Componentes logarítmicas de espacios de foliaciones proyectivas

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Resumen

En esta tesis se aborda un estudio de formas logarítmicas en el espacio proyectivo y de grados arbitrarios, desde un punto de vista relativo a la teoría de foliaciones algebraicas. El problema original que motiva este trabajo es el estudio de componentes irreducibles del espacio de moduli de foliaciones algebraicas proyectivas con singularidades, y la descripción de la geometría de dicho espacio. De modo específico, se prueba la estabilidad de q*-formas logarítmicas proyectivas de manera general para el caso en que* $q = 1$*, y con algunas hipótesis adicionales sobre los grados de las componentes del divisor que las definen para el caso* $q = 2$*.*

De modo más detallado, y en primera instancia, se desarrolla una nueva prueba algebraica del resultado de estabilidad de este tipo de formas para el caso de grado uno. Además, se demuestra que las componentes que determinan en el espacio de foliaciones correspondiente son genéricamente reducidas como esquemas. Este resultado otorga una mejora al teorema de estabilidad ya conocido dado por Omegar Calvo Andrade, cuya prueba involucra métodos de naturaleza puramente topológica. Asimismo, se describen diversos aspectos de la parametrización racional que las define, como la caracterización de su base locus y la prueba de su inyectividad genérica. También, será abordado el problema de su posible racionalidad como variedades algebraicas.

El segundo aspecto importante de esta tesis radica en obtener una generalización de estos resultados a formas de grados superiores, para espacios de foliaciones en codimensiones arbitrarias. Para ello se desarrolla como primera medida, una correcta caracterización de las formas logarítmicas que definen foliaciones en el espacio proyectivo n-dimensional, con una estrecha relación con el conocido haz de formas logarítmicas. A continuación, mediante una adaptación y generalización de los métodos utilizados para el caso de grado uno, se prueba la estabilidad de 2-formas logarítmicas bajo ciertas hipótesis sobre los grados de los polinomios que las definen. Esto permite deducir la existencia de numerosas nuevas componentes irreducibles (y genéricamente reducidas como esquema) del espacio de moduli de foliaciones correspondiente.

Palabras clave: Formas logarítmicas - Foliaciones proyectivas.

Logarithmic components of spaces of projective foliations

Abstract

This thesis is concerned with the study of projective logarithmic forms of arbitrary degrees, in the setting of the theory of algebraic foliations. The original problem which motivates this work is the study of the irreducible components of the moduli space of algebraic projective singular foliations, and the description of their geometry. Specifically, we use algebraic methods to prove the stability of projective logarithmic q-forms when $q = 1$, and when $q = 2$ with some additional assumptions on the degree of the components of the divisor that defines them.

In the second chapter, we develop a new proof of the stability result for polynomial logarithmic differential one-forms. Also, we deduce that the corresponding irreducible components of the space of foliations are generically reduced as schemes. This last fact provides a non-trivial improvement to the well-known stability result due to Omegar Calvo Andrade. Likewise, various aspects of the parametrization defining these components are described. For example, we describe its base locus and prove its generic injectivity. Also, we analyze their possible rationality as algebraic varieties.

Another important aspect of this work is related to obtain a generalization of these results to logarithmic forms of higher degrees and spaces of foliations of arbitrary codimension. To achieve this purpose, we develop a correct characterization of those logarithmic forms which define foliations over the n-dimensional projective space. These results are closely related to the well-known sheaf of logarithmic forms. Finally, improving the methods used in the case of degree one, we prove the stability of logarithmic 2-forms under certain assumptions on the degree of the polynomials that define them. This result allows us to deduce the existence of many new irreducible components of the corresponding moduli space of foliations.

Key words: Logarithmic forms- Projective foliations.

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Contents

Introducción

Tema y objetivos generales:

El tema de investigación en el que se enmarca la presente tesis es el estudio de la geometría y las componentes del espacio de moduli de foliaciones dentro de una variedad algebraica proyectiva. Se centra en el estudio de formas diferenciales que determinan una distribución integrable con singularidades, y sus posibles deformaciones. Nuestro objetivo específico es probar la estabilidad de q-formas logarítmicas (con $q \in \mathbb{N}$) en el espacio proyectivo, y demostrar que constituyen componentes del espacio de foliaciones en codimensión q.

De igual modo, nos resulta de interés el estudio de propiedades de esta familia de q-formas, como sus tipos de singularidades y la geometría de la componente que determinan. Nos proponemos caracterizar el base locus de la parametrización que las define, y determinar si son componentes reducidas o no del espacio de moduli correspondiente.

Antecedentes históricos:

Un problema que ha inspirado diversas áreas de la matemática es el que se conoce como el *Problema de Pfaff*. Este tiene sus origines en la teoría de ecuaciones diferenciales en derivadas parciales de primer orden, originada por Lagrange a mediados del siglo XVIII. Hasta ese entonces, el concepto de *integrabilidad* refería a encontrar soluciones de la ecuación general

$$F(x_1, \dots, x_n, \tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n}, z) = 0,$$

del tipo $z = \phi(x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, a partir de sistemas de ecuaciones diferenciales ordinarias. Luego, a principios del siglo XIX, el problema de la integración de ecuaciones diferenciales no lineales fue abordado por Johann Friedrich Pfaff, quien logró darle un enfoque brillante y original. Su idea se basaba en considerar el problema totalmente general de "*integrar*", desde una nueva perspectiva, una ecuación diferencial *total* del tipo:

$$\omega = A_1(x)dx_1 + \cdots + A_n(x)dx_n. \tag{0.0.1}$$

Como puede verse en [49], uno de los hechos fundamentales que descubrió Pfaff es que la resolución de toda ecuación diferencial en derivadas parciales de primer orden en m variables, puede reducirse a la integrabilidad de una ecuación diferencial total en $2m$ variables. De este modo, resolviendo el problema mas general de integración 0.0.1 se lograba deducir, como caso particular, algunos resultados obtenidos por Lagrange en el caso más clásico.

El concepto adecuado de intregrabilidad refiere a obtener un cierto numero de ecuaciones:

$$\phi_i(x_1, \ldots, x_n) = C_i \quad \forall i = 1, \ldots, k,$$

definidas por funciones $\{\phi_i\}_{i=1}^k$ independientes, tales que en los puntos que las satisfacen se cumplan en cierto sentido las condiciones determinadas por $\omega = 0$. Este concepto de integrabilidad se conecta, en teorías más actuales, con el de subvariedad integral (ver por ejemplo [2]).

Cabe destacar también, que dicho problema de integración de Pfaff ha motivado trabajos de matemáticos como Jacobi, Kronecker, Clebsch, etc. Para más datos del desarrollo histórico del *Problema de Pfaff*, se sugiere consultar [33]. En particular, a continuación resaltamos principalmente las contribuciones realizadas por F.G. Frobenius.

En su famoso trabajo [23], Frobenius realiza un aporte sustancial al problema de integrabilidad de Pfaff, basado principalmente en el trabajo realizado por sus predecesores en los temas anteriormente mencionados. De modo concreto, formuló un criterio para la integración completa de cualquier sistema de ecuaciones de Pfaff, desde un enfoque algebraico y muy elegante. Este criterio (ver por ejemplo [33, Teorema 9.2]) representa los inicios de lo que hoy se conoce como Teorema de integrabilidad de Frobenius (ver apéndice de [6]), y ha sentado las bases de diversos enfoques actuales como la Teoría de Foliaciones, la Teoría de Sistemas Diferenciales Exteriores, o más en general, la aplicación de métodos algebraicos y geométricos a la resolución de ecuaciones diferenciales. En lenguaje moderno, la condición que estableció para que una ecuación de Pfaff ω sea completamente integrable es:

$$\omega \wedge d\omega = 0,$$

involucrando a la derivada exterior (clasicamente denominada derivada covariante) de la forma. Esta condición, que en realidad es local, asegura que por casi por todos los puntos pasa una única variedad integral de ω.

El problema de Pfaff también ha inspirado un famoso trabajo de Élie Cartan, continuando con los desarrollos de sus directores S. Lie y G. Darboux. Este artículo mencionado resultó sentar las bases de sus desarrollos en el cálculo de formas diferenciales exteriores, en la estructura de grupos continuos y, más en general, en sus desarrollos en topología algebraica. Para más referencias sobre estos aspectos, consultar el capítulo 11 de [33] y el 3 de [2].

Los métodos algebraicos y geométricos abordados por todos los matemáticos anteriormente mencionados, han motivado el desarrollo de diversas teorías a lo largo del siglo XX, que continúan actualmente. Entre ellas destacamos la Teoría de Foliaciones. Intuitivamente, una foliación puede pensarse como una descomposición de una variedad en subvariedades de igual dimensión (llamadas hojas), que localmente se agrupan como si fueran las hojas de un libro. De acuerdo con el mencionado Teorema de Frobenius, una ecuación de Pfaff completamente integrable determina una foliación cuyas hojas son de codimensión uno.

Como se menciona en la introducción de [6], la Teoría de Foliaciones, como se la conoce actualmente, tiene sus orígenes en los trabajos de C. Ehresmann y C. Reeb en los años 1940', y ha tomado distintos enfoques a lo largo del siglo XX. A modo de ejemplo, en su tesis doctoral dirigida por Ehresmann, Reeb resolvió afirmativamente un problema propuesto por H. Hopf sobre la existencia de campos de vectores X en la esfera S^3 tales que: $X \cdot rot(X) \equiv 0$. Según el Teorema de Frobenius, esta pregunta era equivalente a preguntarse por la existencia de una foliación de dimensión dos sobre la esfera tres dimensional. Este tipo de problemas ha motivado un gran desarrollo del área de la topología diferencial en relación con la Teoría de Foliaciones.

Otro de los grandes matemáticos que deseamos destacar en este desarrollo histórico es Solomon Lefschetz (1884-1972), debido a su importante aporte en aspectos topológicos y geométricos de ecuaciones diferenciales. Luego de sus famosos trabajos en topología general y geometría algebraica, y a partir de 1944, dedico su trabajo al estudio de ecuaciones diferenciales. En este contexto, en 1958 lideró un grupo de investigación dentro del *Research Institute for Advanced Studies (RIAS)* en Baltimore. Este se convirtió en uno de los grupos más destacados en la investigación en las ecuaciones diferenciales no lineales. Finalmente, dejó RIAS en 1964 para formar el *Lefschetz Center for Dynamical Systems* en la Universidad de Brown, Rhode Island. Específicamente, consideramos de gran relevancia sus aportes en la teoría geométrica de ecuaciones diferenciales (ver [39]), en lo que hoy se conoce como la teoría de sistemas dinámicos y en su introducción de conceptos de naturaleza topológica en estos últimos, como por ejemplo el de *estabilidad* (ver a modo de ejemplo [40]).

A partir de la década de '1970, motivado por problemas de dinámica compleja (relacionados con trabajos de matemáticos como J. Milnor y S. Smale) y por estudios de ecuaciones diferenciales en los números complejos, ha sido de interés el estudio de foliaciones holomorfas en distintas variedades. Destacamos los desarrollos y escuelas originadas por matemáticos como C. Camacho, A. Lins-Neto, P. Sad, R. Moussu, D.Cerveau, J.F Mattei, X. Gomez-Mont, entre otros.

En este contexto, uno de los elementos centrales de estudio ha sido el de formas diferenciales proyectivas que definen foliaciones (formas completamente integrables, según lo descripto anteriormente). Asimismo, resulta de interés sus tipos de singularidades, la geometría y dinámica de sus hojas, y la posible estabilidad del tipo de foliaciones que definen. Muchos de los problemas abordados por estos autores tienen su raíz común en el problema de clasificación de foliaciones holomorfas. Referimos a [8], [9] y a [45] para una perspectiva más detallada de los orígenes de este problema, y de la Teoría de Foliaciones en el caso holomorfo.

En relación con el problema de clasificación señalado y el estudio de familias de foliaciones estables, muchos de los matemáticos mencionados anteriormente han conseguido resultados en variedades complejas, mediantes métodos de naturaleza topológica y elementos clásicos de geometría compleja. Entre estos trabajos destacamos los realizados por A. Lins-Neto y X. Gomez-Mont [25], J.P. Jouanolou [35], D.Cervau y C.F Mattei [8], C. Camacho y A. Lins-Neto [5], D.Cerveau y A. Lins-Neto [9]-[10], O. Calvo Andrade [3], entre otros.

Desde otra perspectiva, diversas herramientas de geometría algebraica han sido utilizadas para el estudio foliaciones y formas integrables sobre variedades. Resaltamos, en primer instancia, el trabajo realizado por Jouanolou en [35] sobre formas de Pfaff algebraicas, cuyo enfoque ha sido útil para el presente trabajo. En relación con este último señalamos un trabajo de E. Ghys [24] que también consideramos relevante.

Con este mismo enfoque mencionado, destacamos los desarrollos e ideas de F. Cukierman - J.V. Pereira [14] y posteriormente F. Cukierman - J. V. Pereira - I. Vainsencher [15], sobre el estudio de componentes irreducibles de espacios de moduli de foliaciones proyectivas. En el primero de ellos, los autores introducen un método infinitesimal basado en perturbaciones de primer orden para el estudio de estabilidad de formas diferenciales integrables en espacios algebraicos. Su idea radica en caracterizar el espacio tangente a una forma dada en un familia específica y probar que cubre todas las direcciones posibles del espacio de foliaciones que la contiene. De ese modo, pueden deducir que la familia de formas involucradas constituye una componente irreducible dentro de dicho espacio. Esta idea novedosa también ha sido utilizada en el segundo trabajo mencionado (ver Proposición 3.1 y Teorema 3.1 de [15]), y es la herramienta fundamental que será utilizada en la presente tesis.

Además, esta herramienta de naturaleza algebraica resulta un aporte sustancial en el área de interés y sienta las bases para trabajos futuros. Incluso, este enfoque permite estudiar propiedades de la geometría de las componentes determinadas teniendo también en cuenta su estructura de esquema. A modo de ejemplo, destacamos que, además de los resultados obtenidos en esta tesis, será utilizada en un contexto esquemático general dentro de un trabajo que se encuentra en preparación por F. Cukierman y C. Massri [13].

Antecedentes más detallados del tema de investigación:

Teniendo en cuenta lo explicado anteriormente, el problema que motiva la presente tesis es encontrar una generalización común a dos tipos de componentes irreducibles ya conocidas (tipos de familias estables de formas diferenciales proyectivas integrables), una dentro del espacio de foliaciones proyectivas de codimensión uno y la otra sobre el de foliaciones dadas en una codimensión arbitraria $q \in \mathbb{N}$. Estas formas diferenciales involucradas son conocidas como las 1-formas logarítmicas (ver el trabajo de O. Calvo-Andrade [3] para el estudio de su estabilidad), y las q-formas racionales (ver Cukierman-Pereira-Vainsencher [15]). Dicho de otro modo, el problema radica en definir fórmulas adecuadas para las "q-formas logarítmicas" en el contexto de la Teoría de Foliaciones, que generalicen los casos anteriores (ver fórmulas 0.0.2 y 0.0.3 más abajo), y probar su estabilidad mediante los métodos infinitesimales introducidos en [14].

Para explicar más detalladamente el problema, comencemos con un breve resumen de los conceptos y definiciones relativas a los espacios de foliaciones proyectivas.

Sea ω una 1-forma diferencial en el espacio afín complejo de dimensión $n+1$, cuyos coeficientes son polinomios homogéneos de grado $d-1$, i.e. formas del tipo:

$$\omega = \sum_{i=0}^{n} A_i dz_i.$$

Suponemos, además, que ω desciende al espacio proyectivo, es decir, que se anula en su contracción con el campo radial de Euler:

$$i_R(\omega) = 0,$$

donde $R = \sum_{i=1}^{n+1} z_i \frac{\partial}{\partial z_i}$. Denotamos por $\mathcal{F}_1(d, \mathbb{P}^n)$ al conjunto de tales 1-formas (módulo constante multiplicativa), que además satisfacen la condición de integrabilidad:

$$\omega \wedge d\omega = 0.$$

Por el Teorema de Frobenius, una tal ω define una foliación (singular) de codimensión uno. Por cada punto no singular (puntos donde la forma no se anula) pasa una única variedad integral de esa codimensión. Un problema central del área consiste en determinar las componentes irreducibles de la variedad algebraica $\mathcal{F}_1(d,\mathbb{P}^n)$, denominada como el espacio de moduli de foliaciones proyectivas singulares de grado d y codimensión uno. Además, resulta relevante estudiar la geometría de sus componentes: dimensión, grado, singularidades, relaciones de incidencia, etc.

Con mayor generalidad, es de interés considerar los problemas análogos para foliaciones singulares de cualquier codimensión. Denotamos por $\mathcal{F}_q(d,\mathbb{P}^n)$ al espacio de moduli de foliaciones singulares de codimensión q y grado d en el espacio proyectivo de dimensión n. Estas consistenten de los sub-haces coherentes saturados del fibrado tangente de rango q y grado d, cerrados por corchete de Lie. Pero, además, también pueden ser caracterizadas por q-formas diferenciales proyectivas con ciertas propiedades que aseguren su integrabiliad.

En primera instancia referimos a la introducción de [9], a [17] y a [15] para una definición más precisa de estos espacios de foliaciones, y para un resumen de las algunas componentes conocidas. A continuación describimos en mayor detalle algunos antecedentes de la teoría de foliaciones en variedades algebraicas.

En referencia al problema de la caracterización de las componentes irreducibles de $\mathcal{F}_1(d,\mathbb{P}^n)$, resumimos los primeros resultados conocidos. Este problema fue resuelto por J. P. Jouanolou [35] en los casos $d = 2$ (una componente irreducible) y $d = 3$ (dos componentes irreducibles). El caso $d = 4$ fué resuelto en [9] donde se demuestra que el número de componentes es seis (racionales, logarítmicas, pull-back y excepcional). En este mismo artículo, Cerveau y Lins Neto plantean también la pregunta natural sobre la estructura del cociente de estas componentes irreducibles bajo la acción del grupo general lineal proyectivo.

Por otro lado, y para hacer mención a los antecedentes de formas logarítmicas desde el punto de vista de su conexión con espacios de foliaciones, puede verse en primera instancia como en [8] los autores deducen que en el espacio de moduli de foliaciones en el caso afín las únicas componentes posibles son logarítmicas. Incluso, en este último trabajo, dejan como pregunta qué sucede en el caso de sistemas intersección completa definidos por $q > 1$ ecuaciones de Pfaff.

En relación a formas logarítmicas que definen foliaciones en variedades proyectivas, Omegar Calvo demuestra en [3] que en una variedad proyectiva con ciertas propiedades estas formas constituyen componentes irreducibles del espacio de moduli correspondiente. Para esto, el autor dá una caracterización de dichas formas, algunas propiedades de su holonomía y singularidades, y mediante métodos de naturaleza topológica muestra su estabilidad por perturbaciones. Como ya hemos destacado, este trabajo es muy relevante dado que forma parte del tipo de componentes o formas que deseamos generalizar.

De modo más específico, fijamos enteros positivos n, m y una m-tupla de grados $\mathbf{d} = (d_1,\dots,d_m)$. Luego, una 1-forma diferencial proyectiva ω es logarítmica (de tipo $\mathbf{d}$) si puede expresarse como:

$$\omega = \left(\prod_{i=1}^{m} F_i\right)\sum_{i=1}^{m}\lambda_i\frac{dF_i}{F_i}, \tag{0.0.2}$$

para ciertos polinomios $F_1,\dots,F_m$ de grados respectivos determinados por $\mathbf{d}$, y donde $\lambda = (\lambda_1,\dots,\lambda_m)$ es un elemento de $\mathbb{C}^m$ que satisface: $\sum_{i=1}^{m}\lambda_i d_i = 0$. Con esta notación y la introducida anteriormente, si $d = \sum_{i=1}^m d_i$ entonces $[\omega] \in \mathcal{F}_1(d, \mathbb{P}^n_{\mathbb{C}})$. Incluso, si definimos por $\mathcal{L}_1(\mathbf{d}, n)$ al espacio de tales 1-formas logarítmicas, en [3] fue demostrado que este espacio determina una componente irreducible (para cada $\mathbf{d}$) de $\mathcal{F}_1(d, \mathbb{P}^n_{\mathbb{C}})$.

En relación a otras propiedades geométricas de foliaciones logarítmicas, podemos citar [16], en donde los autores prueban una caracterización completa de sus espacios de singularidades.

Por último, y en conexión con el estudio de la geometría del espacio $\mathcal{F}_q(d, \mathbb{P}^n)$, mencionamos el trabajo de A. De Medeiros [17] como referencia de las nociones iniciales involucradas. Este artículo contiene un desarrollo del concepto de q-formas diferenciales integrables y localmente descomponibles fuera del singular, que son el objeto que describe adecuadamente a las foliaciones de codimensiones superiores.

Sobre el estudio de componentes del espacio de foliaciones en codimensiones arbitrarias, en [15] puede verse como los autores prueban la existencia de nuevas componentes (en este caso de $\mathcal{F}_q(d, \mathbb{P}^n)$) asociadas a las llamadas q-formas racionales, para cada $q \in \mathbb{N}$. Además, estas generalizan a las componentes racionales ya estudiadas en codimensión uno (ver [25]). Para su correcta definición, se debe fijar una tira de $q+1$ grados $\mathbf{d} = (d_0, \dots, d_q)$. En este caso, se dice que ω es una q-forma diferencial proyectiva racional de tipo $\mathbf{d}$ si puede describirse como:

$$\omega = i_R(dF_0 \wedge \cdots \wedge \dots dF_q) = \left(\prod_{i=0}^m F_i\right) \sum_{i=0}^m (-1)^i d_i \frac{dF_0}{F_0} \wedge \cdots \wedge \frac{d\hat{F}_i}{F_i} \wedge \cdots \wedge \frac{dF_q}{F_q}, \tag{0.0.3}$$

para ciertos polinomios homogéneos $F_1, \dots F_q$ de grados respectivos dados por $\mathbf{d}$. Como ya mencionamos, este trabajo también utiliza el método infinitesimal introducido previamente en [14].

Recordamos que las componentes determinadas por estas q-formas racionales son una referencia importante para esta tesis, debido a que las q-formas logarítmicas que deseamos definir deben corresponderse a un caso más general.

Contribuciones del presente trabajo:

De modo concreto, en la presenta tesis se desarrolla una prueba alternativa y de naturaleza algebraica del resultado de estabilidad de 1-formas logarítmicas demostrado por Calvo Andrade en [3]. Nuestros resultados se basan en la técnicas introducidas en los trabajos [14] y [15]. Además, esta prueba permite deducir que las componentes que determinan estas formas en el espacio de moduli correspondiente son genéricamente reducidas, lo cual no era conocido hasta el momento.

De modo más general, se desarrollan las principales definiciones de formas logarítmicas de mayor grado que determinan foliaciones de codimensión arbitraria. Además, se prueba un teorema original de estabilidad de 2-formas logarítmicas para cierto tipo de grados, utilizando una adaptación de los métodos usados para el caso de grado uno.

Consideramos que estos resultados obtenidos sientan las bases de la demostración del caso general para formas logarítmicas de grados arbitrarios, e incluso para su extensión a variedades más generales (como el caso tórico por ejemplo). Esto último determinaría una generalización común a los resultados de estabilidad de 1-formas logarítmicas [3] y q-formas racionales [15], que además, destacaría características geométricas y algebraicas sobre las nuevas componentes encontradas.

Resumen de los capítulos:

En el capítulo 1, se elabora un desarrollo de las principales definiciones asociadas a estos espacios de moduli, como su construcción en variedades generales y algunas propiedades básicas del espacio algebraico que definen. También, se elabora un resumen de los principales resultados conocidos. Se trata de un breve capítulo introductorio que sirve de marco de referencia teórico del resto del trabajo. Además, se realiza un análisis de la correcta definición de formas que definen foliaciones en codimensiones superiores (asociadas al grado de la forma), comparándola con el concepto de sistemas de Pfaff integrables.

En el siguiente capítulo (2), se aborda un estudio completo de 1-formas logarítmicas, con un enfoque particular en su estudio como formas integrables que definen foliaciones. El principal resultado del capítulo es una prueba algebraica de la estabilidad de 1-formas logarítmicas, que permite deducir que las componentes irreducibles que definen en el espacio de moduli correspondiente son, además, genéricamente reducidas (sin elementos nilpotentes). Este resultado forma parte de un trabajo en colaboración ([11]) junto con los Dres. Fernando Cukierman y Cesar Massri, que se encuentra en su etapa final de redacción. En dicho trabajo, también se muestra una caracterización del *base locus* de la parametrización que define a estas formas, su inyectividad genérica y el estudio algunas características geométricas, como su posible racionalidad como variedades algebraicas.

En referencia al capítulo propiamente dicho, se desarrollará una versión extendida del trabajo anterior, con algunas explicaciones y tratamientos diferentes a los descriptos en el artículo. Además, se analizarán algunas otra propiedades relativas a este tipo de formas y componentes, como su lugar singular, posibles núcleos , factores integrantes, hojas algebraicas, etc.

Por último, en este primer caso de formas logarítmicas de grado uno, aparece una condición importante sobre los grados del divisor que las define, que hace variar la dificultad de nuestras demostraciones. Dicho de otro modo, la prueba que realizamos es sustancialmente más sencilla en el caso en que los grados de los polinomios que definen a estas formas son balanceados (ver definiciones 2.5.29 en el capítulo 2 y 4.4.11 en el 4). Esta distinción entre balanceado y no balanceado, será muy importante en los sucesivos análisis de estabilidad para formas logarítmicas de grados superiores. Cabe destacar también que los métodos e ideas desarrolladas en este capítulo servirán como marco de referencia para lo desarrollado en el capítulo 4.

A lo largo del capítulo 3, se desarrolla el estudio de una generalización de un resultado de

Jouanolou descripto en [35], sobre la posible anulación de la fórmula que describe a las formas logarítmicas. En este trabajo mencionado, en especial en su segundo capítulo, el autor estudia los conceptos de integrales primeras racionales y de factores integrantes para formas de Pfaff proyectivas, en virtud de realizar un desarrollo del posible número de soluciones algebraicas. Asimismo este resultado que deseamos generalizar resulta un lema importante en el contexto de estos desarrollos mencionados. De manera adicional, todos estos aspectos, que resultaron útiles en el caso de formas de grado uno en el segundo capítulo, resultan también relevantes para su posterior adaptación y análisis sobre formas de mayores grados.

Por otro lado, dado que este resultado de Jouanolou indicado fue clave para la determinación del *base locus* de la parametrización de 1-formas logarítmicas, se elaboró una generalización para formas de mayor grado y para variedades más generales. Esto último corresponde a un trabajo ([12]) conjunto con Fernando Cukierman, que se encuentra en etapa de redacción .

Se espera que este último resultado elaborado sea importante para la descripción del *base locus* de la parametrización correspondiente a formas logarítmicas de mayor grado, así como también para el estudio de hojas algebraicas intersección completa.

En particular, destacamos que, como consecuencia de lo desarrollado en este capítulo, se obtiene una caracterización adecuada de las secciones globales del haz de formas logarítmicas de grados arbitrarios en variedades proyectivas con ciertas características. En el siguiente capítulo, esto será de vital importancia para la descripción de aquellas formas logarítmicas que definen foliaciones.

Para finalizar, en el capítulo 4, se realiza el correspondiente análisis de estabilidad para formas logarítmicas de grado dos. Asimismo, de manera más general, se desarrolla una descripción adecuada de cuáles son las formas logarítmicas (de grados arbitrarios) que definen foliaciones en codimensiones superiores, en conexión con las descripciones elaboradas en los capítulos 1 y 3.

La técnica global utilizada para deducir la estabilidad es exactamente la misma que para el caso de 1-formas: describir adecuadamente una parametrización que las define y probar la suryectividad genérica de su diferencial. Es decir, el método se basa en un cálculo explícito de su espacio tangente dentro del espacio de moduli correspondiente.

Cabe destacar también, que esta generalización a formas de mayor grado, es sustancialmente más sencilla en el caso de grado dos, dado que una de las ecuaciones que define el espacio de moduli correspondiente es más simple que en casos superiores. Incluso, la prueba realizada asume la hipótesis de balanceabilidad (generalizada a este caso), que hace más sencillas muchas de las demostraciones, en particular las relativas a la prueba de suryectividad del diferencial.

Chapter 1

The moduli spaces of foliations

1.1 Introducción y resumen en español

En este capítulo introductorio se desarrollarán las principales definiciones y construcciones relativas a foliaciones algebraicas singulares en variedades complejas o algebraicas, y sus repectivos espacios de moduli en codimensiones arbitrarias. Además, se analizarán ejemplos y se revisarán las principales componentes irreducibles conocidas.

De modo específico, dada una variedad compleja (o algebraica) M, una foliación de codimensión uno está dada por una familia de 1-formas holomorfas (o algebraicas) $\{\omega_\alpha\}$ definidas en un cubrimiento abierto $\{U_\alpha\}$ de M, y que satisfacen la condición de integrabilidad de Frobenius:

$$\omega_\alpha \wedge d\omega_\alpha = 0.$$

Además, se pide que cumplan con una condición de compatibilidad:

$$\omega_\alpha = f_{\alpha\beta}\omega_\beta$$

en cada intersección doble $U_\alpha \cap U_\beta \neq \emptyset$, para una función nunca nula $f_{\alpha\beta}$. Esta familia de funciones holomorfas (o regulares) cumple con una adecuada condición de cociclo, y puede ser pensada como un elemento $f = [f_{\alpha\beta}] \in H^1(M, \mathcal{O}^*)$. De este modo, puede ser asociada a un fibrado de línea holomorfo (o algebraico) que denotaremos por $\mathcal{L}$.

Por el clásico Teorema de Frobenius, cada ω_α define una foliación de codimensión uno en el abierto U_α, y cada función $f_{\alpha\beta}$ nos da información respecto de cómo se pegan las respectivas hojas en cada intersección doble, para luego definir la foliación globalmente.

Por lo tanto, y en conclusión, una foliación se corresponde con secciones del haz de 1-formas torcidas por el fibrado de línea, $\Omega^1_M \otimes \mathcal{L}$, que además cumplan con la condición de integrabilidad:

$$\omega \wedge d\omega = 0 \in H^0(M, \Omega^3_M(\mathcal{L} \otimes \mathcal{L})).$$

Además, se define el lugar singular de la foliación (S_ω) como los puntos p donde la forma definida es nula, i.e. $S_\omega = \{p \in M :\ \omega(p) = 0\}$. Por lo tanto, las hojas de la foliación quedarán definidas en $M - S_\omega$, y son típicamente no algebraicas.

Dada dos familias $\{\omega_\alpha\}$ y $\{\mu_\alpha\}$ de formas integrables definidas sobre el mismo cubrimiento, se corresponden con la misma foliación cuando existen funciones locales g_α nunca nulas tales que: $\omega_\alpha = g_\alpha \mu_\alpha$. Esto en particular implica que el fibrado de línea correspondiente es el mismo para ambas. Así, una vez fijado el fibrado, queda definida una relación de equivalencia $\sim$ en el espacio $H^0(M, \Omega^1_M(\mathcal{L}))$, que representa adecuadamente a las foliaciones singulares definidas sobre M (asociadas a $\mathcal{L}$).

Recordemos que si la variedad M es compacta, $H^0(M, \Omega^1(\mathcal{L}))$ es un espacio vectorial complejo de dimensión finita. En este caso, además, dos formas ω y μ definen la misma foliación si y solo si existe una constante $\lambda \in \mathbb{C}^*$ tal que: $\omega = \lambda\mu$.

En este contexto, se define el espacio de moduli de foliaciones singulares sobre una variedad compleja (o algebraica) compacta M asociadas a un fibrado de linea $\mathcal{L}$ como:

$$\mathcal{F}_1(\mathcal{L}, M) = \{[\omega] \in \mathbb{P}H^0(M, \Omega^1_M(\mathcal{L})) : \omega \wedge d\omega = 0 \text{ and } codim(S_\omega) \geq 2\}.$$

En general, si además asumimos $\dim_{\mathbb{C}}(M) \geq 3$, $\mathcal{F}_1(\mathcal{L}, M)$ es una subvariedad algebraica de $\mathbb{P}H^0(M, \Omega^1_M(\mathcal{L}))$, típicamente con singularidades (ver [3]). Un problema principal del área consiste en la determinación y posterior estudio de las componentes irreducibles de este espacio. A continuación describimos las principales componentes conocidas y algunos casos completamente clasificados en grados bajos, en el caso particular del espacio proyectivo que es de nuestro interés.

Si $M = \mathbb{P}^n_{\mathbb{C}}$, los fibrados de línea quedan caracterizados por $\mathcal{O}_{\mathbb{P}^n}(d)$, para cada $d \in \mathbb{Z}$. En este caso, el espacio de moduli correspondiente, ahora denotado por $\mathcal{F}_1(d, \mathbb{P}^n)$, describe las foliaciones proyectivas singulares de grado d. Además, geométricamente, el número $d-2$ describe el número de tangencias entre una copia genérica de $\mathbb{P}^1$ con las hojas de la foliación definida.

En referencia al problema de la caracterización de las componentes irreducibles de estos espacios de moduli, resumimos los primeros resultados conocidos en grados bajos. Este problema fue resuelto por J. P. Jouanolou [35] en los casos $d = 2$ (una componente irreducible) y $d = 3$ (dos componentes irreducibles). El caso $d = 4$ fue resuelto en [9], donde se demuestra que el número de componentes es seis (racionales, logarítmicas, pull-back y excepcional). Por otro lado, en grados más altos, el problema de la clasificación de estas componentes sigue abierto, aunque hay diversos estudios de componentes conocidas.

Durante este capítulo, también se desarrollará una caracterización de los espacios de moduli de foliaciones singulares de codimensiones arbitrarias. En primera instancia referimos a [17] para la definición del objeto correcto que parametriza este tipo de foliaciones. Es decir, en un principio es natural pensar que estas foliaciones deberían estar descriptas por sistemas integrables de q 1-formas. Sin embargo, con un ejemplo sencillo en codimension una menos que la del espacio, puede verse que este concepto sería muy restrictivo para el tipo de singularidades admisibles.

El concepto adecuado que será introducido es el de q-formas integrables y localmente descomponibles fuera de su lugar singular. De modo específico, y nuevamente asumiendo que $M = \mathbb{P}^n_{\mathbb{C}}$, vamos a considerar q-formas torcidas $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ (para $d > q$) que cumplan con la ecuación de descomponibilidad local

$$i_V(\omega) \wedge \omega = 0 \quad \text{para todo } V \in \bigwedge^{q-1} \mathbb{C}^{n+1}, \tag{1.1.1}$$

y con la ecuación de intregrabilidad generalizada

$$(1.1.2) \qquad i_V(\omega) \wedge d\omega = 0 \quad \text{para todo } V \in \bigwedge^{q-1} \mathbb{C}^{n+1}.$$

De este modo, es natural definir al espacio de foliaciones projectivas singulares de codimensión q como:

$$\mathcal{F}_q(d, \mathbb{P}^n) = \{[\omega] \in \mathbb{P}H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d)) : \omega \text{ satisface } 1.1.1,\ 1.1.2 \text{ y } codim(S_\omega) \geq 2\}.$$

Asimismo, se elaborará un breve resumen de algunas componentes conocidas de estos espacios, y una revisión de algunos trabajos que son de nuestro interés. En particular destacamos a las q-formas racionales, que resultarán un caso particular de las q-formas logarítmicas que deseamos describir. Se sugiere ver los corolarios finales del capítulo, para una revisión de estos artículos mencionados.

1.2 Summary

In this introductory chapter, we develop the first definitions and constructions related to singular algebraic foliations on complex or algebraic varieties and their respective moduli spaces. Also, we present some examples and known irreducible components of these spaces. We pay particular attention to describe the correct objects and equations which define this spaces, especially in the case of foliations of higher codimension.

1.3 Codimension one case

We begin with some general comments about foliations of codimension one in a complex manifold M of dimension n. Alternatively, we can also think the same definitions for a differentiable or an algebraic manifold.

Let Ω^1_M be the fiber bundle of 1-forms, and Ω^*_M be the exterior algebra of forms over M. In a geometric setting, a regular foliation $\mathcal{F}$ of codimension one can be thought as a decomposition of the variety M into disjoint subvarieties of such codimension (called leaves). Locally, for a fixed system of coordinates, these leaves look like parallel hyperplanes.

From another point of view, we can think that the leaves are determined by fixing locally at every point $p \in M$ a non-vanishing 1-form $\omega \in (\Omega^1_M)_p$. The idea is that the tangent vectors of the leaf passing through p should satisfy the equations imposed by this 1-form. This last fact introduces the following general definition:

Definition 1.3.1. For a given subsheaf $\mathcal{J} \subset \Omega^1_M$, we will say that $\mathcal{J}$ has rank q if it is locally generated by q 1-forms independent at every point. In others words, for $p \in M$ there exist an open set U containing p, such that $\mathcal{J}|_U \subset \Omega^1_M|_U$ is generated by q 1-forms $\omega_1, \ldots, \omega_q$ with

$$\omega_1(p) \wedge \cdots \wedge \omega_q(p) \neq 0 \qquad \forall p \in U.$$

In addition, we write $\mathcal{I} = \langle \mathcal{J} \rangle$ for the ideal generated by $\mathcal{J}$ in the exterior differential algebra Ω^*_M.

The previous definition is related to the concept of foliations of codimension q, where the expected leaves have such codimension. Note that for $q = 1$, it coincides with the description previously introduced.

Moreover, the requirement for a local 1-form ω to have solutions in codimension one (leaves of the corresponding local foliation) is the Frobenius's integrability condition:

$$\omega \wedge d\omega = 0.$$

In other words, the necessary and sufficient condition for complete integrability of a Pfaff equation is given by the Frobenius theorem (see for example section 3 in the appendix of [6]). In this case, we say that the form ω is integrable. Also, a subsheaf $\mathcal{J}$ of rank one is said to be integrable if $\mathcal{J}$ is locally generated by an integrable non-vanishing 1-form. This concept is equivalent to require $\mathcal{I}$ being closed under the exterior derivative of the algebra Ω^*_M. Similarly, for an ideal $\mathcal{I}$ associated to a higher rank subsheaf, the needed condition to define foliations of higher codimension (complete integrability) is $d\mathcal{I} \subset \mathcal{I}$. These ideals are in general referred as *exterior differential systems.* For more details of these definitions see [2].

Definition 1.3.2. A regular foliation $\mathcal{F}$ of codimension 1 in M is determined by an ideal $\mathcal{I}$ of Ω^*_M generated by an integrable subsheaf $\mathcal{J} \subset \Omega^1_M$ of rank 1.

Remark 1.3.3. In the differentiable case, the topological obstruction for a variety to have regular foliations is associated with the possible vanishing of its Euler characteristic. In other words, for a differentiable manifold M such that $\chi(M) = 0$, there no exist any regular foliation of codimension one (see [50]).

In the holomorphic context, the problem is related to the vanishing of certain Chern's classes of the normal bundle of the foliation (see for instance [1]). Also, as a consequence, it is obtained that the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ has no regular foliations.

Now, we introduce a more general definition of foliation and allow these objects to have *singularities*. Regarding the previous introduction, the difference will be that now the local 1-forms defining the foliation are allowed to have a particular type of zeros. More generally, the local system of q-forms (see definition 1.3.1) is not request to have constant rank.

Let X be a complex manifold, where we are particular interested in complex algebraic smooth varieties. Also, fix an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Gamma}$ of X, which depending on our interest can be considered holomorphic or algebraic. Then, a codimension one singular foliation $\mathcal{F}$ on X can be described as follows.

With the same idea as above, we consider a family of local 1-forms $\{\omega_\alpha\}$, where each $\omega_\alpha \in \Omega^1_X(U_\alpha)$ is required to not be identically zero and to satisfy the Frobenius condition:

$$\omega_\alpha \wedge d\omega_\alpha = 0.$$

Moreover, we add the following compatibility requirement:

$$\omega_\alpha = f_{\alpha\beta}\ \omega_\beta \quad \text{in} \quad U_{\alpha\beta} = U_\alpha \cap U_\beta, \tag{1.3.1}$$

for some non-vanishing function $f_{\alpha\beta} \in \mathcal{O}^*_X(U_{\alpha\beta})$. These last equations ensure that the local leaves defined by each of the local forms glue together in the corresponding intersection. In addition, the null points of the forms ω_α and ω_β in $U_{\alpha\beta}$ are exactly the same.

Definition 1.3.4. With the previous notation, we define the singular set of the foliation $\mathcal{F}$ by:

$$S_{\mathcal{F}} = \{p \in X : \omega_\alpha(p) = 0,\ \text{if}\ p \in U_\alpha\}.$$

On the other hand, note that the non-vanishing functions $\{f_{\alpha\beta}\}$ satisfy the cocycle conditions

$$f_{\alpha\beta} \ = \ f_{\alpha\gamma}\ f_{\gamma\beta} \quad \text{in}\ U_{\alpha\beta\gamma},$$

and so they determine an object of the cohomology group $H^1(X, \mathcal{O}_X^*)$. If L denotes the line bundle represented by the previous cocycle, the formula 1.3.1 shows that the family $\{\omega_\alpha\}$ determine a well defined twisted 1-form, i.e. a section

$$\omega \in H^0(X, \Omega_X^1 \otimes L).$$

In order to understand when two of these families define the same foliation, consider another family of integrable 1-forms $\{\omega'_\alpha\}$ with cocycle conditions induced by $\{f'_{\alpha\beta}\} \in H^1(X, \mathcal{O}_X^*)$. We say that $\{\omega_\alpha, f_{\alpha\beta}\}$ and $\{\omega'_{\alpha\beta}, f'_{\alpha\beta}\}$ are equivalent if there exist a family of functions $\{g_\alpha\}$ fulfilling:

$$\omega_\alpha = g_\alpha \omega'_\alpha \quad \forall \alpha,$$

where each g_α is an element of $\mathcal{O}_X^*(U_\alpha)$, In addition, the previous equality implies that:

$$f'_{\alpha\beta} = g_\alpha f_{\alpha\beta} g_\beta^{-1}.$$

So, in this case, the cocycles $\{f_{\alpha\beta}\}$ and $\{f'_{\alpha\beta}\}$ define the same line bundle L. Furthermore, if we assume that $f_{\alpha\beta} = f'_{\alpha\beta}$, the non-vanishing functions $\{g_\alpha\}$ define an element $g \in H^0(X, \mathcal{O}_X^*)$.

From now on, fix a line bundle L of X, or more precisely, its corresponding class in the Picard group $Pic(X)$. In conclusion, a singular foliation $\mathcal{F}$ is determined by a global twisted form

$$\omega \in H^0(X, \Omega_X^1(L)),$$

which satisfies the integrability condition

$$\omega \wedge d\omega = 0 \in H^0(X, \Omega_X^3(L \otimes L)).$$

Moreover, two twisted forms define the same foliation (are equivalent) if they are related to the action of a non-vanishing global function $g \in H^0(X, \mathcal{O}_X^*)$.

Definition 1.3.5. For $L \in Pic(X)$, we define the space of codimension one singular foliations on X associated to L by the set:

$$\mathcal{F}_1(L, X) = \{[\omega] \in H^0(X, \Omega_X^1(L))/_\sim : \omega \wedge d\omega = 0\}.$$

Sometimes the condition $codim S_\omega \geq 2$ is also added. This is equivalent to require the form to be irreducible in the following sense.

Definition 1.3.6. An element $\omega \in H^0(X, \Omega_X^1(L))$ is said to be irreducible if there no exist $\omega' \in H^0(X, \Omega_X^1(L'))$ and $f \in H^0(X, \mathcal{O}_X(L''))$ (where $L = L' \otimes L''$) such that: $\omega = f\,\omega'$.

Now let us state two simple known results associated with these spaces.

Remark 1.3.7. If X is compact, the space $H^0(X, \Omega_X^1(L))$ is a finite dimensional vector space. Moreover, two forms ω_1 and ω_2 are equivalent if there exist a constant $\lambda \in \mathbb{C}^*$ such that:

$$\omega_1 = \lambda\, \omega_2.$$

So, in conclusion, we get: $\mathcal{F}_1(L, X) \subset \mathbb{P}H^0(X, \Omega_X^1(L))$.

Proposition 1.3.8. If $n = \dim_{\mathbb{C}}(X) \geq 3$ the integrability equation is non-trivial, and $\mathcal{F}_1(L, X)$ determines an algebraic subvariety of $\mathbb{P}H^0(X, \Omega^1_X(L))$ (in general with singularities).

Proof. See [26] pp.133. □

For our particular purposes, we are interested in the projective case: $X = \mathbb{P}^n$. In this case, $Pic(X) = \mathbb{Z}$, and every line bundle is isomorphic to one associated to a twisted invertible sheaf of the type $\mathcal{O}_{\mathbb{P}^n}(d)$ for some $d \in \mathbb{Z}$ (see [31, Corollary 6.17]). So, for every integer $d \in \mathbb{N}_{\geq 2}$ we get the following moduli space:

$$\mathcal{F}_1(d, \mathbb{P}^n) = \{[\omega] \in \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) : \omega \wedge d\omega = 0 \quad \text{and } codim(S_\omega) \geq 2\}.$$

Notice that for the other possible integers there are no global sections in the sheaf $\Omega^1_{\mathbb{P}^n}(d)$.

Now, to understand better the previous moduli space, we will describe in more detail the involved equations in homogeneous coordinates. For this purpose, we need to characterize the space $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$. This is possible because of the so called Euler sequence (see [31, Theorem 8.13]):

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}^{n+1}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

After considering the tensor product of the sequence by $\mathcal{O}_{\mathbb{P}^n}(d)$, we can reinterpret the obtained result in homogeneous coordinates and assign to every element $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$, a homogeneous affine form in $\mathbb{C}^{n+1}$ of the type:

$$\omega = \sum_{i=0}^{n} A_i(z)\, dz_i,$$

for some homogeneous polynomials $\{A_i\}$ of degree $d-1$ which satisfy the equation: $\sum_{i=0}^{n} A_i z_i = 0$. Also, this equation can be expressed in terms of the vanishing of the contraction of the form by the radial Euler field $R = \sum z_i \frac{\partial}{\partial z_i}$, i.e. $i_R(\omega) = 0$.

In other words, the previous description can be interpreted by the following. Consider a codimension one singular foliation on $\mathbb{P}^n$, denoted by $\mathcal{F}$, and let π be the natural projection from $\mathbb{C}^{n+1} - 0$. Then, the pullback foliation $\mathcal{F}^* = \pi^*(\mathcal{F})$ can be described by an integrable affine 1-form $\omega = \sum_{i=0}^{n} A_i dz_i$, where its coefficients are homogeneous polynomials of the same degree. Moreover, they must satisfy the equation $\sum A_i z_i = 0$.

In addition, let us give a geometric interpretation of the previous requirements for the affine form. First, according to the Harstog's extension theorem, the form ω can be extended to $\mathbb{C}^{n+1}$. Also, it is clear that not every form in $\mathbb{C}^{n+1}$ determines a foliation which corresponds to the pullback of a foliation on $\mathbb{P}^n$. The exact condition required is that the form (or the foliation) needs to be invariant by the homotheties of $\mathbb{C}^{n+1}$, and this will imply the homogeneity of the involved polynomials and also the Euler descend condition.

Summarily, the moduli space $\mathcal{F}_1(d, \mathbb{P}^n)$ is an algebraic subvariety of

$$\mathbb{P}H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d)) \simeq \left\{ [\omega] = \sum_{i=0}^{n} A_i(z)\, dz_i : A_i \in \mathbb{C}[z_0, \ldots, z_n]_{d-1} \ \forall i \ \text{ and } \sum_{i=0}^{n} A_i(z)\, z_i = 0 \right\}$$

determined by the $\binom{n+1}{3}$ quadratic equations induced by the integrability condition: $\omega \wedge d\omega = 0$. In addition, we have the condition $codimS_\omega \geq 2$, which can be reinterpreted as the requirement for the polynomials $A_0, \ldots, A_n$ to do not have a common factor.

Remark 1.3.9. Note that for $n = 2$, the integrability equation is trivially satisfied because it corresponds to the vanishing of a twisted 3-form in $\mathbb{P}^2$. So $\mathcal{F}_1(d, \mathbb{P}^2)$ is always a Zariski open subset of a linear variety.

A significant problem related to these moduli spaces is to characterize their irreducible components. We present a summary of the cases in which the total number of them is known.

Remark 1.3.10. With the previous notation, the number of irreducible components of $\mathcal{F}_1(d, \mathbb{P}^n)$ is known in the following cases.

- For $d = 2$ there is only one component. It was proved by Jouanolou in [35].
- For $d = 3$ there are two. One is of the type "rational " and the other is "logarithmic". See also [35] or the introduction of [9].
- For $d = 4$ there are six total components, and it is the last case completely understood. Two of them are of the type "rational ", two "logarithmic", one is the pullback of all the foliations of degree 4 on $\mathbb{P}^2$ (see remark 1.3.9) and last one is exceptional. This work due to D. Cerveau and A.L. Neto can be found at [9].

Furthermore, there are some known components in other cases. We highlight the logarithmic ones, which will be treated in detail in the next chapter. For more information, we refer [3], where the author proves that the spaces of logarithmic forms determine irreducible components of the moduli space of foliations on a complex projective manifold X with $\dim_{\mathbb{C}} X \geq 3$, and such that: $H^1(X, \mathbb{C}) = 0$.

1.4 General cases

In this section, we introduce a short digression about the moduli spaces of foliations of higher codimension. We suggest the article [17] for a global perspective of this task. This work contains most of the definition we will propose next.

First, we need to determine which is the right object to describe singular foliations of codimension $q \in \mathbb{N}_{>1}$ on a complex or algebraic variety (see the introduction of [17]). There are two natural possibilities: systems of q integrable 1-forms or q-forms with some desirable properties.

According to the following remark, the systems of 1-forms do not seem to be the right object because of their possible singularities.

Remark 1.4.1. Let X be an n-dimensional variety with $n > 2$, and select $n-1$ integrable 1-forms on X. The points where these forms are linearly dependent (singular set of such system) is generically a codimension two subvariety. So the foliations which correspond to the orbits of vector fields could never appear in association with these type of systems.

Now, motivated by the above reason, we introduce the concept of integrable q-forms which are *locally decomposable off its singular set*. This concept will generalize the definitions given in the first section for the case of 1-forms. Also, we write again M for any complex smooth manifold.

Definition 1.4.2. We say $\omega \in \Omega^q_M$ is locally decomposable off its singular set if for every $x \in M - \{p \in M : \omega(p) = 0\}$ there exist an open neighborhood, and q local 1-forms $\{\omega_i\}_{i=1}^q$ such that:

$$\omega = \omega_1 \wedge \cdots \wedge \omega_q.$$

Definition 1.4.3. A form ω which is locally decomposable off its singular set, is also said to be integrable if every form ω_i which locally decomposes ω is integrable, i.e. with the notation of the above definition we get: $\omega_i \wedge d\omega_i = 0$.

The following statements are useful equivalences for the previously introduced concepts. For more details see the original work [17] or the introduction of [15].

Proposition 1.4.4. For $\omega \in \Omega^q_M$ the following conditions are equivalent:

1. ω is locally decomposable off its singular set.
2. $\omega(x)$ has rank q or 0 at every point $x \in M$.
3. $i_{V_I}(\omega) \wedge \omega = 0$; for every local frame $\{V_1, \dots V_n\}$ and every subset I of size $|I| = q - 1$.

Proposition 1.4.5. For $\omega \in \Omega^q_M$ locally decomposable off the singular set, the following conditions are equivalent:

1. ω is also integrable.
2. $i_{V_I}(\omega) \wedge d\omega = 0$; for every local frame $\{V_1, \dots V_n\}$ and every subset I of size $|I| = q - 1$.
3. $\ker(d\omega(x)) \subset \ker(\omega(x))$; for every $x \in M - \{x \in M : \omega(x) = 0\}$.

Now, the desired definition for singular foliations on a complex algebraic smooth variety X can be stated with a similar setting to that introduced at the end of the previous section.

For a fixed line bundle L on X, a singular codimension q foliation on X with transitions functions determined by the cocycle of L, can be described by a twisted q-form $\omega \in H^0(X, \Omega^q_X(L))$ which satisfies the following conditions:

1. $i_{V_I}(\omega) \wedge \omega = 0$; for every local frame $\{V_1, \dots V_n\}$ and every subset I of size $|I| = q - 1$.
2. $i_{V_I}(\omega) \wedge d\omega = 0$; for every local frame $\{V_1, \dots V_n\}$ and every subset I of size $|I| = q - 1$.
3. $codim(S_\omega = \{x \in X : \omega(x) = 0\}) \geq 2$.

In addition we need to set the same equivalence relation as in the case of foliations of codimension one. So, we will work with projective classes of forms in $\mathbb{P}H^0(X, \Omega^q_X(L))$ which satisfy the previous equations.

Finally, in our case of interest: $X = \mathbb{P}^n$, we can give a more detailed description of the corresponding moduli space of foliations.

Fix an integer $d \in \mathbb{N}_{>q}$, and consider the projective classes of twisted differential q-forms $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ which satisfies 1, 2 and 3 from above. Also, they can be described by homogeneous q-forms on $\mathbb{C}^{n+1}$ of the type:

$$\omega = \sum_{\substack{I \subset \{0,\dots,n\} \\ |I|=q}} A_I(z) dz_{i_1} \wedge \cdots \wedge dz_{i_q},$$

for some homogeneous polynomials $\{A_I\}$ of degree $d-q$, which also satisfy $\binom{n+1}{q-1}$ equations to ensure that: $i_R(\omega) = 0$ (where R denotes the Euler field).

In conclusion, we need to consider homogeneous affine q-forms as above which also satisfies both the Plücker decomposability condition

$$i_V(\omega) \wedge \omega = 0 \quad \text{for every } V \in \bigwedge^{q-1} \mathbb{C}^{n+1} \tag{1.4.1}$$

and the integrability condition

$$i_V(\omega) \wedge d\omega = 0 \quad \text{for every } V \in \bigwedge^{q-1} \mathbb{C}^{n+1}. \tag{1.4.2}$$

Note that the constant vectors V can be replaced by local rational vector fields. After all, it is natural to set the space of codimension q singular foliations of degree d on $\mathbb{P}^n$ as

$$\mathcal{F}_q(d, \mathbb{P}^n) = \{[\omega] \in \mathbb{P}H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d)) : \omega \text{ satisfies 1.4.1, 1.4.2 and } codim(S_\omega) \geq 2\}.$$

To end this introductory chapter, we will refer some known facts about irreducible components of these spaces. In particular, we want to emphasize three articles which are considered relevant in the setting of the present work. However, it is remarkable that for $d > 1$ and $q > 2$ no irreducible components of $\mathcal{F}_q(d, \mathbb{P}^n)$ are known so far.

Remark 1.4.6 (Foliations with split tangent sheaf)**.**
In [14], the authors deduced that the set of singular holomorphic foliations on projective spaces with split tangent sheaf and a right singular set are open in the space of holomorphic foliations.

Also, they exhibit some previously unknown irreducible components of the spaces of singular holomorphic foliations, which are induced by the action of Lie sub-algebras of $\mathrm{Aut}(\mathbb{P}^n) \simeq \mathfrak{sl}(n+1, \mathbb{C})$. As applications, the work also presents a generalization of a result by Camacho-Lins Neto about linear pull-back foliations, and give certain criteria for the rigidity of L-foliations of codimension $q \geq 2$.

Remark 1.4.7 (Rational foliations of codimension q)**.**
As it was announced, there are not many known irreducible components of the space $\mathcal{F}_q(d, \mathbb{P}^n)$ with $q \geq 2$. One of them is presented at [15], where the authors show that the singular holomorphic foliations of higher codimension induced by dominant quasi-homogeneous rational maps fill

out irreducible components of the desired space. Moreover, these foliations can be described by homogeneous forms of the type:

$$\omega = i_R(dF_0 \wedge \cdots \wedge dF_q) \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d)),$$

for some homogeneous polynomials F_i of respective degrees d_i (with $\sum_{i=0}^q d_i = d$). Such form satisfies the equations of $\mathcal{F}_q(d, \mathbb{P}^n)$. and define a foliation tangent to the fibers of the map:

$$\begin{aligned} \mathbb{P}^n &\dashrightarrow \mathbb{P}^q \\ x = (x_0 : \cdots : x_n) &\longmapsto (F_0^{e_0}(x) : \cdots : F_q^{e_q}(x)). \end{aligned}$$

Also, we consider important to keep in mind these components, because our primary goal is to obtain a common generalization between them and the usual logarithmic 1-forms (see [3]). In general, the logarithmic q-forms which define a codimension q foliation will depend on the selection of m polynomials $F_1, \ldots, F_m$. And in the case when $m = q + 1$ our definitions must coincide with the formulas given in this remark,

Remark 1.4.8 (Complete intersection and codimension two holomorphic foliations (local case))**.** The final remark is related to a recent article [10] of D. Cervau and A.L. Neto. In such work, the author study codimension two foliations and distributions, and deduce a local stability result for foliations which are complete intersection. In addition, they present an overview of problems related to the singular locus and a classification of homogeneous foliations of small degree.

It is relevant to take into account this article because the concept of a complete intersection foliation is close related to our definition of logarithmic q-forms (see chapter 4).

Chapter 2

Logarithmic 1-forms: stability and geometry of the components

2.1 Introducción y resumen en español

En este capítulo estudiaremos las principales propiedades de 1-formas logarítmicas en espacios proyectivos, primordialmente en relación a sus posibles conexiones con los espacios de moduli de foliaciones algebraicas proyectivas. El resultado principal es una prueba algebraica de la estabilidad de 1-formas logarítmicas, deduciendo además que las correspondientes componentes son reducidas (sin elementos nilpotentes), y mostrando algunas características de su geometría.

A modo de resumen de los antecedentes del tema, puede verse en primera instancia como en [8] los autores deducen que en el espacio de moduli de foliaciones del espacio afín n-dimensional, las únicas componentes posibles son logarítmicas. De forma resumida, en dicho volumen demuestran que si una 1-forma de Pfaff integrable ω es no degenerada, es decir, que su contracción con el campo radial de Euler $R = \sum z_i \frac{\partial}{\partial z_i}$ es no nula, entonces admite un factor integrante polinomial F, que, por definición, va a satisfacer que:

$$d(\frac{\omega}{F}) = 0.$$

Como puede verse también en [8], esto esencialmente va a implicar que ω esta en la clausura de alguna componente logarítmica del espacio. Un resultado similar a este último descripto va a ser mencionado en este capítulo para formas en el espacio proyectivo (ver proposición 2.4.27). En este caso es posible establecer una relación entre formas logarítmicas proyectivas y las formas de Pfaff algebraicas proyectivas que admiten al menos un factor integrante.

Con respecto a formas logarítmicas que definen foliaciones en variedades proyectivas, Omegar Calvo demuestra en [3] que en una variedad proyectiva con ciertas propiedades, estas constituyen componentes irreducibles del espacio de moduli correspondiente. Para esto, el autor da una caracterización de dichas formas, algunas propiedades de su holonomía y singularidades, y mediante métodos de naturaleza topológica muestra su estabilidad por perturbaciones.

En conexión a este último antecedente, el resultado principal de este capítulo (Teorema 4.4.1) es una demostración algebraica de la estabilidad de 1-formas logarítmicas proyectivas. La técnica básica utilizada consiste en determinar explícitamente las deformaciones de primer orden de una 1-forma logarítmica genérica. Esta nueva demostración implica además que estas componentes logarítmicas son reducidas (sin nilpotentes), lo cual constituye una adición substancial al resultado puramente topológico de [3].

Además de la prueba algebraica de estabilidad, el objetivo general del capítulo es una descripción adecuada de la geometría de estas componentes irreducibles. A modo de resumen, nos proponemos estudiar las características del morfismo racional que las define, como por ejemplo su base locus y su posible inyectividad genérica. Esto permitirá transferir ciertas propiedades del espacio de parámetros a nuestras componentes, como por ejemplo su racionalidad (o uniracionalidad) como variedades.

A lo largo del capítulo, y en la primera instancia, se repasarán las definiciones básicas de formas logarítmicas en el contexto de variedades algebraicas complejas suaves. Además, se mostrarán algunas propiedades elementales como su correcta caracterización sobre espacios proyectivos, su relación con el conocido haz de formas logarítmicas definido por Deligne (ver [18]), y algunas propiedades de interés en el contexto de la Teoría de Foliaciones: singularidades, factores integrantes y hojas algebraicas.

De modo más específico, la parametrización que define a las formas logarítmicas proyectivas, quedará determinada por la siguiente fórmula:

$$\rho : \mathcal{P}_1(\mathbf{d}) := \mathbb{P}(\mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^{m} \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))) \dashrightarrow \mathcal{F}_1(d, \mathbb{P}^n)$$
$$(\lambda = (\lambda_1 : \cdots : \lambda_m), (F_i)_{i=1}^m) \longmapsto \omega = (\prod_{i=1}^{m} F_i) \sum_{j=1}^{m} \lambda_j \frac{dF_j}{F_j},$$

donde $\mathbf{d} = (d_1, \ldots, d_m)$ es una m-tupla de grados fija, tal que $\sum_{i=1}^m d_i = d$. Donde, además, las constantes (λ_j) pertenecen a $\mathbb{P}\mathbb{C}^m_{\mathbf{d}}$, que denota la proyectivización del espacio lineal de constantes en $\mathbb{C}^m$ que cumplen con la ecuación lineal: $\sum_{i=1}^m d_i\lambda_i = 0$.

EL morfismo ρ previamente introducido, es racional, y se corresponde con la proyectivización de la aplicación multilineal

$$\phi : \mathbb{C}^m_{\mathbf{d}} \times \prod_{i=1}^{m} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i)) \longrightarrow H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d)).$$

En relación con estos morfismos, cabe destacar que nos resulta de interés el estudio del lugar de ceros de ϕ, i.e. $\mathcal{K}(\mathbf{d}) = \phi^{-1}(0)$, y su correspondiente proyectivización que determina el *base locus* de la parametrización natural ρ. Esto último es de relevancia para el análisis y descripción de características geométricas de las componentes que quedarán determinadas.

El resultado principal que describe los elementos de $\mathcal{K}(\mathbf{d})$ se encuentra en la proposición 2.4.1. Para probar esto último será de vital importancia la utilización de un lema descripto por Jouanolou en [35], utilizado para su estudio de hojas algebraicas de ecuaciones de Pfaff proyectivas. Básicamente, este resultado establece que si:

$$\sum_{i=1}^{k} \gamma_j \frac{df_j}{f_j} = 0,$$

para ciertas constantes $\gamma_j \in \mathbb{C}$ y polinomios irreducibles distintos $\{f_j\}_{j=1}^k$, entonces necesariamente cada $\gamma_j = 0$ para todo índice j.

En esta misma sección, probaremos que esta parametrización natural ρ es genéricamente inyectiva, lo cual será un elemento clave para deducir propiedades geométricas de las componentes logarítmicas.

A continuación se probará el resultado principal de estabilidad de formas logarítmicas anunciado (Teorema 2.5.10). Para esto, será importante describir explícitamente el diferencial de esta parametrización y dar algunas caracterizaciones de los elementos que están en su imagen.

Vamos a denotar por $\mathcal{L}_1(\mathbf{d}, n)$ a la clausura Zariski de la imagen de la parametrización ρ. Explícitamente se probará que este variedad determina una componente irreducible y genéricamente reducida (sin nilpotentes) del espacio de moduli de f oliaciones algebraicas proyectivas de codimensión uno.

La demostración de este resultado principal será una consecuencia de la proposición 2.5.11, en la cual se prueba que la derivada de la parametrización ρ es genéricamente suryectiva. En función de esto, probaremos que si una forma $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))/(\omega)$ es una perturbación de primer orden de una forma logarítmica

$$\omega = \sum_{i=1}^{m} \lambda_i (\prod_{j \neq i} F_j) dF_i,$$

que probaremos que es equivalente a que satisfaga la ecuación

$$\omega \wedge d\alpha + \alpha \wedge d\omega = 0$$

entonces, dicha forma α se corrresponde con un elemento que está en la imagen del diferencial de la parametrización ρ. Para este fín, será necesario estudiar la ecuación de perturbación anterior bajo la restricción a los distintos estratos de una filtración del lugar singular de ω. En función de este análisis, y si ω está definida por polinomios $\{F_i\}_{i=1}^m$, se necesitarán obtener generadores adecuados de los ideales que definen a los estratos asociados al divisor $(F_1 \cdot \cdots \cdot F_m = 0)$, definidos como:

$$X^k = \bigcup_{I:|I|=k} X_I = (F_{i_1} = \cdots = F_{i_k} = 0) \qquad 1 \leq k \leq m.$$

Es importante observar tambien que X^2 se corresponde con la componente de codimensión dos del lugar singular de ω. A lo largo del trabajo, generadores y resoluciones de estos ideales serán requeridos de maneras más generales, por lo tanto este tipo de cuentas se encuentran explicadas y desarrolladas en un apéndice externo al capítulo.

Es importante destacar que las técnicas e ideas desarrolladas a lo largo de este capítulo sirven también de marco de referencia para su posterior generalización a formas logarítmicas de mayores grados.

2.2 Summary

Throughout this chapter, we study the main properties of logarithmic 1-forms on projective spaces, concerning their connection with the moduli spaces of projective algebraic singular foliations. The main result is an algebraic version of the stability result for 1-logarithmic forms, which also allows us to deduce that the corresponding components are generically reduced (without nilpotent elements). Also, some aspects of the geometry of these components will be described.

2.3 Basic definitions: logarithmic 1-forms

We start recalling the usual definition of logarithmic 1-forms on the general setting of complex manifolds. The following explanations are in a closed relation to the well-known sheaf of logarithmic forms (see for instance [18]).

Let X be a complex manifold, where we are particular interested in smooth projective algebraic varieties. Let $L_1, \dots, L_m$ be a finite number of line bundles on X, and fix a global section $F_i \in H^0(X, \mathcal{O}(L_i))$, for each index $i \in \{1, \dots, m\}$.

Consider $L = L_1 \otimes \cdots \otimes L_m$ the associated product line bundle, and write $\{s^i_{jk}\} \in H^1(\mathcal{U}, \mathcal{O}^*)$ for the Čech-cocycles defining each L_i over an open covering $\mathcal{U} = \{U_j\}_{j\in J}$ of X. With this setting, it is possible to give a definition of logarithmic regular forms (see for instance [3]). In adittion, the following condition will be required.

Definition 2.3.1. An m-tuple $\lambda = (\lambda_1, , \dots, \lambda_m) \in \mathbb{C}^m$ is said to satisfy the L-condition if:

$$\sum_{i=1}^{m} \lambda_i \frac{ds^i_{jk}}{s^i_{jk}} = 0 \qquad \forall j, k$$

In general, it will be convenient to use the following further notation:

$$F = \prod_{i=1}^{m} F_i \in H^0(X, \mathcal{O}(L))$$
$$\hat{F}_j = \prod_{\substack{i=1 \\ (i\neq j)}}^{m} F_i \in H^0(X, \mathcal{O}(L_1 \otimes, \dots, \otimes \hat{L}_j \otimes, \dots, \otimes L_m))$$

Proposition 2.3.2. If $\lambda = (\lambda_1, , \dots, \lambda_m) \in \mathbb{C}^m$ satisfies the L-condition, then the expression

$$\omega = F \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i$$

is a well-defined twisted global form, i.e. $\omega \in H^0(X, \Omega^1_X(L))$. Also it is integrable: $\omega \wedge d\omega = 0$.

Proof. Assume that each global section $F_i \in H^0(X, \mathcal{O}(L_i))$ is defined by a bunch of local regular functions $\{f^j_i = F_i|_{U_j}\}_{j\in J}$ such that: $f^j_i = s^i_{jk} f^k_i$ (on every non-empty intersection $U_{jk} = U_j \cap U_k$).

The following equality for the restrictions of the form ω to the possible intersections holds:

$$(\omega|_{U_j})|_{U_k} = \left(\prod_{i=1}^{m} f_i^j\right)\sum_{i=1}^{m}\lambda_i \frac{df_i^j}{f_i^j} =$$

$$= \left(\prod_{i=1}^{m} s_{jk}^i\right)\left(\prod_{i=1}^{m} f_i^k\right)\sum_{i=1}^{m}\lambda_i \frac{d\left(s_{jk}^i f_i^k\right)}{s_{jk}^i f_i^k} = \left(\prod_{i=1}^{m} s_{jk}^i f_i^k\right)\sum_{i=1}^{m}\lambda_i \frac{ds_{jk}^i}{s_{jk}^i} + \left(\prod_{i=1}^{m} s_{jk}^i\right)\left(\prod_{i=1}^{m} f_i^k\right)\sum_{i=1}^{m}\lambda_i \frac{df_i^k}{f_i^k} =$$

$$= \left(\prod_{i=1}^{m} s_{jk}^i\right)\left(\prod_{i=1}^{m} f_i^k\right)\sum_{i=1}^{m}\lambda_i \frac{df_i^k}{f_i^k} = \left(\prod_{i=1}^{m} s_{jk}^i\right)(\omega|_{U_k})|_{U_j}$$

So these local forms determines a well defined object $\omega \in H^0(X, \Omega_X^1(L))$. Moreover, according to the usual logarithmic derivative formula, we obtain:

$$\frac{d\omega}{F} - \frac{dF}{F} \wedge \frac{\omega}{F} = d(\frac{\omega}{F}) = \sum_{i=1}^{m}\lambda_i d(\frac{dF_i}{F_i}) = 0.$$

Finally, note that the integrability equation can be deduced from $d\omega = \frac{dF}{F} \wedge \omega$.

□

Now, we present the following definition:

Definition 2.3.3. A twisted global form $\omega \in H^0(X, \Omega_X^1(L))$ is said to be logarithmic of type $L_1 \otimes \cdots \otimes L_m$ if there exist sections $F_i \in H^0(X, \mathcal{O}(L_i))$ (for $i = 1 \ldots m$), and an m-tuple of constants $\lambda = (\lambda_1, \ldots, \lambda_m) \in C^m$ satisfying the L-condition, fulfilling:

$$\omega = \sum_{i=1}^{m}\lambda_i \hat{F}_i dF_i.$$

The space of such logarithmic 1-forms is going to be denoted by $l_1(L_1 \otimes \cdots \otimes L_m, X)$.

Remark 2.3.4. The L-condition is equivalent to require:

$$\sum_{i=1}^{m}\lambda_i c(\mathcal{L}_i) = 0,$$

where $c(\mathcal{L}_i)$ denotes the Chern's class associated to the line bundle L_i (see also [3]). This type of descriptions and conditions will be picked up in the following chapter.

Corollary 2.3.5. Every logarithmic form ω of type $L_1 \otimes \cdots \otimes L_m$ determines a foliation on X with the cocycle conditions determined by L (see definition 1.3.5 of the previous chapter). In other words, the following inclusion holds:

$$(l_1(L_1 \otimes \cdots \otimes L_m, X))/_\sim \subset \mathcal{F}_1(L, X).$$

In addition, for X compact, we get:

$$\mathbb{P}(l_1(L_1 \otimes \cdots \otimes L_m, X)) = \{[\omega] \in \mathbb{P}(H^0(X, \Omega_X^1(L))) : \omega \in l_1(L_1 \otimes \cdots \otimes L_m, X)\} \subset \mathcal{F}_1(L, X).$$

Let us complete the picture for this type of forms on projective spaces. We aim to relate these last definitions to the so-called sheaf of logarithmic differential forms.

From now on, we set $X = \mathbb{P}^n$. In this case, the Picard group of equivalent classes of line bundles correspond to $\mathbb{Z}$, via the isomorphism determined by the powers of the hyperplane bundle $\mathbb{H}$. The following constructions are supported on the characterization of the line bundles over $\mathbb{P}^n$ associated to the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$, where d runs over all the integers.

For every positive integer d, the space of foliations $\mathcal{F}_1(\mathbb{H}^d, \mathbb{P}^n)$ corresponds to projective classes of twisted forms $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)))$ which also satisfy the so-called integrability equation:

$$\omega \wedge d\omega = 0.$$

Moreover, the space $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ can be described in homogeneous coordinates by polynomial affine forms $\omega \in H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})$ of the type:

$$\omega = \sum_{i=0}^{n} A_i(z_0, \dots, z_n) dz_i,$$

for some homogeneous polynomials $(A_i)_{i=0}^n$ of degree $d-1$. We will say that this forms are homogeneous of total degree d. Also, it is required to select polynomials which satisfy the condition:

$$\sum_{i=0}^{n} A_i(z_0, \dots, z_n) z_i = 0.$$

This last equation, known as the *Euler equation*, can be reinterpret by:

$$i_R(\omega) = 0,$$

where R denotes the radial Euler field $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$. This condition ensures that the pullback of the twisted porjective form to every line in $\mathbb{C}^{n+1}$ (passing through 0) vanishes.

In conclusion, the desired moduli space can be characterized by:

$$\mathcal{F}_1(d, \mathbb{P}^n) = \{\omega \in \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) : \omega \wedge d\omega = 0\,,\ \operatorname{codim}(S_\omega) \geq 2\}.$$

Geometrically, the number $d-2$ refers to the number of tangencies of the foliation's leaves with a generic line in $\mathbb{P}^n$.

On the other hand, the logarithmic projective forms of type $\mathbb{H}^d = \mathbb{H}^{d_1} \otimes \cdots \otimes \mathbb{H}^{d_m}$, now referred as of type $\mathbf{d} = (d_1, \dots, d_m) \in \mathbb{N}^m$, correspond to the forms described by:

$$\omega = \left(\prod_{i=1}^{m} F_i\right) \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = \sum_{i=1}^{m} \lambda_i \left(\prod_{j \neq i} F_j\right) dF_i,$$

for some homogeneous polynomials $(F_i)_{i=1}^m$ of respective degrees $(d_i)_{i=1}^m$ and $\lambda = (\lambda_i)_{i=1}^m \in \mathbb{C}^m$. The vector λ should also satisfy the condition ($\mathbb{H}^d$ condition):

$$\sum_{i=1}^{m} \lambda_i d_i = 0.$$

According to these definitions, let us establish some useful notations.

Remark 2.3.6. From now on, the space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e))$ of homogeneous polynomials of degree e will be denoted by S_e. Also, for simplicity we will use the following further notation:

$$F = \prod_{i=1}^{m} F_i \in S_d \quad , \quad \hat{F}_i = \prod_{j \neq i} F_j \in S_{d-d_i}.$$

More generally, for each multi-index $I \subset \{1, \ldots, m\}$, we set:

$$\hat{F}_I = \prod_{j \notin I} F_j \in S_{d-\sum_{i \in I} d_i}.$$

With this notation, the logarithmic projective 1-forms of type $\mathbf{d}$ can be described by:

$$\omega = \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i, \tag{2.3.1}$$

for a vector λ satisfying $\lambda \cdot \mathbf{d} = 0$, and $F_i \in S_{d_i}$, for each index $i \in \{1, \ldots, m\}$. The space of such projective 1-forms in $\mathbb{P}^n$ will be denoted by $l_1(\mathbf{d}, n)$.

Proposition 2.3.7. For every logarithmic form $\omega \in l_1(\mathbf{d}, n)$ described by the formula 2.3.1, the following properties hold:

- ω is homogeneous of total degree d and $i_R(\omega) = 0$, i.e. $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$.
- $\omega \wedge d\omega = 0$.

In conclusion, the projectivization of the corresponding logarithmic algebraic space satisfies the following inclusion:

$$\mathbb{P}l_1(\mathbf{d}, n) \subset \mathcal{F}_1(d, \mathbb{P}^n).$$

Proof. This result is a particular case of proposition 2.3.2. It is clear that the formula 2.3.1 determines a polynomial affine form, which is also homogeneous of total degree d. Moreover, we can also check that ω belongs to $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ by performing the Euler equation:

$$i_R(\omega) = \omega = \sum_{i=1}^{m} \lambda_i \hat{F}_i i_R(dF_i) = F(\sum_{i=1}^{m} \lambda_i d_i) = 0.$$

□

We end this section describing the connection between the logarithmic forms of type $\mathbf{d}$ with the well-known sheaf of logarithmic forms over $\mathbb{P}^n$. The following definitions are briefly explained only for our prompt purposes. For more details, see the next chapter 3, or the references [18] and [48].

Consider an effective divisor $\mathcal{D} = \sum_{i=1}^{m} \mathcal{D}_j$ over $\mathbb{P}^n$, which is assumed to be simple normal crossing, i.e. a divisor with non singular irreducible components $\mathcal{D}_1, \ldots, \mathcal{D}_m$ intersecting each other transversely (see for instance p.449 of [28] or [31]). We write

$$i : U = \mathbb{P}^n - \mathcal{D} \hookrightarrow \mathbb{P}^n,$$

and consider the sheaf $\Omega^q_{\mathbb{P}^n}(*\mathcal{D}) = \lim_k \Omega^q_{\mathbb{P}^n}(k\mathcal{D})$ of meromorphic q-forms with poles only over $\mathcal{D}$. Also, it can be seen that this last sheaf coincides with $i_*(\Omega^q_U)$.

Definition 2.3.8. We say that a local section ω of $\Omega^q_{\mathbb{P}^n}(*\mathcal{D})$ has *logarithmic poles* if ω and $d\omega$ have at most simple poles along $\mathcal{D}$. The space of these forms constitutes a subsheaf

$$\Omega^q_{\mathbb{P}^n}(log(\mathcal{D})) \subset \Omega^q_{\mathbb{P}^n}(*\mathcal{D}),$$

also known as the sheaf of logarithmic q-forms along $\mathcal{D}$ over $\mathbb{P}^n$.

Now, we define the residue associated with a logarithmic one form, which is sometimes called as the Poincaré residue.

Fix an index $j \in \{1, \dots, m\}$ and take any local section ω of $\Omega^1_{\mathbb{P}^n}(log(\mathcal{D}))$. If we consider a regular function f_j which locally defines $\mathcal{D}_j$, then the form ω can be decomposed by:

$$\omega = g_j \frac{df_j}{f_j} + \mu,$$

where μ is not divided by $\frac{df_j}{f_j}$. The local function $g_j|_{\mathcal{D}_j}$ is well defined as a section of $\mathcal{O}_{\mathcal{D}_j}$ and it does not depend on the selection of f_j.

Definition 2.3.9. We define the residue of the form ω over $\mathcal{D}_j$ according to the following function:

$$\Omega^1_{\mathbb{P}^n}(log(\mathcal{D})) \xrightarrow{res_j} (i_j)_*(\mathcal{O}_{\mathcal{D}_j})$$
$$\omega \mapsto g_j|_{\mathcal{D}_j}.$$

Now, we use the notation of proposition 2.3.7 and remark 2.3.6, and we assume that the homogeneous polynomials $\{F_i\}_{i=1}^m$ are irreducible and distinct, and also that the divisor

$$\mathcal{D}_F = (F = 0) = \bigcup_{j=1}^m (F_j = 0),$$

has simple normal crossings. Using this conditions, the following exact sequence of sheaves holds:

$$0 \to \Omega^1_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}(log\mathcal{D}) \xrightarrow{\oplus res_j} \bigoplus_{j=1}^m (i_j)_*(\mathcal{O}_{\mathcal{D}_j}) \to 0. \tag{2.3.2}$$

See [22] at p.13 for the corresponding proof and [48] at p.93 for a more general version.

Proposition 2.3.10. With the notations above, every element $\eta \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(log(\mathcal{D}_F)))$ can be described in homogeneous coordinates by:

$$\eta = \sum_{i=1}^m \lambda_i \frac{dF_i}{F_i},$$

for some vector $\lambda \in \mathbb{C}^m_{\mathbf{d}} = \{\lambda \in \mathbb{C}^m : \lambda \cdot \mathbf{d} = 0\}$. Moreover, the correspondence $\lambda \mapsto \eta$ is bijective.

Proof. Consider the long exact sequence in cohomology associated to the sequence 2.3.2:

$$0 \to 0 \to H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(log\mathcal{D})) \to H^0(\mathbb{P}^n, \bigoplus_{j=1}^m (i_j)_*(\mathcal{O}_{\mathcal{D}_j})) \simeq \mathbb{C}^m \xrightarrow{\delta} H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \simeq \mathbb{C} \to \dots$$

Note that if a global logarithmic form ω has the vector $\{\lambda_i\}_{i=1}^m$ as its corresponding residues, then $\omega - \sum_{j=1}^m \lambda_j \frac{dF_j}{F_j}$ has all its residues equal to zero. So this form vanishes because it corresponds to a global section of the sheaf of regular differential 1-forms over $\mathbb{P}^n$. Finally, the condition $\lambda \cdot \mathbf{d} = 0$ is deduced by making explicit the kernel of the connection morphism δ. □

2.4 The natural parametrization

In agreement with the definitions of the previous section, we can determine a multilinear map which parameterizes the set $l_1(\mathbf{d}, n)$:

$$\phi : \mathbb{C}^m_{\mathbf{d}} \times \prod_{i=1}^{m} S_{d_i} \longrightarrow H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$$
$$(\lambda_i)_{i=1}^m, (F_i)_{i=1}^m \longmapsto \omega = \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i.$$

So, by definition $l_1(\mathbf{d}, n)$ corresponds to the image of ϕ.

The map ρ determined by the projectivization of ϕ has image on the moduli space $\mathcal{F}_1(d, \mathbb{P}^n)$, and is considered as our natural parametrization. Also, from now on, we denote by $\mathcal{P}_1(\mathbf{d})$ the natural projectivization of the domain of ϕ. Explicitly, we get:

$$\rho : \mathcal{P}_1(\mathbf{d}) = \mathbb{P}(\mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^{m} \mathbb{P}S_{d_i} \dashrightarrow \mathcal{F}_1(d, \mathbb{P}^n),$$

which is only a rational map, since it is not well defined over the point where ϕ vanishes.

This rational parametrization allows us to define the announced logarithmic varieties which determine irreducibles components of $\mathcal{F}_1(d, \mathbb{P}^n)$.

Definition 2.4.1. We denote by:

$$\mathcal{L}_1(\mathbf{d}, n) = \overline{\operatorname{im} \rho} \subset \mathcal{F}_1(\mathbf{d}, \mathbb{P}^n),$$

the Zariski closure of the image of the rational parametrization ρ (on any appropriate open set where ρ it is well defined). Moreover, this space coincides with the expected definition of the projectivization: $\mathbb{P}l_1(d, n)$.

For future developments, we need to set certain generic conditions over the space of parameters.

Definition 2.4.2. We say that a parameter $(\lambda, (F_i)_{i=1}^m) \in \mathcal{P}_1(\mathbf{d})$ is generic if the following conditions are verified:

1. $\lambda_i + (r-1)\lambda_j \neq 0 \quad \forall i \neq j \in \{1 \ldots m\}, \ \forall r \in \mathbb{N}_0.$

2. Each F_i is an irreducible polynomial.

3. For each multi-index $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ and every point $x \in \mathbb{P}^n$ such that $F_{i_1}(x) = \cdots = F_{i_k}(x) = 0$, the following holds

$$d_x F_{i_1} \wedge \cdots \wedge d_x F_{i_k} \neq 0.$$

   We denote by $\mathcal{U}_1(\mathbf{d})$ the space of generic parameters, i.e. the elements of $\mathcal{P}_1(\mathbf{d})$ which satisfy the previous conditions.

Remark 2.4.3. The above conditions ensure that $codim(S_\omega) \geq 2$ (see for instance [16]). So, technically, we need to restrict ρ to the set of generic parameters to guarantee that this parametrization has the image in $\mathcal{F}_1(d, \mathbb{P}^n)$.

Observe that the conditions 2 and 3 from above are equivalent to say that the divisor $\mathcal{D}_F = (F = \prod_{i=1}^m F_i = 0)$ is simple normal crossing. Also, we assume that for every selection of k polynomials $F_{i_1}, \ldots, F_{i_k}$, its common zeros define a smooth complete intersection projective subvariety.

In addition, these assumptions determine a Zariski open subset of $\mathcal{P}_1(\mathbf{d})$. To show this fact, for each $k \leq n$ and every multi-index $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$, we consider the space:

$$NC^c(I, n) = \{((F_{i_j})_{j=1}^k, x) \in \prod_{j=1}^k \mathbb{P}S_{d_{i_j}} \times \mathbb{P}^n : F_{i_1}(x) = \cdots = F_{i_k}(x) = 0 \;\; , \;\; d_xF_{i_1} \wedge \cdots \wedge d_xF_{i_k} = 0\}.$$

Proposition 2.4.4. The space $NC^c(I, n)$ is a projective irreducible subvariety of $\prod_{j=1}^k \mathbb{P}S_{d_{i_j}} \times \mathbb{P}^n$ of dimension:

$$\dim(NC^c(I, n)) = \dim(\prod_{j=1}^k \mathbb{P}S_{d_{i_j}}) - 1,$$

and so the projection map $\pi_1 : NC^c(I, n) \to \prod_{j=1}^k \mathbb{P}S_{d_{i_j}}$ could not be dominant.

Proof. First, we can assume that $I = \{1, \ldots, k\}$. Next consider $NC^c(I, n)$ as an incident variety via the projection morphisms:

$$\begin{array}{ccccc} & & NC^c(I,n) & & \\ & \swarrow^{\pi_1|_{NC^c}} & & \searrow^{\pi_2|_{NC^c}} & \\ \prod_{i=1}^k \mathbb{P}S_{d_i} & & & & \mathbb{P}^n \end{array}$$

Now observe that all the fibers of π_2 are isomorphic and has codimension equals to:

$$codim(\pi_2^{-1}(x)) = k + (n - k + 1) = n + 1,$$

considered as subvarieties of the space of polynomials $\prod_{i=1}^k \mathbb{P}S_{d_i}$.

To justify this last claim and without loss of generality, we can assume that the point x corresponds to $(1 : 0 : \cdots : 0)$. So the k selected polynomials can be regarded as general polynomials over $\mathbb{C}^n$ passing through 0. Also, the $k \times n$ Jacobian matrix of $(F_1, \ldots, F_k)$ at 0 can not have maximal rank. To count the number of conditions imposed, observe that there are k conditions over the independent terms of the polynomials for having a root at 0, and $(n - k + 1)$ conditions over the coefficient of the linear terms to ensure that the matrix $J(F_1, \ldots, F_k)(0) \in \mathbb{C}^{k \times n}$ has rank less or equal than $k - 1$. Recall that the space of matrices in $\mathbb{C}^{k \times n}$ which has rank less or equal than some $r \leq k$ has codimension $(n - r)(k - r)$. In conclusion, since the conditions are independent because they are considered over distinct coefficients of the set of polynomials, we have proved that each fiber $\pi_2^{-1}(x)$ has the announced codimension.

The morphism $\pi_2 : NC^c(I, n) \to \mathbb{P}^n$ is clearly dominant and has all isomorphic fibers, and so the space $NC^c(I, n)$ is an irreducible projective variety with pure dimension equals to:

$$\dim(NC^c(I, n)) = \dim(\pi_2^{-1}(x)) + \dim(\mathbb{P}^n) = D - (n + 1) + n = D - 1,$$

where D denotes the dimension of $\prod_{i=1}^{k} \mathbb{P}S_{d_i}$. For this reason, the morphism $\pi_1 : NC^c(I,n) \to \prod_{i=1}^{k} \mathbb{P}S_{d_i}$ could not be dominant. □

For every m-tuple of irreducible polynomials $(F_i)_{i=1}^m$ which does not satisfy the condition 3 from 2.4.2, there exists an integer $k \leq min(n,m)$ and a choice of indexes $i_1, \ldots, i_k$ such that the fiber of $(F_{i_1}, \ldots, F_{i_k})$ by π_1 is not empty. In consequence we obtain:

Corollary 2.4.5. The space $\mathcal{U}_1(\mathbf{d})$ of generic parameters is a non-empty open Zariski subset of $\mathcal{P}_1(\mathbf{d})$.

Proof. It is clear that the conditions imposed in 1) determine a non-empty subset of the projective space $\mathbb{P}(\mathbb{C}^m_{\mathbf{d}})$.

On the other hand, it is well known that the assumption of each F_i being irreducible corresponds to an open Zariski subset of $\prod_{i=1}^{m} \mathbb{P}S_{d_i}$. Also, note that the last condition (3) corresponds to the complement of

$$\bigcup_{\substack{I:|I|=k \\ 1\leq k\leq min\{m,n\}}} \overline{\pi_1(NC^c(I,n))} \times \prod_{j\notin I} \mathbb{P}S_{d_j}$$

in the corresponding product space of homogeneous polynomials. Finally, each component $\overline{\pi_1(NC^c(I,n))}$ is an hypersurface of $\prod_{i=1}^{k} \mathbb{P}S_{d_{i_k}}$, and so its complement is an open algebraic subset which is trivially non-empty. □

We will say that a logarithmic one form $[\omega] \in \mathcal{L}_1(\mathbf{d}, n)$ is generic if it can be written by $\omega = \rho(\lambda, (F_i))$, for certain generic parameter $p = (\lambda, (F_i))$. Also, using the proposition 2.3.10, we can state the following result:

Proposition 2.4.6. The rational map ρ is well defined on $\mathcal{U}_1(\mathbf{d})$. In other words the open set $\mathcal{U}_1(\mathbf{d})$ does not intersect the base locus of ρ.

Proof. Consider parameters $(\lambda, \underline{F}) \in \mathbb{C}^m_{\mathbf{d}} \times \prod_{i=1}^{m} S_{d_i}$ in the kernel of the multilinear map ϕ, i.e.

$$\sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i = 0.$$

If we assume that $([\lambda], [\underline{F}]) \in \mathcal{U}_1(\mathbf{d})$, then the divisor $\mathcal{D}_F$ turns out to be simple normal crossing (conditions 3 and 4 from 2.4.2). Finally, we divide the above equality by F, and apply the proposition 2.3.10 to deduce $\lambda = 0$. Although this vector is not allowed in our projective space of parameters $\mathcal{P}_1(\mathbf{d})$, and so $\mathcal{U}_1(\mathbf{d})$ does not intersect the base locus of ρ. □

In the next subsection, we will state a more precise result to understand the points which belong to the base locus of our natural parametrization.

Finally, we introduce another generic condition which is usually considered for this type of forms. It is connected with the possible number of algebraic leaves that a logarithmic form might have. Although, this new complex open condition is not algebraic.

Definition 2.4.7. We take a complex open subset of $\mathcal{U}_1(\mathbf{d})$, denoted by $\mathcal{V}_1(\mathbf{d})$, by adding the following generic assumption to the previous conditions:

4. Not all the quotients λ_i/λ_j are rational, i.e. $\lambda \notin \mathbb{PQ}^m$.

Next, we describe some useful known facts connected to this new condition. Bur first, we need to remember the definition of an algebraic solution.

Definition 2.4.8. An element $f \in S_e$ is said to be an algebraic solution of a projective Pfaff equation $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ if f divides the homogeneous 2-form $\omega \wedge df \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$. Moreover, when ω is integrable we refer to f as an algebraic leaf of the foliation induced.

In addition, we recall the concept of rational first integral.

Definition 2.4.9. A rational global function $H = F/G$ (for some homogeneous polynomials of the same degree $F, G \in S_e$) is said to be a rational first integral of $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ if it satisfies:

$$dH \wedge \omega = 0.$$

The following known results are concerned with the problem of when a homogeneous projective form (or more particularly a logarithmic form) has an infinite number of algebraic solutions.

Proposition 2.4.10. If we fix a form $\omega \in H^0(P^n, \Omega^1_{\mathbb{P}^n}(d))$ (not necessarily integrable), then it has an infinite number of algebraic solutions if and only if it admits a rational first integral.

Proof. See [35, Theorem 3.3]. □

Proposition 2.4.11. Let $\omega = \phi(\lambda, (F_i))$ be a generic logarithmic form of type $\mathbf{d}$, then ω has a finite number of algebraic solutions if and only if $\lambda \notin \mathbb{P}(\mathbb{Q}^m)$.

Proof. See [35, Proposition 3.7.8]. □

2.4.1 Base locus

Our purpose is to make a description of the points where the rational parametrization ρ is not well defined. Let $\mathcal{K}(\mathbf{d})$ be the kernel of the multilinear morphism ϕ, which is clearly an affine algebraic variety. We attend to describe its elements and moreover its irreducible components.

If we denote by π the natural projection from the domain of ϕ to its projectivization $\mathcal{P}_1(\mathbf{d})$, then the base locus of ρ is described by:

$$B(\mathbf{d}) = \{\pi(\lambda, (F_i)) \in \mathcal{P}_1(\mathbf{d}) : (\lambda, (F_i)) \in \mathcal{K}(\mathbf{d}),\ \lambda \neq 0 \text{ and } F_i \neq 0\ \ \forall i\}.$$

This space is well defined because the vanishing condition for the multi-linear map ϕ does not depend of the representative of the projective class.

Note that the elements of the type $(0, (F_i))$ and $(\lambda, (F_1, \ldots, F_{i-1}, 0, F_{i+1}, \ldots, F_m))$ belong to the space $\mathcal{K}(\mathbf{d})$, but they are not admitted in $B(\mathbf{d})$. So, we want to determine the non trivial elements $(\lambda, (F_i))$ fulfilling:

$$\sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i = 0.$$

The characterization of the non-trivial components of $\mathcal{K}(\mathbf{d})$ will be supported by the following algebraic result which can be found in Jouanolou's book [35].

Lemma 2.4.12 (Jouanolou lemma)**.** Let $F_1, \dots, F_m$ be irreducible distinct polynomials in $\mathbb{C}[z_0, \dots, z_n]$, and $\lambda \in \mathbb{C}^m$. If we suppose given the relation

$$\sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = 0,$$

then necessarily $\lambda_i = 0$ for all $i = 1 \dots m$.

Proof. See [35, Lemma 3.3.1]. □

Observe that for homogeneous polynomials $F_1, \dots, F_m$ the condition of being irreducible and distinct is less restrictive than the requirement of $\mathcal{D}_F$ being simple normal crossing. This means that the last result weakens the hypothesis given at 2.3.10 for the injectivity of the assignment $\lambda \mapsto \frac{\omega}{F}$.

Now, pick a general element $(\lambda, (F_i)) \in \mathcal{K}(\mathbf{d})$ and suppose that $F_i \neq 0$ for all $i = 1, \dots, m$. Select irreducible polynomials $G_1, \dots, G_r$ decomposing simultaneously the homogeneous polynomials $(F_i)_{i=1}^m$, i.e.

$$F_i = \prod_{j=1}^{m'} G_j{}^{e_{ij}} \qquad \forall i = 1, \dots, m.$$

We write $[\mathbf{e}] = (e_{ij}) \in \mathbb{N}_0^{m \times m'}$ to denote the matrix of the previously introduced powers. Moreover, it is also clear that these new irreducible polynomials can be chosen homogeneous, and so $G_j \in S_{d'_j}$ for $j = 1 \dots m'$ and new degrees $d'_1, \dots, d'_{m'}$. Using matrix notation, the relation between the involved degrees can be expressed by $[\mathbf{e}].\mathbf{d}' = \mathbf{d}$. Next, if we divide the equality

$$\sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i = 0$$

by $F = \prod_{i=1}^m F_i$, and replace the decomposition of each F_i, we get:

$$\sum_{i=1}^{m} \sum_{j=1}^{m'} \lambda_i \frac{d(G_j^{e_{ij}})}{G_j^{e_{ij}}} = \sum_{j=1}^{m'} \left(\sum_{i=1}^{m} \lambda_i e_{ij} \right) \frac{dG_j}{G_j} = 0. \tag{2.4.1}$$

From the previous lemma 2.4.12 we deduce: $[\mathbf{e}]^t.\lambda = 0$.

On the other hand, for every partition

$$\mathbf{d} = [\mathbf{e}].\mathbf{d}'$$

and every vector $\lambda \in \mathbb{C}^m$ in the complex kernel of the entire matrix $[\mathbf{e}]^t$, using the expression introduced in the first term of the formula 2.4.1, we can construct a family of elements in $\mathcal{K}(\mathbf{d})$. Note that we have not required $\lambda \cdot \mathbf{d} = 0$ because it is a consequence of:

$$\lambda \cdot \mathbf{d} = \lambda \cdot ([\mathbf{e}].\mathbf{d}') = ([\mathbf{e}]^t.\lambda) \cdot \mathbf{d}' = 0 \cdot \mathbf{d}' = 0.$$

All these remarks allow us to construct a morphism

$$\psi_{(\mathbf{d}',[\mathbf{e}])} : \ker([\mathbf{e}]^t) \times \prod_{j=1}^{m'} S_{d'_i} \longrightarrow \mathcal{K}(\mathbf{d}) \subset \mathbb{C}^m_{\mathbf{d}} \times \prod_{i=1}^{m} S_{d_i}$$

whose second factor corresponds to a Segre-Veronese map:

$$\nu_{(\mathbf{d},\mathbf{d}',[\mathbf{e}])} : \prod_{j=1}^{m'} S_{d'_j} \to \prod_{i=1}^{m} S_{d_i}.$$

We write $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$ for the image of $\psi_{(\mathbf{d}',[\mathbf{e}])}$ and $\Delta(\mathbf{d})$ for the set of all partitions of $\mathbf{d}$ by a matrix $[\mathbf{e}] \in \mathbb{N}_0^{m\times m'}$ and a new vector of degrees $\mathbf{d}'$. Also, the previous argument allows us to deduce the following result.

Proposition 2.4.13. With the previous notation, $\mathcal{K}(\mathbf{d})$ can be described by the union:

$$\mathcal{K}(\mathbf{d}) = \bigcup_{(\mathbf{d}',[\mathbf{e}])\in\Delta(\mathbf{d})} \mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} \cup \left\{(\lambda, \underline{F}) \in \mathcal{K}(\mathbf{d}) : \prod_{i=1}^{m} F_i = 0\right\}.$$

We recall that an element of $\mathcal{K}(\mathbf{d})$ is said to be trivial if it corresponds to some $(\lambda, (F_i)) \in \mathcal{K}(\mathbf{d})$ where the constant parameter λ vanishes or some of the polynomials $\{F_i\}$ is equal to zero. We write

$$T(\mathbf{d}) = \{(\lambda, \underline{F}) \in \mathcal{K}(\mathbf{d}) : \lambda = 0 \text{ or } \prod_{i=1}^{m} F_i = 0\}.$$

Remark 2.4.14. Note that: $\mathcal{K}(\mathbf{d})_{(\mathbf{d},Id)} = \{0\} \times \prod_{i=1}^{m} S_{d_i} \subset T(\mathbf{d})$.

Now, observe that each variety $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$ is clearly irreducible. So we need to determine when there are inclusions among these spaces to characterize the irreducible components of $\mathcal{K}(\mathbf{d})$. For this purpose, we perform some remarks.

Remark 2.4.15. Consider an element $(\mathbf{d}', [\mathbf{e}]) \in \Delta(\mathbf{d})$ such that the matrix $[\mathbf{e}]$ has a column of zeros. Also, denote by $(\tilde{\mathbf{d}}', [\tilde{\mathbf{e}}]) \in \Delta(\mathbf{d})$ the element which corresponds to removing the entire null column of $[\mathbf{e}]$ and the corresponding degree of $\mathbf{d}'$. The following equality holds:

$$\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} = \mathcal{K}(\mathbf{d})_{(\tilde{\mathbf{d}}',[\tilde{\mathbf{e}}])}.$$

Remark 2.4.16. For every $(\mathbf{d}' = (d'_1, \dots, d'_{m'}), [\mathbf{e}]) \in \Delta(\mathbf{d})$, the action of the group $\mathbb{S}_{m'}$ which permutes at the same time the degrees and the respective columns of $[\mathbf{e}]$ gives rise to another element in $\Delta(\mathbf{d})$. Although, the associated irreducible varieties in $\mathcal{K}(\mathbf{d})$ are trivially the same.

We will assume that the vector of degrees $\mathbf{d}'$ is sorted ascending, and if two or more degrees are the same, we will also pick some fixed order in the corresponding columns of $[\mathbf{e}]$.

We define:

$$\tilde{\Delta}(\mathbf{d}) = \{(\mathbf{d}', [\mathbf{e}]) \in \Delta(\mathbf{d}) : d'_1 \leq \cdots \leq d'_{m'},\ \ \ker([\mathbf{e}]^t) \neq 0 \ \text{ and } \ [e] \text{ does not have null columns}\}.$$

With a slight abuse of notation, we take into consideration the description made at the remark 2.4.16 to select in $\tilde{\Delta}(\mathbf{d})$ only one of the possible permutations for every element with repeated degrees. In other words we also use $\tilde{\Delta}(\mathbf{d})$ to consider the quotient by the action of the symmetric group $\mathbb{S}_m$.

Proposition 2.4.17. Consider an element $(\lambda, (F_i)) \in \mathcal{K}(\mathbf{d})$ such that $\prod_{i=1}^{m} F_i \neq 0$. Unless a possible permutation (see the remark 2.4.16 above), there exist a unique partition $(\mathbf{d}', [e]) \in \tilde{\Delta}(\mathbf{d})$ with $(\lambda, (F_i)) \in \mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$ and such that the polynomials $G_1, \ldots, G_{m'}$ decomposing the original polynomials are all irreducible and distinct. We express this induced partition by a function:

$$\chi : \mathcal{K}(\mathbf{d}) \to \tilde{\Delta}(\mathbf{d})$$
$$(\lambda, \underline{F}) \mapsto (\mathbf{d}', [\mathbf{e}]) \ \text{(with the hypothesis explained above).}$$

With this notation, we get: $(\lambda, \underline{F}) \in \mathcal{K}(\mathbf{d})_{\chi(\lambda,\underline{F})}$.

Proof. For every $(\lambda, \underline{F}) \in \mathcal{K}(\mathbf{d})$ we can select irreducible polynomials $\{G_j\}_{j=1}^{m'}$ decomposing simultaneously the polynomials $\{F_i\}_{i=1}^{m}$. This selection gives rise to an element $(\mathbf{d}', [\mathbf{e}]) \in \tilde{\Delta}(\mathbf{d})$ such that the selected parameters belong to $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$. Notice that the matrix $[e]$ has not null columns because each selected irreducible polynomial G_j appears with a positive power in at least one of the decompositions of the original polynomials.

From the other hand, suppose that $(\lambda, \underline{F})$ is an element in $(\mathcal{K}(\mathbf{d})_{([\mathbf{d}',[\mathbf{e}])} \cap \mathcal{K}(\mathbf{d})_{(\mathbf{d}'',[\mathbf{h}])})$ for two partitions in $\tilde{\Delta}(\mathbf{d})$ and such that:

$$F_i = \prod_{j=1}^{m'} G_j^{e_{ij}} = \prod_{k=1}^{m''} H_k^{h_{ik}},$$

for two families $\{G_j\}$ and $\{H_k\}$ of irreducible and distinct polynomials. If for every $j \in \{1, \ldots, m'\}$ there exist an index $k = \tau(j)$ such that $G_j = H_{\tau(j)}$ and $e_{ij} = h_{i\tau(j)}$, then the two partitions $(\mathbf{d}', [\mathbf{e}])$ and $(\mathbf{d}', [\mathbf{h}])$ define the same element in $\tilde{\Delta}(\mathbf{d})$ (see remark 2.4.16). Otherwise, $e_{ij} = 0$ for all $i = 1, \ldots, m$, which is an absurd because the matrix $[e]$ has not null columns. □

Corollary 2.4.18. Suppose given $(\lambda, \underline{F}) \in \mathcal{K}(\mathbf{d})$ such that the entire matrix associated to $\chi(\lambda, \underline{F})$ has trivial kernel, we necessarily get $\lambda_i = 0 \ \forall i$.

This last result extends Jouanolou's lemma 2.4.12, which correspond to the case of $[\mathbf{e}] = Id$ and $F_i = G_i$ for all $i = 1 \ldots m$.

Finally, we introduce the concept of sub-partition.

Definition 2.4.19. We say that an element $(\mathbf{d}' = (d'_1, \ldots, d'_{m'}), [\mathbf{e}]) \in \tilde{\Delta}(\mathbf{d})$ is a sub-partition of $(\mathbf{d}'' = (d''_1, \ldots, d''_{m''}), [\mathbf{h}]) \in \tilde{\Delta}(\mathbf{d})$ if there exist an entire matrix $[\mathbf{a}] \in \mathbb{N}_0^{m'' \times m'}$ such that:

- $[\mathbf{a}] \cdot \mathbf{d}' = \mathbf{d}''$
- $[\mathbf{e}] = [\mathbf{h}] \cdot [\mathbf{a}]$

and also $\ker([\mathbf{e}]^t) = \ker([\mathbf{h}]^t)$. We use the notation $(\mathbf{d}', [\mathbf{e}]) \precsim (\mathbf{d}'', [\mathbf{h}])$ to express this relation.

Remark 2.4.20. If $(\mathbf{d}', [\mathbf{e}]) \precsim (\mathbf{d}'', [\mathbf{h}])$, then we have the inclusion:

$$\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} \subset \mathcal{K}(\mathbf{d})_{(\mathbf{d}'',[\mathbf{h}])}.$$

Proposition 2.4.21. The irreducible components of the space $\mathcal{K}(\mathbf{d})$ are the subvarieties $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$ associated to all the elements $(\mathbf{d}',[\mathbf{e}]) \in \tilde{\Delta}(\mathbf{d})$ which are maximal with respect to the order $\lesssim$, and the trivial components given by:

$$\mathbb{C}^m_{\mathbf{d}} \times S_{d_1} \times \cdots \times S_{d_{i-1}} \times \{0\} \times S_{\mathbf{d}_{i+1}} \times \cdots \times S_{d_m}.$$

Proof. For every degree d we write U_d for the open subset of S_d which consist of irreducible polynomials. Next we consider the space

$$U(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} = \left\{ (\lambda, \underline{F}) \in \mathcal{K}(\mathbf{d}) : \prod_{i=1}^{m} F_i \neq 0 \text{ and } \chi(\lambda, \underline{F}) = (\mathbf{d}', [\mathbf{e}]) \right\},$$

which corresponds to the image of the map $\psi_{(\mathbf{d}',[\mathbf{e}])}$ restricted to $\ker([\mathbf{e}]^t) \times \prod_{i=1}^{m'} U_{d'_i}$. Also note that $U(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$ is an open dense subset of the irreducible variety $\mathcal{K}(\mathbf{d})_{(\mathbf{d},[\mathbf{e}])}$. In addition, if $(\mathbf{d}',[\mathbf{e}])$ and $(\mathbf{d}'',[\mathbf{h}])$ are two different elements in $\tilde{\Delta}(\mathbf{d})$, then as a consequence of proposition 2.4.17 we get:

$$U(\mathbf{d})_{([\mathbf{d}',[\mathbf{e}])} \cap U(\mathbf{d})_{(\mathbf{d}'',[\mathbf{h}])} = \emptyset.$$

Now, suppose that $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} \subset \mathcal{K}(\mathbf{d})_{(\mathbf{d}'',[\mathbf{h}])}$. The assumption in particular implies that $\ker([\mathbf{e}]^t) \subset \ker([\mathbf{h}]^t)$. Also, we can pick an element $(\lambda, \underline{F}) \in U(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} \cap \mathcal{K}(\mathbf{d})_{(\mathbf{d}'',[\mathbf{h}])}$. This allows us to select irreducible homogeneous polynomials $G_1, \ldots, G_{m'}$ of respective degrees $d'_1, \ldots, d'_{m'}$, and polynomials $H_1, \ldots, H_{m''}$ of degrees $d''_1, \ldots, d''_{m''}$ such that:

$$F_i = \prod_{j=1}^{m'} G_j^{e_{ij}} = \prod_{k=1}^{m''} H_k^{h_{ik}} \qquad \forall i = 1, \ldots m.$$

If we consider the irreducible factors associated to the polynomials $\{H_k\}$, then we can deduce that they belong to the set $\{G_j\}_{j=1}^{m'}$. In addition, we figure out the existence of an entire matrix $[\mathbf{a}] \in \mathbb{N}_0^{m'' \times m'}$ satisfying $[\mathbf{a}] \cdot \mathbf{d}' = \mathbf{d}''$ and:

$$\prod_{j=1}^{m'} G_j^{e_{ij}} = \prod_{k=1}^{m'} \prod_{j=1}^{m''} G_j^{h_{ik} a_{kj}} = \prod_{j=1}^{m'} G_j^{\sum_{k=1}^{m''} h_{ik} a_{kj}} \qquad \forall i = 1, \ldots m.$$

Finally, the above equality implies that $\mathbf{e} = [\mathbf{h}] \cdot [\mathbf{a}]$. This last deduction also shows that $\ker([\mathbf{h}]^t) \subset \ker([\mathbf{e}]^t)$, and so the equality holds. In conclusion, we have proved that $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} \subset \mathcal{K}(\mathbf{d})_{(\mathbf{d}'',[\mathbf{h}])}$ if and only if $(\mathbf{d}',[\mathbf{e}]) \lesssim (\mathbf{d}'',[\mathbf{h}])$. To end the proof observe that the components of the type $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$ only intersect the trivial components $\mathbb{C}^m_{\mathbf{d}} \times S_{d_1} \times \cdots \times S_{d_{i-1}} \times \{0\} \times S_{\mathbf{d}_{i+1}} \times \cdots \times S_{d_m}$ at the null point $(\lambda = 0, \underline{F} = 0)$. □

2.4.2 Generic injectivity

Across this section, we deal with the possible generic injectivity of the parametrization ρ. This result is correct assuming that the vector $\mathbf{d}$ has not repeated degrees. However, in the general case, the rational parametrization is only a generically finite map.

First, let us characterize some equivalent definitions for an algebraic map to be generically injective.

Proposition 2.4.22. Let $f : X \to Y$ be a dominant morphism between two algebraic varieties defined over an algebraically closed field k. Also X is assumed to be irreducible. Then, the following conditions are equivalent:

i) There exist a non-empty open subset $U \subset X$ such that $f|_U$ is injective.

ii) There exist a non-empty open subset $V \subset Y$ such that $f|_{f^{-1}(V)}$ is injective.

Proof. At first $ii) \Rightarrow i)$ is trivial. For the other implication, take $Z = X - U$ which is a proper closed subset of X. Moreover, since f is dominant and injective when restricted to U, according to the dimension fiber theorem (see for instance section 8 of chapter I in [44]), we get:

$$\dim(Z) < \dim(X) = \dim(Y).$$

The reason is that each fiber $f^{-1}(f(x))$ has dimension zero for every $x \in U$. Finally, the Zariski closure of $f(Z)$ is a proper closed subset which allows us to take $V = Y - \overline{f(Z)}$. □

To give an appropriated setting for the ideas involved in the proof of the generic injectivity, let us recall the definition of integrating factor.

Definition 2.4.23. An element $G \in S_e$ is said to be an algebraic integrating factor of degree e of $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ if the following equation holds:

$$G\, d\omega = \omega \wedge dG$$

The equation described in the last definition can be also thought as $d(\frac{\omega}{G}) = 0$, which in fact only makes sense when $e = d$.

Remark 2.4.24. A simple calculation shows that for a given $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$, the space $\mathcal{IF}_\omega(e)$ of integrating factors of a fixed degree e is a linear subspace of S_e.

Proposition 2.4.25. If $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ admits an integrating factor of any degree, then it is integrable, i.e. $\omega \wedge d\omega = 0$.

Proof. It immediately follows from: $\omega \wedge d\omega = -\frac{dG}{G} \wedge \omega \wedge \omega = 0$. □

Proposition 2.4.26. For every logarithmic form $\omega \in l_1(\mathbf{d}, n)$, the polynomial $F \in \mathcal{IF}_\omega(d)$, i.e. is an integrating of degree d.

Proof. Observe every logarithmic factor $\frac{dF_i}{F_i}$ is closed under the exterior derivative operator and so:

$$d\left(\frac{\omega}{F}\right) = \sum_{i=1}^{m} \lambda_i d\left(\frac{dF_i}{F_i}\right) = 0$$

□

The following result is a sort of converse implication of the previous proposition.

Proposition 2.4.27. Take $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ and suppose it admits an integrating factor F of degree d. Also, assume that the divisor define by the zero locus of F is simple reduced normal crossing. Then, ω is a logarithmic form, i.e. $\omega \in l_1(\mathbf{d}, n)$ for some $\mathbf{d} \in \mathbb{N}^m$.

Proof. Let $\mathcal{D}$ be the simple reduced normal crossing divisor defined by $F = \prod_{i=1}^m F_i$, where each F_i is irreducible. Next consider the rational form $\eta = \frac{\omega}{F} \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(*\mathcal{D}))$, and observe that η has simple poles on $\mathcal{D}$ and moreover $d\eta = 0$. So, this form defines a global section of the sheaf $\Omega^1_{\mathbb{P}^n}(log(\mathcal{D}))$. Due to the characterization given for $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(log\mathcal{D}))$ at 2.3.10 we deduce

$$\eta = \sum_{i=1}^m \lambda_i \frac{dF_i}{F_i},$$

and so $\omega \in l_1(\mathbf{d}, n)$ as claimed. □

Now, we construct an incident variety which controls the number of possible integrating factors of a certain degree. This space will be the key to deduce the possible generic injectivity for logarithmic 1-forms. Consider

$$\chi_1(d,n) = \{(F,\omega) \in \mathbb{P}S_d \times \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) : F d\omega = \omega \wedge dF\},$$

which consist on pairs $([F],[\omega])$ where $F \in \mathcal{IF}_\omega(d)$. When there is no confusion, the notation [] for the respective projective classes will be avoid.

In addition, note that the remark 2.4.25 in particular implies that the image of th second projection $\pi_2|_{\chi_1(d,n)}$ is contained in the moduli space $\mathcal{F}_1(d,n)$.

Proposition 2.4.28. There exist a non-empty open subset $H_1 \subset \chi_1(d,n)$, such that the restricted projection map $\pi_2|_{H_1}$ is injective.

Proof. The basic idea is to use the upper semi-continuity of the fiber dimension on the morphism $\pi_2|_{\chi_1(d,n)}$ (see [29, Theorem 13.1.3] for a general version of this result). In agreement with this theorem, the set

$$H_1 = \{x = (F,\omega) \in \chi_1(d,n) : \dim_x(\pi_2^{-1}(\omega)) = 0\}$$

is an open subset of $\chi_1(d,n)$. The notation $\dim_x$ refers to the Krull dimension of the local ring associated to x. Also note that the algebraic spaces involved, $\chi_1(d,n)$ and $\mathcal{F}_1(d,n)$, are not required to be irreducible. First, we prove that H_1 is non-empty. According to the usual notation, select a logarithmic form of type $\mathbf{d}$

$$\omega = \sum_{i=1}^m \lambda_i \hat{F}_i dF_i \in l_1(\mathbf{d}, n),$$

where $\lambda \notin \mathbb{P}\mathbb{Q}^m$. Then, necessarily $F = \prod F_i$ will be the unique integrating factor. This last remark immediately follows from 2.4.10 and 2.4.11, and the observation that if ω admits two integrating factors F and G of degree d, then necessarily F/G is a rational first integral of ω.

Now, observe that if a form ω has two different integrating factors then every linear combination of them is also an integrating factor (see remark 2.4.24). This ensures that the fiber $\pi_2^{-1}(\omega) = \pi_2^{-1}(\pi_2(x))$ has a unique element, for every $x = (F,\omega) \in H_1$, which in particular implies the injectivity result announced.

□

The following proposition states the generic injectivity of the parametrization ρ in the case where $\mathbf{d}$ has non repeated degrees.

Proposition 2.4.29. Let $\mathbf{d}$ be a vector of degrees such that $d_i \neq d_j \ \forall i, j$. Then, the rational ρ which parametrizes the logarithmic variety $\mathcal{L}_1(\mathbf{d}, n)$ is generically injective.

Proof. First consider the morphism

$$\tilde{\rho} : \mathbb{PC}^m_{\mathbf{d}} \times \prod_{i=1}^{m} \mathbb{P}S_{d_i} \to \chi_1(d, n)$$

$$(\lambda, (F_i)) \mapsto (F, \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i),$$

which is a rational map, well defined on the open set $\mathcal{U}_1(\mathbf{d})$. Also note that the natural parametrization ρ factors by $\tilde{\rho}$:

$$\begin{array}{ccc} \mathbb{PC}^m_{\mathbf{d}} \times \prod_{i=1}^{m} \mathbb{P}S_{d_i} & \xrightarrow{\tilde{\rho}} & \chi_1(d, n) \\ & \searrow^{\rho} & \downarrow \pi_2 \\ & & \mathcal{F}_1(d, \mathbb{P}^n) \end{array}$$

With the notation of proposition 2.4.28 consider the open subset H_1 of $\chi_1(d, n)$, on which π_2 is injective. We will see that ρ restricted to the open algebraic set

$$H_1(\mathbf{d}) = \tilde{\rho}^{-1}(H_1) \cap \mathcal{U}_1(\mathbf{d})$$

is injective. It is important to remark that in the proof of 2.4.28 we have also proved that for every vector of degrees $\mathbf{d}$ the previously defined open set $H_1(\mathbf{d})$ is non-empty.

In addition, denote by $U \subset \prod_{i=1}^{m} \mathbb{P}S_{d_i}$ the open subset where the polynomial of each factor is irreducible. It is also clear that the first coordinate of the map $\tilde{\rho}$:

$$\tilde{\rho}_1 : U \to \mathbb{P}S_d$$

$$(F_i)_{i=1}^m \mapsto F = \prod_{i=1}^{m} F_i$$

is injective. The reason is that the polynomials $F_1, \ldots, F_m$ are considered to be irreducibles, and all the degrees involved are distinct.

Then, using the injectivity of $\tilde{\rho}_1$ and the lemma 2.4.12, it is easy to check the morphism $\tilde{\rho}$ is injective when restricted to $H_1(\mathbf{d})$. In conclusion, due to the injectivity of $\tilde{\rho}$ on $H_1(\mathbf{d})$ and of π_2 restricted to H_1, the same holds for $\rho = \pi_2 \circ \tilde{\rho}$.

□

Remark 2.4.30. With the notation of the previous result, and a similar proof, it is possible to state the same result for the multilinear map ϕ.

Now, we state a more general result, without any condition imposed on $\mathbf{d}$. We consider convenient to keep in mind the notation introduced at the previous proof.

Proposition 2.4.31. The natural parametrization $\rho : \mathcal{U}_1(\mathbf{d}) \to \mathcal{F}_1(d,n)$ is always a generically finite map, i.e. all its fiber has dimension zero when we consider the restriction to an appropriated open subset of $\mathcal{U}_1(\mathbf{d})$.

Proof. Let $\omega = \phi(\lambda, \underline{F})$ in the image of a generic parameter, and let us prove its corresponding fiber is finite. For this purpose, we can use again the result 2.4.28 and the same notation as in the previous proof. We select the non empty algebraic open set $H_1 \subset \chi_1(d,n)$ on which π_2 is injective, and consider $(\lambda, \underline{F}) \in \mathcal{U}_1(\mathbf{d}) \cap \tilde{\rho}^{-1}(H_1)$.

On the other hand, each fiber of $s : U \subset \prod_{i=1}^m \mathbb{P}S_{d_i} \to \mathbb{P}S_d$ at $F = \prod_{i=1}^m F_i$ is finite. Note that the elements of such fibers correspond to all the possible permutations of the irreducible factors F_i associated to repeated degrees. According to this last fact and the lemma 2.4.12, it is clear that the fiber of $\tilde{\rho} = (s, \rho)$ at (F, ω) is finite. We can end the proof by noting that $\phi = \pi_2 \circ \tilde{\phi}$. □

Corollary 2.4.32. The domain of the parametrization ρ could be redefined by taking the quotient of $\mathcal{P}_1(\mathbf{d})$ by the action of a product of symmetric groups, which are associated to all the possible permutations between homogeneous polynomials of the same degree. Formally, if we realign $\mathbf{d}$ by:

$$d_{i_1^1} = \cdots = d_{i_{k_1}^1} < d_{i_1^2} = \cdots = d_{i_{k_2}^2} < \cdots < d_{i_1^r} = \cdots = d_{i_{k_r}^r},$$

then ρ is generically injective defined over the space:

$$\mathbb{P}\mathbb{C}_{\mathbf{d}}^m \times \prod_{i=1}^m \mathbb{P}S_{d_{i_1^j}}^{(k_j)}.$$

The notation used above is the usual for the symmetric product of a projective space.

Finally, we perform an alternative injectivity result, with a slightly different proof from the given above. The important advantage respect to the previous proposition is that we can exhibit the open set where the restriction of the parametrization is injective. However, this open space is not algebraic. Also, we need to recall the definition of the complex open set $\mathcal{V}_1(\mathbf{d})$ given at 2.4.7. This set is an open complex subset of $\mathcal{U}_1(\mathbf{d})$ determined by the extra condition: $\lambda \notin \mathbb{P}\mathbb{Q}^m$.

Proposition 2.4.33. The natural parametrization ρ restricted to $\mathcal{V}_1(\mathbf{d})$ is injective in the case where $\mathbf{d}$ has non-repeated degrees. Furthermore $\rho|_{\mathcal{V}_1(\mathbf{d})}$ is always a finite map.

Proof. Suppose that there exist two elements $(\lambda, (F_i))$ and $(\beta, (G_j))$ selected on $\mathcal{V}_1(\mathbf{d})$ with:

$$\omega = \sum_{i=1}^m \lambda_i \hat{F}_i dF_i = \sum_{j=1}^m \beta_j \hat{G}_j dG_j. \tag{2.4.2}$$

Now, observe that the associated polynomials $F, G \in S_d$ are both integrating factors of the projective form ω. If it were true $F \neq G$, then ω would admit a rational first integral F/G. But this is not possible according to the results 2.4.10 and 2.4.11, and the assumption of $\lambda \notin \mathbb{P}\mathbb{Q}^m$. So, we deduce

$$F = \prod_{i=1}^m F_i = \prod_{j=1}^m G_j = G,$$

which in particular implies that the irreducible polynomials F_i and G_i must be equal for every index i (if **d** has not repeated degrees). In the other case, if **d** allows repeated degrees, the entire m-tuples of polynomials (F_i) and (G_i) must coincide after a possible permutation between the elements. Also write τ for such permutation, or the identity if **d** has non-repeated elements.

To conclude the proof observe that if we divide the whole equation 2.4.2 by F, and use the lemma 2.4.12, we also obtain: $\lambda_i = \beta_{\tau(i)}$ for $i = 1, \ldots, m$. □

2.5 Infinitesimal stability of logarithmic one forms

Our primary purpose is to show that the space of logarithmic forms $\mathcal{L}_1(\mathbf{d}, n)$ is an irreducible component of the space $\mathcal{F}_1(d, n)$, also generically reduced according to its scheme structure. Since the proofs of these main results are supported on tangent space calculations, we need to develop a correct characterization of the Zariski tangent space of $\mathcal{F}_1(d, n)$ at logarithmic form ω, and also to describe the derivative of ρ.

2.5.1 Zariski tangent spaces and the derivative of the parametrization

First, we will remember the definition of the Zariski tangent space in the general context of abstract algebraic varieties or schemes. See [21, I.2.2] for an overview of this task.

Definition 2.5.1. Fix a point x at a given algebraic variety (or scheme) X over a field k. Denote by $\mathcal{M}_x$ the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Then, the Zariski tangent space of X at x, $\mathcal{T}_x X$, is set as the vector space over the residue field $k(x)$ that is dual to $\mathcal{M}_x / \mathcal{M}_x^2$.

Example 2.5.2. If $X \subset \mathbb{A}^n$ is defined by a system of polynomial equations $F_j(x) = 0$ for $j \in J$, then the Zariski tangent space at $x^0 = (x_1^0, \ldots, x_n^0) \in X$ is defined by the linear system of equations:

$$\sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i}(x^0)(x_i - x_i^0) = 0.$$

Remark 2.5.3. If we consider a complex vector space $\mathbb{V}$ and write $X = \mathbb{P}\mathbb{V}$ for its associated projective space (where $\pi_{\mathbb{V}}$ denote the induced projection), then it is common to identify the Zariski tangent space of X at a given point $p = \pi_{\mathbb{V}}(x)$ with:

$$\mathcal{T}_p X = \mathbb{V}/\langle x \rangle.$$

In addition, consider a projective subvariety Y of X defined by homogeneous equations

$$G_1(x) = \cdots = G_r(x) = 0.$$

With a slight abuse of notation, the Zariski tangent space $\mathcal{T}_p Y$ at $p = \pi_{\mathbb{V}}(x)$, can be identify with all the elements $x' \in \mathbb{V}/\langle x \rangle$ such that:

$$G_1(x + \varepsilon x') = \cdots = G_r(x + \varepsilon x') = 0 \mod \varepsilon^2.$$

In agreement with the above remarks, let us describe the Zariski tangent spaces of all the spaces involved in the natural parametrization ρ.

Proposition 2.5.4. Select $\lambda \in \mathbb{P}(\mathbb{C}^m_{\mathbf{d}})$ and polynomials $\underline{F} = (F_i) \in \prod_{i=1}^m \mathbb{P}S_{d_i}$, and then let $\omega \in \mathcal{F}_1(d, \mathbb{P}^n)$ be the image of $(\lambda, \underline{F})$ by ρ. The tangent elements $\lambda' \in \mathcal{T}_\lambda \mathbb{P}(\mathbb{C}^m_{\mathbf{d}})$, $F'_i \in \mathcal{T}_{F_i} \mathbb{P}S_{d_i}$ (for each possible index i) and $\alpha \in \mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$, can be respectively characterized by:

1) $\lambda' \in (\mathbb{C}^m - \{0\})/_{\langle\lambda\rangle}$ such that: $\mathbf{d} \cdot \lambda' = 0$.

2) $F'_i \in (S_{d_i} - \{0\})/_{\langle F_i \rangle}$.

3) $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))/_{\langle\omega\rangle}$ such that: $\alpha \wedge d\omega + \omega \wedge d\alpha = 0$.

Proof. It will be an immediate consequence of the description made for the Zariski tangent space to a projective variety, in addition to observe that:

$$(\lambda + \varepsilon\lambda') \cdot \mathbf{d} = 0 \mod \varepsilon^2 \iff \lambda' \cdot \mathbf{d} = 0,$$

and

$$(\omega + \varepsilon\alpha) \wedge d(\omega + \varepsilon\alpha) = 0 \mod \varepsilon^2 \iff \alpha \wedge d\omega + \omega \wedge d\alpha = 0.$$

□

The equation established at 3) will be referred as the **perturbation equation** needed for a homogeneous form α to be a Zariski tangent vector of $\mathcal{F}_1(d, \mathbb{P}^n)$ at ω.

Through the rest of the chapter, we keep the notation established in this last remark. In other words, α refers to a first order perturbation of a logarithmic form ω, and $(\lambda', (F'_i))$ to tangent vectors of the space of parameters $\mathcal{P}(\mathbf{d})$ at $(\lambda, \underline{F})$ (in concordance with the representation given at the proposition 2.5.4).

Otherwise recall from 4.3.7 the definition of the natural parametrization ρ:

$$\rho : \mathcal{P}_1(\mathbf{d}) = \mathbb{P}(\mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^m \mathbb{P}S_{d_i} \dashrightarrow \mathcal{F}_1(d, \mathbb{P}^n)$$
$$([\lambda], \underline{F} = ([F_i])) \longmapsto [\omega] = \left[\sum_{i=1}^m \lambda_i \hat{F}_i dF_i\right].$$

From now on, the notation [] will be avoided. Next, we describe the derivative of this rational parametrization at a given point. Moreover, we will keep the notation of the remark 2.3.6.

Proposition 2.5.5. For $(\lambda, \underline{F}) \in \mathcal{P}_1(\mathbf{d})$ and $\omega = \rho(\lambda, \underline{F}) \in \mathcal{L}_1(\mathbf{d}, n) \subset \mathcal{F}_1(d, \mathbb{P}^n)$, the derivative

$$d\rho_{(\lambda,\underline{F})} : \mathcal{T}_\lambda \mathbb{P}(\mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^m \mathcal{T}_{F_i} \mathbb{P}S_{d_i} \longrightarrow \mathcal{T}_\omega F_1(d, \mathbb{P}^n),$$

can be calculated by multi-linearity as:

$$d\rho_{(\lambda,\underline{F})}(\lambda', (F'_i)) = \sum_{i=1}^m \lambda'_i \hat{F}_i dF_i + \sum_{i \neq j} \lambda_i \hat{F}_{ij} F'_j dF_i + \sum_{i=1}^m \lambda_i \hat{F}_i dF'_i.$$

Proof. It is a consequence of the descriptions from the proposition 2.5.4 and a simple multi-linearity argument over the parametrization ϕ, whose projectivization corresponds to ρ. □

Remark 2.5.6. According to the trivial inclusion $\mathrm{im}(d\rho_{(\lambda,\underline{F})}) \subset \mathcal{T}_\omega\mathcal{F}_1(d,n)$, every element in the image of the differential $\alpha = d\rho_{(\lambda,\underline{F})}(\lambda', (F_i'))$ is an homogeneous projective 1-form of total degree d which satisfies the perturbation equation:

$$\alpha \wedge d\omega + \omega \wedge d\alpha = 0.$$

Formally, we can summarize this observation by:

$$\mathrm{im}(d\rho_{(\lambda,\underline{F})}) \subset \mathcal{T}_\omega\mathcal{F}_1(d,\mathbb{P}^n) = \{\alpha \in\ H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))/_{\langle\omega\rangle} : \alpha \wedge d\omega + \omega \wedge d\alpha = 0\}.$$

As it was announced, one significant result is the proof of the reverse inclusion from the one given in the above remark, which also corresponds to show the surjectivity of the differential of the natural parametrization. First, we need to fix again some notation we will further use.

As usual we write $\mathcal{D}_F$ for the divisor defined by the zero locus of $F = \prod_{i=1}^m F_i$, where each F_i is an irreducible homogeneous polynomial. From now on, we assume that $m \geq 3$, and use X_i to denote the components defined by:

$$X_i = \{x \in \mathbb{P}^n : F_i(x) = 0\}.$$

More generally, for every $I \subset \{1, \ldots, m\}$ we set:

$$X_I = \{x \in \mathbb{P}^n : F_i(x) = 0 \ \forall i \in I\}.$$

In addition, for every $k \in \mathbb{N}_{\leq m}$, we refer to the following variety:

$$X^k_{\mathcal{D}_F} = \bigcup_{\substack{I\subset\{1,\ldots,m\}\\ |I|=k}} X_I$$

as the codimension k stratum associated with $\mathcal{D}_F$.

Through this chapter we shall especially use $X^1_{\mathcal{D}_F}$, $X^2_{\mathcal{D}_F}$ and $X^3_{\mathcal{D}_F}$. Also, these spaces will be important in the subsequent chapters for studying logarithmic forms of higher degree.

Now, let us relate this spaces with the tangent vectors of $\mathcal{F}_1(d,\mathbb{P}^n)$ at a logarithmic form ω. The following result can be found at [16].

Proposition 2.5.7. For every projective logarithmic form $\omega \in \mathcal{L}(\mathbf{d}, n)$, its singular locus $S_\omega = \{x \in \mathbb{P}^n : \omega(x) = 0\}$ can be described as a union:

$$S_\omega = X^2_{\mathcal{D}_F} \cup R,$$

where R is a finite set.

With the notation of proposition 2.5.5, we can state the following remark:

Remark 2.5.8. Every element in the image of $d\rho_{(\lambda,\underline{F})}$ can be described as a sum:

$$\tilde{\alpha} = d\rho_{(\lambda,\underline{F})}(\lambda', (F_i')) = \left(\sum_{i=1}^{m} \lambda_i' \hat{F}_i dF_i\right) + \left(\sum_{i\neq j} \lambda_i \hat{F}_{ij} F_j' dF_i + \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i'\right) = \alpha_1 + \alpha_2,$$

where both $\alpha_1 = d\rho_{(\lambda,\underline{F})}(\lambda', 0)$ and $\alpha_2 = d\rho_{(\lambda,\underline{F})}(0, (F_i'))$ satisfy separately the perturbation equation, but vanish on different strata associated to the divisor $\mathcal{D}_F$, i.e.

- $\alpha_1|_{X^2_{\mathcal{D}_F}} = 0$
- $\alpha_2|_{X^3_{\mathcal{D}_F}} = 0$

Moreover, the whole image $\tilde{\alpha}$ always vanishes on $X^3_{\mathcal{D}}$.

Finally, we want to emphasize another characterization of the tangent vectors at ω. Specifically, we describe the perturbation equation in a meromorphic setting.

Proposition 2.5.9. Consider $\omega \in \mathcal{L}_1(\mathbf{d}, n)$ and $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$. With the usual notation, if we write $\eta = \frac{\omega}{F} \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(log(\mathcal{D}_F)))$ and $\beta = \frac{\alpha}{F} \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(*\mathcal{D}_F))$, then the following conditions are equivalent:

i. $\omega \wedge d\alpha + \alpha \wedge d\omega = 0$

ii. $d\beta \wedge \eta = 0$

iii. $d(\beta \wedge \eta) = 0$

Proof. To see $ii \Leftrightarrow iii$, we need to use that the logarithmic meromorphic form η is closed under the exterior derivative:

$$d\eta = \sum_{i=1}^{m} \lambda_i d(\tfrac{dF_i}{F_i}) = 0.$$

The rest of the implications are deduced by a straight forward calculation, working on the open dense subset $(F \neq 0)$. □

2.5.2 Main results

As it was announced, we will show that the logarithmic varieties $\mathcal{L}_1(\mathbf{d}, n)$ determine irreducible component of the moduli space $\mathcal{F}_1(d, \mathbb{P}^n)$. Also, some aspects related to the geometry of these components will be treated.

The main result of this chapter can be summarized in the following theorem:

Theorem 2.5.10. Let $n, m, d \in \mathbb{N}_{\geq 3}$ and fix $\mathbf{d} = (d_1, \ldots, d_m)$ any partition of d. Then, $\mathcal{L}_1(\mathbf{d}, n)$ is a generically reduced and irreducible component of the space $\mathcal{F}_1(d, \mathbb{P}^n)$. It is birrational equivalent to $\mathcal{P}_1(\mathbf{d})$ in the case that $d_i \neq d_j$ (for all selection of i, j), or to a finite quotient of $\mathcal{P}_1(\mathbf{d})$ when $\mathbf{d}$ has repeated degrees. Moreover, $\mathcal{F}_1(d, \mathbb{P}^n)$ is reduced at the points of $\rho(\mathcal{U}_1(\mathbf{d}))$ and also smooth at the points of $\rho(\mathcal{V}_1(\mathbf{d}))$.

As it was explained before, the novelty in the above theorem, which is beside the method of its proof, is what concerns the scheme structure over a generic point. In addition, it set the background to extend these results for higher degree logarithmic forms. See theorem 4.4.1 which states a version of this theorem for foliations of codimension two.

The key in the proof of the above result is the surjectivity of the derivative of the natural parametrization ρ. Summarily, the theorem 2.5.10 will be implied by the next proposition combined with some arguments of scheme theory.

Proposition 2.5.11. Let $n, m, d \in \mathbb{N}_{\geq 3}$ and fix $\mathbf{d} = (d_1, \ldots, d_m)$ a partition of d defined as before. For every element $(\lambda, \underline{F}) \in \mathcal{U}_1(\mathbf{d}) \subset \mathcal{P}_1(\mathbf{d})$, take $\omega = \rho(\lambda, \underline{F})$ as its associated generic logarithmic 1-form of type $\mathbf{d}$. Then, the derivative

$$d\rho_{(\lambda,\underline{F})} : \mathcal{T}_\lambda \mathbb{P}(\mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^{m} \mathcal{T}_{F_i} \mathbb{P}S_{d_i} \longrightarrow \mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$$

is surjective. In other words, the reverse of the inclusion described at the remark 2.5.6 is valid.

Remark 2.5.12. The hypothesis $m, d \in \mathbb{N}_{\geq 3}$ is not necessary. It is sufficient to assume that the numbers are in $\mathbb{N}_{\geq 2}$. However, we will keep that assumption towards to not change the notation in the principal proofs, in which sometimes we need to consider three different indexes in $\{1, \ldots, m\}$. In the other case, the arguments are simpler than the presented in this work. So in conclusion, we use the hypothesis $m \geq 3$ to ensure that the variety $X^3_{\mathcal{D}_F}$ is non-empty.

Furthermore, note that the case $m = 2$ corresponds to the well known rational components (see for instance [15]).

In the sake of clarity, since the hard part of the arguments lies in the last proposition's proof, we will perform the proof of the above theorem assuming that the result **??** is correct. Afterward, a more extensive development of this last proposition will be approached in the following sections.

Remark 2.5.13. The idea behind the following proof is essentially the same as the one used in [15, Theorem 2.1] (see p. 8) and also in [14, Theorem 1] (see pp. 14 and 15).

Proof of theorem 2.5.10.

First, recall that $\mathcal{L}_1(\mathbf{d}, n)$ is defined as the Zariski closure of the image of the rational parametrization ρ, which is well defined on the Zariski open set $\mathcal{U}_1(\mathbf{d})$. So $\mathcal{L}_1(\mathbf{d}, n) = \overline{\rho(\mathcal{U}_1(\mathbf{d}))}$ is a projective irreducible variety. According to the results 2.4.29 and 2.5.11 (generic injectivity of ρ and the surjectivity of its differential), these varieties are birationaly equivalent to the space of parameters $\mathcal{P}_1(\mathbf{d})$ if $\mathbf{d}$ has non repeated degrees, or to a symmetrization of $\mathcal{P}_1(\mathbf{d})$ in the other case.

Let us consider $\mathcal{F}_1(d, \mathbb{P}^n)_{red}$ as the reduced scheme structure associated to $\mathcal{F}_1(d, \mathbb{P}^n)$. Since the space of parameters $\mathcal{P}_1(\mathbf{d})$ is a reduced variety, the natural parametrization factors according to the diagram:

$$\begin{array}{ccc} \mathcal{U}_1(\mathbf{d}) & \xrightarrow{\ \rho\ } & \mathcal{F}_1(d, \mathbb{P}^n) \\ & \searrow^{\tilde{\rho}} & \uparrow{\scriptstyle i_{red}} \\ & & \mathcal{F}_1(d, \mathbb{P}^n)_{red} \end{array} .$$

In other words, the logarithmic variety $\mathcal{L}_1(\mathbf{d},n)$ is generically reduced, and the image of ρ can be thought inside of $\mathcal{F}_1(d,\mathbb{P}^n)_{red}$. This last property can be also found at EGA I (see [30, 5.1.5]). The algebraic idea behind this last fact is that any ring map $A \xrightarrow{f} B$, where B has not nilpotent elements, must factor through the quotient of A by its nilradical ideal, i.e by the space $A_{red} = A/nil(A)$.

Observe that for a given point $(\lambda, \underline{F}) \in \mathcal{U}_1(\mathbf{d})$, the proposition 2.5.11 implies that the derivative:

$$d\rho_{(\lambda,\underline{F})} : \mathcal{T}_{(\lambda,\underline{F})}\mathcal{U}_1(\mathbf{d}) \longrightarrow \mathcal{T}_\omega\mathcal{F}_1(d,\mathbb{P}^n)$$

is surjective, and also factors through $\mathcal{T}_\omega\mathcal{L}_1(\mathbf{d},n) \subset \mathcal{T}_\omega\mathcal{F}_1(d,\mathbb{P}^n)_{red} \subset \mathcal{T}_\omega\mathcal{F}_1(d,\mathbb{P}^n)$. Then, we can deduce:

$$\mathcal{T}_\omega\mathcal{L}_1(\mathbf{d},n) = \mathcal{T}_\omega(\mathcal{F}_1(d,\mathbb{P}^n))_{red} = \mathcal{T}_\omega\mathcal{F}_1(d,\mathbb{P}^n).$$

Moreover, using the usual argument of generic smoothness, we can assume:

$$\dim(T_\omega\mathcal{F}_1(d,\mathbb{P}^n)_{red}) = \dim(\mathcal{F}_1(d,\mathbb{P}^n)_{red}) = \dim(\mathcal{F}_1(d,\mathbb{P}^n)),$$

and

$$\dim(\mathcal{T}_\omega\mathcal{L}_1(\mathbf{d},n)) = \dim(\mathcal{L}_1(\mathbf{d},n)).$$

In conclusion, it follows that $\dim(\mathcal{L}_1(\mathbf{d},n)) = \dim(\mathcal{F}_1(d,\mathbb{P}^n))$, and so $\mathcal{L}_1(\mathbf{d},n)$ is an irreducible component of the moduli space $\mathcal{F}_1(d,\mathbb{P}^n)$, also reduced at the points of $\rho(\mathcal{U}_1(\mathbf{d}))$. □

The argument used in the previous proof holds in a general setting according to the following lemma. This result will take part of a work in progress due to Cukierman F. and Massri C. ([13]).

Lemma 2.5.14. Let X be a reduced and irreducible scheme. Let $f : X \to Y$ be a morphism, let $x \in X$ be a smooth point and let $y = f(x) \in Y$.

If $df(x) : TX(x) \to TY(y)$ is surjective, then $\overline{f(X)}$ is a reduced and irreducible component of Y.

With the notation of corollary 2.4.32, a more precise statement about the geometry of $\mathcal{L}_1(\mathbf{d},n)$ is the following:

Corollary 2.5.15. If $\mathbf{d}$ has non-repeated degrees, $\mathcal{L}_1(\mathbf{d},n)$ is birrational equivalent to $\mathcal{P}_1(\mathbf{d})$. In the other case, it is birrational equivalent to:

$$\mathbb{P}\mathbb{C}^m_{\mathbf{d}} \times \prod_{i=1}^m \mathbb{P}S^{(k_j)}_{d_{i_1^j}}.$$

Moreover, the following aspects about the rationality of these components are remarkable.

Corollary 2.5.16. The irreducible logarithmic components $\mathcal{L}_1(\mathbf{d},n)$ are rational varieties in the case of non-repeated degrees and unirational in general.

Finally, it is possible to deduce the dimension of the irreducible logarithmic varieties.

Corollary 2.5.17. With the notation of the above theorem, the dimension of the logarithmic irreducible components can be calculated by:

$$\dim(\mathcal{L}_1(\mathbf{d},n)) = \sum_{i=1}^m \binom{d_i+n}{d_i} - 2.$$

2.5.3 Suryectivity of the differential of the parametrization

In this section, we state a complete proof of the surjectivity result for the differential of the natural parametrization ρ (see proposition 2.5.11). This result will complete the proof of the main theorem 2.5.10, related with the stability of projective logarithmic 1-forms.

First, we need to state some technical lemmas, which are similar to that used in the corresponding proof of the same surjectivity result for logarithmic 2-forms at chapter 4. Moreover, we consider necessary to properly describe them at this moment, looking for all the proofs of the main results to be self-contained in this chapter.

The setup and notation required for these lemmas are the same as that considered immediately up to this moment. We deal with a logarithmic form of type $\mathbf{d} = (d_1, \ldots, d_m)$, defined by

$$\omega = \rho(\lambda, \underline{F}) \in \mathcal{L}_1(\mathbf{d}, n) \qquad (\text{for } (\lambda, \underline{F}) \in \mathcal{U}_1(\mathbf{d})).$$

According to the conditions assumed on $\mathcal{U}_1(\mathbf{d})$ (see 2.4.2), the divisor $\mathcal{D}_F = (F = 0)$ is simple normal crossings, so the associated strata

$$X^k_{\mathcal{D}} = \bigcup_{I:|I|=k} X_I = (F_{i_1} = \cdots = F_{i_k} = 0)$$

defines a codimension k projective subvariety, and each stratum X_I is also a smooth complete intersection. In general, we will use a slight abuse of notation and write $X^k_{\mathcal{D}}$ and X_I to also denote the corresponding varieties defined by the same polynomials on the affine cone $\mathbb{C}^{n+1}$.

In addition, we set the notation $H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})(d)$ for the homogeneous piece of degree d of the graded $\mathbb{C}[z_0, \ldots, z_n]$-module $H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})$, which is trivially characterized by elements like:

$$\alpha = \sum_{i=0}^{n} A_i(z) dz_i,$$

for some selected homogeneous polynomials $A_0, \ldots, A_n$ of degree $d-1$. These forms are also referred as homogeneous affine forms of total degree d.

Finally, when there is no confusion, we avoid denoting the set where the numerical index belongs. In general, most of the indexes will be considered in the set $\{1, \ldots, m\}$.

Lemma 2.5.18 (Vanishing lemma)**.** Let $\alpha \in H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})(e)$ be a homogeneous affine form satisfying:

$$\alpha|_{X^2_{\mathcal{D}}} = 0,$$

and also recall that we are assuming $d = \sum_{i=1}^m d_i$. Then, there exist a family $\{\alpha_j\}_{j=1}^m$ of homogeneous affine 1-forms of respective total degrees $(e - d + d_j)_{j=1}^m$ such that:

$$\alpha = \sum_{j=1}^{m} \hat{F}_j \alpha_j$$

If a degree $e - d + d_j$ is strictly negative, the corresponding form α_j is considered as zero.

Corollary 2.5.19. The conclusion of the previous lemma also holds for every homogeneous projective 1-form $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(e))$ vanishing when restricted to $X^2_{\mathcal{D}}$.

Proof. Both the lemma and the corollary immediately follow as a particular case of the propositions A.0.22 and A.0.21 treated at the final appendix. □

With the same idea as in the previous lemma, we state below another vanishing lemma, which describes forms vanishing on some determined components of $X^2_{\mathcal{D}_F}$. Also, recall that we are assuming that the divisor $\mathcal{D}_F$ has at least three components, and so $X^2_{\mathcal{D}_F}$ is non-empty.

Lemma 2.5.20 (Second vanishing lemma)**.** For every fixed index $i \in \{1, \dots, m\}$, if we select $\beta_i \in H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})(e)$ (the same conclusion will hold for $\beta_i \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(e))$) satisfying:

$$\beta_i|_{X_{ij}} = 0 \quad \forall j \neq i,$$

there exist homogeneous affine forms $\tilde{\beta}_i$ of degree $(e + d_i - d)$ and γ_i of degree $(e - d_i)$ such that:

$$\beta_i = \hat{F}_i \tilde{\beta}_i + F_i \gamma_i.$$

Again, if some degree is strictly negative, the corresponding form is going to be considered as zero.

Proof. The argument is exactly the same as the used in the proof of A.0.22. If we define the following subvariety:

$$X^2_{\mathcal{D}}(i) = \bigcup_{j \neq i} X_{ij},$$

the hypothesis assumed precisely implies that: $\beta_i|_{X^2_{\mathcal{D}}(i)} = 0$. So, observe that $X^2_{\mathcal{D}}(i) = X^2_{\mathcal{D}} \cap (F_i = 0)$, in particular implies that its associated ideal corresponds to:

$$\mathcal{I}_{X^2_{\mathcal{D}}(i)} = \langle \hat{F}_k \rangle_{k=1}^m + \langle F_i \rangle = \langle \hat{F}_i \rangle + \langle F_i \rangle$$

(see A.0.20 for more details). Finally, the result follows from applying a correct short exact sequence as in A.0.22, and a simple degree calculation. □

Now, using the normal crossing condition for $\mathcal{D}_F$, and the condition 3) described at 2.4.2, we can state a result which allows us to divide forms by $dF_1, \dots, dF_m$ over the different components of the strata $\{X^k_{\mathcal{D}}\}_k$. This result is a variant of the Saito's version of the DeRham's division lemma turned up for our purposes (see for instance [46]). Moreover, it is similar to that stated at [15, Lemma 2.2].

Before entering the statement, we introduce a short digression about the restriction of forms.

Remark 2.5.21 (The restriction of the sheaf of forms)**.** Let X be a smooth complex algebraic variety and $Y \xrightarrow{i} X$ a closed subvariety, whose corresponding sheaf of ideals will be denoted by $\mathcal{I}_Y$.

The following are usual exact sequences of sheaves for the restriction to Y:

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Y \simeq (i_Y)_*(\mathcal{O}_Y) \to 0 \tag{2.5.1}$$

$$0 \to (\mathcal{I}_Y)|_Y \to (\mathcal{O}_X)|_Y \to \mathcal{O}_Y \to 0, \tag{2.5.2}$$

and if Ω^q_X is considered to be locally free, then we also have:

$$0 \to \Omega^q_X \otimes \mathcal{I}_Y \to \Omega^q_X \to \Omega^q_X \otimes (\mathcal{O}_X/\mathcal{I}_Y) \to 0 \tag{2.5.3}$$

$$0 \to (\Omega^q_X \otimes \mathcal{I}_Y)|_Y \to (\Omega^q_X)|_Y \to ((\Omega^q_X)|_Y) \otimes \mathcal{O}_Y \to 0. \tag{2.5.4}$$

Observe that the even numbered sequences are of $\mathcal{O}_X|_Y$-modules. Also, the middle term of 2.5.3 is the sheaf-theoretical restriction of Ω^q_X to Y (its stalks are the same, but it is only supported on Y). Also, the right term of that sequence can be considered as the analytic restriction, sometimes denoted by $\Omega^q_X|_a Y$ (for more details see [27] at pp.20). In the context of algebraic geometry, this last restriction corresponds by definition to the inverse image of the sheaf, i.e.

$$i^*(\Omega^q_X) = \Omega^q_X|_Y \otimes \mathcal{O}_Y.$$

Also, note that this sheaf is of $\mathcal{O}_Y$-modules. For more details, it can be consulted Hartshorne's book at chapter 2 ([35]). It is important to notice that the sections of this sheaf are not the same as the pullback of the elements of Ω^q_X to Y. About this last, and in order to compare this two constructions, we can remember another useful known exact sequence:

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \to (\Omega^1_X)|_Y \to \Omega^1_Y \to 0,$$

which is also named as the "second exact sequence" for the Kahler differentials (see also [31]).

From now on, and when there is no confusion, we will use the common restriction symbol $|_Y$ to denote the "inverse image" or "analytic" restriction.

Finally, using this announced notation, it is possible to identify

$$i_*((\Omega^q_X)|_Y) \simeq \Omega^1_X \otimes (\mathcal{O}_X/\mathcal{I}_Y).$$

And with this setting the first sequence of 2.5.3 can be written back as:

$$0 \to \Omega^q_X \otimes \mathcal{I}_Y \to \Omega^q_X \xrightarrow{|_Y} i_*((\Omega^q_X)|_Y) \to 0, \tag{2.5.5}$$

where the morphism of the right corresponds to the usual restriction of a form to the points of Y.

Now, coming back to our purposes, we can state the announced division lemma, which is a useful variant of Saito's lemma.

Lemma 2.5.22 (Division lemma)**.** Assume that $n \geq 2$ and $q \leq n-1$. Consider $\alpha \in H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})(e)$, a multi-index $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, m\}$ and an integer $1 \leq j \leq q$. If the form α satisfies:

$$(\alpha \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_j})|_{X_I} = 0,$$

then there exist j homogeneous polynomials $G_{i_1}, \ldots, G_{i_j}$ of respective degrees $e - d_{i_1}, \ldots, e - d_{i_j}$ (or zero if the expected degree is negative) such that:

$$\alpha|_{X_I} = (\sum_{k=1}^{j} G_{i_k}\, dF_{i_k})|_{X_I}.$$

The same conclusion will hold for every projective form $\beta \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(e))$ satisfying the condition equivalent to that assumed on α.

This result corresponds to a particular case of the division lemma stated in chapter 4 at 4.4.3. Although to obtain a self-contained development of the topics in this section, we describe now the proof of this case.

Proof. For simplicity, denote by $X = \mathbb{C}^{n+1}$ and let Y be the subvariety defined as the zero locus of the selected polynomials i.e. $Y = (F_{i_1} = \cdots = F_{i_q} = 0)$. It can be noticed that Y corresponds to the affine cone of the projective variety usually denoted by X_I.

First, note that Ω^1_X is a free sheaf generated on global sections by $dz_0, \ldots, dz_n$, i.e.

$$\Omega^1_X = \mathcal{O}_X \cdot dz_0 \oplus \cdots \oplus \mathcal{O}_X \cdot dz_n$$

Furthermore, the restriction $\Omega^1_X|_Y$ is an $\mathcal{O}_Y$-module freely generated on global sections according to:

$$\Omega^1_X|_Y = \mathcal{O}_Y \cdot dz_0|_Y \oplus \cdots \oplus \mathcal{O}_Y \cdot dz_n|_Y.$$

Now recall that the divisor defined by $\mathcal{D}_F$ is simple normal crossing, and so according to condition 3) of 2.4.2, we know:

$$d_x F_{i_1} \wedge \cdots \wedge d_x F_{i_j} \neq 0 \quad \forall x \in Y - \{0\}.$$

This fact in particular implies that 0 is the unique singularity of $dF_{i_1}|_Y \wedge \cdots \wedge dF_{i_j}|_Y$. In other words, if we consider the global decomposition:

$$dF_{i_1}|_Y \wedge \cdots \wedge dF_{i_j}|_Y = \sum_{0 \leq l_1 \leq \cdots \leq l_j \leq n} a_{l_1 \ldots l_j} dz_{l_1}|_Y \wedge \cdots \wedge dz_{l_j}|_Y,$$

then the ideal $\mathcal{A}$ generated by the coefficients $\{a_{l_1 \ldots l_j}\}$ has a hight depth, and in particular, it is greater or equal than one. So, we can to apply the Saito's main theorem in [46] to the free module determined by the global sections of $\Omega^1_X|_Y$. This last remark allows us to divide 1-forms which vanishes against the wedge product by $dF_{i_1}|_Y \wedge \cdots \wedge dF_{i_j}|_Y$. We can conclude the existence of global polynomials $\tilde{G}_{i_1}, \ldots, \tilde{G}_{i_j}$ in the coordinate ring of Y such that:

$$\alpha|_Y = \sum_{k=1}^{j} \tilde{G}_{i_K} dF_{i_k}|_Y.$$

To end the argument, we just need to observe that each of the polynomials $\{\tilde{G}_{i_k}\}$ can be selected homogeneous of corresponding degrees $\{e - d_{i_K}\}$. The reason is that the coordinated ring of Y is graded, and the elements $\alpha|_Y$ and all the forms $dF_{i_j}|_Y$ are homogeneous. Finally, since all the morphisms like:

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \xrightarrow{|_{X_j}} H^0(X_I, \mathcal{O}_{X_I}(j)),$$

are always surjective, we can infer the existence of homogeneous global polynomials $G_{i_1}, \ldots, G_{i_j}$ such that: $G_{i_k}|_Y = \tilde{G}_{i_K}$. Moreover these will be the desired polynomials.

It is also remarkable that the surjectivity of the previous restriction map is a consequence of the vanishing of the cohomology groups $H^1(\mathbb{P}^n, \mathcal{I}_{X_I}(j))$ (see for instance p. 6 of [37]). The important property behind this is that the varieties $\{X_I\}$ are supposed to be smooth complete intersections.

□

The following result is the last technical lemma we need. It will be important in order to deduce the existence of some polynomials $F'_1, \dots, F'_m$ performing the expected formula for every Zariski tangent vector of $\mathcal{F}_1(d, \mathbb{P}^n)$ at ω (see remark 2.5.8).

Lemma 2.5.23 (Fundamental lemma). Fix $n \in \mathbb{N}_{\geq 3}$, a vector of degrees $\mathbf{d} = (d_1, \dots, d_m)$ as usual (with $d = \sum_{i=1}^m d_i$) and an index $j \in \{1, \dots, m\}$. Also, consider a family of homogeneous polynomials $\{B_{ij}\}_{i \neq j}$ in $n+1$ variables, all of them of degree d_j, and which satisfy the relations:

$$B_{ij}|_{X_{ijk}} = B_{kj}|_{X_{ijk}}.$$

Then, for each $j \in \{1, \dots, m\}$ there exist a homogeneous polynomial $F'_j \in S_{d_j}$ such that:

$$B_{ij}|_{X_{ij}} = F'_j|_{X_{ij}} \qquad \forall i \neq j.$$

Moreover, on the restriction to X_j, the following equality holds:

$$B_{ij}|_{X_j} = (F'_j + F_i \tilde{B}_{ij})|_{X_j} \qquad \forall i \neq j,$$

where $\tilde{B}_{ij}$ is another homogeneous polynomial of degree $d_j - d_i$ (or the zero polynomial if this expected degree is negative).

Proof. Fix an index j, and consider the projective subvariety X_j defined by the zero locus of F_j. Also, denote by $\mathcal{D}_{(j)}$ the divisor on X_j defined by:

$$\mathcal{D}_{(j)} = (\hat{F}_j|_{X_j} = 0) = \bigcup_{i \neq j} (F_i = 0) \cap X_j.$$

Also, the ideal sheaf associated to this divisor can be described by: $\mathcal{I}_{\mathcal{D}_{(j)}} = \langle \hat{F}_j|_{X_j} \rangle$. In addition, the lemma's hypotheses imply that the restricted polynomials $B_{ij}|_{X_j}$ give rise to a well defined object:

$$B_{(j)} = \{B_{ij}|_{X_j}\} \in H^0(\mathcal{D}_{(j)}, \mathcal{O}_{\mathcal{D}_{(j)}}(d_j)).$$

If we consider the usual exact sequence of $\mathcal{D}_{(j)}$ as a subvariety of X_j (in this case twisted by $\mathcal{O}_{X_j}(d_j)$):

$$0 \to \mathcal{I}_{\mathcal{D}_{(j)}}(d_j) \to \mathcal{O}_{X_j}(d_j) \to i_*(\mathcal{O}_{\mathcal{D}_{(j)}}(d_j)) \to 0,$$

and note that $\mathcal{I}_{\mathcal{D}_{(j)}} \simeq \mathcal{O}_{X_j}(d_j - d)$, then we can describe the induced long exact sequence by

$$0 \to H^0(X_j, \mathcal{O}_{X_j}(2d_j - d)) \to H^0(X_j, \mathcal{O}_{X_j}(d_j)) \to H^0(\mathcal{D}_{(j)}, \mathcal{O}_{\mathcal{D}_{(j)}}(d_j)) \to H^1(X_j, \mathcal{O}_{X_j}(2d_j - d)) \to \dots$$

An important assertion is that the first arrow corresponds to the assignment $(G \mapsto G\hat{F}_j)$.

In order to finish the proof, we need to show that $H^1(X_j, \mathcal{O}_{X_j}(2d_j - d)) = 0$. This last fact can be deduced from the long exact sequence on cohomology associated to the tensor product of

$$0 \to \mathcal{I}_{X_j} \simeq \mathcal{O}_{\mathbb{P}^n}(-d_j) \to \mathcal{O}_{\mathbb{P}^n} \to (i_{X_j})_*(\mathcal{O}_{X_j}) \to 0, \tag{2.5.6}$$

by $\mathcal{O}_{\mathbb{P}^n}(2d_j - d)$, which let us deduce the short exact sequence:

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2d_j - d)) \to H^1(X_j, \mathcal{O}_{X_j}(2d_j - d)) \to H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_j - d)).$$

So the desired vanishing property follows from the usual knowledge about sheaf cohomology of $\mathcal{O}_{\mathbb{P}^n}(d)$ (see for example the Bott's formulas in [42]) and the hypothesis of $n \geq 3$. In conclusion, we have deduced that the morphism:

$$H^0(X_j, \mathcal{O}_{X_j}(d_j)) \xrightarrow{|_{\mathcal{D}_{(j)}}} H^0(\mathcal{D}_{(j)}, \mathcal{O}_{\mathcal{D}_{(j)}}(d_j))$$

is surjective. With exactly the same idea, consider again the sequence 2.5.6, but now twisted by $\mathcal{O}_{\mathbb{P}^n}(d_j)$, from which it is possible to deduce that the restriction map:

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_j)) \xrightarrow{|_{X_j}} H^0(X_j, \mathcal{O}_{X_j}(d_j))$$

is also surjective. In conclusion, we deduce the existence of a global homogeneous polynomial $F'_j \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_j))$ whose double restriction satisfies:

$$(F'_j|_{X_j})|_{\mathcal{D}_{(j)}} = B_{(j)}.$$

In particular, we obtain: $F'_j|_{X_{ij}} = B_{ij}|_{X_{ij}}$, as claimed. And also the following holds

$$B_{ij}|_{X_j} = F'_j|_{X_j} + (F_i \tilde{B}_{ij})|_{X_j}$$

for some new introduced homogeneous polynomials $\{\tilde{B}_{ij}\}_{i \neq j}$.

It is remarkable that we need to use a surjectivity result for some restriction's maps, which depends on the vanishing of certain cohomology groups. In general, every projectively normal variety X has the desired property: $H^1(\mathbb{P}^n, \mathcal{I}_X(j)) = 0$ for $j \in \mathbb{Z}$ (see [37] or [19]). Moreover, it is always true that every smooth complete intersection is projectively normal (see for example exercise 8.4 of chapter II in [31]). □

Remark 2.5.24. Alternatively, the conclusion of the above lemma could be the following equality for the new selected polynomial F'_j :

$$B_{ij} = F'_j + F_i \tilde{B}_{ij} + c_j F_j,$$

for some complex constant c_j. Furthermore, if we are working in the space $S_{d_j}/\langle F_j \rangle$, the equality holds without the factor $c_j F_j$.

Beginning of the proof

Now, we are ready to complete the proof of the main results introduced before, proposition 2.5.11 and theorem 2.5.10.

According to the constructions described in the present chapter, what we need to prove is that for every fixed projective class of a logarithmic form of type **d**:

$$\omega = \rho(\lambda, (F_i)_{i=1}^m) \in \rho(\mathcal{U}_1(\mathbf{d})) \subset \mathcal{L}_1(\mathbf{d}, n),$$

and every projective homogeneous form $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ satisfying the perturbation equation:

$$\alpha \wedge d\omega + \omega \wedge d\alpha = 0,$$

there exist:

- $\lambda' = (\lambda'_1, \dots, \lambda'_m) \in \mathbb{C}^m/\langle\lambda\rangle$ such that $\sum_{i=1}^m \lambda'_i d_i = 0$
- $F'_i \in S_{d_i}/\langle F_i\rangle$,

fulfilling:

$$\alpha = d\rho_{(\lambda,\underline{F})}(\lambda', \underline{F'}) = \underbrace{\sum_{i=1}^m \lambda'_i \hat{F}_i dF_i}_{d\rho_{(\lambda,\underline{F})}(\lambda',0)} + \underbrace{\sum_{i\neq j} \lambda_i \hat{F}_{ij} F'_j dF_i + \sum_{i=1}^m \lambda_i \hat{F}_i dF'_i}_{d\rho_{(\lambda,\underline{F})}(0,\underline{F'})}.$$

For such purpose, and considering the remark 2.5.8, our first objective is to show that every α satisfying the perturbation equation must vanish on the stratum $X^3_{\mathcal{D}_F}$.

Proposition 2.5.25. If $\alpha \in \mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$ is Zariski tangent vector at $\omega = \rho(\lambda, \underline{F}) \in \rho(\mathcal{U}_1(\mathbf{d}))$, then necessarily: $\alpha|_{X^3_{\mathcal{D}}} = 0$. Moreover, there exist a family of homogeneous polynomials $\{A_{ij}\}_{i\neq j}$ of respective degrees: $\deg(A_{ij}) = d_j$, fulfilling the formula:

$$\alpha = \sum_{i\neq j} A_{ij} \hat{F}_{ij} dF_i + \tilde{\alpha},$$

where $\tilde{\alpha} \in H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})$ is an homogeneous affine form of total degree d such that: $\tilde{\alpha}|_{X^2_{\mathcal{D}}} = 0$.

Proof. Consider the perturbation equation for α:

$$\alpha \wedge d\omega + \omega \wedge d\alpha = 0,$$

and also observe that the exterior derivative of ω can be described by:

$$d\omega = \sum_{i\neq j} \lambda_i \hat{F}_{ij} dF_j \wedge dF_i.$$

In addition, recall that the stratum $X^2_{\mathcal{D}}$ is contained on the singular set of ω, and so for every pair of indexes i, j (fixed from now), on the restriction to X_{ij}, the perturbation equation reduces to:

$$(\alpha \wedge (\lambda_j - \lambda_i)\hat{F}_{ij} dF_i \wedge dF_j)|_{X_{ij}} = 0.$$

Now considering the conditions imposed on $\mathcal{U}_1(\mathbf{d})$, which allow us to cancel the term $(\lambda_i - \lambda_j)\hat{F}_{ij}$, and using the division lemma 2.5.22, we deduce the existence of homogeneous polynomials satisfying:

$$\alpha|_{X_{ij}} = (G_{ij} dF_i + G_{ji} dF_j)|_{X_{ij}}.$$

Moreover, on every intersection $X_{ijk} = X_{ij} \cap X_{jk}$ the different descriptions of $\alpha|_{X_{ijk}}$ from above must coincide:

$$\alpha|_{X_{ijk}} = (G_{ij} dF_i + G_{ji} dF_j)|_{X_{ijk}} = (G_{ik} dF_i + G_{ki} dF_k)|_{X_{ijk}} = (G_{jk} dF_j + G_{kj} dF_k)|_{X_{ijk}}. \tag{2.5.7}$$

Next, remember that the divisor $\mathcal{D}_F$ is normal crossing, and so at every point of $x \in X_{ijk}$ the restricted 1-forms $dF_i|_{X_{ijk}}$, $dF_j|_{X_{ijk}}$ and $dF_k|_{X_{ijk}}$ are independent as elements of $(\Omega^1_{\mathbb{P}^n}|_{X_{ijk}})_x$. Also, note that $\alpha(x)$ belongs to the intersection:

$$\langle d_x F_i, d_x F_j\rangle_{\mathbb{C}} \cap \langle d_x F_j, d_x F_k\rangle_{\mathbb{C}} \cap \langle d_x F_i, d_x F_k\rangle_{\mathbb{C}} = 0.$$

According to the above observations, we are able to deduce that every involved polynomial at 2.5.7 should vanish on the restriction to X_{ijk}. In particular, we obtain:

$$G_{ij}|_{X_{ijk}} = 0 \quad \forall k \neq i, j.$$

Another way to deduce this last conclusion is considering the wedge product by $dF_j|_{X_{ijk}} \wedge dF_k|_{X_{ijk}}$ on the equations of 2.5.7, and applying directly the condition 3) from 2.4.2.

Furthermore, using the assumptions imposed on $F_1, \dots, F_m$, the variety $\bigcup_{k\neq i,j} X_k \cap X_{ij}$ can be defined on X_{ij} as the zero locus of the principal ideal $\langle \hat{F}_{ij}|_{X_{ij}} \rangle$. So it is possible to deduce the existence of a homogeneous polynomials A_{ij} fulfilling:

$$(G_{ij})|_{X_{ij}} = (\hat{F}_{ij}A_{ij})|_{X_{ij}}.$$

This last fact can be thought as a variant for functions of the second vanishing lemma 2.5.20. In addition, we need to use again the surjectivity of some restriction maps like

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e)) \xrightarrow{|_{X_{ij}}} H^0(X_{ij}, \mathcal{O}_{X_{ij}}(e)).$$

Finally, we can vary the fixed pair of indexes to construct a family $\{A_{ij}\}_{i\neq j}$ which satisfies the corresponding above decomposition. To finish the proof, observe that α and the form $\sum_{i\neq j} \hat{F}_{ij}A_{ij}dF_i$ has the same restriction to each piece X_{ij}, and so their difference corresponds to an affine homogeneous form of total degree d that vanishes on $X^2_{\mathcal{D}_F}$. □

Next, we can deduce that every tangent vector is a sum of something in the image of $d\rho$ and another tangent vector vanishing on a lower codimensional stratum.

Proposition 2.5.26. With the notation and hypotheses of proposition 2.5.25, there exist an m-tuple of homogeneous polynomials $(F'_i)_{i=1}^m \in \prod_{i=1}^m S_{d_i}$ and a projective homogeneous form $\varepsilon \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ such that:

$$\alpha = d\rho_{(\lambda,\underline{F})}(0, (F'_i)_{i=1}^m) + \varepsilon,$$

where also $\varepsilon \in \mathcal{T}_\omega\mathcal{F}_1(d, \mathbb{P}^n)$ and $\varepsilon|_{X^2_{\mathcal{D}_F}} = 0$.

Proof. According to the above proposition 2.5.25, we get the following decomposition:

$$\alpha = \sum_{i\neq j} A_{ij}\hat{F}_{ij}dF_i + \tilde{\alpha}, \tag{2.5.8}$$

where each homogeneous polynomial A_{ij} has degree d_j, and also the affine homogeneous form $\tilde{\alpha}$ vanishes on $X^2_{\mathcal{D}_F}$. Next, we can use the restriction lemma 2.5.18 to deduce the existence of an m-tuple of homogeneous affine forms $(\tilde{\alpha}_k)_{k=1}^m$ (of respective degrees $(d_k)_{k=1}^m$) decomposing $\tilde{\alpha}$ as:

$$\tilde{\alpha} = \sum_{k=1}^m \hat{F}_k\tilde{\alpha}_k.$$

The main idea is to replace these expressions for α and $\tilde{\alpha}$ into the perturbation equation, in order to deduce certain conditions which in some sense, will imply that $A_{ij} = \lambda_i F'_j$, for some new introduced homogeneous polynomials.

First, we need to consider the following expressions for the exterior derivative of ω and α. Also, recall that we avoid denoting the set where the numerical index belong.

$$d\omega = \sum_{i\neq j} \lambda_i \hat{F}_{ij} dF_j \wedge dF_i \tag{2.5.9}$$

$$d\alpha = \sum_{i\neq j\neq k} \hat{F}_{ijk} A_{ij} dF_k \wedge dF_i + \sum_{i\neq j} \hat{F}_{ij} dA_{ij} \wedge dF_i + \sum_{k\neq l} \hat{F}_{kl} dF_l \wedge \tilde{\alpha}_k + \sum_{k} \hat{F}_k d\tilde{\alpha}_k. \tag{2.5.10}$$

For simplicity, we are not going to place both formulas into the perturbation equation at the same time, so let us separately consider the two terms of this equation, namely by:

$$\underbrace{\alpha \wedge d\omega}_{\text{Term 1}} + \underbrace{\omega \wedge d\alpha}_{\text{Term 2}} = 0.$$

We start working out the first term:

$$\alpha \wedge d\omega = \sum_{\substack{k\neq l\\ i\neq j}} A_{kl} \hat{F}_{kl} \lambda_i \hat{F}_{ij}\, dF_k \wedge dF_j \wedge dF_i + \sum_{\substack{k\\ i\neq j}} \hat{F}_k \lambda_i \hat{F}_{ij}\, \tilde{\alpha}_k \wedge dF_j \wedge dF_i$$

Note that in the first sum we can also assume that $k \neq i, j$, which allows to replace each factor $\hat{F}_{kl}\hat{F}_{ij}$ by $\hat{F}_l \hat{F}_{ijk}$. Now, fix an index k_0 and take the restriction of the first term to the subvariety X_{k_0}:

$$\alpha \wedge d\omega = \hat{F}_{k_0} \left(\sum_{i\neq j\neq k} \lambda_i A_{k k_0} \hat{F}_{ijk} dF_i \wedge dF_j \wedge dF_k + \sum_{i\neq j} \lambda_i \hat{F}_{ij} \tilde{\alpha}_{k_0} \wedge dF_j \wedge dF_i \right) \qquad (\text{over } X_{k_0}).$$

Obviously, this is not the optimal formula because, a priori, the terms where the index k_0 is not selected must vanish. However, for our prompt purposes, it will be convenient enough.

On the other hand, observe that $\omega|_{X_{k_0}} = (\lambda_{k_0} \hat{F}_{k_0} dF_{k_0})|_{X_{k_0}}$. So, according to the formula expressed for $d\alpha$ at 2.5.10, a possible (not optimal) formula for the restriction of *Term 2* to X_{k_0} is the following:

$$\begin{aligned}\omega \wedge d\alpha = \hat{F}_{k_0} &\left(\sum_{i\neq j} \lambda_{k_0} \hat{F}_{ij} dA_{ij} \wedge dF_i \wedge dF_k + \sum_{i\neq j\neq k} \lambda_{k_0} \hat{F}_{ijk} A_{ij} dF_i \wedge dF_k \wedge dF_{k_0} \right) + \\ + \hat{F}_{k_0} &\left(\sum_{k\neq l} \lambda_{k_0} \hat{F}_{kl} dF_{k_0} \wedge dF_l \wedge \tilde{\alpha}_k + \lambda_{k_0} \hat{F}_{k_0} dF_{k_0} \wedge d\tilde{\alpha}_{k_0} \right) \qquad (\text{over } X_{k_0}).\end{aligned}$$

At this moment, we need to sum the two restricted terms obtained, and cancel the corresponding factor $\hat{F}_{k_0}|_{X_{k_0}}$ which is present everywhere and is not the zero function of the integral ring $H^0(X_{k_0}, \mathcal{O}_{X_{k_0}})$. Next, fix two more indexes i_0, j_0, and restrict the new equation to $X_{i_0 j_0 k_0}$ where most of the sum's terms vanish. After all, we obtain:

$$(-\lambda_{i_0} A_{j_0 k_0} + \lambda_{j_0} A_{i_0 k_0}) \hat{F}_{i_0 j_0 k_0} dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0} = 0 \qquad (\text{over } X_{i_0 j_0 k_0}),$$

which due to the normal crossing condition assumed on $\mathcal{D}_F$, let us deduce: $(-\lambda_{i_0} A_{j_0 k_0} + \lambda_{j_0} A_{i_0 k_0}) = 0$. So if we consider the family of homogeneous polynomials $\{B_{ij}\}_{i\neq j}$ defined by:

$$B_{ij} = \frac{A_{ij}}{\lambda i} \qquad \forall i \neq j,$$

we know they satisfy the relations $B_{ij} = B_{kj}$ on X_{ijk} (for every possible selection of indexes). Now, according to the fundamental lemma 2.5.23, there exist homogeneous polynomials $F'_1, \dots, F'_m$ of respective degrees $d_1, \dots, d_m$ such that:

$$(B_{ij})|_{X_{ij}} = (\frac{A_{ij}}{\lambda_i})|_{X_{ij}} = (F'_j)|_{X_{ij}} \quad \forall i \neq j. \tag{2.5.11}$$

Finally, we must combine the formula 2.5.8 for α and 2.5.11 from above, to deduced that α and the form

$$\sum_{i \neq j} \lambda_i F'_j \hat{F}_{ij} dF_i + \sum_k \lambda_k \hat{F}_k dF'_k = d\rho_{(\lambda, \underline{F})}(0, (F'_i)_{i=1}^m)$$

has the same restriction to each piece of $X^2_{\mathcal{D}} = \bigcup_{i \neq j} X_{ij}$. We finally take

$$\varepsilon := \alpha - d\rho_{(\lambda, \underline{F})}(0, (F'_i)_{i=1}^m),$$

which vanishes when restricted to $X^2_{\mathcal{D}}$.Moreover, since α and $d\rho_{(\lambda, \underline{F})}(0, (F'_i)_{i=1}^m)$ are tangent vectors at ω, then ε also belongs to $\mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$. □

In conclusion, we have reduced the surjectivity problem to the Zariski tangent vectors ε of $\mathcal{F}_1(d, \mathbb{P}^n)$ at ω satisfying $\varepsilon|_{X^2_{\mathcal{D}}} = 0$.

In addition, taking the remark 2.5.8 into consideration, the form ε is expected to be related to a perturbation of the coefficients λ. In other words, we would predict ε equals to $d\rho_{(\lambda, \underline{F})}((\lambda'_i)_{i=1}^m, 0)$ for some tangent vector $\lambda' \in \mathcal{T}_\lambda \mathbb{P}\mathbb{C}^m_{\mathbf{d}}$. This last fact is going to be true only under certain assumptions on $\mathbf{d}$ (balanced case). Furthermore, for $\mathbf{d}$ non-balanced, we need to add another term associated with a new selection of perturbed polynomials $(F'_i)_{i=1}^m$.

Before the beginning of the distinction between the general and the balanced case, we perform another common proposition for tangent vectors vanishing on the restriction to $X^2_{\mathcal{D}}$.

The next result is a slight more general statement than the needed in the short term, but it will also be used for the non-balanced case.

Proposition 2.5.27. Fix a natural number $r \in \mathbb{N}$, and consider $\varepsilon \in \mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$, a tangent vector at $\omega = \rho(\lambda, \underline{F}) \in \rho(\mathcal{U}_1(\mathbf{d}))$ of the form:

$$\varepsilon = \sum_{i=1}^m (\hat{F}_i)^r \varepsilon_i,$$

for some homogeneous affine 1-forms $\varepsilon_1, \dots, \varepsilon_m$ of the correct degree (if it is attainable, or zero in any other case). Then, the following equalities are necessarily satisfied:

$$(\varepsilon_i \wedge dF_i \wedge dF_j)|_{X_{ij}} = 0 \quad \forall i \neq j.$$

Proof. The idea is the same as in the previous proposition, we need to restrict the perturbation equation in a correct order to the codimension two components X_{ij}.

First, observe that the exterior derivative of ε satisfies:

$$d\varepsilon = \sum_{k \neq l} r(\hat{F}_k)^{r-1} \hat{F}_{kl}\, dF_l \wedge \varepsilon_k + \sum_k (\hat{F}_k)^r d\varepsilon_k.$$

So if we fix an index i_0 and use that $\varepsilon|_{X_{i_0}} = (\hat{F}_{i_0})^r \varepsilon_{i_0}$, then the restriction of the perturbation equation to X_{i_0} has the following possible expression:

$$(\hat{F}_{i_0})^r \left(\sum_{i \neq j} \lambda_i \hat{F}_{ij} \varepsilon_{i_0} \wedge dF_j \wedge dF_i + \sum_{k \neq l} \lambda_{i_0} r \hat{F}_{kl} dF_{i_0} \wedge dF_l \wedge \varepsilon_k + \lambda_{i_0} \hat{F}_{i_0} dF_{i_0} \wedge d\varepsilon_{i_0} \right) = 0 \qquad \text{over } X_{i_0}.$$

Now, we are able to cancel the term $(\hat{F}_{i_0})^r$, because it is not the zero element of the corresponding integral coordinate ring of the variety X_{i_0}. After doing this, fix another index j_0 and take the restriction of the equation to $X_{i_0 j_0}$, in order to deduce:

$$(\lambda_{j_0} + (r-1)\lambda_{i_0}) \hat{F}_{i_0 j_0} \varepsilon_{i_0} \wedge dF_{i_0} \wedge dF_{j_0} = 0.$$

Finally, according to the conditions imposed over the open set $\mathcal{U}_1(\mathbf{d})$, we can take out the term $(\lambda_{j_0} + (r-1)\lambda_{i_0}) \hat{F}_{i_0 j_0}$, and arrive to the expected result.

□

Remark 2.5.28. The case $r = 1$ of the previous proposition is going to be used soon as a first approach to understand the Zariski tangent vectors at ω which vanish on $X^2_{\mathcal{D}}$. Furthermore, the cases $r > 1$ are going to be important in the non-balanced case.

In relation with this last comment, notice that, for every $r \in \mathbb{N}$ fixed, the expected degrees of $(\varepsilon_k)_{k=1}^m$ should be:

$$\deg(\varepsilon_k) = d - r(d - d_k) = d_k - (r-1) \sum_{j \neq k} d_j.$$

The balanced concept will be related to the sign of the above degrees.

The balanced case

In order to end the proof of proposition 2.5.11, across this section we set up the final part assuming certain extra conditions on the associated degrees $\mathbf{d}$.

Definition 2.5.29. We say that an m-tuple of degrees $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$ is balanced if for each fixed index $k \in \{1, \ldots, m\}$ the following holds:

$$d_k < \sum_{j \neq k} d_j.$$

Remark 2.5.30. Alternatively, let $\mathcal{D} = \sum_{j=1}^m \mathcal{D}_i = (F_i = 0)$ be a divisor over $\mathbb{P}^n$ of degree d, whose irreducible components $\{\mathcal{D}_i\}$ are defined by irreducible homogeneous polynomials $F_1, \ldots, F_m$ of respective degrees $\mathbf{d} = (d_1, \ldots, d_m)$, with $\sum_{i=1}^m d_i = d$. We will also say that the divisor $\mathcal{D}$ is balanced if its associated vector of degrees $\mathbf{d}$ is balanced.

Now, we are able to prove (assuming the condition of $\mathbf{d}$ being balanced) that every tangent vector at $\omega = \rho(\lambda, \underline{F}) \in \rho(\mathcal{U}_1(\mathbf{d}))$ which vanishes on $X^2_{\mathcal{D}_F}$ is related to a perturbation of the coefficients λ.

Proposition 2.5.31. With the notation of proposition 2.5.25, if $\varepsilon \in \mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$ vanishes on $X^2_{\mathcal{D}_F}$ and the vector $\mathbf{d}$ is balanced, then there exist constants $(\lambda'_1, \dots, \lambda'_m) \in \mathbb{C}^m_{\mathbf{d}}/\langle\lambda\rangle$ such that:

$$\varepsilon = d\rho_{\lambda,\underline{F}}(0, (\lambda'_i)_{i=1}^m) = \sum_{i=1}^m \lambda'_i \hat{F}_i dF_i \in \mathcal{L}_1(\mathbf{d}, n).$$

Proof. According to the vanishing lemma 2.5.18, the form ε can be described by: $\varepsilon = \sum_{i=1}^m \hat{F}_k \varepsilon_k$, for some homogeneous affine forms $\varepsilon_1, \dots, \varepsilon_m$ of respective degrees $d_1, \dots, d_m$. So we can use the previous proposition 2.5.27 (for $r = 1$), to deduce :

$$(\varepsilon_i \wedge dF_i \wedge dF_j)|_{X_{ij}} = 0 \qquad \forall i \neq j.$$

Now, due to the division lemma 2.5.22, there exist homogeneous polynomials of the correct degree (or the zero polynomials in any other case) fulfilling:

$$(\varepsilon_i)|_{X_{ij}} = (A^i_{ij} dF_i + A^i_{ji} dF_j)|_{X_{ij}} \qquad \forall i \neq j.$$

Next, fix three indexes i, j, k, and consider the respecting above decompositions for ε_i on X_{ij} and X_{ik}, which must coincide on their intersection X_{ijk}. Precisely, we get

$$(A^i_{ij} dF_i + A^i_{ji} dF_j)|_{X_{ijk}} = (A^i_{ik} dF_i + A^i_{ki} dF_k)|_{X_{ijk}}.$$

Since $\mathcal{D}_F$ is assumed to be simple normal crossing, the particular condition 3) at 2.4.2 implies:

$$A^i_{ij} = A^i_{ik} \quad \text{and} \quad A^i_{ji} = A^i_{ki} = 0 \qquad (\text{all over } X_{ijk}). \tag{2.5.12}$$

In particular, we have proved that each of the homogeneous polynomials like A^i_{ji} vanishes when restricted to X_{ijk} for $k \neq i, j$. Since the polynomials selected are all irreducible, distinct and with normal crossing, then necessarily exist a new family of homogeneous polynomials $\{B^i_{ji}\}_{i\neq j}$ such that:

$$(A^i_{ji})|_{X_{ij}} = (\hat{F}_{ij} B^i_{ji})|_{X_{ij}} \quad \forall i \neq j.$$

Upcoming we need to take into account the degree of all the involved polynomials. So first note that the polynomials of the family $\{A^i_{ij}\}_{i\neq j}$ are all of degree zero because ε_i and dF_i has always the same degree. From 2.5.12, we deduce that these constants only depends on its upper index, and we write

$$\lambda'_i = A^i_{ij} \in \mathbb{C} \qquad \text{for } i = 1, \dots, m.$$

On the other hand, the degrees of the polynomials $\{B^i_{ji}\}_{i\neq j}$ should fulfill:

$$\deg(B^i_{ji}) = \deg \varepsilon_i - d_j - (d - d_i - d_j) = d_i - \sum_{k\neq i} d_k,$$

so they are all strictly negative according to the balanced assumption on $\mathbf{d}$. Hence we have:

$$(\varepsilon_i - \lambda'_i dF_i)|_{X_{ij}} = 0 \qquad \forall i \neq j.$$

Finally, we need to use the second vanishing lemma 2.5.20, in this case to deduce:

$$\varepsilon_i - \lambda_i' dF_i = \hat{F}_i \tilde{\varepsilon}_i + F_i \gamma_i,$$

for some new introduced homogeneous 1-forms. Again using the balanced condition, observe that the degree of each of the forms $\{\tilde{\varepsilon}_i\}$ must be negative and so they do not take part of the last formula. Furthermore, the forms $\gamma_1, \ldots, \gamma_m$ will be taken as zero because they corresponds to homogeneous affine 1-forms of total degree 0. In conclusion, we deduce that:

$$\varepsilon = \sum_{i=1}^{m} \lambda_i' \hat{F}_i dF_i$$

as claimed. Notice that the condition of λ' lying on the space $\mathbb{C}^m_{\mathbf{d}}$ is deduced from imposing the descend condition to ε, i.e. $i_R(\varepsilon) = 0$. □

Corollary 2.5.32. If $\varepsilon \in \mathcal{T}_\omega \mathcal{F}_1(\mathbf{d}, n)$ vanishes on $X^2_{\mathcal{D}_F}$, then $\varepsilon \in l_1(\mathbf{d}, n)$, i.e. it is another logarithmic form of type $\mathbf{d}$.

Remark 2.5.33. In the case where $\mathbf{d}$ is balanced, combining 2.5.26 and 2.5.31 we arrive to a complete proof of the surjectivity result stated at proposition 2.5.11. So, as a conclusion, according to the theorem 2.5.10, we have proved that $\mathcal{L}_1(\mathbf{d}, n)$ (with $\mathbf{d}$ balanced) corresponds to an irreducible component of the moduli space $\mathcal{F}_1(\mathbf{d}, \mathbb{P}^n)$.

The non-balanced case

With the notation of the previous sections, from now on we need to deal with the problem of characterizing the Zariski tangent vectors ε vanishing on the restriction to $X^2_{\mathcal{D}_F}$, but with the additional condition that the divisor $\mathcal{D}_F$ is not balanced. As it was announced, in this case it is not going to be true that ε must be another logarithmic form of type $\mathbf{d}$.

We start describing a simple fact associated to non-balanced vectors of degrees, in order to restrict the possible number of unbalanced degrees.

Proposition 2.5.34. If $\mathbf{d} = (d_1, \ldots, d_m)$ is a non-balanced m-tuple of degrees, then there exist an unique index $i_0 \in \{1, \ldots, m\}$ such that:

$$d_{i_0} \geq \sum_{j \neq i_0} d_j.$$

Proof. By definition it is clear the existence of at least one element d_{i_0} satisfying the condition of the remark. Moreover, the uniqueness is an immediate consequence of noting that: $d_{i_0} \geq d/2$. □

According to the propositions 2.5.26 and 2.5.34, we can assume that our tangent vector satisfy: $\varepsilon|_{X^2_{\mathcal{D}}} = 0$. Also, for simplicity we can assume that $d_1 \geq \sum_{j>1} d_j$.

Proposition 2.5.35. Select $\mathbf{d}$ with the above conditions, and take $\varepsilon \in \mathcal{T}_\omega \mathcal{F}_1(\mathbf{d}, \mathbb{P}^n)$ as a tangent vector at $\omega = \rho(\lambda, \underline{F}) \in \rho(\mathcal{U}_1(\mathbf{d}))$, which also vanishes on $X^2_{\mathcal{D}_F}$.

Then, there exist constants $\lambda' = (\lambda'_1, \dots, \lambda'_m) \in \mathbb{C}^m$, a set of homogeneous polynomials $\{B_j\}_{j>1}$ (all of the same degree $d_1 - \sum_{j>1} d_j$) and an homogeneous affine 1-form γ_1 (with the same total degree as the polynomials), such that the following description holds:

$$\varepsilon = \sum_{i=1}^{m} \lambda'_i \hat{F}_i \, dF_i \; + \; \sum_{j>1} \hat{F}_1 \hat{F}_{1j} B_j \, dF_j \; + \; (\hat{F}_1)^2 \gamma_1.$$

Proof. The beginning of this proof is exactly the same as in proposition 2.5.31. Since ε is an homogeneous projective form vanishing on $X^2_{\mathcal{D}_F}$, it necessarily decomposes as $\varepsilon = \sum_{k=1}^{m} \hat{F}_k \varepsilon_k$. According to the lemma 2.5.27 (for $r = 1$) each of these new introduced forms satisfy:

$$(\varepsilon_i \wedge dF_i \wedge dF_j)|_{X_{ij}} = 0 \qquad \forall i \neq j.$$

Next by the division lemma 2.5.22, select homogeneous polynomials $\{A^i_{ij}\}_{i \neq j}$ and $\{A^i_{ji}\}_{i \neq j}$ of the correct degree (or zero in any other case) fulfilling:

$$(\varepsilon_i)|_{X_{ij}} = (A^i_{ij} dF_i + A^i_{ji} dF_j)|_{X_{ij}} \qquad \forall i \neq j.$$

Now, we need to compare the previous decompositions for ε_i (with i fixed) on X_{ij} and X_{ik}, restricted to their intersection X_{ijk}. Also, we must take into account the corresponding homogeneous degrees of all the polynomials and the forms involved, in order to deduce:

$$A^i_{ij} = A^i_{ik} = \lambda'_i \in \mathbb{C} \quad \text{and} \quad (A^i_{ji})|_{X_{ijk}} = (A^i_{ki})|_{X_{ijk}} = 0 \qquad \forall i \neq j \neq k.$$

So every polynomial A^i_{ji} vanishes on X_{ijk} for all $k \neq i, j$. In consequence, there exist another homogeneous polynomial B^i_{ji} such that:

$$(A^i_{ji})|_{X_{ij}} = (\hat{F}_{ij} B^i_{ji})|_{X_{ij}}.$$

Notice that the expected degree of these new homogeneous polynomials should be:

$$\deg(B^i_{ji}) = d_i - \sum_{r \neq i} d_r.$$

Since $\mathbf{d}$ is not balanced and the only unbalanced degree is assumed to be d_1, we set $e_1 = d_1 - \sum_{j>1} d_j \geq 0$. Summarily, the following conditions hold for these new introduced polynomials:

$$B^1_{j1} \in S_{e_1} \quad \forall j \neq 1 \qquad \text{and} \qquad B^i_{ji} = 0 \quad \forall i \neq 1, \; \forall j \neq i.$$

Finally, for each $i > 1$, the forms ε_i and $\lambda'_i \, dF_i$ coincide on the codimension two subvariety $X^2_{\mathcal{D}_F}(i) = \bigcup_{j \neq i} X_{ij}$. The same holds for the forms ε_1 and

$$\lambda'_1 \, dF_1 + \sum_{j>1} \hat{F}_{ij} B^1_{j1} \, dF_j,$$

on $X^2_{\mathcal{D}_F}(1)$. From the second vanishing lemma 2.5.20, we are able to introduce new homogeneous affine forms fulfilling respectively:

$$\varepsilon_1 = \lambda'_1 \, dF_1 + \sum_{j>1} \hat{F}_{ij} B^1_{j1} \, dF_j + \hat{F}_1 \, \tilde{\varepsilon}_1 + F_1 \gamma_1,$$

$$\varepsilon_i = \lambda'_i \, dF_i + \hat{F}_i \, \tilde{\varepsilon}_i + F_i \, \gamma_i \qquad \forall i > 1.$$

To conclude the proof, observe that all the forms $\gamma_1, \ldots, \gamma_m$ must be equal to zero, because they correspond to homogeneous 1-forms of total degree 0. Furthermore, the forms $\{\tilde{\varepsilon}_i\}_{i>1}$ do not take part of the formula due to the special assumption on $\mathbf{d}$. To respect the notation of the statement we write: $B_j := B^1_{j1}$, and this ends the proof. □

The next proposition shows that $\lambda' \in \mathbb{C}^m_{\mathbf{d}}$, , and so that necessarily $\sum_{i=1}^m \lambda'_i \hat{F}_i dF_i$ descends to the projective space.

Proposition 2.5.36. With the same notation as in proposition 2.5.35, the following conditions hold:

- $\sum_{i=1}^m \lambda'_i\, d_i = 0$
- $\sum_{j>1} d_j B_j + i_R(\gamma_1) = 0$

Proof. It is an immediate consequence of:

$$0 = i_R(\varepsilon) = \hat{F}_1 \left(F_1 \left(\sum_{i=1}^m \lambda'_i d_i \right) + \hat{F}_1 \left(\sum_{j>1} d_j B_j + i_R(\gamma_1) \right) \right),$$

and noting that the polynomials $F_1, \ldots, F_m$ are irreducible and distinct. □

Corollary 2.5.37. Again with the same notation and hypothesis as in proposition 2.5.35, the exact conclusion is that $\lambda' = (\lambda'_i) \in \mathcal{T}_\lambda \mathbb{C}^m_{\mathbf{d}}$ and:

$$\varepsilon = d\rho_{(\lambda,\underline{F})}((\lambda'_i), 0) \; + \; \sum_{j>1} \hat{F}_1 \hat{F}_{1j} B_j \, dF_j \; + \; (\hat{F}_1)^2 \gamma_1 = d\rho_{(\lambda,\underline{F})}((\lambda'_i), 0) + \tilde{\varepsilon},$$

where also $\tilde{\varepsilon}$ belongs to $\mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$.

From now on, we need to deal with tangent vectors of the form:

$$\tilde{\varepsilon} = \sum_{j>1} \hat{F}_1 \hat{F}_{1j} B_j \, dF_j \; + \; (\hat{F}_1)^2 \gamma_1,$$

and deduce they belong to the image of the differential of the parametrization ρ. Since the final part of the proof will depend on a recursive argument, the following proposition is going to express a characterization for Zariski tangent vectors a little more general than the described above.

Proposition 2.5.38. Let us assume that

$$e_1^r := d_1 - r \sum_{j>1} d_j \geq 0,$$

for some $r \in \mathbb{N}$. Also, let $\varepsilon_{(r)} \in \mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$ be a Zariski tangent vector of the form:

$$\varepsilon_{(r)} = \sum_{j>1} (\hat{F}_1)^r \hat{F}_{1j} B_j \, dF_j \; + \; (\hat{F}_1)^{r+1} \gamma_r,$$

for some homogeneous polynomials $B_2, \ldots, B_m$ and γ_r an homogeneous affine form, all of degree e_1^r. Then, there necessarily exist an homogeneous polynomial $F_1^{(r)}$ of degree e_1^r, and another homogeneous projective form $\varepsilon_{(r+1)}$, such that:

$$\varepsilon_{(r)} = d\rho_{(\lambda,\underline{F})}(0, ((\hat{F}_1)^r) F_1^{(r)}, 0, \ldots, 0) \; + \; (\hat{F}_1)^{r+1} \varepsilon_{(r+1)}.$$

Proof. As in the proof of proposition 2.5.26, we need to deduce some correct equations for $\{B_j\}_{j>1}$ in order to apply the fundamental lemma 2.5.23.

We start describing the most extensive term of the perturbation equation $\omega\wedge d\varepsilon_{(r)}+\varepsilon_{(r)}\wedge d\omega = 0$. Notice that in the first term of

$$d\varepsilon_{(r)} = \sum_{\substack{j>1\\k>1}} r(\hat{F}_1)^{r-1}\hat{F}_{1k}\hat{F}_{1j}B_j dF_k\wedge dF_j + \sum_{k\neq j>1}(\hat{F}_1)^r\hat{F}_{1jk}B_j dF_k\wedge dF_j + \\ + \sum_{j>1}(\hat{F}_1)^r\hat{F}_{1j}dB_j\wedge dF_j + \sum_{k>1}(r+1)(\hat{F}_1)^r\hat{F}_{1k}dF_k\wedge\gamma_r + (\hat{F}_1)^{r+1}d\gamma_r$$

we can also assume that the indexes k and j are distinct, and this allows us to the replace $\hat{F}_{1k}\hat{F}_{1j}$ by $\hat{F}_1\hat{F}_{1jk}$. Moreover, since $\omega|_{X_1} = \lambda_1\hat{F}_1 dF_1$, we can perform a description of the first term of the perturbation equation restricted to the subvariety X_1:

$$(\omega\wedge d\varepsilon_{(r)})|_{X_1} = (\hat{F}_1)^{r+1}\left(\sum(r+1)\hat{F}_{1jk}\lambda_1 B_j dF_1\wedge dF_k\wedge dF_j\right) + \\ + (\hat{F}_1)^{r+1}\left(\sum_{j>1}\lambda_1\hat{F}_{1j}dF_1\wedge dB_j\wedge dF_j + \sum_{k>1}(r+1)\hat{F}_{1k}dF_1\wedge dF_k\wedge\gamma_r\right)$$

On the other hand, using the following expression

$$(d\omega)|_{X_1} = \sum_{i>1}\lambda_i\hat{F}_{1i}dF_1\wedge dF_i + \sum_{k>1}\lambda_1\hat{F}_{1k}dF_k\wedge dF_1,$$

we can also perform a correct formula for the second term of the perturbation equation on X_1:

$$(\varepsilon_{(r)}\wedge d\omega)|_{X_1} = (\hat{F}_1)^{r+1}\left(\sum_{i\neq j>1}\hat{F}_{1ij}B_j\lambda_i dF_j\wedge dF_1\wedge dF_i + \sum_{j\neq k>1}\hat{F}_{1jk}B_j\lambda_1 dF_j\wedge dF_k\wedge dF_1\right) + \\ + (\hat{F}_1)^{r+1}\left(\sum_{i>1}\lambda_i\hat{F}_{1i}\gamma_r\wedge dF_1\wedge dF_i + \sum_{k>1}\lambda_1\hat{F}_{1k}\gamma_r\wedge dF_k\wedge dF_1\right).$$

Now, we need to add the two terms obtained to perform the entire restricted perturbation equation. Also, note that it is possible to cancel the factor $(\hat{F}_1)^{r+1}$, which is present in all the terms. After doing this, if we fix two more indexes j,k and restrict the equation obtained to X_{ijk}, we get

$$\left((r\lambda_1+\lambda_j)B_k - (r\lambda_1+\lambda_k)B_j\right)\hat{F}_{1jk}\,dF_1\wedge dF_j\wedge dF_k = 0 \quad \text{over } X_{1jk}$$

This in particular implies that:

$$\frac{B_j}{r\lambda_1+\lambda_j} = \frac{B_k}{r\lambda_1+\lambda_k} \qquad \text{over } X_{1jk} \quad (\forall j\neq k).$$

So we are able to apply the fundamental lemma 2.5.23 to these family of homogeneous polynomials (all of degree e_1^r) and deduce the existence of another homogeneous polynomial $F_1^{(r)}\in S_{e_1^r}$ such that:

$$(F_1^{(r)})|_{X_{1j}} = \left(\frac{B_j}{r\lambda_1+\lambda_j}\right) \quad \forall j>1.$$

In conclusion, for every index $j > 1$, there exist two more homogeneous polynomials of the correct degree G_j and H_j fulfilling:

$$B_j = (r\lambda_1 + \lambda_j)F_1^{(r)} + F_1 G_j + F_j H_j.$$

Note that the corresponding degree of G_j should be $d_1 - r\sum_{k>1} d_k - d_1 < 0$, and so it must be taken as zero. Finally, we obtain the following formula for our tangent vector $\varepsilon_{(r)}$:

$$\varepsilon_{(r)} = \sum_{j>1} (\hat{F}_1)^r \hat{F}_{1j}(r\lambda_1 + \lambda_j)F_1^{(r)} dF_j + \sum_{j>1} (\hat{F}_1)^{r+1} H_j dF_j + (\hat{F}_1)^{r+1}\gamma_r \tag{2.5.13}$$

In addition, observe that if we take an element in the image of the differential by setting $\lambda' = 0$ and $\underline{F'} = ((F_1)^r F_1^{(r)}, 0, \dots, 0)$, its corresponding formula is the following:

$$\begin{aligned} d\rho_{(\lambda,\underline{F})}(0, ((\hat{F}_1)^r F_1^{(r)}, 0, \dots, 0)) &= \sum_{j>1} \lambda_j \hat{F}_{j1}(\hat{F}_1)^r F_1^{(r)} dF_j \ + \ \lambda_1 \hat{F}_1 d\left((\hat{F}_1)^r F_1^{(r)}\right) = \\ &= \sum_{j>1} (\hat{F}_1)^r \hat{F}_{1j}(r\lambda_1 + \lambda_j)F_1^{(r)} dF_j \ + \ \lambda_1 (\hat{F}_1)^{r+1} d\left(F_1^{(r)}\right). \end{aligned}$$

Furthermore, it is clear that if we add and subtract a suitable term on the formula 2.5.13, we attain:

$$\varepsilon_{(r)} = d\rho_{(\lambda,\underline{F})}(0, ((\hat{F}_1)^r F_1^{(r)}, 0, \dots, 0)) + (\hat{F}_1)^{r+1}\varepsilon_{(r+1)},$$

as claimed. It can be notice that since $\varepsilon_{(r)}$ and $d\rho_{(\lambda,\underline{F})}(0, ((\hat{F}_1)^r F_1^{(r)}, 0, \dots, 0))$ are projective forms, the same condition holds for $\varepsilon_{(r+1)}$. □

Now we are able to state the end of the whole proof of the surjectivity result for **d** non-balanced.

Proposition 2.5.39. Assume that **d** is not balanced and let r be the maximal integer such that

$$d_1 \geq r \sum_{j>1} d_j.$$

Then, for every tangent vector $\varepsilon \in \mathcal{T}_\omega \mathcal{F}_1(d, \mathbb{P}^n)$ such that $\varepsilon|_{X^2_{\mathcal{D}}} = 0$, there exist $\lambda' \in \mathcal{T}_\lambda \mathbb{C}^m_{\mathbf{d}}$ and homogeneous polynomials $F_1^{(1)}, \dots, F_1^{(r)}$, fulfilling:

$$\varepsilon = d\rho_{(\lambda,\underline{F})}(\lambda', 0) + \sum_{k=1}^{r} d\rho_{(\lambda,\underline{F})}(0, ((\hat{F}_1)^k F_1^{(k)}, 0, \dots, 0)).$$

Proof. The proof is based on an iterative argument. According to the proposition 2.5.35 and corollary 2.5.37, we already know that ε can be described by:

$$\varepsilon = d\rho_{(\lambda,\underline{F})}(\lambda', 0) \ + \ \sum_{j>1} \hat{F}_1 \hat{F}_{1j} B_j \, dF_j \ + \ (\hat{F}_1)^2 \gamma_1,$$

for some homogeneous polynomials $\{B_j\}_{j>1}$ of the correct degree, and γ_1 an homogeneous affine form. According to the previous proposition 2.5.38 (for $r = 1$), it can be figured out the existence of another homogeneous polynomial $F_1^{(1)}$ of degree $d_1 - \sum_{j>1} d_j$ satisfying:

$$\varepsilon = d\rho_{(\lambda,\underline{F})}(\lambda', 0) + d\rho_{(\lambda,\underline{F})}(0, (\hat{F}_1)F_1^{(1)}, 0, \dots, 0) \ + \ (\hat{F}_1)^2 \varepsilon_{(2)}.$$

Note that $\varepsilon_{(2)}$ is another homogeneous form also of degree $d_1 - \sum_{j>1} d_j$, and $(\hat{F}_1)^2\varepsilon_{(2)} \in \mathcal{T}_\omega\mathcal{F}_1(d, \mathbb{P}^n)$.

More generally, in proposition 2.5.27 we have proved that every Zariski tangent vector at ω of the form $(\hat{F}_1)^k\varepsilon_{(k)}$ necessarily satisfies:

$$(\varepsilon_{(k)} \wedge dF_1 \wedge dF_j)|_{X_{ij}} = 0 \quad \forall j > 1.$$

Once more we need to use the division lemma 2.5.22 and take into consideration only the admissible positive degrees, to deduce:

$$\varepsilon_{(k)} = C_j^{(k)} dF_j \quad \text{over } X_{1j}.$$

For every index $j > 1$, $C_j^{(k)}$ is an homogeneous polynomial which also vanishes on X_{1jk} for $k \neq 1, j$. The reason is that at every point of the intersection $X_{1j} \cap X_{1k} = X_{1jk}$, the forms dF_j and dF_k are independent (according to the condition 3) at 2.4.2), which can also be reinterpreted as the assumption of $\mathcal{D}_F$ being simple normal crossing. This conditions imply the existence of another homogeneous polynomial $B_j^{(k)}$ such that:

$$(C_j^{(k)})|_{X_{1j}} = (\hat{F}_{1j}B_j^{(k)})|_{X_{1j}}.$$

Also, note that the restriction of $\varepsilon_{(k)}$ to the codimension two subvariety $X^2_{\mathcal{D}}(1) = \bigcup_{j>1} X_{1j}$ coincides with the restriction of the form:

$$\sum_{j>1} \hat{F}_{1j}B_j^{(k)} dF_j.$$

Next we can apply the second vanishing lemma 2.5.20 and a degree argument to deduce a global correct formula for the entire form $(\hat{F}_1)^k\varepsilon_{(k)}$:

$$(\hat{F}_1)^k\varepsilon_{(k)} = \sum_{j>1} (\hat{F}_1)^k \hat{F}_{1j} B_j^{(k)} dF_j + (\hat{F}_1)^{k+1}\varepsilon_{(k+1)}, \tag{2.5.14}$$

where $\varepsilon_{(k+1)}$ is another new homogeneous affine form.

Finally, to end the proof, we need to iterate this last process. Start from $k = 2$, use the previous proposition 2.5.38 on the formula 2.5.14, and repeat the same argument until the degree of the corresponding form $\varepsilon_{(k+1)}$, which corresponds to $d_1 - k\sum_{j>1} d_j$, becomes negative.

□

Corollary 2.5.40. In conclusion if **d** is not balanced and $d_1 > r\sum_{j>1} d_j$ (with r maximal), then every Zariski tangent vector $\alpha \in \mathcal{T}_\omega\mathcal{F}_1(d, \mathbb{P}^n)$ can be described by:

$$\alpha = d\rho_{(\lambda,\underline{F})}(\lambda', 0) + \sum_{k=1}^{r} d\rho_{(\lambda,\underline{F})}(0, (\hat{F}_1)^k F_1^{(k)}, 0, \dots, 0) + d\rho_{(\lambda,\underline{F})}(0, (0, F_2', \dots, F_m')).$$

This concludes the proof of the surjectivity result 2.5.11 for these types of m-tuples of degrees.

Corollary 2.5.41. Based on the corollaries 2.5.33 and 2.5.40, we can perform a complete proof of 2.5.11 and deduce the generic surjectivity of the differential of the natural parametrization ρ. In addition, this ends with the proof on the main theorem 2.5.10.

Chapter 3

Logarithmic q-forms and the extended Jouanolou's lemma

3.1 Introducción y resumen en español

A lo largo de este capítulo se desarrollan distintas versiones de un lema importante sobre 1-formas logarítmicas establecido por Jouanolou en [35], para el estudio de soluciones algebraicas de ecuaciones de Pfaff proyectivas. Este resultado será generalizado a formas logarítmicas de grados arbitrarios, primero en un contexto afín y proyectivo clásico, y luego a variedades más generales. Además, se presentarán los cálculos relativos a la explicitación de las secciones globales del conocido haz de formas logarítmicas, que será de vital importancia en el siguiente capítulo.

De manera prelimar, se introducirá un resumen de las principales definiciones relativas al conocido haz de formas logarítmicas sobre un divisor $\mathcal{D}$ dado en una variedad algebraica suave X, denotado por $\Omega^\bullet_X(\log \mathcal{D})$. Este se corresponde con secciones de formas racionales con singularidades solo en $\mathcal{D}$, tales que ellas y su derivada exterior tienen a lo sumo polos simples en $\mathcal{D}$. También, será descripta su teoría de residuos asociada (ver [18]), y una filtración del haz de formas logarítmicas que será de interés (ver definición 3.3.3). Estos elementos descriptos serán de gran ultilidad a lo largo del capítulo.

En primera instancia, en el contexto del estudio del lema particular a abordar, se comenzará realizando una análisis del resultado original de Jouanolou. Este establece que si $\{f_1, \dots, f_m\}$ es una familia de polinomios irreducibles distintos en $k[x_1, \dots, x_n]$ (con k algebraicamente cerrado), y $\lambda_1, \dots, \lambda_m$ son constantes en k, entonces la forma

$$\sum_{i=1}^{m} \lambda_i \frac{df_i}{f_i}$$

es identicamente nula si y solo $\lambda_j = 0$ para todo índice $j = 1 \dots m$. Se presentará la demostración de este resultado en virtud de observar que la idea principal se basa en utilizar un residuo clásico. Esto último será de relevancia para las generalizaciones a realizar.

De manera similar, también mostraremos cómo el resultado original es extensible a variedades afines, considerando funciones regulares $f_1, \ldots, f_k$ (irreducibles y distintas) en el anillo coordenado asociado a la variedad. Incluso, utilizando las definiciones de la sección introductoria, se mostrará otra versión de este resultado para 1-formas en el contexto de variedades algebraicas proyectivas suaves. La prueba se basará en la utilización de un residuo también clásico:

$$res_k : \Omega^1_X(Log(\mathcal{D})) \longrightarrow j^k_*(\mathcal{O}_{\mathcal{D}_k}),$$

que en general es denominado residuo de Poincaré. Este último resultado se resume a continuación. Si consideramos un divisor $\mathcal{D}$ en un variedad X con las características anteriores y $h^{1,0}_X = 0$, cuyas componente irreducibles, que denotamos por $\mathcal{D}_k$, tienen cruzamientos normales, entonces una forma logarítmica global $\eta \in H^0(X, \Omega^1_X(log(\mathcal{D})))$ es la sección nula si y solo si:

$$Res_k(\eta) = 0.$$

Si bien esta implicación es admisible para variedades más generales, en algún sentido tiene hipótesis mas fuertes sobre las funciones que definen a la forma en cuestión. En el caso en que el divisor esté determinado por los ceros de funciones regulares globales (homogéneas en el caso projectivo), estaríamos asumiendo, además de que sean algebraicamente distintos, también un cierto tipo de cruzamiento más restrictivo.

Cabe destacar también que las descripciones anteriores se basarán en la siguiente sucesión exacta larga de cohomología:

$$0 \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^0(X, \Omega^1_X(\log \mathcal{D})) \xrightarrow{\oplus res_k} \bigoplus_{i=1}^{m} H^0(\mathcal{D}_i, \mathcal{O}_{\mathcal{D}_i}) \xrightarrow{\delta} H^1(X, \Omega^1_X) \ldots.$$

Esto nos permite, por un lado deducir la inyectividad de la aplicación residuo bajo la suposición $h^{0,1}_X = 0$, y por otro, realizar en general una descripción de las seciones globales en $H^0(X, \Omega^1_X(\log \mathcal{D}))$. Esto último se encuentra resumido en la proposición 3.4.11.

Luego de finalizar el análisis del caso básico para 1-formas, se estudiará en las siguientes secciones su generalización al caso de formas de grados superiores.

En principio se utilizará la teoría de residuos ya mencionada, nuevamente para el caso de divisores con cruzamientos normales y formas de grados arbitrarios. En este caso si tomamos un divisor efectivo con k componentes irreducibles, notado por: $\mathcal{D} = \bigcup_{j=1}^{k} \mathcal{D}_j$, entonces estos residuos se definen para formas de grado q y dependen de la elección de un multi-índice $I \subset \{1, \ldots, k\}$:

$$Res_I : \Omega^\bullet_X(log(\mathcal{D})) \longrightarrow \Omega^\bullet_{\mathcal{D}_I}(log(\mathcal{D}(I)))[-k].$$

Esta aplicación tendrá su imagen en el sub-haz de formas regulares correspondiente si consideramos una filtración adecuada del haz de formas de logarítmicas, i.e.

$$Res_I : W_k(\Omega^\bullet_X(log(\mathcal{D})) \longrightarrow \Omega^\bullet_{\mathcal{D}_I}[-k]).$$

El resultado principal será nuevamente deducir que una sección global del haz de formas logarítmicas sera nula si y solo todos sus residuos se anulan (ver Teorema 3.5.4). Cabe destacar también que será necesario asumir una condición de anulación cohomológica sobre la variedad, que llameremos a lo largo del capítulo “free of global forms”. Formalmente, pediremos la anulación de los siguientes números de Hodge: $h_X^{p,0} = 0$ para $1 \leq p \leq \dim(X)$.

A modo de ejemplo, si utilizamos este resultado anterior para formas logarítmicas de grado q en el contexto del espacio proyectivo clásico ($X = \mathbb{P}^n_{\mathbb{C}}$), se obtiene que si $f_1, \ldots, f_m$ son polinomios polinomios homogéneos que se cruzan normalmente y $\lambda \in \bigwedge^q \mathbb{C}^m$ es un multivector de constantes, entonces la forma logarítmica descripta en coordenadas homogéneas por:

$$\sum_{I:|I|=q} \lambda_I \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{df_{i_q}}{f_{i_q}} \tag{3.1.1}$$

es completamente nula si y solo si cada componente del multivector se anula: $\lambda_I = 0$.

Además, de la demostración del resultado general se desprende como son exactamente todas las secciones globales del haz de formas logarítmicas de grados superiores. Este hecho, que se encuentra resumido en las proposiciones 3.5.6 y 3.5.9 , resultará importante para caracterizar a las formas logarítmicas proyectivas de mayor grado que definen foliaciones de codimensiones superiores, lo cual será utilizado en el próximo capítulo. Por ejemplo, en el caso del espacio proyectivo, se podrá deducir que todas las secciones globales del haz de formas logarítmicas son del tipo 3.1.1.

Por otro lado, en la última sección, abordaremos el estudio de un resultado similar al lema original en el contexto del espacio afín n-dimensional, pero para formas de grados superiores y sin asumir cruzamientos normales. Se utilizará una hipótesis adecuada sobre la manera en que se cruzan los polinomios en cuestión, que generalice lo asumido en el caso $q = 1$.

En este caso la idea se basará, nuevamente, en considerar un pullback adecuado (como en la demostración original), y utilizar una teoría de residuos adecuada. para deducir la nulidad de las constantes correspondientes. Para esto será introducida la teoría de símbolos (o residuos) de Grothendieck, y las principales propiedades que usaremos. El resultado principal será el siguiente:

Theorem 1. Sea $A = k[x_1, \ldots, x_n]$ y $K = k(x_1, \ldots, x_n)$, y fijemos una m-tupla de polinomios $(F_i)_{i=1}^m \subset A$. Supongamos que para cada multi-índice $I \subset \{1, \ldots, m\}$ de tamaño q, los polinomios $F_{i_1}, \ldots, F_{i_q}$ tienen una solución común. Además, también asumamos que para todo $J \subset \{1, \ldots, m\}$ ahora de tamaño $q + 1$, los polinomios $F_{j_1}, \ldots, F_{j_{q+1}}$ se cortan propiamente (no necesariamente de mandera no vacía). Asimismo, fijemos constantes $\{a_I\}_{I:|I|=q}$. Entonces, una q-forma logarítmica regular del tipo

$$\omega = \sum_{I:|I|=q} a_I \hat{F}_I dF_I = F. \sum_{I:|I|=q} a_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}} \in H^0(k^n, \Omega^q_{k^n})$$

se anula sobre todo el espacio si y solo si $a_I = 0$ para cada I posible.

Como puede observarse, la hipótesis fundamental del resultado para q-formas es que cada vez que seleccionamos q de los polinomios en cuestión se crucen de manera no vacía, y que de a $q+1$ se crucen de manera propia, es decir tan solo con la dimensión correcta. Con respecto al lema original, en el caso $q = 1$, la hipótesis de k algebraicamente cerrado asegura la primera condición necesaria, y el hecho de que sean irreducibles distintos asegura la segunda (los cruzamientos propios).

3.2 Summary

In this chapter, we develop distinct versions of an important lemma for logarithmic forms due to Jouanolou (see [35]), used for studying algebraic solutions of projective Pfaff equations. This result is going to be generalized to logarithmic forms of arbitrary degrees on classical affine and projective varieties. Moreover, we use these techniques to exhibit the global sections of the well-known sheaf of logarithmic forms. This last fact is going to be important in the next chapter to define correct formulas for regular logarithmic forms that represent foliations of higher codimension.

3.3 The sheaf of logarithmic forms: definitions and residues

First, we recall the principal definitions and properties of the sheaf of logarithmic forms of arbitrary degrees and its classical residues. It is remarkable that this sheaf considered on the variety $\mathbb{P}^n$ has been already used at the previous chapter, specially at section 2.3. In the sake of clarity and in order to present a self-contained development, we make a brief resume of the topic.

Fix a smooth complex algebraic variety X of dimension n and let $\mathcal{D} = \sum_{i=1}^{m} \mathcal{D}_i$ be an effective divisor defined over X, where each $\mathcal{D}_i$ is a non singular irreducible component of $\mathcal{D}$.

Recall that the sheaf of logarithmic q-forms can be defined as a subsheaf of the sheaf of meromorphic forms with poles on $\mathcal{D}$. Concretely, we write $i : UX - \mathcal{D} \hookrightarrow X$, and consider:

$$\Omega^q_X(*\mathcal{D}) = \varinjlim_k \Omega^q_X(k \cdot \mathcal{D}) = i_*(\Omega^q_U).$$

Then, for every open set $V \subset X$, we consider:

$$\Omega^q_X(\log \mathcal{D})(V) = \{\alpha \in \Omega^q_X(*\mathcal{D})(V) : \alpha \text{ and } d\alpha \text{ has simple poles along } \mathcal{D}\}.$$

Moreover, this subsheaf determines a subcomplex:

$$(\Omega^\bullet_X(\log \mathcal{D}), d) \hookrightarrow (\Omega^\bullet_X(*\mathcal{D}), d). \tag{3.3.1}$$

From now on, unless otherwise stated, $\mathcal{D}$ is assumed to be simple normal crossing.

Proposition 3.3.1. Some properties of $\Omega^\bullet_X(\log \mathcal{D})$ are:

i. As a sheaf: $\Omega^q_X(\log \mathcal{D}) = \bigwedge^q \Omega^1_X(\log \mathcal{D})$.

ii. $\Omega^q_X(\log \mathcal{D})$ is a locally free sheaf of rank $\binom{n}{q}$.

Proof. See [22] for a complete proof. In addition, we want to emphasize some aspects of this proof. In particular, we can describe a free system of generators of $\Omega^q_X(\log \mathcal{D})$ at a given point $x \in X$. Fix local coordinates $f_1, \ldots, f_n$ in p, and assume the divisor is defined by the zero locus of $f_1, \ldots, f_s$. Then $\frac{df_1}{f_1}, \ldots, \frac{df_s}{f_s}, df_{s+1}, \ldots, df_n$ is a free system of generators of $\Omega^1_X(\log \mathcal{D})_p$. Furthermore, a basis for the case $q > 1$ can be performed using i). □

One of the most important results related to this subsheaf is that the natural inclusion 3.3.1 is a quasi-isomorphism. This result is due to Deligne (see for instance [18]), and its corresponding proof depends on a well defined residue theory. See [18] or [48] for more details. The next objective is to summarize some principal aspects of these residues.

As an introduction, fix $p \in X$ and V an open neighborhood with coordinates $(f_1, \ldots, f_n)$ in which $\mathcal{D}$ has equation $f_1 = \cdots = f_s = 0$ (where also each component $\mathcal{D}_k$ corresponds to $(f_k = 0)$). According to the description stated at the proof of the above proposition, every local section $\eta \in \Omega^q_X(\log \mathcal{D})(V)$ can be decomposed as:

$$\eta = \eta' \wedge \frac{df_k}{f_k} + \mu, \tag{3.3.2}$$

with η' and μ not containing df_k in its representation. We write $i_k : \mathcal{D}_k \hookrightarrow X$, and denote by $\mathcal{D}(k)$ the divisor on $\mathcal{D}_k$ traced out by $\mathcal{D}$. In this context, we are able to present the usual definition of the residue map over the component $\mathcal{D}_k$.

Definition 3.3.2. With the previous notation. For every local section η of $\Omega^q_X(\log \mathcal{D})$, the element $\eta'|_{\mathcal{D}_k}$ is well defined and does not depend on the representation given. This gives rise to a well define residue map:

$$\begin{aligned} res^q_k : \Omega^q_X(\log \mathcal{D}) &\longrightarrow \Omega^{q-1}_{\mathcal{D}_k}(\log \mathcal{D}(k)) \\ \eta &\longmapsto \eta'|_{\mathcal{D}_k}. \end{aligned}$$

As a special case we get the residues $res_k : \Omega^1_X(\log \mathcal{D}) \to \mathcal{O}_{\mathcal{D}_k}$, that we have already introduced at 2.3.9 in chapter 2, which are usually named as the Poincaré's residues of the form.

For these introduced residues, we have the following exact sequences of sheaves, including the described at 2.3.2. For more details, see for instance [22].

Proposition 3.3.3. The following sequences are exact.

- $0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log \mathcal{D}) \xrightarrow{\oplus res_j} \bigoplus_{j=1}^m \mathcal{O}_{\mathcal{D}_j} \longrightarrow 0$
- $0 \longrightarrow \Omega^{q-1}_X(\log \mathcal{D} - \mathcal{D}_k) \longrightarrow \Omega^q_X(\log \mathcal{D}) \xrightarrow{res^q_k} \Omega^{q-1}_{\mathcal{D}_k}(\log \mathcal{D}(k)) \longrightarrow 0$

Now, we want to describe a correct background to understand a generalization of the above residue theory and the previous exact sequences. The subsequent constructions are given with more details at chapter II of [18]. We begin with the description of the well-known Deligne's filtration of the sheaf $\Omega^q_X(\log \mathcal{D})$. Consider

$$W_m(\Omega^q_X(\log \mathcal{D})) = \begin{cases} 0 & \text{if } m < 0 \\ \Omega^{q-m}_X \wedge \Omega^m_X(\log \mathcal{D}) & \text{if } 0 \le m \le q \\ \Omega^q_X(\log \mathcal{D}) & \text{if } m \ge q \end{cases} \tag{3.3.3}$$

Next, we present the notation for a correct description of the required residues. We write

- $\mathcal{D}_I = \mathcal{D}_{i_1} \cap \cdots \cap \mathcal{D}_{i_k}$ for $I = \{i_1 \ldots i_k\}$
- $\mathcal{D}(I) = \sum_{j \notin I} \mathcal{D}_i \cap \mathcal{D}_j$ view as a divisor on $\mathcal{D}_I$
- $j_I = \mathcal{D}_I \hookrightarrow X$

and set forth:

- $\mathcal{D}^k = \coprod_{I:|I|=k} \mathcal{D}_I$
- $j_k = \mathcal{D}^k \hookrightarrow X$

Since $\mathcal{D}$ is a simple normal crossing divisor each $\mathcal{D}_I$ is a complete intersection subvariety of X of codimension $|I|$. Also, each $\mathcal{D}^k$ corresponds to the normalization of the union of these subvarieties.

Now we are able to define the announced residues for each multi-index I with $|I| = k$:

$$Res_I : \Omega^{\bullet}_X(log(\mathcal{D})) \longrightarrow \Omega^{\bullet}_{\mathcal{D}_I}(log(\mathcal{D}(I)))[-k]$$

The local definition of these map is very similar to the used at 3.3.2. So we will keep the same notation. Fix $I = \{i_1, \ldots, i_k\}$ and set $(f_1, \ldots, f_n)$ local coordinates at $p \in \mathcal{D}_I$, with $\mathcal{D}_{i_j} = (f_j = 0)$. Then, any local section η of $\Omega^q_X(\log \mathcal{D})$ can be described by:

$$\eta = \eta' \wedge \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} + \mu,$$

where η' has at most simple poles along $\mathcal{D}_j$ with $j \notin I$, and μ is not divisible by $\frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m}$. The assignment

$$\eta \in \Omega^q_X(log(\mathcal{D})) \longmapsto \eta'|_{\mathcal{D}_I} \in \Omega^{q-k}_{\mathcal{D}_I}(log(\mathcal{D}(I)))$$

is well defined and not depends on the local coordinates chosen. This construction corresponds to the definition of the residue Res_I.

Remark 3.3.4. If μ is a local section of $\Omega^p_X(log(\mathcal{D}))$ with weight less or equals to k, and I is a multi-index of size k, then $Res_I(\mu)$ is a regular form of the subvariety $\mathcal{D}_I$. So, the previously defined residue map restricts to:

$$Res_I : W_k(\Omega^{\bullet}_X(\log \mathcal{D}) \longrightarrow \Omega^{\bullet}_{\mathcal{D}_I}[-k]).$$

In addition, if μ has weight at most $k-1$, its residue vanish (η coincides with μ in the above decomposition). According to these remarks the corresponding residue map is well defined on the quotient $Gr^W_k(\Omega^{\bullet}_X(\log \mathcal{D}))$.

The next result corresponds to the generalization of the exact sequences exhibit at proposition 3.3.3, and is the key to calculate the global sections of the sheaf of logarithmic q-forms. Moreover, it would be a first approach to generalize the Jouanolou's lemma for logarithmic forms of higher degree.

Proposition 3.3.5. Every residue map Res_I defined over $Gr^W_k(\Omega^{\bullet}_X(\log \mathcal{D}))$ is surjective, and the total residue map

$$Res_k := \bigoplus_{I:|I|=k} Res_I : Gr^W_k(\Omega^{\bullet}_X(\log \mathcal{D})) \longrightarrow (j_k)_*(\Omega^{\bullet}_{\mathcal{D}^k}[-k])$$

is an isomorphism.

Proof. See [48, Lemma 4.6], or [18] for an extended overview. □

3.4 Jouanolou's lemma: the case of 1-forms

Let us start reminding the basic lemma used by J.P Jouanolou to prove a characterization of when a Pfaff equation admits infinite algebraic solutions, and other related facts about algebraic Pfaff equations (see [35, Lemma 3.3.1]). Also, this lemma was the key result to explore the base locus of the natural parametrization of the logarithmic 1-forms of type **d** in chapter 2 (see 2.4.12).

Lemma 3.4.1. Set the notation $A = k[x_1, \dots, x_n]$ and $K = k(x_1, \dots .x_n)$, and let P be a system of irreducible elements of A. Suppose given elements $\{\lambda_i\}_{i=1}^{s} \subset k$ and $\{f_i\}_{i=1}^{s} \subset P$ such that:

$$\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i} = 0 \in \Omega^1_{K/k} = \Omega^1_{A/k} \otimes_A K$$

then necessarily $\lambda_i = 0$ for all $i = 1 \dots s$.

In order to prepare the background for the subsequent generalizations, we describe the original proof of the statement.

Proof. Assume handed out a relation:

$$\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i} = 0 \ \ (\lambda_i \in k, f_i \in P)$$

It is sufficient to show that $\lambda_1 = 0$. Consider the hypersurfaces Y_i with equations $(f_i = 0)$ in the affine n-dimensional space $\mathbb{A}^n$, and fix a point $a \in Y_1 - \bigcup_{j \neq 1} Y_j$. This is possible because the polynomials are irreducible and distinct. Now, take an affine line D passing through a and not contained in Y_1. The basic idea is to consider the pull back of our equation to $\mathbb{A}^1$. For this purpose, fix a parametrization of D:

$$\phi : \mathbb{A}^1 \to \mathbb{A}^n \quad \text{with} \quad \phi(0) = a.$$

Then, if we write $g_i = \phi^*(f_i) = f_i \circ \phi$, it is clear that the following equation holds:

$$\phi^*(\frac{df_i}{f_i}) = \frac{g_i'}{g_i} dt.$$

According to the relation given, it follows that:

$$\lambda_1 \frac{g_1'}{g_1} + \lambda_2 \frac{g_2'}{g_2} + \cdots + \lambda_s \frac{g_s'}{g_s} = 0.$$

Since $a \in Y_1 - \bigcup_{j \neq 1} Y_j$, it is also clear that the functions g_i'/g_i are all regular at 0 (for $i \neq 1$), and the logarithmic function g_1'/g_1 has a simple pole. Finally, take the usual residue to the above equation to conclude

$$res(g_1'/g_1, 0)\lambda_1 = 0,$$

and $\lambda_1 = 0$ as claimed. □

With exactly the same idea, it is possible to state the same result for algebraic affine varieties.

Corollary 3.4.2. Let X be an affine algebraic variety, which is non-singular and rational. We write A for its associated coordinated ring, and K for its rational function field. Also let P be a system of irreducible elements of A, and select constants $(\lambda_i)_{i=1}^s \subset k$ and distinct elements $(f_i)_{i=1}^s \subset P$ fulfilling:

$$\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i} = 0 \in \Omega^1_{A/k} \otimes_A K,$$

then $\lambda_i = 0$ for $i = 1, \ldots, s$.

Proof. The argument is the same as the used in the previous proposition. Note that an element $a \in Y_1 - \bigcup_{j\neq 1} Y_j$ exists because the irreducible hypersurfaces are all distinct. Also, the selection of the affine line D passing through a and not contained in Y_1 depends on a dimension argument and the fact that the variety is rational. The end of the proof follows in the same way as before. □

Now it is possible to describe a slightly more general result using the residues introduced in the preceding section. Moreover, it is substantial for understanding the global sections of the logarithmic sheaf for affine varieties and also set the background to possible generalizations.

Proposition 3.4.3. Take a simple normal crossing effective divisor $\mathcal{D} = \sum_{i=1}^m \mathcal{D}_i$ over $\mathbb{C}^n$, where each component $\mathcal{D}_i$ is defined by the zero locus of an irreducible polynomial $f_i \in \mathbb{C}[x_1, \ldots, x_n]$. For every global logarithmic form $\eta \in H^0(\mathbb{C}^n, \Omega^1_{\mathbb{C}^n}(\log \mathcal{D}))$, there exist polynomials $g_1, \ldots, g_m$ and a regular form $\mu \in H^0(\mathbb{C}^n, \Omega^1_{\mathbb{C}^n})$ such that:

$$\eta = \sum_{i=1}^{m} g_i \tfrac{df_i}{f_i} + \mu.$$

In addition each polynomial g_i is unique considered modulo f_i, i.e. the class $[g_i] \in C[x_1, \ldots, x_n]/_{\langle f_i \rangle}$ only depends on η.

Proof. It is a consequence of the first short sequence described at 3.3.3. Consider the long exact sequence on cohomology associated to obtain:

$$0 \longrightarrow H^0(\mathbb{C}^n, \Omega^1_{\mathbb{C}^n}) \longrightarrow H^0(\mathbb{C}^n, \Omega^1_{\mathbb{C}^n}(\log \mathcal{D})) \xrightarrow{\oplus res_k} \bigoplus_{k=1}^{m} \mathbb{C}[x_1, \ldots, x_n]/_{\langle f_k \rangle} \longrightarrow 0$$

Next, for every logarithmic form η whose residues correspond to the classes $\{[g_i]\}_{i=1}^m$, it follows that η and $\sum_{i=1}^m g_i \frac{df_i}{f_i}$ has the same residues, and so their difference is a regular 1-form as claimed. □

Let us state some useful consequences of the above proposition.

Corollary 3.4.4. [*Jouanolou's lemma for 1-forms - Normal crossing version for the affine space*] Let $f_1, \ldots f_m \in \mathbb{C}[x_1, \ldots, x_n]$ be irreducible polynomials with normal crossings, and suppose given the relation:

$$\sum_{i=1}^{m} g_i \tfrac{df_i}{f_i} = 0,$$

for some polynomials $g_1, \ldots, g_m$. Then necessarily: $[g_i] = 0 \in \mathbb{C}[x_1, \ldots, x_n]/_{\langle f_i \rangle}$ for all i.

Remark 3.4.5. With the notation of the previous corollary, suppose given a relation like:

$$\sum_{i=1}^{m} g_i \frac{df_i}{f_i} = 0.$$

Due to the preceding result each g_i belongs to the ideal $\langle f_i \rangle$, i.e. $g_i = q_i \cdot f_i$ (for $i = 1, \dots, m$) and we can deduce:

$$\sum_{i=1}^{m} q_i \frac{\partial f_i}{\partial x_j} = 0 \;\; \forall j = 1, \dots, n.$$

With matrix notation, we get

$$(q_i) \cdot Jf = 0,$$

where Jf denotes the Jacobian matrix of the multi-valuated function $f = (f_1, \dots, f_n)$.

Remark 3.4.6. The proposition 3.4.3 and its consequences can be stated for an affine variety over $\mathbb{C}$. The difference is we need to replace the ring of polynomials by the corresponding coordinate ring of the introduced variety. To keep the same proofs, we assume that this coordinate ring is a DFU. This condition ensures that any divisor can be defined by the zero loci of a single regular global function. Also, the assumption is very restrictive and is equivalent to require the variety to be normal and with trivial class group (see for instance [31, Propositions 1.12 and 6.2]).

Finally, we describe the same problem on classical projective varieties. Note that the statement presented at 2.3.10, which describe the global sections of the sheaf of logarithmic 1-forms on $\mathbb{P}^n$, also corresponds to a version of the above Jouanolou's lemma for such projective space.

Proposition 3.4.7. Let $F_1, \dots F_m \in \mathbb{C}[x_0, \dots, x_n]$ be irreducible homogeneous polynomials with respective degrees $d_1, \dots, d_m$. Assume that the divisor on $\mathbb{P}^n$ defined by $\mathcal{D} = \sum_{i=1}^{m}(F_i = 0)$ has normal crossing. Every element $\eta \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(log(\mathcal{D})))$ can be described in homogeneous coordinates by:

$$\eta = \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i}$$

for some vector $\lambda \in \mathbb{C}^m_{\mathbf{d}} = \{\lambda \in \mathbb{C}^m : \lambda \cdot \mathbf{d} = 0\}$. Moreover the correspondence $\lambda \mapsto \eta$ is bijective.

Corollary 3.4.8 (*Jouanolou's lemma for 1-forms - Normal crossing version for the projective space*)**.** With the notations above, consider the relation:

$$\sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = 0 \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\log \mathcal{D})) \subset H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(*\mathcal{D})),$$

for some constants $\lambda_1, \dots, \lambda_m$ satisfying $\lambda_1 d_1 + \cdots + \lambda_m d_m = 0$. Then necessarily: $\lambda_i = 0$ for all i.

Remark 3.4.9. The last result can be thought as a particular case of the Jouanolou's lemma for the affine space 3.4.4.

Remark 3.4.10. The previous results correspond to a description in homogeneous coordinates of the long exact sequence in cohomology associated to 3.3.3. Also they are based on describe the irreducible divisors by the zero locus of an homogeneous element of the graded coordinate ring.

From now on, consider X as a smooth complex projective variety (assumed irreducible and embedded on $\mathbb{P}^N$), defined by the zero locus of an homogeneous ideal $\mathcal{I}_X \subset \mathbb{C}[X_o, \dots, X_N]$. Set S_X its associated graded homogeneous coordinate ring and $\mathbb{C}(X)$ its field of rational functions. Fix again a simple normal crossing effective divisor $\mathcal{D} = \sum_{i=1}^m \mathcal{D}_i$ on X. Our goal now is to describe the global sections of $\Omega^1_X(\log \mathcal{D})$ without taking into consideration homogeneous coordinates.

With the same idea as in the first chapter (see for example proposition 2.3.2), fix global sections

$$F_i \in H^0(X, \mathcal{O}_X(\mathcal{D}_i)) \; i = 1, \dots, m,$$

of the associated line bundles, and construct global logarithmic forms.

In this case, the long exact sequence on cohomology associated to 3.3.3 can be described by:

$$0 \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^0(X, \Omega^1_X(\log \mathcal{D})) \xrightarrow{\oplus res_k} \bigoplus_{i=1}^m H^0(\mathcal{D}_i, \mathcal{O}_{\mathcal{D}_i}) \xrightarrow{\delta} H^1(X, \Omega^1_X) \dots$$

Since $H^0(\mathcal{D}_i, \mathcal{O}_{\mathcal{D}_i}) \simeq \mathbb{C}$, the residues are again constants. In addition, the dimension of the global logarithmic forms will depend on the Hodge numbers: $h^{1,0}_X$ and $h^{1,1}_X$.

Proposition 3.4.11. With the same notation as above, the global sections $\eta \in H^0(X, \Omega^1_X(\log \mathcal{D}))$ can be characterized by:

$$\eta = \sum_{i=1}^m \lambda_i \frac{dF_i}{F_i} + \omega,$$

where ω is a global regular 1-form on X. Moreover, note that the residues $\lambda = (\lambda_i)$ are constants in $\ker(\delta)$ (i.e they satisfy the $h^{1,1}_X$ equations imposed by δ) and only depend on η.

Proof. Let us introduce some notation to understand the description made at the result. Because, a priori, the notation $\frac{dF_i}{F_i}$ seems to be confusing. The idea is similar to that used at 2.3.2. We state some expected global sections of $\Omega_X(\log \mathcal{D})$, and use the above long exact sequence (or a dimension count argument) to show that they are all the possible sections.

Set $\mathcal{U} = \{U_\alpha\}$ as an open covering of X, where each $\mathcal{D}_i \cap \mathcal{U}_\alpha$ is defined by a regular function $f_i^\alpha \in \mathcal{O}(U_\alpha)$, this is:

$$\mathcal{D}_i|_{U_\alpha} = (f_i^\alpha).$$

Indeed, we have isomorphisms of sheaves (trivializations) of $\mathcal{O}_{U_\alpha}$-modules

$$s^i_\alpha : \mathcal{O}_X(\mathcal{D}_i)|U_\alpha = \frac{1}{f_i^\alpha}\mathcal{O}_{U_\alpha} \xrightarrow{\cong} \mathcal{O}_{U_\alpha}$$
$$g \longmapsto g\, f_i^\alpha,$$

and also, $s^i_{\alpha\beta} = s^i_\alpha \circ (s^i_\beta)^{-1} = \frac{s^i_\alpha}{s^i_\beta}$ is the cocycles associated the line bundle $\mathcal{O}_X(\mathcal{D}_i)$ for every selected index. Moreover, each global section $F_i \in H^0(X, \mathcal{O}(\mathcal{D}_i))$ can be described in local coordinates by $s^i_\alpha(F_i|_{U_\alpha}) = h^\alpha_i f^\alpha_i$. Just for simplicity we will consider $h^\alpha_i = 1$. This section can be thought as the rational function $1 \in \mathbb{C}(X)$, in the interpretation of $H^0(X, \mathcal{O}_X(\mathcal{D}_i))$ as a subgroup of the rational field $\mathbb{C}(X)$. In other words, consider $F_i = \{s^i_\alpha(F_i|_{U_\alpha}) = f^\alpha_i\}$ which trivially satisfy the cocycle conditions imposed by $\{s^i_{\alpha\beta}\}$.

Then, every form of the type:

$$\mu = \sum_{i=1}^m \lambda_i \frac{dF_i}{F_i} \in H^0(X, \Omega^1_X(\log \mathcal{D})) \hookrightarrow H^0(X, \Omega^1_X(\mathcal{D}), \tag{3.4.1}$$

is well defined as a rational form. The proof of this fact is similar to that used at 2.3.2, and can be summarized by the following argument:

$$(\mu|_{U_\alpha})|_{U_\beta} = \sum_{i=1}^m \lambda_i \frac{df^\alpha_i}{f^\alpha_i} =$$

$$= \sum_{i=1}^m \lambda_i \frac{d\left(s^i_{\alpha\beta} f^\beta_i\right)}{s^i_{\alpha\beta} f^\beta_i} = \sum_{i=1}^m \lambda_i \frac{ds^i_{\alpha\beta}}{s^i_{\alpha\beta}} + \sum_{i=1}^m \lambda_i \frac{df^\beta_i}{f^\beta_i} =$$

$$= \sum_{i=1}^m \lambda_i \frac{df^\beta_i}{f^\beta_i} = (\mu|_{U_\beta})|_{U_\alpha}$$

So we can think on μ as the form determined by the family of local forms $\{\mu_\alpha = \sum_{i=1}^m \lambda_i \frac{df^\alpha_i}{f^\alpha_i}\}_\alpha$, which are forms that coincide in every possible intersection. In this case, we are using again that the constants λ must satisfy the conditions:

$$\sum_{i=1}^m \lambda_i \frac{ds^i_{\alpha\beta}}{s^i_{\alpha\beta}} = 0,$$

previously named by $\mathcal{O}(\mathcal{D})$-conditions. These are the usual conditions satisfied for the residues of a logarithmic form (See for instance the introduction of [3]). Now, since $s^i_{\alpha\beta} = \frac{s^i_\alpha}{s^i_\beta}$, they can be reinterpreted as:

$$\sum_{i=1}^m \lambda_i \left(\frac{df^\alpha_i}{f^\alpha_i} - \frac{df^\beta_i}{f^\beta_i} \right) = 0.$$

This last argument describe the kernel of the connection morphism δ. In addition, since each component $\mathcal{D}_i$ is locally defined by $\{f^\alpha_i\}$, it is clear than the residues of μ are $(\lambda_i)_{i=1}^m$. Finally, suppose given a global logarithmic form $\eta \in H^0(X, \Omega^1(\log \mathcal{D}))$ whose residues corresponds to $\lambda = (\lambda_i)_{i=1}^m$, then it is clear that the constructed forms μ and η has the same residues, and so $\eta - \mu$ is in the kernel of the total residue map, and hence is a regular global form of X as claimed. □

Corollary 3.4.12. [*Jouanolou's lemma for 1-forms - Normal crossing version for smooth complex projective varieties*] Assume $h_X^{1,0} = 0$ and consider the relation:

$$\sum_{i=1}^{m} \lambda_i \tfrac{dF_i}{F_i} = 0 \in H^0(X, \Omega_X^1(\log \mathcal{D})) \subset H^0(X, \Omega_X^1(*\mathcal{D})),$$

for some constants $\lambda \in \ker(\delta)$. Then necessarily: $\lambda_i = 0$ for all i.

Corollary 3.4.13. In addition, if $h_X^{1,0} = 0$ then $H^0(X, \Omega_X^1(\log \mathcal{D})) \simeq \ker(\delta) \subset \mathbb{C}^m$.

Remark 3.4.14. In a more restrictive setting, $\mathcal{D}_i$ could be regarded as the zero locus of an homogeneous irreducible element $F_i \in (S_X)_{d_i}$. Assume again that $h_X^{1,0} = 0$.

In this case, towards to describe everything in the homogeneous coordinates determined by the fixed embedding on $\mathbb{P}^N$, every global logarithmic form η can be described by:

$$\eta = \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i},$$

where now, the conditions imposed by $\ker(\delta)$ can be thought as the "descend" equations for the form η (a priori considered in the cone over X) to be well defined.

We end this section with some comments on the hypotheses that appeared.

Proposition 3.4.15. Every complete intersection in a projective space and every unirational variety has the property: $h^{1,0} = 0$.

Proof. The first remark is a consequence of the Lefschetz hyperplane theorem (See for instance 3.1.B at [38]). On the other hand, since $h^{1,0}$ is a birrational invariant, the condition clearly holds for rational varieties. The argument for the unirational case depends on the inclusion $f^*\Omega_X^m \to \Omega_U^m$ (where U denotes the open set where is defined a generically finite dominant map from $\mathbb{P}^n$ to our variety X). When the map is birational, this inclusion is obvious. When the map is only finite, this is only true in characteristic zero. □

Remark 3.4.16. Clearly, not every irreducible effective divisor can be described by the zero locus of an homogeneous element in S_X. This will be true if we assume for example that S_X is a UFD (which is the case of $\mathbb{P}^n$). The condition is equivalent to assume X to be projectively normal and with class group $cl(X) \simeq \mathbb{Z}$ (see [31, Exercise 6.3]).

Remark 3.4.17. For a local version of these results on a complete factorial algebraic variety see section *II* of [7].

3.5 The case of higher degree logarithmic forms

Through this lecture we attend the problem of extend the results of the previous section to logarithmic forms of an arbitrary degree. The aim is to understand the global section of the sheaf of logarithmic q-forms defined at section 3.3, and also state a version of the Jouanolou's lemma for these type of forms.

3.5.1 Using Deligne's filtration and residues

In this section we will use again the Deligne's residues to describe the global sections of the sheaf of logarithmic forms over a smooth complex projective algebraic variety X of dimension n.

We will keep the same notation and definitions used at the first section 3.3. We also assume that the effective divisor $\mathcal{D} = \sum_{i=1}^{m} \mathcal{D}_i$ is simple normal crossing. In addition, we need to pick a family of global sections:

$$F_i \in H^0(X, \mathcal{O}(\mathcal{D}_i)) \; i = 1, \ldots, m$$

of the line bundles associated to the smooth irreducible components $\{\mathcal{D}_{i_s}\}$.

Alternatively, we might suppose that each $\mathcal{D}_i$ is defined by the zero locus of an homogeneous element of a coordinate ring S_X, in order to describe the desired sections in homogeneous coordinates. We avoid from now the problem of when this last fact happens for every effective irreducible divisor, and left this discussion to the description made at the end of the previous section.

As in the case of 1-forms, the results are simpler if we consider certain vanishing hypothesis on the ambient variety.

Definition 3.5.1. We say that X is free of global forms if:

$$h_X^{p,0} = dim(H^0(X, \Omega_X^p)) = 0 \qquad \text{for} \quad 1 \leq p \leq n-1.$$

Remark 3.5.2. Since the Hodge numbers $h_X^{p,0}$ are birrational invariants the previous property holds on every rational variety. Moreover according to the description made at the proposition 3.4.15 it is also true for unirational varieties.

Now, we show that if X is free of global forms then the same holds for every complete intersection subvariety. In particular this is going to be true for every piece of the strata induced by a normal crossing divisor.

Proposition 3.5.3. Assume X is free of global forms. With the notation of the section 3.3, for each multi-index I of size k (with $k \leq n-2$) the following vanishing property holds:

$$h_{\mathcal{D}_I}^{q,0} = dim(H^0(\mathcal{D}_I, \Omega_{\mathcal{D}_I}^q)) = 0 \qquad \text{for} \quad 1 \leq q \leq n-k-1,$$

i.e. each $\mathcal{D}_I$ is free of global forms.

Proof. According to the Lefschetz hyperplane theorem for Hodge groups (see [38]), if Y is a non-singular ample effective divisor on X, then the restriction map:

$$\gamma_{p,q} : H^q(X, \Omega_X^p) \longrightarrow H^q(Y, \Omega_Y^p)$$

is an isomorphism for $p+q \leq n-2$. This implies that our result is true for every smooth hypersurface of X. Therefore the proposition's proof ends by applying this last observation one to one for each subvariety $F_{i_k} = 0$ viewed as a smooth hypersurface of $\mathcal{D}_{\{i_1,\ldots,i_{k-1}\}}$.

□

At this time we are able to describe a general statement for logarithmic q-forms. The result can be thought as a generalization of 3.4.12.

Theorem 3.5.4. Assume that X is free of global forms. Then for a global logarithmic q-form $\eta \in H^0(X, \Omega^q_X(\log \mathcal{D}))$ with $1 \leq q \leq min\{n-1, m\}$, the following conditions are equivalent:

a) $\eta = 0 \quad \in H^0(X, \Omega^q_X(\log \mathcal{D}))$, the form equals the zero section.

b) $Res_I(\eta) = 0 \quad \in H^0(\mathcal{D}_I, \mathcal{O}_{\mathcal{D}_I})$, for all multi-index I with $|I| = q$.

In other words, the statement ensures that the total residue map Res_q is injective in global sections.

Proof. Using the result stated at 3.3.5, the following exact sequence of sheaves holds:

$$0 \longrightarrow W_{q-1}(\Omega^q_X(\log \mathcal{D})) \longrightarrow \Omega^q_X(\log \mathcal{D}) \longrightarrow (j_q)_*(\Omega^0_{\mathcal{D}^q}) \longrightarrow 0.$$

Also, observe that the global sections of the right term are described by the structure sheaf of the subvarieties $\mathcal{D}_I$ with multi-index of size q:

$$H^0(X, (j_q)_*(\Omega^0_{\mathcal{D}^q})) = \bigoplus_{\substack{I \subset \{1,\dots,m\}: \\ |I|=q}} H^0(\mathcal{D}_I, \mathcal{O}_{\mathcal{D}_I}).$$

Next consider the corresponding long exact sequence on cohomology to obtain:

$$(3.5.1) \qquad 0 \to H^0(X, W_{q-1}(\Omega^q_X(log\mathcal{D}))) \to H^0(X, \Omega^q_X(log\mathcal{D})) \xrightarrow{\oplus Res_I} \bigoplus_{I:|I|=q} H^0(\mathcal{D}_I, \mathcal{O}_{\mathcal{D}_I}) \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} H^1(X, W_{q-1}(\Omega^q_X(log\mathcal{D}))) \longrightarrow \cdots$$

To complete the proof it is sufficient to show $H^0(X, W_{q-1}(\Omega^q_X(log\mathcal{D}))) = 0$. For this purpose we will perform a more general statement:

$$(3.5.2) \qquad H^0(X, W_j(\Omega^q_X(log\mathcal{D}))) = 0 \quad \forall j = 0 \dots q-1,$$

and proceed by induction on j. The first case is connected with the assumption of X to be free of global forms, i.e.:

$$H^0(X, W_0(\Omega^q_X(\log \mathcal{D}))) = H^0(X, \Omega^q_X) = 0$$

Now assume the statement is proved for $j-1$. Due to the proposition 3.3.5, if we take into consideration the exact sequence:

$$0 \longrightarrow W_{j-1}(\Omega^q_X(\log \mathcal{D})) \longrightarrow W_j(\Omega^q_X(\log \mathcal{D})) \longrightarrow (j_j)_*(\Omega^{q-j}_{\mathcal{D}^j}) \longrightarrow 0,$$

Then, the desired result is obtained by looking at the associated long exact sequence on cohomology and by the remark:

$$H^0(X, (j_j)_*(\Omega^{q-j}_{\mathcal{D}^j})) = \bigoplus_{I:|I|=j} H^0(\mathcal{D}_I, \Omega^{q-j}_{\mathcal{D}_I}) = 0 \quad \forall j = 1 \dots q-1.$$

Note that the last assertion is an immediate consequence of the proposition 3.5.3. □

Remark 3.5.5. Note that in the last proof we have only used that: $h_X^{j,0} = 0$ for $j = 1, \ldots, q$. The idea behind this fact is the following. According to the Lefschetz hyperplane theorem, if Y is a smooth complete intersection in X of codimension k, the following equality for their Hodge numbers holds:

$$h_X^{j,0} = h_Y^{j,0} \quad \forall j < n-k.$$

Global sections of the sheaf of logarithmic forms:

According to the previous development, we are able to compute the global section of the sheaf of logarithmic forms. It will be be an immediate consequence of the cohomology argument made at the proof of 3.5.4. Specially, it follows from the exact sequence 3.5.1 and the vanishing cohomology deduced from 3.5.2.

As in the proof of proposition 3.4.11, it will be important to remain fixed the global sections $F_i \in H^0(X, \mathcal{O}_X(\mathcal{D}_i))$ (for every $i \in \{1, \ldots, m\}$. Furthermore, if we denote by $f_i^\alpha \in \mathcal{O}_X(U_\alpha)$ the image of $F_i|_{U_\alpha}$ by a trivialization map on an open covering $\mathcal{U}$, it is going to be assumed that each regular function f_i^α defines locally the component $\mathcal{D}_i|_{U_\alpha}$. In addition, remember that the non vanishing functions $s^i_{\alpha\beta} = \frac{f_i^\alpha}{f_i^\beta}$ define the cocycles of the line bundles $\mathcal{O}_X(\mathcal{D}_i)$.

Proposition 3.5.6. For every global section of the sheaf of logarithmic forms

$$\eta \in H^0(X, \Omega_X^q(\log \mathcal{D})),$$

there exist constants $\lambda = (\lambda_I)_I$ such that η can be described by the form:

$$\eta = \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=q}} \lambda_I \frac{dF_I}{F_I} = \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=q}} \lambda_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}},$$

assuming that λ satisfy certain equations imposed by the kernel of the map δ introduced at 3.5.1. Moreover, there is a one to one correspondence with these type of constants.

Proof. The idea of the proof is the same as that used in proposition 2.3.2. We will described certain forms with the desired properties and check they correspond to all the possible global sections of the sheaf of logarithmic q-forms.

With this purpose in mind, recall the long exact sequence deduced from 3.5.1 and 3.5.2:

$$0 \to H^0(X, \Omega_X^q(log \mathcal{D})) \xrightarrow{\oplus Res_I} \bigoplus_{I:|I|=q} H^0(\mathcal{D}_I, \mathcal{O}_{\mathcal{D}_I}) \xrightarrow{\delta} H^1(X, W_{q-1}(\Omega_X^q(log \mathcal{D}))) \longrightarrow \cdots \tag{3.5.3}$$

Since X is a smooth projective complex variety and each $\mathcal{D}_i$ is a smooth projective irreducible divisor, it is clear that $H^0(\mathcal{D}_I, \mathcal{O}_{\mathcal{D}_I}) \simeq \mathbb{C}$ for every selected multi-index I.

With a slight abuse of notation, if $\lambda \in \mathbb{C}^{\binom{m}{q}}$ satisfies the equations imposed by the kernel of δ then the rational form:

$$\mu = \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I|=q}} \lambda_I \frac{dF_I}{F_I} = \sum_{\{i_1 < \cdots < 1_q\}} \lambda_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}}$$

is well defined on X and its residues equals to λ. Now, we prove that μ is well defined, so consider the local forms determined in U_α by the formula

$$\sum_{i_1<\dots i_q} \lambda_I \frac{f_i^\alpha}{f_i^\alpha} \wedge \cdots \wedge \frac{df_i^\alpha}{f_i^\alpha},$$

which by definition coincides exactly with $\mu|_{U_\alpha}$. Then, in every intersection $U_{\alpha\beta}$, the following equality holds:

$$(\mu|_{U_\alpha})|_{U_\beta} = \sum_{i_1<\dots<i_q} \lambda_I \frac{df_{i_1}^\alpha}{f_{i_1}^\alpha} \wedge \cdots \wedge \frac{df_{i_q}^\alpha}{f_{i_q}^\alpha} =$$

$$= \sum_{i_1<\dots<i_q} \lambda_I \frac{d(s_{\alpha\beta}^{i_1} f_{i_1}^\alpha)}{s_{\alpha\beta}^{i_1} f_{i_1}^\alpha} \wedge \cdots \wedge \frac{d(s_{\alpha\beta}^{i_q} f_{i_q}^\alpha)}{s_{\alpha\beta}^{i_q} f_{i_q}^\alpha} = \sum_{i_1<\dots<i_q} \lambda_I \left(\frac{d(s_{\alpha\beta}^{i_1}}{s_{\alpha\beta}^{i_1}} + \frac{df_{i_1}^\alpha}{f_{i_1}^\alpha} \right) \wedge \cdots \wedge \left(\frac{d(s_{\alpha\beta}^{i_q}}{s_{\alpha\beta}^{i_q}} + \frac{df_{i_q}^\alpha}{f_{i_q}^\alpha} \right) =$$

$$= \tilde{\mu}_{\alpha\beta} + \sum_{i_1<\dots<i_q} \lambda_I \frac{df_{i_1}^\beta}{f_{i_1}^\beta} \wedge \cdots \wedge \frac{df_{i_q}^\beta}{f_{i_q}^\beta} = \tilde{\mu}_{\alpha\beta} + (\mu|_{U_\beta})|_{U_\alpha}.$$

It is important to observe that the q-forms introduced, with the notation $\{\tilde{\mu}_{\alpha\beta}\}$, satisfy the cocycle conditions: $\tilde{\mu}_{\alpha\beta} + \tilde{\mu}_{\beta\gamma} = \tilde{\mu}_{\alpha\gamma}$. Also, in each of the terms of $\tilde{\mu}_{\alpha\beta}$ are involved at most $q-1$ of the forms $\frac{df_1^\alpha}{f_1^\alpha}, \dots, \frac{df_m^\alpha}{f_m^\alpha}$. So, it is clear it determines an element:

$$\{\tilde{\mu}_{\alpha\beta}\} \in H^1(X, W_{q-1}(\Omega_X^q(log\mathcal{D}))),$$

which only depends on the vector λ. It is not hard to show that the vanishing of this element is exactly the condition imposed by the kernel of the connection morphism δ. This assumption extend the L-condition imposed in the case of forms of degree 1. Formally, we need to use the following property obtained at the proof of the theorem 3.5.4:

$$H^0(X, W_{q-1}(\Omega_X^q(\log \mathcal{D}))) = 0,$$

to deduce that the forms $\{(\mu|_{U_\beta})|_{U_\alpha}\}$ glue together to define a global Logarithmic q-form.

Finally, the entire result follows from the injectivity of the total residue map (see the above long exact sequence or the previous proposition 3.5.4). □

Our final purpose is to understand the global sections of $\Omega_X^q(\log \mathcal{D})$ in the case of $X = \mathbb{P}^n$. Specially, we want to complete its description in homogeneous coordinates. To that end, we will compute the cohomology group $H^1(X, W_{q-1}(\Omega_X^q(log\mathcal{D})))$ and make a correct interpretation of the equations imposed by the kernel of δ.

Lemma 3.5.7. Assume $q < n-1$. With the above notations, the following computations holds:

$$H^1(\mathbb{P}^n, W_j(\Omega_{\mathbb{P}^n}^q(log\mathcal{D}))) = 0 \quad \forall j = 1, \dots, q-2.$$

In addition, the group $H^1(\mathbb{P}^n, W_{q-1}(\Omega^q_{\mathbb{P}^n}(log\mathcal{D})))$ has a natural injective morphism to:

$$\bigoplus_{I:|I|=q-1} H^1(\mathcal{D}_I, \Omega^1_{\mathcal{D}_I}) \simeq \mathbb{C}^{\binom{m}{q-1}}.$$

Proof. The case $q = 1$ is trivial and it was explained at 3.4.7, so we shall assume $q > 1$. At first we consider important to take into account the proof of proposition 3.5.4. In that proof we have already shown that:

$$H^0(\mathbb{P}^n, W_j(\Omega^q_{\mathbb{P}^n})) = 0 \quad \forall j = 1, \ldots, q-1. \tag{3.5.4}$$

Note that, in this particular case, the ambient variety $X = \mathbb{P}^n$ has the following characterization for its Hodge numbers:

$$h^{p,q}_{\mathbb{P}^n} = \delta_{pq}.$$

So due to the Lefschetz hyperplane theorem (see [38]) it is also true that:

$$h^{p,q}_{\mathcal{D}^j} = \delta_{pq} \quad \forall p, q : p + q < n - j, \tag{3.5.5}$$

for every codimension j subvariety associated to the the divisor $\mathcal{D}$. Now, we use again the short exact sequences:

$$0 \longrightarrow W_{j-1}(\Omega^q_X(\log \mathcal{D})) \longrightarrow W_j(\Omega^q_X(\log \mathcal{D})) \longrightarrow (j_j)_*(\Omega^{q-j}_{\mathcal{D}^j}) \longrightarrow 0 \quad \text{(for every } j),$$

to prove that:

$$H^1(\mathbb{P}^n, W_j(\Omega^q_{\mathbb{P}^n})) = 0 \quad \forall j < q-1.$$

We proceed by induction on j. The first case ($j = 0$) is an immediate consequence of the assumption: $h^{q,1}_{\mathbb{P}^n} = 0$. Next use the long exact sequence on cohomology associated to the above sequence, and apply 3.5.4 to deduce:

$$0 \to H^1(\mathbb{P}^n, W_{j-1}(\Omega^q_{\mathbb{P}^n})) \to H^1(\mathbb{P}^n, W_j(\Omega^q_{\mathbb{P}^n})) \to H^1(\mathcal{D}^j, \Omega^{q-j}_{\mathcal{D}^j}).$$

The first term vanish by the inductive hypothesis and the last due to the result 3.5.5. In particular, observe we are using the restrictions $q < n-1$ and $j < q-1$. Finally, we need to consider the long exact sequence for the case $j = q-1$ which turns out to be:

$$0 \to H^1(\mathbb{P}^n, W_{j-1}(\Omega^q_{\mathbb{P}^n})) \to \bigoplus_{I:|I|=q} H^1(\mathcal{D}_I, \Omega^1_{\mathcal{D}_I}) \to H^2(\mathbb{P}^n, W_{q-2}(\Omega^q_{\mathbb{P}^n}(\log \mathcal{D}))) \ldots,$$

and ends the proof. □

Corollary 3.5.8. Now assume $q < n-k$. With a similar proof, we can prove that:

$$H^k(\mathbb{P}^n, W_j(\Omega_{\mathbb{P}^n}(\log \mathcal{D}))) = 0 \quad \forall j = 1, \ldots, q-k-1.$$

And in addition, $H^k(\mathbb{P}^n, W_{q-k}(\Omega_{\mathbb{P}^n}(\log \mathcal{D})))$ can be always consider as a subspace of

$$\bigoplus_{I:|I|=q-k} H^1(\mathcal{D}_I, \Omega^1_{\mathcal{D}_I}) \simeq \mathbb{C}^{\binom{m}{q-k}}.$$

Example 3.5.9. [Logarithmic sections on the complex projective space.] According to the above results, we have constructed a certain exact sequence which describes in homogeneous coordinates the global projective logarithmic forms:

$$\begin{array}{ccccccc} 0 \longrightarrow H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(\log \mathcal{D})) & \xrightarrow{Res_q} & \bigoplus_{I:|I|=q} H^0(\mathcal{D}_I, \mathcal{O}_{\mathcal{D}_I}) & \xrightarrow{\delta} & H^1(\mathbb{P}^n, W_{q-1}(\Omega_{\mathbb{P}^n}(\log \mathcal{D}))) & \cdots \\ & & \downarrow \simeq & \searrow & \downarrow & \\ & & \mathbb{C}^{\binom{m}{q}} & & \mathbb{C}^{\binom{m}{q-1}} & \end{array}$$

Now consider the global rational q-form defined by:

$$\eta = \sum_{I:|I|=q} \lambda_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}},$$

for some multi-vector of constants $\lambda \in \bigwedge^q(\mathbb{C}^m)$ which satisfies $i_{\mathbf{d}}(\lambda) = 0$. It is clear that it is a well defined rational form over $\mathbb{P}^n$ since it satisfies the descend condition:

$$i_R(\eta) = \sum_I (i_{\mathbf{d}}\lambda)_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}} = 0,$$

where R denotes the Euler radial field. Moreover, η has simple poles over $\mathcal{D} = (F = F_1 \cdot \cdots \cdot F_m = 0)$ and $d\eta = 0$, so it is clear that it corresponds to a global logarithmic q-form (in homogeneous coordinates). Finally, by dimension count's argument on the above diagram, we can conclude that these formulas describe all the possible global logarithmic projective forms for the fixed divisor $\mathcal{D}$.

This last example is going to be very important in the next chapter for the definition of those logarithmic regular q-forms which induce singular algebraic foliations of higher codimension.

3.5.2 The extended Jouanolou's lemma: using Grothendieck symbols

As we could see in the previous section, if we assume that the divisor is simple normal crossing, both the global sections of the sheaf of logarithmic form and the adapted version of the Jouanolou's lemma can be well described.

On the other hand, note that the original Jouanolou's lemma 3.4.1 for affine 1-forms only use that the polynomials in the formula are irreducible and distinct, which is a weaker condition than assume that they have normal crossings.

Now, we present a new version of the Jouanolou lemma for the affine space and arbitrary degree forms, with less restrictive assumptions on the involved polynomials. This result is a generalization for higher degree forms of the original lemma 3.4.1. The idea behind will be similar to that used in the original proof: consider a pull-back of the form to a correct subvariety and take a suitable residue. In this case, we need to use another general residue theory: the Grothendieck symbols.

Since the construction of these symbols is embedded in a formidable global duality theory (see for instance [32]), we will avoid its formal definition and only introduce its notation and some useful properties we need for the sequel.

Definition 3.5.10. Let $X \to Y$ be a smooth morphism of schemes of relative dimension n. Also, let $F_1, \ldots, F_n \in H^0(X, \mathcal{O}_X)$ such that the closed subscheme Z of X defined by the ideal sheaf $\mathcal{I} = \langle F_1, \ldots, F_n \rangle$ is finite. For every global relative n-form, $\eta \in H^0(X, \Omega^n_{X/Y})$, it is possible to define its residue symbol as a global function of Y:

$$Res_{X/Y}\begin{bmatrix} \eta \\ F_1, \ldots, F_n \end{bmatrix} \in H^0(Y, \mathcal{O}_Y)$$

Proposition 3.5.11. Some properties of the residue symbols are the following:

1) (Restriction) Let $X' \xrightarrow{i} X$ be a complete intersection smooth subvariety, defined by functions $S_1, \ldots, S_p \in H^0(X, \mathcal{O}_X)$. Also assume that X is of relative dimension $n + p$. Take $F_1, \ldots, F_n \in H^0(X, \mathcal{O}_X)$, and let $F'_1, \ldots, F'_n$ be their restrictions to X'. For every $\omega \in H^0(X, \Omega^n_{X/Y})$ the following holds:
$$Res_{X'/Y}\begin{bmatrix} i^*\omega \\ F'_1, \ldots, F'_n \end{bmatrix} = Res_{X/Y}\begin{bmatrix} \omega \\ F_1, \ldots, F_n, S_1, \ldots, S_p \end{bmatrix}$$

2) (Trace formula) Consider $F_1, \ldots, F_n$ and G in $H^0(X, \mathcal{O}_X)$. Then, we get:
$$Res_{X/Y}\begin{bmatrix} GdF_1 \wedge \ldots, \wedge dF_n \\ F_1, \ldots, F_n \end{bmatrix} = Tr_{Z/Y}(G|_Z)$$

3) (Duality) If $\eta \in \sum_{i=1}^n \langle F_i \cdot H^0(X, \Omega^n_{X/Y}) \rangle$, then necessarily:
$$Res_{X/Y}\begin{bmatrix} \eta \\ F_1, \ldots, F_n \end{bmatrix} = 0.$$
Conversely, if $Res_{X/Y}\begin{bmatrix} f\eta \\ F_1, \ldots, F_n \end{bmatrix} = 0$ for all $f \in H^0(X, \mathcal{O}_X)$, then:
$$\eta \in \sum_{i=1}^n \langle F_i \cdot H^0(X, \Omega^n_{X/Y}) \rangle$$

Remark 3.5.12. For an algebraic approach of these symbols see [34].

In the sake of clarity, we describe the announced result for logarithmic regular q-forms over the n-dimensional affine space k^n, with $1 \leq q \leq n$.

Theorem 3.5.13. Set again the notation $A = k[x_1, \ldots, x_n]$ and $K = k(x_1, \ldots, x_n)$, and also fix an m-tuple $(F_i)_{i=1}^m$ of elements of A. Suppose that for every multi-index $I \subset \{1, \ldots, m\}$ of size q, the polynomials $F_{i_1}, \ldots, F_{i_q}$ has a common solution. Also, assume that for every J of size $q + 1$, the polynomials $F_{j_1}, \ldots, F_{j_{q+1}}$ meets properly (not necessarily non-empty). In addition, pick constants $\{a_I\}_{I:|I|=q}$. Then, a q-logarithmic regular form of the type

$$\omega = \sum_{I:|I|=q} a_I \hat{F}_I dF_I = F. \sum_{I:|I|=q} a_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}} \in H^0(k^n, \Omega^q_{k^n})$$

equals to the zero section if and only if $a_I = 0$ for all I of size q.

Proof. Fix $I_0 = \{i_1, \dots, i_q\}$ and denote by $\tilde{X}_{I_0}$ a component of codimension q of the variety $X_{I_0} = (F_{i_1} = \cdots = F_{i_q} = 0)$. We will show that $a_{I_0} = 0$.

Now, fix a point $p \in X_{I_0}$ with the additional hypothesis

$$F_k(p) \neq 0 \quad \forall k \notin I_0,$$

which is an admissible condition because every selection of $q+1$ polynomials meet properly. Then, we can choose global linear functions $S_1, \dots, S_{n+1-q}$ whose associated hyperplanes pass through p and meet $\tilde{X}_{I_0}$ with the correct dimension, i.e. assume that the ideal $\mathcal{J} = \langle F_{i_1}, \dots . F_{i_q}, S_1, \dots, S_{n+1-q}\rangle$ is finite. In addition, we can also suppose that p is the unique point of the variety defined by $\mathcal{J}$.

From now on, we will work with the form:

$$\omega_S = \omega \wedge dS_1 \wedge \cdots \wedge dS_{n+1-q} = \sum_{I:|I|=q} a_I \hat{F}_I dF_I \wedge dS_1 \wedge \cdots \wedge dS_{n+1-q},$$

which according to the theorem's hypothesis vanishes completely, i.e. $\omega_S = 0$. Next, we will use the Grothendieck's residue symbols to deduce $a_{I_0} = 0$. For this purpose, denote by

$$Y_{I_0} = \bigcap_{j=1}^{n+1-q} (S_j = 0),$$

and use the restriction property from 3.5.11 and the linearity of the symbol, to get:

$$0 = Res_{\mathbb{C}^{n+1}} \begin{bmatrix} \omega_S \\ F_{i_1}, \dots, F_{i_q}, S_1, \dots, S_{n+1-q} \end{bmatrix} = Res_{Y_{I_0}} \begin{bmatrix} i^*(\omega) \\ F'_{i_1}, \dots, F'_{i_q} \end{bmatrix} = \tag{3.5.6}$$

$$= \sum_{I:|I|=q, I \neq I_0} a_I Res_{Y_{I_0}} \begin{bmatrix} i^*(\hat{F}_I dF_I) \\ F'_{i_1}, \dots, F'_{i_q} \end{bmatrix} + a_{I_0} Res_{Y_{I_0}} \begin{bmatrix} i^*(\hat{F}_{I_0} dF_{I_0}) \\ F'_{i_1}, \dots, F'_{i_q} \end{bmatrix}, \tag{3.5.7}$$

where i refers to the inclusion map $i : Y_{I_0} \hookrightarrow \mathbb{C}^{n+1}$ and $F'_{i_j} = i^*(F_{i_j})$ for $j = 1 \dots q$.

In addition, observe that if $I \neq I_0$, then there exist $i_k \in I_0 - I$, and so

$$i^*(\hat{F}_I dF_I) = F'_{i_k} i^*(\hat{F}_{I \cup \{i_k\}} dF_I) \in H^0(Y_{I_0}, \sum_j F'_{i_j} \cdot \Omega^q_{Y_{I_0}}).$$

According to the duality property at 3.5.11, we deduce

$$Res_{Y_{I_0}} \begin{bmatrix} i^*(\hat{F}_I dF_I) \\ F'_{i_1}, \dots, F'_{i_q} \end{bmatrix} = 0 \quad \text{for every } I \neq I_0$$

In conclusion, using the trace formula at 3.5.11, we obtain:

$$0 = a_{I_0} Res_{Y_{I_0}} \begin{bmatrix} i^*(\hat{F}_{I_0} dF_{I_0}) \\ F'_{i_1}, \dots, F'_{i_q} \end{bmatrix} = a_{I_0} Res_{Y_{I_0}} \begin{bmatrix} \hat{F'_{I_0}} dF'_{i_1} \wedge \cdots \wedge dF'_{i_q} \\ F'_{i_1}, \dots, F'_{i_q} \end{bmatrix} = a_{I_0} Tr_Z(\hat{F'_{I_0}}|_Z),$$

where Z denotes the variety defined by the ideal $\langle F'_{i_1}, \dots, F'_{i_q}\rangle$ in Y_{I_0}. Finally, observe that the variety Z coincides with that defined by the ideal $\mathcal{J}$. So Z is supported at p and

$$Tr_Z(\hat{F'_{I_0}}|_Z) = \hat{F}_{I_0}(p) = \prod_{k \notin I_0} F_k(p) \neq 0.$$

In consequence $a_{I_0} = 0$ as claimed.

□

Chapter 4

Stability of higher degree logarithmic forms

4.1 Introducción y resumen en español

En este último capítulo abordaremos un estudio de q-formas logarítmicas proyectivas con $q > 1$, en el contexto de foliaciones algebraicas singulares de codimensiones superiores.

En primera instancia, y en relación con el concepto ya definido para el caso de 1-formas en el capítulo 2, se describirá el concepto de q-forma logarítmica de tipo $\mathbf{d}$. *Además, se caracterizará aquellas q-formas logarítmicas que definen foliaciones en codimensión q, utilizando las ecuaciones del espacio de moduli correspondiente* $\mathcal{F}_q(d, \mathbb{P}^n)$ *(introducidas en el capítulo 1).*

El resultado principal del capítulo será la prueba de estabilidad de 2-formas logarítmicas, asumiendo que el vector de grados $\mathbf{d} = (d_1, \ldots, d_m)$ *que las define cumple una condición de balanceabilidad (concepto de 2-balanceado). Esto permite determinar nuevas componentes irreducibles del espacio* $\mathcal{F}_2(d, \mathbb{P}^n)$*, que además son reducidas de acuerdo a su estructura de esquema.*

De modo más detallado, repasamos los principales resultados del capítulo. De acuerdo con el cálculo establecido en el capítulo anterior para las secciones globales del conocido haz de formas logarítmicas $\Omega^q_{\mathbb{P}^n}$ (ver ejemplo 3.5.9), resulta natural definir a las q-formas logarítmicas regulares de tipo $\mathbf{d}$ por la fórmula:

$$\omega = \sum_{I:|I|=q} \lambda_I \hat{F}_I dF_{i_1} \wedge \cdots \wedge dF_{i_q} = F \left(\sum_{I:|I|=q} \lambda_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}} \right) \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d)),$$

donde se requiere que $\lambda = (\lambda_I) \in \bigwedge^q \mathbb{C}^m$ satisfaga la ecuación $i_{\mathbf{d}}(\lambda) = 0$, y los polinomios $F_1, \ldots, F_m$ son elegidos homogéneos y de grados respectivos $d_1, \ldots, d_m$.

En primera instancia, es necesario determinar condiciones sobre estas formas logarítmicas regulares para que satisfagan las ecuaciones de descomponibilidad local e integrabilidad (ver ecuaciones 1.4.1 y 1.4.2 en en capítulo 1), y de ese modo determinen foliaciones algebraicas singulares. En función de esto último, y si denotamos nuevamente por $\mathbb{C}^m_{\mathbf{d}}$ a los vectores $v \in \mathbb{C}^m$ tales que

$\sum v_i d_i = 0$, vamos a necesitar considerar en nuestras fórmulas a constantes pertenecientes al siguiente espacio de Grassmannianas:

$$\lambda \in Gr(q, \mathbb{C}^m_{\mathbf{d}}) \subset \mathbb{P}\left(\bigwedge^q \mathbb{C}^m_{\mathbf{d}}\right).$$

Estas pueden ser caracterizadas, via la inmersión de Plücker, por la clase proyectiva de elementos $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q$, donde cada factor $\lambda_i \in \mathbb{C}^m_{\mathbf{d}}$ (ver proposición 4.3.4). De este modo, para cada vector de grados $\mathbf{d}$ fijo, queda determinada una parametrización natural de las formas logarítmicas de tipo $\mathbf{d}$ que definen foliaciones:

$$\rho : \mathcal{P}_q(\mathbf{d}) := Gr(q, \mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^{m} \mathbb{P}S_{d_i} \dashrightarrow \mathcal{F}_q(d, \mathbb{P}^n) \subset \mathbb{P}H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d)).$$

En este caso, denotaremos por $\mathcal{L}_q(\mathbf{d}, n)$ a la clausura Zariski de la imagen de este morfismo racional, y nos referiremos a estas variedades irreducibles como variedades logarítmicas en codimensión q. En general, vamos a considerar el caso $q = 2$, dado que los resultados principales serán obtenidos para 2-formas logarítmicas.

Antes de enunciar lo teoremas principales, se realizará un análisis de varios aspectos de la parametrización racional ρ. En primera instancia, serán caracterizadas algunas subvariedades irreducibles del base locus de ρ (notado por $\mathcal{B}_q(\mathbf{d})$), para esto ver proposición 4.3.9. Además, se determinará un abierto adecuado ($\mathcal{U}_2(\mathbf{d})$) del espacio de parámetros, disjunto con $\mathcal{B}_2(\mathbf{d})$, donde queda bien definido este morfismo racional (ver proposición 4.3.6).

Por otro lado, también serán descriptos algunos aspectos de su posible inyectividad genérica. Al igual que en el caso de 1-formas, podremos determinar un abierto donde ρ queda bien definido y es inyectivo (o finito, dependiendo de si $\mathbf{d}$ tiene grados repetidos), utilizando una adecuada variedad de incidencia que estará nuevemente relacionada con la cantidad de factores integrantes de una 2-forma. Sin embargo, no podremos probar que la construcción de lugar siempre a un abierto no vacío. En la sección 4.3.2 dejamos una conjetura sobre estos resultados, y una posible demostración del resultado de inyectividad (o finitud de las fibras), para ello ver la proposición 4.3.15.

El resultado principal del capítulo es la prueba de estabilidad de 2-formas logarítmicas, asumiendo que el vector $\mathbf{d}$ es 2-balanceado. Esta última condición se corresponde con:

$$d_i + d_j < \sum_{k \neq i,j} d_k \qquad \forall i, j \in \{1, \ldots, m\},$$

y es un extensión de la hipótesis de 1-balanceado, asumida en algunas secciones del capítulo 2. El teorema principal, que resume el resultado anunciado, es el siguiente:

Theorem 4.1.1. Fijemos $n, m \geq 4$, y un vector de grados $\mathbf{d} = (d_1, \ldots, d_m)$ que sea 2-balanceado, cumpliendo $d = \sum_{i=1}^m d_i$. Luego, la variedad $\mathcal{L}_2(\mathbf{d}, n)$ es una componente irreducible del espacio de moduli $\mathcal{F}_2(d, \mathbb{P}^n)$, que resultará, además, reducida sobre los puntos de $\rho(\mathcal{U}_2(\mathbf{d}))$.

Este teorema será una consecuencia de la suryectividad del diferencial de la parametrización (ver proposición 4.4.2), junto con algunos argumentos clásicos de la Teoría de esquemas.

4.2 Summary

Across this chapter, we perform a study of projective logarithmic q-forms (with $q > 1$) in the setting of singular algebraic foliations of higher codimension.

First, we describe the concept of logarithmic q-forms of type **d**. *This definition also corresponds to a generalization of that given in the case of 1-forms at chapter 2. In addition, we characterize those logarithmic q-forms which define foliations of codimension q, based on the equations described for $\mathcal{F}_q(d, \mathbb{P}^n)$ (see for example the definitions given at chapter 1). In general, we pay particular attention to logarithmic 2-forms, because in that case, we can perform a proof of their infinitesimal stability. For a more general perspective of foliations of higher codimension see 1, [15] and [17].*

The main result established in this chapter is the proof of the stability of logarithmic 2-forms, assuming certain conditions on the vector of degrees **d**. *The assumption will be the vector* **d** *being 2-balanced. This work allows us to determine new irreducible components of the space $\mathcal{F}_2(d, \mathbb{P}^n)$ (see Theorem 4.4.1), also generically reduced according to its scheme structure.*

4.3 Basic definitions

4.3.1 Logarithmic q-forms and projective foliations of codimension q.

Recall from the end of chapter 1 that the moduli space $\mathcal{F}_q(d, \mathbb{P}^n)$ of projective foliations of codimension $q \in \mathbb{N}$ and degree $d \in \mathbb{N}_{>q}$ can be described by twisted projective forms $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ which satisfies the Plücker's decomposability condition:

$$i_v(\omega) \wedge \omega = 0 \qquad \forall v \in \bigwedge^{q-1} \mathbb{C}^{n+1} \tag{4.3.1}$$

and the so called integrability condition

$$i_v(\omega) \wedge d\omega = 0 \qquad \forall v \in \bigwedge^{q-1} \mathbb{C}^{n+1}. \tag{4.3.2}$$

Also the elements v can be considered as local frames $\{v_1 \dots v_{q-1}\}$ on the affine cone of the ambient space $\mathbb{P}^n$, or as local rational fields. For more details consult [17]. In addition, remember that we can think on the elements $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ as homogeneous affine forms on $\mathbb{C}^{n+1}$:

$$\omega = \sum_{I=\{i_1<\dots,i_q\}} A_I(z) dz_{i_1} \wedge \dots \wedge dz_{i_q},$$

whose coefficients $\{A_I\}_I$ are homogeneous polynomials of degree $d - q$. Moreover, we require some linear equations over its coefficients in order to ensure that the form ω descends to the projective space. Specifically, the form must satisfy:

$$i_R(\omega) = 0 \in H^0(\mathbb{C}^{n+1}, \Omega^{q-1}_{\mathbb{C}^{n+1}}). \tag{4.3.3}$$

Observe that for $q = 2$ the decomposability condition 4.3.1 is slightly simpler than in the general case, because is equivalent to:

$$i_v(\omega \wedge \omega) = 0 \qquad \forall v \in \bigwedge^{k-1} \mathbb{C}^{n+1},$$

and so it can be replaced by:

$$\omega \wedge \omega = 0. \tag{4.3.4}$$

In conclusion, the space of codimension k singular foliations on $\mathbb{P}^n$ of degree d is described by:

$$\mathcal{F}_k(d, \mathbb{P}^n) = \{\omega \in \mathbb{P}(H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d))) : \omega \text{ satisfies } 4.3.1\ , 4.3.2 \text{ and } codim(S_\omega) \geq 2\}.$$

Alternatively, it can be described in homogeneous coordinates by:

$$\mathcal{F}_k(d, \mathbb{P}^n) = \Big\{\omega = \sum A_I(z) dz_I \in \mathbb{P}H^0(\mathbb{C}^{n+1}, \Omega^q_{\mathbb{C}^{n+1}}) :$$
$$\text{each } A_I \in S_{d-q} \text{ and the form } \omega \text{ satisfies: } 4.3.3\ , 4.3.1\ , 4.3.2 \text{ and } codim(S_\omega) \geq 2\Big\}.$$

Where also the equation 4.3.2 can be replaced by 4.3.4 if $q = 2$.

In a geometric complex setting, the forms satisfying these last equations define singular holomorphic foliations on $\mathbb{P}^n$, whose leaves are of codimension q. In other words they define a regular holomorphic foliations outside the singular set $S_\omega = \{p \in \mathbb{P}^n : \omega(p) = 0\}$. Moreover the number of tangencies of a generic line with the leaves of these foliations is $d - q$.

On the other hand we want to define correct formulas for regular logarithmic q-forms which define foliation according to the previous equations. We start setting up the notation required. It will be the same notation that the used in the first chapter to describe the variety $\mathcal{L}_1(n, \mathbf{d})$.

Fix a degree $d \in \mathbb{N}$, which is going to be related to the geometric degree of the foliation, and choose a partition of d by an m-tuple of degrees $\mathbf{d} = (d_1, \dots, d_m) \in \mathbb{N}^m$ satisfying $\sum_{i=1}^m d_i = d$.

Let $\mathcal{D} = \sum_{i=1}^m \mathcal{D}_i = (F_i = 0)$ be a divisor defined by homogeneous polynomials:

$$F_i \in S_{d_i} = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i)) \quad (i = 1, \dots, m).$$

The desired formula is supported on the characterization obtained at 3.5.9 for the global sections of $\Omega^q_{\mathbb{P}^n}$ (assuming $\mathcal{D}$ normal crossing), and the relation between this sheaf and the regular logarithmic 1-forms of type $\mathbf{d}$. With these ideas in mind, a regular logarithmic q-form of type $\mathbf{d}$ will be an element $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ defined by the formula:

$$\omega = F\left(\sum_{I:|I|=q} \lambda_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}}\right) = \sum_{\{I:|I|=q\}} \lambda_I \hat{F}_I dF_I = \sum_{I:|I|=q} \lambda_I \left(\prod_{i \notin I} F_i\right) dF_{i_1} \wedge \cdots \wedge dF_{i_q}, \tag{4.3.5}$$

where the vector of constants can be considered as an element $\lambda \in \bigwedge^q \mathbb{C}^m$, which satisfies some linear equations to ensure the descent condition for the form. Recall from 3.5.9 that this condition is equivalent to the require:

$$i_{\mathbf{d}} \lambda = 0,$$

i.e. the vector λ should be a cycle in the Koszul complex associated to the vector $(d_1, \ldots, d_m)$. In conclusion, our definition of regular logarithmic q-forms corresponds to the regularization of the classical meromorphic logarithmic q-forms.

Remark 4.3.1 (Notation)**.** We shall further use the notation $I : |I| = q$, specially for sum's indexes, for multi-indexes $I = (i_1, \ldots, i_q) \in \{1, \ldots, m\}^m$ of size q. In addition, $i_1 < \cdots < i_q$ or $\{i_1, \ldots, i_q\}$ will be used when we want to fix an special unique order or simply when we do not want to take into account the order.

The first important question related to these forms is when they define foliations of codimension q, i.e. when the forms given by the formula 4.3.5 satisfy the equations which define $\mathcal{F}_q(d, \mathbb{P}^n)$. Let us start with the Plücker's decomposability condition 4.3.1 for the case of $q = 2$.

Proposition 4.3.2. Let $F_1, \ldots, F_m$ be homogeneous polynomials such that the divisor $\mathcal{D} = (F = F_1 \cdot \cdots \cdot F_m = 0)$ is simple normal crossings. Consider the logarithmic 2-form of type **d** defined by the formula 4.3.5 for some vector of constants $\lambda \in \bigwedge^2 \mathbb{C}^m$. Then, the following equations are equivalent:

i. $\omega \wedge \omega = 0$

ii. $\lambda \wedge \lambda = 0$.

Proof. First, if we suppose:

$$\omega \wedge \omega = F \left(\sum_{i<j} (\lambda \wedge \lambda)_{ij} \tfrac{dF_i}{F_i} \wedge \tfrac{dF_j}{F_j} \right) = 0,$$

the result follows from applying proposition 3.5.9 (or 3.5.4) which establishes the injectivity of the total residues map. The other implication is trivial, but also note that $\lambda \wedge \lambda = 0$ implies the existence of $\lambda_1, \lambda_2 \in \mathbb{C}^m$ such that: $\lambda = \lambda_1 \wedge \lambda_2$. In this case, if we consider the meromorphic logarithmic forms defined by:

$$\eta_j = \sum_{i=1}^{m} (\lambda_j)_i \tfrac{dF_i}{F_i} \quad j = 1, 2,$$

then it is clear that: $\eta = \frac{\omega}{F} = \eta_1 \wedge \eta_2$. So the meromorphic logarithmic 2-form η is globally decomposable. □

Remark 4.3.3. With the same idea as in the previous proof, it is also true that if we consider a q-vector of the form $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q \in \bigwedge^q \mathbb{C}^m$, the associated logarithmic q-form of type **d** satisfy the Plücker's decomposability equations. The reason is that under this assumption the associated meromorphic q-form is globally decomposable. In other words:

$$\eta = \tfrac{\omega}{F} = \eta_1 \wedge \cdots \wedge \eta_q,$$

and therefore:

$$i_v(\omega) \wedge \omega = \tfrac{1}{F^2} i_v(\eta) \wedge \eta = 0.$$

Taking into consideration the above results, in view of obtaining logarithmic q-forms which define foliations of codimension q, we require:

$$\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q \in Gr(q, \mathbb{C}^m).$$

Where this grassmannian space is going to be considered as a projective algebraic subset of $\mathbb{P}(\bigwedge^q \mathbb{C}^m)$ via the Plücker embedding:

$$\iota : Gr(q, \mathbb{C}^m) \longrightarrow \mathbb{P}(\bigwedge^q \mathbb{C}^m)$$
$$\mathrm{span}(\mu_1, \ldots, \mu_q) \longmapsto [\mu_1 \wedge \cdots \wedge \mu_q]$$

With a slight abuse of notation we will write λ or $\lambda_1 \wedge \cdots \wedge \lambda_q$ for the elements of the grassmannian space $Gr(q, \mathbb{C}^m)$, and (λ_I) for its corresponding antisymmetric coordinates. It is not hard to show that every logarithmic q-form of type $\mathbf{d}$, with constants λ selected on this grassmannian space, satisfies the equations defining $\mathcal{F}_q(d, \mathbb{P}^n)$.

Proposition 4.3.4. Select an element $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q \in Gr(q, \mathbb{C}^m)$ such that $i_{\mathbf{d}}(\lambda) = 0$. Also consider $(F_i)_{i=1}^m$ a family of homogeneous polynomials of corresponding degrees $(d_i)_{i=1}^m$. Then, the form:

$$\omega = \sum_{\{I:|I|=q\}} \lambda_I \hat{F}_I dF_I = \sum_{\{I:|I|=q\}} \lambda_I \left(\prod_{i \neq i_j} F_j \right) dF_{i_1} \wedge \cdots \wedge dF_{i_q}$$

is a twisted projective form which satisfies the equations 4.3.1 and 4.3.2. In other words, its projective class corresponds to a point of the moduli space of singular projective foliations of degree d and codimension q , i.e. $[\omega] \in \mathcal{F}_q(d, \mathbb{P}^n)$.

Proof. It only remains to prove that ω satisfies the integrability equation: $i_v(\omega) \wedge d\omega = 0$. Note that:

$$d\omega = \tfrac{dF}{F} \wedge \omega,$$

and also observe that ω satisfies: $i_v(\omega) \wedge \omega = 0$. And so the integrability equation immediately follows from this two last equations. □

The following result characterize in an useful way the condition $i_{\mathbf{d}}(\lambda) = 0$.

Lemma 4.3.5. Consider $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q \in Gr(q, \mathbb{C}^m) \subset \mathbb{P} \bigwedge^q(\mathbb{C}^m)$. Then, the following equations are equivalent:

i) $i_{\mathbf{d}}\lambda = 0$

ii) $i_{\mathbf{d}}\lambda_i = 0, \quad \forall i = 1, \ldots, q.$

In conclusion the following equality holds:

$$\{\lambda \in Gr(q, \mathbb{C}^m) : i_{\mathbf{d}}\lambda = 0\} = Gr(q, \mathbb{C}^m_{\mathbf{d}}),$$

where $\mathbb{C}^m_{\mathbf{d}}$ denotes the space of vectors $\lambda \in \mathbb{C}^m$ such that: $i_{\mathbf{d}}(\lambda) = \sum_{i=1}^m \lambda_i d_i = 0$.

Proof. The equivalence is deduced from the formula:

$$i_{\mathbf{d}}\lambda = \sum_{j=1}^{m} (-1)^j \; i_{\mathbf{d}}(\lambda_j) \; \lambda_1 \wedge \cdots \wedge \hat{\lambda}_j \wedge \cdots \wedge \lambda_q,$$

and the fact that the vectors $\{\lambda_1 \wedge \cdots \wedge \hat{\lambda}_j \wedge \cdots \wedge \lambda_q\}_j$ are linearly independent. □

Now we are able to define in a correct way the logarithmic varieties associated to the regular logarithmic q-forms and their corresponding parametrizations. We denote by $l_q(\mathbf{d}, n)$ the algebraic set of logarithmic q-forms of type $\mathbf{d}$:

$$l_q(\mathbf{d}, n) = \{\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) : \omega = \sum_{I:|I|=q} \lambda_I \hat{F}_I dF_I, \text{ for some } \lambda \in \bigwedge^q \mathbb{C}^m_{\mathbf{d}} \text{ and } F_i \in S_{d_i}\},$$

which also coincides with the image of the multi-linear map:

$$\begin{aligned} \phi : (\mathbb{C}^m_{\mathbf{d}})^q \times \prod_{i=1}^{m} S_{d_i} &\longrightarrow H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d)) \\ (\lambda = (\lambda_1, \ldots, \lambda_q), (F_i)) &\longmapsto \sum_{I:|I|=q} (\lambda_1 \wedge \cdots \wedge \lambda_q)_I \hat{F}_I dF_I, \end{aligned} \tag{4.3.6}$$

In order to work in the setting of the moduli space of codimension q foliations, we need to take a projectivization of the previous objects. Formally, it is possible to define a natural parametrization for those projective logarithmic q-forms of type $\mathbf{d}$ that determine foliations, as follows:

$$\begin{aligned} \rho : \mathcal{P}_q(\mathbf{d}) := Gr(q, \mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^{m} \mathbb{P}(S_{d_i}) &\dashrightarrow \mathcal{F}_q(d, \mathbb{P}^n) \subset \mathbb{P}(H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))) \\ (\lambda = (\lambda_1 \wedge \cdots \wedge \lambda_q), (F_i)_{i=1}^m) &\longmapsto \quad [\omega] = \left[\sum_{I:|I|=q} \lambda_I \hat{F}_I dF_I \right]. \end{aligned} \tag{4.3.7}$$

Also observe that ρ is only a rational map, because it is not well defined on parameters which give rise to forms whose respective logarithmic formula vanishes completely.

We define the logarithmic variety $\mathcal{L}_q(\mathbf{d}, n)$ by the Zariski closure of the image of ρ:

$$\mathcal{L}_q(\mathbf{d}, n) = \overline{\operatorname{im}(\rho)},$$

which also coincides with $\mathbb{P}l_q(\mathbf{d}, n)$, i.e. with the expected definition of the projectivization of the algebraic variety $l_q(\mathbf{d}, n)$.

Therefore, $\mathcal{L}_q(\mathbf{d}, n)$ is a projective irreducible variety, and when $q = 2$ (in addition with some hypothesis over the fixed degrees $\mathbf{d}$) we will see that it is an irreducible component of the space $\mathcal{F}_2(d, \mathbb{P}^n)$. So from now on, we will further assume $q = 2$.

As in the case of logarithmic 1-forms of type $\mathbf{d}$, we will assume some generic conditions in the space of parameters. See for instance the conditions established at 2.4.2. In particular the assumptions over the homogeneous polynomials $F_1, \dots, F_m$ will be the same as in that case. We will consider irreducible polynomials with smooth simple normal crossings.

Specifically, we define an appropriate algebraic subset of $\mathcal{P}_2(\mathbf{d})$ by taking:

$$\begin{aligned}(4.3.8) \qquad \mathcal{U}_2(\mathbf{d}) = \{(\lambda, (F_i)_{i=1}^m) \in \mathcal{P}_2(\mathbf{d}) :\ & \lambda_{ij} \neq 0,\ \lambda_{ij} - \lambda_{ik} + \lambda_{jk} \neq 0;\ \lambda_{ij} - \lambda_{ik} - \lambda_{jk} \neq 0; \\ & \text{and } F_1, \dots, F_m \text{ satisfy the conditions 3. and 4. from 2.4.2}\}.\end{aligned}$$

This set has the desired properties, summarized in the following remark.

Proposition 4.3.6. The set $\mathcal{U}_2(\mathbf{d})$ is a non-empty algebraic open subset of $\mathcal{P}_2(\mathbf{d})$.

Proof. The grassmannian variety $Gr(2, \mathbb{C}^m_{\mathbf{d}})$ can not be covered by any union of linear hypersurfaces, and so the conditions

$$\lambda_{ij} \neq 0 \quad \text{and} \quad \lambda_{ij} - \lambda_{jk} + \lambda_{ik} \neq 0 \quad ; \quad \lambda_{ij} - \lambda_{ik} - \lambda_{jk} \neq 0 (\forall i, j, k \in \{1, \dots, m\})$$

determine a non-empty algebraic open subset of $Gr(2, \mathbb{C}^m_{\mathbf{d}})$.

On the other hand, we have already prove that the conditions 3. and 4. from 2.4.2 determines a non-empty algebraic subset of the product $\prod_{i=1}^m \mathbb{P}S_{d_i}$. For the corresponding proof of this fact see the corollary 2.4.5. The principal idea was based on the result 2.4.4, which states that the set:

$$NC^c(I, n) = \{((F_{i_j})_{j=1}^k, x) \in \prod_{j=1}^k \mathbb{P}S_{d_{i_j}} \times \mathbb{P}^n : F_{i_1}(x) = \cdots = F_{i_k}(x) = 0 \ \ , \ \ d_x F_{i_1} \wedge \cdots \wedge d_x F_{i_k} = 0\}$$

is an algebraic subvariety of the product $\prod_{j=1}^k \mathbb{P}S_{d_{i_j}} \times \mathbb{P}^n$ and has dimension one less than the space $\prod_{j=1}^k \mathbb{P}S_{d_{i_j}}$. So the first projection map restricted to $NC^c(I, n)$ could not be dominant. □

Remark 4.3.7. The space of parameters $\mathcal{P}_2(\mathbf{d})$ is an algebraic irreducible subvariety of dimension:

$$\dim(\mathcal{P}_2(\mathbf{d})) = \sum_{i=1}^m \binom{n + d_i}{d_i} + m - 6.$$

Moreover, note that this dimension is greater or equal than $2m - 6$, which is non negative because we always assume that $m \geq q = 2$.

4.3.2 Base locus and the fibers of the parametrization

Our goal now is to study the base locus of the rational parametrization ρ. In addition, we want to describe its **possible** generic injectivity.

Set $\mathcal{K}_q(\mathbf{d})$ as the algebraic set where the multilinear map ϕ vanishes, i.e. $\mathcal{K}_q(\mathbf{d}) = \phi^{-1}(0)$. In other words, we are just considering a vector $\lambda \in (\mathbb{C}^m_{\mathbf{d}})^q$ and homogeneous polynomials $(F_i)_{i=1}^m \in \prod_{i=1}^m S_{d_i}$ such that the following q-form vanishes:

$$\sum_{I : |I| = q} (\lambda_1 \wedge \cdots \wedge \lambda_q)_I \hat{F}_I dF_I = 0$$

With the previous notation, we describe the base locus of the parametrization ρ according to:

$$\mathcal{B}_q(\mathbf{d}) = \{(\pi(\lambda_1 \wedge \cdots \wedge \lambda_q), (\pi(F_i))) \in \mathcal{P}_q(\mathbf{d}) : (\lambda, (F_i)) \in \mathcal{K}_q(\mathbf{d}) \quad , \lambda_1 \wedge \cdots \wedge \lambda_q \neq 0 \quad , F_i \neq 0 \ \forall i\}.$$

The variety $\mathcal{B}_q(\mathbf{d})$ describes exactly the points of the parameter space $\mathcal{P}_q(\mathbf{d})$ in which the rational map ρ is not well defined.

A first approach to describe $\mathcal{B}_q(\mathbf{d})$ is to observe that it does not intersect the open algebraic set of generic parameters $\mathcal{U}_q(\mathbf{d})$.

Proposition 4.3.8. The space $\mathcal{B}_q(\mathbf{d})$ is a projective variety which satisfies: $\mathcal{B}_q(\mathbf{d}) \cap \mathcal{U}_q(\mathbf{d}) = \emptyset$.

Proof. First observe that if $(\lambda, (F_i)) \in \mathcal{B}_q(\mathbf{d}) \cap \mathcal{U}_q(\mathbf{d})$, then the meromorphic form associated to $\omega = \phi((\lambda_i), (F_i))$ also vanishes. In other words, the following relation holds:

$$\eta = \frac{\omega}{F} = \sum_{I:|I|=q} \lambda_I \frac{dF_{i_1}}{F_{i_1}} \wedge \cdots \wedge \frac{dF_{i_q}}{F_{i_q}} = 0.$$

Moreover, note that $\eta \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(\log(F = 0)))$. So according to the definition of $\mathcal{U}_q(\mathbf{d})$ and the results stated at 3.5.4 and 3.5.9, we deduce that necessarily $\lambda = 0$, which is an absurd and concludes the proof. □

Another remarkable fact about $\mathcal{B}_q(\mathbf{d})$ (or $\mathcal{K}_q(\mathbf{d})$) is related to the possibility of construct elements using the description made for the case of 1-forms (see section 2.4.1).

Recall we could construct a morphism:

$$\psi_{(\mathbf{d}',[\mathbf{e}])} : \ker([\mathbf{e}]^t) \times \prod_{j=1}^{m'} S_{d'_i} \longrightarrow \mathcal{K}_1(\mathbf{d}) \subset \mathbb{C}^m_{\mathbf{d}} \times \prod_{i=1}^{m} S_{d_i},$$

whose first factor corresponds to the natural inclusion, and the second to a Segre-Veronese map:

$$\nu_{(\mathbf{d},\mathbf{d}',[\mathbf{e}])} : \prod_{j=1}^{m'} S_{d'_j} \to \prod_{i=1}^{m} S_{d_i}.$$

According to the result 2.4.13, the union of the varieties $\mathcal{K}(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])}$ which corresponds to the image of the previous morphism $\psi_{(\mathbf{d}',[\mathbf{e}])}$, cover the entire set $\mathcal{K}_1(\mathbf{d})$. Where also remember that $(\mathbf{d}', [\mathbf{e}]) \in \Delta(\mathbf{d})$ runs over the set of all partitions of $\mathbf{d}$ by a matrix $[\mathbf{e}] \in \mathbb{N}_0^{m\times m'}$ and a new entire vector $\mathbf{d}'$.

On the other hand, observe that for $((\lambda_i), (F_i)) \in (\mathbb{C}^m_{\mathbf{d}})^q \times \prod_{i=1}^m S_{d_i}$ the condition

$$\phi((\lambda_i), (F_i)) = \omega = 0$$

is equivalent to:

$$\frac{\omega}{F} = \eta = \eta_1 \wedge \cdots \wedge \eta_q = 0.$$

Where each form η_i is the (meromorphic) logarithmic 1-form associated to the coefficients $\lambda_i = ((\lambda_i)_j) \in \mathbb{C}^m_{\mathbf{d}}$. So it is clear that the vanishing of any of these forms $\eta_1, \ldots, \eta_q$ implies the vanishing of the entire form η. Then we obtain the following result.

Proposition 4.3.9. The space $\mathbb{C}_{\mathbf{d}}^{q-1} \times \mathcal{K}_1(\mathbf{d})_{\mathbf{d}',[\mathbf{e}]}$ has q natural inclusions in $\mathcal{K}_q(\mathbf{d})$ and determine irreducible subvarieties.

Remark 4.3.10. In terms of the base locus ρ, we can construct the following subvariety:

$\mathcal{B}_q(\mathbf{d})_{(\mathbf{d}',[\mathbf{e}])} = \{(\pi(\lambda_1 \wedge \cdots \wedge \lambda_q), (\pi(F_i))) \in \mathcal{P}_q(\mathbf{d}) : (\lambda, (F_i)) \in \mathbb{C}_{\mathbf{d}}^{q-1} \times \mathcal{K}_1(\mathbf{d})_{\mathbf{d}',[\mathbf{e}]} \quad , \lambda \neq 0 \quad , F_i \neq 0\}.$

This construction can be described precisely as follows. Choose a partition of $\mathbf{d}$ by: $[\mathbf{e}] \cdot \mathbf{d}' = \mathbf{d}$. If we select homogeneous polynomials $F_i' \in S_{d_i'} - \{0\}$ and a non-trivial vector λ_1 in the kernel of the integer matrix $[\mathbf{e}]$ (if there exists), then we can construct elements in the base locus $\mathcal{B}_q(\mathbf{d})$. Obviously, if the matrix $[\mathbf{e}]$ has trivial kernel, the associated set constructed is empty. Also select $q-1$ vectors $\lambda_2, \ldots, \lambda_q \in \mathbb{C}_{\mathbf{d}}^m$ such that:

$$\lambda = \lambda_1 \wedge \cdots \wedge \lambda_q \neq 0,$$

and define the following polynomials:

$$F_i = \prod_{j=1}^{m'} (F_j')^{e_{ij}} \in S_{d_i}.$$

Finally $([\lambda], ([F_i])) \in Gr(q, \mathbb{C}_{\mathbf{d}}^m) \times \prod_{i=1}^m \mathbb{P}S_{d_i}$ determines an element of $\mathcal{B}_q(\mathbf{d})$.

The natural and still open question if these constructions covers the entire set $K_q(\mathbf{d})$. In the particular case of $q = 2$ the question seems to be easier. We are looking forward two vectors λ_1 and λ_2 in $\mathbb{C}_{\mathbf{d}}^m$ (not parallel), and homogeneous polynomials such that:

$$\eta_1 \wedge \eta_2 = \left(\sum_{i=1}^m (\lambda_1)_i \frac{dF_i}{F_i}\right) \wedge \left(\sum_{j=1}^m (\lambda_2)_j \frac{dF_j}{F_j}\right) = 0.$$

Where we also include the additional hypotheses that $\eta_1 \neq 0$ and $\eta_2 \neq 0$, because in such other cases, the characterization holds from the previous development.

Now, we proceed to describe the possible generic injectivity of the rational parametrization ρ. This will be approached for $q = 2$. As in the case of logarithmic 1-forms, it will depend on whether $\mathbf{d}$ has repeated degrees. It would be noticed that the pretended approach is going to be related to the possible number of integrating factors. In other words, for a given homogeneous 2-form ω, we consider homogeneous polynomials F of the same degree as ω, such that:

$$d\left(\frac{\omega}{F}\right) = 0,$$

Let us define an incidence space which seems to be useful to understand the possible generic injectivity announced. So we consider:

$$\mathcal{X}_2(d, n) = \{(F, \omega) \in \mathbb{P}(S_d) \times \mathbb{P}(H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))) : d(\tfrac{\omega}{F}) = 0 \quad , \quad \omega \wedge \omega = 0\}$$

Also, notice that the second projection map defined on this space has image in the moduli space of foliations of degree d, i.e. $\mathcal{F}_2(d, \mathbb{P}^n)$. In addition, the image of this map will cover the union of all the possible logarithmic two forms of this degree, in the sense of the following result:

Proposition 4.3.11. If $(F,\omega) \in \mathcal{X}_2(d,n)$ then ω also satisfy the integrability condition 4.3.2, and hence:

$$\pi_2(\mathcal{X}_2(d,n)) \subset \mathcal{F}_2(d,\mathbb{P}^n).$$

Moreover, if $(F=0)$ defines a simple normal crossing divisor of degree d in $\mathbb{P}^n$, then also $\omega \in \mathcal{L}_2(\mathbf{d},n)$ where $\mathbf{d}$ is the vector of degrees associated to the irreducibles components of F.

Proof. For the first part observe that $d(\frac{\omega}{F})=0$ is equivalent to:

$$d\omega = \frac{dF}{F}\wedge\omega.$$

Also ω satisfies the decomposability condition $i_v(\omega)\wedge\omega = 0$. Then, the integrability condition follows from the previous two equations fulfilled by ω.

For the second remark of the statement, the idea is exactly the same as the used in proposition 2.4.27. Let $\mathcal{D}$ be the simple normal crossing divisor associated to $(F=0)$, and denote by $\{F_i\}_{i=1}^m$ the irreducible (simple) factors of F. Then, observe that the rational form $\frac{\omega}{F}$ has simple poles along $\mathcal{D}$, and moreover $d(\frac{\omega}{F})=0$. So this form determines a well defined section of $H^0(\mathbb{P}^n,\Omega^2_{\mathbb{P}^n}(\log\mathcal{D}))$. According to the characterization given at 3.5.9, we deduce $\frac{\omega}{F}$ is of the type:

$$\frac{\omega}{F} = \sum_{i,j}\lambda_{ij}\frac{dF_i}{F_i}\wedge\frac{dF_j}{F_j},$$

and so $\omega\in\mathcal{L}_2(\mathbf{d},n)$ as claimed. □

In this context, we are able to construct an open set in which the corresponding restriction of π_2 is injective. Unfortunately, it was not possible to show in general that this space is not empty. The problem behind this, is that we have not a general theory supporting the possible number of algebraic solutions of generic logarithmic q-forms as in the case of 1-forms (see for example the alternative argument 2.4.33).

Although, we have good evidence to affirm that the determined open set would be non-empty. In addition, we think that the extended Jouanolou's lemma established at 3.5.13 will be the central key for the needed results.

Let us explain the desired construction. The idea would be the same as that used in the case of 1-forms. We will use the upper semicontinuity of the fiber's dimension applied on the map $\pi_2|_{\mathcal{X}_2(d,n)}$ (see [29, Theorem 13.1.3]). Consider the open subset $\mathcal{H}_2\subset\mathcal{X}_2(d,n)$ defined by:

$$\mathcal{H}_2 = \{x=(F,\omega)\in\mathcal{X}_2(d,n) : \dim_x(\pi_2^{-1}(\omega))=0\}$$

Moreover, take into consideration the following natural map defined by:

$$\tilde{\rho}:\mathcal{P}_2(\mathbf{d}) \longrightarrow \mathcal{X}_2(d,n) \tag{4.3.9}$$

$$(\lambda_1\wedge\lambda_2,(F_i)_{i=1}^m) \longmapsto \left(F=\prod_{i=1}^m F_i\,,\ \omega=\rho(\lambda,(F_i))=\sum_{i,j}(\lambda_1\wedge\lambda_2)_{ij}\hat{F}_{ij}dF_i\wedge dF_j\right) \tag{4.3.10}$$

Once more, observe that the map $\tilde{\rho}$ is only rational. Although, it is well defined on the open set of generic parameters $\mathcal{U}_2(\mathbf{d})$.

Proposition 4.3.12. The restriction of the morphism $\pi_2 : \mathcal{X}_2(n,d) \to \mathcal{F}_2(d,\mathbb{P}^n)$ to $\mathcal{H}_2$ is injective.

Proof. Fix an element $x = (F,\omega) \in \mathcal{H}_2$, and suppose that there exist at least two elements (F,ω) and (G,ω) that belong to the fiber $\pi_2^{-1}(\omega)$. In addition, note that the condition for an homogeneous polynomial H of degree d to be and integrating factor of ω, i.e. $d(\frac{\omega}{H}) = 0$, is equivalent to:

$$Hd\omega = dH \wedge \omega.$$

Then, it is clear that

$$(\alpha F + \beta G, \omega) \in \pi_2^{-1}(\omega) \qquad \forall(\alpha : \beta) \in \mathbb{P}^1_{\mathbb{C}}.$$

In conclusion, it would exist at least an irreducible component of $\pi_2^{-1}(\omega)$ passing through x with dimension greater or equal to one. This last implies that the local dimension at that point satisfies: $\dim_x(\pi_2^{-1}(\omega)) \geq 1$. So that fiber can not contain two different elements, and this ends the proof. □

Finally, the expected injectivity result for ρ could be proven if we can construct at least one element $(\lambda = \lambda_1 \wedge \lambda_2, (F_i)_{i=1}^m) \in \mathcal{U}_2(\mathbf{d})$ such that:

$$\tilde{\rho}(\lambda, (F_i)_{i=1}^m) \in \mathcal{H}_2.$$

See proposition 4.3.15 for more details. In other words, we are looking for a generic logarithmic 2-form of type $\mathbf{d} = (d_1, \ldots, d_m)$:

$$\omega = \sum_{i,j} (\lambda_1 \wedge \lambda_2)_{ij} \hat{F}_{ij} dF_i \wedge dF_j,$$

such that $F = \prod_{i=1}^m F_i$ is the unique homogeneous polynomial of degree d satisfying: $d(\frac{\omega}{F}) = 0$.

Set a logarithmic 2-form ω with the previous formula, and denote by ω_1 and ω_2 the logarithmic 1-forms of type $\mathbf{d}$ associated respectively to the vectors λ_1 and λ_2 in $\mathbb{C}^m_{\mathbf{d}}$. Furthermore, we write η, η_1 and η_2 for the meromorphic logarithmic forms considered as the quotients of the previous forms by the polynomial F. Then, we can state the following equivalences.

Proposition 4.3.13. With the above notation, for $G \in S_d$ the following conditions are equivalent:

1. $d(\frac{\omega}{G}) = 0$
2. $(GdF - FdG) \wedge \omega = 0.$
3. $d(\frac{F}{G}) \wedge \omega = 0.$
4. $(\frac{dF}{F} - \frac{dG}{G}) \wedge \eta_1 \wedge \eta_2 = 0.$
5. $d(\frac{F}{G}) \wedge \omega_1 \wedge \omega_2 = 0$

Proof. They are all straight forward calculations supported on the equivalence:

$$d(\tfrac{\omega}{H}) = 0 \iff d\omega = \frac{dF}{F} \wedge \omega,$$

applied to the polynomials F and G. □

Remark 4.3.14. Note that if any of the logarithmic 1-forms of type **d** given by ω_1 and ω_2 admits another integrating factor G, then $\frac{F}{G}$ will determine a rational first integral of the corresponding form ω_i, i.e.

$$d(\frac{F}{G}) \wedge \omega_i = 0.$$

Also, according to the previous proposition, the polynomial G will be also another integrating factor for ω. So it is expected that we should assume that $\lambda_i \notin \mathbb{PQ}^m$ for $i = 1$ and $i = 2$, which ensure that each ω_i has a unique integrating factor (See propositions 2.4.10 and 2.4.11).

Our conjecture is that for generic parameters $(\lambda = \lambda_1 \wedge \lambda_2, (F_i)) \in \mathcal{U}_2(\mathbf{d})$ such that $\lambda_1, \lambda_2 \notin \mathbb{PQ}^m$, the polynomial F will be the unique integrating factor of degree d of the logarithmic 2-form $\omega = \frac{1}{F}\omega_1 \wedge \omega_2$.

In conclusion, the key of our approach for the generic injectivity depends on the non-trivial fact is to determine when a logarithmic 2-form has more than one integrating factor of degree d.

We end this section with a possible injectivity result which depends on our conjecture. It is based on the following commutative diagram:

$$\begin{array}{ccc} \mathcal{U}_2(\mathbf{d}) & \xrightarrow{\tilde{\rho}} & \chi_2(d,n) \\ & \searrow^{\rho} & \downarrow^{\pi_2} \\ & & \mathcal{F}_2(d,\mathbb{P}^n) \end{array} .$$

In addition, if we could prove the existence of the previous announced logarithmic 2-form of type **d**, then it will follow that the open set $\mathcal{H}_2$ is not empty, and furthermore the same will hold for the open set $\mathcal{U}_2(\mathbf{d}) \cap \tilde{\rho}^{-1}(\mathcal{H}_2)$. In this context, we can state the following result.

Proposition 4.3.15. Assume that $\tilde{\mathcal{U}}_2 := \mathcal{U}_2(\mathbf{d}) \cap \tilde{\rho}^{-1}(\mathcal{H}_2) \neq \emptyset$. Then, if the vector of degrees **d** has non-repeated elements, the natural parametrization $\rho : \mathcal{P}_2(\mathbf{d}) \dashrightarrow \mathcal{L}_2(\mathbf{d}, n) \subset \mathcal{F}_2(d, \mathbb{P}^n)$ restricted to $\tilde{\mathcal{U}}_2$ is injective. Moreover, for **d** general, the restriction to that open set is a fine map.

Proof. We follow the same idea as in the corresponding proof for 1-forms. For every $\omega = \rho(\lambda, (F_i))$, the associated polynomial $F = \prod_{i=1}^m F_i$ (up to constants) will be unique. The rest of the proof depends on an algebraic argument over the irreducible polynomials $F_1, \ldots, F_m$. □

4.3.3 The Zariski tangent space of $\mathcal{F}_2(d, \mathbb{P}^n)$.

In order to perform the infinitesimal proof of the surjectivity result 4.4.2, we need to establish an appropriate characterization of the Zariski tangent space of the corresponding moduli space of foliations. We will keep the same notation of the section 2.5.1 in chapter 2.

Remember that for a classical projective space $X = \mathbb{PV}$, it is common to identify its Zariski tangent space at a given point $x = \pi(v)$ with:

$$\mathcal{T}_x X = \mathbb{V}/_{\langle v \rangle}.$$

Also remember that $\mathcal{F}_2(d,\mathbb{P}^n)$ is a closed subscheme of the projective space $\mathbb{P}(H^0(\mathbb{P}^n,\Omega^2_{\mathbb{P}^n}(d)))$ defined by the equations 4.3.1 (integrability condition) and 4.3.4 (decomposability condition for 2-forms). So, with a slight abuse of notations, the Zariski tangent space $\mathcal{T}_\omega\mathcal{F}_2(d,\mathbb{P}^n)$ can be represented by homogeneous projective forms $\alpha \in H^0(\mathbb{P}^n,\Omega^2_{\mathbb{P}^n}(d))/_{\langle\omega\rangle}$ satisfying the following equations:

$$\begin{aligned}(\omega+\epsilon\alpha)\wedge(\omega+\epsilon\alpha) &= 0 \quad \mathrm{mod}(\epsilon^2)\\ (i_v\omega+\epsilon i_v\omega)\wedge(d\omega+\epsilon d\alpha) &= 0 \quad \mathrm{mod}(\epsilon^2) \text{ and } \forall v\in\mathbb{C}^{n+1}.\end{aligned}$$

This last description is closely related to the following idea: an element α in the Zariski tangent space at $\omega \in \mathcal{F}_2(d,\mathbb{P}^n)$ determines an infinitesimal first order perturbation of ω, so the form $\omega+\epsilon\alpha$ should satisfy "at first order" the equations 4.3.1 and 4.3.4.

Remark 4.3.16. Fix an element $\omega \in \mathcal{F}_2(d,\mathbb{P}^n)$. A simple calculation shows $\alpha \in H^0(\mathbb{P}^n,\Omega^2_{\mathbb{P}^n}(d))/_{\langle\omega\rangle}$ belongs to the Zariski tangent space of $\mathcal{F}_2(d,\mathbb{P}^n)$ at ω if and only if it fulfills the following conditions:

$$\alpha\wedge\omega = 0 \tag{4.3.11}$$

$$(i_v\omega\wedge d\alpha)+(i_v\alpha\wedge d\omega) = 0 \quad \forall v\in\mathbb{C}^{n+1}. \tag{4.3.12}$$

We refer to the first equation as the **locally decomposability perturbation equation** and to the second as the **integrability perturbation equation**. In addition, it is remarkable that the vectors v involved in the previous equations can be also consider as local rational fields.

In conclusion, the desired Zariski tangent space can be described by:

$$\mathcal{T}_\omega\mathcal{F}_2(d,\mathbb{P}^n) = \{\alpha\in H^0(\mathbb{P}^n,\Omega^2_{\mathbb{P}^n}(d))/_{\langle\omega\rangle} : \alpha \text{ satisfies } 4.3.11 \text{ and } 4.3.12\}.$$

4.3.4 The derivative of the natural parametrization.

As it was announced in the previous sections, one of our main purposes is to show that the differential of the natural parametrization ρ, which defines the logarithmic components $\mathcal{L}_2(n,\mathbf{d})$, is surjective. With this idea in mind, it is important to describe the derivative of the rational map ρ.

Remember that $S_e = H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(e))$ denotes the space of homogeneous polynomials of degree e. With the identifications made at the previous section, we can describe the Zariski tangent space of its corresponding projectivization by:

$$\mathcal{T}_{\pi(F)}\mathbb{P}(S_e) = S_e/_{\langle F\rangle}.$$

On the other hand let $\lambda = \lambda_1\wedge\lambda_2$ be an element of the grassmannian space $Gr(2,\mathbb{C}^m_{\mathbf{d}})$ (described by the image of the Plücker embedding into $\mathbb{P}\bigwedge^2\mathbb{C}^m_{\mathbf{d}}$). Then, the Zariski tangent space of $Gr(2,\mathbb{C}^m_{\mathbf{d}})$ at λ is characterized by the antisymmetric constant vectors $\lambda' \in \bigwedge^2(\mathbb{C}^m_{\mathbf{d}})/_{\langle\lambda\rangle}$ such that:

$$\lambda'\wedge\lambda = 0.$$

Also observe that this is the same equation as 4.3.11, which was associated to the decomposability condition for a first order perturbation. Even more, the vectors λ' which satisfy the above equation, are in correspondence with the choice of two vectors $\lambda'_1,\lambda'_2 \in \mathbb{C}^m_{\mathbf{d}}/_{\langle\lambda_1,\lambda_2\rangle}$ fulfilling:

$$\lambda' = \lambda'_1\wedge\lambda_2+\lambda_1\wedge\lambda'_2 \tag{4.3.13}$$

From now on, we will write λ' to denote an element of the tangent space:

$$\mathcal{T}_\lambda Gr(2, \mathbb{C}^m_{\mathbf{d}}) = \{\lambda' \in \bigwedge^2 (\mathbb{C}^m_{\mathbf{d}})/_{\langle\lambda\rangle} : \lambda' \wedge \lambda = 0\}.$$

Consequently with the characterizations given for the tangent spaces involved in the tangent of the parameter space $\mathcal{P}_2(\mathbf{d})$, we can state the following remark, which describes the derivative wanted.

Remark 4.3.17. For a given point $(\lambda = \lambda_1 \wedge \lambda_2, (F_i)_{i=1}^m) = (\lambda, \underline{F}) \in \mathcal{P}(\mathbf{d})$, recall the formula defining the natural parametrization ρ:

$$\omega = \rho((\lambda, \underline{F})) = \sum_{\{i,j\}} \lambda_{ij} \hat{F}_{ij} dF_i \wedge dF_j$$

Also remember that we usually avoid the notation [] in all the corresponding projective classes. Then, the derivative

$$d\rho(\lambda, \underline{F}) : \mathcal{T}_\lambda Gr(2, \mathbb{C}^m_{\mathbf{d}}) \times \prod_{i=1}^m S_{d_i}/_{\langle F_i \rangle} \longrightarrow \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$$

is calculated by multilinearity as

$$d\rho(\lambda, \underline{F})(\lambda', (F'_1, \ldots, F'_m)) = \underbrace{\sum_{i \neq j} \lambda'_{ij} \hat{F}_{ij} dF_i \wedge dF_j}_{\alpha_1} + \underbrace{\sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} F'_k dF_i \wedge dF_j + 2 \sum_{i \neq j} \lambda_{ij} \hat{F}_{ij} dF'_i \wedge dF_j}_{\alpha_2} \tag{4.3.14}$$

Also, recall that λ' is used to denote an element of the form 4.3.13. Furthermore, ω_1 and ω_2 are used to separate the perturbations in the image of the differential which are associated to the following different partial derivatives:

$$\begin{aligned} \alpha_1 &= d\rho(\lambda, \underline{F})(\lambda', (0, \ldots, 0)) \\ \alpha_2 &= d\rho(\lambda, \underline{F})(0, (F'_1, \ldots, F'_m)) \end{aligned}$$

It is important to remark that the forms α_1 and α_2 are both first order perturbations of ω. In other words, each α_i satisfies the equations 4.3.11 and 4.3.12. On the other hand, these two forms vanish on different codimensional strata associated to the divisor $\mathcal{D}_F = (F = 0)$. This will be summarized in the following result:

Remark 4.3.18. As usual, if $X^k_{\mathcal{D}_F}$ denotes the codimension k subscheme associated to the divisor $\mathcal{D}_F$ described by:

$$X^k_{\mathcal{D}_F} = \bigcup_{I:|I|=k} X_I = (F_{i_1} = \ldots F_{i_K} = 0),$$

then, note that α_1 vanishes on $X^3_{\mathcal{D}_F}$ and α_2 over $X^4_{\mathcal{D}_F}$ (but not over $X^3_{\mathcal{D}_F}$).

Corollary 4.3.19. Every element $\tilde{\alpha}$ which belong to the image of the differential must satisfy:

$$\tilde{\alpha}|_{X^4_{\mathcal{D}_F}} = 0.$$

These last conclusions will be important at the time of proving the generic surjectivity of the natural parametrization. The reason is that, in some sense, they give a way of distinguish separately perturbations associated to a variation of the constants "λ" and variations of the polynomials "F_i".

4.4 Infinitesimal stability of a generic logarithmic 2-form

4.4.1 Main results

Now we establish the proof of the stability of generic logarithmic 2-forms in the space of codimension two projective foliations. It will be important to remember the definition of the open conditions stated over the space of parameters, which in particular let us define the set $\mathcal{U}_2(\mathbf{d})$ (4.3.8).

Moreover, we need to add an extra condition for the vector of degrees $\mathbf{d}$, which is an extended assumption than the balanced condition introduced for 1-forms. Concretely, we use the hypothesis of $\mathbf{d}$ being 2-balanced according to the definition 4.4.11 given at section 4.4.2.

The desired result can be summarized in the following theorem:

Theorem 4.4.1. Set $n, m \geq 4$, and fix a 2-balanced vector of degrees $\mathbf{d} = (d_1, \dots, d_m)$ with $d = \sum_i d_i$. The variety $\mathcal{L}_2(\mathbf{d}, n)$ is an irreducible component of the moduli space $\mathcal{F}_2(d, \mathbb{P}^n)$. Moreover $\mathcal{F}_2(d, \mathbb{P}^n)$ is reduced at the points of $\rho(\mathcal{U}_2(\mathbf{d}))$.

The proof of this theorem will be implied by the surjectivity of the derivative of the natural parametrization (proposition below), combined with some basic arguments of scheme theory.

Proposition 4.4.2. Suppose $n, m \geq 4$ and fix a 2-balanced vector $\mathbf{d} = (d_1, \dots, d_m)$ such that $d = \sum_i d_i$. Let $(\lambda_1 \wedge \lambda_2, (F_i)_{i=1}^m) = (\lambda, \underline{F}) \in \mathcal{U}_2(\mathbf{d})$ and $\omega = \rho(\lambda, \underline{F})$. Then, the derivative:

$$d\rho(\lambda, \underline{F}) : \mathcal{T}_\lambda Gr(2, \mathbb{C}^m) \times \prod_{i=1}^{m} S_{d_i}/\langle F_i \rangle \longrightarrow \mathcal{T}_\omega \mathcal{F}_2(d, \mathbb{P}^n)$$

is surjective.

With the same strategy as the one used in the main results of the chapter 2, we will write down first the argument for the Theorem 4.4.1, assuming that the proposition 4.4.2 is correct.

The idea behind the proof of the above theorem is basically the same as the used on the proof of the main Theorem 2.5.10, but with the difference that in this case we can deduce less properties than before, due to the non-possible generic injectivity of the parametrization ρ. For more details, we also refer to the remark 2.5.13 and the lemma 2.5.14.

Proof theorem 4.4.1. Remember that the variety $\mathcal{L}_2(\mathbf{d}, n)$ is defined as the Zariski closure of the image of the rational map: $\rho : \mathcal{P}_2(\mathbf{d}) \dashrightarrow \mathcal{F}_2(n, \mathbf{d})$, which is well defined over $\mathcal{U}_2(\mathbf{d})$. So $\mathcal{L}_2(n, \mathbf{d}) = \overline{\rho(\mathcal{U}_2(\mathbf{d}))}$ is an irreducible projective variety generically smooth and reduced.

Now, let us consider $\mathcal{F}_2(d, \mathbb{P}^n)_{red}$ as the reduced scheme structure associated to the moduli space $\mathcal{F}_2(d, \mathbb{P}^n)$. In addition, the map ρ factors through this reduced space according to the diagram:

$$\begin{array}{ccc} \mathcal{U}_2(\mathbf{d}) & \xrightarrow{\ \rho\ } & \mathcal{F}_2(d, \mathbb{P}^n) \\ & \searrow^{\tilde{\rho}} & \uparrow i_{red} \\ & & \mathcal{F}_2(d, \mathbb{P}^n)_{red} \end{array}$$

We again refer to [30, 5.1.5]. Next, select an element $(\lambda, \underline{F}) \in \mathcal{U}(\mathbf{d})$ and ω its respective image by ρ, then according to proposition 4.4.2 the derivative $d_{(\lambda,\underline{F})}\rho : \mathcal{T}_{(\lambda,\underline{F})}\mathcal{P}(\mathbf{d}) \to \mathcal{T}_\omega \mathcal{F}_2(n, d)$ is surjective, and factors through:

$$\mathcal{T}_\omega \mathcal{L}_2(n, \mathbf{d}) \subset \mathcal{T}_\omega(\mathcal{F}_2(n, d))_{red} \subset \mathcal{T}_\omega \mathcal{F}_2(n, d)$$

In conclusion, it follows that: $\mathcal{T}_\omega \mathcal{L}_2(n, \mathbf{d}) = \mathcal{T}_\omega(\mathcal{F}_2(n, d))_{red} = \mathcal{T}_\omega \mathcal{F}_2(n, d)$, for every element $\omega \in \rho(\mathcal{U}_2(\mathbf{d}))$. The proof ends with the same argument as the stated in the proof of Theorem 2.5.10.

Finally, it follows that $\mathcal{L}_2(\mathbf{d}, n)$ is an irreducible component of $\mathcal{F}_2(d, \mathbb{P}^n)$, which is also generically reduced. □

4.4.2 Suryectivity of the derivative of the natural parametrization

In order to complete the proof of theorem 4.4.1, we need to set up the demonstration of proposition 4.4.2. This is going to be organized in several steps with its respective lemmas and remarks. Before the beginning of the argument, we need to perform the principal technical lemmas needed.

The setup required is identical as before: $\mathcal{D} = \sum_{i=1}^m \mathcal{D}_i$ is going to be a simple normal crossing divisor fixed on $\mathbb{P}^n$, and of degree $d = \sum_i d_i$. In particular it is described in homogeneous coordinates by the union of the zero locus of irreducible elements $F_i \in S_{d_i}$ (for $i = 1, \ldots, m$), which has a normal type crossing. Alternatively, for each index $i \in \{1, \ldots, m\}$ we could also consider the elements F_i as regular global functions over the affine cone $\mathbb{C}^{n+1}$.

Similarly, we say that a form is a homogeneous affine k-form of **total degree** d if it can be described by:

$$\mu = \sum_{I:|I|=k} A_I dz_{i_1} \wedge \cdots \wedge dz_{i_k}, \tag{4.4.1}$$

for some homogeneous polynomials A_I of degree $d - k$. Furthermore, recall the homogeneous projective k-forms, $\alpha \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d))$, can be characterized by the same above formula, but also α must satisfy the descend condition: $i_R(\alpha) = 0$.

The fixed hypotheses on the polynomials $F_1, \ldots, F_m$ (see 4.3.8) will be of particular importance in the sequel. Especially the following one:

($*$) *for each multi-index $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ and every point $x \in X_I = (F_{i_1}(x) = \cdots = F_{i_k}(x) = 0) \subset \mathbb{P}^n$, the following holds*

$$d_x F_{i_1} \wedge \cdots \wedge d_x F_{i_k} \neq 0.$$

In addition, remember that the associated strata

$$X^k_{\mathcal{D}} = \bigcup_{I:|I|=k} X_I = (F_{i_1} = \cdots = F_{i_k} = 0)$$

defines a codimension k projective subvariety, and each stratum X_I is also a smooth complete intersection.

The next lemma is similar to that described at [15, Lemma 2.2], and also it is a generalization of the lemma 2.5.22. Again it will be a consequence of Saito's lemma in [46].

Lemma 4.4.3 (Division lemma)**.** Assume $n \geq 3$. Consider integers j and q, with $1 \leq j \leq q \leq n-2$ and fix $I = \{i_1, \ldots, i_q\}$ a multi-index of size q. For $k = 1$ or $k = 2$, suppose $\mu \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d))$ satisfies:

$$(\mu \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_j})|_{X_I} = 0.$$

Then, there exist a j-tuple of forms $\{\gamma_l\}_{l=1}^j$, where each $\gamma_l \in H^0(\mathbb{C}^{n+1}, \Omega^{k-1}_{\mathbb{C}^{n+1}})$ is a homogeneous affine form of total degree $d - d_{i_l}$, and satisfy:

$$\mu|_{X_I} = \left(\sum_{l=1}^{j} \gamma_l \wedge dF_{i_l}\right)\Big|_{X_I}$$

In addition, the same conclusion holds for affine forms $\mu \in H^0(\mathbb{C}^{n+1}, \Omega^k_{\mathbb{C}^{n+1}})$ which are homogeneous of total degree d, and satisfy the above assumption.

This result is in fact true for arbitrary integers $k \leq n-q$, but it is going to be only performed on the cases we need. Moreover, we consider important to keep in mind the notation and the idea behind the proof for future applications, specially for the study of logarithmic q-forms of higher degrees.

Proof. The proof of the entire result will be essentially the same as the developed for the corresponding proof of the particular case 4.4.3.

We write $X = \mathbb{C}^{n+1}$, and use $Y \subset X$ to denote the affine cone over the codimension q complete intersection subvariety fixed and expressed by X_I. Also we write $\mathcal{I}_Y$ for its associated ideal, described by: $\mathcal{I}_Y = \langle F_{i_k} \rangle_{k=1}^q$.

It is important to recognize the notation and the usuals exact sequences for the restriction to Y of the sheaves of forms. So we refer to the digression made at 2.5.21 to set the notation and the properties used. In addition, remember that $\Omega^1_X|_Y$ is an $\mathcal{O}_Y$-module freely generated by the global sections $dz_0|_Y, \ldots, dz_n|_Y$.

Moreover, note that the assumption $(*)$ implies that the unique singularity (in Y) of the j-form:

$$dF_{i_1}|_Y \wedge \cdots \wedge dF_{i_j}|_Y,$$

is the point zero. Now, consider the ideal $\mathcal{A}$ generated by the coefficients $\{a_{l_1 \ldots l_j}\}$ determined by the decomposition:

$$dF_{i_1}|_Y \wedge \cdots \wedge dF_{i_j}|_Y = \sum_{0 \leq l_1 < \cdots < l_j \leq n} a_{l_1 \ldots l_j} dz_{i_1} \wedge \cdots \wedge dz_{l_j}.$$

The depth of this ideal $\mathcal{A}$ is hight, in particular it is greater or equal than 3. As a comment, observe that the depth of $\mathcal{A}$ is greater or equal than the dimension of Y, which is exactly $n+1-q$. For this reason is that we announce we are able to state the result for $k \leq n-q$.

In conclusion, we are able to apply Saito's lemma ([46]) to divide 1-forms and 2-forms restricted to the subvariety Y. According to this last fact, and the hypothesis assume over the homogeneous form $\mu|_Y \in H^0(Y, \Omega^k_X|_Y)$, there exist homogeneous restricted global $k-1$-forms $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_j$ in $H^0(Y, \Omega^{k-1}_X|_Y)$ such that:

$$\mu|_Y = \sum_{k=1}^{j} \tilde{\gamma}_k \wedge dF_{i_k}|_Y.$$

Furthermore, according to the grading induced by the coordinate ring of Y, note that these new introduced forms $\tilde{\gamma}_{i_1}, \ldots, \tilde{\gamma}_{i_j}$ can be selected homogeneous of the corresponding degrees $d-d_{i_1}, \ldots, d-d_{i_j}$. To end the proof we just need to construct global homogeneous forms whose restriction to Y coincides with the previous obtained forms.

The case $k = 1$ has already been proved in the particular lemma 2.5.22. Its corresponding proof is the following. Each function $\tilde{\gamma}_k$ (for $k = 1, \ldots, j$) can be considered as an element of the cohomology group $H^0(X_I, \mathcal{O}_{X_I}(d-d_{i_k}))$. Then recall that the existence of an homogeneous polynomial $\gamma_k \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-d_{i_k}))$ such that: $\gamma_k|_Y = \tilde{\gamma}_k$, depends on the vanishing of the groups $H^1(\mathbb{P}^n, \mathcal{I}_{X_i}(d-d_{i_k}))$ (for all these needed selected degrees). Moreover these cohomology groups always vanish since the projective subvarieties determined by X_I are complete intersections.

For the next case ($k = 2$) the proof is essentially the same. Each of the 1-forms $\tilde{\gamma}_1, \ldots \tilde{\gamma}_j$ can be described by:

$$\tilde{\gamma}_k = \sum_{l=0}^{n} A^k_l dz_l|_Y \quad \forall k = 1, \ldots, j,$$

where the elements $\{A^k_l\}_{k,l}$ can be again considered as homogeneous elements belonging to cohomology groups like $H^0(X_I, \mathcal{O}_{X_I}(d-d_{i_k}-1))$. Applying the same reasoning as in the previous case, it is clear we can construct the desired homogeneous global affine 1-forms $\gamma_1, \ldots, \gamma_j$ with the restriction property announced. Note that these new introduced global homogeneous affine forms not necessarily satisfy the descend condition to the projective space: $i_R(\gamma) = 0$.

□

At this time, we will describe lemmas which characterize homogeneous affine and projective k-forms, whose restrictions over the points of varieties of the type $X^k_{\mathcal{D}}$ vanish. For a more detailed development of these type of results we refer to the appendix A. The following lemma is stated for the specific cases we need in the sequel.

Lemma 4.4.4 (Vanishing lemma)**.** Set $\alpha \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d))$, and let $\beta \in H^0(\mathbb{C}^{n+1}, \Omega^k_{\mathbb{C}^{n+1}})$ be an homogeneous affine form of total degree d.

a) Suppose $\beta|_{X^2_{\mathcal{D}}} = 0$, then there exist a family of affine k-forms $\{\beta_j\}$ homogeneous of respective total degrees $(d-d_j)^m_{j=1}$ such that:

$$\beta = \sum_{j=1}^{m} \hat{F}_j \, \beta_j$$

b) Suppose $\alpha|_{X^3_{\mathcal{D}}} = 0$, then there exist a family of homogeneous affine k-forms $\{\alpha_{i,j}\}_{i,j=1}^m$ such that:

$$\alpha = \sum_{\substack{i,j=1 \\ (i<j)}}^{m} \hat{F}_{ij}\ \alpha_{ij}$$

Alternatively on b) we will use the notation $\alpha_{\{i,j\}}$ to not make reference to the order between the two distinct indexes selected. Also in both cases, when the total degree of a new form introduced is strictly negative it will be evidently considered as zero.

Proof. It corresponds to a particular case of the propositions A.0.22 and A.0.21 (see appendix A). The key of this proof (see the proposition A.0.20) is to show the elements of the type:

$$\hat{F}_J, \quad J \subset \{1, \dots, m\} : |J| = k - 1,$$

are generators of the homogeneous ideal associated to each codimension k subvariety $X^k_{\mathcal{D}}$. □

Lemma 4.4.5 (Fundamental lemma)**.** Fix a family of homogeneous polynomials on $n+1$ variables $\{B_{ijk}\}_{i,j,k\in\{1\dots m\}}$, symmetric on the first two indexes ($B_{ijk} = B_{jik}$). We assume that their respective degrees are: $\deg(B_{ijk}) = d_k$. Also suppose that these polynomials satisfy the relations:

$$B_{ijl}(x) = B_{ikl}(x) = B_{jkl}(x) \qquad \forall x \in X_{ijkl} = (F_i = F_j = F_k = F_l = 0).$$

Then, for each index $k \in \{1, \dots, m\}$, there exist a polynomial F'_k (also of degree d_k) such that, for each selected indexes $i.j$, the following decomposition holds:

$$B_{ijk}(x) = F'_k(x) + \hat{F}_{jk}\tilde{B}_{ijk} \qquad \forall x \in X_{jk} = (F_j = F_k = 0).$$

Where $\tilde{B}_{ijk}$ is another homogeneous polynomials of degree $d_k - \sum_{i\neq j,k} d_i$. If this last number is negative the corresponding polynomial will be treated as zero.

Proof. In general, the subsequent varieties involved are considered as projective subvarieties immersed in $\mathbb{P}^n$. Select two fixed indexes k, l, and write $X = X_{kl}$. We denote by $\mathcal{D}_X$ the divisor $\mathcal{D}$ restricted to X, i.e. $\mathcal{D}_X = \bigcup_{i\neq k,\, l}(F_i|_X = 0)$. Now, according to the following equality over X:

$$B_{jkl} = B_{ikl},$$

the family of polynomials $\{B_{ikl}\}_{i\neq k,\, l}$ determines a well defined object in $H^0(\mathcal{D}_X, \mathcal{O}_{\mathcal{D}_X}(d_l))$. Now, consider the usual short exact sequence of sheaves:

$$0 \to \mathcal{I}_{\mathcal{D}_X} \to \mathcal{O}_X \to i_*(\mathcal{O}_{\mathcal{D}_X}) \to 0$$

and consider its tensor product by $\mathcal{O}_X(d_l)$, to take into account the long exact sequence:

$$0 \to H^0(X, \mathcal{I}_{\mathcal{D}_X}(d_l)) \to H^0(X, \mathcal{O}_X(d_l)) \to H^0(\mathcal{D}_X, \mathcal{O}_{\mathcal{D}_X}(d_l)) \to H^1(X, \mathcal{I}_{\mathcal{D}_X}(d_l)) \to \dots.$$

In addition, observe that $\mathcal{D}_X = (\hat{F}_{k\,l} = 0) \cap X$ and so according with the hypotheses assumed over the polynomials $\{F_i\}$, we get:

$$\mathcal{I}_{\mathcal{D}_X}(d_l) = \mathcal{O}_{\mathbb{P}^n}(d_l - (d - d_k - d_l))|_X = \mathcal{O}_X(2d_l + d_k - d).$$

With this identification ,the first morphism of the above long sequence corresponds to:

$$H^0(X, \mathcal{O}_X(2d_l + d_k - d)) \to H^0(X, \mathcal{O}_X(d_l))$$
$$G \mapsto G.\hat{F}_{kl}$$

Also, note that $H^1(X, \mathcal{O}_X(2d_l + d_k - d)) = 0$, which implies that the second morphism of this sequence is surjective. This last vanishing property used is true since X is a complete intersection over $\mathbb{P}^n$ (see for instance [37]). Summarily, we have obtained the following short exact sequence:

$$0 \to H^0(X, \mathcal{O}_x(2d_l + d_k - d)) \to H^0(X, \mathcal{O}_X(d_l)) \to H^0(\mathcal{D}_X, \mathcal{O}_{\mathcal{D}_X}(d_l)) \to 0. \tag{4.4.2}$$

Because of that, we conclude the existence of an element $\tilde{H}_{kl} \in H^0(X, \mathcal{O}_X(d_l))$ whose restriction to $\mathcal{D}_X$ equals to the element $\{B_{ikl}\}_{i \neq k,\,l}$. Moreover, with a similar argument, it is easy to see that the restriction:

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_l)) \xrightarrow{|_X} H^0(X, \mathcal{O}_X(d_l))$$

is surjective. So we can choose a global homogeneous polynomial H_{kl} such that:

$$H_{kl} = B_{ikl} \quad \text{over } X_{ikl} \quad , \forall i \neq k,\, l.$$

Also, notice that $(H_{kl}|_X - B_{ikl}|_X)|_{\mathcal{D}_X} = 0$ (for all $i \neq k,\ , l$), and so according to the sequence 4.4.2, we can construct for each i a global homogeneous polynomial $\tilde{B}_{ikl}$ fulfilling:

$$B_{ikl} = H_{kl} + \hat{F}_{kl}\tilde{B}_{ikl}.$$

In addition $H_{kl} = B_{ikl} = B_{kil} = H_{il}$ over X_{ikl}. For this reason, we have construct a family of homogeneous polynomials $\{H_{kl}\}$ all of degree d_l such that $H_{kl} = H_{il}$ on the triple intersection X_{ikl}. Then, for the more basic fundamental lemma stated at 2.5.23, we conclude the existence an homogeneous polynomial F'_l of degree d_l such that:

$$H_{kl} = F'_l + \hat{F}_l\tilde{H}_{kl} \quad \text{over } X_l.$$

The polynomials F'_l and the family $\tilde{B}_{ikl}$ satisfy the condition inferred by the present lemma. □

We end this part of the section with an useful construction.

Remark 4.4.6 (Field's with the δ-hypothesis and a short digression about Logarithmic vector fields)**.** For the rest of the chapter, including various proofs, we need to consider certain rational fields, dual (in some sense) to the 1-forms $\{dF_i\}$. We keep the same notations and hypotheses on the homogeneous polynomials $\{F_i\}_{i=1}^m$ involved in the definition of the logarithmic regular forms.

The desired construction is the following:

Fix an integer $k < n$, and select a multi-index $I = \{i_1, \dots, i_k\} \subset \{1, \dots, m\}$. For each $j \in \{1, \dots, k\}$ we need to select a rational vector field $V_I^{i_j}$ such that:

$$i_{V_I^{i_j}}(dF_{i_l}) = \delta_{jl} \quad \forall l \in \{1, \dots, k\}, \tag{4.4.3}$$

over the points of X_I (or alternatively of its affine cone $Y = (F_{i_1} = \cdots = F_{i_k} = 0)$).

At first note that outside the closed subset of points where the k-form $dF_{i_1} \wedge \cdots \wedge dF_{i_k}$ vanishes, we can construct rational vector fields with the property 4.4.3. This is for example the construction needed in the proof of [15, Proposition 3.1], where the Authors deal with the same equations for the moduli space of foliations with higher codimension. The problem with these rational fields is that we can not ensure that we are able to evaluate them at the points of the subvariety X_I. So the problem is to understand when we are able to remove the functions like F_i of the denominators of our rational fields. This seems to be hard and technical, but the affirmative answer would be again related to the word "logarithmic".

The correct objects needed in this case are the so called *Logarithmic vector fields*. The formal definition of the logarithmic derivations over an hypersurface $\mathcal{D}$ (with associated ideal $\mathcal{I}$) of a complex algebraic variety X, is the following:

$$Der_X(-\log \mathcal{D})_p = \{\chi \in (Der_X)_p : \chi(\mathcal{I}_p) \subset \mathcal{I}_p\}$$

In other words, if h_p is the local defining equation of $\mathcal{D}$ at a point p, we are dealing with derivations χ such that: $\chi h_p \in \langle h_p \rangle \mathcal{O}_{X,p}$.

In the case where $\mathcal{D}$ is the union of smooth subvarieties, which are normal crossing, this notion can be consulted at [18] and [36]. And for a treatment in singular and non-normal crossings cases we refer to [47].

As it can be seen at [47], the modules:

$$\Omega^1_X(\log \mathcal{D})_p \quad \text{and} \quad Der_X(-\log \mathcal{D})_p$$

are reflexive $\mathcal{O}_{X,p}$-modules, dual to each other.

Coming back to our purposes, as it can be also seen at [43], the simple normal crossing case provides a simple example of the renowned Saito's critterion ([47] pp.270). With the same notation as in 3.3.1, fix local coordinates $f_1, \dots, f_n$ in p, and assume the divisor is defined by the zero locus of $f_1, \dots, f_s$. Then $\frac{df_1}{f_1}, \dots, \frac{df_s}{f_s}, df_{s+1}, \dots, df_n$ is a free system of generators of $\Omega^1_X(\log \mathcal{D})_p$. Moreover the local fields:

$$\delta_1 = f_1 \cdot \frac{\partial}{\partial f_1}, \dots, \delta_s = f_s \cdot \frac{\partial}{\partial f_s}, \delta_{s+1} = \frac{\partial}{\partial f_{s+1}}, \dots, \delta_n = \frac{\partial}{\partial f_n}$$

determines a basis of $Der_X(-\log \mathcal{D})_p$, dual to the usual basis for the local logarithmic 1-forms.

In conclusion, for the simple normal crossing projective divisor $\mathcal{D}_F$ determined by the zero locus

$$(F = F_1 \cdot \cdots \cdot F_m = 0),$$

it is possible to construct, locally on each point of the component X_I, fields $V_I^{i_j}$ with the property 4.4.3. We just need to consider the well defined field $\frac{1}{F_{i_j}}\delta_j$ (according to the notation of the previous paragraph).

This last digression ends our technical tools needed for the proof of the surjectivity result 4.4.2.

Beginning of the proof of proposition 4.4.2

Note: The hypothesis of **d** *being 2-balanced will be introduced only when necessary.*

Let $(\lambda_1 \wedge \lambda_2, (F_i)_{i=1}^m) = (\lambda, \underline{F}) \in \mathcal{U}_2(\mathbf{d})$, and ω its image by ρ, which is described by the formula:

$$\omega = \sum_{\{i,j\}} \lambda_{ij} \hat{F}_{ij} dF_i \wedge dF_j.$$

Take $\alpha \in \mathcal{T}_\omega(\mathcal{F}_2(d, \mathbb{P}^n))$ a Zariski tangent vector of $\mathcal{F}_2(d, \mathbb{P}^n)$ at ω, i.e. a projective form $\alpha \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))/_{\langle\omega\rangle}$ satisfying the equations 4.3.11 and 4.3.12. What we want to prove is there exist an element $(\lambda', (F_i')_{i=1}^m) \in T_{(\lambda,\underline{F})}\mathcal{U}_2(\mathbf{d})$ such that:

$$d\rho(\lambda, \underline{F})(\lambda', (F_i')_{i=1}^m) = \alpha.$$

We consider important to keep in mind the notations introduced at 4.3.4, in special such for λ' lying on the corresponding tangent space to λ on the grassmannian space $Gr(2, \mathbb{C}^m_{\mathbf{d}})$. Taking into consideration the formula 4.3.14, the problem is equivalent to show that:

$$\alpha = \sum_{i \neq j} \lambda'_{ij} \hat{F}_{ij} dF_i \wedge dF_j + \sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} F'_k dF_i \wedge dF_j + 2 \sum_{i \neq j} \lambda_{ij} \hat{F}_{ij} dF'_i \wedge dF_j.$$

In the sake of clarity, we will separate the proof in several remarkable steps related to the possible vanishing of the perturbation α over each stratum $X^k_{\mathcal{D}_F}$.

Our first step is to prove that α must vanish on $X^4_{\mathcal{D}_F}$:

Proposition 4.4.7 (Step 1)**.** If α is a tangent vector at ω, then $\alpha|_{X^4_{\mathcal{D}_F}} = 0$. Moreover the following decomposition holds:

$$\alpha = \sum_{i \neq j \neq k} \hat{F}_{ijk} A_{ijk} dF_i \wedge dF_j + \varepsilon$$

for some homogeneous polynomials $(A_{ijk})_{i \neq j \neq k}$ of the correct degree (or the zero polynomials in any other case). Also, this family of polynomials can be selected antisymmetric on the first two indexes, i.e. $A_{ijk} = -A_{jik}$. In addition, the introduced form $\varepsilon \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ is a homogeneous affine form of total degree d satisfying: $\varepsilon|_{X^3_{\mathcal{D}_F}} = 0$.

Proof. At first we need to observe that $X^3_{\mathcal{D}_F}$ is contained in the singular set of ω, i.e. $\omega|_{X^3_{\mathcal{D}_F}} = 0$. So, it is clear $i_v(\omega)|_{X_{ijk}} = 0$, for each vector v and every component X_{ijk} of $X^3_{\mathcal{D}_F}$. In addition, for i, j, k fixed, $d\omega$ can be described in each of these components by:

$$d\omega|_{X_{ijk}} = ((\lambda_{ij} - \lambda_{ik} + \lambda_{jk}) \hat{F}_{ijk} dF_i \wedge dF_j \wedge dF_k)|_{X_{ijk}}.$$

According to these remarks the perturbation equation:

$$i_v(\alpha) \wedge d\omega + i_v(\omega) \wedge d\alpha = 0$$

reduces to:

$$i_v\alpha \wedge ((\lambda_{ij} - \lambda_{ik} + \lambda_{jk})\hat{F}_{ijk} dF_i \wedge dF_j \wedge dF_k) = 0 \quad \text{over } X_{ijk}.$$

Notice that sometimes we will use in some equations the phrase "over" instead the restriction symbol. Now, since the element $\hat{F}_{ijk}$ is not zero on the corresponding graded coordinated ring associated to X_{ijk}, and taking into consideration the conditions imposed on λ in the definition of $\mathcal{U}_2(\mathbf{d})$, we get the simple equation:

$$i_v\alpha \wedge dF_i \wedge dF_j \wedge dF_k = 0 \quad \text{over } X_{ijk}. \tag{4.4.4}$$

Alternatively, we can work out the equations on the open set determined by $\tilde{X}_{ijk} = X_{ijk} - X^4_{\mathcal{D}_F}$. This last deduction combined with the next general property for the inner product by a field

$$\begin{aligned} & i_v(\alpha \wedge dF_i \wedge dF_j \wedge dF_k) = \\ &= i_v\alpha \wedge dF_i \wedge dF_j \wedge dF_k + \alpha \wedge (v(F_i)dF_j \wedge dF_k - v(F_j)dF_i \wedge dF_k + v(F_k)dF_i \wedge dF_j), \end{aligned}$$

let us deduce:

$$i_v(\alpha \wedge dF_i \wedge dF_j \wedge dF_k) = 0 \quad \text{over } X_{ijk} \quad \forall v : v(F_i) = v(F_j) = v(F_k) = 0$$

Also if we take into consideration certain local bases (turned up for this particular case) at the points of $\tilde{X}_{ijk}$, which were described at the remark 4.4.6, we can obtain:

$$\alpha \wedge dF_i \wedge dF_j \wedge dF_k = 0 \quad \text{over } X_{ijk}.$$

Now, we can apply the division lemma 4.4.3 in order to get the decomposition:

$$\alpha = \gamma_i \wedge dF_i + \gamma_j \wedge dF_j + \gamma_k \wedge dF_k \quad \text{over } X_{ijk}, \tag{4.4.5}$$

for some homogeneous forms $\gamma_l \in H^0(\mathbb{C}^{n+1}, \Omega^1_{\mathbb{C}^{n+1}})$ of total degree $d - d_l$.

Next, we need to use the helpful constructed fields with the property 4.4.3 developed in the remark 4.4.6. So, we ought to choose a local rational vector field Y^j_{ijk} such that:

$$(i_{Y^j_{ijk}}(dF_j))|_{X_{ijk}} = 1 \quad ; \quad (i_{Y^j_{ijk}}(dF_i))|_{X_{ijk}} = (i_{Y^j_{ijk}}(dF_k))|_{X_{ijk}} = 0$$

With a slight abuse of notation we are considering the entire restriction to X_{ijk}, where formally the conditions hold locally at every point. The important fact behind this abuse, is that the conclusion is not going to depend on the local fields chosen, and will be valid at every point of X_{ijk}. For simplicity we also write $Y_j = Y^j_{ijk}$. Now, the equation 4.4.4 combined with the decomposition 4.4.5 imply the following:

$$\begin{aligned} 0 &= i_{Y_j}\alpha \wedge dF_i \wedge dF_j \wedge dF_k = \\ &= (i_{Y_j}(\gamma_i)dF_i + i_{Y_j}(\gamma_j)dF_j + i_{Y_j}(\gamma_k)dF_k + \gamma_j) \wedge dF_i \wedge dF_j \wedge dF_k = \\ &= \gamma_j \wedge dF_i \wedge dF_j \wedge dF_k, \end{aligned}$$

always over the points of X_{ijk}. Moreover, we could permute the indexes and deduce the same condition for γ_i and γ_k. Now, apply again the lemma 4.4.3 to decompose each homogeneous affine 1-form γ_l, and achieve the formula:

$$\alpha = G_{ijk}dF_i \wedge dF_j + G_{jki}dF_j \wedge dF_k + G_{ikj}dF_i \wedge dF_k \quad \text{over } X_{ijk}, \tag{4.4.6}$$

for some homogeneous polynomials $\{G_{rst}\}_{r,s,t}$ of the correct degree.

Next, fix another index l and compare the previous decompositions on their restriction to the intersection $X_{ijkl} = X_{ijk} \cap X_{jkl} \cap X_{ikl}$:

$$\begin{aligned}(\alpha)|_{X_{ijkl}} &= (G_{ijk}dF_i \wedge dF_j + G_{jki}dF_j \wedge dF_k + G_{ikj}dF_i \wedge dF_k)|_{X_{ijkl}} \\ &= (G_{jkl}dF_j \wedge dF_k + G_{klj}dF_k \wedge dF_l + G_{jlk}dF_j \wedge dF_l)|_{X_{ijkl}}\end{aligned}$$

Since the divisor $\mathcal{D}_F$ is normal crossing, at every point of X_{ijkl}, the restricted forms $dF_i|_{X_{ijkl}}$, $dF_j|_{X_{ijkl}}$, $dF_k|_{ijkl}$ and $dF_l|_{ijkl}$ are independent as elements of $(\Omega^1_{\mathbb{C}^{n+1}}|_{X_{ijkl}})_x$. Therefore, we get a lot of conditions for the polynomials collected, in particular observe that necessarily:

$$(G_{ijk})|_{X_{ijkl}} = 0 \quad \forall l \neq i, j, k.$$

Then, for every selection of indexes i, j, k, we can deduce the existence of an homogeneous polynomial A_{ijk} with the property: $G_{ijk} = \hat{F}_{ijk}A_{ijk}$. This fact is supported of the hypotheses assumed over the polynomials $F_1, \ldots, F_m$. Moreover, it is easy to check both families of polynomials $\{G_{ijk}\}$ and $\{A_{ijk}\}$ must satisfy the antisymmetric conditions announced in the first two indexes.

Finally, from 4.4.6 it follows that α and $\sum_{i,j,k} \hat{F}_{ijk}A_{ijk}dF_i \wedge dF_j$ has the same restriction to each component X_{ijk}. This in particular implies that the form $\varepsilon = \alpha - \sum_{i,j,k} \hat{F}_{ijk}A_{ijk}dF_i \wedge dF_j$ is a homogeneous affine form of total degree d which vanishes when restricted to $X^3_{\mathcal{D}_F}$. □

The next step is related to finding the expected polynomials "F_i'" (remember the middle term of the formula 4.3.14). Taking into consideration the last proposition, we need to pick up some correct equations for the polynomials $\{A_{ijk}\}$ to deduce that A_{ijk}/λ_{ij} only depends on the index k.

Proposition 4.4.8 (Step 2)**.** With the notation of proposition 4.4.7, for each selection of distinct indexes i, j, k denote by $B_{ijk} = A_{ijk}/\lambda_{ij}$ a slight correction of the polynomial A_{ijk}. Then, these new polynomials $\{B_{ijk}\}^m_{i,j,k=1}$ necessarily satisfy the relations:

$$B_{ijl}(x) = B_{jkl}(x) = B_{ikl}(x) \quad \forall x \in X_{ijkl}$$

Proof. First we need to describe better the factor ε, introduced in the previous step for the description of the perturbation α. So, according to the restriction lemma 4.4.4, we get:

$$\varepsilon = \sum_{i<j} \hat{F}_{ij}\varepsilon_{ij},$$

for some homogeneous affine forms $\{\varepsilon_{ij}\}$. From now on, we develop a similar argument to that used at Step 1 and work out the perturbation equation for our purposes. We will fix in some order four indexes i_0, j_0, k_0, l_0 towards to deduce the desired conditions on the component $X_{i_0j_0k_0l_0}$.

Fix j_0 and l_0, and use again the lemma 4.4.6 to select a local rational vector field $Y_{j_0} = Y^{j_0}_{j_0 l_0}$, now with the property:

$$i_{Y_{j_0}}(dF_{j_0}) = 1 \quad i_{Y_{j_0}}(dF_{l_0}) = 0 \ \text{ over } \ X_{j_0 l_0}.$$

Also, we need to use again a slight abuse of notation and work on the entire restriction to $X_{j_0 l_0}$, when formally the conditions holds locally. The abuse is supported on the fact that the obtained equations are not going to depend on the selected field, and will be valid at all the points. For easiness we will work separately over the two terms of the restricted perturbation equation:

$$(4.4.7) \qquad \underbrace{(i_{Y_{j_0}}(\alpha) \wedge d\omega)|_{X_{j_0 l_0}}}_{\text{First term}} + \underbrace{(i_{Y_{j_0}}(\omega) \wedge d\alpha)|_{X_{j_0 l_0}}}_{\text{Second term}} = 0.$$

In order to develop a manageable expression of the first term notice that:

$$i_{Y_{j_0}}(\alpha) = \sum_{i \neq j \neq k} 2\hat{F}_{ijk} A_{ijk} i_{Y_{j_0}}(dF_i) dF_j + \sum_{i<j} \hat{F}_{ij} i_{Y_{j_0}}(\varepsilon_{ij}).$$

So , the elements of the first term on their restriction to the subvariety $X_{j_0 l_0}$, can be describe by:

$$\begin{aligned}
(i_{Y_{j_0}}(\alpha))|_{X_{j_0 l_0}} = (&\sum_k 2\hat{F}_{j_0 l_0 k} A_{l_0 j_0} i_{Y_{j_0}}(dF_{j_0}) dF_{l_0} + \sum_k 2\hat{F}_{j_0 l_0 k} A_{j_0 l_0 k} \overbrace{i_{Y_{j_0}}(dF_{l_0})}^{=0} dF_{j_0} & + \\
&+ \sum_i 2\hat{F}_{i j_0 l_0} A_{i j_0 l_0} i_(Y_{j_0})(dF_i) dF_{j_0} + \sum_i 2\hat{F}_{i j_0 l_0} A_{i l_0 j_0} i_{Y_{j_0}}(dF_i) dF_{l_0} & + \\
&+ \sum_j 2\hat{F}_{j_0 j l_0} A_{j_0 j l_0} i_{Y_{j_0}}(dF_{j_0}) dF_j + \sum_j 2\hat{F}_{j_0 j l_0} A_{l_0 j j_0} \underbrace{i_{Y_{j_0}}(dF_{l_0})}_{=0} dF_j & + \\
&+ \hat{F}_{j_0 l_0} i_{Y_{j_0}}(\varepsilon_{j_0 l_0})) |_{X_{j_0 l_0}}
\end{aligned}$$

and

$$(d\omega)|_{X_{j_0 l_0}} = (\sum_{i \neq j \neq k} \lambda_{ij} \hat{F}_{ijk} dF_i \wedge dF_j \wedge dF_k)|_{X_{j_0 l_0}} = (\chi_{j_0 l_0} \wedge dF_{j_0} \wedge dF_{l_0})|_{X_{j_0 l_0}}.$$

In addition, the form $\chi_{j_0 l_0}$ represents a certain new introduced homogeneous affine form, which is going to be used only to express that the restricted form $(d\omega)|_{X_{j_0 l_0}}$ is in the direction of $dF_{j_0} \wedge dF_{l_0}$. Having said that, we get a description of the whole first term on its restriction to $X_{j_0 l_0}$:

(*First Term:*)

$$i_{Y_{j_0}}(\alpha) \wedge d\omega = \left(\sum_{\substack{i \neq j \neq k \neq r \\ (\& r \neq j_0, l_0)}} 2\lambda_{ij} A_{j_0 r l_0} \underbrace{\hat{F}_{j_0 l_0 r} \hat{F}_{ijk}}_{= \hat{F}_{j_0 l_0} \hat{F}_{ijkr}} dF_r \wedge dF_i \wedge dF_j \wedge dF_k \right) + 2\hat{F}_{j_0 l_0} i_{Y_{j_0}}(\varepsilon_{\{j_0, l_0\}}) \wedge d\omega.$$

On the other hand, we ought to make the same description for the other term of the formula 4.4.7. In this case, observe that over $X_{j_0 l_0}$ the following holds:

$$(i_{Y_{j_0}}(\omega))|_{X_{j_0 l_0}} = (\sum_{i \neq j} 2\lambda_{ij} \hat{F}_{ij} i_{Y_{j_0}}(dF_i)\, dF_j)|_{X_{j_0 l_0}} = (2\lambda_{j_0 l_0} \hat{F}_{j_0 l_0} dF_{l_0})|_{X_{j_0 l_0}}.$$

Also, the other component ($d\alpha$) can be described by:

$$d\alpha = \sum_{i\neq j\neq k\neq r} A_{ijk}\hat{F}_{ijkr} dF_i \wedge dF_j \wedge dF_r + \sum_{i\neq j\neq k} \hat{F}_{ijk} dA_{ijk} \wedge dF_i \wedge dF_j + \\ + \sum_{\substack{i\neq j\neq k \\ i<j}} \hat{F}_{ijk} dF_k \wedge \varepsilon_{ij} + \sum_{i<j} \hat{F}_{ij} d\varepsilon_{ij}.$$

For the moment, we are not going to describe better this term on the restriction to the subvariety $X_{j_0 l_0}$. For our purposes, it is sufficient to take the wedge product of the two components as they were written above, and deduce the following formula:

(Second term:)

$$i_{Y_{j_0}}(\omega) \wedge d\alpha = 2\lambda_{j_0 l_0}\hat{F}_{j_0 l_0} dF_{l_0} \wedge \left(\sum_{i\neq j\neq k\neq r} A_{ijk}\hat{F}_{ijkr} dF_i \wedge dF_j \wedge dF_r \right) + \\ + 2\lambda_{j_0 l_0}\hat{F}_{j_0 l_0} dF_{l_0} \wedge \left(\sum_{i\neq j\neq k} \hat{F}_{ijk} dA_{ijk} \wedge dF_i \wedge dF_j + \sum_{\substack{i\neq j\neq k \\ i<j}} \hat{F}_{ijk} dF_k \wedge \varepsilon_{ij} + \sum_{i<j} \hat{F}_{ij} d\varepsilon_{ij} \right)$$

With a view to complete the description of the equation 4.4.7, it is needed to add the *first and the second* terms obtained. In addition, we are also allowed to cancel the polynomial factor $\hat{F}_{j_0 l_0}$ repeated in all summands. After that, consider the restriction to $X_{i_0 j_0 k_0 l_0}$, where most of the sum's terms involved vanish, and finally obtain:

$$0 = \sum_{\substack{i\neq j\neq k\neq r \\ r\neq j_0, l_0}} 2\lambda_{ij} A_{j_0 r l_0}\hat{F}_{ijkr} dF_r \wedge dF_i \wedge dF_j \wedge dF_k + \sum_{i\neq j\neq k\neq r} 2\lambda_{j_0 l_0} A_{ijk}\hat{F}_{ijkr} dF_{l_0} \wedge dF_i \wedge dF_j \wedge dF_r \ ,$$

over all the points of $X_{i_0 j_0 k_0 l_0}$.

Obviously, this is not the optimal formula. At once it is clear that only survive the terms whose indexes i, j, k, r coincides in some order with i_0, j_0, k_0, l_0. After doing all the possible assignments of i_0, j_0, k_0, l_0 into both sum's indexes, we get a more useful equation:

$$\left((\lambda_{j_0 k_0} + \lambda_{k_0 l_0})A_{j_0 i_0 l_0} + (\lambda_{i_0 j_0} + \lambda_{l_0 i_0})A_{j_0 k_0 l_0} + \lambda_{j_0 l_0} A_{i_0 k_0 l_0}\right)\hat{F}_{i_0 j_0 k_0 l_0} dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0} \wedge dF_{l_0} = 0$$

Since the restriction of the global forms dF_{i_0},dF_{j_0},dF_{k_0} and dF_{l_0} to the subvariety $X_{i_0 j_0 k_0 l_0}$ are independent, it is possible to deduce:

$$(\lambda_{j_0 k_0} + \lambda_{k_0 l_0})A_{j_0 i_0 l_0} + (\lambda_{i_0 j_0} + \lambda_{l_0 i_0})A_{j_0 k_0 l_0} + \lambda_{j_0 l_0} A_{i_0 k_0 l_0} = 0 \quad \text{over} \quad X_{i_0 j_0 k_0 l_0}$$

According to the definition of the polynomials $\{B_{ijk}\}$, and if we recall the order in which they where selected the fixed indexes, we have produced an equation $[Eq_{j_0 l_0 i_0 k_0}]$ for these new polynomials:

$$(\lambda_{j_0 k_0}\lambda_{j_0 i_0} + \lambda_{k_0 l_0}\lambda_{j_0 i_0})B_{j_0 i_0 l_0} + (\lambda_{i_0 j_0}\lambda_{j_0 k_0} + \lambda_{l_0 i_0}\lambda_{j_0 k_0})B_{j_0 k_0 l_0} + \lambda_{j_0 l_0}\lambda_{i_0 k_0} B_{i_0 k_0 l_0} = 0$$

In order to end this step, notice that any permutation of the fixed indexes gives rise to another linear equation over $X_{i_0 j_0 k_0 l_0}$ for the polynomials $\{B_{ijk}\}$. So, it is sufficient to prove that the linear system produced in this way let us deduce the desired equations.

For this purpose, figure out from 4.4.7 that the original family of polynomials $\{A_{ijk}\}$ are anti-symmetric on the first two indexes, i.e. $A_{ijk} = -A_{jik}$. Using this fact, the equation $[2Eq_{j_0l_0i_0k_0} + Eq_{i_0l_0j_0k_0} + Eq_{k_0l_0i_0j_0}]$ can be exactly described by:

$$\underbrace{(\lambda_{i_0j_0} - \lambda_{i_0k_0} + \lambda_{j_0k_0})}_{\neq 0}(B_{i_0j_0l_0} - B_{j_0k_0l_0}) = 0 \quad \text{over } X_{i_0j_0k_0l_0}.$$

Also notice that the above linear combination of coefficients does not vanish because of the conditions assumed on the definition of $\mathcal{U}_2(\mathbf{d})$. Finally, the other equalities follow from appropriated permutations of the selected indexes, and this completes the proof of step 2.

□

Now, we are able to apply the fundamental lemma 4.4.5, in order to deduce that every tangent vector at ω can be decompose as a perturbation in the image of the differential of the parametrization (associated to some polynomials $\{F'_k\}$ and vanishing on $X^4_{\mathcal{D}_F}$). in addition to another perturbation that vanishes on $X^3_{\mathcal{D}_F}$. This development falls into the description made at the remark 4.3.18, and can be summarized in the following step.

Proposition 4.4.9 (Step 3)**.** Let $\omega = \rho(\lambda, \underline{F})$ be a logarithmic form in the image by ρ of the open set $\mathcal{U}_2(\mathbf{d})$. For a given $\alpha \in \mathcal{T}_\omega\mathcal{F}_2(d, \mathbb{P}^n)$, there exist a family of homogeneous polynomials $\{F'_i\}_{i=1}^m$ of respective degrees $(d_i)_{i=1}^m$, and a projective homogeneous form $\tilde{\alpha} \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))$ which is another tangent vector at ω, such that:

$$\alpha = \sum_{i\neq j\neq k} \lambda_{ij}\hat{F}_{ijk}F'_k dF_i \wedge dF_j + \sum_{i<j} 2\lambda_{ij}\hat{F}_{ij}dF'_i \wedge dF_j + \tilde{\alpha} = d\rho(\lambda, \underline{F})(0, (F'_i)_{i=1}^m) + \tilde{\alpha},$$

where also $\tilde{\alpha}|_{X^3_{\mathcal{D}_F}} = 0$.

Proof. At first combine the steps 1 and 2 of the previous two propositions to get the following decomposition for α:

$$\alpha = \sum_{i\neq j\neq k} \hat{F}_{ijk}A_{ijk}dF_i \wedge dF_j + \varepsilon.$$

And also recall that the family of polynomials $\{B_{ijk} = \frac{A_{ijk}}{\lambda_{ij}}\}$ satisfy the relations

$$B_{ijl} = B_{jkl} = B_{ikl},$$

on their restriction to the subvariety X_{ijkl}. Now, according to the fundamental lemma 4.4.5, there exists a family of homogeneous polynomials $F'_1, \ldots, F'_m$ of respective degrees $d_1, \ldots, d_m$ such that:

$$B_{ijk} = F'_k + \hat{F}_{jk}\tilde{B}_{ijk}$$

over the points of X_{jk}, for some new homogeneous polynomials $\{\tilde{B}_{ijk}\}$ of the correct degree. So, it is clear that the forms α and

$$\sum_{i\neq j\neq k} \lambda_{ij}\hat{F}_{ijk}F'_k dF_i \wedge dF_j + \sum_{i\neq j\neq k} \lambda_{ij}\hat{F}_{ijk}\hat{F}_{jk}\tilde{B}_{ijk}dF_i \wedge dF_j$$

has the same restriction to the every component of $\mathcal{X}^3_{\mathcal{D}} = \bigcup X_{ijk}$. Furthermore, since the second term of the previous above form vanishes on $\mathcal{X}^3_{\mathcal{D}}$, we can write:

$$\alpha = \sum_{i\neq j\neq k} \lambda_{ij}\hat{F}_{ijk}F'_k dF_i \wedge dF_j + \tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ vanishes on $X^3_{\mathcal{D}}$. Finally, we need to add and subtract a suitable term to obtain the desired formula:

$$\alpha = \sum_{i\neq j\neq k} \lambda_{ij}\hat{F}_{ijk}F'_k dF_i \wedge dF_j + \sum_{i<j} 2\lambda_{ij}\hat{F}_{ij}dF'_i \wedge dF_j + \tilde{\alpha} = d\rho(\lambda, \underline{F})(0, (F'_i)_{i=1}^m) + \tilde{\alpha},$$

where $\tilde{\alpha}$ is another polynomial affine form such that $\tilde{\alpha}|_{X^3_{\mathcal{D}}} = 0$.

Since $d\rho(\lambda, \underline{F})(0, (F'_i)_{i=1}^m)$ is an homogeneous projective form tangent to ω, i.e descends to the projective space $(i_R(d\rho(\lambda, \underline{F})(0, (F'_i)_{i=1}^m)) = 0)$ and satisfies the equations 4.3.11 and 4.3.12, the same characteristics hold on $\tilde{\alpha}$. □

In conclusion, from now on, we need to deal with the same problem as in the beginning, and characterize the elements $\tilde{\alpha} \in \mathcal{T}_\omega\mathcal{F}(d, \mathbb{P}^n)$ but with the additional vanishing condition: $\tilde{\alpha}|_{X^3_{\mathcal{D}}} = 0$.

In advanced, taking the expression 4.3.14 into consideration, the form $\tilde{\alpha}$ is expected to be related to a perturbation of the coefficients λ (its the term α_1 of the formula). As in the case of 1-forms, this last is going to be true only assuming certain extra conditions over the vector **d** (balanced case).

The following proposition set the background to end the proof in the balanced case, and also it is useful to understand the possible trouble if **d** do not satisfy the desired assumptions.

Proposition 4.4.10 (Step 4)**.** If $\tilde{\alpha} \in \mathcal{T}_\omega\mathcal{F}(d, \mathbb{P}^n)$ is a Zariski tangent vector at ω and vanishes when restricted to $X^3_{\mathcal{D}_F}$, then:

$$\tilde{\alpha} = \sum_{i\neq j} \hat{F}_{ij}\tilde{\alpha}_{ij}.$$

Where $\{\tilde{\alpha}_{ij}\}$ represent some homogeneous affine forms such that $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}$.

Moreover, for each ordered selection of distinct indexes i, j, k, there exist a constant $\lambda'_{ij} \in \mathbb{C}$ and a homogeneous polynomials B^{ij}_{ik} and B^{ij}_{jk} of the correct degree (or zero in any other case) such that:

$$\tilde{\alpha}_{ij} = \lambda'_{ij}dF_i \wedge dF_j + \hat{F}_{ijk}(B^{ij}_{ik}dF_i \wedge dF_k + B^{ij}_{jk}dF_j \wedge dF_k) \quad \text{over } X_{ijk}.$$

Proof. At first we apply the restriction lemma 4.4.4 to the form $\tilde{\alpha}$, but, just for simplicity, we need to take a slight correction of the obtained forms in order to make them symmetric on the selected indexes. In other words, we can express:

$$\tilde{\alpha} = \sum_{i\neq j} \hat{F}_{ij}\tilde{\alpha}_{ij},$$

where $\{\tilde{\alpha}_{ij}\}$ is as a family of homogeneous affine forms such that $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}$.

The following argument is similar to that used at the proof of 4.4.7, but adapted to each $\tilde{\alpha}_{ij}$. Once more fix two indexes i_0, j_0. According to the remark 4.4.6, select a rational local vector field $Y_{j_0} = Y^{i_0 j_0}_{j_0}$ with the condition 4.4.3. (with the same abuse of notation as usual).

Now, the two terms of the perturbation's equation 4.3.12 associated to $\tilde{\alpha}$ can be described on the whole space by:

$$i_{Y_{j_0}}\tilde{\alpha}\wedge d\omega = \sum_{\substack{l\neq r\\ i\neq j\neq k}} \lambda_{ij}\hat{F}_{lr}\hat{F}_{ijk} i_{Y_{j_0}}(\tilde{\alpha}_{lr})\wedge dF_i\wedge dF_j\wedge dF_k$$

$$i_{Y_{j_0}}\omega\wedge d\tilde{\alpha} = \sum_{\substack{l\neq r\\ i\neq j\neq k}} 2\lambda_{lr}\hat{F}_{lr}\hat{F}_{ijk} i_{Y_{j_0}}(dF_l)\, dF_r\wedge dF_k\wedge\tilde{\alpha}_{ij} + \sum_{\substack{l\neq r\\ i\neq j}} 2\lambda_{lr}\hat{F}_{lr}\hat{F}_{ij} i_{Y_{j_0}}(dF_l)\, dF_r\wedge d\tilde{\alpha}_{ij}$$

Next, restrict these terms to $X_{i_0j_0}$ where most of them vanish. Also, remember that the obtained equations are not going to depend on the selected local fields, and will be valid at all the points of $X_{i_0j_0}$. For simplicity, we are not going to write the optimal formula, it will be described without making all the possible assignments of the two indexes fixed into the term's indexes i, j, k. With those considerations, and using the properties of Y_{j_0}, we get:

$$(i_{Y_{j_0}}\tilde{\alpha}\wedge d\omega)|_{X_{i_0j_0}} = \sum_{i\neq j\neq k} 2\lambda_{ij}\hat{F}_{i_0j_0}\hat{F}_{ijk}\, i_{Y_{j_0}}(\tilde{\alpha}_{i_0j_0})\wedge dF_i\wedge dF_j\wedge dF_k$$

$$(i_{Y_{j_0}}\omega\wedge d\tilde{\alpha})|_{X_{i_0j_0}} = \left(\sum_{i\neq j\neq k} 2\lambda_{j_0i_0}\hat{F}_{i_0j_0}\hat{F}_{ijk}\, dF_{i_0}\wedge dF_k\wedge\tilde{\alpha}_{ij}\right) + 4\lambda_{j_0i_0}\hat{F}_{i_0j_0}\hat{F}_{i_0j_0} dF_{i_0}\wedge d\tilde{\alpha}_{i_0j_0}$$

Following the same idea as in the step 1, and after adding the two terms from above, we could remove the polynomial $\hat{F}_{i_0j_0}$ which is present in all sums, and then restrict the equation to $X_{i_0j_0k_0}$:

$$\begin{aligned}(4.4.8)\qquad & 2(\lambda_{i_0j_0}-\lambda_{i_0k_0}+\lambda_{j_0k_0})\hat{F}_{i_0j_0k_0} i_{Y_{j_0}}(\tilde{\alpha}_{i_0j_0})\;\; dF_{i_0}\wedge dF_{j_0}\wedge dF_{k_0}+\\ & + 4\lambda_{j_0i_0}\hat{F}_{i_0j_0k_0}\;\; dF_{i_0}\wedge dF_{k_0}\wedge\tilde{\alpha}_{i_0j_0} + 4\lambda_{j_0i_0}\hat{F}_{i_0j_0k_0}\;\; dF_{i_0}\wedge dF_{j_0}\wedge\tilde{\alpha}_{i_0k_0} = 0\end{aligned}$$

Note that if we take the product by $(\wedge dF_{j_0})|_{X_{i_0j_0k_0}}$ to the previous equation, we deduce:

$$\tilde{\alpha}_{i_0j_0}\wedge dF_{i_0}\wedge dF_{j_0}\wedge dF_{k_0} = 0 \quad \text{over} \;\; X_{i_0j_0k_0}.$$

Also, the same conclusion holds for the forms $\tilde{\alpha}_{i_0k_0}$ and $\tilde{\alpha}_{j_0k_0}$. Again, according to the division lemma 4.4.3, it is possible to decompose each of the previous forms on the directions of dF_{i_0}, dF_{j_0} and dF_{k_0}. For example, we obtain the following decomposition for $\tilde{\alpha}_{i_0j_0}$:

$$(\tilde{\alpha}_{i_0j_0})|_{X_{i_0j_0k_0}} = (\mu^{(i_0j_0)k_0}_{k_0}\wedge dF_{k_0} + \mu^{(i_0j_0)k_0}_{j_0}\wedge dF_{j_0} + \mu^{(i_0j_0)k_0}_{i_0}\wedge dF_{i_0})|_{X_{i_0j_0k_0}}$$

for some homogeneous affine forms $\{\mu^{(i_0j_0)k_0}_{l}\}_l$, whose respective degree equals to $d_i + d_j - d_l$ (for every index l).

What we want to prove now is that these new forms, restricted to $X_{i_0j_0k_0}$, are also on the directions of dF_{i_0}, dF_{j_0} and dF_{k_0}. For this purpose, select a new rational local vector field $Z_{j_0} = Y^{i_0j_0k_0}_{j_0}$ with the corresponding property deduced from 4.4.6. Then, observe that this is an extended assumption to that used at the beginning of the proof (for Y_{j_0}), however it allows us to deduce the same equation (4.4.8)for the components of $\tilde{\alpha}$. The formal proof of this last fact is almost the same as the developed for the field Y_{j_0}. It is important to respect the index's order in how we restrict to $X_{i_0 j_0 k_0}$, in order to be able to cancel the term $\hat{F}_{i_0 j_0}$. In conclusion we obtain:

$$(\lambda_{i_0 j_0} - \lambda_{i_0 k_0} + \lambda_{j_0 k_0})\hat{F}_{i_0 j_0 k_0} i_{Z_{j_0}}(\tilde{\alpha}_{i_0 j_0})\ dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0} + \\ + 2\lambda_{j_0 i_0}\hat{F}_{i_0 j_0 k_0}\ dF_{i_0} \wedge dF_{k_0} \wedge \tilde{\alpha}_{i_0 j_0} + 2\lambda_{j_0 i_0}\hat{F}_{i_0 j_0 k_0}\ dF_{i_0} \wedge dF_{j_0} \wedge \tilde{\alpha}_{i_0 k_0} = 0$$

Now, if we replace the above decomposition for $\tilde{\alpha}_{i_0 j_0}$ (and $\tilde{\alpha}_{i_0 k_0}$) on this last equation, and use the properties of Z_{j_0}, we finally obtain:

$$((\lambda_{i_0 j_0} - \lambda_{j_0 k_0} + \lambda_{i_0 k_0})\mu_{j_0}^{(i_0 j_0) k_0} - 2\lambda_{i_0 j_0}\mu_{k_0}^{(i_0 k_0) j_0}) \wedge dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0} = 0.$$

Our purpose is to show every form represented by

$$\beta_l^{(i_0 j_0) k_0} = \mu_l^{(i_0 j_0) k_0} \wedge dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0},$$

for $l \in \{i_0, j_0, k_0\}$, equals to zero on its restriction to $X_{i_0 j_0 k_0}$. Also, we need to treat with forms of the type $\beta_l^{(i_0 k_0) j_0}$ and $\beta_l^{(j_0 k_0) i_0}$, and additionally prove they vanish. According to this new notation we can rewrite the above equation as:

$$Eq(I_{j_0 i_0 k_0}): \quad (\lambda_{i_0 j_0} - \lambda_{j_0 k_0} + \lambda_{i_0 k_0})\beta_{j_0}^{(i_0 j_0) k_0} - 2\lambda_{i_0 j_0}\beta_{k_0}^{(i_0 k_0) j_0} = 0,$$

where $Eq(I_{j_0 i_0 k_0})$ refers to the order in which the indexes were selected to deduce the equation. So, $Eq(I_{j_0 i_0 k_0})$ was associated to the process summarized by the contraction with Z_{j_0}, then by the restriction to $X_{i_0 j_0}$, and finally to $X_{i_0 j_0 k_0}$.

In conclusion, if we permute the selected indexes, we could construct a linear system of equations for the forms of type β, towards to deduce the desired property. For example, if we take:

$$Eq(I_{j_0 i_0 k_0}) + Eq(I_{k_0 i_0 j_0}): \quad (\lambda_{i_0 j_0} - \lambda_{j_0 k_0} - \lambda_{i_0 k_0})(\beta_{j_0}^{(i_0 j_0) k_0} - \beta_{k_0}^{(i_0 k_0) j_0}) = 0,$$

we deduce $\beta_{k_0}^{(i_0 k_0) j_0} = \beta_{j_0}^{(i_0 j_0) k_0}$ over $X_{i_0 j_0 k_0}$. With this deduction and the equation $Eq(I_{j_0 i_0 k_0})$ we obtain part of the desired conditions:

$$\beta_{j_0}^{(i_0 j_0) k_0} = 0 \quad \text{over} \quad X_{i_0 j_0 k_0}.$$

Since $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}$, the same process (with the indexes j_0 and i_0 permuted) implies that:

$$(\beta_{i_0}^{(j_0 i_0) k_0})|_{X_{i_0 j_0 k_0}} = (\beta_{i_0}^{(i_0 j_0) k_0})|_{X_{i_0 j_0 k_0}}$$

also vanishes.

It only rest to show: $\beta_{k_0}^{(i_0 j_0) k_0} = 0$. Although, this fact could not be imply by the equations of type I. It will follow from something we have not done before, propose the equation 4.3.12 associated to a rational vector field $Z_{k_0} = Y_{k_0}^{i_0 j_0 k_0}$, and apply the same argument as in the deduction of 4.4.8 described on this proof: first restrict the equation to $X_{i_0 j_0}$, then cancel the polynomial factor $\widehat{F_{i_0 j_0}}$, and in the final step restrict to $X_{i_0 j_0 k_0}$. The equation which hold after this process is the following:

$$2(\lambda_{i_0 j_0} - \lambda_{i_0 k_0} + \lambda_{j_0 k_0})\hat{F}_{i_0 j_0 k_0} i_{k_0}(\tilde{\alpha}_{i_0 j_0}) \wedge dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0} = 0 \quad \text{over} \quad X_{i_0 j_0 k_0}.$$

According to the conditions established in $\mathcal{U}_2(\mathbf{d})$, we get

$$(\mu_{k_0}^{(i_0j_0)k_0} \wedge dF_{i_0} \wedge dF_{j_0} \wedge dF_{k_0})|_{X_{i_0j_0k_0}} = 0,$$

as claimed.

Now we are able to apply again the division lemma, in this case to the forms of type $\mu_l^{(i_0j_0)l}$. After a correct regroupement we obtain:

$$\tilde{\alpha}_{i_0j_0} = A^{i_0j_0}_{i_0j_0k_0} dF_{i_0} \wedge dF_{j_0} + A^{i_0j_0}_{i_0k_0j_0} dF_{i_0} \wedge dF_{k_0} + A^{i_0j_0}_{j_0k_0i_0} dF_{j_0} \wedge dF_{k_0} \quad \text{over } X_{i_0j_0k_0}, \tag{4.4.9}$$

where each the new polynomials of type A are homogeneous of the correct positive degree (or the zero polynomials for negative possible degrees). Observe that we are close to the expected formula, we just need to deduce some extra conditions for these new introduced polynomials.

Notice that if we fix another index l_0, then the different possible formulas for $\tilde{\alpha}_{i_0j_0}$ in the intersection $X_{i_0j_0k_0l_0}$ must coincide, and so it is valid:

$$\begin{aligned}\tilde{\alpha}_{i_0j_0} &= A^{i_0j_0}_{i_0j_0k_0} dF_{i_0} \wedge dF_{j_0} + A^{i_0j_0}_{i_0k_0j_0} dF_{i_0} \wedge dF_{k_0} + A^{i_0j_0}_{j_0k_0i_0} dF_{j_0} \wedge dF_{k_0} = \\ &= A^{i_0j_0}_{i_0j_0l_0} dF_{i_0} \wedge dF_{j_0} + A^{i_0j_0}_{i_0l_0j_0} dF_{i_0} \wedge dF_{l_0} + A^{i_0j_0}_{j_0l_0i_0} dF_{j_0} \wedge dF_{l_0}\end{aligned}$$

Once more we need to use that the restricted forms $dF_r|_{X_{i_0j_0k_0l_0}}$ (for $r \in \{i_0, j_0, k_0, l_0\}$) are independent as elements of $\Omega^1_{\mathbb{C}^{n+1}}|_{X_{i_0j_0k_0l_0}}$. This allows us to deduce exactly the conditions required over the polynomials of type A:

$$A^{i_0j_0}_{i_0j_0k_0} = A^{i_0j_0}_{i_0j_0l_0} \quad ; \quad A^{i_0j_0}_{i_0k_0j_0} = A^{i_0j_0}_{j_0k_0l_0} = 0 \qquad (\text{over } X_{i_0j_0k_0l_0}. \tag{4.4.10}$$

Finally, it is time to bring into consideration the homogeneous degree of each term on the above equation 4.4.9. Since the degree of $\tilde{\alpha}_{i_0j_0}$ equals to $d_{i_0} + d_{j_0}$, it is easy to check that $deg(A^{i_0j_0}_{i_0j_0k_0}) = 0$. So we have found the desired constants, defined by:

$$\lambda'_{i_0j_0} = A^{i_0j_0}_{i_0j_0k_0} = A^{i_0j_0}_{i_0j_0l_0} \in \mathbb{C}.$$

Using the second condition obtained below at 4.4.10, and with the same argument we have used on several times (for example on the proof of 4.4.7), we also deduce:

$$A^{i_0j_0}_{i_0k_0j_0} = \hat{F}_{i_0j_0k_0} B^{i_0j_0}_{i_0k_0}; \quad A^{i_0j_0}_{j_0k_0i_0} = \hat{F}_{i_0j_0k_0} B^{i_0j_0}_{j_0k_0} \quad \text{over } X_{i_0j_0k_0},$$

for some homogeneous polynomials B whose degrees adjust to the following formula:

$$\begin{aligned} deg(B^{i_0j_0}_{i_0k_0}) &= A^{i_0j_0}_{i_0k_0j_0} - (d - d_{i_0} - d_{j_0} - d_{k_0}) = \\ &= deg(\tilde{\alpha}_{i_0j_0}) - (d_{i_0} + d_{k_0}) - (d - d_{i_0} - d_{j_0} - d_{k_0}) = 2d_{j_0} + d_{i_0} - d \end{aligned}$$

□

Observe that if the formula obtained in the last proposition were true in the whole space $\mathbb{P}^n$, and the polynomials of type B were equal to zero, we would deduce that $\tilde{\alpha}$ should be only associated to a perturbation of the coefficients ("λ'").

In the case of logarithmic 2-forms of type **d** the concept of "balanced" will be again related to the degrees obtained at the end of the last proposition's proof. Moreover, it will be a more restrictive concept than the used for 1-forms.

The balanced assumption

Towards to finish the proof of proposition 4.4.2, and according to the previous discussion, we need to set up certain definition in order to restrict the possible degrees lying on $\mathbf{d}$.

For a general natural number k we can define the concept of a k-balanced vector of degrees.

Definition 4.4.11. We say that an m-tuple of degrees $\mathbf{d} = (d_i, \ldots, d_m) \in \mathbb{N}^m$ is k-balanced if for each selected multi-index $I \subset \{1, \cdots, m\}$ of size $|I| = k$, the following inequality holds:

$$\sum_{i \in I} d_i = d_{i_1} + \cdots + d_{i_l} < \sum_{l \notin I} d_l.$$

Remark 4.4.12. If a vector $\mathbf{d}$ is k-balanced then it is k'-balanced for all k' lower than k. Furthermore, notice that the following more precise inequality holds for every multi-index I of size k and every selection of a sub-multi-index $J \subset I$:

$$\sum_{j \in J} d_j < \sum_{l \notin I} d_l$$

Although, the converse of this fact is not true (not even for large m). For example, take:

$$\mathbf{d} = (1, 2, \ldots, m-2, \tfrac{(m-2)(m-3)}{2}, \tfrac{(m-2)(m-3)}{2}).$$

This vector is 1-balanced, but trivially not 2-balanced for every m.

Remark 4.4.13. Alternatively, let $\mathcal{D} = \sum_{i=1}^m \mathcal{D}_i = (F_i = 0)$ be a divisor over $\mathbb{P}^n$ of degree d, whose irreducible components $\mathcal{D}_i$ are defined by homogeneous polynomials F_i of respective degrees d_i where $d = \sum_{i=1}^m d_i$. We will say the divisor is **k-balanced** if the vector $\mathbf{d} = (d_i)_{i=1}^m$ is k-balanced.

Example 4.4.14. If all the possible degrees are all equal to 1, i.e. $\mathbf{d} = (1, \ldots, 1)$, then $\mathbf{d}$ is k-balanced if and only if $2k < m$.

At last, we want to emphasize some aspects towards to understand the difficulty behind the non-balanced case.

Remark 4.4.15. Unlike the case of 1-balanced vectors, it is no more true that if a vector $\mathbf{d}$ is not k-balanced, then there exist a unique selection of k-indexes which is greater or equal than the rest. In general, the possible number of unbalanced k-tuples is completely out of control.

At this time, we are prepared to give a possible end to the proof of the surjectivity of the natural parametrization. From now on, we assume $\mathcal{D}_F$ is a 2-balanced divisor.

Let us recall the situation described at the last proposition 4.4.10 (Step 4). If $\tilde{\alpha} = \sum_{i,j} \hat{F_{ij}} \tilde{\alpha}_{ij}$ is a first order perturbation of ω which vanishes on $X^3_{\mathcal{D}_F}$, then we have the following description:

$$\tilde{\alpha}_{ij} = \lambda'_{ij} dF_i \wedge dF_j + \hat{F_{ijk}} (B^{ij}_{ikj} dF_i \wedge dF_k + B^{ij}_{jki} dF_j \wedge dF_k) \quad \text{over} \quad X_{ijk}. \tag{4.4.11}$$

Also, remember that we want to deduce: $\tilde{\alpha}_{ij} = \lambda'_{ij} dF_i \wedge dF_j$. In other words, what we will prove (assuming $\mathbf{d}$ balanced) is that $\tilde{\alpha} = d\rho(\lambda, \overline{F})((\lambda'_i)_{i=1}^m, 0))$.

The following lemma is really close to the formula we want to conclude.

Lemma 4.4.16 (Step 5)**.** We use the same notation of proposition 4.4.10. Also, assume that the vector **d** is 2-balanced. Then, there exist an homogeneous affine form $\beta \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ of total degree d, and such that $\beta|_{X^2_{\mathcal{D}}} = 0$, which fulfill the following formula:

$$\tilde{\alpha} = \sum_{i\neq j} \lambda'_{ij} \hat{F}_{ij} dF_i \wedge dF_j + \beta.$$

Proof. The previous proposition 4.4.10 (Step 4) determines a specific decomposition for $\tilde{\alpha}$, which involves some homogeneous affine forms denoted by $\tilde{\alpha}_{ij}$ of respective total degrees $d_i + d_j$, and whose restriction to X_{ijk} satisfy:

$$\tilde{\alpha}_{ij} = \lambda'_{ij} dF_i \wedge dF_j + \hat{F_{ijk}}(B^{ij}_{ikj} dF_i \wedge dF_k + B^{ij}_{jki} dF_j \wedge dF_k).$$

Also, the degree of the homogeneous polynomials of type B perform the following:

$$deg(B^{ij}_{ikj}) = d_j - \sum_{k\neq i,j} d_k \qquad deg(B^{ij}_{jki}) = d_i - \sum_{k\neq i,j} d_k.$$

Since **d** is 2-balanced, and in agreement with the remark 4.4.12, we deduce the terms associated to the polynomials B^{ij}_{ikj} and B^{ij}_{jki} cannot take part in the formula, because they should be of negative homogeneous degree. So, for i, j and k fixed the important conclusion is the following:

$$\tilde{\alpha}_{ij} = \lambda'_{ij} dF_i \wedge dF_j \quad \text{over} \ \ X_{ijk}.$$

Now observe that the form $(\tilde{\alpha}_{ij} - \lambda'_{ij} dF_i \wedge dF_j)|_{X_{ij}}$ vanishes on each component of the divisor $(\hat{F}_{ij}|_{X_{ij}} = 0)$, defined on the variety X_{ij}. With a slight modification of the proof of the vanishing lemma 4.4.4, it can be seen that:

$$\tilde{\alpha}_{ij} = \lambda'_{ij} dF_i \wedge dF_j + \hat{F}_{ij}\beta_{ij} \quad \text{over} \ \ X_{ij},$$

for a certain form $\beta_{ij} \in H^0(\mathbb{C}^{n+1}, \Omega^2_{\mathbb{C}^{n+1}})$ also homogeneous. Next, we can compute the possible degree of β_{ij}:

$$deg(\beta_{ij}) = deg(\tilde{\alpha}_{ij}) - deg(\hat{F}_{ij}) = d_i + d_j - (d - d_i - d_j) = d_i + d_j - \sum_{k\neq i,j} d_k.$$

These degrees are all strictly negative according to the balanced assumption on **d**. We deduce the term associated to every β_{ij} does not take part of the formula, and can be considered as zero. In order to finish the proof observe $\tilde{\alpha}$ and

$$\sum_{i\neq j} \lambda_{ij} \hat{F}_{ij} dF_i \wedge dF_j$$

has the same restriction to each component X_{ij}, and therefore their difference $\beta = \tilde{\alpha} - \sum \lambda_{ij} \hat{F}_{ij} dF_i \wedge dF_j$ vanishes when restricted to $X^2_{\mathcal{D}_F}$. □

With the notation and hypotheses of the above lemma, observe that $\tilde{\alpha}$ and $\sum_{i\neq j}\lambda'_{ij}\hat{F}_{ij}dF_i\wedge dF_j$ satisfy the integrability perturbation equation 4.3.12, and so the same holds for β. Therefore, we reduce the problem to study the possible homogeneous affine forms which satisfy the perturbation equation and vanish on the restriction to $X^2_{\mathcal{D}}$.

Note that we are being careful to not say β determines a first order perturbation of ω, because, a priori, it is not known if the form associated to λ' satisfy the locally decomposability perturbation equation 4.3.11. We will deal with this problem later.

Lemma 4.4.17 (Step 6)**.** Assume again that $\mathbf{d}$ is 2-balanced. If $\beta\in H^0(\mathbb{C}^{n+1},\Omega^2_{\mathbb{C}^{n+1}})$ is an homogeneous affine 2-form of degree d, satisfying the integrability perturbation equation:

$$i_v(\beta)\wedge d\omega + i_v(\omega)\wedge d\beta = 0 \qquad \forall v\in\mathbb{C}^{n+1},$$

and $\beta|_{X^2_{\mathcal{D}}}=0$, then necessarily $\beta=0$.

Proof. The idea of this proof is essentially the same as in the step 4 (4.4.10). The advantage will be that the involved form vanishes on a lower codimensional stratum. Therefore, since $\beta|_{X^2_{\mathcal{D}}}=0$, it is possible to describe it using the vanishing lemma 4.4.4 by:

$$\beta=\sum_l \hat{F}_l\beta_l.$$

For each possible index l, $\beta_l\in H^0(\mathbb{C}^{n+1},\Omega^2_{\mathbb{C}^{n+1}})$ is homogeneous of total degree d_l.

Now, fix two distinct indexes $i_0, j_0\in\{1,\ldots,m\}$, and use the construction made at the remark 4.4.6, to choose a rational local vector field $Y_{j_0}=Y^{j_0}_{i_0j_0}$ with the following δ-hypothesis on the restriction to the points of $X_{i_0j_0}$:

$$i_{Y_{j_0}}(dF_{i_0})=0 \quad ; \quad i_{Y_{j_0}}(dF_{j_0})=1.$$

In this case, the terms of the integrability perturbation equation can be described by:

$$i_{Y_{j_0}}(\beta)\wedge d\omega=\sum_{\substack{i\neq j\neq k\\ l}}\lambda_{ij}\hat{F}_{ijk}\hat{F}_l i_{Y_{j_0}}(\beta_l)\wedge dF_i\wedge dF_j\wedge dF_k$$

$$i_{y_{j_0}}(\omega)\wedge d\beta=\sum_{\substack{i\neq j\\ k\neq l}}2\lambda_{ji}\hat{F}_{ij}\hat{F}_{kl}i_{Y_{J_0}}(dF_j)dF_i\wedge dF_k\wedge\beta_l+\sum_{\substack{i\neq j\neq k\\ l}}2\lambda_{ji}\hat{F}_{ij}\hat{F}_l i_{Y_{j_0}}(dF_j)dF_i\wedge d\beta_l$$

The first important thing to observe is that if we take the restriction of this integrability perturbation equation to the point of $X_{i_0j_0}$, we obtain:

$$\lambda_{j_0i_0}(\hat{F}_{i_0j_0})^2\,dF_{i_0}\wedge dF_{j_0}\wedge\beta_{i_0}=0 \qquad (\text{over } \ X_{i_0j_0}).$$

According to the hypotheses assumed on the open set $\mathcal{U}_2(\mathbf{d})$, the restricted polynomial $\lambda_{j_0i_0}(\hat{F}_{i_0j_0})^2$ is not zero, and then we are able to remove it from above. Once more, the division lemma 4.4.3 applied to β_{i_0}, implies the existence of homogeneous affine 1-forms $\mu^{i_0}_{i_0j_0}$ and $\mu^{i_0}_{j_0i_0}$ such that:

$$\beta_{i_0}=\mu^{i_0}_{i_0j_0}\wedge dF_{i_0}+\mu^{i_0}_{j_0i_0}\wedge dF_{j_0} \qquad \text{over} \quad X_{i_0j_0}.$$

Next,since dF_{i_0} and $\tilde{\tilde{\alpha}}_{i_0}$ has the same total degree, we deduce $\mu^{i_0}_{i_0 j_0} = 0$.

Select a new index k_0, and figure out that the above decompositions for β_{i_0}, associated to the subvarieties $X_{i_0 j_0}$ and $X_{i_0 k_0}$, must coincide on the restriction to $X_{i_0 j_0 k_0}$. So, we obtain:

$$\beta_{i_0} = \mu^{i_0}_{j_0 i_0} \wedge dF_{j_0} = \mu^{i_0}_{k_0 i_0} \wedge dF_{k_0} \quad \text{over} \quad X_{i_0 j_0 k_0}.$$

Moreover taking the wedge product by dF_{k_0} in the above equalities it is possible to deduce:

$$\mu^{i_0}_{j_0 i_0} \wedge dF_{j_0} \wedge dF_{k_0} = 0.$$

Using again the division lemma, there exist an homogeneous polynomial $A_{i_0 j_0 k_0}$ (possibly zero) satisfying:

$$\beta_{i_0} = A_{i_0 j_0 k_0} dF_{j_0} \wedge dF_{k_0} \quad \text{over} \quad X_{i_0 j_0 k_0}.$$

With the usual argument that we have been repeating throughout this work, we need to show that the polynomial $\hat{F}_{i_0 j_0 k_0}$ divides $A_{i_0 j_0 k_0}$. For each index $l \notin \{i_0, j_0, k_0\}$, in agreement with the following equality over $X_{i_0 j_0 k_0 l}$:

$$A_{i_0 j_0 k_0} dF_{j_0} \wedge dF_{k_0} = A_{i_0 j_0 l} dF_{j_0} \wedge dF_l$$

and due to the normal crossing condition of $\mathcal{D}_F$, we obtain:

$$A_{i_0 j_0 k_0}|_{X_{i_0 j_0 k_0 l}} = 0.$$

So, it is possible to deduce the existence of a new polynomial $\tilde{A}_{i_0 j_0 k_0}$ (possibly zero) fulfilling:

$$\beta_{i_0} = \hat{F}_{i_0 j_0 k_0} \tilde{A}_{i_0 j_0 k_0} dF_{j_0} \wedge dF_{k_0} \quad \text{over} \quad X_{i_0 j_0 k_0}.$$

Furthermore, we could compute the possible degree of this new polynomial by:

$$deg(\tilde{A}_{i_0 j_0 k_0}) = d_{i_0} - (d - d_{i_0} - d_{j_0} - d_{k_0}) - d_{j_0} - d_{k_0} = d_{i_0} - \sum_{l \neq i_0} d_l,$$

and note that it is negative since the divisor is also 1-balanced (deduced by the remark 4.4.12). As a conclusion and after all the possible permutations of the indexes i_0,j_0 and k_0, we attain the following condition for each homogeneous form β_i:

$$\beta_i|_{X_{ijk}} = 0 \quad \forall \ j, k \neq i.$$

This last fact is going to be the key to prove the vanishing of the forms β. With this purpose, consider the following suitable subvariety defined by:

$$X^3_{(i)} = \bigcup_{j,k \neq i} X_{ijk} \subset X^3_{\mathcal{D}_F}.$$

From where it is cleat that β_i vanishes on $X^3_{(i)}$. Also, recall that the ideal associated to $X^3_{\mathcal{D}}$ can be characterized by $\mathcal{I}^3_{\mathcal{D}} = \langle \hat{F}_{jk} \rangle_{\{j,k\}}$ (see A.0.20), and then observe that $X^3_{(i)} = X^3_{\mathcal{D}} \cap (F_i = 0)$. So it is clear that its associated ideal corresponds to:

$$\mathcal{I}_{X^3_{(i)}} = \langle \hat{F}_{ij} \rangle_{j \neq i}.$$

With a slightly modified proof of the restriction lemma (using the variety $X^3_{(i)}$ instead of $X^3_{\mathcal{D}}$), it is possible to deduce the existence of affine 2-forms $\{\beta_{ij}\}_{j\neq i}$ also homogeneous (possible zero) such that:

$$\beta_i = \sum_{j\neq i} \hat{F}_{ij}\beta_{ij}.$$

The possible degrees of these new introduced forms can be stated by:

$$deg(\beta_{ij}) = d_i - (d - d_i - d_j) = d_i - \sum_{l\neq i,j} d_l.$$

Finally, observe again that they are all negative since **d** is 2-balanced. This last imply each form β_i equals to zero over the whole space, and then we conclude the vanishing of the entire form β, which brings to a close the proof of this lemma.

□

Finally, we can summarize last step of the proof of proposition 4.4.2.

Proposition 4.4.18 (Step 7)**.** We will keep the same notation as usual, and assume that the vector **d** is 2-balanced. Let $\omega = \sum \lambda_{ij}\hat{F}_{ij}dF_i \wedge dF_j \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))$ a logarithmic 2-form of type **d**, where also $\lambda \in Gr(2, \mathbb{C}^m_{\mathbf{d}})$. If $\tilde{\alpha}$ is a Zariski tangent vector of the moduli space $\mathcal{F}_2(d, \mathbb{P}^n)$ at ω, and also $\tilde{\alpha}|_{X^3_{\mathcal{D}}} = 0$, then there exist $\lambda' \in \mathcal{T}_\lambda Gr(2, \mathbb{C}^m_{\mathbf{d}})$ fulfilling:

$$\tilde{\alpha} = \sum_{i\neq j} \lambda'_{ij}\hat{F}_{ij}dF_i \wedge dF_j$$

Proof. The previous lemmas 4.4.16 and 4.4.17, imply that the form $\tilde{\alpha}$ can be described by:

$$\tilde{\alpha} = \sum_{i\neq j} \lambda'_{ij}\hat{F}_{ij}dF_i \wedge dF_j \in H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(d))/_{\langle\omega\rangle},$$

where the antisymmetric matrix of constants (λ'_{ij}) can be thought as an element $\lambda' \in \bigwedge^2(\mathbb{C}^m_{\mathbf{d}})/_{\langle\lambda\rangle}$. Now, recall from section 4.3.4 that we can interpret the elements $\lambda' \in \mathcal{T}_\lambda(Gr(2, \mathbb{C}^m_{\mathbf{d}})$ as vectors $\lambda' \in \bigwedge^2(\mathbb{C}^m_{\mathbf{d}})/_{\langle\lambda\rangle}$ which satisfy:

$$\lambda \wedge \lambda' = 0.$$

In conclusion, it only rest to prove that the element λ' obtained at the beginning, satisfies this last equation. With this purpose, we just need to use in a correct way that $\tilde{\alpha}$ fulfill the decomposability perturbation equation:

$$\tilde{\alpha} \wedge \omega = 0,$$

which according to the representation attained for α is equivalent to:

$$0 = (\sum_{i\neq j} \lambda'_{ij}\hat{F}_{ij}dF_i \wedge dF_j) \wedge (\sum_{k\neq l} \lambda_{kl}\hat{F}_{kl}dF_k \wedge dF_l).$$

Then, if we divide the expression by F^2, we get:

$$0 = \sum_{i\neq j\neq k\neq l} (\lambda' \wedge \lambda)_{ijkl}\frac{dF_i}{F_i} \wedge \frac{dF_j}{F_j} \wedge \frac{dF_k}{F_k} \wedge \frac{dF_l}{F_l}.$$

Finally, according to the extended Jouanolou lemma 3.5.4 for normal crossing divisors, we obtain the desired result $\lambda' \wedge \lambda = 0$, as claimed.

□

Corollary 4.4.19. Based on the combination of the propositions 4.4.9 and 4.4.18 (steps 3 and 7), we can conclude the whole proof of the surjectivity result for the derivative of the natural parametrization ρ (prop. 4.4.2), and then we can deduce our main result: theorem 4.4.1.

Appendix A

Restriction of forms to spaces associated to normal crossing divisors

Let $\mathcal{D} = \sum_{i=1}^{m} \mathcal{D}_i$ be a simple normal crossing effective divisor of degree d over $\mathbb{P}^n$, whose irreducible components $\mathcal{D}_i$ are defined by the zero locus of an homogeneous polynomial F_i of degree d_i. We also write $d = \sum d_i$.

In order to study the restriction of homogeneous forms to certain spaces associated to $\mathcal{D}$, let us remark some simple facts about the differential forms which are going to be involved. The space $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ of global twisted projective forms of degree d can be characterized using the pullback by the projection map $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$.

Explicitly, for $d \geq q$ an element $\alpha \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}.}(d))$ is described in homogeneous coordinates by the affine q-forms on $\mathbb{C}^{n+1}$ of the type:

$$\alpha = \sum_{I \subset \{0,\ldots,n\} : |I| = q} A_I(z_0, \ldots, z_n) dz_{i_1} \wedge \cdots \wedge dz_{i_q}, \tag{A.0.1}$$

where each A_I is a homogeneous polynomial of degree $d - q$. In addition, we require $\binom{n+1}{q-1}$ extra polynomial equations on this family of polynomials in order to ensure the descend condition of form, i.e. $i_R(\alpha) = 0$ (where R denotes the radial Euler field). In other words, we say that the affine q-form α descends to the projective space.

In conclusion, we refer to the forms of type A.0.1 as **homogeneous affine q-forms of total degree** d, or alternatively as elements $H^0(\mathbb{C}^{n+1}, \Omega^q_{\mathbb{C}^{n+1}})(d)$. And we refer to those forms which also satisfies the descend condition $i_R(\alpha) = 0$ as **homogeneous projective q-forms of total degree** d, or alternatively as elements of $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$.

We write $X^k_{\mathcal{D}}$ for the codimension k projective subvariety associated to the divisor $\mathcal{D}$ defined by:

$$X^k_{\mathcal{D}} = \bigcup_{I : |I| = k} X_I = (F_{i_1} = \ldots F_{i_k} = 0) \quad \overset{i_k}{\hookrightarrow} \quad \mathbb{P}^n.$$

An appropriated set of generators for the homogeneous ideals of $X^k_{\mathcal{D}}$ can be described by:

Proposition A.0.20. For each $k \in \{1, \dots, m\}$, the homogeneous ideals I_k associated to $X^k_{\mathcal{D}}$ is:

$$I_k = \langle \hat{F}_J = \prod_{i \notin J} F_i \rangle_{\substack{J \subset \{1,\dots,m\} \\ |J|=k-1}}$$

Proof. For each $J \subset \{1, \dots, m\}$ of size k the homogeneous ideal I_J associated to X_J is clearly generated by the polynomials $F_{j_1}, \dots, F_{j_K}$. Also, since $X^k_{\mathcal{D}} = \bigcup_{J:|J|=k} X_J$, the ideal I_k is trivially described by:

$$I_k = \bigcap_{J:|J|=k} \langle F_{j_1}, \dots, F_{j_k} \rangle.$$

For every integer k fixed, we will proceed by induction on the total number of polynomials m. In addition, $m \geq k$ is required. The base case ($m = k$) follows immediately from the definitions given. Now, suppose the result is true for $m - 1$ polynomials with $m - 1 \geq k$, and let us show that:

$$\bigcap_{J \subset \{1,\dots,m\}:|J|=k} \langle F_{j_1}, \dots, F_{j_k} \rangle = \langle \hat{F}_J \rangle_{\substack{J \subset \{1,\dots,m\}: \\ |J|=k-1}}.$$

The direct inclusion is always clear and does not require the inductive argument. For the reverse inclusion, first observe that using the inductive hypothesis and pulling away the multi-indexes of the intersection which do not contain m, we obtain:

$$I_k = \langle \hat{F}_{J \cup \{m\}} \rangle_{\substack{J \subset \{1,\dots,m-1\}: \\ |J|=k-1}} \cap \bigcap_{\substack{J \subset \{1,\dots,m-1\}: \\ |J|=k-1}} \langle F_{j_1}, \dots, F_{j_{k-1}}, F_m \rangle.$$

For every homogeneous polynomial P in I_k, there exist homogeneous polynomials H_J for each multi-index J of size $k - 1$, fulfilling:

$$P = \sum_{J:|J|=k-1} H_J \hat{F}_{J \cup \{m\}}.$$

Moreover, for every $J_0 \subset \{1, \dots, m - 1\}$ of size $k - 1$ fixed, the class $[P]$ in the quotient ring

$$\mathbb{C}[z_0, \dots, z_n] \big/ \langle F_{j_1}, \dots, F_{j_{k-1}}, F_m \rangle$$

equals to zero, and so $0 = [P] = [H_{J_0}][\hat{F}_{J_0 \cup \{m\}}]$. Finally, since the quotient ring considered is integral and $F_l \notin \langle F_{j_1}, \dots, F_{j_{k-1}}, F_m \rangle$ for every index $l \notin J_0$ (distinct of m), we get $[H_{J_0}] = 0$.

□

We will also write $\mathcal{I}_k$ for the ideal sheaf associated to each projective subvariety $X^k_{\mathcal{D}}$. By the last proposition this sheaf is generated on global sections by the elements of the type $\hat{F}_J$ for every multi-index J of size $k - 1$.

From now on, we want to study the restriction of homogeneous projective forms to this family of subvarieties, and in particular characterize those whose restriction vanish. For this purpose, consider the usual short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_k \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow i_k^*(\mathcal{O}_{X^k_{\mathcal{D}}}) \longrightarrow 0,$$

take the tensor product of the sequence by the twisted sheaf $\Omega^q_{\mathbb{P}^n}(d)$ to get

$$0 \longrightarrow \Omega^q_{\mathbb{P}^n}(d) \otimes \mathcal{I}_k \longrightarrow \Omega^q_{\mathbb{P}^n}(d) \longrightarrow \Omega^q_{\mathbb{P}^n}(d)|_{X^q_{\mathcal{D}}} \longrightarrow 0, \tag{A.0.2}$$

and finally take into consideration the related long exact sequence on cohomology:

$$0 \longrightarrow H^0(\Omega^q_{\mathbb{P}^n}(d) \otimes \mathcal{I}_k) \longrightarrow H^0(\Omega^q_{\mathbb{P}^n}(d)) \longrightarrow H^0(\Omega^q_{\mathbb{P}^n}(d)|_{X^q_{\mathcal{D}}}) \longrightarrow H^1(\Omega^q_{\mathbb{P}^n}(d) \otimes \mathcal{I}_k) \longrightarrow \dots \tag{A.0.3}$$

This last sequence allows us to think the elements $\eta \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d) \otimes \mathcal{I}_k)$ as homogeneous projective q-forms of total degree d such that: $\eta|_{X^q_{\mathcal{D}}} = 0$. The next propositions give a first description of those forms vanishing on this type of restrictions. For simplicity we shall assume that $d_i > q$ for $i = 1, \dots, m$.

Proposition A.0.21. For every $\eta \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(d))$ satisfying $\eta|_{X^k_{\mathcal{D}}} = 0$, there exist homogeneous affine q-forms $\{\gamma_J\}_{J:|J|=k-1}$ on $\mathbb{C}^{n+1}$ of respecting total degrees $(e_J = \sum_{i \notin J} d_i)_{J:|J|=k-1}$, such that:

$$\eta = \sum_{J:|J|=k-1} \hat{F}_J \gamma_J, \tag{A.0.4}$$

with the extra condition $\sum_{J:|J|=k-1} \hat{F}_J i_R(\gamma_J) = 0$.

On the other hand, we get the same result for homogeneous affine forms. We use the notation $C(X^k_{\mathcal{D}})$ to denote the affine cone over the projective variety $X^k_{\mathcal{D}}$.

Proposition A.0.22. For every homogeneous affine q-form on $\mathbb{C}^{n+1}$ of total degree d satisfying $\eta|_{C(X^k_{\mathcal{D}})} = 0$, there exist homogeneous affine q-forms $\{\gamma_J\}_{J:|J|=k-1}$ of respective total degrees $(e_J = \sum_{i \notin J} d_i)_{J:|J|=k-1}$, and such that:

$$\eta = \sum_{J:|J|=k-1} \hat{F}_J \gamma_J,$$

Remark A.0.23. In the case where some degree d_i is lower or equal than q, the corresponding form η_i can be considered as zero.

Proof. According to the characterization given at first part of this appendix, it is sufficient to prove the statement for homogeneous affine q-forms, and separately deduce the extra condition.

Specifically, let $\eta \in H^0(\mathbb{C}^{n+1}, \Omega^q_{\mathbb{C}^{n+1}})$ be an homogeneous affine q-form with the hypotheses of A.0.22. Also, denote by I_k the ideal associated to $C(X^k_{\mathcal{D}})$ (and $\mathcal{I}_k$ the respecting ideal sheaf). By the above proposition, this ideal coincides with $\langle \hat{F}_J \rangle_{J:|J|=k-1}$.

Now, consider the same exact sequence as A.0.2, but turned up for our purposes on the affine cone $C(X^k_{\mathcal{D}})$. Properly, it is related to the pullback of the sequence A.0.2 by the projection morphism π. So, we can express:

$$0 \to \Omega^q_{\mathbb{C}^{n+1}} \otimes \mathcal{I}_k \to \Omega^q_{\mathbb{C}^{n+1}} \to (i_k)_*(\Omega^q_{\mathbb{C}^{n+1}}|_{C(X^k_{\mathcal{D}})}) \to 0.$$

Again, we need to consider the associated long exact on cohomology, and use the fact

$$H^0(\mathbb{C}^{n+1}, \Omega^q_{\mathbb{C}^{n+1}} \otimes \mathcal{I}_k) \simeq \Omega^q_{\mathbb{C}[z_0,\dots,z_n]} \otimes I_k,$$

to describe the affine q-form η by:

$$\eta = \sum_{J:|J|=k-1} \hat{F}_J \gamma_J$$

for some affine forms $\gamma_J \in H^0(\mathbb{C}^{n+1}, \Omega^q_{\mathbb{C}^{n+1}})$. Now, we need to take into consideration the grading induced on the exterior algebra $\Omega^*_{\mathbb{C}[z_0,\dots,z_n]}$, to regard each γ_J as an homogeneous affine form of total degree $\sum_{l\notin J} d_l$.

Finally, if the original form η is an homogeneous projective q-form, then it also satisfies the condition $i_R(\eta) = 0$, and this fact imposes to the obtained formula the additional requirement:

$$\sum_{J:|J|=k-1} \hat{F}_J i_R(\gamma_J) = 0.$$

□

Remark A.0.24. The relation between the propositions A.0.21 and A.0.22, and the corresponding required short exact sequences, is based on the double complex which corresponds to the tensor product of the Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

by

$$0 \longrightarrow \mathcal{I}_k \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow i_k^*(\mathcal{O}_{X^k_{\mathcal{D}}}) \longrightarrow 0,$$

and the corresponding twist by $\mathcal{O}_{\mathbb{P}^n}(d)$.

Remark A.0.25. In the previous propositions, we particularly use the description made at the result A.0.20, which implies that the ideal I_k receives a graded epimorphism:

$$\bigoplus_{I:|I|=k-1} S[-\hat{d}_I] \twoheadrightarrow I_k.$$

It can be notice that our conclusions do not take into account the relations between the generators of the ideal I_k, which are also related to the different expressions like A.0.4 that η allows. In the sequel, we deal with this problem and the resolution of the corresponding ideals.

As it was announced, we want to describe the resolution of the corresponding ideals I_k, and provide better descriptions of the space of projective forms vanishing when restricted to $X^k_{\mathcal{D}}$. We start with the case $k = 2$.

Note that the ideal $\mathcal{I}_2$ is generated by the set $\{\hat{F}_i\}$, and also that between these elements we have the simple relations:

$$R_{i,j}: \qquad F_i \hat{F}_i - F_j \hat{F}_j = 0.$$

In addition, observe that $R_{i,j} = R_{1,j} - R_{1,i}$, and also, that the set of relations $\{R_{1,i}\}$ are independent. Next, we can prove the following result.

Theorem A.0.26. The codimension two subvariety $X^2_{\mathcal{D}}$ of $\mathbb{P}^n$ is arithmetically Cohen-Macaulay, and

$$0 \longrightarrow (\mathcal{O}_{\mathbb{P}^n}(-d))^{m-1} \xrightarrow{A} \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-\hat{d}_i) \xrightarrow{B} \mathcal{I}_2 \longrightarrow 0$$

is a graded resolution of the corresponding ideal. The arrow B is defined by the vector of generators $(\hat{F}_1, \dots, \hat{F}_m)$, and the second arrow A is defined by the $(m \times (m-1))$-matrix of relations:

$$A = \begin{pmatrix} F_1 & F_1 & \dots & F_1 & F_1 \\ -F_2 & 0 & \dots & \dots & 0 \\ 0 & -F_3 & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -F_m \end{pmatrix}$$

Moreover, this short exact sequence can be reinterpreted in terms of the associated graded modules of the involved sheaves, as the following sequence:

$$0 \to S[-d]^{m-1} \xrightarrow{A} \bigoplus_{i=1}^{m} S[-\hat{d}_i] \xrightarrow{B} I_2 \to 0,$$

where S refers to the polynomial graded ring $\mathbb{C}[z_0, \dots, z_n]$.

Proof. First, note that the principal minors of size $(m-1) \times (m-1)$ of the matrix A corresponds exactly with the generators of the ideal I_2. Furthermore, it is a classical result for projective varieties of codimension two, the fact that the announced short exact sequence is a free resolution of I_2, and also that the ideals described is such way determines an arithmetically Cohen-Macaulay variety (see for instance chapter 18 of [19] or the [20, Theorem 3.2] for a general version of the Hilbert-Burch theorem). Moreover, this resolution is its classical associated resolution of length two.

On the other hand, the Buchsbaum–Eisenbud theorem provides another way of proving the exactness of the introduced sequence (see [20, Theorem 3.3]). We just need to observe that the points where the map defined by the matrix A fails to be injective corresponds to the codimension one variety $X^1_{\mathcal{D}}$. □

Remark A.0.27. The assumption of $\mathcal{D}$ being simple normal crossing is not really necessary for the previous theorem. We only need to assume the necessary hypotheses to deduce that $X^2_{\mathcal{D}}$ can be described by the zero locus of the ideal $\langle \hat{F}_i \rangle_{i=1}^m$. Technically, we can assume that every selection of three of the polynomials $\{F_i\}_{i=1}^m$ define a regular sequence on the corresponding ring of polynomials. So, we have deduced that under these assumptions, the codimension two set of singularities of a logarithmic form is always an arithmetically Cohen-Macaulay projective variety. This is a particular case of the result stated at [4], for the Kupka singularities of a codimension one foliation.

Now, we can use the resolution of the ideal I_2 to describe better the sheaf $\Omega^q_{\mathbb{P}^n}(d) \otimes \mathcal{I}_2$. Again, for simplicity we shall assume that $d_i > q$ for $i = 1, \dots, m$. More precisely, consider the tensor product of the Theorem's resolution by $-\otimes\Omega^q_{\mathbb{P}^n}(d)$, to get:

$$0 \longrightarrow (\Omega^q_{\mathbb{P}^n})^{m-1} \longrightarrow \bigoplus_{i=1}^m \Omega^q_{\mathbb{P}^n}(d_i) \longrightarrow \Omega^q_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2 \longrightarrow 0,$$

and take into account the long exact sequence on cohomology:

$$\text{(A.0.5)}\qquad \begin{array}{ccccccc} 0 \longrightarrow \bigoplus_{i=1}^m H^0(\Omega^q_{\mathbb{P}^n}(d_i)) & \xrightarrow{\phi} & H^0(\Omega^q_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2) & \xrightarrow{\delta^2_q} & H^1(\Omega^q_{\mathbb{P}^n}))^{m-1} & \longrightarrow & \dots \\ (\eta_i)_{i=1}^m & \longmapsto & \sum_{i=1}^m \hat{F}_i\eta_i & & & & \end{array}$$

In the previous development we were needed to use that the sheaf $\Omega^q_{\mathbb{P}^n}$ has no global sections. In addition, according to:

$$H^1(\Omega^q_{\mathbb{P}^n}) = \begin{cases} 0 & \text{if } q>1 \\ \simeq \mathbb{C} & \text{if } q=1 \end{cases}$$

we are able to deduce two different results that depend on whether $q>1$.

Proposition A.0.28. The space of projective q-forms of degree d vanishing on the restriction to $X^2_{\mathcal{D}}$, $H^0(\Omega^q_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)$, can be characterized for $q>1$ by the isomorphism:

$$\phi : \bigoplus_{i=1}^m H^0(\Omega^q_{\mathbb{P}^n}(d_i)) \xrightarrow{\sim} H^0(\Omega^q_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2),$$

So for each $\eta\in H^0(\Omega^q_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)\hookrightarrow H^0(\Omega^q_{\mathbb{P}^n}(d))$ there exist a unique m-tuple of homogeneous projective q-forms $(\eta_i)_{i=1}^m$ of degrees (d_i) such that: $\eta=\sum_{i=1}^m \hat{F}_i\eta_i$.

Proof. It follows from the above long exact sequence 3.5.1 and the previous observation about the Hodge numbers $h^{q,1}_{\mathbb{P}^n}$. □

Corollary A.0.29. Recall that we are assuming $d_i>q$ for all index i. Then, according to the Bott's formula (see [42]), we are able to calculate the dimension of the space of global forms vanishing on $X^2_{\mathcal{D}}$, according to:

$$\dim(H^0(\Omega^q_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)) = \sum_{i=1}^m \binom{n+d_i-q}{d_i}\binom{d_i-1}{q}$$

Corollary A.0.30. The previous proposition complements the result A.0.21 with the following non trivial feature. Fix $q>1$. If we consider a global homogeneous q-form η on $\mathbb{C}^{n+1}$ of the type

$$\eta=\sum \hat{F}_i\gamma_i,$$

then $i_R(\eta)=0$ if and only if $i_R(\gamma_i)=0$ for all i.

On the other hand, observe that when $q=1$ the morphism ϕ from A.0.5 is not surjective. Also, the space $Ker(\delta^2_1)$ determines conditions on a homogeneous form at $H^0(\Omega^1_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)$ to be of the type: $\sum\hat{F}_i\eta_i$ for some homogeneous projective forms η_i.

Note that for a given $\eta \in H^0(\Omega^1_{\mathbb{P}^n}(d) \otimes \mathcal{I}_2) \subset H^0(\Omega^1_{\mathbb{P}^n}(d))$, we know there exist homogeneous affine 1-forms $(\gamma_i)_{i=1}^m$ of respecting degrees (d_i), which decompose η by the formula:

$$\eta = \sum_{i=1}^{m} \hat{F}_i \gamma_i,$$

and also satisfy $i_R(\eta) = \sum \hat{F}_i i_R(\gamma_i) = 0$. Since each $i_R(\gamma_i)$ is an homogeneous polynomial, the previous equation establish a relation between the generators of the ideal I_2. Therefore, according to the theorem A.0.26, there exist unique constants $c_1^\eta, \ldots, c_{m-1}^\eta$ such that the relation $i_R(\eta) = 0$ can be expressed by $\sum c_j^\eta R_{1j}$.

It is easy to check that the connection morphism δ_1^2 is described by the assignment $\eta \mapsto c_j^\eta$. In other words, after composing δ_1^2 with the isomorphism $H^1(\Omega^1_{\mathbb{P}^n})^{m-1} \simeq \mathbb{C}^{m-1}$, we obtain:

$$\begin{aligned} H^0(\Omega^q_{\mathbb{P}^n}(d) \otimes \mathcal{I}_2) &\xrightarrow{\mu_1^2} \mathbb{C}^{m-1} \\ \eta &\mapsto (c_i^\eta)_{i=1}^{m-1}. \end{aligned}$$

In addition, the exactness of the sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \bigoplus_{i=1}^m H^0(\Omega^1_{\mathbb{P}^n}(d_i)) & \xrightarrow{\phi} & H^0(\Omega^1_{\mathbb{P}^n}(d) \otimes \mathcal{I}_2) & \xrightarrow{\mu_1^2} & \mathbb{C}^{m-1} & \longrightarrow \ldots \\ & (\eta_i)_{i=1}^m & \longmapsto & \sum_{i=1}^m \hat{F}_i \eta_i & & & \\ & & & \eta & \longmapsto & (c_i^\eta)_{i=1}^{m-1} & \end{array}$$

agree with the following observation. The trivial relation $((c_j^\eta) = 0)$ corresponds exactly with forms γ_i which are all projective, i.e. $i_R(\gamma_i) = 0$.

In conclusion, we have proved the next result.

Proposition A.0.31. The space of projective 1-forms vanishing on $X^2_{\mathcal{D}}$, can be characterize by the short exact sequence:

$$0 \longrightarrow \bigoplus_{i=1}^{m} H^0(\Omega^1_{\mathbb{P}^n}(d_i)) \xrightarrow{\phi} H^0(\Omega^1_{\mathbb{P}^n}(d) \otimes \mathcal{I}_2) \xrightarrow{\mu_1^2} \mathbb{C}^{m-1} \longrightarrow 0$$

with ϕ and μ_1^2 are defined as before. So its dimension equals to:

$$\dim(H^0(\Omega^1_{\mathbb{P}^n}(d) \otimes \mathcal{I}_2)) = m - 1 + \sum_{i=1}^{m} (d_i - 1) \binom{n + d_i - 1}{d_i}.$$

Proof. It is an immediate consequence of A.0.5 and the following Bott's formulas for $k \in \mathbb{Z}$ (see for instance [42]):

$$h^1(\Omega^1_{\mathbb{P}^n}(k)) = 0 \quad ; \quad h^0(\Omega^1_{\mathbb{P}^n}(k)) = \begin{cases} (k-1)\binom{n+h-1}{k} \text{ if } k > 1 \\ 0 \quad \text{otherwise} \end{cases}$$

□

With a slight abuse of notation, observe that we could proved the following isomorphism:

$$H^0(\Omega^1_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)/\bigoplus_{i=1}^{m}H^0(\Omega^1_{\mathbb{P}^n}(d_i))\simeq\mathbb{C}^{m-1}.$$

So, in order to end this description, we need to set some canonical classes of that quotient corresponding for each vector of constants $(c_i)\in\mathbb{C}^{m-1}$. For this purpose, recall that for a given $\eta=\sum\hat{F}_i\gamma_i\in H^0(\Omega^1_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)$ we have the additional condition

$$\sum\hat{F}_i i_R(\gamma_i)=0,$$

which can be thought as a relation expressed by the product:

$$(i_R(\gamma_1),\dots,i_R(\gamma_i),\dots,i_R(\gamma_m)).(\hat{F}_1,\dots,\hat{F}_i,\dots,\hat{F}_m)^t=0.$$

The constants $(c_i^\eta)_{i=1}^{m-1}$ were defined to express the last relation as a combination of the linearly independent relations R_{1i}, i.e.

$$\begin{aligned}(i_R(\gamma_1),\dots,i_R(\gamma_i),\dots,i_R(\gamma_m)) &= \sum_{i=2}^{m}c_{i-1}^\eta(F_1,0,\dots,0,-F_i,0,\dots,0)=\\ &=((\sum_{i=2}^{m}c_{i-1}^\eta)F_1,-c_1F_2,\dots,-c_{m-1}F_m).\end{aligned}$$

The previous equality gives us equations relating the factors (γ_i) with the vector of constants (c_i^η). So, if we fix any constant $c=(c_i)\in\mathbb{C}^{m-1}$, observe that:

$$\eta_c=\frac{(\sum_{i=1}^{m-1}c_i)}{d_1}\hat{F}_1dF_1+\sum_{j=2}^{m}\frac{-c_{j-1}}{d_j}\hat{F}_jdF_j$$

is an element of $H^0(\Omega^1_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)$, whose "$\gamma_i$" are not projective forms, and also satisfies $\mu_1^2(\eta_c)=c$ (according to the previous equations). Then, these forms can be used as a canonical representative of each class. In conclusion, we have the following result.

Proposition A.0.32. For every element $\eta\in H^0(\Omega^1_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)$, there exist unique homogeneous projective 1-forms $(\eta_i)\in\bigoplus_{i=1}^m H^0(\Omega^1_{\mathbb{P}^n}(d_i))$ and $c\in\mathbb{C}^{m-1}$, such that:

$$\eta=\eta_c+\sum_{i=1}^{m}\hat{F}_i\eta_i$$

Corollary A.0.33. Using the previous proposition and notation, we can construct the following identification:

$$H^0(\Omega^1_{\mathbb{P}^n}(d)\otimes\mathcal{I}_2)\simeq H^0(\Omega^1_{\mathbb{P}^n}(log\mathcal{D}))\oplus\bigoplus_{i=1}^{m}H^0(\Omega^1_{\mathbb{P}^n}(d_i))$$

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