) <_G l_A(Q)$ . As  $l_A(Q) \notin P$ , there is a vertex  $w \in P \cap B$  such that  $t(w) < s(l_A(Q))$  or  $t(l_A(Q)) < s(w)$ . As  $l_A(P) <_G l_A(Q)$ , then  $s(l_A(P)) < s(l_A(Q)) < t(l_A(Q))$ . Note that  $I_w$  intersects  $I_{l_A(P)}$  and then,  $s(w) < t(l_A(P))$ .

If  $t(l_A(Q)) < s(w)$  we have that  $s(l_A(P)) < s(l_A(Q)) < t(l_A(Q)) < s(w) < t(l_A(P))$  which implies that  $I_{l_A(P)}$  contains  $I_{l_A(Q)}$ , but  $G$  is a proper interval bigraph. So  $t(w) < s(l_A(Q))$ .

Also, as  $l_A(Q) \in Q$ ,  $s(l_A(Q)) < t(f_B(Q))$ . Then,  $t(w) < t(f_B(Q))$  and, by Observation 7.7,  $s(w) < s(f_B(Q))$ . As  $s(f_B(P)) \leq s(u)$ , for every  $u \in P \cap B$ ,  $s(f_B(P)) \leq s(w) < s(f_B(Q))$ . So  $f_B(P) <_G f_B(Q)$ .

2. Similar to 1. Just change the roles of  $A$  and  $B$ .

□

Observe that, as Lemma 7.4, Lemma 7.8 states that the edges  $f_A(P)l_B(P)$  and  $f_A(Q)l_B(Q)$  do not cross. But in this case they also do not have one vertex in common unless they are the same edge. The same for edges  $l_A(P)f_B(P)$  and  $l_A(Q)f_B(Q)$ .

The next lemma is used to show that the vertices of a biclique in each part are all consecutive.

**Lemma 7.9.** *Let  $G = (A \cup B, E)$  be a proper interval bigraph with order  $<_G$ . Let  $u, v \in A$  and  $w, x \in B$  such that  $u \leq_G v$  and  $w \leq_G x$ . If  $ux, vw \in E$ , then  $yz \in E$  for all  $u \leq_G y \leq_G v$  and all  $w \leq_G z \leq_G x$ .*

*Proof.* Let  $y \in A$  and  $z \in B$  be such that  $u \leq_G y \leq_G v$  and  $w \leq_G z \leq_G x$ . As  $ux \in E$ , (1)  $s(x) < t(u)$ ; as  $vw \in E$ , (2)  $s(v) < t(w)$ ; as  $u \leq_G y \leq_G v$ , (3)  $s(u) \leq s(y) \leq s(v)$ ; as  $w \leq_G z \leq_G x$ , (4)  $s(w) \leq s(z) \leq s(x)$ ; from (2) and (3) it follows that (5)  $s(y) < t(w)$ ; from (3), (4) and Observation 7.7 it follows that (6)  $t(u) \leq t(y)$  and (7)  $t(w) \leq t(z)$ ; from (1) and (6) it follows that (8)  $s(x) < t(y)$  and then (from (4) and (8)) we get (9)  $s(z) < t(y)$ ; from (5) and (7), we get (10)  $s(y) < t(z)$ . Finally, by (9) and (10),  $I_y$  and  $I_z$  intersect and then  $yz \in E$ .  $\square$

**Observation 7.10.** *Given a proper interval bigraph  $G = (A \cup B, E)$  with order  $<_G$ , let  $P$  be a biclique of  $G$ . Every vertex  $u$  such that  $f_A(P) \leq_G u \leq_G l_A(P)$  is in  $P$ . That is,  $P \cap A$  is a set of consecutive vertices. The same for set  $P \cap B$ .*

An ordering  $<$  of the vertices of a bipartite graph  $H$  with bipartition  $(X, Y)$  has the *strong ordering property* if  $u, v \in X$ ,  $w, x \in Y$ ,  $ux, vw \in E(H)$ ,  $u < v$  and  $w < x$  then  $uw, vx \in E(H)$ .

Spinrad, Brandstädt and Stewart proved that a bipartite graph  $G$  is a bipartite permutation graph if and only if there is an ordering of the vertices of  $G$  with the strong ordering property [SBS87, Theorem 1]. As  $\mathcal{BPG} = \mathcal{PIB}$  (proved by Hell and Huang [HH04, Theorem 3.4]), it also holds for  $\mathcal{PIB}$ . By Lemma 7.9, for a proper interval bigraph  $G$ , the order  $<_G$  has the strong ordering property.

**Lemma 7.11.** *Let  $G = (A \cup B, E)$  be a proper interval bigraph with order  $<_G$ . Let  $X \subseteq A$  and  $Y \subseteq B$  be subsets of consecutive vertices.  $S = X \cup Y$  is a biclique of  $G$  if and only if  $l_G(f_A(S)) = l_B(S)$ ,  $l_G(f_B(S)) = l_A(S)$ ,  $f_G(l_A(S)) = f_B(S)$  and  $f_G(l_B(S)) = f_A(S)$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $S$  is a biclique of  $G$ .

By Observation 7.10, since  $f_A(S)$  and  $l_B(S)$  are adjacent, it follows that  $l_B(S) \leq_G l_G(f_A(S))$ . Suppose that  $l_G(f_A(S)) = v$  with  $l_B(S) <_G v$ . As  $l_A(S)$  and  $l_B(S)$  are adjacent, then by Lemma 7.9, all vertices of  $X$  are adjacent to  $v$ . Thus  $S \cup \{v\}$  induces a complete bipartite graph and  $S$  is not a biclique (as it is not maximal). Therefore  $v = l_B(S)$ . The other equalities follow analogously.

( $\Leftarrow$ ) Suppose that  $l_G(f_A(S)) = l_B(S)$ ,  $l_G(f_B(S)) = l_A(S)$ ,  $f_G(l_A(S)) = f_B(S)$  and  $f_G(l_B(S)) = f_A(S)$ .

It follows that  $S$  is a complete bipartite subgraph of  $G$ . Suppose that  $S$  is not maximal. So there is a vertex  $u$  such that  $S \cup \{u\}$  is a complete bipartite subgraph of  $G$ . Without loss of generality, suppose  $u \in A$ . Then  $u$  is before or after the vertices of  $X$  (according to order  $<_G$ ). Suppose  $u$  is before the vertices of  $X$ . So  $f_G(l_B(S)) \leq u < f_A(S)$ , which contradicts the hypothesis. The case in which  $u$  is after the vertices of  $X$  is analogous. We conclude that  $S$  is a biclique of  $G$ .  $\square$

## 7.4. The relationship between $KB(G)$ and $S(G)$

In this section we present an operator that transforms a proper interval bigraph  $G$  into another graph  $S(G)$ , the simplification graph, in which we can identify the bicliques of  $G$ .

We use the graph  $S(G)$  to give a characterization of biclique graphs of proper interval bigraphs.

Let  $G = (A \cup B, E)$  be a proper interval bigraph with order  $<_G$ . The set of *extremal edges* of  $G$  is the set  $T(G) = \{uv \in E(G) \mid u = f_G(v) \text{ and } v = l_G(u)\}$ .

The *simplification graph* of  $G$  is the graph  $S(G)$  with  $T(G)$  as its vertex set and, for edges  $a, b \in T(G)$ , with  $a = uv, b = wx, u, w \in A, v, x \in B$ , there is an edge  $ab$  in  $S(G)$  if and only if  $(u <_G w, x <_G v)$  or  $(u = w)$  or  $(v = x)$ . That is, if the edges  $a$  and  $b$  cross or have a vertex in common. See Figure 7.2 for an example of  $S(G)$ .

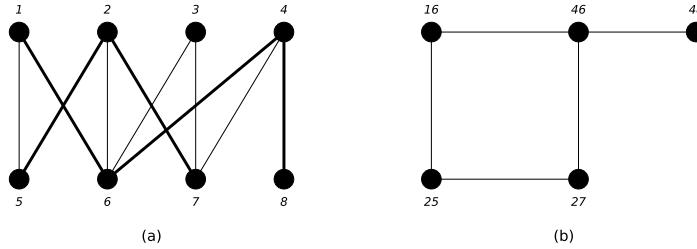


Figura 7.2: (a) Graph  $G$  with the edges of  $T(G)$  (thicker); (b) simplified graph  $S(G)$  of  $G$ .

**Lemma 7.12.** *Let  $G$  be a proper interval bigraph. There is a bijection between the edges of  $S(G)$  and the bicliques of  $G$ .*

*Proof.* Let  $G = (A \cup B, E)$  be a proper interval bigraph with order  $<_G$ .

Let  $X \in A$  and  $Y \in B$  be sets of consecutive vertices. Let  $P = X \cup Y$ . By Lemma 7.11,  $P$  is a biclique of  $G$  if and only if  $l_G(f_A(P)) = l_B(P)$ ,  $l_G(f_B(P)) = l_A(P)$ ,  $f_G(l_A(P)) = f_B(P)$  and  $f_G(l_B(P)) = f_A(P)$ . This is equivalent to affirming that  $a = f_A(P)l_B(P)$  and  $b = l_A(P)f_B(P)$  are in  $T(G)$  and cross or  $f_A(P) = l_A(P)$  or  $f_B(P) = l_B(P)$ . Therefore  $ab \in E(S(G))$ .

Now suppose  $ab \in E(S(G))$ . Suppose also that  $a = uv, b = wx, u, w \in A, v, x \in B$ , and  $u \leq_G w$ . As  $a$  and  $b$  are in  $T(G)$ , then  $l_G(u) = v, l_G(x) = w, f_G(w) = x$  and  $f_G(v) = u$ . Let  $X = \{a' \in A \mid u \leq_G a' \leq_G w\}$  and  $Y = \{b' \in B \mid x \leq_G b' \leq_G v\}$ . By Lemma 7.11,  $S = X \cup Y$  is a biclique of  $G$ .  $\square$

To prove that  $S(G)$  is a proper interval bigraph we need some definitions.

Consider the set  $D(G)$  of vertices of degree 2 in  $G[T(G)]$ . Note that the vertices of  $D(G)$  do not have a false-twin vertex. Let  $G'$  be the graph obtained from  $G$  by adding, for each  $v \in D(G)$ , a false-twin vertex  $v'$ .

Observe that, by the addition of false-twins, the edges of  $T(G)$  that have a vertex in common are substituted in  $T(G')$  by edges that cross and the other crossings are preserved. Then,  $S(G) \cong S(G')$ .

Also note that the bicliques of  $G'$  are associated to the crossing of two edges of  $T(G')$ .

There is an example at Figure 7.3. The graph  $G'$  associated with the graph  $G$  (of Figure 7.2 (a)) is presented in Figure 7.3 (a). Note that vertices 2, 4 and 6 are in  $D(G)$  and then they are duplicated in  $G'$ . The edges of  $T(G')$  are thicker at Figure 7.3 (a) and at Figure 7.3 (b) the graph  $G'[T(G')]$  is presented. The simplified graph  $S(G) \cong S(G')$  is the same graph presented in Figure 7.2 (b).

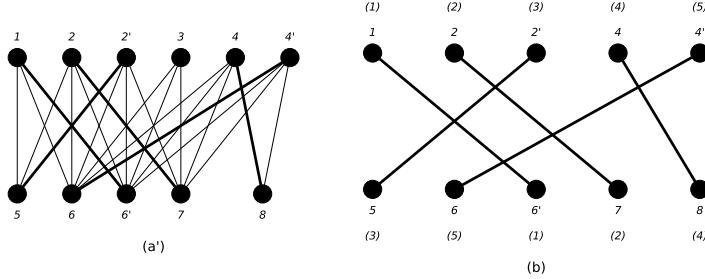


Figura 7.3: (a) graph  $G'$ ; (b) edges of  $T(G')$ .

**Lemma 7.13.** *Given a proper interval bigraph  $G$ ,  $S(G)$  is a proper interval bigraph. Also, for every proper interval bigraph  $H$  there is a proper interval bigraph  $G$  such that  $H \cong S(G)$ .*

*Proof.* Let  $T_f = \{a = uv \in T(G) \mid u \in A \text{ and } f_G(u) = v\}$  and  $T_l = \{a = uv \in T(G) \mid u \in A \text{ and } l_G(u) = v\}$ . By Lemmas 7.4, 7.8 and 7.11, edges of the same set ( $T_f$  or  $T_l$ ) neither cross nor have a vertex in common. So,  $S(G)$  is a bipartite graph.

Now consider the graph  $G'$  defined as above.

Define a bijection  $\psi: T(G) \rightarrow T(G')$  such that  $\psi(a) = a$ , for every  $a \in T(G) \cap T(G')$ ; and  $\psi(a) = a'$ , for every  $a = \{u, v\} \in T(G) \setminus T(G')$ , with  $v \in D(G)$  and  $a' = \{u, v'\}$  ( $v'$  is the false-twin of  $v$ ). See, at Figure 7.3 (a), the false-twin vertex  $2'$  is added and the edge  $\{2, 5\}$  is changed to  $\{2', 5\}$ . The edge  $\{2, 7\}$  is not affected.

We prove that if two edges have a common vertex in  $S(G)$ , then the associated edges of  $S(G')$  do cross in  $G'$ . Let  $ab \in E(S(G))$  be such that  $a = uv$  and  $b = wv$  where  $u <_G w$ . Also, suppose  $a \notin T(G')$ . Then,  $\psi(a) = a' = uv'$  and  $\psi(b) = b$ , where  $v'$  is the false-twin of  $v$ . Observe that  $f_G(v) = u$  and  $l_G(v) = w$ , since  $a, b \in T(G)$  and  $u <_G w$ . So

$l_G(u) = v$  and  $f_G(w) = v$ , and then  $l_{G'}(u) = v'$  and  $v <_{G'} v'$ . So  $a'$  and  $b$  do cross in  $G'$  and  $\psi(a)\psi(b) \in E(S(G'))$ .

Observe that edges  $a, b$  cross in  $G$  or have a vertex in common if and only if  $\psi(a), \psi(b)$  cross in  $G'$ . We conclude that  $\psi$  is an isomorphism and then  $S(G) \cong S(G')$ .

Let  $n = |T(G')|$ . Label the vertices of  $V(G'[T(G')]) \cap V_A(G')$  from 1 to  $n$  according to order  $<_{G'}$ . Label the vertices of  $V(G'[T(G')]) \cap V_B(G')$  with the same label of its neighbor in  $G'[T(G')]$ . At Figure 7.3 (b), giving labels 1, 2, 3, 4 and 5 to the vertices 1, 2, 2', 4 and 4', respectively, yields the labels 3, 5, 1, 2 and 4 to the vertices 5, 6, 6', 7 and 8. Note that there is an edge  $ab$  in  $S(G')$  if and only if the labels of the vertices of  $a$  and  $b$  in  $V_A(G')$  are in reverse order than the labels of the vertices of  $a$  and  $b$  in  $V_B(G')$ , that is,  $a$  and  $b$  cross in  $G'$ . So  $S(G')$  (and also  $S(G)$ ) is a bipartite permutation graph. Consequently,  $S(G)$  is a proper interval bigraph.

Now, consider a proper interval bigraph  $H$  with  $n$  vertices. As  $H$  is also a bipartite permutation graph, there are two permutations,  $\pi_1$  and  $\pi_2$ , of the vertices of  $H$  such that  $uv \in E(H)$  if and only if  $\pi_1(u) < \pi_1(v)$  and  $\pi_2(v) < \pi_2(u)$ .

Define a bipartite graph  $G = (A \cup B, E)$  as follows. Let  $A = \{u_A \mid u \in V(H)\}$  and  $B = \{u_B \mid u \in V(H)\}$ . The edge set has the edges connecting  $u_A$  and  $u_B$ , for every  $u \in V(H)$ , and edges added to satisfy the properties of Lemma 7.9. That is,  $E = T \cup U$ , where  $T = \{u_Au_B \mid u \in V(H)\}$  and  $U = \{u_Av_B \mid \exists u, v, w, x \in V(H) \text{ such that } (\pi_1(w) \leq \pi_1(u) \leq \pi_1(x)) \text{ and } (\pi_2(x) \leq \pi_2(v) \leq \pi_2(w))\}$ . Define order  $<_G$  over  $V(G)$  such that  $u_A <_G v_A$  if and only if  $\pi_1(u) < \pi_1(v)$ ; and  $u_B <_G v_B$  if and only if  $\pi_2(u) < \pi_2(v)$ . The order  $<_G$  has the strong ordering property, so  $G$  is a bipartite permutation graph (also a proper interval bigraph).

Note that the set of extremal edges of  $G$ ,  $T(G) = T$  and that there is a bijection between  $T(G)$  and  $V(H)$  (by definition of  $T$ ). Also note that the edges of  $T(G)$  cross if and only if the corresponding vertices of  $H$  are adjacent. That is,  $S(G) \cong H$ .  $\square$

Since the bicliques of a proper interval bigraph  $G$  are associated with the edges of the simplified graph  $S(G)$ , there exists a close relation between  $L(S(G))$  and  $KB(G)$ .

**Theorem 7.14.** *Given a proper interval bigraph  $G$ ,  $KB(G) \cong (L(S(G)))^2$ .*

*Proof.* By Lemma 7.12, there is a bijection  $\phi: V(L(S(G))) \rightarrow V(KB(G))$ . Let  $a, b \in E(S(G))$  such that  $a = uv$  and  $b = wx$ . Note that  $u, v, w$  and  $x$  are edges of  $T(G)$ , that  $u$  and  $v$  cross or have a vertex in common and that  $w$  and  $x$  cross or have a vertex in common.

If  $ab \in E(L(S(G)))$ ,  $a$  and  $b$  represent two crossings of edges of  $T(G)$  that share an edge (of  $T(G)$ ). That is, they represent two bicliques of  $G$  that share an edge of  $T(G)$ , so these bicliques intersect and then  $\phi(a)\phi(b) \in E(KB(G))$ .

If  $ab \notin E(L(S(G)))$ ,  $a$  and  $b$  do not intersect, and the two crossings do not share any edge. These two crossings represent two bicliques in  $G$  that do not have an edge of  $T(G)$  in common.

If these two bicliques intersect, then one of  $u$  or  $v$  crosses one of  $w$  or  $x$  (in  $T(G)$ ) (by Lemma 7.2). Suppose that  $u$  and  $x$  cross in  $T(G)$ . So  $c = ux \in E(S(G))$  and then,  $ac$  and  $cb$  are edges of  $L(S(G))$ . That means, an edge of  $KB(G)$  that is not in  $L(S(G))$  is incident to two vertices at distance 2 in  $L(S(G))$ .

On the other hand, if the bicliques represented by  $a$  and  $b$  do not intersect, that is,  $\phi(a)\phi(b) \notin E(KB(G))$ , then there is no biclique that shares an edge of  $T(G)$  with both. So, the distance from  $\phi(a)$  to  $\phi(b)$  is greater than 2.

Therefore,  $\phi$  is an isomorphism and then  $KB(G) \cong (L(S(G)))^2$ .  $\square$

In Figure 7.4 we can see the cases for which there are edges in  $KB(G)$  and not in  $L(S(G))$ . In Figures 7.4 (a) and (c) the edges of  $T(G)$  with 1 or 2 additional crossings and in Figures 7.4 (b) and (d) the associated graph  $KB(G)$ . The thicker and dashed edges represent the edges that are not in  $L(S(G))$  but are in  $KB(G)$ .

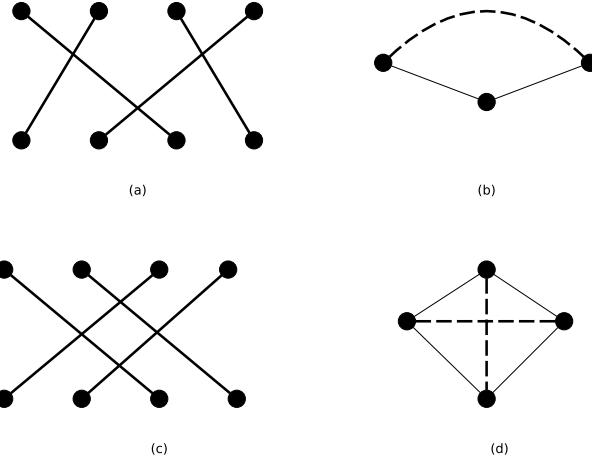


Figura 7.4: (a)  $T(G)$  with 1 new crossing; (b) associated graph  $KB(G)$  of (a); (c)  $T(G)$  with 2 new crossings; (d) associated graph  $KB(G)$  of (c).

Moreover, we use the result of Theorem 7.14 to give a characterization of the class of biclique graphs of proper interval bigraphs.

**Theorem 7.15.**  $KB(\mathcal{PIB}) = (L(\mathcal{PIB}))^2$

*Proof.* Let  $F \in (L(\mathcal{PIB}))^2$ , so there is a proper interval bigraph  $H$  such that  $F \cong (L(H))^2$ . By Lemma 7.13, there is a  $G \in \mathcal{PIB}$  such that  $H \cong S(G)$  and then, by Theorem 7.14,  $(L(H))^2 \cong KB(G)$ . Thus  $F \cong KB(G)$ , and hence  $F \in KB(\mathcal{PIB})$ .

Let  $F \in KB(\mathcal{PIB})$ , so there is a proper interval bigraph  $G$  such that  $F \cong KB(G)$ . Then, by Theorem 7.14,  $F \cong (L(S(G)))^2$  and, by Lemma 7.13,  $F \in (L(\mathcal{PIB}))^2$ .  $\square$

Theorem 7.15 does not give a polynomial time recognition algorithm. However, when we restrict the problem to the subclass of *proper interval bigraphs having acyclic simplification graph* ( $\mathcal{PIB}\text{-ASG}$ ), we obtain a characterization that leads to a polynomial time recognition algorithm of the class  $KB(\mathcal{PIB}\text{-ASG})$ .

Given a proper interval graph  $H$ , let  $(v_1, v_2, \dots, v_n)$  be the vertices of  $H$  in the unique<sup>1</sup> perfect elimination ordering [BLS99, Definition 1.2.2, p. 50]. Let  $(C_1, C_2, \dots, C_k)$  be the ordering of the cliques of  $H$  such that the first vertex of  $C_i$  is before the first vertex of  $C_j$  if and only if  $i < j$ .

A *1-proper interval graph* ( $1\text{-PIG}$ ) is a proper interval graph with the cliques  $(C_1, C_2, \dots, C_k)$  in the above order are such that  $|C_i \cap C_{i+1}| > 1$ , for  $1 \leq i \leq k-1$ , and  $|C_i \cap C_{i+2}| = 1$ , for  $1 \leq i \leq k-2$ .

**Theorem 7.16.** *A graph  $H$  is a biclique graph of some  $G \in \mathcal{PIB}\text{-ASG}$  if and only if  $H$  is a 1-proper interval graph.*

*Proof.* Let  $G \in \mathcal{PIB}\text{-ASG}$ . Note that every acyclic proper interval bigraph is a caterpillar<sup>2</sup>. So, if  $H \cong KB(G)$ , then  $H \cong (L(S(G)))^2$  and  $(L(S(G)))^2$  is a 1-proper interval graph. At Figure 7.5 there is a caterpillar ( $S(G)$ ) and the square of its line graph.

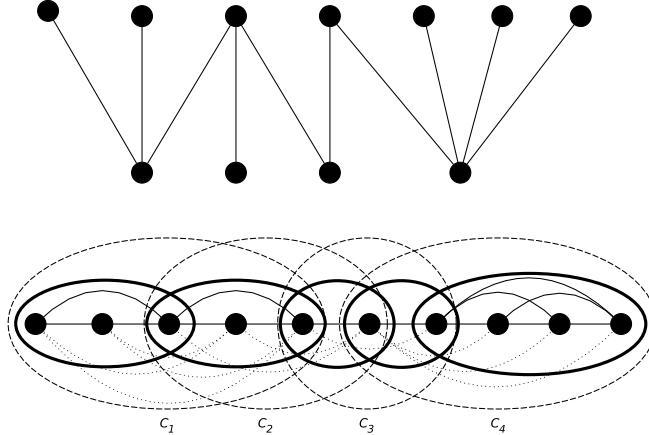


Figura 7.5:  $S(G)$  (a caterpillar) and  $(L(S(G)))^2$  with the edges of  $L(S(G))$  solid (horizontally and above) and the edges added by the square operation dotted (below). The cliques of  $(L(S(G)))^2$ ,  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  (dashed) and the cliques of  $L(S(G))$  (thick) are marked.

<sup>1</sup>If desconsidering twin vertices.

<sup>2</sup>A caterpillar graph is a tree such that there is a central path and the other vertices are leafs adjacent to that path

On the other hand, let  $H$  be a 1-proper interval graph with ordered cliques  $(C_1, C_2, \dots, C_k)$ .

Let  $z_i$  be the first vertex of  $C_{i+1}$ , for  $1 \leq i \leq k-1$ . Let  $z_k$  be the last vertex of  $C_{k-1}$ . Let  $a_i = C_i \cap C_{i+1}$ , for  $1 \leq i \leq k-1$ . Let  $a_0 = (C_1 \setminus a_1) \cup \{z_1\}$  and  $a_k = (C_k \setminus a_{k-1}) \cup \{z_k\}$ .

The graph  $Q = H[a_0] \cup H[a_1] \cup \dots \cup H[a_k]$  is such that  $Q^2 \cong H$ , that is  $Q$  is a square root of  $H$ .

Note that  $\mathcal{C} = \{H[a_0], \dots, H[a_k]\}$  is a family of cliques of  $Q$  such that every vertex of  $Q$  belongs to one or two of these sets, these cliques cover the edges of  $Q$  and the pairwise intersection between these cliques is either empty or has size one. That is,  $\mathcal{C}$  is a clique partition of the edges of  $Q$ . Then  $Q$  is the line graph of some graph  $R$  [Har69, Theorem 8.4, p. 74].

Thus,  $H \cong (L(R))^2$ .

Observe that  $R$  is a caterpillar, and then, it follows that  $R$  is a proper interval bigraph. So, by Lemma 7.13, there exists a proper interval bigraph  $G$  such that  $R \cong S(G)$ . Consequently  $H \cong (L(S(G)))^2 \cong KB(G)$ . So  $H$  is the biclique graph of some  $G \in \mathcal{PIB}\text{-ASG}$ .  $\square$

As a corollary, we obtain an algorithm for solving the problem of recognizing biclique graphs of  $\mathcal{PIB}\text{-ASG}$ .

**Theorem 7.17.** *The problem of deciding if a given graph  $H$  is the biclique graph of some  $G \in \mathcal{PIB}\text{-ASG}$  is polynomially solvable.*

*Proof.* By Theorem 7.16 we just need to check if  $H$  is a 1-proper interval graph. This can be done in polynomial time.  $\square$

# 8 Grafo biclique-completo

Como mencionamos en la introducción, dentro del estudio de un operador  $H$ , se estudian las clases operador-inversa, es decir, se considera el problema de decidir si un grafo pertenece a la clase  $H^{-1}(\mathcal{C})$ , donde  $\mathcal{C}$  es una clase de grafos. Este estudio consiste en caracterizar esa clase, o dar un algoritmo polinomial de reconocimiento o probar la **NP-completitud** del problema.

Varias clases fueron estudiadas en el contexto del operador clique. En particular, la clase clique inversa de la clase de grafos completos.

Siguiendo las mismas ideas, en este trabajo estudiamos la clase biclique inversa de los completos y obtenemos un resultado análogo al del operador clique, probando que decidir si un grafo pertenece a  $KB^{-1}(\text{completos})$  es co-NP-completo.

Decimos que un grafo es *biclique-completo* si su grafo biclique es completo.

Llamamos al problema de decidir si un grafo es biclique-completo *Problema KBCOMPLETE*.

## 8.1. $KB^{-1}(\text{complete})$

In this section we study the *KBCOMPLETE* problem, that is, the problem of, given a graph  $G$ , decide if  $KB(G)$  is a complete graph, i.e, if  $G$  has a complete-biclique graph. We remark that since a graph can have an exponential number of bicliques,  $KB(G)$  can not be built in polynomial time. For that reason, we do not know if the problem is even in **NP**, since *a priori*, a solution would be the list of bicliques.

Then, we prove that this problem is co-NP-complete. The fact that it is in co-NP follows from the fact that a negative certificate consists of two bicliques that do not intersect. To prove that it is co-NP complete, we use a reduction from satisfiability problem. Moreover, we prove that the problem is still co-NP-complete when  $G$  is a bipartite graph.

The *SAT* problem consists of determining if there exists a valuation (an assignment of TRUE or FALSE to the variables) that satisfies a given Boolean formula (conjunctions of one or more disjunctions of one or more literals).

**Teorema 8.1.** *The KBCOMPLETE problem is co-NP-complete.*

*Proof.* We prove the result by given a reduction from *SAT* to *KBCOMPLETE*. Consider an instance of *SAT*:

Let  $\{L_i, 1 \leq i \leq p\}$  be the family of clauses,  $X_1, \dots, X_n$  variables, each  $L_i$  with  $q_i$  literals, and  $I$  a conjunctive normal form, i.e.,  $I = (l_1^1 \vee l_2^1 \vee \dots \vee l_{q_1}^1) \wedge (l_1^2 \vee l_2^2 \vee \dots \vee l_{q_2}^2) \wedge \dots \wedge (l_1^p \vee l_2^p \vee \dots \vee l_{q_p}^p)$ .

We construct a bipartite graph  $V(G) = (V_1, V_2, E)$  such that  $KB(G)$  contains two biclique that do not intersect if and only if the given instance of *SAT* is satisfiable.

The part  $V_1$  is formed by the union of two sets  $X$  and  $C$ , plus two vertices  $u_1, u_2$ .

- $X$ : the set  $X$  contains, for each variable  $X_i$ , two vertices  $x_i, \bar{x}_i$ .
- The set  $C$  contains, for each clause  $L_i$ , a vertex  $c_i$

The part  $V_2$  is formed by the union of two sets  $X'$  and  $B$ , plus three vertices  $\alpha, u'_1$  y  $u'_2$ .

- $X'$ : the set  $X'$  contains, for each variable  $X_i$ , two vertices  $x'_i, \bar{x}'_i$ .
- The set  $B$  contains, for each variable  $X_i$ , a vertex  $b_i$

Now we define the adjacencies:

- Every vertex  $x_i \in X$  is adjacent to  $x'_j \in X'$  for every  $j$ , and to  $\bar{x}'_j \in X'$ , for  $j \neq i$  (i.e,  $x_i$  is adjacent to  $X' - \{\bar{x}'_i\}$ ).
- Every vertex  $\bar{x}_i \in X$  is adjacent to  $\bar{x}'_j \in X'$  for every  $j$ , and to  $x'_j \in X'$ , for  $j \neq i$  (i.e,  $\bar{x}_i$  is adjacent to  $X' - \{x'_i\}$ ).
- For every  $i$ , the vertices  $x_i, \bar{x}_i \in X$  are adjacent to  $b_j \in B$  for every  $j \neq i$  (ie.  $x_i$  and  $\bar{x}_i$  are adjacent to the vertices of  $B - b_i$ ).
- For every  $i$ , the vertices  $x_i, \bar{x}_i \in X$  are adjacent to  $u'_1$ .
- Every  $c_i \in C$ , is adjacent to  $x'_t$  if and only if  $l_j^i \neq X_t$  for every  $1 \leq j \leq q_i$  and  $c_i$  is adjacent to  $\bar{x}'_t$  if and only if  $l_j^i \neq \bar{X}_t$ , that is,  $c_i$  is adjacent to every vertex of  $X'$  that represents a literal that is not in the corresponding clause.
- Every  $c_i \in C$ , is adjacent to every  $b_j \in B$ .
- Every  $c_i \in C$ , is adjacent to  $\alpha, u'_1$  y  $u'_2$ .
- vertex  $u_1$  is adjacent to every vertex of  $X'$ , every vertex of  $B$ ,  $\alpha$  and  $u'_1$ .
- vertex  $u_2$  is adjacent to every vertex of  $B$ ,  $\alpha$  and  $u'_2$ .

We conclude the construction of  $G$ . Next, we prove that if  $\nu$  is a  $1 - 0$  valuation that satisfies the clauses of  $I$ , then there exist two bicliques in  $G$  that do not intersect.

Let  $X_{true} = \{x_i \text{ such that } \nu(X_i) = 1\}$ ,  $\bar{X}_{true} = \{\bar{x}_i \text{ such that } \nu(X_i) = 0\}$ ,  $X'_{true} = \{x'_i \text{ such that } \nu(X_i) = 1\}$ , and  $\bar{X}'_{true} = \{\bar{x}'_i \text{ such that } \nu(X_i) = 0\}$ . Observe that  $x_i \in X_{true}$  if and only if  $x'_i \in X'_{true}$  if and only if  $\bar{x}_i \notin \bar{X}_{true}$  if and only if  $\bar{x}'_i \notin \bar{X}'_{true}$ . Then, we affirm that  $B_1 = \{X_{true}, \bar{X}_{true}, u_1\} \cup \{X'_{true}, \bar{X}'_{true}, u'_1\}$  is a biclique in  $G$ . Clearly  $B_1$  is a complete bipartite subgraph of  $G$ . We prove that it is maximal. Let  $B_3$  be a biclique that contains  $B_1$ . Then  $u_2, u'_2, \alpha \notin B$ . Since  $\nu$  is a valuation, for every  $i$ , either  $x_i \in B_1$  or  $\bar{x}_i \in B_1$ . Then, for every  $i$ ,  $b_i \notin B_3$ . Finally, let  $c_i \in C$ . Since  $\nu$  satisfies  $C_i$  there is a literal  $l_j^i$  that is true. If  $l_j^i = X_j$  then  $\nu(X_j) = 1$  and then  $x_j \in B_1$ . If  $l_j^i = \bar{X}_j$  then  $\nu(X_j) = 0$  and then  $\bar{x}_j \in B_1$ . In any case, there is a vertex in  $B_1$  not adjacent to  $c_i$  and therefore  $c_i \notin B_3$ . Then  $B_3 = B_1$ .

On the other hand,  $B_2 = \{C, u_2\} \cup \{\alpha, B, u'_2\}$  is a biclique in  $G$ . Then,  $G$  contains two bicliques  $B_1$  and  $B_2$  that do not intersect.

Conversely, we prove that if  $G$  contains two disjoint bicliques, then there exists a valuation  $\nu$  that satisfies clauses of  $I$ .

Let  $B_1, B_2$  be such bicliques. Clearly, exactly one of them (suppose  $B_2$ ) contains  $u_2$ . Otherwise,  $u'_1 \in B_1, B_2$  since  $u'_1$  is adjacent to  $V_1 - \{u_2\}$ .

We list the bicliques that contain  $u_2$ . Since  $N(u_2) \subset N(c_i)$  for every  $c_i \in C$ , every biclique that contains  $u_2$  also contains  $C$ .

- $B_{21} = \{C, \{u_2\}\} \cup \{\{\alpha\}, B, \{u'_2\}\}$
- $B_{22} = \{C, \{u_1\}, \{u_2\}\} \cup \{\{\alpha\}, B\}$
- $B_2^{i_1..i_k} = \{Y_{i_1..i_k}, C, \{u_1\}, \{u_2\}\} \cup \{B^{i_1..i_k}\}$  where  $Y_{i_1..i_k} = X - \{x_{i_1}..x_{i_k}, \bar{x}_{i_1}..\bar{x}_{i_k}\}$ , and  $B^{i_1..i_k} = \{b_{i_1}..b_{i_k}\}$

Since  $C, u_2 \notin B_1$  and  $N(u'_1) = V_1 - \{u_2\}$ ,  $N(u'_2) = \{u_2\} \cup C$ , then  $u'_1 \in B_1$  and  $u'_2 \notin B_1$ . It follows that  $u_1 \in B_1$ . Therefore,  $B_2 \neq B_{22}$ . Suppose  $B_1$  contains  $\alpha$ . Then the unique biclique that contains  $\alpha$  and does not contain  $u_2$  and any vertex of  $C$  is  $\{u_1\} \cup \{u'_1\}, \{\alpha\}, X', B\}$ . But this biclique intersects every biclique containing  $u_2$ . Then  $\alpha \notin B_1$ .

It follows that  $B_1$  is of the form  $B_1^{i_1..i_k} = \{Y_{i_1..i_k}, \{u_1\}\} \cup \{Y'_{i_1..i_k}, \{u'_1\}, B^{i_1..i_k}\}$  where  $Y'_{i_1..i_k} = \{x'_{i_1}..x'_{i_k}, \bar{x}'_{i_1}..\bar{x}'_{i_k}\}$  or  $B_1 \cap B = \emptyset$  and  $B_1 = B^{Y, Y'} = \{Y, \{u_1\}\} \cup \{Y', \{u'_1\}\}$  where  $Y \subset X, Y' \subset X'$ . Also for every  $i$ ,  $x'_i \notin Y' \vee \bar{x}'_i \notin Y'$ , otherwise  $x_i, \bar{x}_i \notin B^{Y, Y'}$  and therefore  $b_i \in B^{Y, Y'}$ .

Finally, observe that every biclique of the form  $B_1^{i_1..i_k}$  intersects  $B_2^{i_1..i_k}$  and  $B_{22}$  in  $u_1$ , and intersects  $B_{21}$  in some vertex of  $B$ . Then,  $B_1$  is of the form  $B_1 = B^{Y, Y'} = \{Y, \{u_1\}\} \cup \{Y', \{u'_1\}\}$ .

But this implies that  $B_2 = B_{21}$ , since  $B^{i_1..i_k}$  and  $B_{22}$  constrain  $u_1$ .

We define the valuation  $\mu$  as  $\mu(X_i) = 1$  if and only  $x'_i \in Y'$  and  $\mu(X_i) = 0$  if  $\overline{x'_i} \in Y'$ . Recall that if  $x'_i \in Y'$  then  $\overline{x'_i} \notin Y'$  and then  $\mu$  is well defined.

We need to prove that  $\mu$  is a valuation that satisfies every clause  $L_i$  of  $I$ . Let  $L_i$  be a clause. Suppose that for every literal  $l$  of  $L_i$ ,  $l \neq X_j$  and  $l \neq \overline{X_k}$  for every  $x'_j$  and  $\overline{x'_k}$  of  $Y'$ . Then, vertex  $c_i$  is adjacent to every vertex of  $Y'$  and then  $c_i \in B_1$ , what is not possible. Then, there exists a vertex  $x'_j$  or  $\overline{x'_k}$  in  $Y'$  such that  $L_i$  contains a literal  $l = X_j$  or  $l = \overline{X_k}$ . In either case,  $\mu(l) = 1$  and  $L_i$  is satisfied.

We finish the proof showing that if there are two disjoint bicliques in  $G$ , then  $\mu$  is a TRUE valuation.

□

## 9 Conclusiones

Creemos que este trabajo constituye un buen aporte al estudio de problemas clásicos de grafos, adaptados al contexto de bicliques. Especialmente a una mejor comprensión del grafo biclique, lo que podría ayudar al problema de reconocimiento general de un grafo biclique.

El principal aporte es el estudio del operador biclique aplicado a algunas clases de grafos, fundamentalmente a la clase Split. Para esta clase de grafos (biclique de split) obtuvimos muchas propiedades, entre las que podemos destacar la prueba de la Conjetura de Groshaus-Montero [GM17] y cotas para su conectividad. Además resolvimos el problema de reconocimiento de grafos bicliques para una subclase de los grafos Split: *Split Separables Anidados* donde cada parte contiene al menos 3 vértices, elaborando un algoritmo polinomial de reconocimiento. Para trabajos futuros queda pendiente estudiar más propiedades de los grafos Split, principalmente las que nos ayuden a clasificar el problema de reconocimiento para grafos Split. Recordemos que nuestra intuición es que este problema es NP-hard. Esto se basa en que al estudiar grafo biclique de split, nos encontramos con un problema que, de ser resuelto, obtendríamos un algoritmo polinomial para los grafos split (este problema es el que origina las condiciones para la subclase split separable anidados).

Un estudio similar se realizó al analizar el operador biclique aplicado a la clase bigrafos de intervalo (BI). Si bien no está caracterizada la clase, probamos que  $KB(BI)$  es subconjunto propio de la clase de co-comparabilidad libres de  $K_{1,4}$ . Para una subclase, los grafos bipartitos de permutación (BIP), probamos que  $KB(BIP)$  es igual a la clase de los cuadrados de los grafos línea de BIP, o sea,  $KB(BIP) = (L(BIP))^2$  y para una subclase de BIP encontramos un algoritmo polinomial de reconocimiento para los grafos biclique. Queda como problema abierto caracterizar la subclase  $KB(BI)$  dentro de los co-comparabilidad libres de  $K_{1,4}$ . También, la búsqueda de un algoritmo polinomial para el reconocimiento de los grafos BIP.

También estudiamos el operador biclique aplicado a la clase de los bipartitos cordales. Por un lado caracterizamos la clase de grafos biclique de los grafos bipartitos cordales libres de dominó. Por otro lado, proporcionamos una propiedad para los grafos biclique de los bipartitos cordales que no son necesariamente cordales. Para trabajos futuros queda pendiente buscar algoritmos polinomiales para subclases de los bipartitos cordales y, lo que parece bastante complejo, lograr clasificar el problema de reconocimiento para grafos bipartitos cordales en general.

Otro aporte que hicimos en este sentido es probar que las clases de grafos biclique  $KB(\text{Bipartito } C_4\text{-free})$ ,  $KB(\text{Split})$  y de los grafos clique no tienen ningún subgrafo inducido prohibido. Es decir, todo grafo  $H$  es subgrafo inducido de un grafo de estas clases. En particular, construimos en tiempo polinomial un grafo  $G$  (de cada una de estas clases) del cual  $H$  es subgrafo inducido. Esto muestra que no podemos buscar caracterizaciones de estas clases por subgrafos prohibidos y ratifica la propiedad de no hereditariedad de los grafos biclique y grafos clique, que ya conocíamos.

Por otra parte contribuimos al problema inverso, o sea, el problema de decir qué grafos son grafos biclique de algún grafo de cierta clase  $\mathcal{A}$ . Obtuvimos que cuando  $\mathcal{A}$  es la clase de los grafos completos, decidir si un grafo pertenece a  $KB^{-1}(\mathcal{A})$  es co-NP-completo (lo que es análogo al resultado obtenido para el operador clique). Queda abierto el estudio del reconocimiento de la clase biclique inversa de otras clases de grafos  $\mathcal{A}$ .

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