Multiplicidad de soluciones para ecuaciones tipo Yamabe en variedades

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Multiplicity of solutions for Yamabe-type equations on manifolds

Abstract

Let (M, g) be a closed n-dimensional Riemannian manifold. The Yamabe problem lies in finding a metric conformal to g with constant scalar curvature. The answer is now known to be yes, and it was proved by Yamabe, Trudinger, Aubin and Schoen. The conformal metric $\tilde{g} = u^{p-2}g$ has constant scalar curvature λ if and only if u satisfies the Yamabe equation:

$$\frac{-4(n-1)}{n-2}\Delta_g u + S_g u = \lambda u^{\frac{n+2}{n-2}}$$

where S_g is the scalar curvature of g, Δ_g is the Laplace-Beltrami operator of g and $\lambda \in \mathbb{R}$ is any constant. In the works of Yamabe [40], Trudinger [39], Aubin [3] and Schoen [37] it was proved that the Yamabe equation always has at least one positive solution. We will study mutiplicity results for Yamabe-type equations.

In the first place, we suppose that Ω is a region of $\mathbb{S}^3$ which is invariant by the natural $\mathbb{T}^2$-action and we study the multiplicity of positive solutions of the equation:

$$\Delta_{\mathbb{S}^3} u = -(u^5 + \lambda u) \quad \text{on } \Omega, \tag{1}$$

that vanish on the boundary of Ω, where $\Delta_{\mathbb{S}^3}$ is the Laplace-Beltrami operator of the round metric in $\mathbb{S}^3$. H. Brezis and L. A. Peletier in [14] consider the case in which Ω is invariant by the $SO(3)$-action, namely, when Ω is a spherical cap. We show that the number of solutions of (1) increases as $\lambda \to -\infty$, giving an answer of a particular case of an open problem proposed by H. Brezis and L. A. Peletier in [14].

In the second place, we study a Yamabe-type equation on a product manifold. Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ and $x_0 \in M$ be an isolated local maximum or minimum of the scalar curvature S_g of g. For any positive integer k we prove that if $\epsilon > 0$ is small enough and $q < \frac{n+2}{n-2}$, then the subcritical equation

$$-\epsilon^2 \Delta_g u + (1 + \epsilon^2 \lambda S_g) u = u^q$$

has a positive solution u_k which concentrates around x_0, for those values of λ such that a constant β_λ is non-zero. This provides solutions for the Yamabe equation on Riemannian products $(M \times N, g + \epsilon h)$, where (N, h) is a Riemannian manifold with constant positive scalar curvature.

Multiplicidad de soluciones para ecuaciones tipo Yamabe en variedades

Resumen

Sea (M, g) una variedad riemanniana cerrada de dimensión n. El problema de Yamabe radica en encontrar una métrica conforme a g con curvatura escalar constante. Se sabe que la respuesta es sí, y fue probado por Yamabe, Trudinger, Aubin y Schoen. La métrica conforme $\tilde{g} = u^{p-2}g$ tiene curvatura escalar constante λ si y solo si u satisface la ecuación de Yamabe:

$$\frac{-4(n-1)}{n-2}\Delta_g u + S_g u = \lambda u^{\frac{n+2}{n-2}}$$

donde S_g es la curvatura escalar de g, Δ_g es el operador de Laplace-Beltrami respecto g y λ es cualquier constante en $\mathbb{R}$. En los trabajos de Yamabe [40], Trudinger [39], Aubin [3] y Schoen [37] se prueba que la ecuación de Yamabe siempre tiene al menos una solución positiva. En esta tesis obtenemos resultados sobre multiplicidad de soluciones de ecuaciones tipo Yamabe.

En primer lugar, suponemos que Ω es una región de $\mathbb{S}^3$ que es invariante por la acción natural de $\mathbb{T}^2$ y estudiamos la multiplicidad de soluciones positivas de la ecuación:

$$\Delta_{\mathbb{S}^3} u = -(u^5 + \lambda u) \quad \text{en } \Omega, \tag{2}$$

que se anulen en el borde de Ω, donde $\Delta_{\mathbb{S}^3}$ es el operador de Laplace-Beltrami respecto de la métrica redonda de $\mathbb{S}^3$. H. Brezis y L. A. Peletier en [14] consideran el caso en el que Ω es invariante por $SO(3)$, es decir, cuando Ω es un casquete esférico. En este trabajo mostramos que el número de soluciones de (2) aumenta cuando $\lambda \to -\infty$, dando una respuesta a un caso particular de un problema abierto propuesto por H. Brezis y L. A. Peletier en [14].

En segundo lugar, estudiamos la ecuación de Yamabe en una variedad producto. Sea (M, g) una variedad riemanniana cerrada de dimensión $n \geq 3$ y $x_0 \in M$ sea un máximo o mínimo local aislado de la curvatura escalar S_g de g. Demostramos que para cualquier entero positivo k, si $\epsilon > 0$ es suficientemente chico y $q < \frac{n+2}{n-2}$, entonces la ecuación subcrítica

$$-\epsilon^2 \Delta_g u + (1 + \epsilon^2 \lambda S_g) u = u^q$$

tiene una solución positiva u_k que se concentra alrededor de x_0, para los valores de λ que hacen que cierta constante β_λ no sea cero. Esto proporciona soluciones a la ecuación de Yamabe en productos riemannianos $(M \times N, g + \epsilon h)$, donde (N, h) es una variedad riemanniana con curvatura escalar positiva constante.

Contents

Introduction

A basic question in differential geometry is to find canonical metrics on a given manifold M. For example, if the dimension of M is 2, the Uniformization Theorem states that every simply connected Riemann surface is conformally equivalent to the open unit disk, the complex plane, or the Riemann sphere. Then in a given conformal class one can find a metric of constant Gaussian curvature (for a proof see [19]). In higher dimiensions one would consider the scalar curvature, which is the average curvature of the metric at a point. Recall that two metrics $\tilde{g}$ and g are said to be conformal if $\tilde{g} = e^{2u} g$ for some smooth function u. The Yamabe problem lies in finding for any closed Riemannian manifold (M, g) of dimension $n \geq 3$ a conformal metric $\tilde{g}$ of constant scalar curvature. The Yamabe problem can be viewed as a natural uniformization question for higher dimensions.

Let (M, g) be a smooth, connected and compact Riemannian manifold without boundary. The Yamabe problem can be reduced to the solvability of a certain semilinear elliptic equation. To that end, let us write $\tilde{g} = e^{2u} g$ with $u \in C^\infty(M)$. Let S and $\tilde{S}$ denote the scalar curvatures of (M, g) and $(M, \tilde{g})$ respectively. The relation between them is given by

$$\tilde{S} = e^{-2u}(S + 2(n-1)\,\Delta u - (n-1)(n-2)\,|\nabla u|^2),$$

where Δ is the Laplace-Beltrami operator of the metric g. The above formula simplifies if we put $\tilde{g} = u^{p-2} g$ with $p = \frac{2n}{n-2}$:

$$\tilde{S} = u^{-(p-1)}\,(S\ u - 4\,\frac{n-1}{n-2}\,\Delta u).$$

Hence $\tilde{g}$ has constant scalar curvature λ if and only if u satisfies the *Yamabe equation*:

$$-a\,\Delta u + S\ u = \lambda\, u^{p-1} \tag{3}$$

where $a = a_n = 4\dfrac{n-1}{n-2}$. This can be seen as a nonlinear eigenvalue problem. In fact, the way to prove that the equation $-a\,\Delta u + S\ u = \lambda\, u^q$ has a solution depends strongly on q. When $q = 1$, the equation is just the linear eigenvalue problem for $-a\Delta + S$. When q is close to 1, its behavior is similar to that of the eigenvalue problem. If q is very large however, linear theory is no longer useful. It turns out that the exponent in the Yamabe equation is the critical value below which the equation can be solved by classical methods and above which it may be unsolvable.

Yamabe observed that equation (3) is the Euler-Lagrange equation for the functional

$$Q(\tilde{g}) = \frac{\int_M S_{\tilde{g}} dV_{\tilde{g}}}{(\int_M dV_{\tilde{g}})^{2/p}}.$$

where $\tilde{g}$ is allowed to vary over metrics conformally equivalent to g. To see this, observe that Q can be written as $Q(\tilde{g}) = Q(\phi^{p-2} g) = Q_g(\phi)$, where

$$Q_g(\phi) = \frac{E(\phi)}{||\phi||_p^2},$$

$$E(\phi) = \int_M a\, |\nabla\phi|^2 + S\ \phi^2\ dV_g, \qquad ||\phi||_p = \Big(\int_M |\phi|^p\ dV_g\Big)^{1/p}.$$

Then for any $\psi \in C^\infty(M)$, integration by parts yields

$$\frac{d}{dt} Q_g(\phi + t\psi)\Big|_{t=0} = \frac{2}{||\phi||_p^2} \int_M (-a\,\Delta\phi + S\,\phi + \ ||\phi||_p^{-p}\ E(\phi)\ \phi^{p-1})\ \psi\ dV_{\tilde{g}}.$$

Thus ϕ is a critical point of Q_g if and only if it satisfies the Yamabe equation (3) with $\lambda = E(\phi)/||\phi||_p^p$. Since by Hölder's inequality $|\int_M S\phi^2|$ is bounded by a multiple of $||\phi||_p^2$, it follows that Q_g (and thus Q) is bounded below. We denote by $[g]$ the family of conformal metrics to g, and let

$$\begin{aligned} Y_g(M) &= \inf\{Q(\tilde{g}) : \tilde{g} \in [g]\} \\ &= \inf\{Q_g(\phi) : \phi \text{ a smooth, positive function on } M\}. \end{aligned} \tag{4}$$

This constant $Y_g(M)$ is an invariant of the conformal class $[g]$, called the Yamabe invariant. Its value is central to the analysis of the Yamabe problem.

In 1960, Yamabe claimed to have found a solution to this problem in [40]. However, Yamabe's proof contains an error that was discovered by Neil Trudinger in 1968. Trudinger was able to use the Yamabe's work in [39] but only by introducing further assumptions on the manifold M. In fact, Trudinger showed that there is a positive constant $Y_0(M)$ such that the result is true when the Yamabe invariant satisfies $Y(M) < Y_0(M)$. In 1976, Aubin improved Trudinger's work by showing that $Y_0(M) = Y(\mathbb{S}^n)$ where the n-sphere is equipped with its standard metric. Moreover, Aubin showed in [3] that if M has dimension $n \geq 6$ and is not locally conformally flat, then $Y_g(M) < Y(\mathbb{S}^n)$. The remaining cases had been resolved in 1984 by Schoen [37], thereby completing the solution to the Yamabe problem. This combined work of Yamabe, Trudinger, Aubin and Schoen gives the existence of a constant scalar curvature metric in every conformal class of Riemannian metrics on a compact manifold M without boundary.

Subsequent developments extended this problem to manifolds with boundary and to non-compact manifolds. One fundamental contribution to the solution of the Yamabe problem on manifolds with boundary is due to Escobar in [17].

When $Y_g(M)$ is non-positive, the Yamabe problem has a unique solution among unit volume metrics. However, in general uniqueness does not hold in the positive case. An important result in the positive case was proved by Obata in [28]. The theorem states that if $u > 0$ satisfies (3) on $\mathbb{S}^n$ for $S = n(n-1)$, then $u^{p-2}g_0 = \phi^* g_0$ for a conformal transformation $\phi : \mathbb{S}^n \to \mathbb{S}^n$. These are the only metrics conformal to the standard one on $\mathbb{S}^n$ that have constant scalar curvature.

Multiplicity results for solutions of the Yamabe problem have been proved in many cases. A result by Pollack shows that every positive conformal class $[g]$ can be C^0-approximated by one with any large number of distinct solutions, see [34]. In [11] Brendle constructed smooth examples where the family of solutions to the Yamabe equation is not compact. As another example, consider the product metric on $\mathbb{S}^{n-1}(1) \times \mathbb{S}^1(L)$. In this case, all solutions of the Yamabe equation are rotationally symmetric. If the length L of the $\mathbb{S}^1$-factor is sufficiently small, then the Yamabe equation has a unique solution (which is constant). On the other hand, the Yamabe equation has many non-minimizing solutions if L is large. We refer to [38] for a detailed discussion of this example.

Let (M, g) be any closed Riemannianan manifold with scalar curvature s_g and (N, h) be a Riemannian manifold of constant positive scalar curvature s_h. We are interested in multiplicity results for the Yamabe equation on the Riemannian product $(M \times N, g + \delta h)$, where δ small enough so that the scalar curvature of the product $s_g + \frac{1}{\delta} s_h$ is positive. Most of the known multiplicity results in these situations use bifurcation theory and assume that s_g is constant (see for example [8], [9], [32]). For the case where the manifold is a product of spheres, Henry and Petean obtained multiplicity of solutions by studying the isoparametric hypersufaces, see [21]. Further, in [29] Otoba and Petean proved multiplicity results for the Yamabe equation on total spaces of harmonic Riemannian submersions of constant positive scalar curvature. On the other hand, De Lima, Piccione and Zedda studied in [16] multiplicity of constant scalar curvature metrics in arbitrary products of compact manifolds by bifurcation theory.

The situation when s_g is non-constant was treated by J. Petean in [33], where it is proved that the Yamabe equation on the Riemannian product $(M \times N, g + \delta h)$ has at least $Cat(M) + 1$ solutions with low energy, where $Cat(M)$ denotes the Lusternik–Schnirelmann-category of M.

Throughout this work we will focus our study on the problem of multiplicity of solutions for the Yamabe equation (3) for two particular cases. In the first place, we will study solutions of a Yamabe-type equation which are invariant by the $\mathbb{T}^2$-action in a particular open subset of $\mathbb{S}^3$. Then we will study positive solutions of the Yamabe equation for the product manifold $(M \times N, g + \epsilon^2 h)$, where (M^n, g) is any closed manifold and (N^m, h) is a manifold of constant positive scalar curvature s_h. We will look for solutions that depend only on the manifold M. Thus the equation becomes a subcritical equation on M.

We provide below a brief discussion of both cases.

0.1 The Yamabe equation on an invariant region of $\mathbb{S}^3$.

It is well known that the sphere $\mathbb{S}^3$ with the round metric g has constant positive scalar curvature. We will study the critical elliptic equation on $\mathbb{S}^3$:

$$\Delta_{\mathbb{S}^3} U = -\left(U^5 + \lambda U\right) \tag{5}$$

where $\Delta_{\mathbb{S}^3}$ is the Laplace-Beltrami operator on $\mathbb{S}^3$. Let Ω be a particular open subset of $\mathbb{S}^3$. We look for positive solutions of (5) on Ω such that

$$U = 0 \quad \text{on } \partial\Omega. \tag{6}$$

Problems of this kind have attracted the attention of several researchers with the aim to understand the existence and properties of the solutions.

H. Brezis and L. Nirenberg considered the problem in $\mathbb{R}^3$:

$$\Delta_{\mathbb{R}^3} U = -\left(U^5 + \lambda U\right), U > 0 \text{ in } B_{R^*}, \quad U = 0 \quad \text{on } \partial B_{R^*} \tag{7}$$

where B_{R^*} is the ball of radius R^* of $\mathbb{R}^3$. Using variational techniques, they obtained in [12] necessary and sufficient conditions on the value of λ for the existence of a solution. This solution was shown to be unique by M. K. Kwong and Y. Li in [25]. This is now called the Brezis-Nirenberg problem and there are numerous results about solutions of this problem in different open subsets of $\mathbb{R}^n$.

The case when Euclidean space is replaced by $\mathbb{S}^3$ was considered in [4], [5], [13] and [14]. Let D_{θ^*} be a geodesic ball in the 3-dimensional sphere centered at the North pole with geodesic radius θ^*. Problem (5)-(6) with $\Omega = D_{\theta^*}$ has been investigated by C. Bandle and R. Benguria in [4], C. Bandle and L.A. Peletier in [5] and H. Brezis and L. A. Peletier in [14] in order to identify the range of values of the parameters θ^* and λ for which there exists a solution. It is well-known that the method of moving planes can be applied when $\theta^* < \pi/2$ (which means that the geodesic ball is contained in a hemisphere) to prove that all solutions are radial (see for instance [30] and [23]). The value $\lambda = -3/4$ is special since $\Delta_{\mathbb{S}^3} - 3/4$ is the conformal Laplacian on $\mathbb{S}^3$ and Eq. (5) is then the Yamabe equation: in this case it is known that there are no nontrivial solutions satisfying (6). The cases $\lambda > -3/4$ and $\lambda < -3/4$ present very different features. We will be interested in the second case. In particular, the situation when $\lambda \to -\infty$ studied by H. Brezis and L. A. Peletier in [14]. The main result in [14] reads:

Theorem (H. Brezis and L. A. Peletier). *Given any $\theta^* \in (\pi/2, \pi)$ and any $k \geq 1$, there exists a constant $A_k > 0$ such that for $\lambda < -A_k$, problem (5)-(6) with $\Omega = D_{\theta^*}$ has at least $2k$ positive radial solutions U such that $U(\text{North pole}) \in (0, |\lambda|^{1/4})$.*

This result was extended by C. Bandle and J. Wei in [6, 7] to general dimensions and non-critical exponents. Also when $\theta^* > \pi/2$ the method of moving planes does not work and in [6] the authors establish the existence of positive nonradial solutions. In [7] the authors proved for balls of geodesic radius $\theta^* > \pi/2$ the existence of radially symmetric clustered layer solutions as $\lambda \to -\infty$.

Inspired by the theorem of H. Brezis and L. A. Peletier, we study problem (5)-(6) for the special case where Ω is a torus invariant region of $\mathbb{S}^3$. The spherical caps D_{θ^*} are invariant by the codimension one action of $O(3)$ on $\mathbb{S}^3$. The poles are the singular orbits of the action and the spherical caps are the geodesic tubes around one of the singular orbits. In this paper we will consider the torus action on $\mathbb{S}^3$, which is the another codimension one isometric action.

As in the case of spherical caps studied by Brezis and Peletier, we consider an open set Ω which is the geodesic tube around one of the singular orbits:

$$\Omega = \{\tilde{x} \in \mathbb{S}^3 / \operatorname{dist}(\tilde{x}, \mathbb{S}^1 \times 0) \leq \theta_1\},$$

with $\theta_1 \in (0, \pi/2)$. Note that Ω is a closed subset in $\mathbb{S}^3$ invariant by the $\mathbb{T}^2$-action.

Now from the change of variables that will be detailed in Chapter 1, it follows that if we restrict the original problem to functions which are invariant by the $\mathbb{T}^2$-action, then it is equivalent to finding solutions of:

$$\begin{cases} u''(\theta) + 2\frac{\cos(2\theta)}{\sin(2\theta)}u'(\theta) &= \lambda\left(u(\theta)^5 - u(\theta)\right), \quad u > 0 \quad \text{on } (0, \theta_1), \\ u'(0) &= 0, \\ u(\theta_1) &= 0. \end{cases} \tag{8}$$

We consider positive solutions of (8) with initial value in the interval (0, 1). We first prove a nonexistence theorem:

Theorem 0.1.1. *If $\theta_1 \in (0, \pi/4)$, then there are no solutions of (8) with initial value in the interval* $(0, 1)$.

This means that the solutions of (8) with initial value in the interval $(0, 1)$ do not vanish before $\pi/4$. However, we shall prove the existence of an increasing number of solutions of problem (8) as λ goes to $-\infty$ with initial value in the interval $(0, 1)$, which gives a partial positive answer to the open problem 8.3 proposed by H. Brezis and L. A. Peletier in [14]. Our main result in the first part of this thesis is the following

Theorem 0.1.2. *Given any $k \geq 1$ and any $\theta_1 > \pi/4$, then there exists a constant $A_k > 0$ such that for $\lambda < -A_k$ problem (8) has at least $2k$ solutions with initial value in the interval* $(0, 1)$.

We are also interested to study solutions of the equation invariant by the $\mathbb{T}^2$-action in the whole sphere $\mathbb{S}^3$:

$$\Delta_{\mathbb{S}^3} U = \lambda\left(U^5 - U\right), \quad U > 0 \quad \text{on } \mathbb{S}^3. \tag{9}$$

Positive solutions of (9) are called "ground state" solutions. We have the following result analogous to [14, Theorem 1.6]:

Theorem 0.1.3. *Let $n \geq 1$ and $\lambda \in [-(2n+2)(2n+3), -(2n)(2n+1))$. Then for every $k \in \{1, 2, \ldots, n\}$ there exists at least one solution U_k of problem (9), where $U_k = u_k(\theta)$ has the following propieties:*

1. *u_k has exactly k local maximum on* $(0, \frac{\pi}{2})$,
2. *$u_k(\pi/2 - \theta) = u_k(\theta)$ for $\theta \in (0, \frac{\pi}{2})$,*
3. *$u_k(0) < 1$.*

0.2 The Yamabe equation on a product manifold

Let (M^n, g) be any closed manifold and (N^m, h) a manifold of constant positive scalar curvature s_h. We will be interested in positive solutions of the Yamabe equation for the product manifold $(M \times N, g + \epsilon^2 h)$:

$$-a(\Delta_g + \Delta_{\epsilon^2 h})u + (s_g + \epsilon^{-2} s_h)u = u^{p-1}, \tag{10}$$

with $a = a_{m+n} = \frac{4(m+n-1)}{m+n-2}$, $p = p_{m+n} = \frac{2(m+n)}{m+n-2}$, s_g the scalar curvature of (M^n, g), and ϵ small enough so that the scalar curvature $s_g + \epsilon^{-2} s_h$ is positive. The conformal metric $u^{p-2}(g + \epsilon^2 h)$ then has constant scalar curvature.

We restrict our study to functions that depend only on the first factor, $u : M \to \mathbb{R}$. We normalize h so that $s_h = a_n$. Then u solves the Yamabe equation if and only if (after renormalizing)

$$-\epsilon^2 \Delta_g u + \left(a_n^{-1} s_g \epsilon^2 + 1\right) u = u^{p-1}. \tag{11}$$

Note that $p = p_{m+n} < p_n$. So the problem becomes a subcritical problem on M. We will study the general equation

$$-\epsilon^2 \Delta_g u + \left(\lambda s_g \epsilon^2 + 1\right) u = u^{p-1}, \tag{12}$$

where $\lambda \in \mathbb{R}$. Positive solutions of this equation are the critical points of the functional $J_\epsilon : H^{1,2}(M) \to \mathbb{R}$, given by

$$J_\epsilon(u) = \epsilon^{-n} \int_M \left(\frac{1}{2}\epsilon^2 |\nabla u|^2 + \frac{1}{2}\left(\epsilon^2 \lambda s_g + 1\right) u^2 - \frac{1}{p}(u^+)^p \right) dV_g,$$

where $u^+(x) = \max\{u(x), 0\}$.

We will build solutions of (10) by using the Lyapunov-Schmidt reduction procedure which was applied by several authors. In particular in the articles by Micheletti and Pistoia [26] and Dancer, Micheletti and Pistoia [15] the procedure is used to build solutions of a similar elliptic equation under certain conditions on the scalar curvature. We will apply a similar technique to problem (11).

Let us briefly describe the construction. One first considers what will be called the limit equation in $\mathbb{R}^n$. Recall that for $2 < q < \frac{2n}{n-2}$, $n > 2$, the equation

$$-\Delta U + U = U^{q-1} \text{ in } \mathbb{R}^n \tag{13}$$

has a unique (up to translations) positive solution $U \in H^1(\mathbb{R}^n)$ that vanishes at infinity. Such function is radial and exponentially decreasing at infinity, namely

$$\lim_{|x|\to\infty} U(|x|)|x|^{\frac{n-1}{2}} e^{|x|} = c > 0,$$

$$\lim_{|x|\to\infty} U'(|x|)|x|^{\frac{n-1}{2}} e^{|x|} = -c.$$

See reference [24] for details. We will denote this solution by U in the following.

Note that for any $\epsilon > 0$, the function $U_\epsilon(x) = U(\frac{x}{\epsilon})$, is a solution of

$$-\epsilon^2 \Delta U_\epsilon + U_\epsilon = U_\epsilon^{q-1}.$$

For any $x \in M$ consider the exponential map $\exp_x : T_x M \to M$. Since M is closed we can fix $r_0 > 0$ such that $\exp_x \big|_{B(0,r_0)} : B(0, r_0) \to B_g(x, r_0)$ is a diffeomorphism for any $x \in M$. Here $B(0, r)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius r and $B_g(x,r)$ is the geodesic ball in M centered at x with radius r. Let χ_r be a smooth radial cut-off function such that $\chi_r(z) = 1$ if $z \in B(0, r/2)$, $\chi_r(z) = 0$ if $z \in \mathbb{R}^n \setminus B(0,r)$, $|\nabla\chi_r(z)| < 3/r$ and $|\nabla^2\chi_r(z)| < 2/r^2$. Fix any $r < r_0$. For a point $\xi \in M$ and $\epsilon > 0$ let us define on M the function

$$W_{\epsilon,\xi}(x) = \begin{cases} U_\epsilon(\exp_\xi^{-1}(x))\chi_r(\exp_\xi^{-1}(x)) & if \quad x \in B_g(\xi, r), \\ 0 & otherwise. \end{cases}$$

One considers $W_{\epsilon,\xi}$ as an approximate solution to equation (11) which concentrates around ξ. As $\epsilon \to 0$ $W_{\epsilon,\xi}$ will get more concentrated around ξ and will be closer to an exact solution. Summing up a finite number k of these functions concentrating on different points we have an approximate solution of equation (11): Let $k_0 \geq 0$ be a fixed integer and denote $\xi = (\xi_1, \ldots, \xi_{k_0}) \in M^{k_0}$. Then

$$V_{\epsilon,\bar{\xi}}(x) := \sum_{i=1}^{k_0} W_{\epsilon,\xi_i^\epsilon}$$

is our approximate solution. We will find exact solutions by perturbing these approximate solutions. Let

$$\beta_\lambda := \lambda \int_{\mathbb{R}^n} U^2(z)\,dz - \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U(z)|^2 |z|^2\,dz. \tag{14}$$

It has been proved by A. M. Micheletti and A. Pistoia [27] that for a generic Riemannian metric the critical points of the scalar curvature are non-degenerate and in particular isolated. For all λ such that $\beta_\lambda < 0$ we will show that for small ϵ and any isolated local maximum x_0 of s_g there exists a solution of problem (11), with the points in $\bar{\xi}$ approaching x_0, which is close to $V_{\epsilon,\bar{\xi}}$ in the norm $\| \|_\epsilon$ defined by:

$$\|u\|_\epsilon^2 := \frac{1}{\epsilon^n}\Big(\epsilon^2 \int_M |\nabla_g u|^2\,d\mu_g + \int_M (\epsilon^2 \lambda s_g + 1)\,u^2\,d\mu_g\Big).$$

Analogously, for those λ such that β_λ is positive the same result is obtained by taking an isolated minimum of the scalar curvature instead of a maximum:

Theorem 0.2.1. *Assume that $\beta_\lambda \neq 0$. If $\beta_\lambda < 0$ ($\beta_\lambda > 0$) then let ξ_0 be an isolated local maximum (minimum) point of the scalar curvature S_g. For each positive integer k_0, there exists $\epsilon_0 = \epsilon_0(k_0) > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ there exist points $\xi_1^\epsilon, \ldots, \xi_{k_0}^\epsilon \in M$ such that*

$$\frac{d_g(\xi_i^\epsilon, \xi_j^\epsilon)}{\epsilon} \to +\infty \quad and \quad d_g(\xi_0, \xi_j^\epsilon) \to 0 \qquad as\ \epsilon \to 0, \tag{15}$$

and a solution u_ϵ of problem (11) *such that*

$$\|u_\epsilon - \sum_{i=1}^{k_0} W_{\epsilon,\xi_i^\epsilon}\|_\epsilon \to 0 \qquad as\ \epsilon \to 0.$$

This thesis is organized as follows:

In the next chapter, we give the mathematics needed to fully understand the results mentioned in this introduction. It is divided in three sections, in which a new formulation of the Yamabe equation on $\mathbb{S}^n$, methods for solving nonlinear eigenvalue problems and the Lyapunov-Schmidt reduction, are described.

In Chapter 2 we first study the ground state solutions of a Yamabe-type equation on $\mathbb{S}^3$. Then we prove a result about multiplicity of solutions for that equation on an torus invariant region of $\mathbb{S}^3$ with boundary.

In Chapter 3 we consider the Yamabe equation on a product of two manifold, assuming that the solution depends only on one of those manifolds. We use the Lyapunov-Schmidt reduction to prove a result of multiplicity of solutions for the equation.

Chapter 1

Preliminaries

In this chapter we enunciate several concepts that will be used throughout the work. In the first section we consider the natural $\mathbb{T}^2$-action and we give a new formulation of the Yamabe equation. In Section 1.2 we explain the shooting method and we review the statement of the Sturm Liouville Comparison Theorem. In Section 1.3 we formulate the Lyapunov-Schmidt reduction as it will be applied in Chapter 3.

1.1 $\mathbb{T}^2$-action on $\mathbb{S}^3$

Consider $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and the natural action $\mathbb{T}^2 \times \mathbb{S}^3 \to \mathbb{S}^3$ given by

$$(\alpha, \beta)(x, y, z, w) = (\alpha \cdot (x, y), \beta \cdot (z, w)) \tag{1.1}$$

where $\cdot$ is the complex multiplication. This is an isometric, codimension one, action on $\mathbb{S}^3$ and there are two special orbits: $\mathbb{S}^1 \times \{0\}$ and $\{0\} \times \mathbb{S}^1$. The distance between these two singular orbits is $\pi/2$.

Now we present a change of variables leading to a different formulation of Yamabe equation on $\mathbb{S}^3$. With this aim, we introduce local coordinates in $\mathbb{R}^4$:

$$\begin{cases} x_1 &= r\cos(\theta)\cos(\eta_1), \\ x_2 &= r\cos(\theta)\sin(\eta_1), \\ x_3 &= r\sin(\theta)\cos(\eta_2), \\ x_4 &= r\sin(\theta)\sin(\eta_2), \end{cases} \tag{1.2}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$, $0 \leq \theta < \pi/2$, $0 \leq \eta_1, \eta_2 \leq 2\pi$. In these coordinates, the unit sphere $\mathbb{S}^3$ can be parameterized by $r = 1$, $\{0 \leq \theta \leq \pi/2, 0 < \eta_1, \eta_2 < 2\pi\}$. The round metric g on the 3-sphere in these coordinates is given by

$$ds^2 = d\theta^2 + \cos^2(\theta)d\eta_1^2 + \sin^2(\theta)d\eta_2^2$$

Recall that the Beltrami-Laplace operator on $\mathbb{S}^3$ in local coordinates is given by:

$$\Delta_{\mathbb{S}^3} = \frac{1}{\sqrt{|g|}} \sum_{i=1}^{3} \frac{\partial}{\partial \eta_i} \left(g_{ii}^{-1} \sqrt{|g|} \frac{\partial}{\partial \eta_i} \right). \tag{1.3}$$

Suppose that the function $U : \Omega \to \mathbb{R}$ is invariant by the $\mathbb{T}^2$-action. Then $U(x, y, z, w) = u(\theta)$ for some function $u : [0, \theta_1] \to \mathbb{R}$ and since

$$|g| = \cos^2(\theta) \sin^2(\theta),$$

the Laplace-Beltrami operator on $\mathbb{S}^3$ applied to U takes the form:

$$\begin{aligned} \Delta_{\mathbb{S}^3} U &= \frac{1}{\cos(\theta)\sin(\theta)} \frac{d}{d\theta} \left(\cos(\theta)\sin(\theta) \frac{du}{d\theta} \right) \\ &= u''(\theta) + \left(\frac{\cos(\theta)}{\sin(\theta)} - \frac{\sin(\theta)}{\cos(\theta)} \right) u'(\theta) \qquad (1.4) \\ &= u''(\theta) + 2 \frac{\cos(2\theta)}{\sin(2\theta)} u'(\theta). \end{aligned}$$

We will use this in the next chapter to study the Yamabe equation on a torus invariant region of $\mathbb{S}^3$.

1.2 Methods for solving nonlinear eigenvalue problems

A classical Sturm-Liouville equation is a real second-order linear differential equation of the form

$$(p(x)u'(x))' + q(x)u(x) = \lambda u(x) \tag{1.5}$$

where $p(x)$ and $q(x)$ are given smooth functions on the finite closed interval $[a, b]$ and $\lambda \in \mathbb{R}$. It may be assumed throughout the following, that p is strictly positive on the open interval (a, b). The value of λ is not specified in the equation; finding the values of λ for which there exists a nontrivial (nonzero) solution u of (1.5) satisfying certain boundary conditions is part of the problem called the Sturm-Liouville problem. Such values of λ, when they exist, are called the *eigenvalues* of the boundary value problem defined by (1.5) and the prescribed set of boundary conditions. The corresponding solutions $u(x)$ are the *eigenfunctions* of this problem. Often the Sturm-Liouville equation is defined together with boundary conditions, specifying the value of the solution at the endpoints a and b. For a more elaborated study of the Sturm-Liouville problems we can refer to [22] and [10].

A classical result about the relative position of the zeros of different solutions is the Sturm Liouville Comparison Theorem:

Theorem 1.2.1. (Sturm Liouville Comparison Theorem) *For $i = 1, 2$ set $u_i(x)$ be a nontrivial solution on (a, b) of $(p_i(x)y')' + Q_i(x)y = 0$ where $0 < p_2 \leq p_1$ and $Q_2 \geq Q_1$ on (a, b). Then (strictly) between any two zeros of u_1 lies at least one zero of u_2 except when u_2 is a constant multiple of u_1.*

For a proof see [10]. The most common application is to a Sturm-Liouville system with different eigenvalues λ_i, for $i = 1, 2$. If $p_1 = p_2 = p$ and $Q_i(x) = \lambda_i - q(x)$, then the theorem makes a comparison of the different eigenfunctions of the equation (1.5). We assume that $\lambda_2 > \lambda_1$. The zeros of the eigenfunction of λ_2 then lie between the zeros of the eigenfunction of λ_1. We say that the higher eigenfunction is oscillating 'more rapidly' than the lower eigenfunction.

On the other hand the shooting method is a method for solving a boundary value problem by reducing it to the solution of an initial value problem. The central idea is to replace the boundary value problem under consideration by an initial value problem. Suppose we want to solve a boundary value problem with Dirichlet boundary conditions:

$$\begin{aligned} &u'' = f(t, u, u') \quad \text{on } [a, b], \\ &u(a) = u(b) = 0. \end{aligned} \tag{1.6}$$

Let us suppose $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and locally Lipschitz with respect to u, which guarantees that for any $\alpha \in \mathbb{R}$ the initial value problem

$$\begin{aligned} &u'' = f(t, u, u') \quad \text{on } [a, b], \\ &u(a) = 0, u'(a) = \alpha \end{aligned} \tag{1.7}$$

has a unique solution u_α, defined in a maximal nontrivial interval $I_\alpha = [a, M)$, with $M = M(\alpha) \in (a, +\infty)$. In general, we cannot know if $M(\alpha) > b$, although on the set $\{\alpha : M(\alpha) > b\}$, the function $\alpha \to u_\alpha(b)$ is continuous. This is due to the continuous dependence with respect to the initial values.

It is clear that we are looking for a value α such that $u_\alpha(b)$ is well defined and $u_\alpha(b) = 0$. In other words, we are looking for a zero of the function T defined by

$$T(\alpha) := u_\alpha(b).$$

Due to the continuity mentioned before, it is enough to find an interval $\Lambda = [\alpha_*, \alpha^*]$ such that $T(\alpha)$ is well defined for all $\alpha \in \Lambda$ and, moreover, $T(\alpha_*) \leq 0 \leq T(\alpha^*)$ or vice versa.

For instance, if f is bounded, the solutions of (1.7) are defined in $[a, b]$. Moreover, direct integration of the equation yields

$$u'_\alpha(x) = \alpha + \int_0^x f(s, u_\alpha(s))ds.$$

This says that, if $\alpha \geq \|f\|_\infty$, then

$$u'_\alpha(x) \geq \alpha - x\|f\|_\infty \geq 0$$

for $x \leq b$. Thus u_α is nondecreasing, and hence $T(\alpha) > 0$. In a similar way, if $\alpha < -\|f\|_\infty$, then $T(\alpha) < 0$. Which allows to deduce that $T(\alpha) = 0$ for some $\alpha \in [-\|f\|_\infty, \|f\|_\infty]$. For more information and examples about this method see [2].

1.3 Lyapunov-Schmidt reduction

In this section the main features of the classical Lyapunov-Schmidt reduction are outlined in a form suitable to be used in Chapter 3. The method is broader and is frequently employed in bifurcation theory. Here we will fix notation and give the steps in the precise form needed for later applications. This method is explained in detail in [1].

Let H be a Hilbert space. Let $S \in C^1(H, H)$ be such that $S'(0)$ is not invertible and consider the equation $S(u) = 0$. Suppose that $u = 0$ is a solution for the equation and denote $S' = S'(0)$ the differential of S evaluated at 0. Assume that S' has kernel $K = Ker(S')$ with $\dim K > 0$ and let $K^\perp$ denote the complement of K in H. Then S' has range $Ran(S')$ and this range has a complement $Ran(S')^\perp$ in H. We suppose:

- S' self-adjoint operator,
- S' is zero-index Fredholm.

Then it follows that $K^\perp = Ran(S')$ and $K = Ran(S')^\perp$. Let $\pi^\perp : H \to K^\perp$ be the linear projection onto the range and $\pi : H \to K$ the linear projection onto K. For all $u \in H$, there exists a unique decomposition: $u = v + \phi$, with $v \in K$ and $\phi \in K^\perp$.

Thus applying π and $\pi^\perp$ to $S(u) = 0$ we obtain the following equivalent system:

$$\pi(S(v + \phi)) = 0, \tag{1.8}$$

$$\pi^\perp(S(v + \phi)) = 0. \tag{1.9}$$

The latter is called the auxiliary equation.

The auxiliary equation (1.9) is uniquely solvable in $K^\perp$, locally near 0. Set $T(v,\phi) = \pi^\perp S(v+\phi)$. One has that $T \in C^1(K \times K^\perp, K^\perp)$ and $\partial_\phi T(0,0)$ is the linear map from $K^\perp$ to $K^\perp$ given by

$$\partial_\phi T(0,0)(\phi) = \pi^\perp(S'(0)(\phi)) = \pi^\perp(S'(\phi)) = S'(\phi), \tag{1.10}$$

since $S'(\phi) \in K^\perp$. In other words,$\partial_\phi T(0,0)$ is the restriction of S' to $K^\perp$, and thus it is injective and surjective (as a map from $K^\perp$ to $K^\perp$). Since $K^\perp$ is closed, it follows that $\partial_\phi T(0,0)$ is invertible and a straight application of the Implicit Function Theorem yields the following

Lemma 1.3.1. *There exist neighbourhoods V_0 of $v = 0$ in K , W_0 of $\phi = 0$ in $K^\perp$, and a map $\phi = \phi(v)$ such that*
$\pi^\perp(S(v+\phi)) = 0$ $v \in V_0$ *, and $\phi \in W_0$, if and only if, $\phi = \phi(v)$.*

Then it remains only to solve the problem in finite dimension (1.8).

In Chapter 3 we apply the Liapunov-Schmidt method of finite-dimensional reduction, but we take as "trial solutions" of the equation $S(u) = 0$ the translates $V_{\epsilon,\tilde{\xi}}$ of the solution U of the limit equation (13).

The idea is that the kernel of $S'(V_{\epsilon,\tilde{\xi}})$ should be close to K and then the linear map $\pi \to \pi^\perp(S'((V_{\epsilon,\tilde{\xi}})(\phi))$ should be invertible as a map from $K^\perp$ to $K^\perp$. Then the Inverse Function Theorem could be apply and finally it remains to solve the finite dimensional problem.

Resumen del Capítulo

En este capítulo enunciamos varios conceptos que se utilizarán a lo largo de la tesis. En la primera sección consideramos la acción natural de $\mathbb{T}^2$ en $\mathbb{S}^3$ y damos una formulación diferente de la ecuación de Yamabe.

En la sección 1.2 explicamos el método de shooting, que utilizaremos en esta tesis para resolver un problema de valor límite. La idea central es reemplazar el problema del valor límite considerado por un problema de valor inicial. También enunciamos el Teorema de comparación de Sturm Liouville, que es un resultado clásico sobre la posición relativa de los ceros de diferentes soluciones de una ecuación diferencial ordinaria dada.

En la sección 1.3 formulamos las características principales de la reducción clásica de Lyapunov-Schmidt de una forma adecuada para ser utilizada en el capítulo 3. El método es más amplio de lo que aquí se presenta y se emplea con frecuencia en teoría de bifurcación.

Chapter 2

The Yamabe equation on an invariant region of $\mathbb{S}^3$

2.1 Introduction

In this chapter we will study the following problem:

$$\begin{cases} u''(\theta) + 2\frac{\cos(2\theta)}{\sin(2\theta)}u'(\theta) & = & \lambda\left(u(\theta)^5 - u(\theta)\right), \quad u > 0 \quad \text{on } (0, \theta_1), \\ u'(0) & = & 0 \\ u(\theta_1) & = & 0 \end{cases} \tag{2.1}$$

where $u : [0, \theta_1] \to \mathbb{R}$ and we will prove Theorem 0.1.1, Theorem 0.1.2 and Theorem 0.1.3. Our approach to prove Theorem 0.1.2 mainly relies upon a method that has been successfully used in [14]. First we use this method to show that there exists at least 2 solutions of problem (2.1) with initial value in the interval $(0, 1)$ that have a single spike or maximum. The next step is to prove the theorem in the case $k = 2$ using the same techniques. Finally the theorem follows by induction.

The chapter is organized as follows. In Section 2.2 we will study properties of the ground state solutions and prove Theorem 0.1.3. Section 2.3 contains some results about auxiliary linear problems, that will help us to prove the main theorem in next section. Theorem 0.1.1 will be proved in Section 2.4, as well as Theorem 0.1.2.

A version of this part of the thesis appears in [35].

2.2 Positive solutions on $\mathbb{S}^3$

In this section we present a detailed study of the problem obtained by linearizing Eq. (2.1) around the nontrivial constant solution when $\lambda < 0$.

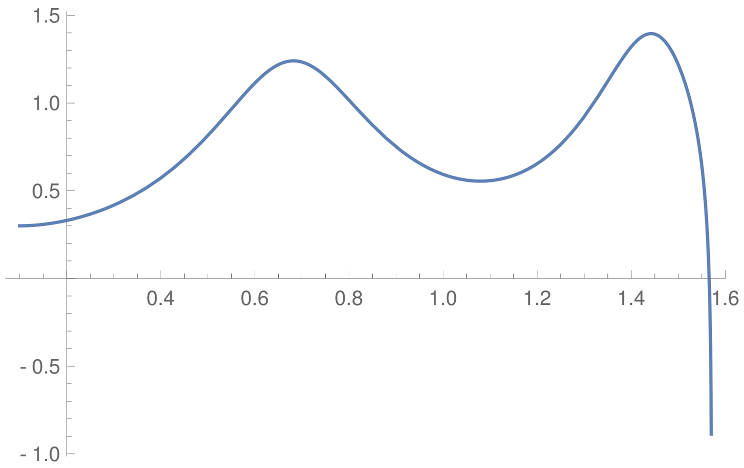

Figure 2.1: Two-spike solution u of problem (2.1) for $\lambda = -25$ and $u(0) = 0.3$

Then we use these results to prove Theorem 0.1.3. Let $\alpha \in (0, 1)$, $\lambda < 0$, and denote by $u_{\alpha,\lambda}(\theta)$ the solution of:

$$\begin{cases} u''(\theta) + 2\frac{\cos(2\theta)}{\sin(2\theta)}u'(\theta) &= \lambda\left(u(\theta)^5 - u(\theta)\right), \\ u(0) &= \alpha, \\ u'(0) &= 0. \end{cases} \tag{2.2}$$

There is a constant solution $u_{1,\lambda} \equiv 1$, and it is important to understand the behavior of the solutions $u_{\alpha,\lambda}(\theta)$ with α close to 1. With this aim, consider the function

$$w_\lambda(\theta) = \frac{d}{d\alpha}\Big|_{\alpha=1} u_{\alpha,\lambda}(\theta).$$

Then w_λ is the solution of the linear problem

$$\begin{cases} w''(\theta) + 2\frac{\cos(2\theta)}{\sin(2\theta)}w'(\theta) &= 4\lambda w, \\ w(0) &= 1, \\ w'(0) &= 0. \end{cases} \tag{2.3}$$

This is the eigenvalue equation for $\Delta_{\mathbb{S}^3}$ restricted to functions invariant by the $\mathbb{T}^2$-action. It can be understood for instance adapting the techniques used by J. Petean in [32] (for the case of radial functions). We will sketch the proofs briefly for completeness.

Let $\lambda_n := -n(n+1)$.

If we denote by $F_c(\varphi)(\theta) = \varphi''(\theta) + 2\frac{\cos(2\theta)}{\sin(2\theta)}\varphi'(\theta) - c\varphi(\theta)$ then by a direct computation:

$$F_c(\cos^k(2\theta)) = (4\lambda_n - c)\cos^k(2\theta) + 4k(k-1)\cos^{k-2}(2\theta).$$

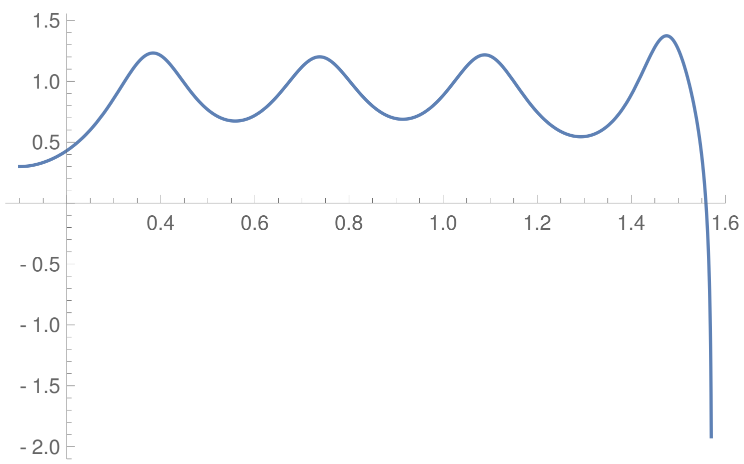

Figure 2.2: Four-spike solution u of problem (2.1) for $\lambda = -100$ and $u(0) = 0.3$

Lemma 2.2.1. *The solution w_{λ_n} of (2.3) is a linear combination of $\cos^{n-2j}(2\theta)$, where $0 \le 2j \le n$.*

It then follows that $w_{\lambda_n}(\pi/2) = (-1)^n$. If n is odd then $w_{\lambda_n}(\pi/4) = 0$ and if n is even then $w'_{\lambda_n}(\pi/4) = 0$.

It follows from Sturm-Liouville theory that the number of zeros of w_λ in $(0, \pi/2)$ is a non-increasing function of λ (< 0). It is then easy to see that:

Lemma 2.2.2. *The solution w_{λ_n} has exactly n zeros in the interval $(0, \frac{\pi}{2})$ and therefore exactly $n-1$ critical points in $(0, \frac{\pi}{2})$.*

And using again Sturm-Liouville theory and the previous comments it follows:

Lemma 2.2.3. *If $\lambda \in [\lambda_{2n+2}, \lambda_{2n})$ then w_λ has exactly n critical points in the interval $(0, \pi/4)$.*

Denote by

$$\tau_1^0(\lambda) < \tau_2^0(\lambda) < \ldots < \tau_n^0(\lambda)$$

the critical points of w_λ in $(0, \pi/2)$. Using the uniform continuity of the solution of problem (2.2) with respect to the initial value α we obtain:

Lemma 2.2.4. *Suppose that w_λ has a critical point $\tau_k^0(\lambda)$ for some $k \geq 1$. Then for $\alpha < 1$ sufficiently close to 1, the solution $u_{\alpha,\lambda}$ has a critical point $\tau_k(\alpha)$ and*

$$\tau_k(\alpha) \to \tau_k^0(\lambda), \text{ as } \alpha \to 1.$$

Remark. $\tau_k = \tau_k(\alpha)$ is a continuous function (where it is defined) and from (2.2) it is easy to see that $u_{\alpha,\lambda}(\tau_j) > 1$ if j is odd, and $u_{\alpha,\lambda}(\tau_j) < 1$ if j is even.

Lemma 2.2.5. *If for any $\alpha \in (0,1)$ the solution u_α of problem (2.2) satisfies $u'_\alpha(\frac{\pi}{4}) = 0$, then $u'_\alpha(\frac{\pi}{2}) = 0$ and $u_\alpha(\theta) = u_\alpha(\frac{\pi}{2} - \theta)$ for $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$.*

Proof. The function $v(\theta) = u_\alpha(\frac{\pi}{2} - \theta)$ for $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$ is also a solution of the equation. Moreover $v(\frac{\pi}{4}) = u_\alpha(\frac{\pi}{4})$ and $v'(\frac{\pi}{2}) = 0 = u'_\alpha(\frac{\pi}{4})$. Therefore $v = u_\alpha$ in $[\frac{\pi}{4}, \frac{\pi}{2})$ and the lemma follows. □

Lemma 2.2.6. *If α is close to zero, then the solution u_α of problem (2.2) has no local extremes on $(0, \frac{\pi}{4})$.*

Proof. For α close to 0 the solution u_α increases slowly in interval $(0, \frac{\pi}{4})$ and stays less than 1 in that interval. Therefore it does not have any local extremes on $(0, \frac{\pi}{4})$. □

Now define:

$$F(u) := \int_0^u \left(s^5 - s\right) ds = \frac{1}{6}u^6 - \frac{1}{2}u^2. \tag{2.4}$$

Then $F(\alpha) < 0$. Note that F has only one positive zero $\sigma := 3^{\frac{1}{4}}$.

Lemma 2.2.7. *If $\tau_j(\alpha) < \frac{\pi}{4}$, then $0 < u_\alpha(\tau_j(\alpha)) < \sigma$.*

To prove this lemma we consider the energy function defined by

$$E_\alpha(\theta) := \frac{(u'_\alpha(\theta))^2}{2} - \lambda\left(\frac{(u_\alpha(\theta))^6}{6} - \frac{(u_\alpha(\theta))^2}{2}\right). \tag{2.5}$$

If u_α is a solution of problem (2.2) then we have

$$E'_\alpha(\theta) = -2\frac{\cos(2\theta)}{\sin(2\theta)}u'_\alpha(\theta)^2.$$

Consequently E_α is decreasing on $[0, \frac{\pi}{4}]$ and $E_\alpha(0) = -\lambda F(\alpha)$.

Proof of Lemma 2.2.7. Since E_α is decreasing on $[0, \frac{\pi}{4}]$ and $0 < \tau_j(\alpha) \leq \frac{\pi}{4}$, it follows that

$$E_\alpha(\tau_j) < E_\alpha(0) = -\lambda F(\alpha).$$

Consequently, since $E_\alpha(\tau_j(\alpha)) = -\lambda F(u_\alpha(\tau_j(\alpha)))$ and $0 < \alpha < 1$ we have that

$$F(u_\alpha(\tau_j(\alpha))) < F(\alpha) < 0.$$

This means that $0 < u_\alpha(\tau_j(\alpha)) < \sigma$, as asserted.

□

Next we define α_k^* as the infimum value of α for which $\tau_k(\alpha)$ exists on $(\alpha, 1)$:

$\alpha_k^* = \inf\{\alpha_0 \in (0,1) :$ for $\alpha \in (\alpha_0, 1)$ u_α has at least k critical points on $(0, \pi/2)\}$.

Lemma 2.2.8. *Suppose that $\tau_k(\alpha)$ exists for some $\alpha < 1$ sufficiently close to 1 so that α_k^* is well defined. Then there exists $\delta > 0$ such that*

$$\tau_k(\alpha) \geq \frac{\pi}{4} \quad \text{if } \alpha \in (\alpha_k^*, \alpha_k^* + \delta).$$

Proof. If $\alpha_k^* = 0$ then the assertion follows from Lemma 2.2.6. Thus we may assume that $\alpha_k^* \in (0,1)$. Suppose there exists a decreasing sequence $\{\alpha_j\}$ such that

$$\tau_k(\alpha_j) < \frac{\pi}{4} \quad \text{and} \quad \alpha_j \to \alpha_k^*.$$

Since the sequences $\{\tau_k(\alpha_j)\}$ and $\{u_{\alpha_j}(\tau_k(\alpha_j))\}$ are bounded by Lemma 2.2.7, it follows that there exist $\tau_k^* \in [0, \frac{\pi}{4}]$ and $u^* \in [0, \sigma]$ such that, taking a subsequence, we may soppose:

$$\tau_k(\alpha_j) \to \tau_k^* \quad \text{and} \quad u_{\alpha_j}(\tau_k(\alpha_j)) \to u^*.$$

If u^* is 1 or 0, then by uniqueness $u_{\alpha_k^*}$ is constant, which contradicts the fact that $\alpha_k^* \in (0,1)$. If $u^* \in (0,1)$, then we use the Implicit Function Theorem with the function $G(\alpha, \theta) = u'_\alpha(\theta)$. Since $\alpha_k^* \neq 0, 1$, it follows that $\frac{d}{d\theta} G(\alpha_k^*, \theta) \neq 0$. But since $G(\alpha_k^*, \theta) = 0$, we have that $\tau_k(\alpha)$ is well defined for all α in a neighbourhood of α_k^*, which contradicts the definition of α_k^*. □

We end this section with the proof of Theorem 0.1.3:

Theorem. *Let $n \geq 1$ and $\lambda \in [-(2n+2)(2n+3), -(2n)(2n+1))$. Then for every $k \in \{1, 2, \ldots, n\}$ there exists at least one solution U_k of problem (9), where $U_k = u_k(\theta)$ has the following propieties:*

1. *u_k has exactly k local maximum on $(0, \frac{\pi}{2})$,*
2. *$u_k(\pi/2 - \theta) = u_k(\theta)$ for $\theta \in (0, \frac{\pi}{2})$,*
3. *$u_k(0) < 1$.*

Proof. Suppose $n \geq 1$ and $\lambda \in [-(2n+2)(2n+3), -(2n)(2n+1))$. Given $k \in \{1, 2, \ldots, n\}$ we will show that

$$\tau_k(\alpha_0) = \frac{\pi}{4}$$

for some α_0 and hence the solution u_{α_0} has k local extremes on $(0, \frac{\pi}{4}]$. Since $u'_{\alpha_0}(\pi/4) = 0$, by Lemma 2.2.5 it follows that u_{α_0} satisfies (i)–(iii) of Theorem 0.1.3.

By Lemmas 2.2.3 and 2.2.4 , since $\lambda \in [\lambda_{2n+2}, \lambda_{2n})$ and α is close to 1, the solution u_α has n local extremes $(0, \pi/4)$. Therefore $\tau_k(\alpha) < \frac{\pi}{4}$. On the other hand, by Lemma 2.2.8 we know that if α is close to α_k^* then $\tau_k(\alpha) \geq \frac{\pi}{4}$. By continuous dependence it follows that there is α_0 such that $\tau_k(\alpha_0) = \frac{\pi}{4}$.

□

2.3 Auxiliary results

In this section we will establish three auxiliary results that we will need to prove our main theorem in next section.

Lemma 2.3.1. *Let $\delta, \kappa > 0$ and K be constants. Then there are constants $\epsilon_1 > 0$ and $\beta > 0$ such that the solution φ_ϵ of*

$$\begin{cases} \epsilon^2\varphi'' - 2\epsilon^2 K\varphi' - \kappa\varphi &= 0, \\ \varphi(\pm\delta) &= 1/2, \end{cases} \tag{2.6}$$

satisfies: $\varphi_\epsilon(0) < e^{-\beta/\epsilon}$ for all $\epsilon \in (0, \epsilon_1)$.

Proof. Note that $\varphi_\epsilon(\theta) = Ae^{c_1\theta} + Be^{c_2\theta}$, with A, B given by

$$A = \frac{1 - e^{2c_2\delta}}{2(e^{c_1\delta} - e^{(2c_2 - c_1)\delta})}, \quad \text{and} \quad B = \frac{1 - e^{2c_1\delta}}{2(e^{c_2\delta} - e^{(2c_1 - c_2)\delta})},$$

where c_1, c_2 are the roots of the equation $\epsilon^2 x^2 - 2\epsilon^2 K x - \kappa = 0$. Then

$$c_1, c_2(\epsilon) = K \pm \sqrt{\mu_\epsilon}$$

where $\mu_\epsilon = K^2 + \kappa/\epsilon^2$. Now it is easy to see that $c_1(\epsilon) \to +\infty$, $c_2(\epsilon) \to -\infty$ as $\epsilon \to 0$ and consequently

$$e^{\sqrt{\mu_\epsilon}\delta}(A + B) \to C, \quad \text{as } \epsilon \to 0,$$

where C is some positive constant. It follows that there are constants $\beta > 0$ and $\epsilon_1 > 0$ such that

$$\varphi_\epsilon(0) = A + B < e^{-\beta/\epsilon}, \quad \text{if } \epsilon < \epsilon_1.$$

□

Now we shall study the behavior of the solutions of the equation

$$Z''(s) + Z(s)^5 - Z(s) = 0, \quad Z'(0) = 0, \tag{2.7}$$

when $s \to -\infty$. To this end consider the following lemma.

Lemma 2.3.2. *Let Z a solution to the Eq. (2.7) such that*

$$Z'(0) = 0, \tag{2.8}$$
$$Z(0) = \alpha, \tag{2.9}$$

with $\alpha > 0$. Then

1. *If* $\alpha < 3^{1/4}$ *and* $\alpha \neq 1$ *then* Z *oscillates around* 1.

2. *If* $\alpha > 3^{1/4}$ *then* Z *vanishes at some* $s < 0$*, and it is positive and increasing in* $(s, 0)$*.*

3. *If* $\alpha = 3^{1/4}$ *then* Z *is increasing in* $(-\infty, 0)$ *and* $\lim_{s\to-\infty} Z(s) = 0$.

Proof. If we multiply the equation (2.7) by Z' and integrate, then we have

$$c = \frac{Z'(s)^2}{2} + \frac{Z(s)^6}{6} - \frac{Z(s)^2}{2}. \tag{2.10}$$

It immediately follows that Z is globally defined and $c = \alpha^6/6 - \alpha^2/2$.

Note that if s_1 is a critical point of Z then

$$c = \frac{Z(s_1)^6}{6} - \frac{Z(s_1)^2}{2}. \tag{2.11}$$

Now if $c \geq 0$, ie $\alpha \geq 3^{1/4}$, there is only one positive value of $Z(s_1)$ which satisfies the previous equation. There are two options: either Z vanishes at some $s_0 < 0$ or $L = \lim_{s\to-\infty} Z(s)$ exists and it is non-negative. Suppose first that Z vanishes at some $s_0 < 0$ and that $Z'(s_0) \neq 0$ because the uniqueness of solutions. Evaluating in (2.10) we get $Z'(s_0)^2/2 = c$ and $c > 0$. Otherwise if there is a $L \geq 0$ such that

$$L = \lim_{s\to-\infty} Z(s). \tag{2.12}$$

Then there is a sequence $s_j \to -\infty$ as $j \to \infty$ such that $Z'(s_j) \to 0$. If we take the limit when $s_j \to -\infty$ to the equation (2.11), then we obtain $L = \alpha$, which is a contradiction because Z is increasing, or $L = 0$, which implies $c = 0$.

Now if if $c < 0$, ie $\alpha < 3^{1/4}$, and Z has a critical point in s_1 there is two possible values of $Z(s_1)$: a minimum less than 1 and a maximum greater than 1. If Z is not oscillating, it remains over or below the value 1. We will show that it is not possible. Suppose Z remains below 1. Then Z is convex and positive, so there is a $0 < L < 1$ that satisfies (2.12). Morover $\lim_{j\to\infty} Z'(s_j) = 0$ and $\lim_{j\to\infty} Z''(s_j) = 0$ for a sequence $s_j \to -\infty$. Taking limit when $s_j \to -\infty$ in (2.7) we have: $\lim_{s_j\to-\infty} Z''(s_j) = L - L^5$. There is a contradiction. If Z is over the value 1, we get a contradiction in a similar way. Therefore Z remains oscillating around 1.

□

Lemma 2.3.3. *Let* $z = z_\epsilon$ *be a solution of the equation*

$$z''(s) + 2\epsilon\frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)}z'(s) + z(s)^5 - z(s) = 0, \tag{2.13}$$

which is positive and increasing on the interval $(\psi(\epsilon), 0)$ *with the initial conditions*

$$z'(0) = 0, \tag{2.14}$$

$$z(0) = u_0(\epsilon), \tag{2.15}$$

and let Z_0 *be the unique solution of problem (2.7)-(2.8) such that* $Z_0(0) = 3^{1/4}$*. Assume that* $\psi(\theta)$ *is a function such that* $\psi(\epsilon) \to -\infty$ *as* $\epsilon \to 0$*. Then*

$$z_\epsilon(s) \to Z_0(s) \quad \textit{and} \quad z'_\epsilon(s) \to Z'_0(s) \quad \textit{when } \epsilon \to 0$$

uniformly over bounded intervals and, in particular,

$$u_0(\epsilon) \to 3^{1/4} \quad \textit{when } \epsilon \to 0. \tag{2.16}$$

Proof. It is knwon that such solutions z_ϵ are uniformly bounded (it can be proved for instance as in [24, Lemma 15]). Since the family of solutions $\{z_\epsilon(s) : 0 < \epsilon < \epsilon_0\}$ is equicontinuous it follows from Arzelà-Ascoli Theorem that

$$z_\epsilon(s) \to Z(s)$$

along a sequence, uniformly on bounded intervals, where Z is a solution of (2.7). But on a large interval the solution Z must be positive and increasing, therefore $Z = Z_0$ by the previous lemma. It then follows that the entire family converges to Z_0. In a similar manner it is proved that $z'_\epsilon(s) \to Z'_0(s)$. □

2.4 Proof of the main theorems

This section is devoted to the proofs of Theorems 0.1.1 and 0.1.2:

Theorem (0.1.1)**.** *If* $\theta_1 \in (0, \pi/4)$*, then there are no solutions of (8) with initial value in the interval* $(0, 1)$.

Theorem (0.1.2)**.** *Given any* $k \geq 1$ *and any* $\theta_1 > \pi/4$*, then there exists a constant* $A_k > 0$ *such that for* $\lambda < -A_k$ *problem (8) has at least* $2k$ *solutions with initial value in the interval* $(0, 1)$.

The proof of Theorem 0.1.1 is based on the techniques used by C. Bandle and R. Benguria in [4].

Proof of Theorem 0.1.1. Multiply Eq. (2.1) by $u'(\theta)$ and integrate over $(0, \theta_1)$. This yields

$$\frac{1}{2}u'(\theta_1)^2 + 2\int_0^{\theta_1} \frac{\cos(2\theta)}{\sin(2\theta)}(u'(\theta))^2\, d\theta = -\lambda F(\alpha). \tag{2.17}$$

If $0 < \theta < \theta_1 < \frac{\pi}{4}$ then $\frac{\cos(2\theta)}{\sin(2\theta)} > 0$. Since $\lambda < 0$, we have a contradiction:

$$0 < \frac{1}{2}u'(\theta_1)^2 + 2\int_0^{\theta_1} \frac{\cos(2\theta)}{\sin(2\theta)}u'(\theta)^2\, d\theta = -\lambda F(\alpha) < 0.$$

□

Now we prove Theorem 0.1.2 for $k = 1$. We shall show that there exist at least two solutions of problem (2.1) with initial value in the interval $(0, 1)$ that have a single spike. Let

$$\epsilon^2 = \frac{1}{|\lambda|}$$

and $\alpha \in (0, 1)$. Then consider the initial value problem

$$\begin{cases} \epsilon^2 u''(\theta) + 2\epsilon^2 \frac{\cos(2\theta)}{\sin(2\theta)} u'(\theta) + u(\theta)^5 - u(\theta) = 0 & \text{in } (0, \theta_1), \\ u > 0 & \text{in } (0, \theta_1), \\ u(0) = \alpha, & \\ u'(0) = 0. & \end{cases} \tag{2.18}$$

We denote the solution by $u_{\alpha,\epsilon}(\theta)$ and define

$$\Theta(\alpha, \epsilon) = \sup\{\theta \in (0, \pi/2) : u_{\alpha,\epsilon} > 0 \quad \text{in } (0, \theta)\}. \tag{2.19}$$

We will show that for ϵ small enough there are two values $\alpha_1, \alpha_2 \in (0, 1)$ such that $\Theta(\alpha_i, \epsilon) = \theta_1$ for $i = 1, 2$ and the solutions $u_{\alpha_1,\epsilon}$ and $u_{\alpha_2,\epsilon}$ have exactly 1 spike on the interval $(0, \theta_1)$. These techniques have been used successfully in [14].

Note that Theorem 0.1.1 implies that $\Theta(\alpha, \epsilon) > \frac{\pi}{4}$. It may happen that the solution does not vanish in the interval $(0, \frac{\pi}{2})$. Therefore we define $\mathcal{A}(\epsilon)$ as the set of values of α for which $u_{\alpha,\epsilon}$ vanishes before $\frac{\pi}{2}$:

$$\mathcal{A}(\epsilon) = \{\alpha \in (0, 1) : 0 < \Theta(\alpha, \epsilon) < \pi/2\}. \tag{2.20}$$

$\mathcal{A}(\epsilon)$ is an open set and if $\alpha \in \mathcal{A}(\epsilon)$ then it follows by uniqueness that

$$u_{\alpha,\epsilon}(\Theta(\alpha, \epsilon)) = 0 \qquad \text{and} \qquad u'_{\alpha,\epsilon}(\Theta(\alpha, \epsilon)) < 0.$$

On the other hand if we fix a $T_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$, then by the Sturm Liouville Comparison Theorem for ϵ small enough the solution w_λ of the linear Eq.

(2.3) has a maximum in $\tau_1^0(\lambda) < T_0$. Hence there exists an initial value $\alpha_0 \in (0,1)$ such that

$$\tau_1(\alpha_0) = T_0.$$

Since α_0 depends on ϵ denote

$$\alpha_0 = \alpha_0(\epsilon); \quad u_\epsilon(\theta) = u_{\alpha_0(\epsilon),\epsilon}(\theta) \quad \text{and} \quad u_0(\epsilon) = u_\epsilon(T_0).$$

In other words, fixed $T_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ and ϵ small enough, we can find a solution u_ϵ of problem (2.18) that reaches its first maximum $u_0(\epsilon)$ at $\theta = T_0$. It is clear that $u_0(\epsilon) > 1$. In the following lemmas we show that for ϵ small enough, $F(u_0(\epsilon)) > 0$, where F is the function defined in (2.4):

$$F(u) = \frac{1}{6}u^6 - \frac{1}{2}u^2.$$

Then, since F is increasing on $(1,\infty)$ and $u_0(\epsilon) > 1$, it follows that $u_0(\epsilon) > \sigma$, where $\sigma = 3^{1/4}$ is the positive zero of F.

Lemma 2.4.1. *There exist constants $A > 0$ and $\epsilon_0 > 0$ such that*

$$F(u_0(\epsilon)) > A\epsilon \quad \text{for } \epsilon < \epsilon_0.$$

Consider the energy function $E_{\alpha_0}(\theta)$ associated with the solution u_{α_0} defined in (2.5) with $\epsilon^2 = \dfrac{1}{|\lambda|}$. It satisfies

$$E_{\alpha_0}(0) = \frac{1}{\epsilon^2}F(\alpha_0) \qquad \text{and} \qquad E_{\alpha_0}(T_0) = \frac{1}{\epsilon^2}F(u_0(\epsilon)). \tag{2.21}$$

Integration of E'_{α_0} over $(0,T_0)$ yields

$$F(u_0(\epsilon)) - F(\alpha_0) = -2\epsilon^2 \int_0^{T_0} \frac{\cos(2\theta)}{\sin(2\theta)} u'_\epsilon(\theta)^2 \, d\theta.$$

Define

$$\begin{aligned} J_1(\epsilon) &= -2\epsilon^2 \int_0^{\frac{\pi}{4}} \frac{\cos(2\theta)}{\sin(2\theta)} u'_\epsilon(\theta)^2 \, d\theta, \\ J_2(\epsilon) &= -2\epsilon^2 \int_{\frac{\pi}{4}}^{T_0} \frac{\cos(2\theta)}{\sin(2\theta)} u'_\epsilon(\theta)^2 \, d\theta. \end{aligned} \tag{2.22}$$

The expression for $F(u_0(\epsilon))$ then becomes

$$F(u_0(\epsilon)) = F(\alpha_0) + J_1(\epsilon) + J_2(\epsilon). \tag{2.23}$$

The following lemmas are used to estimate the terms on the right hand side of (2.23).

Let $\kappa > 0$ be a constant such that

$$s^5 - s + \kappa s < 0 \quad \text{for } 0 < s < 1/2. \tag{2.24}$$

Write $\theta = T_0 + \epsilon s$ and let $z_\epsilon(s) = u_\epsilon(\theta)$. Then z_ϵ solves problem (2.13). By Lemma 2.3.3 we know that if Z_0 is the solution of (2.7)-(2.8) such that $Z(0) = 3^{1/4}$, then there is a $s_0 < 0$ such that $Z_0(s_0) = 1/4$ and hence $z_\epsilon(s_0) = u_\epsilon(T_0 + \epsilon s_0) \to 1/4$ as $\epsilon \to 0$. It follows that for ϵ small enough,

$$u_\epsilon(T_0 + \epsilon s_0) < \frac{1}{2}.$$

Let $t_0 = T_0 + \epsilon s_0$, with ϵ so that $\frac{\pi}{4} < t_0 < T_0$. Since u is increasing on $(0, T_0)$, it yields

$$u_\epsilon(\theta) < \frac{1}{2} \quad \text{and} \quad u_\epsilon^5(\theta) - u_\epsilon(\theta) + \kappa u_\epsilon(\theta) < 0 \tag{2.25}$$

for $0 < \theta < t_0$.

Lemma 2.4.2. *Suppose u_ϵ is a solution of problem (2.18) which is monotone on an interval $[t_1 - \delta, t_1 + \delta] \subset (0, \pi/2)$ and $u_\epsilon(t_1 \pm \delta) < 1/2$. Then there exists a constant $\beta > 0$ and $\epsilon_1 > 0$ such that if $\epsilon \in (0, \epsilon_1)$ then*

$$u_\epsilon(t_1) \le e^{-\frac{\beta}{\epsilon}}.$$

Proof. Suppose that u_ϵ is increasing on $(t_1 - \delta, t_1 + \delta)$ and choose K such that $\frac{\cos(2\theta)}{\sin(2\theta)} + K < 0$ for $\theta \in (t_1 - \delta, t_1 + \delta)$. Let φ_ϵ the solution of problem (2.6) centered in t_1. Let $v = \varphi_\epsilon - u_\epsilon$. Thus the function v satisfies

$$\begin{aligned}
\epsilon^2 v'' - 2\epsilon^2 K v' - \kappa v &= -\epsilon^2 u_\epsilon'' + 2\epsilon^2 K u_\epsilon' + \kappa u_\epsilon \\
&= 2\epsilon^2 (\tfrac{\cos(2\theta)}{\sin(2\theta)} + K) u_\epsilon' + u_\epsilon^5 - u_\epsilon + \kappa u_\epsilon \\
&< 2\epsilon^2 (\tfrac{\cos(2\theta)}{\sin(2\theta)} + K) u_\epsilon' \\
&\le 0,
\end{aligned} \tag{2.26}$$

for $\theta \in (t_1 - \delta, t_1 + \delta)$ because $u_\epsilon' \ge 0$. Moreover $v(t_1 \pm \delta) > 0$. Then it follows from the minimum principle that $v(\theta) > 0$ for all θ in the interval, and in particular for $\theta = t_1$. It follows from Lemma 2.3.1 that there exist $\beta, \epsilon_1 > 0$ such that if $\epsilon < \epsilon_1$

$$u_\epsilon(t_1) < \varphi_\epsilon(t_1) < e^{-\beta/\epsilon}.$$

The case when u_ϵ is decreasing is proved similarly, picking K such that $\frac{\cos(2\theta)}{\sin(2\theta)} + K > 0$. □

Then there exists an interval $(\pi/4 - \delta, \pi/4 + \delta)$ where the solution u_ϵ of problem (2.18) is strictly increasing and so $u_\epsilon(\pi/4 \pm \delta) < 1/2$. From Lemma 2.4.2 it follows that if $\epsilon < \epsilon_1$, then

$$u_\epsilon\,(\pi/4) \le e^{-\frac{\beta}{\epsilon}}. \tag{2.27}$$

Lemma (A). *There exist positive constants A and ϵ_0 such that*

$$|J_1(\epsilon)| < A\epsilon^{-2}e^{-\frac{2\beta}{\epsilon}} \qquad \text{for } \epsilon < \epsilon_0.$$

Proof. Let $\theta < \frac{\pi}{4}$. Integration of Eq. (2.18) over $(0, \theta)$ yields

$$\epsilon^2 u_\epsilon'(\theta) = -2\int_0^\theta \frac{\cos(2s)}{\sin(2s)} u_\epsilon'(s)\, ds + \int_0^\theta (u_\epsilon(s) - u_\epsilon(s)^5)\, ds.$$

Since $u > 0$ on $(0, \pi/4)$ we have

$$\epsilon^2 u_\epsilon'(\theta) < -2\int_0^\theta \frac{\cos(2s)}{\sin(2s)} u_\epsilon'(s)\, ds + \int_0^\theta u_\epsilon(s)\, ds.$$

Note that $\frac{\cos(2s)}{\sin(2s)} > 0$ for $s \in (0, \theta)$ and u_ϵ is increasing on $(0, \pi/4)$. Consequently

$$u_\epsilon'(\theta) < \epsilon^{-2} u_\epsilon(\pi/4)\theta,$$

and by previous remark we have

$$u_\epsilon'(\theta)^2 < \epsilon^{-4} u_\epsilon(\pi/4)^2\theta^2 < \epsilon^{-4} e^{-2\beta/\epsilon}\theta^2,$$

for all $\epsilon < \epsilon_1$. Finally

$$\begin{aligned} |J_1(\epsilon)| &= 2\epsilon^2 \int_0^{\frac{\pi}{4}} \frac{\cos(2\theta)}{\sin(2\theta)} u_\epsilon'(\theta)^2\, d\theta. \\ &< 2\epsilon^{-2} e^{-2\beta/\epsilon} \int_0^{\frac{\pi}{4}} \frac{\cos(2\theta)}{\sin(2\theta)} \theta^2\, d\theta. \end{aligned} \tag{2.28}$$

Let $A := 2\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos(2\theta)}{\sin(2\theta)}\theta^2\, d\theta > 0$. Then we have $|J_1(\epsilon)| < A\epsilon^{-2}e^{-2\beta/\epsilon}$. □

Lemma (B). *There exist constants B and $\epsilon_0 > 0$ such that*

$$J_2(\epsilon) \geq B\epsilon \qquad \text{for } \epsilon < \epsilon_0.$$

Proof. We shall see that

$$B := \liminf_{\epsilon \to 0} \frac{1}{\epsilon} J_2(\epsilon) > 0.$$

Replacing u_ϵ by z_ϵ in (2.22), we have

$$J_2(\epsilon) = -2\epsilon \int_{(\frac{\pi}{4} - T_0)/\epsilon}^0 \frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)} z_\epsilon'(s)^2\, ds.$$

Let Z_0 be the solution of problem (2.7)-(2.8) with $Z(0) = \sigma$. It follows from Lemma 2.3.3 that for any $L > 0$:

$$\int_{-L}^{0} \frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)} z_\epsilon'(s)^2\, ds \to \frac{\cos(2T_0)}{\sin(2T_0)} \int_{-L}^{0} Z_0'(s)^2\, ds,$$

as $\epsilon \to 0$. Note that if $\frac{\frac{\pi}{4} - T_0}{\epsilon} < -L < 0$ then we have that

$$\frac{1}{\epsilon} J_2(\epsilon) \geq -2 \int_{-L}^{0} \frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)} z_\epsilon'(s)^2\, ds.$$

Then

$$\begin{aligned} B := \liminf \tfrac{1}{\epsilon} J_2(\epsilon) &\geq -2 \liminf \int_{-L}^{0} \frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)} z_\epsilon'(s)^2\, ds \\ &= -2 \lim_{\epsilon \to 0} \int_{-L}^{0} \frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)} z_\epsilon'(s)^2\, ds \qquad (2.29) \\ &= -2 \tfrac{\cos(2T_0)}{\sin(2T_0)} \int_{-L}^{0} Z_0'(s)^2\, ds > 0. \end{aligned}$$

□

Lemma (C). *For ϵ small enough, there exists a positive constant C such that*

$$|F(\alpha_0(\epsilon))| \leq C e^{-\frac{2\beta}{\epsilon}}.$$

Proof. Since u_ϵ is increasing on $(0, \pi/4)$ it follows from (2.27) that

$$\alpha(\epsilon) < u_\epsilon(\pi/4) \leq e^{-\frac{\beta}{\epsilon}} \qquad \text{for } \epsilon < \epsilon_1,$$

and since $|F|$ is increasing on $(0, 1)$ it results that for ϵ small enough

$$|F(\alpha(\epsilon))| < |F(e^{-\frac{\beta}{\epsilon}})| \leq \frac{K}{2} e^{-\frac{2\beta}{\epsilon}}.$$

□

From Lemmas (A), (B) and (C) it follows that

$$F(u_0(\epsilon)) = F(\alpha_0) + J_1(\epsilon) + J_2(\epsilon),$$

with

$$|J_1(\epsilon)| < A\epsilon^{-2} e^{-2\beta/\epsilon}, \quad J_2(\epsilon) \geq B\epsilon, \quad |F(\alpha_0(\epsilon))| \leq C e^{-2\beta/\epsilon}$$

for ϵ small enough. This completes the proof of Lemma 2.4.1.

□

Fixed $T_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$, we considered the solution u_ϵ of problem (2.18) that reaches its first maximum at $\theta = T_0$ and we have proved that for ϵ is small enough $u_\epsilon(T_0) > \sigma$. Next we show that the solution hits the θ-axis shortly after T_0.

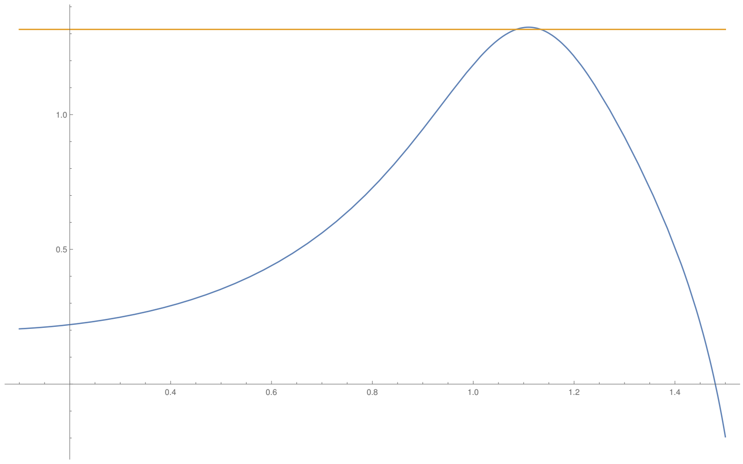

Figure 2.3: One-spike solution u of problem (2.1) with $u_\epsilon(\tau_1) > \sigma$.

Lemma 2.4.3. *There exists a constant $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exists $\tau_\epsilon < \frac{\pi}{2}$ with the following properties:*

$$u_\epsilon(\tau_\epsilon) = 0,$$

and

$$\begin{cases} u'_\epsilon(\theta) > 0 & \text{for } 0 < \theta < T_0, \\ u'_\epsilon(\theta) < 0 & \text{for } T_0 < \theta < \tau_\epsilon. \end{cases} \tag{2.30}$$

Moreover

$$|T_0 - \tau_\epsilon| = O(\sqrt{\epsilon}) \quad \text{when } \epsilon \to 0. \tag{2.31}$$

Proof. Recall that u_ϵ has the following properties at T_0:

$$u_\epsilon(T_0) > \sigma \quad \text{and} \quad u'_\epsilon(T_0) = 0.$$

From Eq. (2.18) it is easy to see that there is a constant $C > 0$ such that

$$u''_\epsilon(\theta) < -\frac{C}{\epsilon^2} \tag{2.32}$$

for all $\theta > T_0$ while $u_\epsilon(\theta) > \sigma$. Integration of (2.32) over (T_0, θ) yields

$$u'_\epsilon(\theta) < -\frac{C}{\epsilon^2}(\theta - T_0).$$

We know that $|u_0(\epsilon)| < M$ for all ϵ small and for some $M > 0$. Then

$$u_\epsilon(\theta) - u_0(\epsilon) < -\frac{C}{2\epsilon^2}(\theta - T_0)^2. \tag{2.33}$$

Since $u_0(\epsilon) > \sigma$ and u is decreasing while $\theta > T_0$ and $u_\epsilon(\theta) > 1$, there exists $\tau_\sigma > T_0$ such that

$$u_\epsilon(\tau_\sigma) = \sigma \quad \text{and} \quad u'_\epsilon(\tau_\sigma) < 0.$$

Taking $\theta = \tau_\sigma$ in (2.33) it follows that

$$\sigma - u_0(\epsilon) < -\frac{C}{2\epsilon^2}(\tau_\sigma - T_0)^2.$$

Finally we have

$$|\tau_\sigma - T_0| = O(\epsilon). \tag{2.34}$$

Next we use the energy function associated with u_ϵ defined in (2.5). If $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, then E'_ϵ satisfies

$$E'_\epsilon(\theta) = -2\frac{\cos(2\theta)}{\sin(2\theta)}u'_\epsilon(\theta)^2 > 0.$$

Consequently integration of $E'_\epsilon(\theta)$ over (T_0, τ_σ) yields

$$0 < \frac{u'_\epsilon(\tau_\sigma)^2}{2} - \frac{1}{\epsilon^2}F(u_0(\epsilon)).$$

Therefore from Lemma 2.4.1 it follows:

$$u'_\epsilon(\tau_\sigma)^2 > \frac{2}{\epsilon^2}F(u_0(\epsilon)) > \frac{A}{\epsilon}. \tag{2.35}$$

Define

$$\tau_\epsilon = \sup\{T_0 < \theta < \frac{\pi}{2} : u_\epsilon > 0 \text{ and } u'_\epsilon < 0 \text{ on } (T_0, \theta)\},$$

and integrate E'_ϵ over (τ_σ, θ) with $\tau_\sigma < \theta < \tau_\epsilon$. Then

$$\frac{u'_\epsilon(\theta)^2}{2} + \frac{1}{\epsilon^2}F(u_\epsilon(\theta)) > \frac{u'_\epsilon(\tau_\sigma)^2}{2}. \tag{2.36}$$

Since $F(u_\epsilon(\theta)) < 0$ and $u'_\epsilon(\tau_\sigma) < 0$, it follows from (2.35) and (2.36) that

$$u'_\epsilon(\theta) < u'_\epsilon(\tau_\sigma) < -\sqrt{\frac{A}{\epsilon}} \quad \text{for } \tau_\sigma < \theta < \tau_\epsilon.$$

Now we have:

$$|\tau_\epsilon - \tau_\sigma| = O(\sqrt{\epsilon}). \tag{2.37}$$

Write

$$|\tau_\epsilon - T_0| = |\tau_\epsilon - \tau_\sigma| + |\tau_\sigma - T_0|. \tag{2.38}$$

Putting the estimates (2.34)-(2.37) into (2.38) we obtain the estimate (2.31).

□

This result allows us to establish the following

Proposition 2.4.4. *For ϵ small enough there exists $\alpha_0 \in \mathcal{A}(\epsilon)$ such that the solution $u_{\alpha_0,\epsilon}$ of problem (2.18) with initial value $\alpha_0(\epsilon)$ has exactly one spike.*

Let $\mathcal{A}_1(\epsilon)$ the connected components of $\mathcal{A}(\epsilon)$ such that the solutions $u_{\alpha,\epsilon}$ with $\alpha \in \mathcal{A}_1(\epsilon)$ have exactly one spike. By the previous proposition for ϵ small enough $\mathcal{A}_1(\epsilon)$ is not empty.

Proposition 2.4.5. *Let $(\alpha_1^-, \alpha_1^+) \subset (0,1)$ be any connected component of $\mathcal{A}_1(\epsilon)$ and let $\Theta(\alpha,\epsilon)$ be as in (2.19). Then*

$$\lim_{\alpha \to \alpha_1^{\pm}} \Theta(\alpha,\epsilon) = \frac{\pi}{2}.$$

Proof. Suppose that the assertion of Proposition 2.4.5 is not true, so that there exists a sequence $\{\alpha_n\}$ which converges to, say α_1^-, such that $\Theta(\alpha_n,\epsilon)$ converges to a point $\theta_\infty < \frac{\pi}{2}$. Then, by continuity $\Theta(\alpha_1^-,\epsilon) = \theta_\infty$ and therefore $\alpha_1^- \in \mathcal{A}(\epsilon)$, which contradicts the definition of α_1^-.

□

This proposition enables us to define:

$$\Theta^1_{min,\epsilon} = \min\{\Theta(\alpha,\epsilon) : \alpha \in \mathcal{A}_1(\epsilon)\}.$$

Proposition 2.4.6.

$$\lim_{\epsilon \to 0} \Theta^1_{min,\epsilon} = \frac{\pi}{4}. \tag{2.39}$$

Proof. In order to prove Proposition 2.4.4 we introduced an arbitrary point $T_0 > \frac{\pi}{4}$. We may choose this point arbitrarily close to $\frac{\pi}{4}$. In Lemma 2.4.3 it has been shown that by choosing ϵ small enough, we can achieve that τ_ϵ is arbitrary close to T_0. Then we have (2.39). □

It follows from Proposition 2.4.6 that, given $\theta_1 \in (\frac{\pi}{4}, \frac{\pi}{2})$, there exists $\epsilon_1 > 0$ such that if $\epsilon < \epsilon_1$, then

$$\frac{\pi}{4} < \Theta^1_{min,\epsilon} < \theta_1.$$

Let $\Gamma_1(\epsilon) = \{(\alpha, \Theta(\alpha,\epsilon)) : \alpha \in (\alpha_1^-, \alpha_1^+)\}$, where (α_1^-, α_1^+) is a connected component of $\mathcal{A}_1(\epsilon)$ such that $\min\{\Theta(\alpha,\epsilon) : \alpha \in (\alpha_1^-, \alpha_1^+)\} < \theta_1$. Hence $\Gamma_1(\epsilon)$ intersects the line $\theta = \theta_1$ at least twice for all $\epsilon < \epsilon_1$. This yields at least two $\alpha_1(\epsilon), \alpha_2(\epsilon) \in \mathcal{A}_1(\epsilon)$ such that $u_{\alpha_1(\epsilon)}, u_{\alpha_2(\epsilon)}$ are solutions of problem (2.18) having exactly one spike, and this completes the proof of Theorem 0.1.2 for the case $k = 1$. In others words, we have proved that for ϵ small enough there are at least two solutions with a single spike.

Now we prove Theorem 0.1.2 for $k = 2$ in a similar way. We shall prove that given any $\theta_1 > \pi/4$ there exists $\epsilon_2 > 0$ such that if $\epsilon < \epsilon_2$, then problem

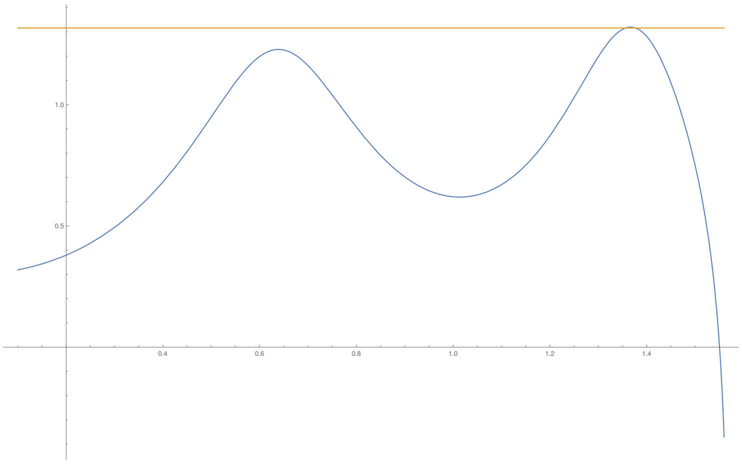

Figure 2.4: Two-spike solution u of problem (2.1) with $u_0(\epsilon) \geq \sigma$

(2.18) has at least two solutions with initial value on $(0, 1)$ that have exactly two spikes.

Repeating the argument we fix $T_0 \in (\frac{\pi}{4}, \theta_1)$. For ϵ small enough we find an initial value $\alpha_0 \in (0, 1)$ such that

$$\tau_3(\alpha_0) = T_0.$$

Write $\alpha_0 = \alpha_0(\epsilon)$; $u_\epsilon(\theta) = u_{\alpha_0(\epsilon)}(\theta)$; $u_0(\epsilon) = u_\epsilon(T_0)$ and $\tau_k(\alpha_0(\epsilon)) = \tau_k(\epsilon)$. Then

$$\tau_1(\epsilon) < \tau_2(\epsilon) < \tau_3(\epsilon) = T_0. \tag{2.40}$$

We have the following results:

Lemma 2.4.7.

$$\limsup_{\epsilon \to 0} \tau_1(\epsilon) \leq \frac{\pi}{4}. \tag{2.41}$$

Proof. Let $\tau_+ = \limsup_{\epsilon \to 0} \tau_1(\epsilon)$. Note that $\tau_+ \leq T_0$ and suppose that $\frac{\pi}{4} < \tau_+ \leq T_0$. Then, repeating the previous argument with T_0 replaced by τ_+, we find that for ϵ small enough, the solution u_ϵ has a zero τ_ϵ in a right neighbourhood of τ_+ and is strictly decreasing on (τ_+, τ_ϵ). Since, by construction, u_ϵ has a local maximum at T_0 for every $\epsilon > 0$, which lies above the line $u = 1$, this is not possible. This completes the proof. □

Lemma 2.4.8. *Let u_ϵ be a 2-spike solution of (2.18) with $\tau_2(\epsilon)$ the second critical point. Then there are constants $\beta > 0$ and $\epsilon_1 > 0$ such that*

$$u_\epsilon(\tau_2(\epsilon)) \leq e^{-\frac{\beta}{\epsilon}} \textit{ for } \epsilon < \epsilon_1.$$

Proof. From (2.40) and (2.41) follows that for any fixed $\epsilon > 0$ there exists $\delta > 0$ (independent of ϵ small enough) such that

$$\begin{aligned} \text{either} \quad |\tau_2(\epsilon) - T_0| &> 2\delta, && (2.42)\\ \text{or} \quad |\tau_2(\epsilon) - \pi/4| &> 2\delta. && (2.43) \end{aligned}$$

Assume that we have (2.42) and let t_1 be such that

$$(t_1 - \delta, t_1 + \delta) \subset (\tau_2(\epsilon), T_0) \quad \text{and} \quad u_\epsilon(t_1 \pm \delta) < 1/2.$$

Then u is increasing on $(t_1 - \delta, t_1 + \delta)$ and it follows from Lemma 2.4.2 that

$$u_\epsilon(\tau_2(\epsilon)) < u_\epsilon(t_1) < e^{-\beta/\epsilon},$$

where β does not depend on ϵ. The case in which δ satisfies (2.43) is analogous (note that the constant 1/2 used in Lemma 2.3.1 and Lemma 2.4.2 can be replaced for any other positive constant, as long as it is independent of ϵ). □

Lemma 2.4.9.

$$\lim_{\epsilon\to 0} \frac{T_0 - \tau_2(\epsilon)}{\epsilon} = \infty.$$

Proof. Note that u_ϵ is a positive solution of the equation

$$u_\epsilon''(\theta) + 2\frac{\cos(2\theta)}{\sin(2\theta)}u_\epsilon'(\theta) = \frac{u_\epsilon(\theta) - u_\epsilon(\theta)^5}{\epsilon^2} \tag{2.44}$$

such that

$$u_\epsilon(\tau_2(\epsilon)) = e^{-\tilde{\beta}/\epsilon} \quad u_\epsilon'(\tau_2(\epsilon)) = 0 \quad u_\epsilon(T_0) = u_0(\epsilon) \quad u_\epsilon'(T_0) = 0.$$

We want to show that $u_\epsilon(\tau_2(\epsilon) + \sqrt{\epsilon}) < 1$, because u_ϵ is increasing in the interval $(\tau_2(\epsilon), T_0)$ and $u_\epsilon(\tau_2(\epsilon)) < 1 < u_\epsilon(T_0)$. This means that u_ϵ cannot catch up $u_0(\epsilon)$ in the interval $(\tau_2(\epsilon), \tau_2(\epsilon) + \sqrt{\epsilon})$. Then

$$\frac{T_0 - \tau_2(\epsilon)}{\epsilon} > \frac{\sqrt{\epsilon}}{\epsilon} \to \infty \quad \text{when } \epsilon \to 0.$$

To see that, consider the linear auxiliary problem:

$$w''(\theta) + 2\frac{\cos(2T_0)}{\sin(2T_0)}w'(\theta) = \frac{w(\theta)}{\epsilon^2} \tag{2.45}$$

with initial conditions

$$w(\tau_2(\epsilon)) = e^{-\tilde{\beta}/\epsilon} \quad w'(\tau_2(\epsilon)) = 0.$$

Then by the Sturm Comparison Theory for all $0 < \theta < T_0$, we have $u_\epsilon(\theta) < w(\theta)$. In particular,

$$u_\epsilon(\tau_2(\epsilon) + \sqrt{\epsilon}) < w(\tau_2(\epsilon) + \sqrt{\epsilon}).$$

Note that $w(\theta) = Ae^{c_1(\theta-\tau_2(\epsilon))} + Be^{c_2(\theta-\tau_2(\epsilon))}$, where c_1, c_2 are the roots of the equation $\epsilon^2 x^2 - 2\epsilon^2 Kx - 1 = 0$, with $K = -\frac{\cos(2T_0)}{\sin(2T_0)}$. Let $\mu_\epsilon = K^2 + \kappa/\epsilon^2$. Then A, B are given by

$$A = \frac{K - \sqrt{\epsilon}}{2\sqrt{\mu_\epsilon}} e^{-\tilde{\beta}/\epsilon}, \qquad B = \frac{K + \sqrt{\epsilon}}{2\sqrt{\mu_\epsilon}} e^{-\tilde{\beta}/\epsilon}.$$

Finally we have

$$w(\tau_2(\epsilon) + \sqrt{\epsilon}) = \epsilon^{-\tilde{\beta}/\epsilon} + (K - \sqrt{\mu_\epsilon})\sqrt{\epsilon} + \frac{K - \sqrt{\epsilon}}{2\sqrt{\mu_\epsilon}} e^{-\tilde{\beta}/\epsilon + 2\sqrt{\mu_\epsilon}\sqrt{\epsilon}}.$$

Consequently, for ϵ small enough $w(\tau_2(\epsilon) + \sqrt{\epsilon}) < 1$.

□

Integration of $E'_\epsilon(\theta)$ over $(\tau_2(\epsilon), T_0)$ yields

$$F(u_0(\epsilon)) - F(u_\epsilon(\tau_2(\epsilon))) = J(\epsilon),$$

where

$$J(\epsilon) = -2\epsilon^2 \int_{\tau_2(\epsilon)}^{T_0} \frac{\cos(2\theta)}{\sin(2\theta)} u'_\epsilon(\theta)^2 d\theta.$$

Then

$$F(u_0(\epsilon)) = F(u_\epsilon(\tau_2(\epsilon))) + J(\epsilon). \tag{2.46}$$

Next we show that there is a constant $A > 0$ such that $F(u_0(\epsilon)) > A\epsilon$ for ϵ enough small.

Lemma ($\tilde{B}$)**.** *There is a constant $C_1 > 0$ such that*

$$J(\epsilon) > C_1\epsilon$$

for ϵ small enough.

Proof. To prove this lemma we may assume that $\tau_2(\epsilon) > \pi/4$, because when $\tau_2(\epsilon) < \pi/4$, the proof operates in the same way as before. Write $\theta = T_0 + \epsilon s$ and $z_\epsilon(s) = u_\epsilon(\theta)$ and replace u by z in J. Then, z_ϵ solves problem (2.13) and

$$J(\epsilon) = -2\epsilon \int_{\frac{\tau_2(\epsilon)-T_0}{\epsilon}}^{0} \frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)} z'_\epsilon(s)^2 ds.$$

It follows from Lemma 2.3.3 and Lemma 2.4.9 that for any $0 < L < (T_0 - \pi/4)/\epsilon$

$$\begin{aligned} C_1 := \liminf \tfrac{1}{\epsilon} J(\epsilon) &\geq -2 \lim_{\epsilon\to 0} \int_{-L}^{0} \frac{\cos(2T_0 + 2s\epsilon)}{\sin(2T_0 + 2s\epsilon)} z_\epsilon'(s)^2 \, ds \\ &= -2 \tfrac{\cos(2T_0)}{\sin(2T_0)} \int_{-L}^{0} Z_0'(s)^2 \, ds > 0. \end{aligned} \tag{2.47}$$

□

The following lemma will be needed in order to complete the proof and follows immediately from Lemma 2.4.8.

Lemma $(\tilde{C})$**.** *There is a constant $C_2 > 0$ such that*

$$|F(u_\epsilon(\tau_2(\epsilon)))| < C_2 e^{-\beta/2\epsilon}$$

for ϵ small enough.

From Lemmas $(\tilde{B})$, $(\tilde{C})$ and the Eq. (2.46) we can see that $F(u_0(\epsilon)) > 0$ for ϵ enough small. Then it follows that $u_0(\epsilon) \geq \sigma$ and we can repeat the argument in Lemma 2.4.3 to prove that u_ϵ has a zero $\tau_\epsilon \in (T_0, \frac{\pi}{2})$ such that $|T_0 - \tau_\epsilon| = O(\sqrt{\epsilon})$. It allows us to establish the following

Proposition 2.4.10. *For ϵ small enough there exists $\alpha_0 \in \mathcal{A}(\epsilon)$ such that the solution $u_{\alpha_0}(\theta)$ of problem (2.18) with initial value α_0 has exactly two spikes, where $\mathcal{A}(\epsilon)$ is the set defined in (2.20).*

Let $\mathcal{A}_2(\epsilon)$ be the connected components of $\mathcal{A}(\epsilon)$ such that the solutions $u_{\alpha,\epsilon}$ with $\alpha \in \mathcal{A}_2(\epsilon)$ have exactly two spikes. The proof of Theorem 0.1.2 for $k = 2$ results from the following propositions.

Proposition 2.4.11. *Let $(\alpha_2^-, \alpha_2^+) \subset (0, 1)$ be any connected component of $\mathcal{A}_2(\epsilon)$ and $\Theta(\alpha, \epsilon)$ as in (2.19). Then*

$$\lim_{\alpha\to\alpha_2^\pm} \Theta(\alpha, \epsilon) = \frac{\pi}{2}.$$

Now we can define:

$$\Theta^2_{min,\epsilon} = \min\{\Theta(\alpha, \epsilon) : \alpha \in \mathcal{A}_2(\epsilon)\}.$$

Proposition 2.4.12.

$$\lim_{\epsilon\to 0} \Theta^2_{min,\epsilon} = \frac{\pi}{4}. \tag{2.48}$$

Then it follows as in the case $k = 1$ that there are at least two $\alpha_1(\epsilon), \alpha_2(\epsilon) \in \mathcal{A}_2(\epsilon)$ such that $u_{\alpha_1(\epsilon)}, u_{\alpha_2(\epsilon)}$ are solutions of problem (2.18) having exactly two spikes, and thus completes the proof of Theorem 0.1.2 for $k = 2$.

Finally, we turn to solutions with k spikes. They are located at the points $\{\tau_{2j-1} : j = 1, 2, \ldots, k\}$. In the construction we fix $\tau_{2k-1} = T_0$ and we show that $\limsup \tau_{2(k-1)-1} \leq \pi/4$. Consequently $F(u_0(\epsilon)) > 0$ and there exists $\alpha_0(\epsilon)$ such that the solution of (2.18) $u_{\epsilon,\alpha_0(\epsilon)}$ has k spikes. This can be done with the methods developed in this section. Let $\mathcal{A}_k(\epsilon)$ be the connected components of $\mathcal{A}(\epsilon)$ which contains the solutions with k spikes. Let $(\alpha_k^-, \alpha_k^+) \subset (0, 1)$ be any connected component of $\mathcal{A}_k(\epsilon)$. Then it can be shown that

$$\lim_{\alpha \to \alpha_k^\pm} \Theta(\alpha, \epsilon) = \frac{\pi}{2}.$$

Now we can define $\Theta^k_{min,\epsilon} = \min\{\Theta(\alpha, \epsilon) : \alpha \in \mathcal{A}_k(\epsilon)\}$ and it turns out that

$$\lim_{\epsilon \to 0} \Theta^k_{min,\epsilon} = \frac{\pi}{4}.$$

It follows that, given $\theta_1 \in (\frac{\pi}{4}, \frac{\pi}{2})$ we have that for ϵ small enough

$$\frac{\pi}{4} < \Theta^k_{min,\epsilon} < \theta_1.$$

Then exactly as in the cases $k = 1$ and $k = 2$ we obtain at least two solutions of problem (2.18) having exactly k spikes. This completes the proof of Theorem 0.1.2.

Resumen del Capítulo

En este capítulo estudiamos el siguiente problema:

$$\begin{cases} u''(\theta) + 2\frac{\cos(2\theta)}{\sin(2\theta)}u'(\theta) &= \lambda\Big(u(\theta)^5 - u(\theta)\Big), \quad u > 0 \quad \text{on } (0, \theta_1), \\ u'(0) &= 0, \\ u(\theta_1) &= 0, \end{cases}$$

donde $u : [0, \theta_1] \to \mathbb{R}$. Primero probamos un teorema de no existencia de soluciones:

Theorem 2.4.13. *Si* $\theta_1 \in (0, \pi/4)$*, entonces no hay soluciones del problema con valor inicial en el intervalo* $(0, 1)$*.*

Luego probamos el teorema que se enuncia a continuación, basándonos principalmente en un método que se ha utilizado con éxito en [14]:

Theorem 2.4.14. *Dado cualquier entero positivo k y cualquier $\theta_1 > \pi/4$, existe una constante $A_k > 0$ tal que para $\lambda < -A_k$ el problema dado tiene al menos $2k$ soluciones con valor inicial en el intervalo* $(0, 1)$.

Primero usamos este método para mostrar que existen al menos 2 soluciones a este problema con valor inicial en el intervalo $(0, 1)$ que tienen un solo pico o máximo. El siguiente paso es probar el teorema en el caso $k = 2$ usando las mismas técnicas. Finalmente el teorema sigue por inducción.

El capítulo está organizado de la siguiente forma. En la sección 2.2 demostramos el teorema de no existencia. La sección 2.3 contiene algunos resultados sobre problemas lineales auxiliares, que nos ayudarán a probar el teorema principal en la siguiente sección.

También estamos interesados en estudiar soluciones de la ecuación invariante por la acción $\mathbb{T}^2$ en toda la esfera $\mathbb{S}^3$. En la sección 2.4 demostramos un resultado sobre multiplicidad de soluciones para este caso especial, junto con la demostración del teorema 2.4.14.

Chapter 3

The Yamabe equation on a product manifold

3.1 Introduction

In this chapter we will use Lyapunov-Schmidt reduction techniques to prove the multiplicity results for the Yamabe equation in Riemannian products, proving Theorem 0.2.1.

Let (M^n, g) be any closed manifold and (N^m, h) a manifold of constant positive scalar curvature s_h. We will be interested in positive solutions of the Yamabe equation for the product manifold $(M \times N, g + \epsilon^2 h)$:

$$-a(\Delta_g + \Delta_{\epsilon^2 h})u + (s_g + \epsilon^{-2} s_h)u = u^{p-1}, \tag{3.1}$$

with $a = a_{m+n} = \frac{4(m+n-1)}{m+n-2}$, $p = p_{m+n} = \frac{2(m+n)}{m+n-2}$, s_g the scalar curvature of (M^n, g), and ϵ small enough so that the scalar curvature $s_g + \epsilon^{-2} s_h$ is positive. The conformal metric $u^{p-2}(g + \epsilon^2 h)$ then has constant scalar curvature.

By restricting our study to solution functions that depend only on the first factor, $u : M \to \mathbb{R}$, and by normalizing h so that $s_h = a$, we note that solving the Yamabe equation is equivalent to solving:

$$-\epsilon^2 \Delta_g u + \left(\frac{s_g}{a}\epsilon^2 + 1\right) u = u^{p-1}. \tag{3.2}$$

We will study the general equation

$$-\epsilon^2 \Delta_g u + \left(\lambda s_g \epsilon^2 + 1\right) u = u^{p-1}, \tag{3.3}$$

where $\lambda \in \mathbb{R}$. Positive solutions of this equation are the critical points of the functional $J_\epsilon : H^{1,2}(M) \to \mathbb{R}$, given by

$$J_\epsilon(u) = \epsilon^{-n} \int_M \left(\frac{1}{2}\epsilon^2 |\nabla u|^2 + \frac{1}{2}\left(\epsilon^2 \lambda s_g + 1\right) u^2 - \frac{1}{p}(u^+)^p\right) dV_g,$$

where $u^+(x) = \max\{u(x), 0\}$.

Our goal is to obtain solutions of the equation for ϵ small. We will build solutions by using the Lyapunov-Schmidt reduction procedure which was applied by several authors (see [15] and [26]).

To explain the construction one first considers what will be called the limit equation in $\mathbb{R}^n$. Recall that for $2 < q < \frac{2n}{n-2}$, $n > 2$, the equation

$$-\Delta U + U = U^{q-1} \text{ in } \mathbb{R}^n \tag{3.4}$$

has a unique (up to translations) positive solution $U \in H^1(\mathbb{R}^n)$ that vanishes at infinity. Such function is radial and exponentially decreasing at infinity, namely

$$\lim_{|x|\to\infty} U(x)|x|^{\frac{n-1}{2}} e^{|x|} = c > 0 \tag{3.5}$$

$$\lim_{|x|\to\infty} |\nabla U(x)|\, |x|^{\frac{n-1}{2}} e^{|x|} = c. \tag{3.6}$$

See reference [24] for details. We will denote this solution by U in the following.

Note that for any $\epsilon > 0$, the function $U_\epsilon(x) = U(\frac{x}{\epsilon})$, is a solution of

$$-\epsilon^2 \Delta U_\epsilon + U_\epsilon = U_\epsilon^{q-1}.$$

We define the following constant associated with the solution of the limit equation U:

$$\beta_\lambda := \lambda \int_{\mathbb{R}^n} U^2(z)\, dz - \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U(z)|^2 |z|^2\, dz. \tag{3.7}$$

The sign of β_λ is fundamental to understand the role of the critical points of the scalar curvature.

For any $x \in M$ consider the exponential map $\exp_x : T_pM \to M$. Since M is closed we can fix $r_0 > 0$ such that $\exp_x\big|_{B(0,r_0)} : B(0, r_0) \to B_g(x, r_0)$ is a diffeomorphism for any $x \in M$. Here $B(0, r)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius r and $B_g(x, r)$ is the ball in M centered at x with radius r with respect to the distance induced by the metric g:

$$d_g(x, y) = \exp_x^{-1}(y).$$

Let χ_r be a smooth cut-off function such that $\chi_r(z) = 1$ if $z \in B(0, r/2)$, $\chi_r(z) = 0$ if $z \in \mathbb{R}^n \setminus B(0, r)$, $|\nabla\chi_r(z)| < 2/r$ and $|\nabla^2\chi_r(z)| < 2/r^2$.

For any $r < r_0$ fixed, a point $\xi \in M$ and $\epsilon > 0$ let us define on M the function

$$W_{\epsilon,\xi}(x) = \begin{cases} U_\epsilon(\exp_\xi^{-1}(x))\chi_r(\exp_\xi^{-1}(x)) & if \quad x \in B_g(\xi, r), \\ 0 & otherwise. \end{cases}$$

We will prove that if $\beta_\lambda < 0$ then there exists a critical point $\overline{\xi}_\epsilon$ of $\overline{J_\epsilon}$. If $\beta_\lambda > 0$ one can prove the same result replacing the isolated local minimum of the scalar curvature by an isolated local maximum. Then we have

Theorem 3.1.1. *Assume that $\beta_\lambda \neq 0$. If $\beta_\lambda < 0$ ($\beta_\lambda > 0$) then let ξ_0 be an isolated local maximum (minimum) point of the scalar curvature S_g. For each positive integer k_0, there exists $\epsilon_0 = \epsilon_0(k_0) > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ there exist points $\xi_1^\epsilon, \ldots, \xi_{k_0}^\epsilon \in M$ such that*

$$\frac{d_g(\xi_i^\epsilon, \xi_j^\epsilon)}{\epsilon} \to +\infty \quad \textit{and} \quad d_g(\xi_0, \xi_j^\epsilon) \to 0, \qquad \textit{as } \epsilon \to 0, \tag{3.8}$$

and a solution u_ϵ of problem (3.3) *such that*

$$\|u_\epsilon - \sum_{i=1}^{k_0} W_{\epsilon,\xi_i^\epsilon}\|_\epsilon \to 0 \qquad \textit{as } \epsilon \to 0.$$

The Yamabe constant of these products tends to the Yamabe constant of the product with $\mathbb{R}^n$. Then the limit of this constant is $m(E)$. In [33] it is proved that the equation has $Cat(M)$ solutions with energy close to $m(E)$.

Note that for the Yamabe equation on $(M \times N, g + \epsilon^2 h)$ the constant β become:

$$\beta = \frac{n+m-2}{4(n+m-1)} \int_{\mathbb{R}^n} U^2(z)\, dz - \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U(z)|^2 |z|^2\, dz. \tag{3.9}$$

So it depends only on n, m. We have not been able to find an analytical proof that $\beta \neq 0$ but we give the numerical computation of β for low values of m and n in [36]. In all cases $\beta < 0$. Then, by the Theorem 0.2.1 we show the multiplicity of metrics of constant scalar curvature in the product manifold $(M \times N, g + \epsilon^2 h)$.

The chapter is organized as follows. In Section 3.2 we will introduce notation and background and discuss the finite dimensional reduction of the problem by the Lyapunov-Schmidt procedure. Then we will construct approximate solutions for the equation. In Sections 3.3 we will explain the asymptotic expansion of the functional energy in terms of ϵ. The Section 3.4 is devoted to prove Theorem 0.2.1 assuming the technical Proposition 3.2.1, which is proved in Section 3.5.

3.2 Approximate solutions and the reduction of the equation

Positive solutions of (3.4) are the critical points of the functional $E : H^1(\mathbb{R}^n) \to \mathbb{R}$,

$$E(f) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla f|^2 + \frac{1}{2} f^2 - \frac{1}{p}(f^+)^p \right) dx.$$

Let $S_0 = \nabla E : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$. $S_0(U) = 0$ and the solution U is non-degenerate in the sense that Kernel$(S_0'(U))$ is spanned by

$$\psi^i(x) := \frac{\partial U}{\partial x_i}(x)$$

with $i = 1, \dots, n$.

Note that for any $\epsilon > 0$, the function $U_\epsilon(x) = U(\frac{x}{\epsilon})$, is a solution of

$$-\epsilon^2 \Delta U_\epsilon + U_\epsilon = U_\epsilon^{p-1}, \tag{3.10}$$

and so it is a critical point of the functional

$$E_\epsilon(f) = \epsilon^{-n} \int_{\mathbb{R}^n} \left(\frac{\epsilon^2}{2} |\nabla f|^2 + \frac{1}{2} f^2 - \frac{1}{p} (f^+)^p \right) dx.$$

If $S_{0\epsilon} = \nabla E_\epsilon$ then Kernel$(S_{0\epsilon}'(U_\epsilon))$ is spanned by the functions

$$\psi^i_\epsilon(x) := \psi^i(\epsilon^{-1} x)$$

with $i = 1, \dots, n$.

Let us define on M the functions

$$Z^i_{\epsilon,\xi}(x) := \begin{cases} \psi^i_\epsilon(\exp_\xi^{-1}(x)) \chi_r(\exp_\xi^{-1}(x)) & \text{if} \quad x \in B_g(\xi, r), \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Let H_ϵ be the Hilbert space $H^1_g(M)$ equipped with the inner product

$$\langle u, v \rangle_\epsilon := \frac{1}{\epsilon^n} \Big(\epsilon^2 \int_M \nabla_g u \nabla_g v \, d\mu_g + \int_M (\epsilon^2 \lambda s_g + 1) \, uv \, d\mu_g \Big),$$

which induces the norm

$$\|u\|^2_\epsilon := \frac{1}{\epsilon^n} \Big(\epsilon^2 \int_M |\nabla_g u|^2 \, d\mu_g + \int_M (\epsilon^2 \lambda s_g + 1) \, u^2 \, d\mu_g \Big).$$

Similarly on $\mathbb{R}^n$ we define the inner product for $u, v \in H^1_\epsilon(\mathbb{R}^n)$

$$\langle u, v \rangle_\epsilon := \frac{1}{\epsilon^n} \Big(\epsilon^2 \int_{\mathbb{R}^n} \nabla u \nabla v \, dz + \int_{\mathbb{R}^n} uv \, dz \Big),$$

which induces the norm

$$\|u\|^2_\epsilon := \frac{1}{\epsilon^n} \Big(\epsilon^2 \int_{\mathbb{R}^n} |\nabla u|^2 \, dz + \int_{\mathbb{R}^n} u^2 \, dz \Big).$$

It is important to note that $\|f_\epsilon\|_\epsilon$ is independent of ϵ, where as before $f_\epsilon(x) = f(\frac{x}{\epsilon})$.

For $\epsilon > 0$ and $\overline{\xi} = (\xi_1, ..., \xi_{k_0}) \in M^{k_0}$ let

$$K_{\epsilon,\overline{\xi}} := \text{ span } \{Z^i_{\epsilon,\xi_j} : i = 1, \dots, n, j = 1, \dots, k_0\}$$

and

$$K^\perp_{\epsilon,\overline{\xi}} := \{\phi \in H_\epsilon : \langle \phi, Z^i_{\epsilon,\xi_j}\rangle_\epsilon = 0, i = 1, \dots, n, j = 1 \dots, k_0\}.$$

Let $\Pi_{\epsilon,\overline{\xi}} : H_\epsilon \to K_{\epsilon,\overline{\xi}}$ and $\Pi^\perp_{\epsilon\overline{\xi}}$: $H_\epsilon \to K^\perp_{\epsilon,\overline{\xi}}$ be the orthogonal projections. In order to solve equation (3.3) we call

$$S_\epsilon = \nabla J_\epsilon : H_\epsilon \to H_\epsilon.$$

Equation (3.3) is then $S_\epsilon(u) = 0$. The idea is that the kernel of $S'_\epsilon(V_{\epsilon,\overline{\xi}})$ should be close to $K_{\epsilon,\overline{\xi}}$ and then the linear map $\phi \mapsto \Pi^\perp_{\epsilon\overline{\xi}} S'_\epsilon(V_{\epsilon,\overline{\xi}})(\phi) : K^\perp \to K^\perp$ should be invertible. Then the Inverse Function Theorem would imply that there is a unique small $\phi = \phi_{\epsilon,\overline{\xi}} \in K^\perp_{\epsilon,\overline{\xi}}$ such that (this is the content of Proposition 3.2.1)

$$\Pi^\perp_{\epsilon,\overline{\xi}}\{S_\epsilon(V_{\epsilon,\overline{\xi}} + \phi)\} = 0. \tag{3.12}$$

And then we have to solve the finite dimensional problem

$$\Pi_{\epsilon,\overline{\xi}}\{S_\epsilon(V_{\epsilon,\overline{\xi}} + \phi)\} = 0. \tag{3.13}$$

Consider the function $\overline{J_\epsilon} : M^{k_0} \to \mathbb{R}$ defined by

$$\overline{J_\epsilon}(\overline{\xi}) := J_\epsilon(V_{\epsilon,\overline{\xi}} + \phi_{\epsilon,\overline{\xi}}) .$$

We will show in Proposition 3.3.1 that (3.13) is equivalent to finding critical points of $\overline{J_\epsilon}$.

Let $\xi_0 \in M$ be an isolated local minimum point of the scalar curvature. Let $k_0 \geq 1$ be a fixed integer. Given $\rho > 0$, $\epsilon > 0$ we consider the open set

$$D^{k_0}_{\epsilon,\rho} := \Big\{\overline{\xi} \in M^{k_0} \quad : \quad d_g(\xi_0, \xi_i) < \rho, i = 1, \dots, k_0, \ \sum_{i \neq j}^{k_0} U_\epsilon\Big(\exp^{-1}_{\xi_i} \xi_j\Big) < \epsilon^2\Big\}. \tag{3.14}$$

Recall $U_\epsilon\Big(\exp^{-1}_{\xi_i} \xi_j\Big) = U\Big(\epsilon^{-1} \exp^{-1}_{\xi_i} \xi_j\Big)$ and that U is a radial, positive, decreasing function. Then if $\overline{\xi}_\epsilon = (\xi_{\epsilon 1}, ..., \xi_{\epsilon k_0}) \in D^{k_0}_{\epsilon,\rho}$ since $\| \exp^{-1}_{\xi_i} \xi_j\| = d_g(\xi_{\epsilon i}, \xi_{\epsilon j})$ we have that

$$\lim_{\epsilon \to 0} \frac{d_g(\xi_{\epsilon i}, \xi_{\epsilon j})}{\epsilon} = +\infty. \tag{3.15}$$

Moreover for any $\delta > 0$ we have

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^2}\, e^{-(1+\delta)\frac{d_g(\xi_{\epsilon i},\xi_{\epsilon j})}{\epsilon}} = 0. \tag{3.16}$$

This follows from (3.5): if we had $a > 0$ and a sequence $\epsilon_i \to 0$ such that

$$e^{-(1+\delta)\frac{d_g(\xi_{\epsilon i},\xi_{\epsilon j})}{\epsilon}} > a\epsilon^2,$$

then

$$e^{-\frac{d_g(\xi_{\epsilon i},\xi_{\epsilon j})}{\epsilon}} > a\epsilon^2 e^{\delta\frac{d_g(\xi_{\epsilon i},\xi_{\epsilon j})}{\epsilon}},$$

and applying (3.5) to $\epsilon^{-1}\exp^{-1}_{\xi_i}\xi_j$ since $U\Big(\epsilon^{-1}\exp^{-1}_{\xi_i}\xi_j\Big) < \epsilon^2$ we get

$$\epsilon^2\Big(\frac{d_g(\xi_{\epsilon i},\xi_{\epsilon j})}{\epsilon}\Big)^{\frac{n-1}{2}} > ce^{-\frac{d_g(\xi_{\epsilon i},\xi_{\epsilon j})}{\epsilon}} > ca\epsilon^2 e^{\delta\frac{d_g(\xi_{\epsilon i},\xi_{\epsilon j})}{\epsilon}},$$

giving a contradiction.

We will prove:

Proposition 3.2.1. *There exists $\rho_0 > 0, \epsilon_0 > 0, c > 0$ and $\sigma > 0$ such that for any $\rho \in (0,\ \rho_0)$, $\epsilon \in (0,\ \epsilon_0)$ and $\overline{\xi} \in D^{k_0}_{\epsilon,\rho}$ there exists a unique $\phi_{\epsilon,\overline{\xi}} = \phi(\epsilon,\ \overline{\xi}) \in K^{\perp}_{\epsilon,\overline{\xi}}$ which solves equation* (3.12) *and satisfies*

$$\|\phi_{\epsilon,\overline{\xi}}\|_\epsilon \le c\Big(\epsilon^2 + \sum_{i\neq j} e^{-\frac{(1+\sigma)d_g(\xi_i,\xi_j)}{2\epsilon}}\Big). \tag{3.17}$$

Moreover, $\overline{\xi} \to \phi_{\epsilon,\overline{\xi}}$ is a C^1- map. Note that by (3.16) $\|\phi_{\epsilon,\overline{\xi}}\|_\epsilon = o(\epsilon)$.

The proof of the proposition is technical and follows the same lines used in previous works, see [15, 18, 26]. For completeness we will sketch the proof following the proof in [15], but we pospone it to Section 3.5. In the next two sections we will prove Theorem 3.1.1 assuming this proposition.

On the Banach space $L^q_g(M)$ consider the norm

$$|u|_{q,\epsilon} := \Big(\frac{1}{\epsilon^n}\int_M |u|^q\, d\mu_g\Big)^{1/q}.$$

Since $2 < p < \frac{2n}{n-2}$ it follows from the usual Sobolev inequalities that there exists a constant c independent of ϵ such that

$$|u|_{p,\epsilon} \le c\|u\|_\epsilon \tag{3.18}$$

for any $u \in H_\epsilon$.

We denote by L^p_ϵ the Banach space $L^p_g(M)$ with the norm $|u|_{p,\epsilon}$. For $p' := \dfrac{p}{p-1}$ the dual space L^{p*}_ϵ is identified with $L^{p'}_\epsilon$ with the pairing

$$\langle \varphi, \psi \rangle = \frac{1}{\epsilon^n} \int_M \varphi\psi,$$

for $\varphi \in L^p_\epsilon$, $\psi \in L^{p'}_\epsilon$.

The embedding $\iota_\epsilon : H_\epsilon \to L^p_\epsilon$ is a compact continuous map and the adjoint operator $\iota^*_\epsilon : L^{p'}_\epsilon \to H_\epsilon$, is a continuous map such that

$$u = \iota^*_\epsilon(v) \Leftrightarrow \langle \iota^*_\epsilon(v), \varphi \rangle_\epsilon = \frac{1}{\epsilon^n} \int_M v\varphi, \; \varphi \in H_\epsilon \Leftrightarrow$$

$$-\epsilon^2 \Delta_g u + (\epsilon^2 \lambda s_g + 1)u = v \text{ (weakly) on } M. \tag{3.19}$$

Moreover for the same constant c in (3.18) we have that

$$\|\iota^*_\epsilon(v)\|_\epsilon \leq c|v|_{p',\epsilon} \tag{3.20}$$

for any $v \in L^{p'}_\epsilon$.

Let

$$f(u) := (u^+)^{p-1}.$$

Note that

$$S_\epsilon(u) = u - \iota^*_\epsilon(f(u)) \, , \; u \in H_\epsilon, \tag{3.21}$$

and we can rewrite problem (3.3) in the equivalent way

$$u = \iota^*_\epsilon(f(u)) \, , \; u \in H_\epsilon. \tag{3.22}$$

Note that a solution to (3.22) is a critical point of J_ϵ and so it is a positive function.

Now we will discuss some estimates related to the approximate solutions. The estimates are similar to ones obtained in [15, 26] and we refer the reader to these articles for details.

The next lemma gives an explicit sense in which $W_{\epsilon,\xi}$ is an approximate solution of equation (3.3):

Lemma 3.2.2. *There exists a constant c and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $\xi \in M$,*

$$\|S_\epsilon(W_{\epsilon,\xi})\|_\epsilon \leq c\epsilon^2.$$

Proof. Let $Y_{\epsilon,\xi} = -\epsilon^2 \Delta_g W_{\epsilon,\xi} + (\epsilon^2 \lambda s_g + 1) W_{\epsilon,\xi}$, so that by (3.19) we get $W_{\epsilon,\xi} = \iota^*_\epsilon(Y_{\epsilon,\xi})$. Then

$$\|S_\epsilon(W_{\epsilon,\xi})\|_\epsilon = \|\iota_\epsilon^*(f(W_{\epsilon,\xi})) - W_{\epsilon,\xi}\|_\epsilon$$

$$= \|\iota_\epsilon^*(f(W_{\epsilon,\xi}) - Y_{\epsilon,\xi})\|_\epsilon \leq c|f(W_{\epsilon,\xi}) - Y_{\epsilon,\xi}|_{p',\epsilon}$$

$$\leq c|f(W_{\epsilon,\xi}) + \epsilon^2 \Delta_g W_{\epsilon,\xi} - W_{\epsilon,\xi}|_{p',\epsilon} + c|\epsilon^2 \lambda s_g W_{\epsilon,\xi}|_{p',\epsilon}.$$

But

$$|W_{\epsilon,\xi}|_{p',\epsilon} = \left(\epsilon^{-n} \int_{B(0,r)} (U_\epsilon \chi_r)^{p'}\right)^{\frac{1}{p'}} \leq c\left(\int_{B(0,r/\epsilon)} (U\chi_r(\epsilon z))^{p'} dz\right)^{\frac{1}{p'}}$$

$$\leq \bar{c}\left(\int_{\mathbb{R}^n} U^{p'} dz\right)^{\frac{1}{p'}} \leq \bar{\bar{c}}.$$

Then

$$|\epsilon^2 s_g W_{\epsilon,\xi}|_{p',\epsilon} \leq C\epsilon^2.$$

In [26, Lemma 3.3] it is proved that

$$|f(W_{\epsilon,\xi}) + \epsilon^2 \Delta_g W_{\epsilon,\xi} - W_{\epsilon,\xi}|_{p',\epsilon} \leq C\epsilon^2,$$

and the lemma follows.

□

Since the function U is radial it follows that if $i \neq j$ then $\langle \psi_\epsilon^i, \psi_\epsilon^j \rangle_\epsilon = 0$. Then it is easy to see that for any $\xi \in M$

$$\lim_{\epsilon\to 0} \langle Z_{\epsilon,\xi}^i,\ Z_{\epsilon,\xi}^j \rangle_\epsilon = \delta_{ij} \int_{\mathbb{R}^n} (|\nabla \psi^l|^2 + (\psi^l)^2)\, dz. \tag{3.23}$$

Let us call $C = \displaystyle\int_{\mathbb{R}^n} (|\nabla \psi^l|^2 + (\psi^l)^2)\, dz$.

Given $\xi \in M$ and normal coordinates $(x_1, ..., x_n)$ around ξ it also follows that

$$\lim_{\epsilon\to 0} \epsilon \left\| \frac{\partial W_{\epsilon,\xi}}{\partial x^k} \right\|_\epsilon = C, \tag{3.24}$$

$$\lim_{\epsilon\to 0} \epsilon \left\langle Z_{\epsilon,\xi}^i, \frac{\partial W_{\epsilon,\xi}}{\partial x^k} \right\rangle_\epsilon = \delta_{ik} C, \tag{3.25}$$

and

$$\lim_{\epsilon\to 0}\epsilon\Big\|\frac{\partial Z^i_{\epsilon,\xi}}{\partial x_k}\Big\|_\epsilon = \int_{\mathbb{R}^n}(|\nabla\frac{\partial\psi^i}{\partial x_k}|^2 + (\frac{\partial\psi^i}{\partial x_k})^2)\,dz. \tag{3.26}$$

The previous estimates deal with one peak approximations. For the multipeak approximate solutions $V_{\epsilon,\bar{\xi}}$ with $\bar{\xi} \in D^{k_0}_{\epsilon,\rho}$ consider normal coordinates $(x^i_1, ..., x^i_n)$ around each ξ_i (i=1,...,k_0). Note that if $i \neq j$:

$$\frac{\partial}{\partial y^j_h}Z^l_{\epsilon,\xi_i(y^i)} = \frac{\partial}{\partial y^j_h}W_{\epsilon,\xi_i(y^i)} = 0. \tag{3.27}$$

Also since the points are appropriately separated by (3.15) and the exponential decay of U (3.5), (3.6), it follows that if $i \neq j$,

$$\langle Z^l_{\epsilon,\xi_j}, \frac{\partial}{\partial y^i_h}W_{\epsilon,\xi_i(y^i)}\rangle_\epsilon = o(1). \tag{3.28}$$

3.3 The asymptotic expansion of $\overline{J}_\epsilon$

For $\bar{\xi} \in D^{k_0}_{\epsilon,\rho}$ we consider the unique $\phi_{\epsilon,\bar{\xi}} = \phi(\epsilon, \bar{\xi}) \in K^{\perp}_{\epsilon,\bar{\xi}}$ given by Proposition 3.2.1 and define as before, $\overline{J}_\epsilon(\bar{\xi}) = J_\epsilon(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}})$. In this section we will prove the following:

Proposition 3.3.1. *For $\bar{\xi} \in D^{k_0}_{\epsilon,\rho}$ we have*

$$\overline{J}_\epsilon(\bar{\xi}) = k_0\alpha + (1/2)\beta_\lambda\,\epsilon^2\sum_{i=1}^{k_0} s_g(\xi_i) - \frac{1}{2}\sum_{i\neq j, i,j=1}^{k_0}\gamma_{ij}U(\frac{\exp^{-1}_{\xi_i}\xi_j}{\epsilon}) + o(\epsilon^2), \tag{3.29}$$

C^0-uniformly with respect to $\bar{\xi}$ in compact sets of $D^{k_0}_{\epsilon,\rho}$ as ϵ goes to zero, where

$$\alpha := \frac{1}{2}\int_{\mathbb{R}^n}|\nabla U(z)|^2\,dz + \frac{1}{2}\int_{\mathbb{R}^n}U^2(z)\,dz - \frac{1}{p}\int_{\mathbb{R}^n}U^p(z)\,dz, \tag{3.30}$$

$$\gamma_{ij} := \int_{\mathbb{R}^n}U^{p-1}(z)e^{\langle b_{ij},z\rangle}dz \tag{3.31}$$

with $|b_{ij}| = 1$ and

$$\beta_\lambda := \lambda\int_{\mathbb{R}^n}U^2(z)\,dz - \frac{1}{n(n+2)}\int_{\mathbb{R}^n}|\nabla U(z)|^2|z|^2\,dz. \tag{3.32}$$

Moreover, if $\bar{\xi}_\epsilon$ is a critical point of $\overline{J_\epsilon}$, then the function $V_{\epsilon,\bar{\xi}_\epsilon} + \phi_{\epsilon,\bar{\xi}_\epsilon}$ is a solution to problem (3.3).

For a point $\xi \in M$ we will identify a geodesic ball around it with a ball in $\mathbb{R}^n$ by normal coordinates. We denote by g_{ij} the expression of the metric g in these coordinates and consider the higher order terms in the Taylor expansions of the functions g_{ij}. We will need the following lemma which is proved for instance in [20]:

Lemma 3.3.2. *In a normal coordinates neighborhood of $\xi_0 \in M$, the Taylor's series of g around ξ_0 is given by*

$$g_{\xi ij}(z) = \delta_{ij} + \frac{1}{3} R_{kijl}(\xi) z_k z_l + O(|z|^3),$$

as $|z| \to 0$. Moreover,

$$g_\xi^{ij}(z) = \delta_{ij} - \frac{1}{3} R_{kijl}(\xi) z_k z_l + O(|z|^3).$$

Furthermore, the volume element on normal coordinates has the following expansion

$$\sqrt{det\ g_\xi(z)} = 1 - \frac{1}{6} R_{kl}(\xi) z_k z_l + O(|z|^3).$$

Lemma 3.3.3. *For $\xi \in M$ and $\epsilon > 0$ small we have*

$$J_\epsilon(W_{\epsilon,\xi}) = \alpha + \frac{\beta_\lambda}{2} \epsilon^2 s_g(\xi) + o(\epsilon^2). \tag{3.33}$$

Proof. By direct computation

$$\begin{aligned} J_\epsilon(W_{\epsilon,\xi}) &= \frac{1}{\epsilon^n} \int_M \Big[\frac{1}{2} \epsilon^2 \big| \nabla_g W_{\epsilon,\xi} \big|^2 + \frac{1}{2} (\epsilon^2 \lambda s_g + 1) W_{\epsilon,\xi}^2 - \frac{1}{p} \big| W_{\epsilon,\xi} \big|^p \Big] \, d\mu_g \\ &= \frac{1}{\epsilon^n} \int_M \Big[\frac{1}{2} \epsilon^2 \big| \nabla_g W_{\epsilon,\xi} \big|^2 + \frac{1}{2} W_{\epsilon,\xi}^2 - \frac{1}{p} \big| W_{\epsilon,\xi} \big|^p \Big] \, d\mu_g \\ &\quad + \frac{1}{\epsilon^n} \int_M \frac{1}{2} \epsilon^2 \lambda s_g(x) W_{\epsilon,\xi}(x)^2 \, d\mu_g = J + I \end{aligned}$$

We first estimate I. Let $x = \exp_\xi(\epsilon z)$ with $z \in B(0, \frac{r}{\epsilon})$. Then doing the change of variables we obtain the expression

$$2I = \epsilon^{-n} \epsilon^2 \lambda \int_{B_g(0, \frac{r}{\epsilon})} s_g(\exp_\xi(\epsilon z)) \Big(U(z) \chi_{r/\epsilon}(\epsilon z) \Big)^2 \sqrt{det\ g_\xi(\epsilon z)} \, \epsilon^n \, dz.$$

By the exponential decay of U (3.5), we have

$$2I = \epsilon^2 \lambda \int_{B_g(0, \frac{r}{\sqrt{\epsilon}})} s_g(\exp_\xi(\epsilon z)) U(z)^2 \sqrt{det\ g_\xi(\epsilon z)} \, dz + o(\epsilon^2).$$

We consider the Taylor's expansions of g and s_g around ξ. For instance

$$s_g(exp_\xi(z)) = s_g(\xi) + \frac{\partial s_g}{\partial z_k}(\xi) z_k + O(|z|^2)$$

as $|z| \to 0$. Therefore if $|z| < \frac{r}{\sqrt{\epsilon}}$ for some fixed r ¿ 0, then

$$s_g(exp_\xi(\epsilon z)) = s_g(\xi) + \frac{\partial s_g}{\partial z_k}(\xi) \epsilon z_k + O(\epsilon).$$

Then

$$2I = \epsilon^2 \lambda s_g(\xi) \int_{B_g(0, \frac{r}{\sqrt{\epsilon}})} U(z)^2 \, dz + o(\epsilon^2).$$

And using again the exponential decay of U we get

$$2I = \epsilon^2 \lambda s_g(\xi) \Big(\int_{\mathbb{R}^n} U^2(z) \, dz \Big) + o(\epsilon^2). \tag{3.34}$$

By Lemma 5.3 of [26] we have

$$J = \alpha - \epsilon^2 s_g(\xi) \frac{1}{6} \int_{\mathbb{R}^n} \Big(\frac{u'(|z|)}{|z|} \Big)^2 z_1^4 \, dz + o(\epsilon^2). \tag{3.35}$$

Here we are using that U is a radial function, $U(z) = u(|z|)$ for a function $u : [0, +\infty) \to \mathbb{R}$, and we identify $|\nabla U(z)| = |u'(|z|)|$. Using polar coordinates to integrate

$$\int_{\mathbb{R}^n} \Big(\frac{u'(|z|)}{|z|} \Big)^2 z_1^4 \, dz = \int_0^{+\infty} \int_{\mathbb{S}^{n-1}(r)} \Big(\frac{u'(r)}{r} \Big)^2 z_1^4 \, dS(y) \, dr$$

$$= \int_0^{+\infty} \Big(\frac{u'(r)}{r} \Big)^2 r^{n-1} \int_{\mathbb{S}^{n-1}} (r z_1)^4 \, dS(y) \, dr = \int_0^{+\infty} (u'(r))^2 r^{n+1} \, dr \int_{\mathbb{S}^{n-1}} z_1^4 \, dS(y).$$

For any homogeneous polynomial $p(x)$ of degree d using the divergence theorem one obtains (see Proposition 28 in [11])

$$\int_{\mathbb{S}^{n-1}} p(x) \, dS(x) = \frac{1}{d(d+n-2)} \int_{\mathbb{S}^{n-1}} \Delta p(x) \, dS(x). \tag{3.36}$$

Then

$$\int_{\mathbb{S}^{n-1}} z_1^4 \, dS(z) = \frac{1}{4(n+2)} \int_{\mathbb{S}^{n-1}} 12 z_1^2 \, dS(z) = \frac{3}{n(n+2)} \int_{\mathbb{S}^{n-1}} \sum_{i=1}^n z_1^2 \, dS(z)$$

$$= \frac{3}{n(n+2)} V_{n-1},$$

where V_{n-1} is the volume of $\mathbb{S}^{n-1}$. Then we get

$$\int_{\mathbb{R}^n} \Big(\frac{u'(|z|)}{|z|}\Big)^2 z_1^4 \, dz = \frac{3}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U|^2 |z|^2 dz$$

and using (3.34), (3.35) the lemma follows.

□

Lemma 3.3.4.

$$\overline{J_\epsilon}(\bar{\xi}) = J_\epsilon(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) = J_\epsilon(V_{\epsilon,\bar{\xi}}) + o(\epsilon^2) \tag{3.37}$$

C^0- uniformly in compact sets of $D^{k_0}_{\epsilon,\rho}$.

Proof. If we let $F(u) = \frac{1}{p}(u^+)^p$ then

$$J_\epsilon(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) - J_\epsilon(V_{\epsilon,\bar{\xi}})$$

$$= \frac{1}{2}\|\phi_{\epsilon,\bar{\xi}}\|_\epsilon^2 + \frac{1}{\epsilon^n}\int_M [\epsilon^2 \nabla_g V_{\epsilon,\bar{\xi}} \nabla_g \phi_{\epsilon,\bar{\xi}} + (\epsilon^2 \lambda s_g + 1) V_{\epsilon,\bar{\xi}} \phi_{\epsilon,\bar{\xi}} - f(V_{\epsilon,\bar{\xi}})\phi_{\epsilon,\bar{\xi}}]$$

$$-\frac{1}{\epsilon^n}\int_M [F(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) - F(V_{\epsilon,\bar{\xi}}) - f(V_{\epsilon,\bar{\xi}})\phi_{\epsilon,\bar{\xi}}].$$

Since $\phi_{\epsilon,\bar{\xi}} \in K^\perp_{\epsilon,\bar{\xi}}$ and it satisfies (3.12)

$$0 = \langle \phi_{\epsilon,\bar{\xi}}, S_\epsilon(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}})\rangle_\epsilon = \langle \phi_{\epsilon,\bar{\xi}}, (V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) - \iota_\epsilon^*(f(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}))\rangle_\epsilon$$

$$= \|\phi_{\epsilon,\bar{\xi}}\|_\epsilon^2 + \frac{1}{\epsilon^n}\int_M [\epsilon^2 \nabla_g V_{\epsilon,\bar{\xi}} \nabla_g \phi_{\epsilon,\bar{\xi}} + (\epsilon^2 \lambda s_g + 1) V_{\epsilon,\bar{\xi}} \phi_{\epsilon,\bar{\xi}} - f(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}})\phi_{\epsilon,\bar{\xi}}.$$

Therefore

$$J_\epsilon(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) - J_\epsilon(V_{\epsilon,\bar{\xi}}) = -\frac{1}{2}\|\phi_{\epsilon,\bar{\xi}}\|_\epsilon^2 + \frac{1}{\epsilon^n}\int_M [f(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) - f(V_{\epsilon,\bar{\xi}})]\phi_{\epsilon,\bar{\xi}}$$

$$-\frac{1}{\epsilon^n}\int_M [F(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) - F(V_{\epsilon,\bar{\xi}}) - f(V_{\epsilon,\bar{\xi}})\phi_{\epsilon,\bar{\xi}}]. \tag{3.38}$$

By Proposition 3.2.1 we get $\|\phi_{\epsilon,\bar{\xi}}\|_\epsilon^2 = o(\epsilon^2)$. By the Mean Value Theorem we get for some $t_1, t_2 \in [0, 1]$

$$\frac{1}{\epsilon^n}\int_M [f(V_{\epsilon,\bar{\xi}} + \phi_{\epsilon,\bar{\xi}}) - f(V_{\epsilon,\bar{\xi}})]\phi_{\epsilon,\bar{\xi}} = \frac{1}{\epsilon^n}\int_M f'(V_{\epsilon,\bar{\xi}} + t_1\phi_{\epsilon,\bar{\xi}})\phi^2_{\epsilon,\xi}, \tag{3.39}$$

and

$$\frac{1}{\epsilon^n}\int_M [F(V_{\epsilon,\bar\xi}+\phi_{\epsilon,\bar\xi}) - F(V_{\epsilon,\bar\xi}) - f(V_{\epsilon,\bar\xi})\phi_{\epsilon,\bar\xi}]$$
$$= \frac{1}{2\epsilon^n}\int_M f'(V_{\epsilon,\bar\xi}+t_2\phi_{\epsilon,\bar\xi})\phi^2_{\epsilon,\bar\xi}. \tag{3.40}$$

Moreover we have for any $t \in [0,1]$

$$\frac{1}{\epsilon^n}\int |f'(V_{\epsilon,\bar\xi}+t\phi_{\epsilon,\bar\xi})|\phi^2_{\epsilon,\bar\xi} \le c\frac{1}{\epsilon^n}\int V^{p-2}_{\epsilon,\bar\xi}\phi^2_{\epsilon,\bar\xi} + c\frac{1}{\epsilon^n}\int \phi^p_{\epsilon,\bar\xi}$$

$$\le c\frac{1}{\epsilon^n}\int \phi^2_{\epsilon,\bar\xi} + c\frac{1}{\epsilon^n}\int \phi^p_{\epsilon,\bar\xi} \le C(\|\phi_{\epsilon,\bar\xi}\|^2_\epsilon + \|\phi_{\epsilon,\bar\xi}\|^p_\epsilon) = o(\epsilon^2). \tag{3.41}$$

In the last inequality we use (3.18) and the last equality follows from Proposition 3.2.1. This proves the lemma.

□

Lemma 3.3.5. *For $\bar\xi \in D^{k_0}_{\epsilon,\rho}$ we have*

$$J_\epsilon(V_{\epsilon,\bar\xi}) = k_0\alpha + \frac{1}{2}\beta_\lambda\,\epsilon^2\sum_{i=1}^{k_0} s_g(\xi_i) - \frac{1}{2}\sum_{i\neq j}\gamma_{ij}U\Big(\frac{\exp^{-1}_{\xi_j}(\xi_i)}{\epsilon}\Big) + o(\epsilon^2) \tag{3.42}$$

Here

$$\gamma_{ij} := \int_{\mathbb{R}^n} U^{p-1}(z)e^{\langle b_{ij},z\rangle}\,dz,$$

where

$$b_{ij} := \lim_{\epsilon\to 0}\frac{\exp^{-1}_{\xi_i}\xi_j}{|\exp^{-1}_{\xi_i}\xi_j|}.$$

Proof.

$$J_\epsilon(V_{\epsilon,\bar\xi}) = J_\epsilon\Big(\sum_{i=1}^{k_0} W_{\epsilon,\xi_i}\Big) =$$

$$\frac{1}{\epsilon^n}\int_M \Big[\frac{1}{2}\epsilon^2\Big|\nabla_g\Big(\sum_{i=1}^{k_0} W_{\epsilon,\xi_i}\Big)\Big|^2 + \frac{1}{2}(\epsilon^2\lambda s_g+1)\Big(\sum_{i=1}^{k_0} W_{\epsilon,\xi_i}\Big)^2 - \frac{1}{p}\Big(\sum_{i=1}^{k_0} W_{\epsilon,\xi_i}\Big)^p\Big]\,d\mu_g$$

$$= \frac{1}{\epsilon^n}\sum_{i=1}^{k_0}\Big[\int_M \frac{1}{2}\epsilon^2\big|\nabla_g W_{\epsilon,\xi_i}\big|^2\,d\mu_g + \frac{1}{2}\int_M(\epsilon^2\lambda s_g+1)\big(W_{\epsilon,\xi_i}\big)^2\,d\mu_g - \frac{1}{p}\int_M\big(W_{\epsilon,\xi_i}\big)^p\,d\mu_g\Big]$$

$$+\frac{1}{\epsilon^n}\sum_{i<j}^{k_0}\Big[\int_M \epsilon^2\nabla_g W_{\epsilon,\xi_i}\nabla_g W_{\epsilon,\xi_j}\,d\mu_g + \int_M(\epsilon^2\lambda s_g+1)W_{\epsilon,\xi_i}W_{\epsilon,\xi_j}\,d\mu_g - \int_M\big(W_{\epsilon,\xi_i}\big)^{p-1}W_{\epsilon,\xi_j}\,d\mu_g\Big]$$

$$-\frac{1}{\epsilon^n}\Big[\frac{1}{p}\int_M\Big(\sum_{i=1}^{k_0}W_{\epsilon,\xi_i}\Big)^p\,d\mu_g-\frac{1}{p}\sum_{i=1}^{k_0}\int_M\Big(W_{\epsilon,\xi_i}\Big)^p\,d\mu_g-\sum_{i<j}^{k_0}\int_M\Big(W_{\epsilon,\xi_i}\Big)^{p-1}W_{\epsilon,\xi_j}\,d\mu_g\Big]$$
$$=: I_1+I_2+I_3. \tag{3.43}$$

By Lemma 3.3.3 we get

$$I_1=k_0\alpha+\frac{\beta_\lambda}{2}\epsilon^2\sum_{i=1}^{k_0}s_g(\xi_i)+o(\epsilon^2).$$

Let us estimate the second term I_2 in (3.43). We claim that $I_2=o(\epsilon^2)$.

$$I_2=\sum_{i<j}^{k_0}\int_M\epsilon^2\lambda s_gW_{\epsilon,\xi_i}W_{\epsilon,\xi_j}\,d\mu_g\;+$$

$$\frac{1}{\epsilon^n}\sum_{i<j}^{k_0}\Big[\int_M\epsilon^2\nabla_gW_{\epsilon,\xi_i}\nabla_gW_{\epsilon,\xi_j}\,d\mu_g+\int_MW_{\epsilon,\xi_i}W_{\epsilon,\xi_j}\,d\mu_g-\int_M\Big(W_{\epsilon,\xi_i}\Big)^{p-1}W_{\epsilon,\xi_j}\,d\mu_g\Big].$$

It is easy to see that the first term is $o(\epsilon^2)$. The second term only involves the W_{ϵ,ξ_j}'s and it is explicitly estimated in [15, Lemma 4.1]: it is shown there that it is of the order of $o(\epsilon^2)$. The term I_3 also only involves the W_{ϵ,ξ_j}'s and it is estimated in [15, Lemma 4.1]. They show

$$I_3=-\frac{1}{2}\sum_{i\neq j}\gamma_{ij}U\Big(\frac{\exp^{-1}_{\xi_j}(\xi_i)}{\epsilon}\Big)+o(\epsilon^2).$$

This proves the lemma.

□

Proof of Proposition 3.3.1. The last two lemmas prove (3.29). We are left to prove that if $\overline{\xi}_\epsilon=(\xi_1,\ \dots\ ,\ \xi_{k_0})$ is a critical point of $\overline{J_\epsilon}$, then the function $V_{\epsilon,\overline{\xi}_\epsilon}+\phi_{\epsilon,\overline{\xi}_\epsilon}$ is a solution to problem (3.3). For $\alpha=1,...,k_0$ and $x^\alpha\in B(0,r)$ we let $y^\alpha=\exp_{\xi_\alpha}(x^\alpha)$ and $\overline{y}=(y^1,\ \dots,\ y^{k_0})\in M^{k_0}$.

Since $\overline{\xi}$ is a critical point of $\overline{J_\epsilon}$,

$$\frac{\partial}{\partial x_i^\alpha}\overline{J_\epsilon}(\overline{y}(x))\Big|_{x=0}=0,\ \text{for}\ \alpha=1,...,k_0,\ \ i=1,...,n. \tag{3.44}$$

We write

$$S_\epsilon(V_{\epsilon,\overline{y}(x)}+\phi_{\epsilon,\overline{y}(x)})=\Pi^\perp_{\epsilon,\overline{y}}S_\epsilon(V_{\epsilon,\overline{y}(x)}+\phi_{\epsilon,\overline{y}(x)})+\Pi_{\epsilon,\overline{y}}S_\epsilon(V_{\epsilon,\overline{y}(x)}+\phi_{\epsilon,\overline{y}(x)}).$$

The first term on the right is of course 0 by the construction of $\phi_{\epsilon,\overline{y}(x)}$. We write the second term as

$$\Pi_{\epsilon,\overline{y}}S_{\epsilon}(V_{\epsilon,\overline{y}(x)} + \phi_{\epsilon,\overline{y}(x)}) = \Sigma_{i,\alpha}C_{\epsilon}^{i,\alpha}Z^{i}_{\epsilon,y^{\alpha}},$$

for some functions $C_{\epsilon}^{i,\alpha} : B(0,r)^{k_0} \to \mathbb{R}$. We have to prove that for each i,α (and $\epsilon > 0$ small), $C_{\epsilon}^{i,\alpha}(0) = 0$. Then fix i,α.

We have

$$0 = \frac{\partial}{\partial x_i^{\alpha}}\overline{J_{\epsilon}}(\overline{y}(x)) = J'_{\epsilon}(V_{\epsilon,\overline{y}(x)} + \phi_{\epsilon,\overline{y}(x)})[\frac{\partial}{\partial x_i^{\alpha}}(V_{\epsilon,\overline{y}(x)} + \phi_{\epsilon,\overline{y}(x)})] =$$

$$= \langle S_{\epsilon}(V_{\epsilon,\overline{y}(x)} + \phi_{\epsilon,\overline{y}(x)}), \frac{\partial}{\partial x_i^{\alpha}}\Big|_{x=0}(V_{\epsilon,\overline{y}(x)} + \phi_{\epsilon,\overline{y}(x)})\rangle_{\epsilon}$$

$$= \langle\sum_{k,\beta} C_{\epsilon}^{k,\beta}(0)Z^{k}_{\epsilon,y^{\beta}}, \frac{\partial}{\partial x_i^{\alpha}}\Big|_{x=0}(V_{\epsilon,\overline{y}(x)} + \phi_{\epsilon,\overline{y}(x)})\rangle_{\epsilon}. \tag{3.45}$$

Since $\phi_{\epsilon,\overline{\xi}(y)} \in K^{\perp}_{\epsilon,\overline{\xi}(y)}$, for any k and β we have that $\langle Z^{k}_{\epsilon,y^{\beta}}, \phi_{\epsilon,\overline{y}(x)}\rangle_{\epsilon} = 0$. Then

$$\liminf_{\epsilon\to 0} |\langle Z^{k}_{\epsilon,y^{\beta}},\ (\frac{\partial}{\partial x_i^{\alpha}}\phi_{\epsilon,\overline{y}(x)})\Big|_{y=0}\rangle_{\epsilon}| = \liminf_{\epsilon\to 0} |-\langle(\frac{\partial}{\partial x_i^{\alpha}}Z^{k}_{\epsilon,y^{\beta}})\Big|_{y=0},\ \phi_{\epsilon,\overline{y}(x)}\rangle_{\epsilon}|$$

$$\leq \liminf_{\epsilon\to 0} \|(\frac{\partial}{\partial x_i^{\alpha}}Z^{k}_{\epsilon,y^{\beta}})\Big|_{y=0}\|_{\epsilon}\cdot\|\phi_{\epsilon,\overline{y}(x)}\|_{\epsilon} = 0, \tag{3.46}$$

where the last equality follows from Proposition 3.2.1 and (3.26).

Now from (3.27)

$$\langle\sum_{k,\beta} C_{\epsilon}^{k,\beta}(0)Z^{k}_{\epsilon,y^{\beta}}, \frac{\partial}{\partial x_i^{\alpha}}\Big|_{x=0}V_{\epsilon,\overline{y}(x)}\rangle_{\epsilon} \tag{3.47}$$

$$= \langle\sum_{k,\beta} C_{\epsilon}^{k,\beta}(0)Z^{k}_{\epsilon,y^{\beta}}, \frac{\partial}{\partial x_i^{\alpha}}\Big|_{x=0}W_{\epsilon,y^{\alpha}(x)}\rangle_{\epsilon} \tag{3.48}$$

$$= \langle\sum_{k} C_{\epsilon}^{k,\alpha}(0)Z^{k}_{\epsilon,y^{\alpha}}, \frac{\partial}{\partial x_i^{\alpha}}\Big|_{x=0}W_{\epsilon,y^{\alpha}(x)}\rangle_{\epsilon} + \langle\sum_{k,\beta\neq\alpha} C_{\epsilon}^{k,\beta}(0)Z^{k}_{\epsilon,y^{\beta}}, \frac{\partial}{\partial x_i^{\alpha}}\Big|_{x=0}W_{\epsilon,y^{\alpha}(x)}\rangle_{\epsilon}. \tag{3.49}$$

It follows from (3.28) that

$$\lim_{\epsilon\to 0}\langle\sum_{k,\beta\neq\alpha} C_{\epsilon}^{k,\beta}(0)Z^{k}_{\epsilon,y^{\beta}}, \frac{\partial}{\partial x_i^{\alpha}}\Big|_{x=0}W_{\epsilon,y^{\alpha}(x)}\rangle_{\epsilon} = 0. \tag{3.50}$$

Also

$$\langle\sum_k C_\epsilon^{k,\alpha}(0)Z_{\epsilon,y^\alpha}^k, \frac{\partial}{\partial x_i^\alpha}\Big|_{x=0} W_{\epsilon,y^\alpha(x)}\rangle_\epsilon = \langle C_\epsilon^{i,\alpha}(0)Z_{\epsilon,y^\alpha}^i, \frac{\partial}{\partial x_i^\alpha}\Big|_{x=0} W_{\epsilon,y^\alpha(x)}\rangle_\epsilon$$

$$+\langle\sum_{k\neq i} C_\epsilon^{k,\alpha}(0)Z_{\epsilon,y^\alpha}^k, \frac{\partial}{\partial x_i^\alpha}\Big|_{x=0} W_{\epsilon,y^\alpha(x)}\rangle_\epsilon.$$

Then it follows from (3.25) that

$$\lim_{\epsilon\to 0} \epsilon\,\langle\sum_k C_\epsilon^{k,\alpha}(0)Z_{\epsilon,y^\alpha}^k, \frac{\partial}{\partial x_i^\alpha}\Big|_{x=0} W_{\epsilon,y^\alpha(x)}\rangle_\epsilon = C_\epsilon^{i,\alpha}(0)C.$$

And then it follows from (3.45) that $C_\epsilon^{i,\alpha}(0) = 0$.

□

3.4 Proof of Theorem 0.2.1

Proof of Theorem 0.2.1. We will prove that if $\bar{\xi}_\epsilon \in \overline{D_{\epsilon,\rho}^{k_0}}$, is such that $\overline{J}_\epsilon(\bar{\xi}_\epsilon) = \max\{\overline{J}_\epsilon(\bar{\xi}) : \bar{\xi} \in \overline{D_{\epsilon,\rho}^{k_0}}\}$, then $\bar{\xi}_\epsilon \in D_{\epsilon,\rho}^{k_0}$. Then by Proposition 3.3.1 $u_\epsilon = V_{\epsilon,\bar{\xi}_\epsilon} + \phi_{\epsilon,\bar{\xi}_\epsilon}$ is a solution to problem (3.3) and $\|u_\epsilon - V_{\epsilon,\bar{\xi}_\epsilon}\| = \|\bar{\phi}_{\epsilon,\bar{\xi}_\epsilon}\| = o(\epsilon)$.

We first construct a particular $\bar{\eta}_\epsilon \in D_{\epsilon,\rho}^{k_0}$. Let $\bar{\eta}_\epsilon = (\eta_1, \eta_2, ..., \eta_k)$, with $\eta_i = \eta_i(\epsilon) = \exp_{\xi_0}(\sqrt{\epsilon}\, e_i)$, for $i \in \{1, 2, ..., k\}$, where $e_1, e_2, ..., e_k$ are distinct points in $\mathbb{R}^n$.

Then, by direct computation, $\bar{\eta}_\epsilon$ verifies the following estimates:
a. $d_g(\xi_0, \eta_i) = \sqrt{\epsilon}\,|e_i|$.
b. $d_g(\eta_i, \eta_j) = |exp_{\eta_i}^{-1}\eta_j| = \sqrt{\epsilon}\,(|e_i - e_j| + o(1))$.
c. $U\left(\frac{\exp_{\eta_i}^{-1}\eta_j}{\epsilon}\right) = o(\epsilon^2)$, since

$$U\left(\frac{\exp_{\eta_i}^{-1}\eta_j}{\epsilon}\right) = U\left(\frac{d_g(\eta_i, \eta_j)}{\epsilon}\right) = U\left(\frac{\sqrt{\epsilon}(|e_i - e_j| + o(1))}{\epsilon}\right) = o(\epsilon^2).$$

We can then see that $\bar{J}(\bar{\eta}_\epsilon) = k_0\alpha + (1/2)\beta_\lambda\epsilon^2 \sum_{i=1}^{k_0} s_g(\eta_i) + o(\epsilon^2)$, by combining (c) and the expansion of $\bar{J}(\bar{\eta}_\epsilon)$ in Proposition 3.3.1.

Note that (a) and (c) imply that, for a fixed $\rho > 0$ and ϵ small enough, $\bar{\eta}_\epsilon = (\eta_1, \eta_2, ..., \eta_k) \in D_{\epsilon,\rho}^{k_0}$.

Now, since $s_g(\xi_0)$ is a local minimum, for ϵ small we have an expansion for $s_g(\eta_i)$:

$$s_g(\eta_i) = s_g(\xi_0) + s_g''(\xi_0)\,(d_g(\xi_0, \eta_i))^2 + o(\sqrt{\epsilon}^3) = s_g(\xi_0) + s_g''(\xi_0)\,\epsilon\,|e_i|^2 + o(\sqrt{\epsilon}^3),$$

so in particular $s_g(\eta_i) = s_g(\xi_0) + o(1)$.

Then we have:

$$\bar{J}(\bar{\eta}_\epsilon) = k_0\alpha + (1/2)\beta_\lambda\epsilon^2\sum_i^{k_0} s_g(\eta_i) + o(\epsilon^2) = k_0\alpha + (1/2)\beta_\lambda\epsilon^2\sum_i^{k_0}(s_g(\xi_0) + o(1)) + o(\epsilon^2),$$

and we obtain

$$\bar{J}(\bar{\eta}_\epsilon) = k_0\alpha + k_0(1/2)\beta_\lambda\epsilon^2 s_g(\xi_0) + o(\epsilon^2). \tag{3.51}$$

Now, since $\bar{\xi}_\epsilon$ is a maximum of $\tilde{J}_\epsilon$ in $\overline{D^{k_0}_{\epsilon,\rho}}$, we have

$$\bar{J}_\epsilon(\bar{\xi}_\epsilon) \geq \bar{J}(\bar{\eta}_\epsilon). \tag{3.52}$$

Applying Proposition 3.3.1 to the left side of (3.52), we get

$$k_0\alpha + \frac{1}{2}\beta_\lambda\epsilon^2\sum_i^{k_0} s_g(\xi_i) - \frac{1}{2}\sum_{i,j=1,i\neq j}^{k_0}\gamma_{ij}\,U\left(\frac{\exp^{-1}_{\xi_i}\xi_j}{\epsilon}\right) \geq k_0\alpha + k_0\frac{1}{2}\beta_\lambda\epsilon^2 s_g(\xi_0) + o(\epsilon^2),$$

that is,

$$\beta_\lambda\epsilon^2\left(k_0 s_g(\xi_0) - \sum_i^{k_0} s_g(\xi_i)\right) + \sum_{i,j=1,i\neq j}^{k_0}\gamma_{ij}\,U\left(\frac{\exp^{-1}_{\xi_i}\xi_j}{\epsilon}\right) \leq o(\epsilon^2). \tag{3.53}$$

Fix ρ, small enough so that ξ_0 is the only minimum of s_g in $B_g(\xi_0,\rho)$. With this choice of ρ we see that, in fact, each term in the left hand side of inequality (3.53) is non-negative and therefore bounded from above by $o(\epsilon^2)$.

Since $d(\xi_0,\xi_i) \leq \rho$, for each i, $1 \leq i \leq k_0$, we have,

$$0 \leq \beta_\lambda\epsilon^2\left(k_0 s_g(\xi_0) - \sum_i^{k_0} s_g(\xi_i)\right) = o(\epsilon^2),$$

that is, since $\beta_\lambda < 0$,

$$0 \geq k_0 s_g(\xi_0) - \sum_i^{k_0} s_g(\xi_i) = o(1). \tag{3.54}$$

It follows that $\lim_{\epsilon\to 0} s_g(\xi_i) = s_g(\xi_0)$. And then, since ξ_0 is the only minimum point of s_g in $B_g(\xi_0,\xi_i)$, we have $\lim_{\epsilon\to 0}\xi_i = \xi_0$. Hence, ϵ small enough implies

$$d_g(\xi_i,\xi_0) < \rho. \tag{3.55}$$

Now, recall that $\gamma_{ij} := \int_{\mathbb{R}^n} U^{p-1}(z)e^{\langle b_{ij},z\rangle}dz$, and that $|b_{ij}| = 1$, for all $i, j \leq k$. This implies that γ_{ij} is bounded from below by a positive constant. We define

$$\gamma := \min\left\{\int_{\mathbb{R}^n} U^{p-1}(z)e^{\langle b,z\rangle}dz : b \in \mathbb{R}^n, |b| = 1\right\} > 0.$$

Then, by (3.53) and (3.54),

$$o(\epsilon^2) \geq \sum_{i,j=1,i\neq j}^{k_0} \gamma_{ij}\, U\left(\frac{\exp_{\xi_i}^{-1}\xi_j}{\epsilon}\right) \geq \sum_{i,j=1,i\neq j}^{k_0} \gamma\, U\left(\frac{\exp_{\xi_i}^{-1}\xi_j}{\epsilon}\right),$$

that is, for ϵ small enough,

$$U\left(\frac{\exp_{\xi_i}^{-1}\xi_j}{\epsilon}\right) < \epsilon^2. \tag{3.56}$$

Of course, (3.56) and (3.55) imply that $\bar{\xi}_\epsilon \in D^{k_0}_{\epsilon,\rho}$.

3.5 Proof of Proposition 3.2.1

In this section we sketch a proof for the finite dimensional reduction, Proposition 3.2.1. A detailed proof in a similar situation can be found in [15, 26].

Proof. Recall the operator

$$S_\epsilon = \nabla J_\epsilon : H_\epsilon \to H_\epsilon.$$

and eq. (3.12)

$$\Pi^\perp_{\epsilon,\bar{\xi}}\{S_\epsilon(V_{\epsilon,\bar{\xi}} + \phi)\} = 0.$$

We may rewrite eq. (3.12) as

$$0 = \Pi^\perp_{\epsilon,\bar{\xi}}\{S_\epsilon(V_{\epsilon,\bar{\xi}}+\phi)\} = \Pi^\perp_{\epsilon,\bar{\xi}}\{S_\epsilon(V_{\epsilon,\bar{\xi}})+S'_\epsilon(V_{\epsilon,\bar{\xi}})\,\phi+\bar{N}_{\epsilon,\bar{\xi}}(\phi)\} = -R_{\epsilon,\bar{\xi}}+L_{\epsilon,\bar{\xi}}(\phi)-N_{\epsilon,\bar{\xi}}(\phi),$$

with the first term being independent of ϕ:

$$R_{\epsilon,\bar{\xi}} := \Pi^\perp_{\epsilon,\bar{\xi}}\{S_\epsilon(V_{\epsilon,\bar{\xi}})\} = \Pi^\perp_{\epsilon,\bar{\xi}}\{i^*_\epsilon[f(V_{\epsilon,\bar{\xi}})] - V_{\epsilon,\bar{\xi}}\},$$

the second term, the linear operator:

$$L_{\epsilon,\bar{\xi}}(\phi) = \Pi^\perp_{\epsilon,\bar{\xi}}\{S'_\epsilon(V_{\epsilon,\bar{\xi}})\,\phi\} = \Pi^\perp_{\epsilon,\bar{\xi}}\{\phi - i^*_\epsilon[f'(V_{\epsilon,\bar{\xi}})\phi]\},$$

and the last term a remainder:

$$N_{\epsilon,\bar{\xi}}(\phi) := \Pi^\perp_{\epsilon,\bar{\xi}}\{\bar{N}_{\epsilon,\bar{\xi}}(\phi)\} = \Pi^\perp_{\epsilon,\bar{\xi}}\{i^*_\epsilon[f(V_{\epsilon,\bar{\xi}} + \phi) - f(V_{\epsilon,\bar{\xi}}) - f'(V_{\epsilon,\bar{\xi}})\phi]\}.$$

Hence, eq. (3.12) can be written as

$$L_{\epsilon,\bar{\xi}}(\phi) = N_{\epsilon,\bar{\xi}}(\phi) + R_{\epsilon,\bar{\xi}}.$$

And then, if L is invertible, we may turn eq. (3.12), into a fixed point problem, for the operator $T_{\epsilon,\bar{\xi}}(\phi) := L^{-1}_{\epsilon,\bar{\xi}}(N_{\epsilon,\bar{\xi}}(\phi) + R_{\epsilon,\bar{\xi}})$.

We start by proving that $L_{\epsilon,\bar{\xi}}$ is in fact invertible, for appropriate $\bar{\xi}$ and ϵ.

Lemma 3.5.1. *There exists* $\epsilon_0 > 0$ *and* $c > 0$*, such that for any* $\epsilon \in (0, \epsilon_0)$ *and* $\bar{\xi} \in M^K$*,* $\bar{\xi} = (\xi_1, \xi_2, ..., \xi_K)$*, such that*

$$\sum_{i,k=1, i\neq k}^{K} U\left(\frac{\exp_{\xi_i}^{-1} \xi_k}{\epsilon}\right) < \epsilon^2,$$

we have,

$$\|L_{\epsilon,\bar{\xi}}(\phi)\|_\epsilon \geq c\|\phi\|_\epsilon,$$

for any $\phi \in K_{\epsilon,\bar{\xi}}^{\perp}$*.*

Proof. We will proceed by contradiction. Suppose that there are sequences $\{\epsilon_j\}_{j\in\mathbb{N}}$, $\epsilon_j \to 0$ and $\{\bar{\xi}_j\}_{j\in\mathbb{N}}$, $\bar{\xi}_j = (\xi_{1_j}, \xi_{2_j}, ..., \xi_{K_j})$, such that

$$\sum_{i,k=1, i\neq k}^{K} U\left(\frac{\exp_{\xi_{i_j}}^{-1} \xi_{k_j}}{\epsilon_j}\right) < \epsilon_j^2,$$

and $\{\phi_j\} \subset K_{\epsilon,\bar{\xi}}^{\perp}$, such that $L_{\epsilon_j,\bar{\xi}_j}(\phi_j) = \psi_j$, with $\|\phi_j\|_{\epsilon_j} = 1$ and $\|\psi_j\|_{\epsilon_j} \to 0$.

Let $\zeta_j := \Pi_{\epsilon_j,\bar{\xi}_j}\{\phi_j - i_{\epsilon_j}^*[f'(V_{\epsilon_j,\bar{\xi}_j})\phi_j]\}$. Hence,

$$\phi_j - i_{\epsilon_j}^*[f'(V_{\epsilon_j,\bar{\xi}_j})\phi_j] = \psi_j + \zeta_j. \tag{3.57}$$

That is, for each j, $\psi_j \in K_{\epsilon_j,\bar{\xi}_j}^{\perp}$ and $\zeta_j \in K_{\epsilon_j,\bar{\xi}_j}$. Now, let $u_j := \phi_j - (\psi_j + \zeta_j)$.

We will prove the following contradictory consequences of the existence of such series:

$$\frac{1}{\epsilon_j^n}\int_M f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2 \, d\mu_g \to 1, \tag{3.58}$$

and

$$\frac{1}{\epsilon_j^n}\int_M f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2 \, d\mu_g \to 0, \tag{3.59}$$

this will prove that such sequences $\{\bar{\xi}_j\}$, $\{\phi_j\}$, $\{\epsilon_j\}$ cannot exist. We start by proving (3.58).

First we note that:

$$\|\zeta_j\|_{\epsilon_j} \to 0 \text{ as } j \to \infty. \tag{3.60}$$

Since $\zeta_j \in K_{\epsilon_j,\bar{\xi}_j}$, let $\zeta_j := \sum_{i=1}^{K}\sum_{k=1}^{n} a_j^{ki} Z_{\epsilon_j,\xi_{i_j}}^k$. For any $h \in \{1, 2, \dots, n\}$, and $l \in \{1, 2, \dots, K\}$ we multiply $\psi_j + \zeta_j$ (eq. (3.57)) by $Z_{\epsilon_j,\xi_{l_j}}^h$, and we find

$$\sum_{i=1}^{K}\sum_{k=1}^{n} a_j^{ki}\langle Z^k_{\epsilon_j,\xi_{i_j}},\ Z^h_{\epsilon_j,\xi_{l_j}}\rangle_{\epsilon_j} \;=\; \langle \phi_j,\ Z^h_{\epsilon_j,\xi_{l_j}}\rangle_{\epsilon_j} - \langle \iota^*_{\epsilon_j}[f'(V_{\epsilon_j,\bar{\xi}_j})\phi_j],\ Z^h_{\epsilon_j,\xi_{l_j}}\rangle_{\epsilon_j}. \tag{3.61}$$

On the other hand, by (3.23),

$$\sum_{i=1}^{K}\sum_{k=1}^{n} a_j^{ki}\langle Z^k_{\epsilon_j,\xi_{i_j}},\ Z^h_{\epsilon_j,\xi_{l_j}}\rangle_{\epsilon_j} = Ca_j^{hl} + o(1),$$

combining this and (3.61):

$$Ca_j^{hl}+o(1) = \frac{1}{\epsilon_j^n}\int_M [\epsilon_j^2\nabla_g Z^h_{\epsilon_j,\xi_{l_j}}\nabla_g\phi_j+(\epsilon_j^2\lambda s_g+1)Z^h_{\epsilon_j,\xi_{j_j}}\phi_j-f'(V_{\epsilon_j,\bar{\xi}_j})\phi_j Z^h_{\epsilon_j,\xi_{l_j}}]\, d\mu_g. \tag{3.62}$$

Let

$$\tilde{\phi}_{l\,j}(z) = \begin{cases} \phi_{l_j}(\exp_{\xi_{l_j}}(\epsilon_j z)\chi_r(\epsilon_j z)) & \text{if } z\in B(0,\ r/\epsilon_j)\ , \\ 0 & \text{otherwise,} \end{cases} \tag{3.63}$$

Then we have that for some constant $\tilde{c}$, $\|\tilde{\phi}_{l\,j}\|_{H^1(\mathbb{R}^n)} \le \tilde{c}\|\tilde{\phi}_{l_j}\|_{\epsilon_j} \le \tilde{c}$. Therefore, we can assume that $\tilde{\phi}_{l\,j}$ converges weakly to some $\tilde{\phi}$ in $H^1(\mathbb{R}^n)$ and strongly in $L^q_{loc}(\mathbb{R}^n)$ for any $q\in[2,p_n)$. Also note that,

$$|\frac{1}{\epsilon_j^n}\int_M \epsilon_j^2 s_g Z^h_{\epsilon_j,\xi_{l_j}}\phi_j\, d\mu_g| \le \epsilon_j^2\, c_1\ |\int_{B\left(0,\frac{r}{\epsilon_j}\right)} \psi^h(z)\chi_r(\epsilon_j z)\tilde{\phi}_{l_j}(z)\ |g_{\xi_{l_j}(\epsilon_j z)}|^{1/2}\, dz|$$

$$= \epsilon_j^2\, c_1\left(\int_{\mathbb{R}^n}\psi^h\,\tilde{\phi}\, dz + o(1)\right) \le \epsilon_j^2\, c_1\left(\int_{\mathbb{R}^n}(\psi^h)^2\, dz\right)^{1/2}\left(\int_{\mathbb{R}^n}\tilde{\phi}^2\, dz\right)^{1/2} + o(\epsilon_j^2)$$

$$\le c_1\, c_2\,\epsilon_j^2\,\|\tilde{\phi}\|_{L^2(\mathbb{R}^n)} + o(\epsilon_j^2) \le c_1\, c_2\, c_3\,\epsilon_j^2 + o(\epsilon_j^2) = o(\epsilon_j), \tag{3.64}$$

where c_1 is an upper bound for s_g, c_2 for $\|\nabla U\|_{L^2(\mathbb{R}^n)}$ and c_3 for $\|\tilde{\phi}\|_{L^2(\mathbb{R}^n)}$.

Then we have, by eqs. (3.62) and (3.64)

$$\begin{aligned} Ca_j^{hl}+o(1) &= \frac{1}{\epsilon_j^n}\int_M [\epsilon_j^2\nabla_g Z^h_{\epsilon_j,\xi_{l_j}}\nabla_g\phi_j + (\epsilon_j^2\lambda s_g+1)Z^h_{\epsilon_j,\xi_{l_j}}\phi_j - f'(V_{\epsilon_j,\bar{\xi}_j})\phi_j Z^h_{\epsilon_j,\xi_{l_j}}]\, d\mu_g \\ &= \frac{1}{\epsilon_j^n}\int_M [\epsilon_j^2\nabla_g Z^h_{\epsilon_j,\xi_{l_j}}\nabla_g\phi_j + Z^h_{\epsilon_j,\xi_{l_j}}\phi_j - f'(V_{\epsilon_j,\bar{\xi}_j})\phi_j Z^h_{\epsilon_j,\xi_{l_j}}]\, d\mu_g + o(\epsilon_j) \\ &= \int_{\mathbb{R}^n}(\nabla\psi^h\nabla\tilde{\phi} + \psi^h\tilde{\phi} - f'(U)\psi^h\tilde{\phi})\, dz + o(1) = o(1). \end{aligned} \tag{3.65}$$

From (3.65), we get that $a_j^{hl}\to 0$ for any $h=1,\cdots,n$, and any $l=1,\cdots,K$ and then (3.60) follows. We are ready to prove (3.58).

Recall that $u_j = \phi_j - (\psi_j + \zeta_j)$, since $\|\phi_j\|_{\epsilon_j} = 1, \|\psi_j\|_{\epsilon_j} \to 0$ and $\|\zeta_j\|_{\epsilon_j} \to 0$ then

$$\|u_j\|_{\epsilon_j} \to 1. \tag{3.66}$$

Moreover, by (3.57) $u_j = \iota^*_{\epsilon_j}[f'(W_{\epsilon_j,\xi_j})\phi_j]$, hence, by (3.19), it satisfies weakly

$$-\epsilon_j^2 \Delta_g u_j + (\epsilon_j^2 \lambda s_g + 1) u_j = f'(V_{\epsilon_j,\bar{\xi}_j}) u_j + f'(V_{\epsilon_j,\bar{\xi}_j})(\psi_j + \zeta_j) \text{ in } M. \tag{3.67}$$

Multiplying (3.67) by u_j, and integrating over M,

$$\|u_j\|^2_{\epsilon_j} = \frac{1}{\epsilon_j^n} \int_M f'(V_{\epsilon_j,\bar{\xi}_j}) u_j^2 \, d\mu_g + \frac{1}{\epsilon_j^n} \int_M f'(V_{\epsilon_j,\bar{\xi}_j})(\psi_j + \zeta_j) u_j \, d\mu_g. \tag{3.68}$$

By Hölder's inequality and eq. (3.18) we can find eq. (3.58):

$$|\frac{1}{\epsilon_j^n} \int_M f'(V_{\epsilon_j,\bar{\xi}_j})(\psi_j + \zeta_j) u_j \, d\mu_g|$$

$$\leq \left(\frac{1}{\epsilon_j^n} \int_M (f'(V_{\epsilon_j,\bar{\xi}_j}) \, u_j)^2 \, d\mu_g \right)^{\frac{1}{2}} \left(\frac{1}{\epsilon_j^n} \int_M (\psi_j + \zeta_j)^2 \, d\mu_g \right)^{\frac{1}{2}}$$

$$\leq c \, \|u_j\|_{\epsilon_j} \, \|\psi_j + \zeta_j\|_{\epsilon_j} = o(1),$$

since $\|\psi_j\|_{\epsilon_j} \to 0$, $\|\zeta_j\|_{\epsilon_j} \to 0$, and $\|u_j\|_{\epsilon_j} \to 1$ as $j \to \infty$. We conclude from eq. (3.68) that $\frac{1}{\epsilon_j^n} \int_M f'(V_{\epsilon_j,\bar{\xi}_j}) u_j^2 \, d\mu_g \to 1$.

Finally, we prove eq. (3.59).

Given $l \in \{1, \cdots, K\}$, we define

$$\tilde{u}_{l_j} = u_j \left(\exp_{\xi_{l_j}}(\epsilon_j z) \right) \chi_r \left(\exp_{\xi_{l_j}}(\epsilon_j z) \right), \; z \in \mathbb{R}^n .$$

Note that $\|\tilde{u}_{l_j}\|^2_{H^1(\mathbb{R}^n)} \leq c \|u_j\|^2_{\epsilon_j} \leq c$. Then, up to a subsequence, $\tilde{u}_{l_j} \to \tilde{u}_l$ weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^q_{loc}(\mathbb{R}^n)$ for any $q \in [2, \; p_n)$, for some $\tilde{u}_l \in H^1(\mathbb{R}^n)$.

We now claim that $\tilde{u}_l$ solves weakly the problem

$$-\Delta \tilde{u}_l + \tilde{u}_l = f'(U) \tilde{u}_l \quad \text{in} \quad \mathbb{R}^n. \tag{3.69}$$

Let $\varphi \in \mathbb{C}_0(\mathbb{R}^n)$. Set $\varphi_j(x) := \varphi\left(\frac{\exp^{-1}_{\xi_{l_j}}(x)}{\epsilon_j} \right) \chi_r \left(\exp^{-1}_{\xi_{l_j}}(x) \right)$, for x in $B(\xi_{l_j}, \epsilon_j \; R) \subset M$. For R big enough such that *supp* $\varphi \subset B(0, R)$ and j big enough such that $B(\xi_{l_j}, \epsilon_j \; R) \subset B(\xi_{l_j}, r)$.

Multiplying (3.67) by φ_j and integrating over M,

$$\frac{1}{\epsilon^n} \int_M \left(\epsilon_j^2 \nabla_g u_j \, \nabla_g \varphi_j + (1 + s_g \lambda \epsilon_j^2) \, u_j \varphi_j \right) d\mu_g$$

$$= \frac{1}{\epsilon_j^n} \int_M f'(V_{\epsilon_j,\bar{\xi}_j})\, u_j\, \varphi_j\, d\mu_g + \frac{1}{\epsilon_j^n} \int_M f'(V_{\epsilon_j,\bar{\xi}_j})(\psi_j + \zeta_j)\, \varphi_j\, d\mu_g.$$

We may rewrite this equation in $\mathbb{R}^n$ by setting $x = \exp_{\xi_{l_j}}(\epsilon_j\, z)$:

$$\int_{B(0,R)} \left(\sum_{s,t=1}^{n} g^{st}_{\xi_{l_j}}(\epsilon_j z) \frac{\partial \tilde{u}_{lj}}{\partial z_s} \frac{\partial \varphi}{\partial z_t} + (1 + s_g \lambda \epsilon_j^2) \tilde{u}_{l_j} \varphi \right) |g_{\xi_{l_j}}(\epsilon_j z)|^{1/2} dz$$

$$= \int_{B(0,R)} f' \left(U(z)\chi_r(\epsilon_j z) + \sum_{i \neq l} U \left(\frac{\exp^{-1}_{\xi_{l_j}} \exp_{\xi_{i_j}}(\epsilon_j z)}{\epsilon_j} \right) \chi_r \left(\exp^{-1}_{\xi_{l_j}} \exp_{\xi_{i_j}}(\epsilon_j z) \right) \right)$$

$$\tilde{u}_{l_j}\, \varphi\, |g_{\xi_{l_j}(\epsilon_j z)}|^{1/2} dz$$

$$+ \int_{B(0,R)} f' \left(U(z)\chi_r(\epsilon_j z) + \sum_{i \neq l} U \left(\frac{\exp^{-1}_{\xi_{l_j}} \exp_{\xi_{i_j}}(\epsilon_j z)}{\epsilon_j} \right) \chi_r \left(\exp^{-1}_{\xi_{l_j}} \exp_{\xi_{i_j}}(\epsilon_j z) \right) \right) \tag{3.70}$$

$$(\tilde{\psi}_j + \tilde{\zeta}_j)\, \varphi\, |g_{\xi_{l_j}}(\epsilon_j z)|^{1/2} dz,$$

where $\tilde{\psi}_j(z) := \psi_j(\exp_{\xi_{l_j}}(\epsilon_j\, z))$ and $\tilde{\zeta}_j(z) := \zeta_j(\exp_{\xi_{l_j}}(\epsilon_j\, z))$ for $z \in B(0, R/\epsilon_j)$.

Note that

$$\int_{B(0,R)} s_g \epsilon_j^2 \tilde{u}_{l_j} \varphi |g_{\xi_{l_j}}(\epsilon_j z)|^{1/2} dz \le c\, \epsilon_j^2 \int_{B(0,R)} \tilde{u}_{l_j} \varphi |g_{\xi_{l_j}}(\epsilon_j z)|^{1/2} dz$$

$$\le c\, \epsilon_j^2 \left(\int_{B(0,R)} \tilde{u}_{l_j}^2 dz \right)^{1/2} \left(\int_{B(0,R)} \varphi^2 |g_{\xi_{l_j}}(\epsilon_j z)| dz \right)^{1/2} \le c\, \epsilon_j^2 \|\tilde{u}_{l_j}\|\|_{H^1(\mathbb{R}^n)} c_2 = o(\epsilon_j).$$

with c an upper bound for s_g, c_2^2 an upper bound for $\int_{B(0,R)} \varphi^2 |g_{\xi_{l_j}(\epsilon_j z)}| dz$. Recall also that u_{l_j} is bounded independently of j in $H^1(\mathbb{R}^n)$.

Hence, taking the limit as $\epsilon_j \to 0$, in (3.70)

$$\int_{\mathbb{R}^n} \left(\sum_{s,t=1}^{n} \delta_{s,t} \frac{\partial \tilde{u}_l}{\partial z_s} \frac{\partial \varphi}{\partial z_t} + \tilde{u}_l \varphi \right) dz = \int_{\mathbb{R}^n} f'\,(U(z))\, \tilde{u}_l\, \varphi\, dz, \tag{3.71}$$

since $\tilde{\psi}_j, \tilde{\zeta}_j \to 0$ strongly in $H^1(\mathbb{R}^n)$. Eq. (3.71) for each $\varphi \in C_0^\infty(\mathbb{R}^n)$, implies the claim that $\tilde{u}_l$ solves weakly eq. (3.69) in $\mathbb{R}^n$.

We now claim that for any $h \in \{1, 2, ...n\}$, $\tilde{u}_l$ satisfies also

$$\int_{\mathbb{R}^n} \left(\nabla \psi^h \nabla \tilde{u}_l + \psi^h \tilde{u}_l \right) dz = 0. \tag{3.72}$$

To prove (3.72) we compute

$$|\langle Z^h_{\epsilon_j,\xi_{l_j}},\ u_j\rangle_{\epsilon_j}| = |\langle Z^h_{\epsilon_j,\xi_{l_j}}, \phi_j - \psi_j - \zeta_j\rangle_{\epsilon_j}| = |\langle Z^h_{\epsilon_j,\xi_{l_j}},\ \zeta_j\rangle_{\epsilon_j}|$$

$$\leq \|Z^h_{\epsilon_j,\xi_{l_j}}\|_{\epsilon_j}\ \ \|\zeta_j\|_{\epsilon_j} = o(1), \tag{3.73}$$

since $\phi_j, \psi_j \in K^{\perp}_{\epsilon_j,\bar{\xi}_j}$ and eq. (3.60). On the other hand, we have

$$\langle Z^h_{\epsilon_j,\xi_{l_j}},\ u_j\rangle_{\epsilon_j} = \frac{1}{\epsilon_j^n}\int_M [\epsilon_j^2\ \nabla_g Z^h_{\epsilon_j,\xi_{l_j}} \nabla_g u_j + (\epsilon_j^2\ \lambda s_g + 1)\ Z^h_{\epsilon_j,\xi_{l_j}}\ u_j]\ d\mu_g.$$

Of course, by Hölder's inequality and eq. (3.18):

$$\left|\frac{1}{\epsilon_j^n}\int_M \epsilon_j^2 \lambda s_g Z^h_{\epsilon_j,\xi_{l_j}} u_j\ d\mu_g\right| \leq c\ \epsilon_j^2 \left|\frac{1}{\epsilon_j^n}\int_M Z^h_{\epsilon_j,\xi_{l_j}} u_j\ d\mu_g\right|$$

$$\leq c\ \epsilon_j^2 \left(\frac{1}{\epsilon_j^n}\int_M (Z^h_{\epsilon_j,\xi_{l_j}})^2\ d\mu_g\right)^{\frac{1}{2}} \left(\frac{1}{\epsilon_j^n}\int_M (u_j)^2\ d\mu_g\right)^{\frac{1}{2}}$$

$$\leq c\ \epsilon_j^2 \left(\int_{B(0,r/\epsilon_j)} \left(\psi^h(z)\,\chi_r(\epsilon_j z)\right)^2 |g_{\xi_{l_j}}(\epsilon_j z)|^{1/2} dz\right)^{\frac{1}{2}} \left(\int_M u_j^2\ d\mu_g\right)^{\frac{1}{2}}$$

$$\leq c\ \epsilon_j^2 \left(\int_{\mathbb{R}^n} |\nabla U|^2 dz + o(1)\right)^{1/2} \ \ \|u_j\|_{\epsilon_j} = o(\epsilon_j),$$

since $\psi^h(z) = \frac{\partial U}{\partial z_h}(z)$, and $\|u_j\|_{\epsilon_j} \to 1$ as $j \to \infty$. Then

$$\langle Z^h_{\epsilon_j,\xi_{l_j}},\ u_j\rangle_{\epsilon_j} = \frac{1}{\epsilon_j^n}\int_M [\ \epsilon_j^2 \nabla_g Z^h_{\epsilon_j,\xi_{l_j}} \nabla_g u_j + Z^h_{\epsilon_j,\xi_{l_j}} u_j\ ]\ d\mu_g + o(\epsilon_j)$$

$$= \int_{B(0,r/\epsilon_j)} [\sum_{s,t=1}^{n} g^{st}_{\xi_{l_j}}(\epsilon_j z) \frac{\partial}{\partial z_s}\left(\psi^h(z)\chi_r(\epsilon_j z)\right) \frac{\partial}{\partial z_t}\left(\tilde{u}_{l_j}(z)\right)$$

$$+\ \psi^h(z)\chi_r(\epsilon_j z)\tilde{u}_{l_j}(z)\ ]\ |g_{\xi_{l_j}}(\epsilon_j z)|^{\frac{1}{2}}\ dz + o(\epsilon_j)$$

$$= \int_{\mathbb{R}^n} \left(\nabla\psi^h \nabla\tilde{u} + \psi^h \tilde{u}\right)\ dz + o(1). \tag{3.74}$$

From (3.73) and (3.74) we prove the claim of eq. (3.72). Therefore, by (3.69) and (3.72) it follows that $\tilde{u} = 0$.

We now prove eq. (3.59). We will estimate $\frac{1}{\epsilon_j^n}\int_M f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2\ d\mu_g$ by partitioning M in various subsets. First we will make estimates in small neighborhoods around each ξ_{l_j}, $l \in \{1, 2, ..., K\}$, using the fact that $\tilde{u}_l = 0$.

Then we will make estimates in the complement of these neighborhoods using the hypothesis that

$$\sum_{i,k=1,i\neq k}^{K} U\left(\frac{\exp^{-1}_{\xi_{i_j}}\xi_{k_j}}{\epsilon_j}\right) < \epsilon_j^2.$$

Let $R_j = \frac{1}{2}\min\{d_g\left(\xi_{l_j},\xi_{m_j}\right), l\neq m\}$. Let $\tilde{M} = \bigcup_{l=1}^{K} B_g(\xi_{l_j},R_j)$. Then

$$\frac{1}{\epsilon_j^n}\int_M f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2 d\mu_g = \frac{1}{\epsilon_j^n}\sum_{l=1}^{K}\int_{B_g(\xi_{l_j},R_j)} f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2 d\mu_g + \frac{1}{\epsilon_j^n}\int_{M\setminus\tilde{M}} f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2 d\mu_g. \tag{3.75}$$

Now, on one hand, for each l, since $\tilde{u}_l = 0$,

$$\frac{1}{\epsilon_j^n}\int_{B_g(\xi_{l_j},R_j)} f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2 d\mu_g$$

$$= \int_{B(0,\ \epsilon_j R_j)} f'\left(U(z)\chi_r(\epsilon_j z) + \sum_{i\neq l}^{K} U\left(\frac{\exp^{-1}_{\xi_{l_j}}\exp_{\xi_{i_j}}(\epsilon_j z)}{\epsilon_j}\right)\chi_r\left(\exp^{-1}_{\xi_{l_j}}\exp_{\xi_{i_j}}(\epsilon_j z)\right)\right)$$

$$\tilde{u}_{l_j}^2(z)\,|g_{\xi_{l_j}}(\epsilon_j z)|^{1/2} dz$$

$$= o(1). \tag{3.76}$$

On the other hand, by Hölder's inequality

$$\frac{1}{\epsilon_j^n}\int_{M\setminus\tilde{M}} f'(V_{\epsilon_j,\bar{\xi}_j})u_j^2 d\mu_g$$

$$\leq \left(\frac{1}{\epsilon_j^n}\int_{M\setminus\tilde{M}}\left(f'(V_{\epsilon_j,\bar{\xi}_j})\right)^{n/2} d\mu_g\right)^{2/n}\left(\frac{1}{\epsilon_j^n}\int_{M\setminus\tilde{M}} u_j^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}}$$

$$\leq c_1\left(\frac{1}{\epsilon_j^n}\int_{M\setminus\tilde{M}}\left((p-1)\sum_{l=1}^{K} W^{(p-2)}_{\epsilon_j,\xi_{l_j}}\right)^{\frac{n}{2}} d\mu_g\right)^{2/n} \quad \|u_j\|^2_{\epsilon_j}$$

$$\leq c_2\left(\frac{1}{\epsilon_j^n}\int_{M\setminus\tilde{M}}\left(\sum_{l=1}^{K} U^{(p-2)}\left(\frac{\exp^{-1}_{\xi_{l_j}}(x)}{\epsilon_j}\right)\chi_r^{(p-2)}\left(\frac{\exp^{-1}_{\xi_{l_j}}(x)}{\epsilon_j}\right)\right)^{\frac{n}{2}} d\mu_g\right)^{2/n}$$

$$\leq c_2\frac{1}{\epsilon_j^2}\sum_{l=1}^{K}\left(\int_{B_g(\xi_{l_j},r)\setminus\tilde{M}} U^{\frac{(p-2)n}{2}}\left(\frac{\exp^{-1}_{\xi_{l_j}}(x)}{\epsilon_j}\right)d\mu_g\right)^{2/n}.$$

$$\leq c_2 \frac{1}{\epsilon_j^2} e^{-(p-2)\frac{R_j}{\epsilon_j}} \sum_{l=1}^{K} \left(\int_{B_g(\xi_{l_j}, r) \setminus \tilde{M}} d\mu_g \right)^{2/n} \leq c_3 \frac{1}{\epsilon_j^2} e^{-(p-2)\frac{R_j}{\epsilon_j}} = o(1). \tag{3.77}$$

Eqs. (3.75), (3.76) and (3.77) prove (3.59), which contradicts (3.58) .

□

Next we study an estimate for the term $R_{\epsilon,\bar{\xi}} = \Pi^{\perp}_{\epsilon,\bar{\xi}}\{i^*_\epsilon[f(V_{\epsilon,\bar{\xi}})] - V_{\epsilon,\bar{\xi}}\}$.

Lemma 3.5.2. *There exist* $\rho_0 > 0$*,* $\epsilon_0 > 0$*,* $c > 0$ *and* $\sigma > 0$ *such that for any* $\rho \in (0, \rho_0)$*,* $\epsilon \in (0, \epsilon_0)$ *and* $\bar{\xi} \in D^{k_0}_{\epsilon,\rho}$*, it holds*

$$\|R_{\epsilon,\bar{\xi}}\|_\epsilon \leq c\Big(\epsilon^2 + \sum_{i \neq j} e^{-\frac{1+\sigma}{2}\frac{d_g(\xi_i,\xi_j)}{\epsilon}}\Big). \tag{3.78}$$

Proof. Let $Y_{\epsilon,\xi} = \epsilon^2 \Delta_g W_{\epsilon,\xi} + (\epsilon^2 \lambda s_g + 1) W_{\epsilon,\xi}$, so that by (3.19): $W_{\epsilon,\xi} = \iota^*_\epsilon(Y_{\epsilon,\xi})$. Hence, if $Y_{\epsilon,\bar{\xi}} := \sum_{i=1}^{k_0} Y_{\epsilon,\xi_i}$, we have

$$-\epsilon^2 \Delta_g V_{\epsilon,\bar{\xi}} + (1 + \epsilon^2 \lambda s_g) V_{\epsilon,\bar{\xi}} = Y_{\epsilon,\bar{\xi}} \quad \text{on } M, \tag{3.79}$$

that is, $V_{\epsilon,\bar{\xi}} = \iota^*_\epsilon(Y_{\epsilon,\bar{\xi}})$. Then, using the estimate in (3.20):

$$\begin{aligned} &\|R_{\epsilon,\bar{\xi}}\|_\epsilon = \|\iota^*_\epsilon(f(V_{\epsilon,\bar{\xi}})) - V_{\epsilon,\bar{\xi}}\|_\epsilon \leq C \mid f(V_{\epsilon,\bar{\xi}}) - Y_{\epsilon,\bar{\xi}} \mid_{p',\epsilon} \\ &\leq C \mid \Big(\textstyle\sum_{i=1}^{k_0} W_{\epsilon,\xi_i}\Big)^{p-1} - \sum_{i=1}^{k_0} W^{p-1}_{\epsilon,\xi_i} \mid_{p',\epsilon} + \mid \sum_{i=1}^{k_0} W^{p-1}_{\epsilon,\xi_i} - Y_{\epsilon,\bar{\xi}_i} \mid_{p',\epsilon}, \end{aligned} \tag{3.80}$$

for some $C > 0$. On one hand, by arguing as in Lemma 3.3 in [15], for some $\sigma > 0$, we get

$$\mid \Big(\sum_{i=1}^{k_0} W_{\epsilon,\xi_i}\Big)^{p-1} - \sum_{i=1}^{k_0} W^{p-1}_{\epsilon,\xi_i} \mid_{p',\epsilon} = o\Big(\sum_{i \neq j} e^{-\frac{1+\sigma}{2}\frac{d_g(\xi_i,\xi_j)}{\epsilon}}\Big). \tag{3.81}$$

On the other hand,

$$\begin{aligned} \mid \sum_{i=1}^{k_0} W^{p-1}_{\epsilon,\xi_i} - Y_{\epsilon,\bar{\xi}_i} \mid_{p',\epsilon} &= \mid \sum_{i=1}^{k_0} \Big(W^{p-1}_{\epsilon,\xi_i} - Y_{\epsilon,\xi_i}\Big) \mid_{p',\epsilon} \\ &\leq \sum_{i=1}^{k_0} \mid W^{p-1}_{\epsilon,\xi_i} - Y_{\epsilon,\xi_i} \mid_{p',\epsilon} . \end{aligned} \tag{3.82}$$

Let $\tilde{Y}_{\epsilon,\xi}(z) = Y_{\epsilon,\xi}(\exp_\xi(z))$ for $z \in B(0, r)$, then

$$\tilde{Y}_{\epsilon,\xi} = -\epsilon^2 \Delta_g W_{\epsilon,\xi} + (1 + \epsilon^2 \lambda s_g) W_{\epsilon,\xi} = -\epsilon^2 \Delta_g (U_\epsilon \chi_r) + (1 + \epsilon^2 \lambda s_g) U_\epsilon \chi_r$$

$$= -\epsilon^2 \chi_r \Delta U_\epsilon + U_\epsilon \chi_r + \epsilon^2 \lambda s_g U_\epsilon \chi_r - \epsilon^2 U_\epsilon \Delta \chi_r - 2\epsilon^2 \langle \nabla U_\epsilon, \nabla \chi_r \rangle$$

$$+\epsilon^2 (g_\xi^{ij} - \delta_{i,j}) \partial_{ij} (U_\epsilon \chi_r) - \epsilon^2 g_\xi^{ij} \Gamma_{ij}^k \partial_k (U_\epsilon \chi_r)$$

$$= \left(U_\epsilon^{p-1} \chi_r - \epsilon^2 U_\epsilon \Delta \chi_r - 2\epsilon^2 \langle \nabla U_\epsilon, \nabla \chi_r \rangle + \epsilon^2 (g_\xi^{ij} - \delta_{i,j}) \partial_{ij} (U_\epsilon \chi_r) - \epsilon^2 g_\xi^{ij} \Gamma_{ij}^k \partial_k (U_\epsilon \chi_r) \right)$$

$$+ \left(\epsilon^2 \lambda s_g U_\epsilon \chi_r \right).$$

Then

$$\left(\frac{1}{\epsilon^n} \int_M (W_{\epsilon,\xi}^{p-1} - Y_{\epsilon,\xi})^{p'} d\mu_g \right)^{\frac{1}{p'}} = \left(\frac{1}{\epsilon^n} \int_{B(0,r)} \left((U_\epsilon(z) \chi_r(z))^{p-1} - \tilde{Y}_{\epsilon,\xi(z)} \right)^{p'} |g_\xi(z)| \, dz \right)^{\frac{1}{p'}}$$

$$\leq c \left(\frac{1}{\epsilon^n} \int_{B(0,r)} \left(U_\epsilon^{p-1} (\chi_r^{p-1} - \chi_r) \right)^{p'} dz \right)^{\frac{1}{p'}} + c\epsilon^2 \left(\frac{1}{\epsilon^n} \int_{B(0,r)} (U_\epsilon \Delta \chi_r)^{p'} dz \right)^{\frac{1}{p'}}$$

$$+c\epsilon^2 \left(\frac{1}{\epsilon^n} \int_{B(0,r)} (\langle \nabla U_\epsilon, \nabla \chi_r \rangle)^{p'} dz \right)^{\frac{1}{p'}}$$

$$+c\epsilon^2 \left(\frac{1}{\epsilon^n} \int_{B(0,r)} \left((g_\xi^{ij} - \delta_{i,j}) \partial_{ij} (U_\epsilon \chi_r) \right)^{p'} dz \right)^{\frac{1}{p'}} + c\epsilon^2 \left(\frac{1}{\epsilon^n} \int_{B(0,r)} \left(g_\xi^{ij} \Gamma_{ij}^k \partial_k (U_\epsilon \chi_r) \right)^{p'} dz \right)^{\frac{1}{p'}}$$

$$+c\epsilon^2 \left(\frac{1}{\epsilon^n} \int_{B(0,r)} \left(s_g U_\epsilon \chi_r \right)^{p'} dz \right)^{\frac{1}{p'}},$$

by Lemma 3.3 in [26], the first five terms in the last inequality are $o(\epsilon^2)$. Meanwhile, for the last term we have:

$$\left(\frac{1}{\epsilon^n} \int_{B(0,r)} \left(\epsilon^2 s_g U_\epsilon \chi_r \right)^{p'} dz \right)^{\frac{1}{p'}} \leq c_1 \, \epsilon^2 \left(\frac{1}{\epsilon^n} \int_{B(0,r)} U_\epsilon^{p'} \chi_r^{p'} dz \right)^{\frac{1}{p'}}$$

$$\leq c_1 \, \epsilon^2 \left(\int_{B(0,\frac{r}{\epsilon})} U^{p'} dz \right)^{\frac{1}{p'}} \leq c_2 \, \epsilon^2.$$

Thus, eq. (3.82) turns into

$$| \sum_{i=1}^{k_0} W_{\epsilon,\xi_i}^{p-1} - Y_{\epsilon,\bar{\xi}_i} |_{p',\epsilon} \leq \sum_{i=1}^{k_0} | W_{\epsilon,\xi_i}^{p-1} - Y_{\epsilon,\xi_i} |_{p',\epsilon} \leq c \, \epsilon^2, \tag{3.83}$$

for some $c > 0$. Eqs. (3.81) and (3.83) imply the estimate of the lemma.

□

As stated above, in order to solve eq. (3.12) we need to find a fixed point for the operator $T_{\epsilon,\bar{\xi}} : K^{\perp}_{\epsilon,\bar{\xi}} \to K^{\perp}_{\epsilon,\bar{\xi}}$ defined by

$$T_{\epsilon,\bar{\xi}}(\phi) = L^{-1}_{\epsilon,\bar{\xi}}(N_{\epsilon,\bar{\xi}}(\phi) + R_{\epsilon,\bar{\xi}}).$$

By Lemma 3.5.1 we have

$$\|T_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} \leq c\Big(\|N_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} + \|R_{\epsilon,\bar{\xi}}\|_{\epsilon}\Big) \tag{3.84}$$

and

$$\|T_{\epsilon,\bar{\xi}}(\phi_1) - T_{\epsilon,\bar{\xi}}(\phi_2)\|_{\epsilon} \leq c\Big(\|N_{\epsilon,\bar{\xi}}(\phi_1)\|_{\epsilon} - \|N_{\epsilon,\bar{\xi}}(\phi_2)\|_{\epsilon}\Big).$$

By (3.18) and (3.20), it holds

$$\|N_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} \leq C \mid f(V_{\epsilon,\bar{\xi}} + \phi) - f(V_{\epsilon,\bar{\xi}}) - f'(V_{\epsilon,\bar{\xi}})\phi \mid_{p',\epsilon} .$$

And by the Mean Value Theorem, there is some $\tau \in (0, 1)$ such that, if $\|\phi_1\|_{\epsilon}$ and $\|\phi_2\|_{\epsilon}$ are small enough,

$$\begin{aligned} &\mid f(V_{\epsilon,\bar{\xi}} + \phi_1) - f(V_{\epsilon,\bar{\xi}} + \phi_2) - f'(V_{\epsilon,\bar{\xi}})(\phi_1 - \phi_2) \mid_{p',\epsilon} \\ &\leq C \mid (f'(V_{\epsilon,\bar{\xi}} + \phi_2 + \tau(\phi_1 - \phi_2)) - f'(V_{\epsilon,\bar{\xi}}))(\phi_1 - \phi_2) \mid_{p',\epsilon} \\ &\leq C \mid f'(V_{\epsilon,\bar{\xi}} + \phi_2 + \tau(\phi_1 - \phi_2)) - f'(V_{\epsilon,\bar{\xi}}) \mid_{\frac{p}{p-2},\epsilon} \mid \phi_1 - \phi_2 \mid_{p',\epsilon} . \end{aligned} \tag{3.85}$$

It follows from [15], Section 3, that

$$\begin{aligned} &\mid f'(V_{\epsilon,\bar{\xi}} + \phi_2 + \tau(\phi_1 - \phi_2)) - f'(V_{\epsilon,\bar{\xi}}) \mid_{\frac{p}{p-2},\epsilon} \mid \phi_1 - \phi_2 \mid_{p',\epsilon} \\ &\leq C\|\phi_1 - \phi_2\|_{\epsilon}. \end{aligned} \tag{3.86}$$

And then we have

$$\|T_{\epsilon,\bar{\xi}}(\phi_1) - T_{\epsilon,\bar{\xi}}(\phi_2)\|_{\epsilon} \leq \|N_{\epsilon,\bar{\xi}}(\phi_1) - N_{\epsilon,\bar{\xi}}(\phi_2)\|_{\epsilon} \leq c\|\phi_1 - \phi_2\|_{\epsilon}, \tag{3.87}$$

for $c \in (0, 1)$, provided $\|\phi_1\|_{\epsilon}$ and $\|\phi_2\|_{\epsilon}$ are small enough.

Hence $T_{\epsilon,\bar{\xi}}$ has a fixed point in a small enough ball in $K^{\perp}_{\epsilon,\bar{\xi}}$, centered at 0.

Moreover, for such fixed point, we have by eq. (3.84),

$$\|\phi_{\epsilon,\bar{\xi}}\|_{\epsilon} = \|T_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} \leq c\Big(\|N_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} + \|R_{\epsilon,\bar{\xi}}\|_{\epsilon}\Big).$$

On the other hand

$$\|N_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} \leq c\|\phi\|_{\epsilon}, \tag{3.88}$$

for ϕ with $\|\phi\|_{\epsilon}$ small enough, since

$$\|N_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} \leq c\left(\|\phi\|^{p-1}_{\epsilon} + \|\phi\|^{2}_{\epsilon}\right),$$

by eq (3.35) in [15].

Hence by Lemma 3.5.2, and inequality (3.88),

$$\|\phi_{\epsilon,\bar{\xi}}\|_\epsilon \le c\Big(\|N_{\epsilon,\bar{\xi}}(\phi)\|_\epsilon + \|R_{\epsilon,\bar{\xi}}\|_\epsilon\Big) \le c_1\|\phi_{\epsilon,\bar{\xi}}\|_\epsilon + c_2\Big(\epsilon^2 + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{d_g(\xi_i,\xi_j)}{\epsilon}}\Big).$$

This implies the estimate of the lemma:

$$\|\phi_{\epsilon,\bar{\xi}}\|_\epsilon \le c_3\Big(\epsilon^2 + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{d_g(\xi_i,\xi_j)}{\epsilon}}\Big).$$

Finally, to prove that the map $\xi \to \phi_{\epsilon,\bar{\xi}}$ is in fact a C^1 map, given ϵ, we use the Implicit Function Theorem applied to the function

$$F(\bar{\xi},\phi) = T_{\epsilon,\bar{\xi}}(\phi) - \phi.$$

As stated above, eq. (3.87) guarantees that there is some $\phi_{\epsilon,\bar{\xi}}$, such that $F(\bar{\xi},\phi_{\epsilon,\bar{\xi}}) = 0$. Also, $T_{\epsilon,\bar{\xi}}(\phi)$ is differentiable, with differentiable inverse $L_{\epsilon,\bar{\xi}}(\phi)$. The Implicit Function Theorem then implies that $\xi \to \phi_{\epsilon,\bar{\xi}}$ is a C^1 map.

□

Resumen del Capítulo

En este capítulo usamos las técnicas de reducción de Lyapunov-Schmidt para probar resultados de multiplicidad de soluciones de la ecuación de Yamabe en una variedad producto.

Sean (M^n, g) una variedad cerrada y (N^m, h) una variedad de curvatura escalar constante positiva s_h. Estamos interesados en soluciones positivas de la ecuación de Yamabe para la variedad producto $(M\times N, g+\epsilon^2 h)$. Bajo ciertas condiciones sobre la solución u, la ecuación de Yamabe en esta variedad resulta:

$$-\epsilon^2\Delta_g u + \left(\frac{s_g}{a}\epsilon^2 + 1\right)u = u^{p-1}, \tag{3.89}$$

donde $a = a_{m+n} = \frac{4(m+n-1)}{m+n-2}$, $p = p_{m+n} = \frac{2(m+n)}{m+n-2}$, s_g es la curvatura escalar de (M^n, g), y ϵ es suficientemente pequeño como para que la curvatura escalar de la variedad producto $s_g + \epsilon^{-2}s_h$ resulte positiva. En este capítulo estudiamos la ecuación

$$-\epsilon^2\Delta_g u + \left(\lambda s_g\epsilon^2 + 1\right)u = u^{p-1},$$

donde λ es cualquier número real. Las soluciones positivas de esta ecuación son los puntos críticos del funcional energía $J_\epsilon : H^{1,2}(M) \to \mathbb{R}$, dado por

$$J_\epsilon(u) = \epsilon^{-n} \int_M \left(\frac{1}{2}\epsilon^2 |\nabla u|^2 + \frac{1}{2}\left(\epsilon^2 \lambda s_g + 1\right) u^2 - \frac{1}{p}(u^+)^p \right) dV_g,$$

con $u^+(x) = \max\{u(x), 0\}$.

Dados cualquier punto $\xi \in M$ y $\epsilon > 0$ construimos soluciones aproximadas $W_{\epsilon,\xi}$ mediante el procedimiento de reducción Lyapunov-Schmidt. Luego probamos el siguiente resultado:

Theorem 3.5.3. *Supongamos que $\beta_\lambda \neq 0$. Si $\beta_\lambda < 0$ ($\beta_\lambda > 0$) sea ξ_0 un punto de máximo (mínimo) local aislado de la curvatura escalar S_g. Para cada entero positivo k_0 existe $\epsilon_0 = \epsilon_0(k_0) > 0$ tal que si $\epsilon \in (0, \epsilon_0)$ entonces existen puntos $\xi_1^\epsilon, \ldots, \xi_{k_0}^\epsilon \in M$ tales que*

$$\frac{d_g(\xi_i^\epsilon, \xi_j^\epsilon)}{\epsilon} \to +\infty \quad \textit{and} \quad d_g(\xi_0, \xi_j^\epsilon) \to 0, \tag{3.90}$$

y existe una solución u_ϵ del problema (3.89) que cumple:

$$\|u_\epsilon - \sum_{i=1}^{k_0} W_{\epsilon,\xi_i^\epsilon}\|_\epsilon \to 0,$$

donde β es una constante que depende de las dimensiones de M y N y de λ.

El capítulo está organizado de la siguiente forma. En la Sección 3.2 presentamos la notación y los antecedentes, y analizamos la reducción dimensional del problema mediante el procedimiento Lyapunov-Schmidt. Luego construimos soluciones aproximadas para la ecuación. En la Sección 3.3 explicamos la expansión asintótica del funcional energía en términos de ϵ. En la sección 3.4 nos dedicamos a probar el teorema anterior asumiendo que una proposición auxiliar es verdadera. Luego demostramos dicha proposición en la Sección 3.5.

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