Homogeneización y diseño óptimo en difusión no local

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(Resumen)

En esta tesis, estudiamos algunos problemas que involucran difusión no local. En la primera parte, obtenemos un resultado de compacidad para la noción de H-convergencia de una familia de problemas monótonos de tipo elípticos, no locales y lineales, por medio del método de Tartar de funciones de prueba oscilantes. En la segunda parte, probamos la existencia de solución para algunos problemas de optimización de forma. Más aún, analizamos la transición de las ecuaciones de estado no locales a las del caso local.

Palabras clave: difusión no local, homogeneización, optimización de forma, Γ-convergencia, capacidad fraccionaria.

Homogenization and optimal design in nonlocal diffusion

(Abstract)

In this thesis, we study some problems involving nonlocal diffusion. In the first part, we obtain a compactness result for the H-convergence of a family of nonlocal and linear monotone elliptic-type problems by means of Tartar's method of oscillating test functions. In the second part, we prove existence results for some shape optimization problems. Moreover, we also analyze the transition from nonlocal to local state equations.

Key words: nonlocal diffusion, homogenization, shape optimization, Γ-convergence, fractional capacity.

Introducción

Para empezar con esta tesis, nos gustaría darle al lector una idea intuitiva de los tres conceptos que aparecen en su título:

- homogeneización,
- diseño óptimo,
- difusión no local.

Empecemos con el primero: **homogeneización**.

Cubrir el piso del cuarto de los niños con alfombra es común para protegerlos cuando juegan y ocasionalmente caen. Cuando vemos una alfombra en una macro-escala, podemos decir que parece ser una cosa única, homogénea. PERO, si nos acercamos lo suficiente, podemos distinguir los espacios entre las diferentes felpas. Por lo que parece heterogénea en una micro-escala.

También, al ver una pared hecha de rocas porosas, puede parecer de textura homogénea cuando la vemos globalmente, en una macro-escala. En contraste, en la micro-escala, parece ser realmente heterogénea.

En ambos casos: alfombra y rocas porosas, podemos decir que las heterogeneidades son demasiado pequeñas comparadas con la totalidad de la dimensión de cada objeto.

Una pregunta posible es: *podemos recolectar información de las propiedades macroscópicas teniendo en cuenta también las microscópicas?*. Este es el objetivo de la **homogeneización**.

Sigamos con el segundo concepto: **diseño óptimo**. Pensemos en una empresa que vende hojas de metal como conductores de electricidad. Podemos asumir que para hacer un producto bueno la empresa debe usar al menos dos materiales:

- el mejor material conductor, pero también el más caro,
- el más barato, pero también el de peor calidad.

Podemos encontrar el diseño óptimo (la forma, la manera de combinar ambos materiales) para fabricar un producto bueno y razonable?

Leyendo entre líneas, podemos decir que un problema de **diseño óptimo** es esencialmente encontrar una *forma* que minimice cierto funcional de costo.

Solo queda un concepto más a discurtir: **difusión no local**.

Los guepardos usualmente cazan sus presas a solo la mitad de su velocidad máxima. Después de cada persecución, un guepardo necesita media hora para recuperar su respiración antes de poder comer. Asumimos que tener una técnica efectiva para cazar es realmente importante para sobrevivir.

Los guepardos comen animales de tamaño chico o mediano, por ejemplo, gacelas. La excelente vista del guepardo lo ayuda a encontrar presas durante el día. PERO, una vez visto el guepardo por las gacelas, las gacelas no esperarán ser atrapadas. Por lo tanto, será más conveniente para el guepardo elegir al azar una dirección, moverse rápidamente en esa dirección y golpear a su presa contra el suelo y luego morder su garganta.

Este tipo de *atropello con fuga* (procedimiento de caza) está relacionado con el concepto de **difusión no local**. Podemos decir que es razonable que los depredadores usen una estategia de *difusión no local* para cazar sus presas más eficientemente.

Una amigable mirada a la difusión no local

A lo largo de la tesis, lidiamos con una familia de operadores no locales. Decimos que $\mathcal{L}$ es un operador no local si debemos saber qué pasa en toda la región cuando solo nos interesa saber el valor en un punto fijo $x \in \mathbb{R}^n$. No importa qué tan lejos un punto $y \in \mathbb{R}^n$ esté del punto fijo x. No alcanza con conocer cómo una función se comporta en un entorno, sino cómo lo hace en toda la región. Solo como para ilustrar, pensemos en la economía global. No importa que tan lejos o cerca estemos de China o Estados Unidos, cualquier decisión económica que tomen, nos afectará, nuestra economía sufrirá las consecuencias.

Un operador local clásico es el laplaciano $-\Delta$. Para u una función suave

$$-\Delta u(x) = -\operatorname{div}(\nabla u(x)) = -\left(\partial^2_{x_1^2} u(x) + \cdots + \partial^2_{x_n^2} u(x)\right).$$

Observemos que solo algunas de las derivadas de segundo orden de u son necesarias para calcular el valor de $-\Delta u(x)$, entonces si solo tenemos información en un entorno de un punto fijo x, será suficiente para llegar al valor de $-\Delta u(x)$.

En contraste, miremos el operador laplaciano fraccionario $(-\Delta)^s$,

$$(-\Delta)^s u(x) = \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy.$$

Más adelante, daremos más detalles. Ahora, solo entendamos al parámetro $s \in (0,1)$ como un exponente fraccionario y a u como una función adecuada. La constante $c(n,s)$ juega un rol clave cuando analizamos el comportamiento asintótico $s \uparrow 1$ de algunos problemas. PERO, por ahora, es solo una *constante de normalización*. Pensemos por ejemplo, el caso $s = \frac{1}{2}$ será como tomar la raíz cuadrada del clásico laplaciano.

Notemos que para calcular $(-\Delta)^s u(x)$ necesitamos saber el valor de $u(z)$ para todo z, no importa qué tan cerca o lejos estén z y x.

Ahora, nos gustaría dar una motivación probabilística del laplaciano fraccionario. Tiene que ver con caminos aleatorios que permiten grandes saltos. Pensamos que es la manera más linda y amigable de introducir este operador no local por primera vez. También el clásico laplaciano tiene una interpretación probabilística similar, pero enfocamos nuestra atención en el caso fraccionario, ya que es nuestro objeto de estudio clave a lo largo de la tesis.

Una motivación probabilística para el laplaciano fraccionario

Un camino aleatorio que permite saltos grandes arbitrarios. Comencemos por describir un proceso probabilístico en el que una partícula se mueve aleatoriamente en el espacio, sujeto a una probabilidad que permite grandes saltos; originando el laplaciano fraccionario.

Hay dos variables a tener en cuenta: $t > 0$ para el **tiempo** y $x \in \mathbb{R}^n$ para la posición **espacial**. Nos referimos a la probabilidad de encontrar la partícula en el punto x a tiempo t como $u(x,t)$. Empezamos decribiendo el proceso con ambas variables discretas. Al final, tomando el límite cuando los *pasos* de tiempo y espacio sean pequeños, llegaremos a la ecuación del calor no local. Fijemos la medida de los pasos: $\tau > 0$ para el paso del tiempo $h > 0$ para el paso del espacio.

Supongamos que la partícula empieza a tiempo t en la posición x. Para comenzar a moverse, la partícula debe elegir al azar una dirección, digamos $v \in \partial B_1$, y un número de pasos, digamos $k \in \mathbb{N}$. Ya que la medida de cada paso es h, la nueva posición a tiempo $t+\tau$ puede ser descripta como $x + khv$.

Para hablar de $u(x, t+\tau)$, la probabilidad de encontrar la partícula en la posición x a tiempo $t+\tau$, es suficiente decidir la probabilidad de elegir una dirección $v \in \partial B_1$ y un número $k \in \mathbb{N}$. Consideremos la distribución uniforme en ∂B_1 y para cada $k \in \mathbb{N}$, denotemos por $a(k)$ la probabilidad de elegir ese k. Entonces, para cada $I \subset \mathbb{N}$, podemos definir

$$P(I) := \sum_{k\in I} a(k).$$

Los grandes saltos están permitidos pero con menos probabilidad que los cortos. Por lo tanto, tomamos $a(k)$ con un decaimiento polinomial, digamos $a(k) \simeq \frac{1}{k^\alpha}$, donde $\alpha > 0$.

Queremos que P también sea una probabilidad en $\mathbb{N}$. Luego, necesitamos $P(\mathbb{N}) = 1$. Como $a(k) \simeq \frac{1}{k^\alpha}$, es suficiente elegir $\alpha > 1$ para la convergencia de la serie. Esto quiere decir, $\alpha = 1 + \beta$, con $\beta > 0$. Puede escribirse como $\alpha = 1 + 2s$ con $s \in (0, 1)$, por ejemplo.

Finalmente, para $I \subset \mathbb{N}$ definimos la probabilidad P como

$$P(I) := c \sum_{k\in I} \frac{1}{k^{1+2s}},$$

donde c es elegida de manera que $P(\mathbb{N}) = 1$.

Ahora, la probabilidad $u(x, t+\tau)$ de encontrar la partícula en la posición x a tiempo $t+\tau$ es la suma de las probabilidades de encontrar la partícula en cualquier otro lugar, digamos en $x + khv$, para alguna dirección $v \in \partial B_1$ y algún número natural $k \in \mathbb{N}$, por la probabilidad de haber elegido esa dirección y ese número natual. Es decir,

$$u(x, t+\tau) = \frac{c}{|\partial B_1|} \sum_{k\in\mathbb{N}} \int_{\partial B_1} \frac{u(x+khv, t)}{|k|^{1+2s}}\, dS_v.$$

Teniendo en cuenta que $\frac{c}{|\partial B_1|}$ es una constante de normalización, por consiguiente, substrayendo $u(x,t)$, obtenemos

$$\begin{aligned} u(x,t+\tau)-u(x,t) &= \frac{c}{|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)}{|k|^{1+2s}}\,dS_v - u(x,t) \\ &= \frac{c}{|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)-u(x,t)}{|k|^{1+2s}}\,dS_v. \end{aligned}$$

Por simetría, podemos cambiar v por $-v$ en la integral de arriba, por lo tanto, tenemos que

$$u(x,t+\tau)-u(x,t) = \frac{c}{|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x-khv,t)-u(x,t)}{|k|^{1+2s}}\,dS_v.$$

Luego, sumando estas dos expresiones, llegamos a

$$u(x,t+\tau)-u(x,t) = \frac{c}{2|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{|k|^{1+2s}}\,dS_v.$$

Ahora, dividiendo por τ, reconocemos una suma de Riemann.

$$\begin{aligned} \partial_t u(x,t) &\simeq \frac{u(x,t+\tau)-u(x,t)}{\tau} \\ &= \frac{c}{2|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{\tau|k|^{1+2s}}\,dS_v \end{aligned}$$

Multiplicando y dividiendo por h^{1+2s}, obtenemos, tomando límite formal,

$$\begin{aligned} \partial_t u(x,t) &\simeq \frac{h^{1+2s}}{\tau}\frac{c}{2|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{\tau|hk|^{1+2s}}\,dS_v \\ &\simeq \frac{h^{2s}}{\tau}\frac{c}{2|\partial B_1|}\int_0^\infty\int_{\partial B_1}\frac{u(x+rv,t)+u(x-rv,t)-2u(x,t)}{|r|^{1+2s}}\,dS_v dr \end{aligned}$$

Se infiere que la relación *adecuada* entre las medidas de los pasos h y τ es $\frac{h^{2s}}{\tau}=\nu$, ν puede ser llamada la constante de difusividad no local. Entonces, usando coordenadas polares,

$$\begin{aligned} \partial_t u(x,t) &\simeq \frac{c}{2|\partial B_1|}\int_{\mathbb{R}^n}\frac{u(x+y,t)+u(x-y,t)-2u(x,t)}{|y|^{n+2s}}\,dy \\ &= -c_{n,s}(-\Delta)^s u(x,t), \end{aligned}$$

donde $c_{n,s}$ es una constante positiva. Al menos formalmente, para pasos pequeños de tiempo y espacio, el proceso probabilístico de arriba aproxima la ecuación del calor no local

$$\partial_t u(x,t)+(-\Delta)^s u(x,t)=0,$$

salvo constantes.

Hemos mostrado cómo este fenómeno no local (camino aleatorio con grandes saltos arbitrarios) se transforma en un operador no local (involucrando el laplaciano fraccionario $(-\Delta)^s$).

Podemos asumir que para tiempo suficientemente grande, el problema se transforma en estacionario, por lo tanto ya no depende del tiempo. También, suponemos que hay una fuente afectando la cantidad de partículas, digamos una cierta función f. Así, llegamos a este tipo de ecuación

$$(-\Delta)^s u = f.$$

En la teoría de probabilidad, el laplaciano fraccionario es un conocido ejemplo que puede ser visto como un generador infinitesimal de procesos de Lévy, en ámbitos más generales. Ver, por ejemplo, [2, 6, 11, 66].

Existen gran cantidad de aplicaciones relacionadas a tipos de problemas no locales. Para nombrar algunas referencias: en física [42, 43, 50, 64, 68, 97], finanzas [3, 65, 84], dinámica de fluidos [30, 34], ecología [58, 67, 77], procesamiento de imágenes [51].

A lo largo de toda la tesis, trabajamos con algunos problemas de Dirichlet:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{0.0.1}$$

donde Ω es un subconjunto abierto acotado de $\mathbb{R}^n$ y $\mathcal{L}$ pertenece a cierta clase de operadores no locales en la que el laplaciano fraccionario es el ejemplo principal. Este tipo de problemas integro-diferenciales surgen naturalmente en el estudio de procesos estocásticos con saltos, como motivamos previamente. Aquellos problemas han sido ampliamente estudiados en las áreas de Probabilidad y Análisis, Ecuaciones en derivadas parciales.

Probamos existencia de solución de los problemas de Dirichlet tratados en esta tesis, a través de cálculo de variaciones, estableciendo una equivalencia entre ser solución débil del problema (0.0.1) y ser minimizante de la energía asociada. Luego, vemos la existencia del minimizante. Resultados de estabilidad y comparación de soluciones también son mostrados.

El lector interesado puede mirar los trabajos de Barles-Imbert [8], Felsinger-Kassmann-Voigt [46], Hoh-Jacob [57], Xiang-Pucci-Squassina-Zhang [96], sobre existencia de soluciones para problemas con operadores más generales. La regularidad interior de las soluciones fue considerada por Bass-Levin [9], Caffarelli-Silvestre [25], Iannizzotto-Mosconi-Squassina [59, 60], Kassmann [61], por ejemplo. Trabajos de regularidad en la frontera de soluciones: Bogdan [13], Grubb [53, 54], Ros-Oton–Serra [80, 79, 81]. Para otras propiedades cualitativas de las soluciones, ver Birkner-López-Wakolbinger [12], Dipierro-Savin-Valdinoci [41], por ejemplo.

Homogeneización

La teoría de homogeneización data desde los trabajos de S. Spagnolo [91], E. De Giorgi y S. Spagnolo [35], I. Babuška [7], A. Bensoussan, J.L. Lions y G. Papanicolaou [10] y E. Sánchez-Palencia [82] entre otros.

En el contexto de ecuaciones lineales elípticas en derivadas parciales, el modelo a estudiar es el límite de $k \to \infty$ de los siguientes problemas

$$\begin{cases} -\operatorname{div}(A_k \nabla u_k) = f & \text{en } \Omega \\ u_k = 0 & \text{en } \partial\Omega, \end{cases} \tag{0.0.2}$$

donde $\Omega \subset \mathbb{R}^n$ es un dominio acotado, $f \in H^{-1}(\Omega)$ y $\{A_k\}_{k\in\mathbb{N}} \subset [L^\infty(\Omega)]^{n\times n}$ es una sucesión de matrices simétricas y uniformemente acotadas.

Como ejemplo modelo, ha sido considerado el caso donde las matrices A_k están dadas en términos de una única matriz A en la forma

$$A_k(x) = A(kx),$$

donde A es periódica, de período 1, en cada variable.

En el marco periódico, el problema límite cuando $k \to \infty$ puede facílmente caracterizarse completamente. Ver [10].

Con el objetivo de lidiar con el caso general, Spagnolo y De Giorgi introdujeron el concepto de G-convergencia, que fue luego generalizado por Murat y Tartar a finales de los 70s y es ahora llamada H-convergencia. Ver [29].

Cuando F. Murat en 1974 estaba estudiando el comportamiento de (0.0.2) cuando $k \to \infty$, uno de los principales inconvenientes que encontró fue el hecho de que el producto de dos sucesiones débilmente convergentes no convergen, en general, al producto de sus límites. Murat venció esta dificultad descubriendo un argumento compensatorio de compacidad conocido como el *div-curl Lema*, denominación sugerida por su tutor, J.L. Lions, debido al hecho de que resulta de un efecto compensatotio. El Lema fue publicado en 1978 [69] y una demostración alternativa fue probada por L. Tartar también en 1978 [93] usando el argumento de compacidad de Hörmander para la inyección de $H_0^1(\Omega)$ en $L^2(\Omega)$. El lema afirma que si consideramos dos sucesiones $\{\psi_k\}_{k\in\mathbb{N}}$ y $\{\phi_k\}_{k\in\mathbb{N}}$ en $[L^2(\Omega)]^n$ tales que

$$\psi_k \rightharpoonup \psi, \quad \text{y} \quad \phi_k \rightharpoonup \phi \quad \text{débil en } [L^2(\Omega)]^n,$$

con la hipótesis adicional de que

$$\operatorname{div}\psi_k \to \operatorname{div}\psi \text{ en } H^{-1}(\Omega), \quad \text{y} \quad \operatorname{curl}\phi_k \to \operatorname{curl}\phi \text{ en } [H^{-1}(\Omega)]^{n\times n},$$

entonces podemos garantizar que $\psi_k \cdot \phi_k \to \psi \cdot \phi$ en el sentido de las distribuciones. Recordemos que el curl de un campo vectorial $\phi \in [L^2(\Omega)]^n$ está definido por

$$\operatorname{curl}\phi = \left(\frac{\partial \phi^i}{\partial x^j} - \frac{\partial \phi^j}{\partial x^i} \right)_{1\le i,j\le n}.$$

El div-curl Lema juega un rol crucial en la teoría de homogeneización. De hecho, basado en este lema, Tartar introdujo en [93, 94] un método dirigido al comportamiento límite de (0.0.2) cuando $k \to \infty$, obteniendo la existencia de una matriz coercitiva $A_0 \in [L^\infty(\Omega)]^{n\times n}$

tal que la sucesión de soluciones $\{u_k\}_{k\in\mathbb{N}}$ de (0.0.2) converge débilmente en $H_0^1(\Omega)$, para una subsucesión, a una función u_0 que es solución del siguiente problema límite *homogeneizado*

$$\begin{cases} -\operatorname{div}(A_0 u_0) = f & \text{en } \Omega \\ u_0 = 0 & \text{en } \partial\Omega. \end{cases} \tag{0.0.3}$$

Más aún, $A_k \nabla u_k \cdot \nabla u_k \to A_0 \nabla u_0 \cdot \nabla u_0$ en el sentido de las distribuciones, ver por ejemplo, [4, 29]. Esto es, la sucesión A_k H-converge a A_0.

En el caso cuasilineal, este tipo de resultados fueron obtenidos por varios autores en los finales de los 80s y los comienzos de los 90s. Al lector interesado damos la referencia de [28, 71] y del libro de G. Dal Maso [31] donde los autores usan métodos de la Γ-convergencia con el fin de lidiar con estos problemas. Ver [17] para el caso periódico. Mencionemos que la Γ-convergencia estudia el comportamiento de los mínimos en problemas variacionales, en el caso especial de funcionales cuadráticos, esto da el comportamiento para problemas elípticos simétricos.

Observamos que en el caso lineal, la H-convergencia y la Γ-convergencia coinciden incluso en el caso no simétrico; fue recientemente demostrado por Ansini, Dal Maso y Zeppieri [5].

Algunos problemas más generales fueron considerados recientemente. Evans, en [44], estudió el caso de homogeneización periódica de ciertas ecuaciones en derivadas parciales elípticas totalmente no lineales y de tipo Hamilton-Jacobi. Posteriormente, Caffarelli, Souganidis y Wang [27] extendieron los resultados de Evans a medios ergódicos estacionarios. En estos artículos la existencia de las ecuaciones homogeneizadas fue demostrada, pero, debido a la generalidad de los mismos, no se puede obtener información adicional sobre la estructura de los problemas límites.

En esta tesis, abordamos el problema de la H-convergencia de la versión no local de (0.0.2) y damos una caracterización del problema límite homogeneizado. Antes de entrar en detalles, repasamos los antecedentes en relación con problemas no locales y su homogeneización.

La teoría de regularidad para ecuaciones integro-diferenciales completamente no lineales, que incluyen al laplaciano fraccionario como un ejemplo trivial, fue estudiada reciente y extensamente. Ver, por ejemplo, [25, 26, 80, 88].

Basados en estos resultados de regularidad para ecuaciones integro-diferenciales completamente no lineales, R. Schwab en [85, 86] extendió los resultados de Evans-Caffarelli, Souganidis-Wang a este marco, pero nuevamente no más información sobre el problema límite fue obtenida. Recordamos que los resultados de Schwab hacen extensivo uso de la periodicidad o de la ergodicidad del problema y el autor no obtiene ningún resultado general de convergencia.

Un trabajo reciente de homogeneización no local en el marco periódico puede encontrarse en [72].

Ahora, describimos brevemente nuestra contribución en Homogeneización en difusión no local.

Sean $0 < \lambda \leq \Lambda < \infty$. Consideremos la familia de núcleos simétricos y acotados

$$\mathcal{A}_{\lambda,\Lambda} = \{a \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \colon a(x,y) = a(y,x),\ \lambda \leq a(x,y) \leq \Lambda \ \text{c.t.p.}\ \}$$

Enfocamos nuestro análisis en una familia general de operadores anisotrópicos lineales de la forma

$$\mathcal{L}_a u(x) := \text{v.p.} \int_{\mathbb{R}^n} a(x,y) \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \quad s \in (0,1),$$

para $a(x,y) \in \mathcal{A}_{\lambda,\Lambda}$. El problema a ser estudiado es el comportamiento cuando $k \to \infty$ de

$$\begin{cases} \mathcal{L}_{a_k} u_k = f & \text{en } \Omega \\ u_k = 0 & \text{en } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{0.0.4}$$

donde $\Omega \subset \mathbb{R}^n$ es un dominio acotado, $f \in L^2(\Omega)$, y $\{a_k\}_{k\in\mathbb{N}}$ denota una sucesión en $\mathcal{A}_{\lambda,\Lambda}$.

Los funcionales de energía asociados están dados por

$$J_{a_k}(v) := \begin{cases} \frac{1}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} a_k(x,y) \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dxdy & \text{si } v \in H_0^s(\Omega) \\ +\infty & \text{en otro caso.} \end{cases}$$

Asumamos que $a_k \overset{*}{\rightharpoonup} a_0$ en $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Probamos que $J_{a_k} \overset{\Gamma}{\to} J_{a_0}$ en $L^2(\Omega)$. Como un corolario inmediato, obtenemos que $u_k \rightharpoonup u_0$ en $H_0^s(\Omega)$, donde u_k es la solución de (0.0.4) y u_0 es la solución de

$$\begin{cases} \mathcal{L}_{a_0} u_0 = f & \text{en } \Omega \\ u_0 = 0 & \text{en } \mathbb{R}^n \setminus \Omega. \end{cases}$$

El núcleo homogeneizado $a_0(x,y)$ hereda la positividad y el hecho de estar acotado de la sucesión $\{a_k(x,y)\}_{k\in\mathbb{N}}$.

Para alcanzar la H-convergencia, no es suficiente la convergencia de soluciones dada como consecuencia de la Γ-convergencia de los funcionales de energía. Falta llegar a la convergencia de los flujos relacionados con la ecuación. Por lo tanto, queremos aplicar el método de Tartar. Con este objetivo, primero probamos una versión no local del div-curl Lema que nos permite lidiar con la sucesión de problemas y encontrar la convergencia de los flujos.

Diseño óptimo

En su forma más general, un problema de optimización de forma puede expresarse como sigue: dado un *funcional de costo* F, y una clase de *dominios admisibles* $\mathcal{A}$, queremos resolver el problema de minimización

$$\min_{A \in \mathcal{A}} F(A). \tag{0.0.5}$$

Este tipo de problemas han sido extensamente estudiados, se originan en distintos campos y aplicaciones, como ya fue descripto anteriormente. La literatura matemática es muy amplia, desde los casos clásicos de problemas isoperimétricos hasta las más recientes aplicaciones incluyendo optimización espectral y elasticidad. Solo para mencionar algunas referencias, sugerimos al lector los libros de Allaire [4], Bucur-Buttazzo [21], Henrot [55], Pironneau [73] y Sokołowski-Zolésio [90], donde una gran cantidad de problemas de optimización de forma son abordados.

Usualmente, el funcional de costo F está dado en términos de una función u_A que es solución de una *ecuación de estado* a ser resuelta en A. Típicamente, esta ecuación de estado es una ecuación diferencial elíptica en derivadas parciales.

Hay solo unos pocos resultados sobre problemas de diseño óptimo de la forma (0.0.5) donde la ecuación de estado involucra un operador no local en lugar de una ecuación diferencial elíptica en derivadas parciales.

Por ejemplo, en [89], los autores extienden la conocida desigualdad de Faber-Krahn al caso fraccionario y como un simple corolario, solucionan el problema (0.0.5) en el caso $F(A) = \lambda_1^s(A)$ donde $\lambda_1^s(A)$ es el primer autovalor del laplaciano fraccionario con condiciones de Dirichlet en $\mathbb{R}^n \setminus A$ y $\mathcal{A}$ es la clase de abiertos de medida (de Lebesgue) fija.

En [18] los autores consideran otra vez la clase $\mathcal{A}$ de abiertos de medida (de Lebesgue) fija y $F(A) = \lambda_2^s(A)$ el segundo autovalor del laplaciano fraccionario con condiciones de Dirichlet en $\mathbb{R}^n \setminus A$. Prueban que el problema (0.0.5) NO tiene solución. De hecho, una sucesión minimizante consiste en bolas de la misma medida donde la distancia de sus centros diverge.

Finalmente, en [48], los autores toman la clase $\mathcal{A}$ de conjuntos medibles de medida fija contenidos en un conjunto abierto fijo Ω y el funcional de costo $F(A) = \lambda_1^s(\Omega \setminus A)$ donde en este caso, $\lambda_1^s(\Omega \setminus A)$ es el primer autovalor del laplaciano fraccionario con condiciones de Dirichlet en A y condiciones de Neumann en $\mathbb{R}^n \setminus \Omega$.

Para otros problemas de optimización de forma donde la ecuación de estado es no local, ver [23, 34, 62, 63, 76], y las referencias allí dentro.

La contribución de esta tesis es probar la existencia de solución para ciertos problemas de diseño óptimo donde la ecuación de estado involucrada está dada en términos de un operador no local particular, que es el laplaciano fraccionario.

Bajo ciertas hipótesis naturales sobre los funcionales de costo, que son similares a aquellas consideradas en [24] y [22] donde se estudió el marco clásico, somos capaces de recuperar los resultados de existencia en el contexto no local. *Rigurosamente* hablando, esas hipótesis son:

- monotonía con respecto a la inclusión de conjuntos y
- semi-continuidad inferior con respecto a una noción de convergencia de dominios adecuada.

Observemos que los resultados de [18] ponen una restricción sobre la clase de dominios admisibles que hay que tener en cuenta si se quiere obtener un resultado positivo. Esto se debe principalmente al hecho de que tomar un dominio con dos componentes conexas y hacer que dichas componentes estén cada vez más lejos hace decrecer la energía no local. Entonces, en el espíritu de [24] nos restringimos a la clase $\mathcal{A}$ de *abiertos* de medida fija contenidos en una *caja* $\Omega \subset \mathbb{R}^n$.

Para funcionales de costo adecuados, probamos existencia de solución de

$$\min\{F_s(A)\colon A \in \mathcal{A}_s(\Omega),\ |A| = c\}, \qquad \text{para } 0 < c < |\Omega| \text{ fija},$$

y también,

$$\min\{F_s(A_1, \dots, A_m)\colon A_i \in \mathcal{A}_s(\Omega),\ A_i \cap A_j = \emptyset \text{ para } i \neq j\},\ \text{ para } m \in \mathbb{N} \text{ fija},$$

donde $\mathcal{A}_s(\Omega)$ es la clase de dominios admisibles.

Más aún, investigamos la conexión entre el marco no local y el clásico, esto es, analizamos el comportamiento cuando el parámetro fraccionario $0 < s < 1$ tiende a 1, probando convergencia de mínimos y de las formas óptimas.

Esquema de la tesis

El Capítulo 1 contiene algunas herramientas preliminares usadas a lo largo de esta tesis. Casi siempre, los resultados no están citados de la manera más general, pero sí en la forma apropiada para nuestros objetivos; aún así algunos de ellos son ligeramente más generales de lo que estrictamente necesitamos. La mayoría son bien conocidos, sin embargo los incluimos aquí por el bien de la completud. A veces, no entraremos en detalle refiriendo al lector a la correspondiente literatura.

El Capítulo 2 abarca los resultados de homogeneización. Obtenemos un resultado de compacidad para la H-convergencia de una familia de operadores no locales de problemas tipo-elípticos por medio de el método de Tartar de funciones test oscilantes.

El Capítulo 3 engloba los resultados de existencia de algunos problemas de optimización de forma. Más aún, analizamos la transición desde las ecuaciones de estado no locales a la local.

Publicaciones incluidas

Los resultados presentados en los Capítulos 2 y 3 han aparecido publicados como artículos científicos. Estos resultados son entendidos como contribuciones individuales unidos como un tema común y todos ellos están publicados o aceptados para publicación en revistas recomendadas. Los capítulos contienen los siguientes artículos:

- *H-convergence result for nonlocal elliptic-type problems via Tartar's method*, Society for Industrial and Applied Mathematics (SIAM) Journal on Mathematical Analysis, **49** (2017), no. 4, 2387-2408. MR 3668594. Julián Fernández Bonder, A. Ritorto y Ariel Martín Salort.

- *A class of shape optimization problems for some nonlocal operators*, to appear in Advances in Calculus of Variations. Julián Fernández Bonder, A. Ritorto y Ariel Martín Salort, `arXiv:1612.08717`.

- *Optimal partition problems for the fractional Laplacian*, to appear in Annali di Matematica Pura ed Applicata, `arXiv:1703.05642`.

Introduction

To begin with this thesis, we would like to give the reader an intuitive idea of the three concepts appearing in its title:

- homogenization,
- optimal design,
- nonlocal diffusion.

Let us start with the first one: **homogenization**.

Covering children's floor bedroom with carpet is common to protect them when they play and occasionally fall down. When we see the carpet in a macro-scale, we may say that it seems to be a unique homogeneous thing. But, if we get closer enough, we can distinguish the spaces between different kind of plush. So that, it looks like a heterogeneous thing in a micro-scale.

As well as seeing a wall made by porous rocks. The wall might look like a homogeneous texture when we see it as a global impression, in a macro-scale. By contrast, it seems to be really heterogeneous in the micro-scale.

BUT, in both cases: carpet and porous rocks, we could say that those heterogeneities are too small compared to the entire dimension of each object.

One possible question is: *Can we gather information of macroscopic properties BUT also taking into account microscopic ones?* That is **homogenization** goal.

Let us carry on the second concept: **optimal design**. Think about a company that sells metal sheets as electricity conductors. We may assume that to make a good product the company should use at least two materials:

- the best conductive material, but also the most expensive,
- the cheapest material, but also the worst as far as quality is concerned.

Can we find out the optimal design (shape, way to combine both materials) to make a reasonable good product?

By reading between lines, we can say an **optimal design** problem is essentially finding out a shape that minimize some cost functional.

There is only one more concept left to discuss: **nonlocal diffusion**.

Cheetahs usually chase their prey at only about half their highest speed. After a chase, a cheetah needs half an hour to catch its breath before it can eat. We may assume that having an effective technique for hunting is really important to survive.

Cheetahs eat small to medium size animals, for instance, gazelles. The cheetah's excellent eyesight helps it find prey during the day. BUT, gazelles will not wait to be killed by a cheetah once they have seen it. So that it will be more convenient for the cheetah to pick up a random direction, move rapidly over there and knock its prey to the ground and then bite its throat.

This kind of hit-and-run hunting procedure is related to the concept of **nonlocal diffusion**. Let us say that it is not unreasonable that predators use a nonlocal diffusion strategy to hunt their prey more effectively.

A nice glance at nonlocal diffusion

Throughout the thesis, we deal with a family of nonlocal operators. We say that $\mathcal{L}$ is a nonlocal operator if we must know what happens in the entire region when we just want to know its value in a fixed point $x \in \mathbb{R}^n$. It does not matter how much far a point $y \in \mathbb{R}^n$ is from the fixed point x. So, it is not enough to know how a function behaves in a neighborhood, we have to know how it behaves in the entire region. Just to illustrate, think about global economy. It does not matter how far or close we are from China or United States, any economical decision they come to, we will be affected, our economy will suffer the consequences of their decisions.

A classical local operator is the Laplacian $-\Delta$. For any u smooth function,

$$-\Delta u(x) = -\operatorname{div}(\nabla u(x)) = -\left(\partial^2_{x_1^2} u(x) + \cdots + \partial^2_{x_n^2} u(x)\right).$$

Observe that just some of the second order derivatives of u are needed to compute the value $-\Delta u(x)$, so if we only have information in a neighborhood of a fixed point x, it will be enough to arrive at the value $-\Delta u(x)$.

In contrast, look at the fractional Laplacian operator $(-\Delta)^s$,

$$(-\Delta)^s u(x) = \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy.$$

Later on, we give more details about it, but now, just only understand the parameter $s \in (0,1)$ as a fractional exponent and let u be a suitable function. The constant $c(n,s)$ plays a key role when we analyze the asymptotic behavior $s \uparrow 1$ of some problems. BUT, from now, it is only a *normalization constant*. Think for instance, the case $s = \frac{1}{2}$ will be like taking the square root of the classical Laplacian operator.

Notice that to compute $(-\Delta)^s u(x)$ we need to know the value of $u(z)$ for every z, it does not matter how much close or far z and x are.

Now, we would like to give a probabilistic motivation for the fractional Laplacian. It has to do with random walks allowing long jumps. We think it is the nicest and most friendly way to meet this nonlocal operator for the first time. Also the classical Laplacian operator has a similar probabilistic interpretation, but we focus our attention on the fractional case, since it is our key object of study along this thesis.

A probabilistic motivation for the fractional Laplacian

A random walk that allows arbitrarily long jumps. Let us begin by describing a probabilistic process in which a particle moves randomly in the space, subject to a probability allowing long jumps; originating naturally the fractional Laplacian operator.

There are two variables to be taken into account: $t > 0$ for **time** and $x \in \mathbb{R}^n$ for **space** position. We refer to the probability of finding the particle at point x at time t as $u(x,t)$. We begin by describing the process with both variables being discrete. At the end, by taking the limit when time and space steps are small, we get to the nonlocal heat equation. Let us fix the measure of the steps: $\tau > 0$ for time step and $h > 0$ for space step.

Suppose the particle starts at time t in the position x. To start moving, the particle should choose randomly one direction, say $v \in \partial B_1$, and a number of steps, say $k \in \mathbb{N}$. Since the measure of each step is h, the new position at time $t+\tau$ can be described as $x + khv$.

To talk about $u(x, t+\tau)$, the probability of finding the particle at position x at time $t+\tau$, it is enough to decide the probability of choosing a direction $v \in \partial B_1$ and a number $k \in \mathbb{N}$. Consider the uniform distribution on ∂B_1 and for each $k \in \mathbb{N}$, denote by $a(k)$ the probability of choosing it. Then, for any $I \subset \mathbb{N}$, we can define

$$P(I) := \sum_{k \in I} a(k).$$

Long jumps are allowed but with less probability than short ones. Therefore, we take $a(k)$ with a polynomial decay, say $a(k) \simeq \frac{1}{k^\alpha}$, where $\alpha > 0$.

We want P to be a probability in $\mathbb{N}$, too. Then, we need $P(\mathbb{N}) = 1$. Since $a(k) \simeq \frac{1}{k^\alpha}$, it is enough to select $\alpha > 1$ for the series convergence. That means, $\alpha = 1 + \beta$, with $\beta > 0$. It can be written as $\alpha = 1 + 2s$ with $s \in (0,1)$, for instance.

Eventually, for $I \subset \mathbb{N}$ define the probability P as

$$P(I) := c \sum_{k \in I} \frac{1}{k^{1+2s}},$$

where c is chosen in such a way that $P(\mathbb{N}) = 1$.

Now, the probability $u(x, t+\tau)$ of finding the particle at position x at time $t+\tau$ is the sum of probabilities of finding the particle somewhere else, say at $x + khv$, for some direction $v \in \partial B_1$ and some natural number $k \in \mathbb{N}$, times the probability of having selected such a direction and such a natural number. That means,

$$u(x, t+\tau) = \frac{c}{|\partial B_1|} \sum_{k \in \mathbb{N}} \int_{\partial B_1} \frac{u(x+khv, t)}{|k|^{1+2s}} \, dS_v.$$

By noticing that $\frac{c}{|\partial B_1|}$ is a normalizing probability constant, hence we subtract $u(x,t)$ and obtain

$$\begin{aligned} u(x,t+\tau)-u(x,t) &= \frac{c}{|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)}{|k|^{1+2s}}\,dS_v - u(x,t) \\ &= \frac{c}{|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)-u(x,t)}{|k|^{1+2s}}\,dS_v. \end{aligned}$$

By symmetry, we can change v by $-v$ in the integral above, so that we get

$$u(x,t+\tau)-u(x,t) = \frac{c}{|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x-khv,t)-u(x,t)}{|k|^{1+2s}}\,dS_v.$$

Then, we can sum up theses two expressions, arrive at

$$u(x,t+\tau)-u(x,t) = \frac{c}{2|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{|k|^{1+2s}}\,dS_v.$$

Now, dividing by τ, we recognize a Riemann sum.

$$\begin{aligned} \partial_t u(x,t) &\simeq \frac{u(x,t+\tau)-u(x,t)}{\tau} \\ &= \frac{c}{2|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{\tau|k|^{1+2s}}\,dS_v \end{aligned}$$

Now, multiply and divide by h^{1+2s}, then take a formal limit to obtain

$$\begin{aligned} \partial_t u(x,t) &\simeq \frac{h^{1+2s}}{\tau}\frac{c}{2|\partial B_1|}\sum_{k\in\mathbb{N}}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{\tau|hk|^{1+2s}}\,dS_v \\ &\simeq \frac{h^{2s}}{\tau}\frac{c}{2|\partial B_1|}\int_0^{\infty}\int_{\partial B_1}\frac{u(x+rv,t)+u(x-rv,t)-2u(x,t)}{|r|^{1+2s}}\,dS_v dr \end{aligned}$$

It is inferred that the *suitable* relation between the step measures h and τ is $\frac{h^{2s}}{\tau}=\nu$, ν can be called nonlocal diffusion constant. Then, use polar coordinates

$$\begin{aligned} \partial_t u(x,t) &\simeq \frac{c}{2|\partial B_1|}\int_{\mathbb{R}^n}\frac{u(x+y,t)+u(x-y,t)-2u(x,t)}{|y|^{n+2s}}\,dy \\ &= -c_{n,s}(-\Delta)^s u(x,t), \end{aligned}$$

where $c_{n,s}$ is a positive constant. At least formally, for small time and space steps, the above probabilistic process approaches a fractional heat equation

$$\partial_t u(x,t) + (-\Delta)^s u(x,t) = 0,$$

up to constants.

We have shown how this nonlocal phenomenon (random walk with arbitrarily long jumps) is transformed into a nonlocal operator (involving the fractional Laplacian $(-\Delta)^s$).

We may assume that for time large enough, the problem becomes stationary, so that it does not depend on time anymore. Also, we suppose there is a source affecting the quantity of particles, let us say a function f. Hence, we arrive at this kind of equation

$$(-\Delta)^s u = f.$$

In the probabilistic theory, the fractional Laplacian operator is a well-known example which can be seen as an infinitesimal generator of Lévy processes, in further generality. See for instance, [2, 6, 11, 66].

There are a lot of applications related to this type of nonlocal problems. To mention some references: in Physics [42, 43, 50, 64, 68, 97], Finance [3, 65, 84], Fluid dynamics [30, 34], Ecology [58, 67, 77], Image processing [51].

Throughout all this thesis, we work with some Dirichlet problems:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{0.0.6}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and $\mathcal{L}$ belongs to some class of nonlocal operators where the fractional Laplacian is the main example. This kind of integral-differential problems arise naturally in the study of stochastic processes with jumps, as we motivate previously. Those problems have been widely studied both in Probability and in Analysis, Partial Differential Equations.

We prove existence of solutions to the Dirichlet problems treated in this thesis through calculus of variation, by establishing a equivalence between being a weak solution to (0.0.6) and minimizing the associated energy . Then, we see the existence of a minimizer. Stability of solutions and comparison results are also shown.

The interested reader could take a look at Barles-Imbert [8], Felsinger-Kassmann-Voigt [46], Hoh-Jacob [57], Xiang-Pucci-Squassina-Zhang [96], for existence of solution to problems involving more general operators. Interior regularity of solutions was considered by Bass-Levin [9], Caffarelli-Silvestre [25], Iannizzotto-Mosconi-Squassina [59, 60], Kassmann [61], for instance. Works on boundary regularity of solutions: Bogdan [13], Grubb [53, 54], Ros-Oton–Serra [80, 79, 81]. For other qualitive properties of solutions, see Birkner-López-Wakolbinger [12], Dipierro-Savin-Valdinoci [41], for instance.

Homogeneization

Homogenization theory dates back to the works of S. Spagnolo [91], E. De Giorgi and S. Spagnolo [35], I. Babuška [7], A. Bensoussan, J.L. Lions and G. Papanicolaou [10] and E. Sánchez-Palencia [82] among others.

In the context of linear elliptic partial differential equations, the model to be studied is the limit as $k \to \infty$ of the following problems

$$\begin{cases} -\operatorname{div}(A_k \nabla u_k) = f & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.7}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $f \in H^{-1}(\Omega)$ and $\{A_k\}_{k\in\mathbb{N}} \subset [L^\infty(\Omega)]^{n\times n}$ is a sequence of symmetric and uniformly coercive matrices.

As a model example, it has been considered the case where the matrices A_k are given in terms of a single matrix A in the form

$$A_k(x) = A(kx),$$

where A is periodic, of period 1, in each variable.

In the periodic setting, the limit problem when $k \to \infty$ can easily be fully characterized. See [10].

In order to deal with the general case, Spagnolo and De Giorgi introduced the concept of G-convergence, that was later generalized by Murat and Tartar in the late 70s and is now called H-convergence. See [29].

When F. Murat in 1974 was studying the behavior of (0.0.7) as $k \to \infty$, one of the main drawbacks he found was the fact that the product of two weakly convergent sequences do not converge, in general, to the product of their limits. Murat overcame this difficulty by developing a compensated compactness argument known as the *div-curl Lemma*, denomination suggested by his advisor, J.L. Lions, due to the fact that it results from a compensation effect. The Lemma was published in 1978 [69] and an alternative proof was provided by L. Tartar also in 1978 [93] by using Hörmander's compactness argument for the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$. The lemma claims that if we consider two sequences $\{\psi_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ in $[L^2(\Omega)]^n$ such that

$$\psi_k \rightharpoonup \psi, \quad \text{and} \quad \phi_k \rightharpoonup \phi \quad \text{weakly in } [L^2(\Omega)]^n,$$

with the additional assumption that

$$\operatorname{div} \psi_k \to \operatorname{div} \psi \text{ in } H^{-1}(\Omega), \quad \text{and} \quad \operatorname{curl} \phi_k \to \operatorname{curl} \phi \text{ in } [H^{-1}(\Omega)]^{n\times n},$$

then we can guarantee that $\psi_k \cdot \phi_k \to \psi \cdot \phi$ in the sense of distributions. Recall that the curl of a vector field $\phi \in [L^2(\Omega)]^n$ is defined as

$$\operatorname{curl} \phi = \left(\frac{\partial \phi^i}{\partial x^j} - \frac{\partial \phi^j}{\partial x^i} \right)_{1\le i,j\le n}.$$

The div-curl Lemma plays a crucial role in homogenization theory. In fact, based on this lemma, Tartar introduced in [93, 94] a method leading to the limiting behavior of (0.0.7) as $k \to \infty$, obtaining the existence of a coercive matrix $A_0 \in [L^\infty(\Omega)]^{n\times n}$ such that the sequence of solutions $\{u_k\}_{k\in\mathbb{N}}$ of (0.0.7) converges weakly in $H^1_0(\Omega)$, up to some subsequence, to a function u_0 which is the solution of the following *homogenized* limit problem

$$\begin{cases} -\operatorname{div}(A_0 u_0) = f & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.8}$$

Moreover, $A_k \nabla u_k \cdot \nabla u_k \to A_0 \nabla u_0 \cdot \nabla u_0$ in the sense of distributions, see for instance [4, 29]. That is, the sequence A_k H-converges to A_0.

In the quasilinear case, this type of results were obtained by several authors in the late 80s and the beginning of the 90s. We refer the interested reader to [28, 71] and to G. Dal Maso's book [31] where the authors use Γ-convergence methods in order to deal with these problems. See [17] for the periodic case. Let us mention that Γ-convergence studies the behavior of minimums in variational problems, so when specialized in quadratic functionals, this gives the behavior for symmetric elliptic problems.

We remark that in the linear case, H-convergence and Γ-convergence where recently shown to coincide even in the non symmetric case by Ansini, Dal Maso and Zeppieri [5].

More general classes of problems were addressed recently. In the case of periodic homogenization of certain Hamilton-Jacobi and fully nonlinear elliptic partial differential equations was studied first by Evans [44]. In the context of fully nonlinear uniformly elliptic equations in stationary ergodic media, the problem was studied by Caffarelli, Sounganidis and Wang [27]. In these papers the existence of homogenized equations is proved, but, due to the generality of these problems, no further information about the structure of the limit problems was obtained.

In this thesis, we address the H-convergence problem to the nonlocal version of (0.0.7) and give a characterization of the homogenized limit problem. Before getting into detail, we review the background regarding nonlocal problems and its homogenization.

The regularity theory for fully nonlinear integro-differential equations, which include the fractional laplacian as a trivial example, was recently extensively studied. See, for instance, [25, 26, 80, 88].

Based in these regularity results for fully nonlinear integral-differential equations, R. Schwab in [85, 86] extended the results of Evans and Caffarelli, Souganidis and Wang to this setting, but again no information on the limit problem is obtained. We recall that the results of Schwab make extensive use either of the periodicity or the ergodicity of the problem and the author does not obtain any general convergence result.

A recent result on nonlocal homogenization in the periodic setting can be found in [72].

Now, we describe briefly our contribution in Homogenization related to nonlocal diffusion.

Let $0 < \lambda \leq \Lambda < \infty$. Consider the family of bounded symmetric kernels

$$\mathcal{A}_{\lambda,\Lambda} = \{a \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \colon a(x,y) = a(y,x),\ \lambda \leq a(x,y) \leq \Lambda \text{ a.e. }\}$$

We focus our analysis to a general family of linear anisotropic operators of the form

$$\mathcal{L}_a u(x) := \text{p.v.} \int_{\mathbb{R}^n} a(x,y) \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy, \quad s \in (0,1),$$

for a given $a(x,y) \in \mathcal{A}_{\lambda,\Lambda}$. The problem to be studied is the behavior as $k \to \infty$ of

$$\begin{cases} \mathcal{L}_{a_k} u_k = f & \text{in } \Omega \\ u_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{0.0.9}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $f \in L^2(\Omega)$ and $\{a_k\}_{k\in\mathbb{N}}$ denotes a sequence in $\mathcal{A}_{\lambda,\Lambda}$.

The associated energy functionals are given by

$$J_{a_k}(v) := \begin{cases} \frac{1}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} a_k(x,y) \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dxdy & \text{if } v \in H_0^s(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Assume $a_k \overset{*}{\rightharpoonup} a_0$ in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We prove that $J_{a_k} \overset{\Gamma}{\to} J_{a_0}$ in $L^2(\Omega)$. As an immediately corollary, we obtain $u_k \rightharpoonup u_0$ in $H_0^s(\Omega)$, where u_k is the solution to (0.0.9) and u_0 is the solution to

$$\begin{cases} \mathcal{L}_{a_0} u_0 = f & \text{in } \Omega \\ u_0 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The homogenized kernel $a_0(x,y)$ inherits the positivity and boundedness of the sequence $a_k(x,y)$.

To achieve the H-convergence, it is not enough the convergence of solutions given by a consequence of the Γ-convergence of the energy functionals. It is remained to arrive at the convergence of flows related to the equation. Therefore, we want to apply Tartar's method. To this aim, we first prove a nonlocal version of the div-curl Lemma that allows us to deal with the sequence of problems and find out the convergence of flows.

Optimal design

In its most general form, a shape optimization problem can be stated as follows: Given a *cost functional* F, and a class of *admissible domains* $\mathcal{A}$, we want to solve the minimization problem

$$\min_{A\in\mathcal{A}} F(A). \tag{0.0.10}$$

These types of problems have been extensively considered, and they arise in many fields and in many applications, as it has been described earlier. The mathematical literature is very wide, from the classical cases of isoperimetrical problems to the most recent applications including elasticity and spectral optimization. Only to mention some references, we refer the reader to the books of Allaire [4], Bucur and Buttazzo [21], Henrot [55], Pironneau [73] and Sokołowski and Zolésio [90], where a huge amount of shape optimization problems are tackled.

Usually, the cost functional F is given in terms of a function u_A which is the solution of a *state equation* to be solved on A. Typically, this state equation is an elliptic partial differential equation.

However, there are only a handful of results of shape optimization problems of the form (0.0.10) where the state equation involves a nonlocal operator instead of an elliptic partial differential equation.

For instance, in [89], the authors extend the well-known Faber-Krahn inequality to the fractional case and as a simple corollary, they solve problem (0.0.10) in the case where $F(A) = \lambda_1^s(A)$ where $\lambda_1^s(A)$ is the first eigenvalue of the Dirichlet fractional laplacian and the class $\mathcal{A}$ is the class of open sets of fixed measure.

In [18] the authors consider again the class $\mathcal{A}$ of open sets of fixed measure and $F(A) = \lambda_2^s(A)$ and prove that problem (0.0.10) does not have a solution. In fact, a minimization sequence of domains consists of a sequence of balls of the same measure where the distance of the centers diverges.

Finally, in [48], the authors take the class $\mathcal{A}$ of measurable sets of fixed measure contained in a fixed open set Ω and the cost functional $F(A) = \lambda_1^s(\Omega \setminus A)$ where in this case, $\lambda_1^s(\Omega \setminus A)$ is the first eigenvalue of the fractional laplacian with Dirichlet condition on A and Neumann condition in $\mathbb{R}^n \setminus \Omega$.

For other recent shape optimization problems where the state equation is nonlocal, see [23, 34, 62, 63, 76], and references therein.

The contribution in this thesis is existence of solutions to some shape optimization problems where the involved state equation is given in terms of a particular nonlocal operator, which is the fractional Laplacian.

Under some natural assumptions on the cost functional, which are similar to those considered in [24] and [22] where the classical setting was studied, we are able to recover existence results in the nonlocal setting. Roughly speaking, these assumptions are:

- monotonicity with respect to the inclusion and
- lower semi-continuity with respect to a suitable defined notion of convergence of domains.

Observe that the results of [18] put a restriction on the classes of admissible domains that one needs to consider if you want to obtain a positive result. This is mainly due to the fact that taking a domain with two connected components and making these components go far away from each other makes the nonlocal energy decrease. So, in the spirit of [24] we restrict ourselves to the class $\mathcal{A}$ of open sets of fixed measure that are contained in a fixed box $Q \subset \mathbb{R}^n$.

For suitable cost functionals, we prove existence of solution to

$$\min\{F_s(A)\colon A \in \mathcal{A}_s(\Omega),\ |A| = c\}, \qquad \text{for fixed } 0 < c < |\Omega|,$$

and also for the partition problem

$$\min\{F_s(A_1, \ldots, A_m)\colon A_i \in \mathcal{A}_s(\Omega),\ A_i \cap A_j = \emptyset \text{ for } i \neq j\}, \text{ for fixed } m \in \mathbb{N},$$

where $\mathcal{A}_s(\Omega)$ is the class of admissible domains.

Furthermore, we also investigate the connection between the nonlocal setting and the classical one, that is, we analyze the behavior when the fractional parameter $0 < s < 1$ goes to 1, proving convergence of the minimums and of the optimal shapes.

Thesis outline

Chapter 1 contains some preliminary tools used throughout this thesis. Almost always, the results are not quoted in the most general form, but in a way that is appropriate to our purposes; nevertheless some of them are actually slightly more general than we strictly need. Most of these results are well known, but we include them here for the sake of completeness. Sometimes, we will not go into details referring the reader to the corresponding literature.

Chapter 2 encompasses the homogenization results. We obtain a compactness result for the H-convergence of a family of nonlocal and linear monotone elliptic-type problems by means of Tartar's method of oscillating test functions.

Chapter 3 addresses the existence results for some shape optimization problems. Moreover, we also analyze the transition from nonlocal to local state equations.

Included publications

The results presented in Chapters 2 and 3 have appeared published as research articles. These results are readable as individuals contributions linked by a common theme and all of them are either published or accepted for publication for publication in refereed journals. The chapters contain the following papers:

- *H-convergence result for nonlocal elliptic-type problems via Tartar's method*, Society for Industrial and Applied Mathematics (SIAM) Journal on Mathematical Analysis, **49** (2017), no. 4, 2387-2408. MR 3668594. Julián Fernández Bonder, A. Ritorto y Ariel Martín Salort.

- *A class of shape optimization problems for some nonlocal operators*, to appear in Advances in Calculus of Variations. Julián Fernández Bonder, A. Ritorto y Ariel Martín Salort, `arXiv:1612.08717`.

- *Optimal partition problems for the fractional Laplacian*, to appear in Annali di Matematica Pura ed Applicata, `arXiv:1703.05642`.

Chapter 1

Preliminaries

To begin with this thesis, we gather some well-known results which are needed for Chapter 2 and 3, where the original contributions are treated. We suggest those who are experts on *fractional setting* to keep on going to next chapter. If any confusion appear, for instance, any not usual notation, the reader can go back to this chapter.

We separate the content in five sections. The first one is dedicated to introduce the spaces we work with and some useful properties. Also, to fix related notations. Some references we have considered are [14, 40, 52, 75]. Secondly, we establish the involved operator, the related Dirichlet problem, existence of solution, among other properties. The third section states some outcomes of fractional capacities, which can be found in their most generally form in [87, 95]. To deal with one of the major goals of this thesis, that means attaining the H-convergence for a sequence of certain linear operators, we need a compactness result for such class. It is the content of fourth section, where we recall some lemmas from [4]. We end this chapter with basic notions about Γ-convergence, taking [31] as the principal reference.

1.1 Spaces we work with and some properties

As we have mentioned before, all the results in this part are well-known and extensively studied. To say some references we have used, see, for instance, [14, 15, 38, 40, 52, 75].

1.1.1 Fractional Sobolev spaces and their dual spaces

Given $0 < s < 1 \leq p < \infty$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined as

$$W^{s,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) \colon \frac{u(x) - u(y)}{|x-y|^{\frac{n}{p}+s}} \in L^p(\mathbb{R}^n \times \mathbb{R}^n) \right\}.$$

The norm in this space is then naturally defined as

$$\|u\|_{s,p} = (\|u\|_p^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

where $\|\cdot\|_p$ is, as usual, the L^p−norm in $\mathbb{R}^n$ and

$$[u]_{s,p} := \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy \right)^{\frac{1}{p}}$$

is the so-called *Gagliardo seminorm.*

The space $W^{s,p}(\mathbb{R}^n)$ with the norm $\|\cdot\|_{s,p}$ is a uniformly convex Banach space. For a uniformly convex normed vector space we understand that for every $0<\varepsilon\le 2$, there exists $\delta>0$ such that for any $u,v\in W^{s,p}(\mathbb{R}^n)$ satisfying $\|u\|_{s,p}=1=\|v\|_{s,p}$, the condition

$$\varepsilon \le \|u-v\|_{s,p} \quad \text{implies that} \quad \|\frac{u+v}{2}\|_{s,p} \le 1-\delta.$$

Consequently, $W^{s,p}(\mathbb{R}^n)$ is a reflexive Banach space.

Moreover, $W^{s,p}(\mathbb{R}^n)$ is a separable Banach space. Indeed, we can include $W^{s,p}(\mathbb{R}^n)$ in a suitable separable space through an isometric:

$$u\in W^{s,p}(\mathbb{R}^n) \mapsto \left(u(x); \frac{u(x)-u(y)}{|x-y|^{\frac{n}{p}+s}} \right) \in L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n\times\mathbb{R}^n).$$

In the case $p=2$, we use the following notations: $W^{s,2}(\mathbb{R}^n)=H^s(\mathbb{R}^n)$, $[\,\cdot\,]_{s,2}=[\,\cdot\,]_s$ and $\|\cdot\|_{s,2}=\|\cdot\|_s$. We get that $(H^s(\mathbb{R}^n),\|\cdot\|_s)$ is a Hilbert space.

It is also easy to see that smooth functions with compact support are contained in $W^{s,p}(\mathbb{R}^n)$. Also, smooth and rapidly decreasing functions belong to $W^{s,p}(\mathbb{R}^n)$. Moreover, $C_c^\infty(\mathbb{R}^n)$ is a dense set in $W^{s,p}(\mathbb{R}^n)$, see a proof in [38, Proposition 4.27].

We want to solve some equations in different subsets of $\mathbb{R}^n$, so we need to introduce some spaces such that the *boundary conditions* of these equations are taken into account.

Given an open set $\Omega\subset\mathbb{R}^n$, we consider

$$W_0^{s,p}(\Omega) := \overline{C_c^\infty(\Omega)} \subset W^{s,p}(\mathbb{R}^n), \tag{1.1.1}$$

where the closure is taken with respect to the $\|\cdot\|_{s,p}$-norm.

In the particular case that the set Ω is Lipschitz, we can characterize $W_0^{s,p}(\Omega)$ as the space of functions in $W^{s,p}(\mathbb{R}^n)$, vanishing outside of Ω.

Theorem 1.1.1 (Corollary 1.4.4.5, [52])**.** *Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz bounded open set. Then, we get the identity $W_0^{s,p}(\Omega)=\{u\in W^{s,p}(\mathbb{R}^n)\colon u=0$ a.e. in $\mathbb{R}^n\setminus\Omega\}$.*

Moreover, if $sp<1$, $W_0^{s,p}(\Omega)=\{u|_\Omega\colon u\in W^{s,p}(\mathbb{R}^n)\}$.

The dual space of $W^{s,p}(\mathbb{R}^n)$ will be denoted by $W^{-s,p'}(\mathbb{R}^n)$. Also, the dual space of $W_0^{s,p}(\Omega)$ will be denoted by $W^{-s,p'}(\Omega)$ as usual. Recall that in these spaces the norm is defined as

$$\|f\|_{-s,p'} := \sup\{\langle f,u\rangle\colon u\in W^{s,p}(\mathbb{R}^n),\ \|u\|_{s,p}=1\}$$

and

$$\|f\|_{-s,p',\Omega} := \sup\{\langle f,u\rangle\colon u\in W_0^{s,p}(\Omega),\ [u]_{s,p}=1\}.$$

Observe that $W^{-s,p'}(\mathbb{R}^n) \subset W^{-s,p'}(\Omega)$ with continuous inclusion.

Notice that that since $\mathcal{D}(\Omega) := C_c^\infty(\Omega) \subset W_0^{s,p}(\Omega)$, the dual space $W^{-s,p'}(\Omega)$ is contained in the space of distributions $\mathcal{D}'(\Omega)$.

In the case $p = 2$, we use the notations $W^{-s,2}(\mathbb{R}^n) = H^{-s}(\mathbb{R}^n)$, $W^{-s,2}(\Omega) = H^{-s}(\Omega)$, $\|\cdot\|_{-s,2} = \|\cdot\|_{-s}$ and $\|\cdot\|_{-s,2,\Omega} = \|\cdot\|_{-s,\Omega}$.

1.1.2 Relation between $[\,\cdot\,]_{s,p}$ and $\|\nabla\cdot\|_p$ for a fixed function

We are interested in connecting in some way the fractional semi-norms $[\,\cdot\,]_{s,p}$ with $\|\nabla\cdot\|_p$, the usual norm in $W_0^{1,p}(\Omega)$.

In this part, ideas from Bourgain-Brezis-Mironescu [14] and Ponce [75] were used.

Let us start with a preliminary lemma.

Lemma 1.1.2. *Let $u \in W^{1,p}(\mathbb{R}^n)$, $1 \le p < \infty$. Then,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy \le \frac{\omega_{n-1}}{p}\left(\frac{1}{1-s}\|\nabla u\|_p^p + \frac{2^p}{s}\|u\|_p^p\right),$$

for every $0 < s < 1$, where ω_{n-1} is the $(n-1)$-dimensional measure of the unit sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$.

Proof. To begin with, observe that for $h \in \mathbb{R}^n$ and $u \in C_c^1(\mathbb{R}^n)$,

$$u(x+h)-u(x) = \int_0^1 \frac{d}{dt}u(x+th)\,dt = \int_0^1 \nabla u(x+th)\cdot h\,dt.$$

Then, we obtain

$$|u(x+h)-u(x)| \le \int_0^1 |\nabla u(x+th)||h|\,dt \le |h|\left(\int_0^1 |\nabla u(x+th)|^p\,dt\right)^{\frac{1}{p}},$$

from we get that

$$\int_{\mathbb{R}^n} |u(x+h)-u(x)|^p\,dx \le |h|^p \int_{\mathbb{R}^n}\int_0^1 |\nabla u(x+th)|^p\,dt\,dx = |h|^p \int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx,$$

where we have used Fubini's Theorem and the integral invariance with respect to translations.

Finally, recall the density of $C_c^1(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$ to obtain

$$\left(\int_{\mathbb{R}^n} |u(x+h)-u(x)|^p\,dx\right)^{\frac{1}{p}} \le |h|\|\nabla u\|_p, \tag{1.1.2}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$.

Now, we rewrite $[u]_{s,p}^p$ separating in two pieces, as follows:

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy =& \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x+h)-u(x)|^p}{|h|^{n+sp}}\,dxdh\\ =& \int_{B_1} \frac{1}{|h|^{n+sp}}\left(\int_{\mathbb{R}^n} |u(x+h)-u(x)|^p\,dx\right)dh\\ &+ \int_{\mathbb{R}^n\setminus B_1} \frac{1}{|h|^{n+sp}}\left(\int_{\mathbb{R}^n} |u(x+h)-u(x)|^p\,dx\right)dh\\ =& I+II.\end{aligned}$$

To estimate I, we use (1.1.2). Indeed,

$$I \le \|\nabla u\|_p^p \int_{B_1} \frac{1}{|h|^{n+sp-p}}\,dh = \|\nabla u\|_p^p \omega_{n-1} \int_0^1 \frac{1}{r^{n+sp-p}} r^{n-1}\,dr = \frac{\omega_{n-1}}{p(1-s)}\|\nabla u\|_p^p.$$

To estimate II, first observe that

$$\int_{\mathbb{R}^n} |u(x+h)-u(x)|^p\,dx \le 2^{p-1}\int_{\mathbb{R}^n} (|u(x+h)|^p + |u(x)|^p)\,dx = 2^p\|u\|_p^p,$$

from where follows

$$II \le 2^p\|u\|_p^p \int_{\mathbb{R}^n\setminus B_1} \frac{1}{|h|^{n+sp}}\,dh = 2^p\|u\|_p^p\omega_{n-1}\int_1^\infty \frac{1}{r^{n+sp}}r^{n-1}\,dr = 2^p\frac{\omega_{n-1}}{sp}\|u\|_p^p.$$

By combining both estimates we obtain the desired result. □

We introduce the following notation:

$$\rho(x) := \begin{cases} C\exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x|<1\\ 0 & \text{if } |x|\ge 1,\end{cases}$$

where $C>0$ is chosen in such a way that $\int_{\mathbb{R}^n}\rho(x)\,dx = 1$. We name ρ *regularizing standard mollifier*. Observe that ρ is a nonnegative radial function, $\rho\in C_c^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\rho) = B_1(0)$.

Let $\varepsilon>0$. From ρ, we construct the *identity aproximations*

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right).$$

These functions verify that $\rho_\varepsilon\in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}(\rho_\varepsilon) = B_\varepsilon(0)$, $\rho_\varepsilon\ge 0$, $\int_{\mathbb{R}^n}\rho_\varepsilon\,dx = 1$.

Let $u\in L^p(\mathbb{R}^n)$. We define the ***ε-regularizations*** as

$$u_\varepsilon(x) := u*\rho_\varepsilon(x) = \int_{\mathbb{R}^n} u(y)\rho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^n} u(x-y)\rho_\varepsilon(y)\,dy.$$

Then, we get that $u_\varepsilon\in L^p(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$, $u_\varepsilon\to u$ in $L^p(\mathbb{R}^n)$ and if u has compact support, then u_ε also has compact support.

Lemma 1.1.3. *Let $u \in L^p(\mathbb{R}^n)$ and $\{u_\varepsilon\}_{\varepsilon>0}$ be the $\varepsilon-$regularizations. Then,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{n+sp}}\,dxdy \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy$$

for every $\varepsilon > 0$ and $0 < s < 1$.

Proof. To begin with this elementary proof, we have that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{n+sp}}\,dxdy = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |u_\varepsilon(x+h)-u_\varepsilon(x)|^p\,dx\right)\frac{dh}{|h|^{n+sp}}. \tag{1.1.3}$$

Now, use Jensen's inequality to arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} |u_\varepsilon(x+h)-u_\varepsilon(x)|^p\,dx &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}(u(x+h-y)-u(x-y))\rho_\varepsilon(y)\,dy\right|^p\,dx\\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(x+h-y)-u(x-y)|^p\rho_\varepsilon(y)\,dy\,dx\\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|u(x+h-y)-u(x-y)|^p\,dx\right)\rho_\varepsilon(y)\,dy\\ &= \int_{\mathbb{R}^n}|u(x+h)-u(x)|^p\,dx,\end{aligned}$$

where we have used the norm invariance with respect to translations and the fact that ρ_ε has integral equal to one.

By combining this inequality with (1.1.3) the outcome is proved. □

Finally, we also have to analyze what happens when we truncate a function to make its support compact.

To this aim, we consider $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ if $x \in B_1(0)$, $\operatorname{supp}(\eta) = B_2(0)$, $0 \leq \eta(x) \leq 1$, $x \in \mathbb{R}^n$ and we define $\eta_k(x) = \eta(\frac{x}{k})$ for each $k \in \mathbb{N}$.

The sequence $\{\eta_k\}_{k\in\mathbb{N}}$ verify that

$$\eta_k \in C_c^\infty(\mathbb{R}^n),\ 0 \leq \eta_k \leq 1,\ \eta_k = 1 \text{ en } B_k(0),\ \operatorname{supp}(\eta_k) = B_{2k}(0),\ |\nabla\eta_k| \leq \frac{\|\nabla\eta\|_\infty}{k}. \tag{1.1.4}$$

Then, given $u \in L^p(\mathbb{R}^n)$, we define the ***truncations*** of u as $u_k = \eta_k u$. We have the following Lemma.

Lemma 1.1.4. *Let $u \in L^p(\mathbb{R}^n)$ and $\{\eta_k\}_{k\in\mathbb{N}}$ be given by (1.1.4). Then, by naming $u_k = \eta_k u$, it holds that*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+sp}}\,dxdy \leq C\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy + \frac{\|u\|_p^p}{s(1-s)}\right)$$

where $C > 0$ depends only of n and p.

Proof. First, we observe that

$$\begin{aligned} |u_k(x)-u_k(y)|^p &\le 2^{p-1}(\eta_k(x)^p|u(x)-u(y)|^p+|u(y)|^p|\eta_k(x)-\eta_k(y)|^p) \\ &\le 2^{p-1}(|u(x)-u(y)|^p+|u(y)|^p|\eta_k(x)-\eta_k(y)|^p). \end{aligned} \tag{1.1.5}$$

Then, we obtain that

$$\begin{aligned} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+sp}}\,dxdy \le & 2^{p-1}\Bigg(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy \\ & + \int_{\mathbb{R}^n}|u(y)|^p\left[\int_{\mathbb{R}^n}\frac{|\eta_k(x)-\eta_k(y)|^p}{|x-y|^{n+sp}}\,dx\right]dy\Bigg). \end{aligned} \tag{1.1.6}$$

On the other hand, we notice that

$$\begin{aligned} \int_{\mathbb{R}^n}\frac{|\eta_k(x)-\eta_k(y)|^p}{|x-y|^{n+sp}}\,dx =& \int_{B_1(y)}\frac{|\eta_k(x)-\eta_k(y)|^p}{|x-y|^{n+sp}}\,dx \\ &+ \int_{\mathbb{R}^n\setminus B_1(y)}\frac{|\eta_k(x)-\eta_k(y)|^p}{|x-y|^{n+sp}}\,dx \\ =& I + II. \end{aligned}$$

Now, we estimate I as follows:

$$I \le \left(\frac{\|\nabla\eta\|_\infty}{k}\right)^p \omega_{n-1}\int_0^1 \frac{1}{r^{n+sp-p}}r^{n-1}\,dr = \left(\frac{\|\nabla\eta\|_\infty}{k}\right)^p \frac{\omega_{n-1}}{p(1-s)}$$

and II in this way:

$$II \le 2^p\|\eta_k\|_\infty^p\omega_{n-1}\int_0^1 \frac{1}{r^{n+sp}}r^{n-1}\,dr = 2^p\frac{\omega_{n-1}}{sp}.$$

By combining these estimates with (1.1.6) we arrive at the desired outcome. □

Remark 1.1.5. The estimate given by Lemma 1.1.4 can be improved. Indeed, we need next inequality instead of that used in (1.1.5), to obtain

$$(a+b)^p \le (1+\delta)a^p + C_\delta b^p$$

where $\delta > 0$ is arbitrary. Then, we get

$$\begin{aligned} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+sp}}\,dxdy \le & (1+\delta)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy \\ & + C_\delta C(n,p)\left(\frac{1}{k(1-s)}+\frac{1}{s}\right)\|u\|_p^p. \end{aligned} \tag{1.1.7}$$

From (1.1.7), we conclude that given $\delta > 0$, there exist $k_0 \in \mathbb{N}$ and $s_0 \in (0,1)$ such that for every $k \ge k_0$ and $s_0 < s < 1$, it ensues

$$\begin{aligned} (1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+sp}}\,dxdy \le & (1+\delta)(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy \\ & + \delta\|u\|_p^p. \end{aligned}$$

Let us see the last ingredient, but not less important, needed to connect the fractional semi-norms $[\,\cdot\,]_{s,p}$ with $\|\nabla\cdot\|_p$, for a fixed function u.

Lemma 1.1.6. *Let $u \in C_c^2(\mathbb{R}^n)$. Then, for every fixed $x \in \mathbb{R}^n$, it holds*

$$\lim_{s\uparrow 1}(1-s)\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy = K(n,p)|\nabla u(x)|^p,$$

where

$$K(n,p) = \frac{1}{p}\int_{\mathcal{S}^{n-1}}|z_1|^p\,dS_z. \tag{1.1.8}$$

Proof. Let $u \in C_c^2(\mathbb{R}^n)$ and name $M = \|\nabla u\|_\infty$.

There exists a constant $C = C(M) > 0$ such that

$$|a^p - b^p| \le C|a-b|, \qquad \text{for every } 0 \le a, b < M.$$

On the other hand, we have these two estimates involving M:

$$a := \frac{|u(x)-u(y)|}{|x-y|} \le M \quad \text{ and } \quad b := \left|\nabla u(x)\cdot\frac{(x-y)}{|x-y|}\right| \le M,$$

from we deduce that

$$\begin{aligned}\left|\frac{|u(x)-u(y)|^p}{|x-y|^p} - \left|\nabla u(x)\cdot\frac{(x-y)}{|x-y|}\right|^p\right| &\le C\left|\frac{|u(x)-u(y)|}{|x-y|} - \left|\nabla u(x)\cdot\frac{(x-y)}{|x-y|}\right|\right| \\ &\le C\frac{|u(x)-u(y)-\nabla u(x)\cdot(x-y)|}{|x-y|} \\ &\le C|x-y|.\end{aligned} \tag{1.1.9}$$

The fact that $u \in C_c^2(\mathbb{R}^n)$ was used in the last inequality.

Now, we analyze $[u]_{s,p}^p$ by splitting it in two pieces: inside and outside the unit ball.

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy &= \int_{B_1(x)}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy + \int_{\mathbb{R}^n\setminus B_1(x)}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy \\ &= I + II.\end{aligned}$$

To bound II, we proceed as follows

$$II \le 2^p\|u\|_\infty^p\int_{\mathbb{R}^n\setminus B_1(x)}\frac{1}{|x-y|^{n+sp}}\,dy = 2^p\|u\|_\infty^p\omega_{n-1}\int_1^\infty\frac{r^{n-1}}{r^{n+sp}}\,dr = \frac{2^p}{sp}\|u\|_\infty^p\omega_{n-1}.$$

Therefore, $(1-s)II \to 0$ when $s \uparrow 1$.

To control I, (1.1.9) allows us to conclude that

$$\begin{aligned}
&\int_{B_1(x)} \left| \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} - \left| \nabla u(x) \cdot \frac{(x-y)}{|x-y|} \right|^p \frac{1}{|x-y|^{n+sp-p}} \right| dy \\
&= \int_{B_1(x)} \left| \frac{|u(x)-u(y)|^p}{|x-y|^p} - \left| \nabla u(x) \cdot \frac{(x-y)}{|x-y|} \right|^p \right| \frac{1}{|x-y|^{n+sp-p}} \, dy \\
&\leq C \int_{B_1(x)} \frac{1}{|x-y|^{n+sp-p-1}} \, dy = C\omega_{n-1} \frac{1}{1+p(1-s)}.
\end{aligned}$$

Accordingly,

$$\lim_{s\uparrow 1} (1-s) \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dy = \lim_{s\uparrow 1} (1-s) \int_{B_1(x)} \left| \nabla u(x) \cdot \frac{(x-y)}{|x-y|} \right|^p \frac{dy}{|x-y|^{n+sp-p}}.$$

By working out in the right-hand side of the previous identity, we realize that

$$\begin{aligned}
\int_{B_1(x)} \left| \nabla u(x) \cdot \frac{(x-y)}{|x-y|} \right|^p \frac{dy}{|x-y|^{n+sp-p}} &= \int_0^1 \left(\int_{\mathcal{S}^{n-1}} |\nabla u(x) \cdot z|^p \, dS_z \right) r^{p-sp-1} \, dr \\
&= \frac{1}{p(1-s)} \int_{\mathcal{S}^{n-1}} |\nabla u(x) \cdot z|^p \, dS_z
\end{aligned}$$

Notice that the last integral is rotationally invariant. So, we consider a rotation R such that $R(\nabla u(x)) = |\nabla u(x)| e_1$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Also denote R^T its transpose. Then, we get

$$\nabla u(x) \cdot R^T z = R(\nabla u(x)) \cdot z = |\nabla u(x)| e_1 \cdot z.$$

By taking into account that, we rewrite the integral as

$$\begin{aligned}
\int_{\mathcal{S}^{n-1}} |\nabla u(x) \cdot z|^p \, dS_z &= \int_{\mathcal{S}^{n-1}} |\nabla u(x) \cdot Rz|^p \, dS_z \\
&= |\nabla u(x)|^p \int_{\mathcal{S}^{n-1}} |e_1 \cdot z|^p \, dS_z \\
&= |\nabla u(x)|^p \int_{\mathcal{S}^{n-1}} |z_1|^p \, dS_z.
\end{aligned}$$

Eventually, it is concluded that

$$\lim_{s\uparrow 1} (1-s) \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dy = K(n,p) |\nabla u(x)|^p,$$

where $K(n,p)$ is given by (1.1.8). □

We have paved the way to make easily the proof of the relation between $[\,\cdot\,]_{s,p}$ and $\|\nabla \cdot\|_p$ for a fixed function, which is the topic of next Theorem.

Theorem 1.1.7 (Theorem 2, [15])**.** *Let $0 < s < 1 < p < \infty$ and $u \in L^p(\mathbb{R}^n)$. Then,*

$$\lim_{s \uparrow 1}(1-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy = K(n,p) \int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx,$$

where $K(n,p)$ appears in (1.1.8)*.*

We consider the term $\int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx$ equal to ∞ if $u \notin W^{1,p}(\mathbb{R}^n)$.

Proof. Thanks to 1.1.6 and Lebesgue Dominated Convergence's Theorem, we only have to show the existence of a dominated integrable function.

Let $u \in C_c^2(\mathbb{R}^n)$ and suppose $\mathrm{supp}(u) \subset B_R(0)$. Define

$$F_s(x) = \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy.$$

Hence, if $|x| < 2R$, we separate $F_s(x)$ to deal with the two different troubles appeared due to the powers which change their behavior outside and inside the unit ball.

$$|F_s(x)| = \int_{B_1(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy + \int_{\mathbb{R}^n \setminus B_1(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy = I + II.$$

Repeating the same techniques we have used in previous lemmas, we arrive at

$$I \leq \frac{\omega_{n-1}}{p(1-s)} \|\nabla u\|_\infty^p \quad \text{and} \quad II \leq \frac{2^p}{sp} \|u\|_\infty^p.$$

It is remained to analyze the case $|x| \geq 2R$:

$$F_s(x) = \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|x-y|^{n+sp}}\, dy = \int_{B_R(0)} \frac{|u(y)|^p}{|x-y|^{n+sp}}\, dy.$$

But $|x-y| \geq |x| - R \geq \frac{1}{2}|x|$. Hence, if in addition we know $\frac{1}{2} < s < 1$, it follows that

$$|F_s(x)| \leq \left(\frac{2}{|x|}\right)^{n+sp} \|u\|_p^p \leq \left(\frac{2}{|x|}\right)^{n+\frac{1}{2}p} \|u\|_p^p.$$

Notice that restricting to the case $\frac{1}{2} < s < 1$ is not a real restriction. Since we are interested is taking the limit $0 < s \uparrow 1$.

So, for $\frac{1}{2} < s < 1$ it ensues

$$|(1-s)F_s(x)| \leq C \left(\chi_{B_R(0)}(x) + \frac{1}{|x|^{n+\frac{1}{2}p}} \chi_{\mathbb{R}^n \setminus B_R(0)}(x) \right) \in L^1(\mathbb{R}^n),$$

where $C > 0$ depends only on n, p and u, but it is independent of s.

Till this moment, we have proved the existence of a dominated integral function. Hence, taking into account the point-wise convergence from Lemma 1.1.6, we conclude the result for functions in $C_c^2(\mathbb{R}^n)$.

To extend the outcome to an arbitrary function $u \in W^{1,p}(\mathbb{R}^n)$, first we remind that $[\,\cdot\,]_{s,p}$ is a semi-norm, so it is nonnegative, homogeneous of one degree and it verifies the triangle inequality.

Let $u \in W^{1,p}(\mathbb{R}^n)$. We have to show that $\lim_{s\uparrow 1}(1-s)^{\frac{1}{p}}[u]_{s,p} = K(n,p)^{\frac{1}{p}}\|\nabla u\|_p$.

Let $\{u_k\}_{k\in\mathbb{N}} \subset C_c^2(\mathbb{R}^n)$ be such that $u_k \to u$ en $W^{1,p}(\mathbb{R}^n)$. Let $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $\|u - u_{k_0}\|_{1,p} < \varepsilon$. Thus,

$$\begin{aligned}\left|(1-s)^{\frac{1}{p}}[u]_{s,p} - K(n,p)^{\frac{1}{p}}\|\nabla u\|_p\right| \leq &(1-s)^{\frac{1}{p}}\left|[u]_{s,p} - [u_{k_0}]_{s,p}\right| \\ &+ \left|(1-s)^{\frac{1}{p}}[u_{k_0}]_{s,p} - K(n,p)^{\frac{1}{p}}\|\nabla u_{k_0}\|_p\right| \\ &+ K(n,p)^{\frac{1}{p}}\left|\|\nabla u_{k_0}\|_p - \|\nabla u\|_p\right| \\ =&I + II + III.\end{aligned}$$

We can control the first term I thanks to Lemma 1.1.2:

$$(1-s)^{\frac{1}{p}}\left|[u]_{s,p} - [u_{k_0}]_{s,p}\right| \leq (1-s)^{\frac{1}{p}}[u-u_{k_0}]_{s,p} \leq C(n,p)\|u-u_{k_0}\|_{1,p} < C(n,p)\varepsilon.$$

Also the third term is controlled.

$$K(n,p)^{\frac{1}{p}}\left|\|\nabla u_{k_0}\|_p - \|\nabla u\|_p\right| \leq K(n,p)^{\frac{1}{p}}\|\nabla u_{k_0} - \nabla u\|_p \leq K(n,p)^{\frac{1}{p}}\|u_{k_0} - u\|_{1,p} < K(n,p)^{\frac{1}{p}}\varepsilon$$

Therefore,

$$\left|(1-s)^{\frac{1}{p}}[u]_{s,p} - K(n,p)^{\frac{1}{p}}\|\nabla u\|_p\right| \leq M_{n,p}\,\varepsilon + \left|(1-s)^{\frac{1}{p}}[u_{k_0}]_{s,p} - K(n,p)^{\frac{1}{p}}\|\nabla u_{k_0}\|_p\right|,$$

for every s, where $M_{n,p} := \max\{C(n,p), K(n,p)^{\frac{1}{p}}\}$. By taking the limit $s \uparrow 1$ and then, $\varepsilon \downarrow 0$ we conclude the result for $u \in W^{1,p}(\mathbb{R}^n)$.

The last step is to show that if $u \in L^p(\mathbb{R}^n)$ verifies

$$\liminf_{s\uparrow 1}(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy < \infty, \tag{1.1.10}$$

then $u \in W^{1,p}(\mathbb{R}^n)$.

By truncating and regularizing as we did in Lemmas 1.1.3 and 1.1.4, we build the family $\{u_{k,\varepsilon}\}_{k\in\mathbb{N},\varepsilon>0}$,

$$u_{k,\varepsilon} = \rho_\varepsilon * (u\eta_k)$$

which satisfies the following properties:

$$u_{k,\varepsilon} \in C_c^\infty(\mathbb{R}^n), \tag{1.1.11}$$

$$\liminf_{s\uparrow 1}(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u_{k,\varepsilon}(x)-u_{k,\varepsilon}(y)|^p}{|x-y|^{n+sp}}\,dxdy < C, \tag{1.1.12}$$

where C is independent of $k \in \mathbb{N}$ and $\varepsilon > 0$. Observe that (1.1.12) is a straightforward consequence of Lemmas 1.1.3 and 1.1.4 and the hypothesis (1.1.10).

Thanks to (1.1.11) and the first part of this theorem, we get

$$K(n,p)\|\nabla u_{k,\varepsilon}\|_p^p = \lim_{s\uparrow 1}(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_{k,\varepsilon}(x)-u_{k,\varepsilon}(y)|^p}{|x-y|^{n+sp}}\,dxdy < C.$$

Hence, the family $\{u_{k,\varepsilon}\}_{k\in\mathbb{N},\varepsilon>0}$ is bounded in $W^{1,p}(\mathbb{R}^n)$. Consequently, there exists a sequence $u_j = u_{k_j,\varepsilon_j}$ where $k_j \to \infty$ and $\varepsilon_j \downarrow 0$ and $\tilde{f} \in W^{1,p}(\mathbb{R}^n)$ such that $u_j \rightharpoonup \tilde{u}$ weakly in $W^{1,p}(\mathbb{R}^n)$.

Moreover, as we already know,

$$\|u_{k,\varepsilon} - u\|_p \to 0 \quad \text{ when } k\to\infty \text{ and } \varepsilon\downarrow 0,$$

we immediately conclude $\tilde{u} = u$ from where $u \in W^{1,p}(\mathbb{R}^n)$. □

1.1.3 Relation between $[\,\cdot\,]_{s,p}$ and $\|\nabla\cdot\|_p$ for a sequence

Let us start with a couple of general lemmas, that seem to be not related with our goal at first sight. Ideas from Bourgain-Brezis-Mironescu [14] and Ponce [75] were used here.

Lemma 1.1.8. *Let (X,μ) be a finite measure space and let $G,H \in L^1(X)$ be such that*

$$(G(x)-G(y))(H(x)-H(y)) \geq 0. \tag{1.1.13}$$

Then,

$$\int_X GH\,d\mu \geq \frac{1}{\mu(X)}\int_X G\,d\mu\int_X H\,d\mu.$$

Proof. It follows immediately from the monotone inequality (1.1.13). Indeed, (1.1.13) is equivalent to

$$G(x)H(x) + G(y)H(y) \geq G(x)H(y) + G(y)H(x). \tag{1.1.14}$$

Now, integrate this inequality with respect to variable x and y to obtain

$$2\mu(X)\int_X GH\,d\mu \geq 2\int_X G\,d\mu\int_X H\,d\mu,$$

which proves the lemma. □

We need to refine Lemma 1.1.8, since we want to apply a similar result to not necessary monotone functions. So that we give now an improved lemma, which fits to our final goal.

Lemma 1.1.9. *Let $g,h\colon (0,1)\to\mathbb{R}_+$ be measurable functions. Assume that $g(t)\leq g(\frac{t}{2})$ for $t\in(0,1)$ and h is a decreasing function. Then, given $r>-1$,*

$$\int_0^1 t^r g(t)h(t)\,dt \geq \frac{r+1}{2^{r+1}}\int_0^1 t^r g(t)\,dt\int_0^1 t^r h(t)\,dt.$$

Proof. The idea of the proof is to use the monotony of g in $\frac{1}{2}$-jumps, then we will be able to apply Lemma 1.1.8.

Let us start by rewriting this expression,

$$\begin{aligned}\int_0^1 t^r g(t)h(t)\,dt &= \sum_{j=0}^{\infty} \int_{\frac{1}{2^{j+1}}}^{\frac{1}{2^j}} t^r g(t)h(t)\,dt \\ &= \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} \int_{\frac{1}{2}}^{1} s^r g(\tfrac{s}{2^j})h(\tfrac{s}{2^j})\,ds \\ &= \int_{\frac{1}{2}}^{1} \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} s^r g(\tfrac{s}{2^j})h(\tfrac{s}{2^j})\,ds,\end{aligned}$$

where we have used the Monotone Convergence Theorem in the last identity.

Observe that by choosing $h(t) \equiv 1$ in the previous identity, we get that

$$\int_0^1 t^r g(t)\,dt = \int_{\frac{1}{2}}^{1} \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} s^r g(\tfrac{s}{2^j})\,ds.$$

Now, we are able to apply Lemma 1.1.8 for $H(j) = h(\frac{s}{2^j})$, $G(j) = g(\frac{s}{2^j})$ and $\mu(\{j\}) = \frac{1}{2^{j(r+1)}}$, to obtain

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} g(\tfrac{s}{2^j})h(\tfrac{s}{2^j}) &\geq \frac{1}{\sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}}} \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} g(\tfrac{s}{2^j}) \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} h(\tfrac{s}{2^j}) \\ &= \left(1 - \frac{1}{2^{r+1}}\right) \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} g(\tfrac{s}{2^j}) \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} h(\tfrac{s}{2^j})\end{aligned}$$

Since h is a decreasing function, we get for $j \geq 1$,

$$\int_{\frac{1}{2^j}}^{\frac{1}{2^{j-1}}} t^r h(t)\,dt \leq h(\tfrac{1}{2^j}) \int_{\frac{1}{2^j}}^{\frac{1}{2^{j-1}}} t^r\,dt = \frac{2^{r+1}-1}{r+1} \frac{1}{2^{j(r+1)}} h(\tfrac{1}{2^j}).$$

Again, thanks to the decreasing function h, we know that $h(\frac{1}{2^j}) \leq h(\frac{s}{2^j})$ for $0 < s < 1$, from we obtain

$$\int_0^1 t^r h(t)\,dt \leq \frac{2^{r+1}-1}{r+1} \sum_{j=0}^{\infty} \frac{1}{2^{j(r+1)}} h(\tfrac{s}{2^j}).$$

Put together both estimates to conclude that

$$\int_0^1 t^r g(t)h(t)\,dt \geq \frac{r+1}{2^{r+1}} \int_0^1 t^r g(t)\,dt \int_0^1 t^r h(t)\,dt,$$

as we wanted to show. □

We give this key inequality relating two fractional semi-norms.

Theorem 1.1.10. *Let $1 < p < \infty$ and $0 < s_1 < s_2 < 1$. Then,*

$$\begin{aligned}(1-s_1)\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+s_1p}}\,dxdy \leq& 2^{(1-s_1)p}(1-s_2)\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+s_2p}}\,dxdy\\ &+\frac{\omega_{n-1}(1-s_1)2^p}{s_1p}\int_{\mathbb{R}^n}|u|^p\,dx,\end{aligned}$$

for every $u \in W^{s_2,p}(\mathbb{R}^n)$. The quantities are thought to be infinitive otherwise.

Proof. Let $u \in L^p(\mathbb{R}^n)$ and $t > 0$. We define

$$\begin{aligned}F(t) &= \int_{\mathcal{S}^{n-1}}\int_{\mathbb{R}^n}|u(x+tw)-u(x)|^p\,dx\,dS_w\\ &= \frac{1}{t^{n-1}}\int_{\{|h|=t\}}\int_{\mathbb{R}^n}|u(x+h)-u(x)|^p\,dx\,dS_h.\end{aligned}$$

This function verifies that

$$\begin{aligned}F(2t) =& \int_{\mathcal{S}^{n-1}}\int_{\mathbb{R}^n}|u(x+2tw)-u(x)|^p\,dx\,dS_w\\ =& \int_{\mathcal{S}^{n-1}}\int_{\mathbb{R}^n}|u(x+2tw)-u(x+tw)+u(x+tw)-u(x)|^p\,dx\,dS_w\\ \leq& 2^{p-1}\left(\int_{\mathcal{S}^{n-1}}\int_{\mathbb{R}^n}|u(x+2tw)-u(x+tw)|^p\,dx\,dS_w\right.\\ &\left.+\int_{\mathcal{S}^{n-1}}\int_{\mathbb{R}^n}|u(x+tw)-u(x)|^p\,dx\,dS_w\right)\\ =& 2^p\int_{\mathcal{S}^{n-1}}\int_{\mathbb{R}^n}|u(x+tw)-u(x)|^p\,dx\,dS_w\\ =& 2^pF(t).\end{aligned}$$

Then, by naming $g(t) = \frac{F(t)}{t^p}$, we get that $g(2t) \leq g(t)$ for every $t > 0$.

Now, observe the following identity,

$$\begin{aligned}\int_{\{|h|<1\}}\int_{\mathbb{R}^n}\frac{|u(x+h)-u(x)|^p}{|h|^{n+sp}}\,dxdh &= \int_0^1\int_{\{|h|=t\}}\int_{\mathbb{R}^n}\frac{|u(x+h)-u(x)|^p}{t^{n+sp}}\,dx\,dS_h\,dt\\ &= \int_0^1\frac{1}{t^{1+sp}}F(t)\,dt \qquad (1.1.15)\\ &= \int_0^1\frac{1}{t^{1-(1-s)p}}g(t)\,dt.\end{aligned}$$

Consider $0 < s_1 < s_2 < 1$. Then,

$$\int_0^1\frac{1}{t^{1-(1-s_2)p}}g(t)\,dt = \int_0^1\frac{1}{t^{1-(1-s_1)p}}g(t)\frac{1}{t^{(s_2-s_1)p}}\,dt.$$

Apply Lemma 1.1.9 for $r = (1 - s_1)p - 1$ and $h(t) = t^{-(s_2 - s_1)p}$, to arrive at

$$\begin{aligned}\int_0^1 \frac{1}{t^{1-(1-s_2)p}} g(t)\, dt \geq & \frac{(1-s_1)p}{2^{(1-s_1)p}} \int_0^1 \frac{1}{t^{1-(1-s_1)p}} g(t)\, dt \int_0^1 \frac{1}{t^{1-(1-s_2)p}}\, dt \\ = & \frac{1}{2^{(1-s_1)p}} \frac{1-s_1}{1-s_2} \int_0^1 \frac{1}{t^{1-(1-s_1)p}} g(t)\, dt.\end{aligned} \tag{1.1.16}$$

From (1.1.15) and (1.1.16) we deduce that

$$\begin{aligned}\frac{(1-s_1)}{2^{(1-s_1)p}} \int_{\{|h|<1\}} \int_{\mathbb{R}^n} \frac{|u(x+h) - u(x)|^p}{|h|^{n+s_1p}}\, dxdh \leq \\ (1-s_2) \int_{\{|h|<1\}} \int_{\mathbb{R}^n} \frac{|u(x+h) - u(x)|^p}{|h|^{n+s_2p}}\, dxdh\end{aligned} \tag{1.1.17}$$

Finally, observe that

$$\begin{aligned}\int_{\{|h|\geq 1\}} \int_{\mathbb{R}^n} \frac{|u(x+h) - u(x)|^p}{|h|^{n+sp}}\, dxdh \leq 2^p \|u\|_p^p \omega_{n-1} \int_1^\infty \frac{1}{t^{1+sp}}\, dt \\ = \frac{\omega_{n-1} 2^p}{sp} \|u\|_p^p.\end{aligned}$$

By combining this last inequality with (1.1.17), we conclude the desired result. □

Now, we are able to prove an analogous outcome to Theorem 1.1.7 which is for a fixed function u, where a sequence of functions varying with $s \in (0, 1)$ is involved.

Theorem 1.1.11 (Theorem 4, [15])**.** *Let $0 < s_k \uparrow 1$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^n)$ be such that*

$$\sup_{k\in\mathbb{N}} (1 - s_k) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+s_kp}}\, dxdy < \infty \quad \textit{and} \quad \sup_{k\in\mathbb{N}} \|u_k\|_p < \infty.$$

Then, there exist a function $u \in L^p(\mathbb{R}^n)$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ such that $u_{k_j} \to u$ in $L^p_{loc}(\mathbb{R}^n)$. Furthermore, $u \in W^{1,p}(\mathbb{R}^n)$ and the following estimation holds

$$K(n, p) \int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx \leq \liminf_{k\to\infty} (1 - s_k) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+s_kp}}\, dxdy,$$

where $K(n, p)$ is given in (1.1.8)*.*

Proof. Thanks to Theorem 1.1.10, the proof is concluded easily.

Let $0 < s_k \uparrow 1$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^n)$ be such that

$$\sup_{k\in\mathbb{N}} (1 - s_k) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+s_kp}}\, dxdy < \infty \quad \text{and} \quad \sup_{k\in\mathbb{N}} \|u_k\|_p < \infty.$$

Fix $0 < t < 1$. By Theorem 1.1.10, the sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{t,p}(\mathbb{R}^n)$ is bounded, so that, by Rellich-Kondrashov's Theorem, there exist a subsequence (we still denote by $\{u_k\}_{k\in\mathbb{N}}$) and a function $u \in L^p(\mathbb{R}^n)$ such that $u_k \to u$ in $L^p_{\text{loc}}(\mathbb{R}^n)$. Furthermore, occasionally by passing to a new subsequence, we can assume that $u_k \to u$ almost everywhere $\mathbb{R}^n$.

By Fatou's Lemma, we get that

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+tp}}\, dxdy \leq \liminf_{k\to\infty} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+tp}}\, dxdy$$

and again by Theorem 1.1.10, we obtain

$$\frac{1-t}{2^{(1-t)p}} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+tp}}\, dxdy \leq \liminf_{k\to\infty}(1-s_k) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+s_kp}}\, dxdy + \frac{\omega_{n-1}2^p(1-t)}{tp} \sup_{k\in\mathbb{N}} \|u_k\|_p^p.$$

Eventually, the result follows by taking the limit $t \uparrow 1$ and by Theorem 1.1.7. □

1.1.4 Poincaré's inequality

Let us give a proof of Poincaré's inequality as a consequence of Theorem 1.1.11.

Theorem 1.1.12. *Let A be the sharp constant in the classical Poincaré's inequality*

$$\int_\Omega |u|^p\, dx \leq A \int_\Omega |\nabla u|^p\, dx \tag{1.1.18}$$

for every $u \in W_0^{1,p}(\Omega)$.

Hence, given $\delta > 0$ there exists $0 < s_0 < 1$ such that

$$\int_\Omega |u|^p\, dx \leq \left(\frac{A}{K(n,p)} + \delta\right)(1-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy, \tag{1.1.19}$$

for every $s_0 \leq s < 1$ and $u \in L^p(\Omega)$. The constant $K(n,p)$ was given in (1.1.8).

Proof. Let us proceed by contradiction. Suppose the statement is false, so that there exist a constant $C > \frac{A}{K(n,p)}$, a sequence $s_j \uparrow 1$ and $\{u_j\}_{j\in\mathbb{N}} \subset L^p(\Omega)$ such that

$$\|u_j\|_p = 1 \quad \text{and} \quad (1-s_j) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_j(x)-u_j(y)|^p}{|x-y|^{n+s_jp}}\, dxdy \leq \frac{1}{C}.$$

By Theorem 1.1.11, by passing occasionally to a new subsequence, there exists $u \in W^{1,p}(\mathbb{R}^n)$ such that $\|u_j - u\|_{p;\Omega} \to 0$ and $u_j \to u$ almost everywhere in $\mathbb{R}^n$. Then, $u \in W_0^{1,p}(\Omega)$, $\|u\|_p = 1$ and, again by Theorem 1.1.11,

$$K(n,p)\|\nabla u\|_p^p \leq \liminf_{j\to\infty}(1-s_j) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_j(x)-u_j(y)|^p}{|x-y|^{n+s_jp}}\, dxdy \leq \frac{1}{C}.$$

This last inequality contradicts the sharpness of the constant A, so that the Theorem is demonstrated. □

Remark 1.1.13. The constant A from (1.1.18) depends only on n, p and Ω. Moreover, in fact, $A = \lambda_p(\Omega)^{-1}$, where $\lambda_p(\Omega)$ is the first eigenvalue of p-Laplacian operator in Ω with homogeneous Dirichlet conditions. Consequently, also the parameter s_0 depends on n, p and Ω.

For generalizations of this inequality, we suggest the article [74].

Immediately, we get next corollary.

Corollary 1.1.14. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $1 < p < \infty$. Then, there exists a constant $C > 0$ depending only on n, p and Ω such that*

$$\|u\|_p^p \leq C(1-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy$$

for every $0 < s < 1$ and $u \in L^p(\Omega)$.

Proof. By Theorem 1.1.12, for $\delta = 1$, there exists $0 < s_0 < 1$ such that

$$\|u\|_p^p \leq \left(\frac{A}{K(n,p)} + 1 \right) (1-s)[u]_{s,p}^p \tag{1.1.20}$$

for every $s_0 \leq s < 1$ and $u \in L^p(\Omega)$. The constant $K(n,p)$ was given in (1.1.8).

Now, for $0 < s < s_0$, we have the following estimate:

$$\begin{aligned}
(1-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy &\geq (1-s_0) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy \\
&\geq (1-s_0) \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}}\, dydx \\
&\geq (1-s_0) \int_\Omega |u(x)|^p \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{|x-y|^{n+sp}}\, dydx \\
&\geq (1-s_0) \int_\Omega |u(x)|^p \int_{\mathbb{R}^n \setminus B_{R_\Omega}(x)} \frac{1}{|x-y|^{n+sp}}\, dydx \\
&= (1-s_0) \frac{\bar{\omega}_n}{sp \operatorname{diam}(\Omega)} \|u\|_p^p
\end{aligned}$$

where $R_\Omega = \operatorname{diam}(\Omega)$, $\bar{\omega}_n$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Thus, we obtain for $0 < s < s_0$

$$\begin{aligned}
\|u\|_p^p &\leq sp \operatorname{diam}(\Omega) (\bar{\omega}_n (1-s_0))^{-1} (1-s)[u]_{s,p}^p \\
&\leq p \operatorname{diam}(\Omega) (\bar{\omega}_n (1-s_0))^{-1} (1-s)[u]_{s,p}^p \\
&= C(n,p,\Omega)(1-s)[u]_{s,p}^p.
\end{aligned}$$

From the previous inequality and (1.1.20), the result follows. □

Now, we deduce fractional Sobolev spaces are ordered. We give here a proof which is a bit more complicated than that you can find in [40, Proposition 2.1], since we are interested in the behavior of such constant. We want to know how it depends on the fractional exponents involved.

Proposition 1.1.15. *Let $1 < p < \infty$ and $0 < t < s < 1$. Then, $W_0^{s,p}(\Omega) \subset W_0^{t,p}(\Omega)$. Moreover,*

$$(1-t)[u]_{t,p}^p \le C(n,t,p,\Omega)(1-s)[u]_{s,p}^p, \tag{1.1.21}$$

for every $u \in W_0^{s,p}(\Omega)$, where

$$\lim_{t\uparrow 1} C(n,t,p,\Omega) = 1.$$

Proof. Let $u \in W_0^{s,p}(\Omega)$ and $0 < t < s < 1$. First, by Theorem 1.1.10, we know that

$$\begin{aligned}(1-t)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+tp}}\,dxdy \le & 2^{(1-t)p}(1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy \\ & + \frac{\omega_{n-1}(1-t)2^p}{tp}\int_{\mathbb{R}^n} |u|^p\,dx,\end{aligned}$$

By Corollary 1.1.14,

$$\begin{aligned}(1-t)[u]_{t,p}^p &\le 2^{(1-t)p}(1-s)[u]_{s,p}^p + \frac{\omega_{n-1}(1-t)2^p}{tp}C(1-s)[u]_{s,p}^p \\ &= \left(2^{(1-t)p} + \frac{\omega_{n-1}(1-t)2^p}{tp}C\right)(1-s)[u]_{s,p}^p,\end{aligned}$$

where $C = C(n,p,\Omega) > 0$. Now, take $C(n,t,p,\Omega) := 2^{(1-t)p} + \frac{\omega_{n-1}(1-t)2^p}{tp}C$ and the result follows.

□

The extension of the Rellich-Kondrachov compactness theorem to the fractional order Sobolev spaces is also well-known.

Theorem 1.1.16. *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure. Then the immersion $W_0^{s,p}(\Omega) \subset L^p(\Omega)$ is compact. That is, if $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ is bounded, then there exists $u \in W_0^{s,p}(\Omega)$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ such that*

$$\|u_{k_j} - u\|_p \to 0 \text{ as } j \to \infty.$$

Proof. Let us see that both hypotheses of Frechet-Kolmogorov's Theorem (the $L^p(\Omega)$-version of Arzelà-Ascoli's Theorem for continuous functions) hold for a bounded sequence in $W_0^{s,p}(\Omega)$.

Let $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ be a bounded sequence. So we must show that $\{u_k\}_{k\in\mathbb{N}}$ is equicontinuous and uniformly bounded in $L^p(\Omega)$.

First, notice that the sequence boundedness in $W_0^{s,p}(\Omega)$ and Poincaré's inequality 1.1.14, immediately imply that the sequence is uniformly bounded in $L^p(\Omega)$.

Secondly, we will prove for $h \in \mathbb{R}^n$,

$$\|v(\cdot + h) - v(\cdot)\|_p \leq C_1 |h|^s [v]_{s,p}, \tag{1.1.22}$$

from where we deduce the equicontinuous condition for our $W_0^{s,p}(\Omega)$-bounded sequence.

To start proving (1.1.22), observe that

$$|h|^n \bar{\omega}_n \|\tau_h v - v\|_p^p = \int_{\mathbb{R}^n} \int_{B_{|h|}(x)} |v(x+h) - v(x)|^p \, dy \, dx, \tag{1.1.23}$$

where $\bar{\omega}_n$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$ and $\tau_h v = v(\cdot + h)$.

Now, use the following elemental inequality

$$|v(x+h) - v(x)|^p \leq 2^{p-1}(|v(x+h) - v(y)|^p + |v(y) - v(x)|^p), \tag{1.1.24}$$

for every $y \in B_{|h|}(x)$.

From (1.1.24), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{B_{|h|}(x)} |v(x+h) - v(x)|^p \, dy \, dx \leq \\
&\qquad 2^{p-1} \int_{\mathbb{R}^n} \int_{B_{|h|}(x)} \frac{|v(x+h) - v(y)|^p}{|x+h-y|^{n+sp}} |x+h-y|^{n+sp} \, dy \, dx \\
&\qquad + 2^{p-1} \int_{\mathbb{R}^n} \int_{B_{|h|}(x)} \frac{|v(x) - v(y)|^p}{|x-y|^{n+sp}} |x-y|^{N+sp} \, dy \, dx \\
&\qquad = 2^{p-1}(I + II).
\end{aligned}$$

The technique we apply in I and II to estimate them from above is similar. For $x \in \mathbb{R}^N$ and $y \in B_{|h|}(x)$, we know that

$$|x-y| \leq |h| \quad \text{and} \quad |x+h-y| \leq |x-y| + |h| \leq 2|h|. \tag{1.1.25}$$

From (1.1.25), we estimate

$$I \leq (2|h|)^{n+sp} \int_{\mathbb{R}^n} \int_{B_{|h|}(x)} \frac{|v(x+h) - v(y)|^p}{|x+h-y|^{n+sp}} \, dy \, dx \leq (2|h|)^{n+sp} \, [v]_{s,p}^p. \tag{1.1.26}$$

Similarly,

$$II \leq |h|^{n+sp} \, [v]_{s,p}^p. \tag{1.1.27}$$

By using (1.1.23), (1.1.26) and (1.1.27) we conclude that

$$\|\tau_h v - v\|_p^p \leq \frac{2^{p-1}(2^{n+sp} + 1)}{\bar{\omega}_n} |h|^{sp} \, [v]_{s,p}^p,$$

which ends the proof. □

1.2 Some nonlocal operators

In this section, we introduce those nonlocal operators we work with throughout the thesis, starting with the fractional Laplacian, which is a particular example of a class defined later. The main reason of studying it in a separated section is that we are really interested in the asymptotic behavior of the constant $c(n,s)$ when $s \uparrow 1$. One of the major reference for the analysis of the fractional constant $c(n,s)$ is [40].

1.2.1 Fractional Laplacian operator

To start with this section, we introduce first the **fractional Laplacian** operator. It will be a particular case of next example, $\mathcal{L}_a$ which will be defined later.

Definition 1.2.1. Given $s \in (0,1)$ we consider the fractional Laplacian, that for smooth functions u is defined as

$$(-\Delta)^s u(x) := c(n,s)\text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

where $c(n,s)$ is a normalization constant.

Remark 1.2.2. We can rewrite the fractional Laplacian of u as

$$(-\Delta)^s u(x) = \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\, dy. \tag{1.2.1}$$

Proof. Let u be a smooth function. Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\, dy &= \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\, dy \\
&= \lim_{\varepsilon\downarrow 0} \left(\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{u(x)-u(x+y)}{|y|^{n+2s}}\, dy + \int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{u(x)-u(x-y)}{|y|^{n+2s}}\, dy \right) \\
&= \lim_{\varepsilon\downarrow 0} \left(\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(w)}{|x-w|^{n+2s}}\, dy + \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\, dy \right) \\
&= 2\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = 2\,\text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,
\end{aligned}$$

where we use changes of variable $w = x+y$ and $z = x-y$. □

The Remark above also shows why it is convenient to put the factor $\frac{1}{2}$ multiplying the constant $c(n,s)$.

Notice that the expression (1.2.1) does not need the *principal value* formulation. Assume $u \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$. Use the Taylor expansion of u in B_1 to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}}\,dy \leq \\ &\leq \|u\|_\infty \int_{\mathbb{R}^n \setminus B_1} \frac{1}{|y|^{n+2s}}\,dy + \int_{B_1} \frac{|D^2u(x)||y|^2}{|y|^{n+2s}}\,dy \\ &\leq \|u\|_\infty \int_{\mathbb{R}^n \setminus B_1} \frac{1}{|y|^{n+2s}}\,dy + \|D^2u\|_\infty \int_{B_1} \frac{1}{|y|^{n+2s-2}}\,dy,\end{aligned}$$

where

$$\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|y|^{n+2s}}\,dy = \omega_{n-1} \int_1^\infty r^{n-1-n-2s}\,dr = \frac{\omega_{n-1}}{2s} < \infty$$

and

$$\int_{B_1} \frac{1}{|y|^{n+2s-2}}\,dy = \omega_{n-1} \int_0^1 r^{n-1-n-2s+2}\,dr = \frac{\omega_{n-1}}{2(1-s)} < \infty.$$

The constant $c(n,s)$ is chosen in such a way that the following identity holds,

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(u)),$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ the Schwartz class of rapidly decreasing and infinitely differentiable functions, where $\mathcal{F}$ denotes the Fourier transform:

$$\mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx;$$

which is the content of next proposition.

Proposition 1.2.3 (Proposition 3.3, [40]). *The fractional Laplacian defined in 1.2.1 satisfies*

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(u))$$

for every $u \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{F}$ denotes the Fourier transform.

Proof. By applying some basic properties of Fourier's transform, we obtain these identities.

$$\begin{aligned}\mathcal{F}((-\Delta)^s u(x)) &= \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \frac{\mathcal{F}(2u(x) - u(x+y) - u(x-y))}{|y|^{n+2s}}\,dy \\ &= \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \hat{u}(\xi) \frac{2 - e^{-2\pi\xi\cdot y} - e^{2\pi\xi\cdot y}}{|y|^{n+2s}}\,dy \\ &= \hat{u}(\xi) c(n,s) \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy\end{aligned}$$

Now, we use the change of variable $z = |\xi| y$ to get

$$\mathcal{F}((-\Delta)^s u(x)) = \hat{u}(\xi) c(n,s) |\xi|^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi \frac{\xi}{|\xi|} \cdot z)}{|z|^{n+2s}} \, dz.$$

Since the right-hand side is rotationally invariant, we consider a rotation R that sends $e_1 = (1, 0, \ldots, 0)$ into $\frac{\xi}{|\xi|}$ and we denote R^T its transpose. Then, by using the change of variables $y = R^T z$ we obtain that

$$\begin{aligned}
|\xi|^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi \frac{\xi}{|\xi|} \cdot z)}{|z|^{n+2s}} \, dz &= |\xi|^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi R e_1 \cdot z)}{|z|^{n+2s}} \, dz \\
&= |\xi|^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi R^T z \cdot e_1)}{|R^T z|^{n+2s}} \, dz \\
&= |\xi|^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi y \cdot e_1)}{|y|^{n+2s}} \, dy \\
&= |\xi|^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi y_1)}{|y|^{n+2s}} \, dy.
\end{aligned}$$

We use one more change of variable $z = 2\pi y$, to arrive at

$$|\xi|^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(2\pi y_1)}{|y|^{n+2s}} \, dy = (2\pi|\xi|)^{2s} \int_{\mathbb{R}^n} \frac{1 - \cos(z_1)}{|z|^{n+2s}} \, dz.$$

Let us now show that this constant

$$\int_{\mathbb{R}^n} \frac{1 - \cos(z_1)}{|z|^{n+2s}} \, dz$$

is finite. Notice that outside the unit ball we get

$$\int_{\mathbb{R}^n \setminus B_1} \frac{|1 - \cos(z_1)|}{|z|^{n+2s}} \, dz \le \int_{\mathbb{R}^n \setminus B_1} \frac{2}{|z|^{n+2s}} \, dz = 2\omega_{n-1} \int_1^\infty \frac{r^{n-1}}{r^{n+2s}} \, dr = \frac{\omega_{n-1}}{s} < \infty.$$

On the other hand, inside the unit ball, we use the Taylor expansion of the cosine function to realize that

$$\int_{B_1} \frac{|1 - \cos(z_1)|}{|z|^{n+2s}} \, dz \le \int_{B_1} \frac{|z|^2}{|z|^{n+2s}} \, dz = \omega_{n-1} \int_0^1 \frac{r^{n-1}}{r^{n+2s-2}} \, dr = \frac{\omega_{n-1}}{2(1-s)} < \infty.$$

Hence, by taking

$$c(n,s) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(z_1)}{|z|^{n+2s}} \, dz \right)^{-1}. \tag{1.2.2}$$

and by gathering all the information, we conclude that

$$\mathcal{F}((-\Delta)^s u(x)) = (2\pi|\xi|)^{2s} \hat{u}(\xi),$$

from the outcome follows. Notice that the normalization constant introduced in the Definition 1.2.1 is now computed in (1.2.2). □

That choice of the constant is consistent in order to recover the usual Laplacian.

Theorem 1.2.4 (Proposition 4.4,[40]). *Let $u \in \mathcal{S}(\mathbb{R}^n)$. Then,*

$$\lim_{s\uparrow 1}(-\Delta)^s u = -\Delta u. \tag{1.2.3}$$

Proof. We already know that

$$\begin{aligned}
-\Delta u(x) &= -\Delta\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\mathcal{F}(u)(\xi)\,d\xi\right)\\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^2\mathcal{F}(u)(\xi)\,d\xi\\
&= \mathcal{F}^{-1}(|\xi|^2\mathcal{F}(u)(\xi)).
\end{aligned}$$

By Proposition 1.2.3, we also know that $(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(u)(\xi))$.

Basically, the point-wise convergence

$$|\xi|^{2s}\mathcal{F}(u)(\xi) \to |\xi|^2\mathcal{F}(u)(\xi)$$

and the dominating integral function $(1+|\xi|^2)\mathcal{F}(u)(\xi) \in L^1(\mathbb{R}^n)$ along with Lebesgue Convergence Theorem, prove 1.2.3.

Notice that $|\xi|^{2s} \le 1$ in the unit ball B_1, and $|\xi|^{2s} \le |\xi|^2$ outside it. □

Asymptotic behavior of $c(n,s)$

Aimed at our purposes in this thesis, it is suitable to analyze the behavior of the normalization constant $c(n,s)$ as $s \uparrow 1$.

Let us begin with changing variables in $\mathbb{R}\times\mathbb{R}^{n-1}$ as follows: $w_1 = z_1$ and $(w_2,\ldots,w_n) = w' = \frac{z'}{|z_1|}$.

$$\begin{aligned}
\int_{\mathbb{R}^n}\frac{1-\cos(z_1)}{|z|^{n+2s}}\,dz &= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\frac{1-\cos(w_1)}{|w_1|^{n+2s}(1+|w'|^2)^{\frac{n}{2}+s}}\,|w_1|^{n-1}dw'dw_1\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\frac{1-\cos(w_1)}{|w_1|^{1+2s}(1+|w'|^2)^{\frac{n}{2}+s}}\,dw'dw_1\\
&= \int_{\mathbb{R}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt\int_{\mathbb{R}^{n-1}}\frac{1}{(1+|w'|^2)^{\frac{n}{2}+s}}\,dw'.
\end{aligned}$$

Now, we split the analysis of asymptotic behavior into two new integrals:

$$\alpha(s) := \int_{\mathbb{R}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt \quad \text{and} \quad \beta(n,s) := \int_{\mathbb{R}^{n-1}}\frac{1}{(1+|w|^2)^{\frac{n}{2}+s}}\,dw. \tag{1.2.4}$$

Proposition 1.2.5 (Proposition 4.1, [40]). *Let $\alpha(s)$ and $\beta(n,s)$ be the functions defined above (1.2.4). Then,*

$$\lim_{s\uparrow 1}(1-s)\alpha(s)=\frac{1}{2} \qquad \text{and} \qquad \lim_{s\uparrow 1}\beta(n,s)=\omega_{n-2}\int_0^\infty \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+1}}\,d\rho,$$

where ω_{n-2} is the $(n-2)$-dimensional measure of the unit sphere $\mathcal{S}^{n-2}\subset\mathbb{R}^{n-1}$.

Proof. Let us start by using polar coordinates in $\beta(n,s)$:

$$\beta(n,s)=\int_{\mathbb{R}^{n-1}}\frac{1}{(1+|w|^2)^{\frac{n}{2}+s}}\,dw=\omega_{n-2}\int_0^\infty\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+s}}\,d\rho.$$

Now, we notice that it is easy to get a dominating integral and a point-wise limit function, which allow us to apply Lebesgue Dominated Convergence's Theorem, to conclude the asymptotic behavior for $\beta(n,s)$. Indeed, for $s\in(0,1)$ and $\rho\geq 0$ it holds

$$\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+s}}\leq\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}}}\in L^1((0,\infty)).$$

To analyze $\alpha(s)$, first, we split the integral into two pieces:

$$\int_{\mathbb{R}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt=\int_{\{|t|<1\}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt+\int_{\{|t|\geq 1\}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt.$$

Let us continue with the term that does not contribute to the limit $s\uparrow 1$. Indeed,

$$0\leq\int_{\{|t|\geq 1\}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt=2\int_1^\infty\frac{1-\cos(t)}{t^{1+2s}}\,dt\leq 4\int_1^\infty\frac{1}{t^{1+2s}}\,dt=\frac{2}{s},$$

hence,

$$\lim_{s\uparrow 1}s(1-s)\int_{\{|t|\geq 1\}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt=0.$$

Now, we analyze the remained term by using the Taylor expansion of cosine to arrive at

$$0\leq\int_{\{|t|<1\}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt-\int_{\{|t|<1\}}\frac{t^2}{2|t|^{1+2s}}\,dt\leq\frac{1}{6}\int_{\{|t|<1\}}\frac{|t|^3}{|t|^{1+2s}}\,dt=\frac{1}{3(3-2s)}.$$

By multiplying by $0<1-s<1$ and taking into account the previous estimation, we find that

$$\begin{aligned}\lim_{s\uparrow 1}(1-s)\alpha(s)&=\lim_{s\uparrow 1}(1-s)\int_{\{|t|<1\}}\frac{1-\cos(t)}{|t|^{1+2s}}\,dt=\lim_{s\uparrow 1}(1-s)\int_{\{|t|<1\}}\frac{t^2}{2|t|^{1+2s}}\,dt\\&=\lim_{s\uparrow 1}(1-s)\int_0^1 t^{1-2s}\,dt=\lim_{s\uparrow 1}\frac{(1-s)}{2(1-s)}=\frac{1}{2}.\end{aligned}$$

□

Theorem 1.2.6 (Cororally 4.2, [40])**.** *Let $c(n,s)$ be the constant defined in* (1.2.2)*. Then,*

$$\lim_{s\uparrow 1}\frac{c(n,s)}{1-s}=\frac{4n}{\omega_{n-1}}, \tag{1.2.5}$$

where ω_{n-1} denotes the $(n-1)$-measure of the unit sphere $\mathcal{S}^{n-1}\subset\mathbb{R}^n$.

Proof. By definition, we know that

$$\frac{c(n,s)}{1-s}=\frac{1}{(1-s)\alpha(s)\beta(n,s)}.$$

Therefore, we apply Proposition 1.2.5 to get next identity:

$$\lim_{s\uparrow 1}\frac{c(n,s)}{1-s}=2\left(\omega_{n-2}\int_0^\infty\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+1}}\,d\rho\right)^{-1}.$$

Our goal is reduced to show that

$$\omega_{n-2}\int_0^\infty\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+1}}\,d\rho=\frac{\omega_{n-1}}{2n}. \tag{1.2.6}$$

The strategy will be define a recursive sequence and use an induction argument with the help of a the well-known behavior of the constant ω_n:

$$\omega_n=\frac{2\pi}{n-1}\omega_{n-2}. \tag{1.2.7}$$

To begin with, let $t\in\mathbb{R}$ be such that $t>n-1$ and define

$$E_n(t):=\int_0^\infty\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{t}{2}}}\,d\rho.$$

The parameter t is chosen to guarantee convergence of the integral. We can rewrite $E_n(t)$ by integrating by parts.

$$E_n(t)=\int_0^\infty\left(\frac{\rho^{n-1}}{n-1}\right)'\frac{1}{(1+\rho^2)^{\frac{t}{2}}}\,d\rho=\frac{t}{n-1}\int_0^\infty\frac{\rho^n}{(1+\rho^2)^{\frac{t+2}{2}}}\,d\rho=\frac{t}{n-1}E_{n+2}(t+2). \tag{1.2.8}$$

Now, we name I_n the quantity:

$$I_n:=E_n(n+2)=\int_0^\infty\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2}{2}}}\,d\rho.$$

Thanks to (1.2.8), we obtain

$$I_n=\frac{n+2}{n-1}E_{n+2}(n+4). \tag{1.2.9}$$

That allows us to find a recursive form to I_n:

$$I_{n+2} = \frac{n-1}{n+2} I_n.$$

We claim that

$$I_n = \frac{\omega_{n-1}}{2n\omega_{n-2}}. \tag{1.2.10}$$

As we say before, we now turn to the induction argument. Let us start by checking the inductive bases are satisfied.

$$I_2 = \int_0^\infty \frac{1}{(1+\rho^2)^2}\, d\rho = \frac{\pi}{4}, \qquad \text{and} \qquad I_3 = \int_0^\infty \frac{\rho}{(1+\rho^2)^{\frac{5}{2}}}\, d\rho = \frac{1}{3}.$$

To prove the inductive step, since (1.2.9), it is enough to show that

$$\frac{\omega_{n+1}}{\omega_n} = \frac{n-1}{n}\frac{\omega_{n-1}}{\omega_{n-2}}, \tag{1.2.11}$$

which easily follows from (1.2.7).

We include the proof of (1.2.7). We just separate the last two variables and use polar coordinates. Indeed, denote by $x = (\tilde{x}, x') \in \mathbb{R}^{n-2} \times \mathbb{R}^2$ and $\bar{\omega}_n$ the Lebesgue measure of the n-dimensional unit ball.

Now, by integrating in $\mathbb{R}^{n-2}$ and then using polar coordinates, we arrive at

$$\begin{aligned}
\bar{\omega}_n &= \int_{\{|x|^2 \le 1\}} dx = \int_{\{|x'| \le 1\}} \left(\int_{\{|\tilde{x}|^2 \le 1 - |x'|^2\}} d\tilde{x} \right) dx' \\
&= \int_{\{|x'| \le 1\}} \left(\int_0^{(1-|x'|^2)^{\frac{1}{2}}} \bar{\omega}_{n-2}\, r^{n-2}\, dr \right) dx' \\
&= \bar{\omega}_{n-2} \int_{\{|x'| \le 1\}} (1 - |x'|^2)^{\frac{n-2}{2}}\, dx' \\
&= 2\pi\bar{\omega}_{n-2} \int_0^1 \rho(1-\rho^2)^{\frac{n-2}{2}}\, d\rho = \frac{2\pi\bar{\omega}_{n-2}}{n}.
\end{aligned}$$

Furthermore, on the other hand, by again using polar coordinates, we get

$$\bar{\omega}_n = \int_{\{|x| \le 1\}} dx = \omega_{n-1} \int_0^1 r^{n-1}\, dr = \frac{\omega_{n-1}}{n-1}.$$

By combining both previous identities, we find the relation

$$\omega_{n-1} = n\bar{\omega}_n = n\frac{2\pi\bar{\omega}_{n-2}}{n} = 2\pi\bar{\omega}_{n-2} = \frac{2\pi\omega_{n-3}}{n-2}.$$

Replace n instead of $n-1$, to rewrite and obtain

$$\omega_n = \frac{2\pi\omega_{n-2}}{n-2},$$

from where we deduce (1.2.11), and then (1.2.10).

Eventually, we come to the conclusion that

$$\lim_{s\uparrow 1} \frac{c(n,s)}{1-s} = \frac{2}{\omega_{n-2} I_n} = \frac{4n}{\omega_{n-1}}.$$

□

We can rewrite Theorem 1.1.7 in the case $p = 2$ as

$$\lim_{s\uparrow 1} \frac{c(n,s)}{2} [u]_s^2 = \|\nabla u\|_2^2. \tag{1.2.12}$$

where $c(n,s) = \frac{1-s}{K(n,2)}$, with $K(n,2)$ defined by (1.1.8).

1.2.2 The $\mathcal{L}_a$ operator

Let us continue with a class of nonlocal operators, involving positive bounded kernels. Most properties are well-known, for instance, they can be found in [1].

We present here the case $p = 2$. The extended version for $1 \leq p < \infty$ can be found in [49].

Given $0 < \lambda < \Lambda < \infty$, we denote by $\mathcal{A}_{\lambda,\Lambda}$ the class

$$\mathcal{A}_{\lambda,\Lambda} := \{a \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \colon a(x,y) = a(y,x),\ \lambda \leq a(x,y) \leq \Lambda \text{ a.e.}\}. \tag{1.2.13}$$

Therefore, for $a \in \mathcal{A}_{\lambda,\Lambda}$ we define the operator $\mathcal{L}_a$ by

$$\mathcal{L}_a u(x) = \text{p.v.} \int_{\mathbb{R}^n} a(x,y) \frac{(u(x) - u(y))}{|x-y|^{n+2s}}\, dy. \tag{1.2.14}$$

Remark 1.2.7. Notice that if we choose $a(x,y) := c(n,s)$ defined in (1.2.2), we obtain the fractional Laplacian, see Definition 1.2.1.

Proposition 1.2.8. *Let $a \in \mathcal{A}_{\lambda,\Lambda}$. Then, $\mathcal{L}_a$ is a well defined operator between $H^s(\mathbb{R}^n)$ and its dual $H^{-s}(\mathbb{R}^n)$ and also between $H_0^s(\Omega)$ and $H^{-s}(\Omega)$. In fact,*

$$\langle \mathcal{L}_a u, v\rangle = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} a(x,y) \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}}\, dxdy. \tag{1.2.15}$$

Proof. Let $u \in H^s(\mathbb{R}^n)$. We want to know how $\mathcal{L}_a u$ acts over $H^s(\mathbb{R}^n)$, as an element from the dual space $H^{-s}(\mathbb{R}^n)$. For $H_0^s(\Omega)$ and $H^{-s}(\Omega)$, it is the same argument.

Let $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Consider

$$\mathcal{L}_a^\varepsilon u(x) := \int_{\{|x-y| \geq \varepsilon\}} a(x,y) \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy.$$

Now, we prove $\mathcal{L}_a^\varepsilon u \in L^2(\mathbb{R}^n)$ for every $\varepsilon > 0$. So, by the boundedness of a and Hölder's inequality,

$$\begin{aligned}|\mathcal{L}_a^\varepsilon u(x)| &\leq \Lambda \int_{\{|x-y|\geq\varepsilon\}} \frac{|u(x)-u(y)|}{|x-y|^{\frac{n+2s}{2}}} \frac{1}{|x-y|^{\frac{n+2s}{2}}}\, dy \\ &\leq \Lambda \left(\int_{\{|x-y|\geq\varepsilon\}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\right)^{\frac{1}{2}} \left(\int_{\{|x-y|\geq\varepsilon\}} \frac{1}{|x-y|^{\frac{n+2s}{2}}}\, dy\right)^{\frac{1}{2}}. \\ &\leq \frac{\Lambda}{\varepsilon^s}\sqrt{\frac{\bar{\omega}_n}{2s}} \left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\right)^{\frac{1}{2}},\end{aligned}$$

where $\bar{\omega}_n$ is the measure of the unit ball in $\mathbb{R}^n$. Then,

$$\int_{\mathbb{R}^n} |\mathcal{L}_a^\varepsilon u(x)|^2\, dx \leq \frac{\Lambda^2}{\varepsilon^{2s}} \frac{\bar{\omega}_n}{2s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dydx = \frac{\Lambda^2}{\varepsilon^{2s}} \frac{\bar{\omega}_n}{2s} [u]_s^2 < \infty,$$

for every $\varepsilon > 0$. Therefore, $\mathcal{L}_a^\varepsilon u \in L^2(\mathbb{R}^n) \subset H^{-s}(\mathbb{R}^n)$. Consequently, for every $v \in H^s(\mathbb{R}^n)$, we know that

$$\begin{aligned}\langle \mathcal{L}_a^\varepsilon u, v\rangle &= \int_{\mathbb{R}^n} \mathcal{L}_a^\varepsilon u(x) v(x)\, dx = \int_{\mathbb{R}^n} \int_{\{|x-y|\geq\varepsilon\}} a(x,y) \frac{(u(x)-u(y))}{|x-y|^{n+2s}}\, dy\, v(x)\, dx \\ &= \int_{\mathbb{R}^n} \int_{\{|x-y|\geq\varepsilon\}} a(x,y) \frac{(u(x)-u(y))v(x)}{|x-y|^{n+2s}}\, dydx \\ &= \int_{\mathbb{R}^n} \int_{\{|x-y|\geq\varepsilon\}} a(y,x) \frac{(u(y)-u(x))v(y)}{|x-y|^{n+2s}}\, dxdy \\ &= -\int_{\mathbb{R}^n} \int_{\{|x-y|\geq\varepsilon\}} a(x,y) \frac{(u(x)-u(y))v(y)}{|x-y|^{n+2s}}\, dydx,\end{aligned}$$

where we use the symmetry of the kernel a. By summing up the first and the last identities, we obtain

$$\langle \mathcal{L}_a^\varepsilon u, v\rangle = \frac{1}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \chi_{\{|x-y|\geq\varepsilon\}}(x,y)\, dydx$$

for every $v \in H^s(\mathbb{R}^n)$. Let us verify that

$$a(x,y) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \chi_{\{|x-y|\geq\varepsilon\}}(x,y) \in L^1(\mathbb{R}^n\times\mathbb{R}^n).$$

Again, thanks to the boundedness of the kernel a and Hölder's inequality, we get

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left| a(x,y) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \chi_{\{|x-y|\geq\varepsilon\}}(x,y)\right| dydx \leq \Lambda [u]_s^2 [v]_s^2.$$

Now, by Dominated Convergence Theorem, the result (1.2.15) follows. Moreover,

$$|\langle \mathcal{L}_a u, v\rangle| \le \frac{\Lambda}{2}[u]_s^2[v]_s^2$$

for every $u, v \in H^s(\mathbb{R}^n)$.

□

Remark 1.2.9. In the non-symmetric case, one has that

$$\begin{aligned}\langle \mathcal{L}_a u, v\rangle =& \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a_{\text{sym}}(x,y)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy \\ &+ \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_{\text{anti}}(x,y)\frac{(u(x)-u(y))}{|x-y|^{n+2s}}v(x)\,dxdy,\end{aligned}$$

where

$$a_{\text{sym}}(x,y) = \frac{a(x,y)+a(y,x)}{2} \quad \text{and} \quad a_{\text{anti}}(x,y) = \frac{a(x,y)-a(y,x)}{2},$$

denote the symmetric and anti-symmetric parts of a respectively.

In order for this operator to be well defined, one needs to impose some extra condition on the anti-symmetric part a_{anti}. For instance,

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|a_{\text{anti}}(x,y)|^2}{|x-y|^{n+2s}}\,dy < \infty.$$

See [46, 83].

In this thesis, we restrict ourselves to the symmetric case.

1.2.3 The Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure and let $a \in \mathcal{A}_{\lambda,\Lambda}$. Given $f \in H^{-s}(\Omega)$ we define the associated Dirichlet problem as

$$\begin{cases}\mathcal{L}_a u = f & \text{in } \Omega\\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega.\end{cases} \tag{1.2.16}$$

We say that $u \in H_0^s(\Omega)$ is a **weak solution** of (1.2.16) if

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy = \langle f, v\rangle,$$

for every $v \in H_0^s(\Omega)$.

Thanks to (1.2.15), this is equivalent to say that $\mathcal{L}_a u = f$ in the sense of distributions.

To prove existence of weak solution to problems of the form (1.2.16), it would be enough to observe that the left-hand-side defines a coercive continuous bilinear form, thanks to the symmetry of the kernel $a(\cdot,\cdot)$. Therefore, by Lax-Milgram Theorem, we obtain existence and uniqueness. Nevertheless, we decide to apply an alternative technique, which allows dealing with nonlinear problems, as it was shown in [49]. To this aim, first, we establish an equivalence with a minimization problem associated. Secondly, we find a minimum by using calculus of variations. Those are the contents of next Propositions 1.2.10 and 1.2.11.

Proposition 1.2.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure, $0 < \lambda \leq \Lambda < \infty$, $a \in \mathcal{A}_{\lambda,\Lambda}$ and $0 < s < 1$ fixed. Then, for any $f \in H^{-s}(\Omega)$, the following statements are equivalent:*

1. *$u \in H_0^s(\Omega)$ is a weak solution of (1.2.16), where $\mathcal{L}_a$ is defined by (1.2.14).*
2. *$\mathcal{J}(u) = \min_{v\in H_0^s(\Omega)} \mathcal{J}(v)$, where $\mathcal{J}\colon H_0^s(\Omega) \to \mathbb{R}$ is defined by*

$$\mathcal{J}(v) = \frac{1}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dxdy - \langle f, v\rangle. \tag{1.2.17}$$

Proof. The proof is standard.

First, we assume (1). Let $v \in H_0^s(\Omega)$, and use $u - v$ as a test function in the weak formulation of (1.2.16) to obtain

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy + \langle f, u-v\rangle.$$

We now write $a(x,y) = (a(x,y))^{\frac{1}{2}}(a(x,y))^{\frac{1}{2}}$ and apply Young's inequality to the right-hand-side to obtain

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy \leq \mathcal{J}(v) + \frac{1}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy + \langle f, u\rangle,$$

from where it follows that $\mathcal{J}(u) \leq \mathcal{J}(v)$ for every $v \in H_0^s(\Omega)$, which proves (2).

Conversely, now assume (2). Let $t \in \mathbb{R}, v \in H_0^s(\Omega)$ and consider $j(t) = \mathcal{J}(u + tv)$. Then, j attains its minimum at $t = 0$. Therefore, $0 = j'(0)$. That is,

$$0 = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy - \langle f, v\rangle.$$

So, u is the weak solution of (1.2.16). □

Proposition 1.2.11. *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure, $0 < \lambda \leq \Lambda < \infty$, $a \in \mathcal{A}_{\lambda,\Lambda}$ and $0 < s < 1$ fixed. Then, for any $f \in H^{-s}(\Omega)$, there exists a unique $u \in H_0^s(\Omega)$ minimizer of $\mathcal{J}$ over $H_0^s(\Omega)$, where $\mathcal{J}$ is defined by (1.2.17).*

Proof. Clearly, $m := \inf_{H_0^s(\Omega)} \mathcal{J} < +\infty$. We will prove $\mathcal{J}$ is bounded from below.

$$\mathcal{J}(v) \geq \lambda[v]_s^2 - \|f\|_{-s}[v]_s \geq (\lambda - \frac{\varepsilon}{2})[v]_s^2 - \frac{C(\varepsilon)}{2}\|f\|_{-s}^2.$$

Choose $0 < \varepsilon < 2\lambda$, thus, $m \neq -\infty$.

Let $\{u_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ be such that $J(u_k) \to m$, as $k \to \infty$. By the previous inequality, we deduce that $\{u_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ is bounded. Then, since $H_0^s(\Omega)$ is a reflexive space, thanks to Alaoglu's theorem, up to a subsequence, there exists $u \in H_0^s(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $H_0^s(\Omega)$. Thus, by the weak lower semi-continuity of $\mathcal{J}$ (recall that $\mathcal{J}$ is convex), we obtain

$$\mathcal{J}(u) \leq \liminf_{k\to\infty} \mathcal{J}(u_k) = m = \inf_{H_0^s(\Omega)} \mathcal{J}.$$

The uniqueness of the minimizer follows by the strict convexity of $\mathcal{J}$. Suppose $m = \mathcal{J}(u) = \mathcal{J}(v)$, $u \neq v$. Then, $m \leq \mathcal{J}(\frac{u+v}{2}) < \frac{\mathcal{J}(u)}{2} + \frac{\mathcal{J}(v)}{2} = m$, which is a contradiction. □

Propositions 1.2.10 and 1.2.11 trivially imply the following.

Corollary 1.2.12. *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure, $0 < \lambda \leq \Lambda < \infty$, $a \in \mathcal{A}_{\lambda,\Lambda}$ and $0 < s < 1$ fixed. Then, for any $f \in H^{-s}(\Omega)$, there exists a unique weak solution $u \in H_0^s(\Omega)$ to* (1.2.16).

Stability of solution is proved in next Proposition.

Proposition 1.2.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure, $0 < \lambda \leq \Lambda < \infty$, $a \in \mathcal{A}_{\lambda,\Lambda}$ and $0 < s < 1$ fixed. Let $f, g \in H^{-s}(\Omega)$ and u, v be the solutions to*

$$\begin{cases} \mathcal{L}_a u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} \mathcal{L}_a v = g & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then,

$$[u - v]_s \leq C(\lambda)\|f - g\|_{-s}.$$

Moreover, if $f \leq g$ in $H^{-s}(\Omega)$, then $u \leq v$ in $\mathbb{R}^n$.

Proof. Consider $u - v$ as a test function. Thus,

$$\langle \mathcal{L}_a u, u - v\rangle = \langle f, u - v\rangle, \quad \langle \mathcal{L}_a v, u - v\rangle = \langle g, u - v\rangle.$$

Then,

$$\langle \mathcal{L}_a u - \mathcal{L}_a v, u - v\rangle = \langle f - g, u - v\rangle \leq \|f - g\|_{-s}[u - v]_s.$$

On the other hand, we can rewrite $\langle \mathcal{L}_a u - \mathcal{L}_a v, u - v\rangle$ to obtain

$$\langle \mathcal{L}_a u - \mathcal{L}_a v, u - v\rangle \geq \lambda[u - v]^2.$$

Now, suppose $f \leq g$ in $H^{-s}(\mathbb{R}^n)$. Consider $(u - v)^+ \in H_0^s(\Omega)$. By using it as a test function in both problems, we obtain $\langle \mathcal{L}_a(u - v), (u - v)^+\rangle \leq 0$.

Denote by $E := \{x \in \mathbb{R}^n \colon u(x) > v(x)\}$. Then, we can rewrite $\langle \mathcal{L}_a u - \mathcal{L}_a v, (u-v)^+\rangle$ in four terms: $E \times E$, $E \times E^c$, $E^c \times E$ and $E^c \times E^c$. The last term does not contribute, since $(u-v)^+ \equiv 0$ in E^c.

By using that $u(x) - v(x) + v(y) - u(y) \geq 0$ for $x \in E, y \in E^c$, it is deduced that

$$\begin{aligned} 0 \geq \langle \mathcal{L}_a u - \mathcal{L}_a v, (u-v)^+\rangle &\geq \lambda \int_E \int_E \frac{|(u-v)(x) - (u-v)(y)|^2}{|x-y|^{n+2s}}\, dxdy \\ &+ 2\lambda \int_{E^c} \int_E \frac{(u(x) - u(y))(u-v)^+(x)}{|x-y|^{n+2s}}\, dxdy \\ &- 2\lambda \int_{E^c} \int_E \frac{(v(x) - v(y))(u-v)^+(x)}{|x-y|^{n+2s}}\, dxdy \\ &\geq \lambda \int_E \int_E \frac{|(u-v)(x) - (u-v)(y)|^2}{|x-y|^{n+2s}}\, dxdy \geq 0. \end{aligned}$$

From it follows that $(u-v)^+ \equiv 0$ in E, so that $u \leq v$ in $\mathbb{R}^n$. □

Proposition 1.2.14. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $0 \leq f \in H^{-s}(\Omega)$. Let $A \subset B \subset \Omega$ be open sets. Consider u, v the solutions to*

$$\begin{cases} \mathcal{L}_a u = f & \text{in } A, \\ u = 0 & \text{in } \mathbb{R}^n \setminus A, \end{cases} \qquad \text{and} \qquad \begin{cases} \mathcal{L}_a v = f & \text{in } B, \\ v = 0 & \text{in } \mathbb{R}^n \setminus B. \end{cases}$$

Then, $u \leq v$ in $\mathbb{R}^n$.

Proof. By Proposition 1.2.13 and $f \geq 0$, we deduce $u, v \geq 0$.

Since $A \subset B$, we get $H_0^s(A) \subset H_0^s(B)$.

Consider $(u-v)^+ \in H_0^s(A)$. By using it as a test function in both problems, we obtain $\langle \mathcal{L}_a u, (u-v)^+\rangle = \langle \mathcal{L}_a v, u - v\rangle$. Then, $\langle \mathcal{L}_a u - \mathcal{L}_a v, (u-v)^+\rangle = 0$. Now, we proceed in the same way of Proposition 1.2.13, to conclude that $u \leq v$ in $\mathbb{R}^n$.

□

1.3 Fractional Capacities

We would like to start this section making clear, as we have said in the beginning of this chapter, that all the presented results in Chapter 1 are well-known. In this case, we gather some properties of the fractional capacities. The reader could find them in the most general form in [87, 95]. On the other hand, we prove some of the results we did not find in the literature, following straightforwardly those proofs were the case $s = 1$ was studied, for instance, [45]

Definition 1.3.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. Given $A \subset \Omega$, for any $0 < s < 1$, we define the Gagliardo $s-$capacity of A relative to Ω as

$$\mathrm{cap}_s(A, \Omega) = \inf \left\{ [u]_s^2 \colon u \in H_0^s(\Omega),\ u \geq 0,\ u \geq 1 \text{ in a neighborhood of } A \right\}.$$

We give here some basic properties needed in Chapter 3.

Lemma 1.3.2 (Proposition 3.6, [95])**.** *Let $A, B \subset \Omega$. Then,*

$$cap_s(A \cup B, \Omega) + cap_s(A \cap B, \Omega) \leq cap_s(A, \Omega) + cap_s(B, \Omega).$$

Proof. Let $u, v \in H_0^s(\Omega)$ be such that $u, v \geq 0$ and $u \geq 1$ in a neighbourhood of A and $v \geq 1$ in a neighbourhood of B. Consider $\max\{u,v\}, \min\{u,v\} \in H_0^s(\Omega)$. Then, $\max\{u,v\}, \min\{u,v\} \geq 0$ and $\max\{u,v\} \geq 1$ in a neighbourhood of $A \cup B$, $\min\{u,v\} \geq 1$ in a neighbourhood of $A \cap B$. In addition,

$$[\max\{u,v\}]_s^2 + [\min\{u,v\}]_s^2 \leq [u]_s^2 + [v]_s^2, \tag{1.3.1}$$

where easily the result follows. Denote by $w := \max\{u,v\}$ and $z := \min\{u,v\}$. We prove that

$$|w(x) - w(y)|^2 + |z(x) - z(y)|^2 \leq |u(x) - u(y)|^2 + |v(x) - v(y)|^2,$$

for $x, y \in \mathbb{R}^n$. It is clear for $x, y \in \{u \geq v\}$ and $x, y \in \{u < v\}$.

Let $x \in \{u \geq v\}, y \in \{u < v\}$. Then, we get

$$\begin{aligned} |w(x) - w(y)|^2 + |z(x) - z(y)|^2 &= |v(x) - u(y)|^2 + |u(x) - v(y)|^2 \\ &= |u(x) - u(y)|^2 + |v(x) - v(y)|^2 + |v(x) - u(y)|^2 + \\ &+ |u(x) - v(y)|^2 - |u(x) - u(y)|^2 - |v(x) - v(y)|^2 \\ &= |u(x) - u(y)|^2 + |v(x) - v(y)|^2 + 2(v(y) - u(y))(v(x) - u(x)) \\ &\leq |u(x) - u(y)|^2 + |v(x) - v(y)|^2. \end{aligned}$$

Using the estimate above, we conclude (1.3.1). □

Next lemma gives a relation between the Lebesgue measure and the s-capacity of a subset $A \subset \Omega$. The proof is easy and follows [45, Section 4.7, Theorem 2 VI], where it was shown with the classical capacity measure ($s = 1$).

Lemma 1.3.3. *For every $A \subset \Omega$, $|A| \leq C(\Omega, s)\, cap_s(A, \Omega)$, where $C(\Omega, s)$ is the Poincaré's constant in $H_0^s(\Omega)$.*

Proof. For every $\varepsilon > 0$, there exists a funciton $u_\varepsilon \in H_0^s(\Omega)$ such that $u_\varepsilon \geq 1$ a.e. in a neighborhood of A and

$$[u_\varepsilon]_s^2 \leq \mathrm{cap}_s(A, \Omega) + \varepsilon.$$

On the other hand, by Poincaré's inequality,

$$|A| = \int_A 1\, dx \leq \int_{\mathbb{R}^n} u_\varepsilon^2\, dx \leq C(\Omega, s)[u_\varepsilon]_s^2 \leq C(\Omega, s)\left(\mathrm{cap}_s(A, \Omega) + \varepsilon\right).$$

Take the limit $\varepsilon \downarrow 0$ to obtain the result. □

1.3.1 s-Quasi-open sets

Definition 1.3.4. We say that a subset A of Ω is a *s-quasi open* set if there exists a decreasing sequence $\{\omega_k\}_{k\in\mathbb{N}}$ of open subsets of Ω such that $\mathrm{cap}_s(\omega_k, \Omega) \to 0$, as $k \to \infty$, and $A \cup \omega_k$ is an open set for all $k \in \mathbb{N}$.

We denote by $\mathcal{A}_s(\Omega)$ the class of all $s-$quasi open subsets of Ω, that is,

$$\mathcal{A}_s(\Omega) := \{A \subset \Omega \colon A \text{ is } s\text{-quasi open } \}.$$

In the case $s = 1$ the definitions are completely analogous with $\|\nabla u\|_2$ instead of $[u]_s^2$.

Now, we prove a key estimate which is a simply remark following the proof of [40, Proposition 2.2]. We are interested in finding a positive constant connecting in some sense $\mathrm{cap}_s(\cdot, \Omega)$ and $\mathrm{cap}_1(\cdot, \Omega)$. But, we also want that this constant does not depend on s. One of our thesis goals is related to analyze the behavior of some problems in the limit case $s \uparrow 1$. So we can assume $0 < \varepsilon_0 < s < 1$ for some ε_0 and that will be enough to obtain this desired and *independent* constant.

As we said before, the proof of next lemma follows [40, Proposition 2.2] and, despite of the similarity, it is included since we want to analyse how the constant depends on s.

Lemma 1.3.5. *Let $\varepsilon_0 > 0$ and $\varepsilon_0 < s < 1$. Then, there exits a constant $C > 0$ such that for every $u \in H_0^1(\Omega)$*

$$(1 - s)[u]_s^2 \leq C\|\nabla u\|_{L^2(\Omega)}^2.$$

and $C = C(\Omega, n, \varepsilon_0)$ does not depend on s.

Proof. Let $u \in H_0^1(\Omega)$. By Lemma 1.1.2, we get

$$(1 - s)[u]_s^2 \leq \frac{\omega_{n-1}}{2}\left(\|\nabla u\|_{L^2(\Omega)}^2 + 4\frac{1-s}{s}\|u\|_2^2\right).$$

Since $\varepsilon_0 < s < 1$, we obtain

$$(1 - s)[u]_s^2 \leq \left(\frac{\omega_{n-1}}{2} + 2\frac{1-\varepsilon_0}{\varepsilon_0}C_1(\Omega, n)\omega_{n-1}\right)\|\nabla u\|_{L^2(\Omega)}^2 = C(\Omega, n, \varepsilon_0)\|\nabla u\|_{L^2(\Omega)}^2,$$

where $C_1(\Omega, n)$ is the constant of classical Poincaré's inequality in $H_0^1(\Omega)$.

□

Automatically, we obtain an estimate relating the s-capacity and the 1-capacity.

Corollary 1.3.6. *Let $\varepsilon_0 > 0$ and $\varepsilon_0 < s < 1$. Then, there exits a constant $C > 0$ such that for every $A \subset \Omega$*

$$(1 - s)\, cap_s(A, \Omega) \leq C\, cap_1(A, \Omega),$$

and $C = C(\Omega, n, \varepsilon_0)$ does not depend on s.

We deduce other useful remark from Lemma 1.3.5: every 1-quasi open set is also an s-quasi open, for $0 < s < 1$.

Remark 1.3.7. For every $0 < s < 1$, $\mathcal{A}_1(\Omega) \subset \mathcal{A}_s(\Omega)$. Moreover, if $0 < s < t \leq 1$, then $\mathcal{A}_t(\Omega) \subset \mathcal{A}_s(\Omega)$.

Proof. Let $A \in \mathcal{A}_1(\Omega)$. There exists a decreasing sequence of open sets $\{G_k\}_{k\in\mathbb{N}}$ such that $A \cup G_k$ is open and $\mathrm{cap}_1(G_k, \Omega) \to 0$ when $k \to \infty$.

Let $0 < s < 1$. By Corollary 1.3.6, $\mathrm{cap}_s(G_k, \Omega) \to 0$, when $k \to \infty$. Then, $A \in \mathcal{A}_s(\Omega)$.

To prove $\mathcal{A}_t(\Omega) \subset \mathcal{A}_s(\Omega)$, use definitions of capacity and Proposition 1.1.15 for $0 < s < t < 1$, and Lemma 1.1.2 for $0 < s < t = 1$.

□

1.3.2 s-Quasi-continuous functions

Working with s-quasi-continuous functions will be more convenient for solving shape optimization problems in Chapter 3. Let us now introduce the definition and some basic properties. For further properties of the s-capacity see [87, 95].

Definition 1.3.8. Let $u\colon \mathbb{R}^n \to \mathbb{R}$. We say u is an *s-quasi continuous function* if there exists a decreasing sequence $\{E_k\}_{k\in\mathbb{N}}$ of open sets such that $u|_{\mathbb{R}^n\setminus E_k}$ is a continuous function for every $k \in \mathbb{N}$ and $\mathrm{cap}_s(E_k, \Omega) \to 0$, when $k \to \infty$.

The following lemmas address some basic properties of s-quasi continuous functions.

Lemma 1.3.9. *Let $u, v\colon \mathbb{R}^n \to \mathbb{R}$ be s-quasi continuous functions. Then, the product $u \cdot v$ is also an s-quasi continuous function.*

Proof. By definition, there exist decreasing sequences $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ of open sets such that $\lim_{k\to\infty} \mathrm{cap}_s(A_k, \Omega) = \lim_{k\to\infty} \mathrm{cap}_s(B_k, \Omega) = 0$ and $u|_{\mathbb{R}^n\setminus A_k}$, $v|_{\mathbb{R}^n\setminus B_k}$ are continuous.

Consider $C_k := A_k \cup B_k$. Then, $\{C_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of open sets such that $\lim_{k\to\infty} \mathrm{cap}_s(C_k, \Omega) = 0$, since $\mathrm{cap}_s(C_k, \Omega) \leq \mathrm{cap}_s(A_k, \Omega) + \mathrm{cap}_s(B_k, \Omega)$ by Lemma 1.3.2. Moreover, $(u \cdot v)|_{\mathbb{R}^n\setminus C_k}$ is continuous. □

If any s-quasi-continuous function is nonnegative almost everywhere, then it is also nonnegative s-quasi everywhere. This is the content of next proposition.

Proposition 1.3.10. *Let $u \in H^s_0(\Omega)$ be an s-quasi-continuous function such that $u \geq 0$ almost everywhere (a.e.), then $u \geq 0$ s-quasi everywhere (s-q.e.).*

Proof. We have to show $\mathrm{cap}_s(\{u < 0\}, \Omega) = 0$. By definition of s-quasi continuity, there exists a decreasing sequence $\{E_k\}_{k\in\mathbb{N}}$ of open sets such that $\mathrm{cap}_s(E_k, \Omega) \to 0$ when $k \to \infty$, the restriction of $u|_{\mathbb{R}^n\setminus E_k}$ is continuous, and $\{u < 0\} \cup E_k$ is an open set for any $k \in \mathbb{N}$.

Let $\varepsilon > 0$. By $\mathrm{cap}_s(E_k, \Omega)$ definition, there exists a function $v_k \in H^s_0(\Omega)$ such that $v_k \geq 1$ in a neighborhood of E_k and $[v_k]^2_s \leq \mathrm{cap}_s(E_k, \Omega) + \varepsilon$. Since $\mathrm{cap}_s(E_k, \Omega) \to 0$, when $k \to \infty$, we conclude $v_k \to 0$ in $H_0^s(\Omega)$. Moreover, since $|\{u<0\}|=0$, we get $v_k \geq 1$ in a neighborhood of $\{u<0\}\cup E_k$. Therefore,

$$\operatorname{cap}_s(\{u<0\},\Omega) \leq \operatorname{cap}_s(\{u<0\}\cup E_k,\Omega) \leq [v_k]_s^2,$$

which implies $\operatorname{cap}_s(\{u<0\},\Omega)=0$. □

Next Theorem allows us to work with s-quasi-continuous functions instead of the fractional Sobolev functions. We say that every $u \in H_0^s(\Omega)$ has a unique *s-quasi continuous representative* $\tilde{u}$, up to a set of zero $\operatorname{cap}_s(\cdot,\Omega)$.

Theorem 1.3.11. *Let $u \in H_0^s(\Omega)$. Then, there exists an s-quasi-continuous function $\tilde{u}$ such that $u=\tilde{u}$ a. e.. Moreover, $\tilde{u}$ is unique up to a set of zero s-capacity.*

Proof. There exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that $u_k \in C_c^\infty(\Omega), u_k \to u$ in $H_0^s(\Omega)$, a.e., up to a subsequence. Occasionally, taking another subsequence, we may assume that

$$\sum_{k=1}^{\infty} 2^{2k}[u_{k+1}-u_k]_s^2 < \infty.$$

Consider the following open sets:

$$E_j := \{|u_{j+1}-u_j| > 2^{-j}\}; \quad A_k := \cup_{j\geq k} E_j.$$

Since $2^j|u_{j+1}-u_j| \geq 1$ in E_j and this function belongs to $H_0^s(\Omega)$, we are able to estimate the $\operatorname{cap}_s(A_k,\Omega)$:

$$\operatorname{cap}_s(A_k,\Omega) \leq \sum_{j\geq k} \operatorname{cap}_s(E_j,\Omega) \leq \sum_{j\geq k} 2^{2j}[u_{j+1}-u_j]_s^2.$$

Then, $\lim_{k\to\infty}\operatorname{cap}_s(A_k,\Omega)=0$.

Now, let us check that $u|_{\mathbb{R}^n\setminus A_k}$ is a continuous function.

For any $x \in \mathbb{R}^n \setminus A_k$, we have $|u_{j+1}(x)-u_j(x)| \leq 2^{-j}$ for all $j \geq k$. Therefore, for k fixed, the restricted function $u_j|_{\mathbb{R}^n\setminus A_k}$ converges uniformly when j goes to infinity. Denote $\tilde{u}$ the limit function of $\{u_j\}_{j\geq k}$. So, we know that the restricted function $\tilde{u}|_{\mathbb{R}^n\setminus A_k}$ is continuous for any $k \in \mathbb{N}$.

To complete the definition of $\tilde{u}$ in the whole $\mathbb{R}^n$, we extend by zero in $\cap_{k\in\mathbb{N}} A_k$. Since $u_k \to u$ a.e., we conclude that $\tilde{u}$ is a cap_s-representative of u, it is s-quasi-continuous by construction.

Uniqueness is a consequence of Proposition 1.3.10. Indeed, suppose $f=u=g$ a.e., where f and g are s-quasi-continuous functions. Then, $f-g=0=g-f$ a.e. Thus, $f=g$ s-q.e. It means that the s-quasi continuous representative is unique up to a set of zero s-capacity. □

Remark 1.3.12. Observe that the sequence $\{u_k\}_{k\in\mathbb{N}}$ we have built in the previous Theorem 1.3.11 also converges to u s-q.e..

Proposition 1.3.13. *Let $\{u_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ and $u \in H_0^s(\Omega)$ be such that $u_k \to u$ in $H_0^s(\Omega)$. Then, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ such that $\tilde{u}_{k_j} \to \tilde{u}$ s-q.e.*

Proof. Choose a subsequence such that

$$\sum_{k=1}^{\infty} 2^{2k}[u_{k+1} - u_k]_s^2 < \infty$$

and consider the following sets

$$E_j := \{|\tilde{u}_{j+1} - \tilde{u}_j| > 2^{-j}\}; \quad A_k := \bigcup_{j \geq k} E_j.$$

Now, the proof follows by the same argument used in Theorem 1.3.11. □

Remark 1.3.14. From this point, we denote by u the s-quasi continuous representative of a function $u \in H_0^s(\Omega)$, instead of $\tilde{u}$; thanks to Theorem 1.3.11.

Lemma 1.3.15. *Let $u \in H_0^s(\Omega)$. Then, $\{u > a\}$ is s-quasi-open for every $a \in \mathbb{R}$.*

Proof. Since u is s-quasi-continuous, there exists a decreasing sequence $\{E_k\}_{k \in \mathbb{N}}$ of open subsets of Ω such that $\mathrm{cap}_s(E_k, \Omega) \to 0$ when $k \to \infty$, and the restricted function $u|_{\mathbb{R}^n \setminus E_k}$ is continuous for every $k \in \mathbb{N}$.

In particular, $\{u|_{\mathbb{R}^n \setminus E_k} > a\}$ is an open set contained in $\mathbb{R}^n \setminus E_k$.

On the other hand, we know that $\{u > \alpha\} = \{u|_{\mathbb{R}^n \setminus E_k} > a\} \cup \{u|_{E_k} > a\}$. Then, $\{u > \alpha\} \cup E_k = \{u|_{\mathbb{R}^n \setminus E_k} > a\} \cup E_k$, since $\{u|_{E_k} > a\} \subset E_k$. □

1.3.3 A particular s-quasi-open set

For every $A \in \mathcal{A}_s(\Omega)$, we define the associated function space

$$H_0^s(A) := \{u \in H_0^s(\Omega) \colon u = 0 \ s\text{-q.e. in } \mathbb{R}^n \setminus A\}. \tag{1.3.2}$$

Remark 1.3.16. Let $A \in \mathcal{A}_s(\Omega)$ and $f \colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then, $f(v)$ belongs to $H_0^s(A)$, for every $v \in H_0^s(A)$.

Indeed, the following inequality

$$|f(v)(x) - f(v)(y)| \leq \|f\|_{\mathrm{Lip}} |v(x) - v(y)| \tag{1.3.3}$$

implies that $[f(v)]_s \leq \|f\|_{\mathrm{Lip}}[v]_s < \infty$. Moreover, by (1.3.3) and the fact that $v = 0$ s-q.e. in $\mathbb{R}^n \setminus A$, we get that $f(v)$ is equal to a constant s-q.e. in $\mathbb{R}^n \setminus A$. Since $f(v) \in H^s(\mathbb{R}^n)$, it must be $f(v) = 0$ s-q.e. in $\mathbb{R}^n \setminus A$. Therefore, $f(v)$ belongs to $H_0^s(A)$.

Immediately from Remark 1.3.16, we obtain the following basic property of $H_0^s(A)$.

Corollary 1.3.17. *Let $A \in \mathcal{A}_s(\Omega)$. Then, $v^+ = \max\{v, 0\}$ and $v^- = \min\{v, 0\}$ belong to $H_0^s(A)$, for every $v \in H_0^s(A)$.*

Given $A \in \mathcal{A}_s(\Omega)$, we denote by $u_A^s \in H_0^s(A)$ the unique weak solution to

$$(-\Delta)^s u_A^s = 1 \quad \text{in } A, \qquad u_A^s = 0 \quad \text{in } \mathbb{R}^n \setminus A. \tag{1.3.4}$$

Remark 1.3.18. Observe also that u_A^s is the unique minimizer of

$$I_s(u) := \frac{c(n,s)}{2}[u]_s^2 - \int_A u\,dx, \tag{1.3.5}$$

in $H_0^s(A)$.

For every $A \in \mathcal{A}_s(\Omega)$, we will show that $A = \{u_A^s > 0\}$ in the sense of $\text{cap}_s(\cdot, \Omega)$. To prove this aim, we need some previous results which are modifications from [32, Lemma 2.1] and [33, Proposition 5.5].

We want to emphasize that the proof of next lemma is completely analogous to that of [32, Lemma 2.1].

Lemma 1.3.19. *Let $A \in \mathcal{A}_s(\Omega)$. Then, there exists an increasing sequence $\{v_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ of nonnegative functions, such that $\sup_{k\in\mathbb{N}} v_k = 1_A$ s-q.e. on Ω.*

Proof. By definition of s-quasi open set, there exists a decreasing sequence $\{V_j\}_{j\in\mathbb{N}}$ of open subsets of Ω such that $A_j := A \cup V_j$ is an open set for every $j \in \mathbb{N}$, and $\text{cap}_s(V_j, \Omega) < \frac{1}{j}$.

Since A_j is an open set, there exists an increasing sequence $\{\varphi_k^j\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ of nonnegative fuctions such that $\{\varphi_k^j\}_{k\in\mathbb{N}}$ converges to 1_{A_j} a.e. Then, by Proposition 1.3.13, we obtain this convergence holds s-q.e.

On the other hand, since $\text{cap}_s(V_j, \Omega) < \frac{1}{j}$, there exists a function $u_j \in H_0^s(\Omega)$ such that $u_j \geq 0$ s-q.e., $u_j \geq 1$ s-q.e. on V_j, and $[u_j]_s^2 < \frac{1}{j}$. This last condition tells us that $u_j \to 0$ s-q.e. on Ω.

Moreover, $\varphi_k^j \leq 1_{A_j} = 1_{A\cup V_j}$ and $u_j \geq 1$ on V_j, imply that $(\varphi_k^j - u_j)^+ \leq 1_A$ s-q.e. Define

$$0 \leq v_k := \sup_{1\leq j\leq k} (\varphi_k^j - u_j)^+ \in H_0^s(\Omega), \quad \psi := \sup_{k\in\mathbb{N}} v_k.$$

Then, $v_k \uparrow \psi \leq 1_A$ s-q.e. Notice that for every $k \geq j$,

$$\psi \geq v_k \geq (\varphi_k^j - u_j)^+ \geq \varphi_k^j - u_j.$$

Thus, taking the limit $k \to \infty$, we obtain $\psi \geq 1_{A_j} - u_j$. Since $A \subset A_j$, $\psi \geq 1 - u_j$ s-q.e. in A. Taking the limit $j \to \infty$, $\psi \geq 1$ s-q.e. in A. That is $\psi \geq 1_A$ s-q.e. □

We prove a density result in $H_0^s(A)$, for $A \in \mathcal{A}_s(\Omega)$, which is similar to [33, Proposition 5.5].

Lemma 1.3.20. *Let $A \in \mathcal{A}_s(\Omega)$. Then, $\{\varphi u_A^s \colon \varphi \in C_c^\infty(\Omega)\}$ is dense in $H_0^s(A)$.*

Proof. In order to prove the lemma, it is sufficient to see that we can approximate any nonnegative function $w \in H_0^s(A)$ with $(-\Delta)^s w \in L^\infty(\Omega)$, since $L^\infty(\Omega)$ is dense in $H^{-s}(\Omega)$ and $w = w^+ - w^-$. Indeed, for an arbitrary function $w \in H_0^s(\Omega)$, we know that $(-\Delta)^s w =: f \in H^{-s}(\Omega)$.

Denote by $f := (-\Delta)^s w$. Then,

$$(-\Delta)^s w \leq \|f\|_{L^\infty(\Omega)} = \|f\|_{L^\infty(\Omega)}(-\Delta)^s u_A^s \quad \text{in } A.$$

By comparison, we obtain $0 \leq w \leq c u_A^s$, where $c := \|f\|_{L^\infty(\Omega)}$.

For every $\varepsilon > 0$, consider $(w - c\varepsilon)^+ \in H_0^s(\Omega)$. Thus,

$$\{(w - c\varepsilon)^+ > 0\} \subset \{u_A^s > \varepsilon\}. \tag{1.3.6}$$

Notice that $u_A^s \in L^\infty(\Omega)$ by [39, Theorem 4.1]. Observe that, using (1.3.6), $\varepsilon < u_A^s \leq \|u_A^s\|_{L^\infty(\Omega)}$ in $\{(w - c\varepsilon)^+ > 0\}$. Then, the function $\frac{(w-c\varepsilon)^+}{u_A^s}$ belongs to $H_0^s(\Omega)$. So, there exists a sequence $\{\varphi_k^\varepsilon\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_k^\varepsilon \to \frac{(w-c\varepsilon)^+}{u_A^s}$ strongly in $H_0^s(\Omega)$, when $k \to \infty$. Therefore, $\varphi_k^\varepsilon u_A^s \to (w - c\varepsilon)^+$ strongly in $H_0^s(\Omega)$, when $k \to \infty$.

On the other hand, $(w - c\varepsilon)^+ \to w$ strongly in $H_0^s(\Omega)$, when $\varepsilon \downarrow 0$.

Consequently, by a diagonal argument, there exist subsequences $\varepsilon_j \downarrow 0$ and $\{\varphi_{k_j}^{\varepsilon_j}\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_{k_j}^{\varepsilon_j} u_A^s \to w$ strongly in $H_0^s(\Omega)$. □

The following proposition is an essential component to relate domains and functions, and it also contributes to the proofs of the principal results Theorems 3.1.18 and 3.2.11. We can say that is the main outcome of this section.

Proposition 1.3.21. *Let $A \in \mathcal{A}_s(\Omega)$. Then, $A = \{u_A^s > 0\}$ in sense of $cap_s(\cdot, \Omega)$. That is, $cap_s(A \triangle \{u_A^s > 0\}, \Omega) = 0$.*

Proof. It is clear that $u_A^s = 0$ s-q.e. on $\mathbb{R}^n \setminus A$. So, $\{u_A^s > 0\} \subset A$.

To see $A \subset \{u_A^s > 0\}$, we use the previous lemmas.

By Lemma 1.3.19, there exists an increasing sequence $\{v_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ of nonnegative functions, such that $\sup_{k\in\mathbb{N}} v_k = 1_A$ s-q.e. on Ω.

For every v_k, by Lemma 1.3.20, there exists a sequence $\{\varphi_j^k\}_{j\in\mathbb{N}} \in C_c^\infty(\Omega)$ such that $\varphi_j^k u_A^s \to v_k$ strongly in $H_0^s(\Omega)$ and s-q.e., when $j \to \infty$. Since $\varphi_j^k u_A^s = 0$ s-q.e. in $\{u_A^s = 0\}$, then $v_k = 0$ s-q.e. in $\{u_A^s = 0\}$. Therefore, $1_A = 0$ s-q.e. in $\{u_A^s = 0\}$, which implies $A \subset \{u_A^s > 0\}$. □

1.4 Compactness for linear operators

In this section we prove a compactness result for linear operators. This results can be extended to nonlinear monotone operators, as the reader could find in [47]. We restrict ourselves to the linear case in this thesis, so we only need to recall [4, Lemmas 1.3.3 and 1.3.4]. They are crucial in the construction of oscillating test functions (see Lemma 2.3.4).

We now have this compactness result for linear operators.

Proposition 1.4.1 (Lemma 1.3.3, [4])**.** *Let X be a separable reflexive Banach space. Let $S_k \colon X' \to X$ be a sequence of linear continuous operators such that*

$$\|S_k\| = \sup_{\|f\|_{X'}=1} \|S_k f\|_X \leq C,$$

where $0 < C < \infty$ is a constant independent of $k \in \mathbb{N}$. Then there exists a subsequence, still denoted by $\{S_k\}_{k\in\mathbb{N}}$, and a limit linear operator S_0 such that

$$S_k f \rightharpoonup S_0 f \quad \textit{weakly in } X$$

for any $f \in X'$. Moreover,

$$\|S_0\| \leq \liminf_{k\to\infty} \|S_k\|.$$

Proof. Let $\mathcal{D}$ be a dense countable subset of X'. Since $\sup_{k\in\mathbb{N}} \|S_k f\| < \infty$, by a standard diagonal argument, there exists a subsequence, that we still denote by $\{S_k\}_{k\in\mathbb{N}}$ such that

$$S_k f \rightharpoonup S_0 f \quad \text{weakly in } X, \tag{1.4.1}$$

for every $f \in \mathcal{D}$.

This defines an operator $S_0 \colon \mathcal{D} \to X$. Let us first see that S_0 can be extended to X' and that $S_k f \rightharpoonup S_0 f$ for every $f \in X'$. In fact, if $f \in X'$, there exists $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{D}$ such that $f_j \to f$ strongly in X' and then

$$\langle g, S_0 f_j - S_0 f_l\rangle = \langle g, S_0 f_j - S_k f_j\rangle + \langle g, S_k f_j - S_k f_l\rangle + \langle g, S_k f_l - S_0 f_l\rangle,$$

so

$$\begin{aligned} |\langle g, S_0 f_j - S_0 f_l\rangle| \leq & |\langle g, S_0 f_j - S_k f_j\rangle| + |\langle g, S_k f_l - S_0 f_l\rangle| \\ & + \sup_{k\in\mathbb{N}} (|\langle g, S_k f_j - S_k f\rangle| + |\langle g, S_k f_l - S_k f\rangle|) \\ < & |\langle g, S_0 f_j - S_k f_j\rangle| + |\langle g, S_k f_l - S_0 f_l\rangle| + \varepsilon, \end{aligned}$$

if $j, l \geq j_0$. Taking the limit, as $k \to \infty$, on the right-hand-side of the former inequality gives that $\{S_0 f_j\}_{j\in\mathbb{N}} \subset X$ is weakly Cauchy, since (1.4.1). Therefore, there exists a point, that we denote by $S_0 f \in X$ such that

$$S_0 f_j \rightharpoonup S_0 f \quad \text{weakly in } X.$$

A completely analogous argument shows that the limit $S_0 f$ is independent of the sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{D}$ and that $S_k f \rightharpoonup S_0 f$ weakly in X for every $f \in X'$. The operator S_0 is clearly linear. Moreover, by the weak lower semicontinuity of the norm, we deduce $\|S_0\| \leq \liminf_{k\to\infty} \|S_k\|$.

□

Next proposition will be useful in Chapter 2, to prove the existence of some *test functions* needed to deal with the H-convergence of certain class of nonlocal operators. The notation used here maybe is not the simplest, but it has to do with notations used in Chapter 2.

Proposition 1.4.2 (Lemma 1.3.4,[4])**.** *Let X be a separable and reflexive Banach space. Let $\alpha, \beta > 0$ be positive constants. Let $\{\hat{\mathcal{L}}_k\}_{k\in\mathbb{N}}$ be a sequence of linear operators $\hat{\mathcal{L}}_k \colon X \to X'$ such that*

$$\langle \hat{\mathcal{L}}_k v, v\rangle \geq \alpha \|v\|_X^2, \quad \textit{for avery } v \in X, \tag{1.4.2}$$

and

$$\langle \hat{\mathcal{L}}_k^{-1} f, f\rangle \geq \beta \|f\|_{X'}^2, \quad \textit{for every } f \in X'. \tag{1.4.3}$$

Then, there exist a subsequence, still denote by $k \in \mathbb{N}$, and a limit linear operator $\hat{\mathcal{L}}_0 \colon X \to X'$ such that (1.4.2)-(1.4.3) *are satisfied and*

$$\hat{\mathcal{L}}_k^{-1} f \rightharpoonup \hat{\mathcal{L}}_0^{-1} f \quad \textit{weakly in } X.$$

Proof. We would like to start by remarking that (1.4.2) and Lax-Milgram Theorem imply the existence of each $\hat{\mathcal{L}}_k^{-1}$.

Let $f \in X'$ and take $v = \hat{\mathcal{L}}_k^{-1} f$ in (1.4.2). Thus,

$$\langle f, \hat{\mathcal{L}}_k^{-1} f\rangle \geq \alpha \|\hat{\mathcal{L}}_k^{-1} f\|_X^2. \tag{1.4.4}$$

By Chauchy-Schwarz inequality, we deduce

$$\|\hat{\mathcal{L}}_k^{-1} f\|_X \leq \frac{1}{\alpha}\|f\|_{X'}.$$

Now, observe that the dual space X' is also a separable reflexive Banach space. Therefore, we can apply Proposition 1.4.1 to the sequence $\{\hat{\mathcal{L}}_k^{-1}\}_{k\in\mathbb{N}}$. So that, there exist a subsequence, still denoted by $\{\hat{\mathcal{L}}_k^{-1}\}_{k\in\mathbb{N}}$, and a limit linear operator $\hat{\mathcal{L}}_0^{-1}$ such that

$$\hat{\mathcal{L}}_k^{-1} f \rightharpoonup \hat{\mathcal{L}}_0^{-1} f \quad \text{weakly in } X.$$

Moreover,

$$\|\hat{\mathcal{L}}_0^{-1}\| \leq \liminf_{k\to\infty} \|\hat{\mathcal{L}}_k^{-1}\|.$$

Notice that we choose the notation $\hat{\mathcal{L}}_0^{-1}$, but it remains to prove that $\hat{\mathcal{L}}_0^{-1}$ is invertible and its inverse operator $\hat{\mathcal{L}}_0$ satisfies (1.4.2)-(1.4.3), too. So, by taking the limit in (1.4.3), we obtain

$$\langle \hat{\mathcal{L}}_0^{-1} f, f\rangle \geq \beta \|f\|_{X'}^2, \quad \text{for every } f \in X'.$$

That means, jointed with Lax-Milgram Theorem, that $\hat{\mathcal{L}}_0^{-1}$ is invertible. Again, taking the limit in (1.4.4) and by using the lower semicontinuity of the norm, we get

$$\langle f, \hat{\mathcal{L}}_0^{-1} f\rangle \geq \alpha \|\hat{\mathcal{L}}_0^{-1} f\|_X^2.$$

The previous property can be rewritten as (1.4.2) for $\hat{\mathcal{L}}_0$, by replacing $f = \hat{\mathcal{L}}_0 v$, where $v \in X$. □

1.5 Γ-convergence

This notion of convergence was introduced by De Giorgi in the 70s (see [36] and [37]) and has been proved to be an extremely useful tool when dealing with the convergence of variational problems. See, for instance Dal Maso's book [31] for a thorough description of the Γ-convergence and its properties and also Braides' book [16] where many different applications of this notion of convergence are shown.

Let us begin by recalling the definition of Γ-convergence.

Definition 1.5.1. Let X be a metric space and let $J_k\colon X \to \bar{\mathbb{R}}$, $k \geq 0$.

We say that J_k $\Gamma-$converges to J_0 if the following two inequalities hold

(liminf inequality) For every $u \in X$ and every sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ such that $u_k \to u$,

$$J_0(u) \leq \liminf_{k\to\infty} J_k(u_k).$$

(limsup inequality) For every $u \in X$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$, $u_k \to u$ such that

$$J_0(u) \geq \limsup_{k\to\infty} J_k(u_k).$$

Throughout this section, X will be a Hilbert space.

The Γ-convergence is stable under continuous perturbations. This is the content of next lemma.

Lemma 1.5.2. *Let $J_k, J, G\colon X \to (-\infty, \infty]$ be such that $J_k \xrightarrow{\Gamma} J$ in X and G is continuous in X. Then, $J_k + G \xrightarrow{\Gamma} J + G$ in X.*

Proof. It is a straightforward consequence of the definition of Γ-convergence and the continuity of G. Indeed, take $u \in X$ and $\{u_k\}_{k\in\mathbb{N}} \subset X$ such that $u_k \to u$ in X. Then, since G is continuous, $G(u) = \lim_{k\to\infty} G(u_k)$. So we get the liminf inequality:

$$J(u) + G(u) \leq \liminf_{k\to\infty} J_k(u_k) + \lim_{k\to\infty} G(u_k) = \liminf_{k\to\infty} (J_k + G)(u_k).$$

On the other hand, for a fixed $u \in X$ such that $J(u) < \infty$, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ such that $u_k \to u$ in X and $J(u) \geq \limsup_{k\to\infty} J_k(u_k)$. Again, by using the continuity of G, $G(u) = \lim_{k\to\infty} G(u_k)$. So we arrive at the limsup inequality:

$$J(u) + G(u) \geq \limsup_{k\to\infty} (J_k + G)(u_k).$$

Both inequalities give us the desired Γ-convergence from $J_k + G$ to $J + G$. □

The main feature of this notion of convergence is the fact that minimizers of J_k converges to those of J_0.

Theorem 1.5.3 (Corollary 7.20,[31])**.** *For $k \geq 0$, let $J_k \colon X \to (-\infty, \infty]$ be such that $J_k \xrightarrow{\Gamma} J_0$ in X. Let u_k be a minimizer of J_k in X. If u is a cluster point of $\{u_k\}_{k\in\mathbb{N}}$, then u is a minimizer of J_0 in X and*

$$J_0(u) = \limsup_{k\to\infty} J_k(u_k).$$

If $u_k \to u$ in X, then u is a minimizer of J_0 in X and

$$J_0(u) = \lim_{k\to\infty} J_k(u_k).$$

We include here the proof of a weaker version of Theorem 1.5.3 that will be enough for us.

Theorem 1.5.4. *For $k \geq 0$, let $J_k \colon X \to (-\infty, \infty]$ be such that $J_k \xrightarrow{\Gamma} J_0$ in X. Assume that for every $\alpha \in \mathbb{R}$, there exists a compact set $K_\alpha \subset X$ such that*

$$\{v \in X \colon J_k(v) \leq \alpha\} \subset K_\alpha \quad \textit{for every } k \in \mathbb{N}.$$

Then, J_0 attains its minimum value over X and

$$\lim_{k\to\infty} \inf_X J_k = \min_X J_0.$$

Furthermore, if u_k is a minimizer of J_k in X and J_0 has a unique minimizer in X, then,

$$\min_X J_0 = J_0(u) = \limsup_{k\to\infty} J_k(u_k),$$

for every u cluster point of $\{u_k\}_{k\in\mathbb{N}}$.

Proof. First, for every $k \in \mathbb{N}$ there exists $v_k \in X$ such that

$$J_k(v_k) \leq \inf_X J_k + \frac{1}{k}. \tag{1.5.1}$$

Without losing generality, we can assume there exists $w_0 \in X$ such that $J_0(w_0) < \infty$. By Γ-convergence definition, there exists a sequence $\{w_k\}_{k\in\mathbb{N}} \subset X$ such that $w_k \to w_0$ in X and

$$\infty > J_0(w_0) \geq \limsup_{k\to\infty} J_k(w_k).$$

Thus, $\sup_{k\in\mathbb{N}} J_k(w_k) < \infty$. As a consequence,

$$J_k(v_k) \leq \inf_X J_k + \frac{1}{k} \leq J_k(w_k) + 1 \leq \sup_{k\in\mathbb{N}} J_k(w_k) + 1 =: \alpha \in \mathbb{R}.$$

For this α, by hypothesis, there exists a compact set K_α in X such that

$$v_k \in \{v \in X \colon J_k(v) \leq \alpha\} \subset K_\alpha \quad \text{for every } k \in \mathbb{N}.$$

Therefore, there exist a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}} \subset \{v_k\}_{k\in\mathbb{N}}$ and $v_0 \in X$ such that $v_{k_j} \to v_0$ in X. By Γ-convergence definition again and (1.5.1), we know that

$$\inf_X J_0 \leq J_0(v_0) \leq \liminf_{j\to\infty} J_{k_j}(v_{k_j}) \leq \liminf_{k\to\infty} \inf_X J_k. \tag{1.5.2}$$

On the other hand, for every $\varepsilon > 0$ there exists $v^\varepsilon \in X$ such that

$$J_0(v^\varepsilon) \leq \inf_X J_0 + \varepsilon. \tag{1.5.3}$$

By Γ-convergence definition, there exists a sequence $\{v_k^\varepsilon\}_{k\in\mathbb{N}}$ such that $v_k^\varepsilon \to v^\varepsilon$ in X and

$$J_0(v^\varepsilon) \geq \limsup_{k\to infty} J_k(v_k^\varepsilon) \geq \limsup_{k\to\infty} \inf_X J_k.$$

Thanks to (1.5.3) and by taking the limit $\varepsilon \downarrow 0$, we obtain

$$\inf_X J_0 \geq \limsup_{k\to\infty} \inf_X J_k. \tag{1.5.4}$$

From the previous inequality and (1.5.2), we conclude the first part of the theorem.

Now, assume u_k is a minimizer of J_k in X and J has a unique minimizer u_0 in X. Then,

$$\min_X J_k = J_k(u_k), \quad \text{for every } k \geq 0.$$

By the first part of the theorem, we know that

$$\lim_{k\to\infty} J_k(u_k) = J_0(u_0). \tag{1.5.5}$$

We will see that every subsequence $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ admits a sub-subsequence which converges to u_0. Then, the whole sequence $\{u_k\}_{k\in\mathbb{N}}$ converges to u_0.

Fix a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$. Thanks to (1.5.5), there exist $\alpha \in \mathbb{R}$ and K_α a compact set such that $\{u_{k_j}\}_{j\in\mathbb{N}} \subset K_\alpha$. Then, there exists a sub-subsequence $\{u_{k_{j_l}}\}_{l\in\mathbb{N}}$ which converges to a point $z_0 \in K_\alpha$. Then,

$$J_0(z_0) \leq \lim_{l\to\infty} J_{k_{j_l}}(u_{k_{j_l}}) = J_0(u_0) = \min_X J_0.$$

Since J_0 has a unique minimizer, we conclude $z_0 = u_0$ and it ends the proof. □

The next example will be a key element in the following chapters.

Example 1.5.5. Consider $X = L^2(\Omega), Y = H_0^s(\Omega)$ for a fixed $\Omega \subset \mathbb{R}^n$ domain, $0 < s < 1$, and

$$J(v) = \begin{cases} \frac{1}{4} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,y) \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dxdy & \text{if } u \in H_0^s(\Omega), \\ \infty & \text{otherwise ,} \end{cases}$$

where $a \in \mathcal{A}_{\lambda,\Lambda}$ defined in 1.2.13.

If we choose $a \equiv 1$, $J(u) = \frac{1}{4}[u]_s^2$ for every $u \in H_0^s(\Omega)$.

Let $\alpha \in \mathbb{R}$. Notice that

$$0 \leq \frac{\lambda}{4}[u]_s^2 \leq J(v) \leq \alpha.$$

Then, for $\alpha < 0$, we get that $\{J \leq \alpha\} = \emptyset$. On the other hand, for $\alpha \geq 0$, we observe that

$$\{v \in L^2(\Omega)\colon J(v) \leq \alpha\} \subset K_\alpha := \left\{v \in L^2(\Omega)\colon [v]_s^2 \leq \frac{4\alpha}{\lambda}\right\},$$

which is a compact set in $L^2(\Omega)$. Indeed, take a sequence $\{v_k\}_{k\in\mathbb{N}} \subset K_\alpha$. Automatically, $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $H_0^s(\Omega)$. Therefore, there exist a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}} \subset \{v_k\}_{k\in\mathbb{N}}$ and a function $v \in H_0^s(\Omega)$ such that $v_{k_j} \rightharpoonup v$ in $H_0^s(\Omega)$. Thus,

$$[v]_s^2 \leq \liminf_{j\to\infty}[v_{k_j}]_s^2 \leq \frac{4\alpha}{\lambda},$$

that means v belongs to K_α. Moreover, by Theorem 1.1.16, we get that $v_{k_j} \to v$ in $L^2(\Omega)$, taking occasionally another subsequence. Finally, K_α is a compact set in $L^2(\Omega)$.

Definition 1.5.6 (Quadratic Form)**.** A functional $J\colon X \to [0,\infty]$ is a nonnegative quadratic form if there exist a linear subspace $Y \subset X$ and a symmetric bilinear form $B\colon Y \times Y \to \mathbb{R}$ such that

$$J(u) = \begin{cases} B(u,u) & \text{if } u \in Y, \\ \infty & \text{otherwise .} \end{cases}$$

Lemma 1.5.7. *Let $J\colon X \to [0,\infty]$ be an arbitrary functional.*

(1) If J is a quadratic form, then

- *(a) $J(0) = 0$*
- *(b) $J(tu) \leq t^2 J(u)$ for every $u \in X$ and $t > 0$.*
- *(c) $J(u+v) + J(u-v) \leq 2(J(u) + J(v))$ for every $u, v \in X$.*

(2) $J\colon X \to [0,\infty]$ satisfies (a)-(c) if and only if J is a quadratic form.

Proof. Let us start by proving (1). Assume J is a quadratic form, so there exist a linear subspace $Y \subset X$ and a bilinear form $B\colon Y \times Y \to \mathbb{R}$ as in Definition 1.5.6. Condition (a) is clear. Take $u \in X$ and $t > 0$. Then, if $J(u) = \infty$ there is nothing to be proved. So that, suppose $J(u) < \infty$ and $u \in Y \subset X$. Thus, $J(u) = B(u,u)$. Hence, $J(tu) = B(tu,tu) = t^2 B(u,u) = t^2 J(u)$, that is condition (b).

Finally, for $u, v \in X$ such that $J(u), J(v) < \infty$, we know that $J(u) = B(u,u)$ and $J(v) = B(v,v)$. Since Y is a subspace of X, $u \pm v$ belongs to Y. Therefore,

$$\begin{aligned} J(u+v) + J(u-v) &= B(u,u) + 2B(u,v) + B(v,v) + B(u,u) - 2B(u,v) + B(v,v) \\ &= 2(J(u) + J(v)). \end{aligned}$$

Thanks to (1), to prove (2) it is enough to show that if J is an arbitrary function satisfying (a)-(c), then J is a quadratic form. We claim that

(A) $J(tu) = t^2 J(u)$ for every $t > 0$ and $u \in X$.

(B) $J(u+v) + J(u-v) = 2J(u) + 2J(v)$ for every $u, v \in X$.

To these aims, first, let us prove that J is even, that means, $J(u) = J(-u)$. Take $u = 0$ in (c), so that $J(v) + J(-v) \leq 2J(-v)$. Hence, $J(-v) \leq J(v)$. By replacing v by $-v$, $J(v) = J(-v)$. Now, suppose there exist $t_0 > 0, u_0 \in X$ such that $J(t_0 u_0) < t_0^2 J(u_0)$. Then, by condition (a) and the fact that J is even, we get that

$$J(t_0 u_0) < t_0^2 J(u_0) = t_0^2 J(-u_0) \geq J(-t_0 u_0) = J(t_0 u_0),$$

which is a contradiction. We have shown (A).

Let us prove (B). Take $u, v \in X$ and define $w = \frac{u+v}{2}$ and $z = \frac{u-v}{2}$. In this way, $u = w + z$ and $v = w - z$. In addition, by (A), $J(w) = \frac{1}{4} J(u+v)$ and $J(z) = \frac{1}{4} J(u-v)$. Thus, by using (c),

$$J(u) + J(v) = J(w+z) + J(w-z) \leq 2(J(w) + J(z)) = \frac{1}{2} J(u+v) + \frac{1}{2} J(u-v),$$

so, use again condition (c) to conclude (B).

Define $Y := \{u \in X \colon J(u) < \infty\}$ and $B \colon Y \times Y \to \mathbb{R}$ as

$$B(u,v) := \frac{1}{4}(J(u+v) - J(u-v)).$$

Thanks to $(a), (A)$ and (B), Y is a linear subspace of X. From (a) and (A), we obtain $B(u,u) = J(u)$ for every $u \in Y$. The symmetric property of B follows from the fact that J is even.

Let us prove that B is a bilinear form in $Y \times Y$. We split the proof in several steps. First, we will see through a chain of equivalences, the simple identity

$$B(u+v, w) = B(u,w) + B(v,w), \text{ for every } u, v, w \in Y. \tag{1.5.6}$$

After that, to see that we can take out scalars, we will start proving with -1, then with any natural number k, any integer, any rational number, till we arrive at the final step: proving for any real number $t \in \mathbb{R}$.

By B definition, it is equivalent to prove that

$$J(u+v+w) - J(u+v-w) = J(u+w) - J(u-w) + J(v+w) - J(v-w),$$

that can be re-written as

$$J(u+v+w) + J(u-w) + J(v-w) = J(u+v-w) + J(u+w) + J(v+w).$$

Since J is even, $J(u-v+w) = J(-u+v-w)$, hence the identity above is equivalent to

$$J(u+v+w) + J(u-v+w) + J(u-w) + J(v-w) = J(u+v-w) + J(-u+v-w) + J(u+w) + J(v+w).$$

Now, use (B)

$$J(u+v+w)+J(u-v+w)=2J(u+w)+2J(v),$$

$$J(u+v-w)+J(-u+v-w)=2J(u)+2J(v-w).$$

By re-writing and using (B) again, we arrive at

$$J(u)+J(v)+J(w)=J(u)+J(v)+J(w),$$

which is clearly satisfied. This conclude the proof of (1.5.6).

Once again, since J is even, we get $B(0,v)=0$ for every $v\in Y$. Thus, $0=B(u-u,v)=B(u,v)+B(-u,v)$. So,

$$B(-u,v)=-B(u,v),\ \text{for every } u,v\in Y. \tag{1.5.7}$$

Now, by induction, thanks to (1.5.6) we obtain $B(ku,v)=kB(u,v)$ for every $k\in\mathbb{N}$. Since (1.5.7), that also holds for $k\in\mathbb{Z}$. Moreover, replacing u by $\frac{u}{k}$ for $k\in\mathbb{Z}\setminus\{0\}$, we get $B(\frac{u}{k},v)=\frac{1}{k}B(u,v)$. Therefore,

$$B(tu,v)=tB(u,v),\ \text{for every } t\in\mathbb{Q}. \tag{1.5.8}$$

Since B is symmetric, from (1.5.6) and (1.5.8), we know that

$$B(tu+v,tu+v)=t^2B(u,u)+2tB(u,v)+B(v,v).$$

Re-writing, we obtain

$$0\le J(tu+v)\le t^2J(u)+2tB(u,v)+J(v)\ \text{for every } u,v\in Y, t\in\mathbb{Q},$$

hence $B(u,v)^2\le J(u)J(v)$ for every $u,v\in Y$. This implies that

$$\begin{aligned} J(u+v)=B(u+v,u+v)&=B(u,u)+2B(u,v)+B(v,v)\\ &\le J(u)+2J(u)^{\frac12}J(v)^{\frac12}+J(v)\\ &=(J(u)^{\frac12}+J(v)^{\frac12})^2, \end{aligned}$$

so $J(u+v)^{\frac12}\le J(u)^{\frac12}+J(v)^{\frac12}$ for every $u,v\in Y$. From this inequality, (a) and (A), it follows that $J^{\frac12}$ is a seminorm on Y. Thus, for every $u,v\in Y$, the functions $t\mapsto J(tu+v)$ and $t\mapsto J(tu-v)$ are continuous on $\mathbb{R}$. By construction of B, also the function $t\mapsto B(tu,v)$ is continuous on $\mathbb{R}$ for every $u,v\in Y$. Therefore, (1.5.8) implies $B(tu,v)=tB(u,v)$ for every $u,v\in Y$ and $t\in\mathbb{R}$. This identity ends the proof of B being a symmetric bilinear form on $Y\times Y$. $\square$

Proposition 1.5.8. *Let $J_k,J\colon X\to(-\infty,\infty]$ be such that $J_k\xrightarrow{\Gamma}J$ in X and J_k is a non negative quadratic form for every $k\in\mathbb{N}$. Then, J is also a non negative quadratic form.*

Proof. Being a quadratic form is equivalent to satisfy (a)-(c) from the previous Lemma 1.5.7.

So, assume J_k verifies (a)-(c) for every $k \in \mathbb{N}$, and let us see they are also satisfied by J.

To see (a) holds for J, observe that by Γ-convergence definition, for each $u \in X$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ such that

$$J(u) \geq \limsup_{k\to infty} J_k(u_k) \geq 0.$$

For every sequence $\{v_k\}_{k\in\mathbb{N}}$ such that $v_k \to 0$ in X, we know that

$$J(0) \leq \liminf_{k\to\infty} J_k(v_k).$$

Now, choose $v_k = 0$ for every $k \in \mathbb{N}$. Then, since $J_k(0) = 0$, we get $J(0) \leq 0$. But, previously, we observe that $J(0) \geq 0$. Therefore, we have proved condition (a).

Let us continue with condition (b). Fix $u \in X$ and $t > 0$. Take the recovery sequence for u, that is, a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that $u_k \to x$ in X and $J(u) \geq \limsup_{k\to\infty} J_k(u_k)$. Then, $\{tu_k\}_{k\in\mathbb{N}}$ is such that $tu_k \to tu$ in X. We know, thanks to the liminf inequality, that

$$J(tu) \leq \liminf_{k\to\infty} J_k(tu_k) = \liminf_{k\to\infty} t^2(u_k) \leq t^2 \limsup_{k\to\infty} J_k(u_k) \leq t^2 J(u),$$

where we have used property (b) of J_k in the identity above.

It is remained to prove property (c). Fix $u, v \in X$. Consider $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ the recovery sequences for u and v respectively. Then, by using property (c) for J_k and Γ-convergence Definition, we obtain

$$\begin{aligned}
J(u+v) + J(u-v) &\leq \liminf_{k\to\infty} J_k(u_k + v_k) + \liminf_{k\to\infty} J_k(u_k - v_k) \\
&\leq \liminf_{k\to\infty} J_k(u_k + v_k) + J_k(u_k - v_k) \\
&\leq 2 \limsup_{k\to\infty} J_k(u_k) + J_k(v_k) \\
&\leq 2 \limsup_{k\to\infty} J_k(u_k) + 2 \limsup_{k\to\infty} J_k(v_k) \\
&\leq 2(J(u) + J(v))
\end{aligned}$$

Since we have prove (a)-(c), J is a quadratic form too. □

Let $J\colon X \to [0, \infty]$ be a quadratic form.The domain of J is the linear subspace of X:

$$D(J) := \{u \in X \colon J(u) < \infty\}.$$

The bilinear form associated to J is the unique symmetric bilinear form $B\colon D(J) \times D(J) \to \mathbb{R}$ such that $J(u) = B(u, u)$ for every $u \in D(J)$.

Denote by $V := \overline{D(J)}$, the closure of $D(J)$ respect to the norm $\|\cdot\|_X$. The operator L associated to J is the linear operator L defined on

$$D(L) := \{u \in D(J) \colon \exists\, f \in V \text{ such that } B(u, v) = \langle f, v\rangle_X \text{ for every } v \in D(J)\},$$

as $Lu = f$, for every $u \in D(L)$, where $\langle \cdot, \cdot \rangle_X$ denotes the scalar product on X. Observe that the uniqueness of f (so that, the well-definition of L), follows from the density of $D(J)$ in V.

Example 1.5.9. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $X = L^2(\Omega)$ and $J \colon L^2(\Omega) \to [0, \infty]$ be the quadratic form

$$J(u) = \begin{cases} \frac{c(n,s)}{2}[u]_s^2 & \text{if } u \in H_0^s(\Omega), \\ \infty & \text{otherwise,} \end{cases}$$

where $c(n, s)$ is defined in (1.2.2). Thus, $B(u, v) = \langle(-\Delta)^s u, v\rangle$ for every $u, v \in H_0^s(\Omega)$.

Then, the associated linear operator L is the fractional Laplacian, and its domain

$$D(L) = \{u \in H_0^s(\Omega) \colon \exists\, f \in L^2(\Omega) \text{ such that } \langle(-\Delta)^s u, v\rangle = \langle f, v\rangle \text{ for every } v \in H_0^s(\Omega)\}.$$

Notice that, here, $\langle\cdot,\cdot\rangle$ denotes the $L^2(\Omega)$-scalar product.

Let $J \colon X \to [0, \infty]$ be a quadratic form. The scalar product $(\cdot,\cdot)_J$ on $D(J)$ is defined by

$$(u, v)_J := B(u, v) + \langle u, v\rangle_X$$

where B is the bilinear form associated to J. The corresponding norm $\|\cdot\|_J$ is given by

$$\|u\|_J = (J(u) + \|u\|_X^2)^{\frac{1}{2}},$$

for every $u \in D(J)$.

Example 1.5.10. Let J be the same quadratic form from Example 1.5.9. Then, the scalar product $(\cdot,\cdot)_J$ coincides with the scalar product of $H^s(\Omega)$, up to a constant, that is,

$$(u, v)_J = c(n, s) \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\, dxdy + \int_\Omega uv\, dx$$

for every $u, v \in H_0^s(\Omega)$.

Proposition 1.5.11. *Let $J \colon X \to [0, \infty]$ be a lower semicontinuous quadratic form and let L be the associated operator on $V = \overline{D(J)}^{\|\cdot\|_X}$. Then, $D(L)$ is dense in $D(J)$ for the $\|\cdot\|_J$ norm. That is, $\overline{D(L)}^{\|\cdot\|_J} = D(J)$.*

Proof. Since we are dealing with Hilbert spaces, it will be enough to prove that if $v \in D(J)$ is such that $(u, v)_J = 0$ for every $u \in D(L)$, then $v = 0$.

Let $v \in D(J)$ be such that $(u, v)_J = 0$ for every $u \in D(L)$. We have to prove $v = 0$.

Observe that $\langle Lu, v\rangle_X = B(u, v)$ for every $u \in D(L)$, thanks to Riesz Representation Theorem. In particular, by taking $u = v$, we obtain $\langle Lv, v\rangle_X = B(v, v) = J(v) \geq 0$. So, L is a positive operator. Clearly, L is also symmetric. Moreover, L is self-adjoint on V, see [31, Theorem 12.13].

Since L is positive and self-adjoint on V, we know that $Im(Id + L) = V$. So, there exists a $w \in D(L)$ such that $v = w + Lw$. Then,

$$\|v\|_X^2 = \langle v, v\rangle_X = \langle w + Lw, v\rangle_X = \langle w, v\rangle_X + \langle Lw, v\rangle_X = \langle w, v\rangle_X + B(w, v) = (w, v)_J = 0,$$

since $w \in D(J)$ and $v \perp D(L)$. Therefore, $v = 0$ as we wanted to prove. □

Resumen del capítulo

El capítulo abarca distintos conocimientos previos, como los espacios en donde se trabaja, algunas de sus propiedades, desigualdades involucradas, la clase operadores que es objeto de estudio, existencia de solución para el problema de Dirichlet involucrando dichos operadores, principios de comparación entre soluciones, estabilidad, etc. Siempre considerando aquellos resultados que son necasarios para los objetivos de esta tesis.

Este capítulo está dividido en cinco partes.

La primera parte describe los espacios de funciones que intervienen a lo largo de la tesis y sus propiedades básicas. Dichos espacios son los Sobolev fraccionarios y sus respectivos espacios duales. Se hace un estudio detallado de la relación existente entre las normas fraccionarias $\|\cdot\|_{W^{s,p}(\Omega)}$ y la norma de los espacios de Sobolev clásicos $\|\cdot\|_{W^{1,p}(\Omega)}$. Esta relación será clave en los resultados del Capítulo 3, en donde se analiza la transición de un problema de optimización de forma involucrando el laplaciano fraccionario (oparador no local) a un problema en el que intervine el laplaciano clásico (operador local).

En la segunda parte de este capítulo, nos dedicamos a introducir en primer lugar el laplaciano fraccionario, que luego será un caso particular de una clase de oparadores más amplia, utilizada en el Capítulo 2. Probamos existencia de solución, estabilidad, principio de comparación de soluciones. El problema tratado a lo largo de la tesis, es el problema de Dirichlet, que también fue estudiado en esta parte del trabajo.

Como tercera parte, tenemos la sección dedicada a las medidas s-capacitarias. Estas medidas nos permiten relajar ciertos problemas clásicos de diseño óptimo, para obtener resultados positivos de existencia. Se listan las herramientas necesarias para los objetivos de la tesis, como la relación que existe entre las medidas s- capacitarias, asociadas a las semi-normas $[\,\cdot\,]_s$, y la medida clásica 1-capacitaria, asociada a la norma $\|\nabla\cdot\|_{L^2(\Omega)}$. Definimos los conjuntos s-cuasi abiertos y las funciones s-cuasi continuas. También, se prueban algunos resultados de convergencia que relacionan las medidas s-capacitarias con la medida de Lebesgue. Por último, se trabaja con el problema de Dirichlet en un conjunto A s-cuasi abierto:

$$(-\Delta)^s u_A = 1 \quad \text{en } A, \quad u_A = 0 \quad \text{en } \mathbb{R}^n \setminus A.$$

Dada la función solución u_A se prueba que el conjunto A coincide con el conjunto de positividad de la solución u_A, en el sentido de la medida s-capacitaria, es decir, difieren en un conjunto de s-capacidad cero. Es decir, $\{u_A > 0\} = A$. Éste es otro de los resultados clave para lidiar con los problemas de diseño óptimo en el Capítulo 3.

En la cuarta parte, recordamos un resultado de compacidad para una sucesión de operadores lineales.

Finalmente, la quinta parte de este capítulo, abarca un resumen de Γ-convergencia. Su definición y su propiedad esencial que relaciona los mínimos valores y los minimizantes de una sucesión de funcionales Γ-convergente. Esta herramienta es fundamental para las contribuciones originales plasmadas en esta tesis. En el Capítulo 2, la Γ-convergencia de los funcionales de energía asociados a una sucesión de problemas, implica casi automáticamente la convergencia débil de la sucesión de soluciones a la solución del problema límite homogeneizado. Si bien éste no es el resultado principal de dicho capítulo, sienta una base cómoda para lidiar con la convergencia de la sucesión de flujos, preparando el terreno para el objetivo principal que es la H-convergencia. Por otro lado, en el Capítulo 3, la Γ-convergencia nos permite establecar una noción de convergencia de espacios de Sobolev fraccionarios, dada una sucesión de s-cuasi abiertos $\{A_k\}_{k\in\mathbb{N}}$. En cierta forma, $\{H_0^s(A_k)\}_{k\in\mathbb{N}}$ *converge* a $H_0^s(\{u > 0\})$, donde $u_{A_k} \to u$ en $L^2(\Omega)$. Más aún, se obtiene un resultado análogo variando además el parámetro $0 < s < 1$. En ambos casos, es fundamental contar con la Γ-convergencia y las propiedades aquí mencionadas.

Chapter 2

Homogenization for nonlocal diffusion

In this chapter, we give our contribution to Homogenization theory in the nonlocal setting. For the sake of simplicity we decide to present the outcomes in the linear case, since all the difficulties appear also in this situation. We refer the reader for the general case, $1 \leq p < \infty$, to [49].

2.1 A nonlocal div-curl Lemma

In this section we prove a nonlocal version of the div-curl Lemma. This will be a fundamental tool in order to use Tartar's method in homogenization. In the classical setting this lemma was proved by Tartar in [93, 94]. Here we do not need the lemma in its full generality. We prove only a special case that will suffice for our purposes. See [4] where a similar approach is made in the classical setting.

We need to introduce some notation and terminology. Given $u \in H^s(\mathbb{R}^n)$, we define its s-gradient as

$$D_s u(x,y) := \frac{u(x)-u(y)}{|x-y|^{\frac{n}{2}+s}}. \tag{2.1.1}$$

Observe that $D_s u \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, for any $u \in H^s(\mathbb{R}^n)$.

Now, given $\phi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, we define its s-divergence as

$$d_s\phi(x) := \text{p.v.} \int_{\mathbb{R}^n} \frac{\phi(x,y)-\phi(y,x)}{|x-y|^{\frac{n}{2}+s}}\, dy. \tag{2.1.2}$$

With this definitions we have $(-\Delta)^s u = \frac{c(n,s)}{2} d_s(D_s u)$. Moreover, if $\mathcal{L}_a$ is given by (1.2.14), we have $\mathcal{L}_a u = \frac{1}{2} d_s(a D_s u)$.

We now need to check that this s-divergence operator is a well defined operator between

$L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$ and that the following *integration by parts formula* holds

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \phi D_s u \, dxdy = \langle d_s\phi, u\rangle, \tag{2.1.3}$$

for every $u \in H^s(\mathbb{R}^n)$ and $\phi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

In order to keep the computations as simple as possible, the following notations will be used: for $\phi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ we denote

$$\phi = \phi(x,y); \tag{2.1.4}$$

$$\phi' = \phi(y,x); \tag{2.1.5}$$

$$\bar{\phi} = \phi(x,x). \tag{2.1.6}$$

Theorem 2.1.1. *Given $\phi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, it follows that $d_s\phi \in H^{-s}(\mathbb{R}^n)$, where $d_s\phi$ is defined in* (2.1.2)*. Moreover, for any $u \in H^s(\mathbb{R}^n)$ the integration by parts formula* (2.1.3) *holds true.*

Proof. Let us define

$$d_s^\varepsilon\phi(x) := \int_{|x-y|\geq\varepsilon} \frac{\phi(x,y) - \phi(y,x)}{|x-y|^{\frac{n}{2}+s}} \, dy.$$

Then, it is easy to see that $d_s^\varepsilon\phi \in L^2(\mathbb{R}^n)$. In fact,

$$\begin{aligned}
|d_s^\varepsilon\phi(x)| &\leq \int_{|x-y|\geq\varepsilon} \frac{|\phi| + |\phi'|}{|x-y|^{\frac{n}{2}+s}} \, dy \\
&\leq \left(\int_{|x-y|\geq\varepsilon} \frac{1}{|x-y|^{n+2s}} \, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (|\phi| + |\phi'|)^2 \, dy\right)^{\frac{1}{2}} \\
&= \left(\frac{\omega_n}{2s\varepsilon^{2s}}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (|\phi| + |\phi'|)^2 \, dy\right)^{\frac{1}{2}},
\end{aligned}$$

where ω_n is the measure of the unit sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$. From this estimate, one immediately obtains

$$\|d_s^\varepsilon\phi\|_2 \leq 2^{\frac{1}{2}} \left(\frac{\omega_n}{2s\varepsilon^{2s}}\right)^{\frac{1}{2}} \|\phi\|_2.$$

So $d_s^\varepsilon\phi \in L^2(\mathbb{R}^n) \subset H^{-s}(\mathbb{R}^n)$, therefore

$$\begin{aligned}
\langle d_s^\varepsilon\phi, u\rangle &= \int_{\mathbb{R}^n} d_s^\varepsilon\phi u \, dx \\
&= \int_{\mathbb{R}^n} \int_{|x-y|\geq\varepsilon} \frac{\phi - \phi'}{|x-y|^{\frac{n}{2}+s}} u(x) \, dy \, dx \\
&= \int_{\mathbb{R}^n} \int_{|x-y|\geq\varepsilon} \phi \frac{u(x)}{|x-y|^{\frac{n}{2}+s}} \, dy \, dx - \int_{\mathbb{R}^n} \int_{|x-y|\geq\varepsilon} \phi' \frac{u(x)}{|x-y|^{\frac{n}{2}+s}} \, dy \, dx \\
&= \int_{\mathbb{R}^n} \int_{|x-y|\geq\varepsilon} \phi \frac{u(x)}{|x-y|^{\frac{n}{2}+s}} \, dy \, dx - \int_{\mathbb{R}^n} \int_{|x-y|\geq\varepsilon} \phi \frac{u(y)}{|x-y|^{\frac{n}{2}+s}} \, dy \, dx \\
&= \int_{\mathbb{R}^n} \int_{|x-y|\geq\varepsilon} \phi(x,y) D_s u(x,y) \, dy \, dx.
\end{aligned}$$

Now we take the limit $\varepsilon \downarrow 0$ and obtain the desired result. $\square$

The next lemma is a crucial step, but first, we need to introduce a definition.

Definition 2.1.2. Given $\{f_k\}_{k\in\mathbb{N}} \subset H^{-s}(\mathbb{R}^n)$ and $f \in H^{-s}(\mathbb{R}^n)$, we say that $f_k \to f$ in $H^{-s}_{\mathrm{loc}}(\mathbb{R}^n)$ if $\|f_k - f\|_{-s,\Omega} \to 0$ for every $\Omega \subset \mathbb{R}^n$ bounded and open.

Lemma 2.1.3. *Let $\phi_k, \phi_0 \in L^2(\mathbb{R}^n\times\mathbb{R}^n)$ be such that $\phi_k \rightharpoonup \phi_0$ weakly in $L^2(\mathbb{R}^n\times\mathbb{R}^n)$. Assume moreover that $d_s\phi_k \to d_s\phi_0$ strongly in $H^{-s}_{loc}(\mathbb{R}^n)$. Then, for every $\varphi \in H^{1,\infty}(\mathbb{R}^n\times\mathbb{R}^n)$, it follows that $d_s(\varphi\phi_k) \to d_s(\varphi\phi_0)$ strongly in $H^{-s}_{loc}(\mathbb{R}^n)$.*

Proof. In the proof the notations (2.1.4)–(2.1.6) will be used.

Observe, to begin with, that

$$\begin{aligned} d_s(\varphi\phi_k) &= \text{p.v.}\int_{\mathbb{R}^n} \frac{\varphi\phi_k - \varphi'\phi_k'}{|x-y|^{\frac{n}{2}+s}}\,dy \\ &= \bar{\varphi}\,d_s\phi_k + \text{p.v.}\int_{\mathbb{R}^n}\left(\frac{\varphi-\bar{\varphi}}{|x-y|^{\frac{n}{2}+s}}\phi_k + \frac{\bar{\varphi}-\varphi'}{|x-y|^{\frac{n}{2}+s}}\phi_k'\right)dy, \end{aligned}$$

for any $k \geq 0$. Clearly, one has

$$\bar{\varphi}\,d_s\phi_k \to \bar{\varphi}\,d_s\phi_0 \text{ strongly in } H^{-s}_{\mathrm{loc}}(\mathbb{R}^n).$$

We now denote, for $k \geq 0$,

$$\begin{aligned} J_k^1 &:= \text{p.v.}\int_{\mathbb{R}^n} \frac{\varphi-\bar{\varphi}}{|x-y|^{\frac{n}{2}+s}}\phi_k\,dy, \\ J_k^2 &:= \text{p.v.}\int_{\mathbb{R}^n} \frac{\bar{\varphi}-\varphi'}{|x-y|^{\frac{n}{2}+s}}\phi_k'\,dy. \end{aligned}$$

From Theorem 1.1.16, the lemma will be proved if we show that

$$J_k^i \rightharpoonup J_0^i \text{ weakly in } L^2_{\mathrm{loc}}(\mathbb{R}^n),\ i = 1, 2.$$

We prove this fact for $i = 1$, the other case is analogous.

Let $v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $K \subset \mathbb{R}^n$ compact, so

$$\int_K J_k^1 v\,dx = \int_{\mathbb{R}^n} J_k^1 v_K\,dx = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \phi_k \frac{\varphi-\bar{\varphi}}{|x-y|^{\frac{n}{2}+s}} v_K(x)\,dxdy,$$

where $v_K = v\chi_K$. Therefore, it suffices to show that $\frac{\varphi-\bar{\varphi}}{|x-y|^{\frac{n}{2}+s}}v_K(x) \in L^2(\mathbb{R}^n\times\mathbb{R}^n)$. But,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} |v_K(x)|^2 \frac{|\varphi-\bar{\varphi}|^2}{|x-y|^{n+2s}}\,dxdy = \int_K |v(x)|^2 \left(\int_{\mathbb{R}^n} \frac{|\varphi(x,y)-\varphi(x,x)|^2}{|x-y|^{n+2s}}\,dy\right)dx$$

and

$$\int_{\mathbb{R}^n} \frac{|\varphi(x,y)-\varphi(x,x)|^2}{|x-y|^{n+2s}}\,dy = \left(\int_{|x-y|<1} + \int_{|x-y|\geq 1}\right) \frac{|\varphi(x,y)-\varphi(x,x)|^2}{|x-y|^{n+2s}}\,dy$$
$$= I + II.$$

For I observe that $|\varphi(x,y)-\varphi(x,x)| \leq \|\nabla\varphi\|_\infty |x-y|$ and so

$$I \leq \|\nabla\varphi\|_\infty^2 \int_{|x-y|<1} \frac{1}{|x-y|^{n+2s-2}}\,dy = \frac{\omega_n}{2(1-s)}\|\nabla\varphi\|_\infty^2,$$

where ω_n is the measure of the unit sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$. Finally, for II,

$$II \leq 4\|\varphi\|_\infty^2 \int_{|x-y|\geq 1} \frac{1}{|x-y|^{n+2s}}\,dy = \frac{4\omega_n}{2s}\|\varphi\|_\infty^2.$$

This completes the proof of the lemma. □

Now we are in position to prove the main result of the section.

Lemma 2.1.4 (Nonlocal Div-Curl Lemma). *Let $\phi_k, \phi_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and let $v_k, v_0 \in H^s(\mathbb{R}^n)$ be such that*

$$\begin{cases} v_k \rightharpoonup v_0 & \text{weakly in } H^s(\mathbb{R}^n), \\ \phi_k \rightharpoonup \phi_0 & \text{weakly in } L^2(\mathbb{R}^n \times \mathbb{R}^n), \\ d_s\phi_k \to d_s\phi_0 & \text{strongly in } H^{-s}_{loc}(\mathbb{R}^n). \end{cases}$$

Then, $\phi_k D_s v_k \to \phi_0 D_s v_0$ in the sense of distributions.

Remark 2.1.5. In this special version of the div-curl Lemma, we are considering $\psi_k = D_s v_k$. In this case, since ψ_k are s-gradients of scalar functions, there is no need for the introduction of the s-curl operator.

Proof. The proof is an easy consequence of the previous lemma. In fact, if $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, from Lemma 2.1.3 and the integration by parts formula (2.1.3) we get

$$\begin{aligned} \lim_{k\to\infty} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \phi_k D_s v_k \varphi\,dxdy &= \lim_{k\to\infty} \langle d_s(\varphi\phi_k), v_k\rangle \\ &= \langle d_s(\varphi\phi_0), v_0\rangle \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \phi_0 D_s v_0 \varphi\,dxdy. \end{aligned}$$

The proof is complete. □

2.2 Pave the way for the H-convergence with a Γ-convergence result

Let $0 < \lambda \leq \Lambda < \infty$ and $\{a_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\lambda,\Lambda}$ (defined by (1.2.13)) be a sequence of positive and bounded symmetric kernels.

Since $0 < \lambda \leq a_k \leq \Lambda$ for every $k \in \mathbb{N}$, up to a subsequence, we can assume that $a_k \overset{*}{\rightharpoonup} a_0$ in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, that is,

$$\lim_{k\to\infty} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k(x,y)g(x,y)\,dxdy = \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_0(x,y)g(x,y)\,dxdy,$$

for every $g \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

We denote the associated nonlocal operators $\mathcal{L}_k := \mathcal{L}_{a_k}$, given by (1.2.14). Then, the sequence $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ define a sequence of energy functionals $\{\mathcal{J}_k\}_{k\in\mathbb{N}}$, given by

$$\mathcal{J}_k(v) = \frac{1}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k(x,y)\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dxdy \tag{2.2.1}$$

for $k \in \mathbb{N}$, defined in $H_0^s(\Omega)$. Thanks to the definition of s-gradient (2.1.1), we can rewrite the functional $\mathcal{J}_k$ as follows

$$\mathcal{J}_k(v) = \frac{1}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k(x,y)(D_s v(x,y))^2\,dxdy.$$

We then define, for $k \in \mathbb{N}$, $J_k \colon L^2(\Omega) \to (-\infty,\infty]$ as

$$J_k(v) := \begin{cases} \mathcal{J}_k(v) & \text{if } v \in H_0^s(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2.2}$$

Theorem 2.2.1. *Let* $0 < \lambda \leq \Lambda < \infty$ *and* $\{a_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\lambda,\Lambda}$. *Let* $\mathcal{L}_k := \mathcal{L}_{a_k}$ *be the operators defined in* (1.2.14).

Then, the associated functionals J_k *given by* (2.2.2) $\Gamma-$*converge to* J_0 *in* $L^2(\Omega)$.

Proof. **Liminf inequality**. Let $\{u_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$ be such that $u_k \to u$ in $L^2(\Omega)$. We want to prove that

$$J(u) \leq \liminf_{k\to\infty} J_k(u_k).$$

Assume $\liminf_{k\to\infty} J_k(u_k) < \infty$. Otherwise, the inequality is trivial. Up to a subsequence, we can also assume $u_k \in H_0^s(\Omega)$ for every $k \in \mathbb{N}$. In addition, by the uniform boundedness of the sequence of kernels, there exists a weak limit function in $H_0^s(\Omega)$. Since $u_k \to u$ in $L^2(\Omega)$, this weak limit function should be u.

Let $0 < \delta < R < \infty$. Consider

$$Q_{R,\delta} := B_R(0) \times B_R(0) \setminus \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \colon |x-y| < \delta\}.$$

Observe that $(D_s u_k)^2 \to (D_s u)^2$ in $L^1(Q_{R,\delta})$. See the definition of s-gradient in (2.1.1).

Using the strong convergence in $L^1(Q_{R,\delta})$ and the weak* convergence of the kernels, we obtain

$$\begin{aligned}\liminf_{k\to\infty} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k (D_s u_k)^2\, dxdy &\geq \liminf_{k\to\infty} \iint_{Q_{R,\delta}} a_k (D_s u_k)^2\, dxdy \\ &\geq \iint_{Q_{R,\delta}} a_0 (D_s u)^2\, dxdy\end{aligned}$$

To finish the liminf inequality, take the limit $R \uparrow \infty$ and $\delta \downarrow 0$. Consequently,

$$\liminf_{k\to\infty} J_k(u_k) = \liminf_{k\to\infty} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k (D_s u_k)^2\, dxdy \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_0 (D_s u)^2\, dxdy = J_0(u).$$

Limsup inequality. Let $u \in L^2(\Omega)$ be such that $J(u) < \infty$. We want to find a recovery sequence $\{u_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$. That means, $u_k \to u$ in $L^2(\Omega)$ and

$$\limsup_{k\to\infty} J_k(u_k) \leq J(u).$$

Notice that taking the constant sequence u will be enough. Since $J(u) < \infty$, the function u belongs to $H_0^s(\Omega)$. Then, we get $(D_s u)^2 \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Using the convergence $a_k \overset{*}{\rightharpoonup} a_0$, we conclude

$$\limsup_{k\to\infty} J_k(u) = \limsup_{k\to\infty} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k (D_s u)^2\, dxdy = \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_0 (D_s u)^2\, dxdy = J(u).$$

□

As an easy consequence of this Γ-convergence in $L^2(\Omega)$, we prove the $H_0^s(\Omega)$-weak convergence of $\{u_k\}_{k\in\mathbb{N}}$ the sequence of solutions to

$$\begin{cases} \mathcal{L}_k u_k = f & \text{in } \Omega \\ u_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.2.3}$$

to a function which is a solution to *the limit problem*, where $\mathcal{L}_k = \mathcal{L}_{a_k}$ defined by (1.2.14). But, first, we need this lemma which guarantees the existence of a weak limit function, so that it is still remained to show that it solves *the limit problem*.

Lemma 2.2.2. *Let $\{u_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ be the sequence of weak solutions to* (2.2.3)*. Then $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $H_0^s(\Omega)$ and therefore, up to some subsequence, there exists $u_0 \in H_0^s(\Omega)$ such that $u_k \rightharpoonup u_0$ weakly in $H_0^s(\Omega)$.*

Proof. The proof is straightforward. In fact, from the properties of the kernel a_k, we have

$$\begin{aligned}\lambda[u_k]_s^2 = \lambda\|D_s u_k\|_2^2 &\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k(x,y)|D_s u_k(x,y)|^2\,dxdy \\ &= 2\langle \mathcal{L}_k u_k, u_k\rangle \\ &= 2\langle f, u_k\rangle \\ &\leq 2\|f\|_{-s}\|D_s u_k\|_2 = 2\|f\|_{-s}[u_k]_s.\end{aligned}$$

Therefore

$$[u_k]_s \leq (2\lambda^{-1}\|f\|_{-s}).$$

From this uniform bound, the rest of the lemma follows. □

Now, it is time to see the weak convergence of the *solution sequence* to a function which is also a solution to the same class of problem.

Corollary 2.2.3. *Let* $0 < \lambda \leq \Lambda$, $\{a_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\lambda,\Lambda}$ *and* $f \in H^{-s}(\Omega)$. *Assume* $a_k \overset{*}{\rightharpoonup} a_0$ *in* $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. *For every* $k \geq 0$, *denote by* u_k *the solution to* (2.2.3), *where* $\mathcal{L}_k = \mathcal{L}_{a_k}$ *defined by* (1.2.14). *Then,* $u_k \rightharpoonup u_0$ *in* $H_0^s(\Omega)$.

Proof. For every $k \geq 0$, by Proposition 1.2.10, $u_k \in L^2(\Omega)$ is also the weak solution to

$$J_k(u_k) - \langle f, u_k\rangle = \inf_{v\in L^2(\Omega)} J_k(v) - \langle f, v\rangle.$$

By Theorem 2.2.1, we know that $J_k \xrightarrow{\Gamma} J_0$ in $L^2(\Omega)$. On the other hand, observe that $v \mapsto \langle f, v\rangle$ is a continuous function in $L^2(\Omega)$. Then, since Γ-convergence is stable under continuous perturbations, we obtain $J_k(\cdot) - \langle f,\cdot\rangle \xrightarrow{\Gamma} J_0(\cdot) - \langle f,\cdot\rangle$ in $L^2(\Omega)$.

By Theorem 1.5.3, the sequence of minimizers $\{u_k\}_{k\in\mathbb{N}}$ converges to u_0 in $L^2(\Omega)$. Moreover, by Lemma 2.2.2, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $H_0^s(\Omega)$, so there exist a subsequence and a weak limit function in $H_0^s(\Omega)$, it should be u_0. Therefore, $u_k \rightharpoonup u_0$ in $H_0^s(\Omega)$.

□

2.3 The highly anticipated H-convergence

Now, let $\{a_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\lambda,\Lambda}$ be a sequence of positive and bounded kernels and let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure. We denote the associated nonlocal operators $\mathcal{L}_k := \mathcal{L}_{a_k}$, given by (1.2.14).

Now, given $f \in H^{-s}(\Omega)$ we denote by $u_k \in H_0^s(\Omega)$ the unique weak solution to (2.2.3). Until now, we know the existence of a subsequence (that we still denote by $\{u_k\}_{k\in\mathbb{N}}$), a function $u_0 \in H_0^s(\Omega)$ and a positive bounded kernel $a_0 \in \mathcal{A}_{\lambda_0,\Lambda_0}$ such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } H_0^s(\Omega)$$

and u_0 is a weak solution to (2.2.3) with $\mathcal{L}_0 = \mathcal{L}_{a_0}$. BUT, we are more ambitious. We want to arrive at the flow convergence, that is,

$$a_k D_s u_k \rightharpoonup a_0 D_s u_0 \quad \text{weakly in } L^2(\Omega).$$

Both of the previous convergences come to the H-convergence definition.

Definition 2.3.1. For any $k \geq 0$ let $0 < \lambda_k \leq \Lambda_k < \infty$ and let $a_k \in \mathcal{A}_{\lambda_k,\Lambda_k}$ be a sequence of kernels. Let us denote by $\mathcal{L}_k$, $k \geq 0$, the associated nonlocal operators given by (1.2.14) with $a = a_k$ respectively.

We say that the sequence $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ H-converges to $\mathcal{L}_0$, if for any $f \in H^{-s}(\Omega)$, the sequence of solutions $\{u_k\}_{k\in\mathbb{N}}$ of

$$\begin{cases} \mathcal{L}_k u_k = f & \text{in } \Omega \\ u_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.3.1}$$

satisfies

$$\begin{aligned} u_k &\rightharpoonup u_0 && \text{weakly in } H_0^s(\Omega) \\ a_k D_s u_k &\rightharpoonup a_0 D_s u_0 && \text{weakly in } L^2(\Omega) \end{aligned}$$

where u_0 is the solution to

$$\begin{cases} \mathcal{L}_0 u_0 = f & \text{in } \Omega \\ u_0 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.3.2}$$

As we have said in the Introduction, this notion of convergence was introduced by Murat and Tartar in [70] generalizing the notion of G-convergences for symmetric operators given by Spagnolo in [91, 92] and De Giorgi and Spagnolo in [35]. All of the above mentioned papers work in the context of linear elliptic PDEs.

As far as we know, this is the first time that this notion is applied to the nonlocal context.

We start with a simple lemmas which ensures us the existence of a $L^2(\mathbb{R}^n \times \mathbb{R}^n)$-weak limit function for the sequence of associated fluxes $\{a_k D_s u_k\}_{k\in\mathbb{N}}$.

Lemma 2.3.2. *Let $\{u_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ be the sequence of weak solutions to* (2.2.3)*. Then the sequence of fluxes $\{\xi_k := a_k D_s u_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is bounded and therefore, up to some subsequence, there exists $\xi_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\xi_k \rightharpoonup \xi_0$ weakly in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. The proof is also straightforward. In fact, from the boundedness of the kernels $\{a_k\}_{k\in\mathbb{N}}$ and from Lemma 2.2.2, we have

$$\begin{aligned} \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\xi_k|^2\,dxdy &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} |a_k D_s u_k|^2\,dxdy \\ &\leq \Lambda^2 \iint_{\mathbb{R}^n\times\mathbb{R}^n} |D_s u_k|^2\,dxdy \\ &\leq (2\Lambda\lambda^{-1})^2 \|f\|_{-s}^2. \end{aligned}$$

The proof is complete. □

The following observation is trivial.

Proposition 2.3.3. *The sequence of operators $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ is uniformly strictly monotone.*

Proof. The proof follows immediately from the operator definition (1.2.14) and the uniform estimate $\lambda \le a_k(x,y) \le \Lambda$ a.e. $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$\langle \mathcal{L}_k u - \mathcal{L}_k v, u - v\rangle \ge \lambda [u-v]_s^2,$$

for every $u, v \in H_0^s(\Omega)$. □

The oscillating test function method of Tartar needs the existence of such test functions. This is the content of next lemma.

Lemma 2.3.4. *Given a sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\lambda,\Lambda}$ and a function $w_0 \in H^s(\mathbb{R}^n)$, there exist a sequence $\{w_k\}_{k\in\mathbb{N}} \subset H^s(\mathbb{R}^n)$ and $g_0 \in H^{-s}(\mathbb{R}^n)$ such that*

$$w_k \rightharpoonup w_0 \quad \textit{weakly in } H^s(\mathbb{R}^n) \tag{2.3.3}$$

$$g_k := \mathcal{L}_k w_k \to g_0 \quad \textit{strongly in } H^{-s}_{loc}(\mathbb{R}^n). \tag{2.3.4}$$

Proof. First, observe that the operators $\mathcal{L}_k \colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ verify the following estimates:

$$\|\mathcal{L}_k u\|_{-s} \le \frac{\Lambda}{2}[u]_s, \tag{2.3.5}$$

$$\langle \mathcal{L}_k u, u\rangle \ge \frac{\lambda}{2}[u]_s^2. \tag{2.3.6}$$

These estimates follow easily from the definitions and Hölder's inequality.

Now, we define the operator $\hat{\mathcal{L}}_k \colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ by $\hat{\mathcal{L}}_k u = \mathcal{L}_k u + u$. From (2.3.5) and (2.3.6), it follows that $\hat{\mathcal{L}}_k$ verifies the estimates

$$\|\hat{\mathcal{L}}_k u\|_{-s} \le \max\left\{\frac{\Lambda}{2}; 1\right\} \|u\|_s, \tag{2.3.7}$$

$$\langle \hat{\mathcal{L}}_k u, u\rangle \ge \min\left\{\frac{\lambda}{2}; 1\right\} \|u\|_s^2. \tag{2.3.8}$$

Proposition 2.3.3 implies the monotonicity of $\hat{\mathcal{L}}_k$. Observe that $\hat{\mathcal{L}}_k$ is continuous on finite-dimensional subspaces of $H^s(\mathbb{R}^n)$, therefore, by (2.3.8) and Lax-Milgram Theorem , $\hat{\mathcal{L}}_k$ admits an inverse, $\hat{\mathcal{L}}_k^{-1}$.

Let us check that the family of operators $\{\hat{\mathcal{L}}_k^{-1}\}_{k\in\mathbb{N}}$ fulfills the hypotheses of Proposition 1.4.1. The operators $\hat{\mathcal{L}}_k^{-1}$ are uniformly strictly monotone since are the inverse of the sequence of uniformly strictly monotone operators $\{\hat{\mathcal{L}}_k\}_{k\in\mathbb{N}}$.

Observe that from (2.3.7) and (2.3.8) one immediately obtains

$$\langle \hat{\mathcal{L}}_k u, u\rangle \ge c\|\hat{\mathcal{L}}_k u\|_{-s}^2, \tag{2.3.9}$$

where $c := \frac{\min\left\{\frac{\lambda}{2};1\right\}}{\left(\min\left\{\frac{\Lambda}{2};1\right\}\right)^2} = c(\lambda, \Lambda)$, which can be written as

$$\langle f, \hat{\mathcal{L}}_k^{-1} f\rangle \geq c\|f\|_{-s}^2$$

for every $f \in H^{-s}(\mathbb{R}^n)$. Consequently, $\{\hat{\mathcal{L}}_k^{-1}\}_{k\in\mathbb{N}}$ is uniformly coercive.

From (2.3.8) it follows that

$$c\|u\|_s^2 \leq \|\hat{\mathcal{L}}_k u\|_{-s}\|u\|_s$$

where $c = \min\left\{\frac{\lambda}{2}; 1\right\}$, that is,

$$\|\hat{\mathcal{L}}_k^{-1} f\|_s \leq c^{-1}\|f\|_{-s}.$$

Since c is independent on k, it follows that $\sup_{k\in\mathbb{N}} \|\hat{\mathcal{L}}_k^{-1} f\|_s < \infty$.

Then, by Proposition 1.4.1, there exist a subsequence of operators, that we still denote by $\{\hat{\mathcal{L}}_k^{-1}\}_{k\in\mathbb{N}}$, and a limit linear uniformly coercive operator $\hat{\mathcal{L}}_0^{-1}\colon H^{-s}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, such that

$$\hat{\mathcal{L}}_k^{-1} f \rightharpoonup \hat{\mathcal{L}}_0^{-1} f \quad \text{weakly in } H^s(\mathbb{R}^n) \text{ for every } f \in H^{-s}(\mathbb{R}^n). \tag{2.3.10}$$

Since $\hat{\mathcal{L}}_0^{-1}$ is continuous on finite subspaces of $H^{-s}(\mathbb{R}^n)$, again, by Lax-Milgram Theorem, $\hat{\mathcal{L}}_0^{-1}$ is invertible, that is, there exists a linear continuous operator $\hat{\mathcal{L}}_0\colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$. Observe that $\hat{\mathcal{L}}_0$ satisfies (2.3.7) and (2.3.8).

Consider $\hat{g}_0 := \hat{\mathcal{L}}_0 w_0 \in H^{-s}(\mathbb{R}^n)$ and define $w_k := \hat{\mathcal{L}}_k^{-1}\hat{g}_0 \in H^s(\mathbb{R}^n)$. Thus, by (2.3.10) we obtain that $w_k \rightharpoonup w_0$ in $H^s(\mathbb{R}^n)$.

Finally, if we denote $g_k := \mathcal{L}_k w_k$, we obtain that

$$g_k = \mathcal{L}_k w_k = \hat{\mathcal{L}}_k w_k - w_k = \hat{g}_0 - w_k.$$

Since $w_k \rightharpoonup w_0$ weakly in $H^s(\mathbb{R}^n)$ it follows that $w_k \to w_0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^n)$, therefore

$$g_k \to \hat{g}_0 - w_0 =: g_0 \text{ strongly in } H^{-s}_{\text{loc}}(\mathbb{R}^n).$$

The proof is complete. □

With all of these preliminaries, we are ready to prove the main result of this section.

Theorem 2.3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure and let $0 < \lambda \leq \Lambda < \infty$. Then, for any sequence $\{a_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\lambda,\Lambda}$, there exists subsequence $\{a_{k_j}\}_{j\in\mathbb{N}} \subset \{a_k\}_{k\in\mathbb{N}}$ and a kernel $a_0 \in \mathcal{A}_{\lambda,\frac{\Lambda^2}{\lambda}}$ such that the sequence of operators $\{\mathcal{L}_{k_j}\}_{j\in\mathbb{N}}$, H-converges to $\mathcal{L}_0$.*

Proof. Consider $w_0(x) = e^{-|x|^2} \in H^s(\mathbb{R}^n)$ and let $\{w_k\}_{k\in\mathbb{N}} \subset H^s(\mathbb{R}^n)$ be the sequence given by Lemma 2.3.4.

Let us denote by $\eta_k = a_k D_s w_k$ and observe that from (2.3.3) and the boundedness of the kernels a_k it follows that

$$\|\eta_k\|_2 \leq \Lambda\|D_s w_k\|_2 = \Lambda[w_k]_s \leq C.$$

Then, there exists a function $\eta_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that, up to a subsequence,

$$\eta_k \rightharpoonup \eta_0 \quad \text{weakly in } L^2(\mathbb{R}^n \times \mathbb{R}^n).$$

Given $\theta \in \mathbb{R}$, we apply Lemma 2.1.4 to the following nonnegative quantity

$$(\xi_k - \theta\eta_k)(D_s u_k - \theta D_s w_k) \geq 0,$$

where, as in Lemma 2.3.2, we note $\xi_k(x, y) = a_k(x, y) D_s u_k(x, y)$.

Therefore,

$$(\xi_k - \theta\eta_k)(D_s u_k - \theta D_s w_k) \to (\xi_0 - \theta\eta_0)(D_s u_0 - \theta D_s w_0) \geq 0, \tag{2.3.11}$$

in the sense of distributions.

Take now $\theta = \theta_t = \frac{(u_0(x) - u_0(y)) - t\theta_0}{w_0(x) - w_0(y)}$, where $\theta_0 \in \mathbb{R}$ and $t > 0$. Observe that θ_t is well defined a.e. in $\mathbb{R}^n \times \mathbb{R}^n$. Therefore, by (2.3.11) we obtain

$$(\xi_0 - \theta_t\eta_0)\theta_0 \geq 0.$$

Since $\theta_0 \in \mathbb{R}$ is arbitrary, we conclude that

$$\xi_0 - \theta_t\eta_0 = 0,$$

for every $t > 0$. Passing to the limit $t \downarrow 0$, we get

$$\xi_0 = \theta_u\eta_0, \tag{2.3.12}$$

where $\theta_u = \frac{u_0(x) - u_0(y)}{w_0(x) - w_0(y)}$.

Now, we obtain

$$\xi_0(x, y) = a_0(x, y) D_s u_0(x, y), \tag{2.3.13}$$

where $a_0(x, y) := \frac{\eta_0(x,y)}{D_s w_0(x,y)}$.

Finally, observe that from (2.2.3) and Lemma 2.3.2, it follows that

$$\frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \xi_0 D_s v \, dxdy = \langle f, v \rangle,$$

for every $v \in H_0^s(\Omega)$. But, by (2.3.13)

$$\xi_0 D_s v = a_0 D_s u_0 D_s v,$$

then, u_0 is the weak solution of (2.3.2).

To conclude the proof of the theorem, it remains to show that $a_0 \in \mathcal{A}_{\lambda, \frac{\Lambda^2}{\lambda}}$, but this is the content of Proposition 2.3.6 that we prove next. □

The next proposition shows the coercivity and boundedness of the homogenized kernel a_0.

Proposition 2.3.6. *Under the same assumptions and notations of Theorem 2.3.5, the homogenized kernel a_0 belongs to the class $\mathcal{A}_{\lambda,\frac{\Lambda^2}{\lambda}}$.*

Proof. First, we prove the boundedness from below $a_0(x,y) \geq \lambda$, a.e. $x, y \in \mathbb{R}^n$. Fix $v_0 \in H^s(\mathbb{R}^n)$ (for instance $v_0(x) = e^{-|x|^2}$) and denote by v_k the solution of

$$\begin{cases} \mathcal{L}_k v_k = \mathcal{L}_0 v_0 & \text{in } \Omega \\ v_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.3.14}$$

By Lemma 2.2.2, $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $H_0^s(\Omega)$. Then, it has a weak limit in $H_0^s(\Omega)$. But, by Theorem 2.3.5, that limit is v_0. Applying the nonlocal div-curl Lemma, Lemma 2.1.4, to the sequences $\{a_k D_s v_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$, we obtain

$$a_k |D_s v_k|^2 \to a_0 |D_s v_0|^2, \tag{2.3.15}$$

in the sense of distributions.

Since $a_k \in \mathcal{A}_{\lambda,\Lambda}$,

$$\lambda \iint_{\mathbb{R}^n\times\mathbb{R}^n} |D_s v_k|^2 \varphi \, dxdy \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k |D_s v_k|^2 \varphi \, dxdy,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\varphi \geq 0$.

Therefore, from (2.3.15) and since the left hand side is weak lower semi-continuous in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, we obtain

$$\lambda \iint_{\mathbb{R}^n\times\mathbb{R}^n} |D_s v_0|^2 \varphi \, dxdy \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_0 |D_s v_0|^2 \varphi \, dxdy.$$

Since $0 \leq \varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is arbitrary, we conclude that

$$\lambda |D_s v_0|^2 \leq a_0 |D_s v_0|^2, \text{ a.e. in } \mathbb{R}^n \times \mathbb{R}^n. \tag{2.3.16}$$

Now, observe that (2.3.16) holds for any $v_0 \in H^s(\mathbb{R}^n)$ and so

$$\lambda \leq a_0 \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}^n,$$

as we wanted to prove.

It remains to prove the boundedness from above $a_0 \leq \frac{\Lambda^2}{\lambda}$ a.e. in $\mathbb{R}^n \times \mathbb{R}^n$.

Take $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be nonnegative and by our hypotheses on the kernel a_k we have

$$\begin{aligned} \iint_{\mathbb{R}^n\times\mathbb{R}^n} |a_k D_s v_k|^2 \varphi \, dxdy &\leq \Lambda^2 \iint_{\mathbb{R}^n\times\mathbb{R}^n} |D_s v_k|^2 \varphi \, dxdy \\ &\leq \frac{\Lambda^2}{\lambda} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a_k |D_s v_k|^2 \varphi \, dxdy. \end{aligned}$$

From this point the proof follows as in the previous case, using the convergence of the fluxes $a_k D_s v_k \rightharpoonup a_0 D_s v_0$ weakly in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

The proof is now complete. □

Remark 2.3.7. All the outcomes of this section can be extended for $1 \leq p < \infty$, as it was shown in [49].

Resumen del capítulo

En este capítulo contamos nuestro aporte en el tema Homogeneización en difusión no local, puede ser encontrado en su versión más general en [49]. Comenzamos con una versión no local del conocido *div-curl Lema*, en un caso particular que encaja con las necesidades que origina el problema a estudiar. Previamente, se definen el s-gradiente y la s-divergencia. Para $u \in H^s(\mathbb{R}^n)$, se define el s-gradiente como

$$D_s u(x,y) := \frac{u(x)-u(y)}{|x-y|^{\frac{n}{2}+s}}.$$

Se observa que $D_s u \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, para toda $u \in H^s(\mathbb{R}^n)$. Para $\phi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, se define la s-divergencia como

$$d_s\phi(x) := \text{v.p.} \int_{\mathbb{R}^n} \frac{\phi(x,y)-\phi(y,x)}{|x-y|^{\frac{n}{2}+s}}\, dy.$$

Con estas definiciones, se tiene que $(-\Delta)^s u = \frac{c(n,s)}{2} d_s(D_s u)$.

Se prueba la siguiente versión del *div-curl Lema*: Dadas $\phi_k, \phi_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ y $v_k, v_0 \in H^s(\mathbb{R}^n)$ tales que

$$\begin{cases} v_k \rightharpoonup v_0 & \text{débil en } H^s(\mathbb{R}^n), \\ \phi_k \rightharpoonup \phi_0 & \text{débil en } L^2(\mathbb{R}^n \times \mathbb{R}^n), \\ d_s\phi_k \to d_s\phi_0 & \text{fuerte en } H^{-s}_{\text{loc}}(\mathbb{R}^n). \end{cases}$$

Se tiene que $\phi_k D_s v_k \to \phi_0 D_s v_0$ en el sentido de las distribuciones.

Dada una sucesión de operadores $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$, donde $\mathcal{L}_k = \mathcal{L}_{a_k}$ para $a_k \in \mathcal{A}_{\lambda,\Lambda}$, definido en (1.2.13), se estudia el *paso al límite* del problema

$$\begin{cases} \mathcal{L}_k u_k = f & \text{in } \Omega \\ u_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

para $f \in H^{-s}(\Omega)$. Como primer paso, se obtiene, a través de la Γ-convergencia de los funcionales de energía asociados, la convergencia débil de las soluciones. Posteriormente, a través de una sucesión de *funciones oscilantes*, se concluye finalmente la H-convergencia:

$$\begin{aligned} u_k &\rightharpoonup u_0 && \text{débil en } H^s_0(\Omega) \\ a_k D_s u_k &\rightharpoonup a_0 D_s u_0 && \text{débil en } L^2(\Omega) \end{aligned}$$

donde u_0 es solución de

$$\begin{cases} \mathcal{L}_0 u_0 = f & \text{in } \Omega \\ u_0 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

y $\mathcal{L}_0 = \mathcal{L}_{a_0}$, donde a_0 hereda de $\{a_k\}_{k\in\mathbb{N}}$ la positividad y el hecho de ser acotado.

Chapter 3

Optimal design for nonlocal diffusion

In this chapter we present our contribution in shape optimization problems involving the fractional Laplacian. The reader could find this results in [47, 78].

3.1 Some existence results

The goal of this section is to prove existence of a minimal shape, that is, a solution to a problem of the form

$$\min_{A\in\mathcal{A}} F(A), \tag{3.1.1}$$

where F is a cost functional and $\mathcal{A}$ is the class of admissible domains.

Assume there exists a notion of set convergence in $\mathcal{A}$, let say ν, that makes $\mathcal{A}$ be a **compact set**. In addition, suppose $F\colon \mathcal{A}\to\mathbb{R}$ is continuous. Then, solving a problem like (3.1.1) is really easy. Indeed, we consider a minimizer sequence $\{A_k\}_{k\in\mathbb{N}}\subset\mathcal{A}$. Since $\mathcal{A}$ is ν-compact, there exist a subsequence $\{A_{k_j}\}_{j\in\mathbb{N}}\subset\{A_k\}_{k\in\mathbb{N}}$ and a set $A_1\in\mathcal{A}$ such that $A_{k_j}\overset{\nu}{\to}A_1$. Finally, by using the continuity of F, we conclude A_1 is a solution to (3.1.1). Moreover, we can relax the hypothesis over F. Since we are interesting in solving a minimization problem, it is enough to consider F be **ν-lower semicontinuous**, and the same argument works.

Inspired in the previous argument, we start this section introducing the class of admissible domains and some notions of set convergence.

Results presented in this section of the Thesis form part of works [47] and [78].

3.1.1 Strong and weak γ_s-convergence

Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz bounded open set . Let $0<s<1$ and consider $\mathcal{A}_s(\Omega)$ the class of s-quasi open subset of Ω, see Definition 1.3.4.

Definition 3.1.1 (Strong γ_s-convergence)**.** Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ and $A \in \mathcal{A}_s(\Omega)$. We say that $A_k \overset{\gamma_s}{\to} A$ if $u^s_{A_k} \to u^s_A$ strongly in $L^2(\Omega)$, where u^s_A is defined in (1.3.4).

Let $m \in \mathbb{N}$, $\{(A^k_1, \ldots, A^k_m)\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \ldots, A_m) \in \mathcal{A}_s(\Omega)^m$.

We say $(A^k_1, \ldots, A^k_m) \overset{\gamma_s}{\to} (A_1, \ldots, A_m)$ if $A^k_i \overset{\gamma_s}{\to} A_i$ for every $i = 1, \ldots, m$.

Remark 3.1.2. This is the fractional version of the $\gamma-$convergence of sets defined in [24].

Definition 3.1.3 (Weak γ_s-convergence)**.** Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$. We say that $A_k \overset{\gamma_s}{\rightharpoonup} A$ if $u^s_{A_k} \to u$ strongly in $L^2(\Omega)$, where $A := \{u > 0\}$.

Let $m \in \mathbb{N}$ and $\{(A^k_1, \ldots, A^k_m)\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$. We say $(A^k_1, \ldots, A^k_m) \overset{\gamma_s}{\rightharpoonup} (A_1, \ldots, A_m)$ if $A^k_i \overset{\gamma_s}{\rightharpoonup} A_i$ for every $i = 1, \ldots, m$.

We follow the same approach and ideas of [24], where the laplacian operator (the case $s = 1$) was involved, in order to obtain a compactness result in $\mathcal{A}_s(\Omega)$ with respect to γ_s-convergence.

Remark 3.1.4. In this chapter, we always consider the s-quasi representative given by Theorem 1.3.11. As well as, equalities and inequalities are thought s-quasi everywhere.

We introduce $\mathcal{K}_s$ defined by

$$\mathcal{K}_s = \{w \in H^s_0(\Omega) \colon w \geq 0\,, (-\Delta)^s w \leq 1 \text{ in } \Omega\}. \tag{3.1.2}$$

We begin giving an idea of the steps we follow to conclude certain set compactness.

Step 1 Given $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$, we consider $u^s_{A_k}$ the solution to (1.3.4). We prove that $\{u^s_{A_k}\}_{k\in\mathbb{N}} \subset \mathcal{K}_s$ and that $\mathcal{K}_s$ is a $\|\cdot\|_{L^2(\Omega)}$-compact set. Then, there exist a subsequence (still denoted with the same index) and a function $u \in \mathcal{K}_s$ such that $u^s_{A_k} \to u$ in $L^2(\Omega)$. Denote $A := \{u > 0\}$. Notice that we **are not able** to conclude $u = u^s_A$.

Step 2 Since u^s_A is also the solution of

$$\max\{w \in H^s_0(\Omega) \colon w \leq 0 \text{ in } \mathbb{R}^n \setminus A,\ (-\Delta)^s w \leq 1 \text{ in } \Omega\}, \tag{3.1.3}$$

we obtain the inequality $u \leq u^s_A$ in $\mathbb{R}^n$.

Step 3 Let $\varepsilon > 0$. Consider $A^\varepsilon := \{u^s_A > \varepsilon\}$. By the same argument from **Step 1**, the sequence $\{u^s_{A_k\cup A^\varepsilon}\}_{k\in\mathbb{N}} \subset \mathcal{K}_s$ and $u^s_{A_k\cup A^\varepsilon} \to u^\varepsilon \in \mathcal{K}_s$ in $L^2(\Omega)$. Next, we prove $u^\varepsilon \leq u^s_A$ in $\mathbb{R}^n$.

Step 4 We obtain the convergence $u^\varepsilon \to u^s_A$ in $L^2(\Omega)$, when $\varepsilon \downarrow 0$.

Step 5 Finally, by a standard diagonal argument, we conclude $u^s_{A_k\cup A^{\varepsilon_k}} \to u^s_A$ in $L^2(\Omega)$. In other words, we obtain an *enlarged* sequence such that $A_k \cup A^{\varepsilon_k} =: \tilde{A}_{k_j} \overset{\gamma_s}{\to} A$.

We start by proving u^s_A is also the solution to (3.1.3), which is the main part of **Step 2**.

Proposition 3.1.5. *For every $A \in \mathcal{A}_s(\Omega)$, it follows that $u_A^s \geq 0$ in $\mathbb{R}^n$ and $(-\Delta)^s u_A^s \leq 1$ in Ω, where u_A^s is defined in* (1.3.4).

Moreover, u_A^s is the solution to (3.1.3).

Proof. Let us define

$$K_A = \{w \in H_0^s(\Omega)\colon w \leq 0 \text{ in } \mathbb{R}^n \setminus A\},$$

and $w_A \in K_A$ the (unique) minimizer of

$$I_s\colon K_A \to \mathbb{R}, \qquad I_s(w) = \frac{c(n,s)}{2}[w]_s^2 - \int_\Omega w\,dx.$$

Observe that, by Stampacchia's Theorem, w_A is characterized by the variational inequality

$$\mathcal{E}(w_A, v - w_A) \geq \int_\Omega (v - w_A)\,dx \quad \forall v \in K_A, \tag{3.1.4}$$

where we denote

$$\mathcal{E}(u,v) := c(n,s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy. \tag{3.1.5}$$

Next, we prove that both functions u_A^s and w_A agree.

The proof is standard. We will use the standard notations of $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$.

Take w_A^+ as test function in the variational inequality (3.1.4) and obtain

$$\begin{aligned} 0 &\leq \int_\Omega w_A^-\,dx \leq \mathcal{E}(w_A, w_A^-) \\ &\leq -c(n,s) \iint_{\{w_A \leq 0\} \times \{w_A \leq 0\}} \frac{(w_A^-(x) - w_A^-(y))^2}{|x-y|^{n+2s}}\,dxdy. \end{aligned}$$

From this inequality one easily conclude that $w_A^- = 0$ in Ω and so, since $w_A \in K_A$, $w_A \in H_0^s(A)$.

Therefore, since, by Remark 1.3.18, u_A^s is the unique minimum of I_s over $H_0^s(A)$ and, since also $u_A^s \in K_A$, $I_s(w_A) \leq I_s(u_A^s)$ the identity $w_A = u_A^s$ in $\mathbb{R}^n$ follows.

Observe that from the maximum principle, Proposition 1.2.13, it follows that $u_A^s \geq 0$ in $\mathbb{R}^n$.

Given $v \in H_0^s(\Omega)$ such that $v \geq 0$, then we get $v \geq 0$. Therefore, $-v$ belongs to K_A. By using it as a test function in (3.1.4) we obtain that

$$\mathcal{E}(u_A^s, -v - u_A^s) = -c(n,s)[u_A^s]_s^2 - \mathcal{E}(u_A^s, v) \geq -\int_\Omega v\,dx - \int_\Omega u_A^s\,dx.$$

Using that $(-\Delta)^s u_A^s = 1$ in A, the last inequality reads as

$$\mathcal{E}(u_A^s, v) \leq \int_\Omega v\,dx.$$

Since $v \in H_0^s(\Omega)$ is nonnegative but otherwise arbitrary, we get that $(-\Delta)^s u_A^s \leq 1$ in Ω.

Finally, if $w \leq 0$ in $\mathbb{R}^n \setminus A$ and $(-\Delta)^s w \leq 1$ in Ω, then

$$(-\Delta)^s w \leq (-\Delta)^s u_A^s \text{ in } A \quad \text{ and } \quad w \leq u_A^s \text{ in } \mathbb{R}^n \setminus A.$$

Hence, by comparison, $w \leq u_A^s$ in $\mathbb{R}^n$. □

According to **Step 1**, given a sequence $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$, we want to conclude that the sequence $\{u_{A_k}^s\}_{k\in\mathbb{N}}$ of solutions to (1.3.4) is contained in $\mathcal{K}_s$. That is a clear consequence of Proposition 3.1.5 and it is the content of next Corollary.

Corollary 3.1.6. *The function u_A^s belongs to $\mathcal{K}_s$ for every $A \in \mathcal{A}_s(\Omega)$, where u_A^s is the solution to* (1.3.4) *and $\mathcal{K}_s$ is defined by* (3.1.2).

The set $\mathcal{K}_s$ defined by (3.1.2) is a compact set in $L^2(\Omega)$.

Proposition 3.1.7. *$\mathcal{K}_s$ is a convex, closed and bounded subset of $H_0^s(\Omega)$. Consequently, $\mathcal{K}_s$ is pre-compact in $L^2(\Omega)$.*

Proof. Clearly, $\mathcal{K}_s$ is a convex set. $\mathcal{K}_s$ is also bounded. Indeed, given $u \in \mathcal{K}_s$, by Hölder and Poincaré's inequalities we get

$$c(n,s)[u]_s^2 \leq \int_\Omega u\,dx \leq |\Omega|^{\frac{1}{2}} \|u\|_{L^2(\Omega)} \leq C|\Omega|^{\frac{1}{2}}[u]_s.$$

In order to see that $\mathcal{K}_s$ is closed, let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{K}_s$ such that $u_k \to u$ in $H_0^s(\Omega)$. For any $k \in \mathbb{N}$ and any $v \in H_0^s(\Omega)$, $v \geq 0$, it holds that

$$\mathcal{E}(u_k, v) \leq \int_\Omega v\,dx,$$

where $\mathcal{E}$ is defined by (3.1.5). Since $\mathcal{E}(\cdot, v)$ is continuous in $H_0^s(\Omega)$ (in fact is weakly continuous), taking the limit as $k \to \infty$ we obtain that $\mathcal{E}(u,v) \leq \int_\Omega v\,dx$, but, since $v \in H_0^s(\Omega)$ is nonnegative but otherwise arbitrary we obtain that $(-\Delta)^s u \leq 1$ in Ω and then $u \in \mathcal{K}_s$. □

Remark 3.1.8. Observe that optimal constant in Poincaré's inequality

$$\|u\|_{L^2(\Omega)}^2 \leq C(\Omega, s)[u]_s^2,$$

has a dependence on s of the form

$$C(\Omega, s) \leq (1-s)C(\Omega).$$

See Corollary 1.1.14.

Therefore, the proof of Proposition 3.1.7 gives that if $u \in \mathcal{K}_s$, then

$$(1-s)[u]_s^2 \leq C, \tag{3.1.6}$$

where C depends on Ω but is independent on $0 < s < 1$.

Remark 3.1.9. Notice that $\mathcal{A}_s(\Omega)$ endowed whit the weak γ_s-convergence is compact. Indeed, given a sequence $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$, by Corollary 3.1.6, we know that $\{u^s_{A_k}\}_{k\in\mathbb{N}} \subset \mathcal{K}_s$. By Proposition 3.1.7, there exist a subsequence $\{u^s_{A_{k_j}}\}_{j\in\mathbb{N}} \subset \{u^s_{A_k}\}_{k\in\mathbb{N}}$ and a function $u \in \mathcal{K}_s$ such that $u^s_{A_{k_j}} \to u$ strongly in $L^2(\Omega)$. Denote by $A := \{u > 0\}$. Then, $A_{k_j} \stackrel{\gamma_s}{\rightharpoonup} A$.

Thanks to the previous Remark 3.1.9, given a sequence $\{A_k\}_{k\in\mathbb{N}} \in \mathcal{A}_s(\Omega)$, we can assume $A_k \stackrel{\gamma_s}{\to} A := \{u > 0\}$, where u is the $L^2(\Omega)$-limit of the associated sequence of solutions $\{u^s_{A_k}\}_{k\in\mathbb{N}}$.

We would like to relate the function spaces $H^s_0(A_k)$ and $H^s_0(A) = H^s_0(\{u > 0\})$. This expected relation between those spaces will be useful to prove **Step 3**. This is the content of next lemma. We decide to omit its proof due to the similarity with Lemma 3.2.7.

Lemma 3.1.10. *Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ be such that $u^s_{A_k} \to u$ in $L^2(\Omega)$, and let $\{w_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $H^s_0(\Omega)$ such that $w_k \in H^s_0(A_k)$ and $w_k \to w$ in $L^2(\Omega)$.*

Then, $w \in H^s_0(\{u > 0\})$.

Given a sequence $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ such that $A_k \stackrel{\gamma_s}{\rightharpoonup} A$, that is, $u^s_{A_k} \to u$ in $L^2(\Omega)$ and $A := \{u > 0\}$, we obtain as an easy consequence of Proposition 3.1.5 the inequality $u \le u^s_A$ in $\mathbb{R}^n$. We want to enlarge the set sequence in such a way that its function $L^2(\Omega)$-limit associated is still less than u^s_A. To be precise, we refer to **Step 3**.

Lemma 3.1.11. *Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ be such that $u^s_{A_k} \to u$ in $L^2(\Omega)$, $u \le u^s_A$ in $\mathbb{R}^n$ and $u^s_{A_k\cup A^\varepsilon} \to u^\varepsilon$ in $L^2(\Omega)$, where $A^\varepsilon := \{u^s_A > \varepsilon\}$ and $\varepsilon > 0$.*

Then, $u^\varepsilon \le u^s_A$ in $\mathbb{R}^n$.

The proof of Lemma 3.1.11 is omitted due to the similarity with Lemma 3.2.8.

We have paved the way for proving the **compactness** result in $\mathcal{A}_s(\Omega)$.

Theorem 3.1.12. *Let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$. Then, there exist a subsequence $\{A_{k_j}\}_{j\in\mathbb{N}} \subset \{A_k\}_{k\in\mathbb{N}}$, an enlarged sequence $\{\tilde{A}_{k_j}\}_{j\in\mathbb{N}}$ and $A \in \mathcal{A}_s(\Omega)$ such that*

$$A_{k_j} \subset \tilde{A}_{k_j}, \quad \text{and} \quad \tilde{A}_{k_j} \stackrel{\gamma_s}{\to} A.$$

Moreover, $|A| \le \liminf_{k\to\infty} |A_k|$.

Proof. Let $u^s_{A_k}$ be the solution to (1.3.4) for A_k. Then, by Proposition 3.1.7, there exist a subsequence (still denoted with the same index) and a function $u \in \mathcal{K}_s$ such that $u^s_{A_k} \to u$ in $L^2(\Omega)$. Denote $A := \{u > 0\}$ and consider u^s_A the solution to (1.3.4) for A. Thanks to Remark 1.3.14 and Lemma 1.3.15, we know that $A \in \mathcal{A}_s(\Omega)$.

Since $u \in \mathcal{K}_s$ and u^s_A is also the solution to (3.1.3), we obtain $u \le u^s_A$ in $\mathbb{R}^n$.

Let $\varepsilon > 0$. Consider $A^\varepsilon := \{u^s_A > \varepsilon\}$ and $u^s_{A_k\cup A^\varepsilon}$ the solution to (1.3.4) for $A_k \cup A^\varepsilon$. Then, by Corollary 3.1.6 and Proposition 3.1.7, there exist a subsequence (still denoted by the same index) and a function $u^\varepsilon \in \mathcal{K}_s$ such that $u^s_{A_k\cup A^\varepsilon} \to u^\varepsilon$ in $L^2(\Omega)$.

By Lemma 3.1.11, we conclude $u^\varepsilon \le u_A^s$ in $\mathbb{R}^n$.

We claim that $(u_A^s - \varepsilon)^+ \le u_{A^\varepsilon}^s$ in $\mathbb{R}^n$. Indeed,

$$(u_A^s - \varepsilon)^+(x) - (u_A^s - \varepsilon)^+(y) = \begin{cases} u_A^s(x) - u_A^s(y) & \text{if } x, y \in A^\varepsilon \\ u_A^s(x) - \varepsilon & \text{if } x \in A^\varepsilon \text{ and } y \notin A^\varepsilon \\ -u_A^s(y) + \varepsilon & \text{if } x \notin A^\varepsilon \text{ and } y \in A^\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $v \in H_0^s(A^\varepsilon)$ such that $v \ge 0$, we get

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{((u_A^s(x) - \varepsilon)^+ - ((u_A^s(y) - \varepsilon)^+)(v(x) - v(y))}{|x-y|^{n+2s}}\,dxdy =$$
$$\iint_{A^\varepsilon \times A^\varepsilon} \frac{(u_A^s(x) - u_A^s(y))(v(x) - v(y))}{|x-y|^{n+2s}}\,dxdy + 2\iint_{A^\varepsilon \times (A^\varepsilon)^c} \frac{(u_A^s(x) - \varepsilon)v(x)}{|x-y|^{n+2s}}\,dydx =$$
$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_A^s(x) - u_A^s(y))(v(x) - v(y))}{|x-y|^{n+2s}}\,dxdy + 2\iint_{A^\varepsilon \times (A^\varepsilon)^c} \frac{(u_A^s(y) - \varepsilon)v(x)}{|x-y|^{n+2s}}\,dydx \le$$
$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_A^s(x) - u_A^s(y))(v(x) - v(y))}{|x-y|^{n+2s}}\,dxdy$$

That is, $(-\Delta)^s(u_A^s - \varepsilon)^+ \le (-\Delta)^s u_A^s = 1 = (-\Delta)^s u_{A^\varepsilon}^s$ in A^ε. Moreover, since $0 = (u_A^s - \varepsilon)^+ = u_{A^\varepsilon}^s$ in $\mathbb{R}^n \setminus A^\varepsilon$, from the comparison principle it follows that $(u_A^s - \varepsilon)^+ \le u_{A^\varepsilon}^s$ in $\mathbb{R}^n$.

We have obtained the following chain of inequalities

$$(u_A^s - \varepsilon)^+ \le u_{A^\varepsilon}^s \le u_{A_k \cup A^\varepsilon}^s.$$

Taking limit as $k \to \infty$ we conclude that

$$(u_A^s - \varepsilon)^+ \le u^\varepsilon \le u_A^s,$$

since $u^\varepsilon \le u_A^s$ in $\mathbb{R}^n$ and $u_{A_k \cup A^\varepsilon}^s \to u^\varepsilon$.

Since $u^\varepsilon \in \mathcal{K}_s$, by (3.1.6), $\{u^\varepsilon\}_{\varepsilon > 0}$ is uniformly bounded in $H_0^s(\Omega)$. Consequently, up to a subsequence, $u^\varepsilon \to u_A^s$ in $L^2(\Omega)$.

By a standard diagonal argument, there exists a sequence $\varepsilon_k \downarrow 0$ such that $u_{A_k \cup A^{\varepsilon_k}}^s \to u_A^s$ in $L^2(\Omega)$.

In conclusion, we have proved that the enlarged sequence $A_k \cup A^{\varepsilon_k} =: \tilde{A}_k$ γ_s-converges to A.

To finish the proof, we have to show that $|A|$ is bounded from above by $\liminf_{k\to\infty} |A_k|$.

For every $\varepsilon > 0$, we have the following inclusion

$$\{u \ge \varepsilon\} \subset \left\{|u - u_{A_k}^s| \ge \frac{\varepsilon}{2}\right\} \cup \left\{u_{A_k}^s \ge \frac{\varepsilon}{2}\right\}.$$

Indeed, let $x \in \mathbb{R}^n$ be such that $|u(x) - u^s_{A_k}(x)| < \frac{\varepsilon}{2}$ and $u^s_{A_k}(x) < \frac{\varepsilon}{2}$. Then,

$$u(x) = u(x) - u^s_{A_k}(x) + u^s_{A_k}(x) \leq |u(x) - u^s_{A_k}(x)| + u^s_{A_k}(x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $x \in \{u < \varepsilon\}$.

By Chebyshev's inequality, Proposition 1.3.21, we obtain

$$\begin{aligned} |\{u \geq \varepsilon\}| &\leq \left|\left\{|u - u^s_{A_k}| \geq \frac{\varepsilon}{2}\right\}\right| + \left|\left\{u^s_{A_k} \geq \frac{\varepsilon}{2}\right\}\right| \\ &\leq \frac{4}{\varepsilon^2}\int_\Omega |u - u^s_{A_k}|^2\,dx + \left|\left\{u^s_{A_k} > 0\right\}\right| \\ &= \frac{4}{\varepsilon^2}\int_\Omega |u - u^s_{A_k}|^2\,dx + |A_k| \\ &\leq \frac{4}{\varepsilon^2}\int_\Omega |u - u^s_{A_k}|^2\,dx + \liminf_{k\to\infty} |A_k|. \end{aligned}$$

Use the convergence $u^s_{A_k} \to u$ in $L^2(\Omega)$, to conclude

$$|\{u \geq \varepsilon\}| \leq \liminf_{k\to\infty} |A_k|$$

for every $\varepsilon > 0$. Finally, observe that

$$\bigcup_{\varepsilon>0}\{u > \varepsilon\} = \{u > 0\}, \qquad \text{and} \qquad \{u > \varepsilon\} \subset \{u > t\} \text{ for } 0 < t < \varepsilon,$$

then $|A| = |\{u > 0\}| = \lim_{\varepsilon\downarrow 0} |\{u > \varepsilon\}| \leq \liminf_{k\to\infty} |A_k|$. □

Remark 3.1.13. It will be useful to emphasize that once we apply Theorem 3.1.12, we obtain a γ_s-limit for an enlarged sequence of domains and we also deduce its characterization: $A = \{u > 0\}$, where u is the $L^2(\Omega)$-limit of $\{u^s_{A_k}\}_{k\in\mathbb{N}}$.

Thanks to the previous Theorem 3.1.12, we can also obtain a compactness result in $\mathcal{A}_s(\Omega)^m$, for fixed $m \in \mathbb{N}$. That is the content of next Corallary.

Corollary 3.1.14. *Let* $\{(A^k_1, \ldots, A^k_m)\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$.

Then, there exist a subsequence $\{(A^{k_j}_1, \ldots, A^{k_j}_m)\}_{j\in\mathbb{N}} \subset \{(A^k_1, \ldots, A^k_m)\}_{k\in\mathbb{N}}$*, an enlarged sequence* $\{(\tilde{A}^{k_j}_1, \ldots, \tilde{A}^{k_j}_m)\}_{j\in\mathbb{N}}$ *and* $(A_1, \ldots, A_m) \in \mathcal{A}_s(\Omega)^m$ *such that*

$$A^{k_j}_i \subset \tilde{A}^{k_j}_i \text{ for every } i = 1, \ldots, m, \qquad \text{and} \qquad (\tilde{A}^{k_j}_1, \ldots, \tilde{A}^{k_j}_m) \xrightarrow{\gamma_s} (A_1, \ldots, A_m).$$

Moreover, $|A_i| \leq \liminf_{k\to\infty} |A^k_i|$*, for* $i = 1, \ldots, m$.

Proof. By Theorem 3.1.12, there exist $A_1 \in \mathcal{A}_s(\Omega)$, a subsequence $\{A^{k_j}_1\}_{j\in\mathbb{N}} \subset \{A^k_1\}_{k\in\mathbb{N}}$ and an enlarged sequence $\{\tilde{A}^{k_j}_1\}_{j\in\mathbb{N}}$ such that

$$A^{k_j}_1 \subset \tilde{A}^{k_j}_1, \qquad \text{and} \qquad \tilde{A}^{k_j}_1 \xrightarrow{\gamma_s} A_1.$$

Now, consider $\{A_2^{k_j}\}_{j\in\mathbb{N}}$ and apply again Theorem 3.1.12. Thus, there exist $A_2 \in \mathcal{A}_s(\Omega)$, a subsequence $\{A_2^{k_{j_l}}\}_{l\in\mathbb{N}} \subset \{A_2^{k_j}\}_{j\in\mathbb{N}}$ and an enlarged sequence $\{\tilde{A}_2^{k_{j_l}}\}_{l\in\mathbb{N}}$ such that

$$A_i^{k_{j_l}} \subset \tilde{A}_i^{k_{j_l}}, \quad \text{and} \quad A_i^{k_{j_l}} \overset{\gamma_s}{\to} A_i \text{ for } i = 1, 2.$$

Repeating this argument and renaming the final subsequence extracted, we obtain the enlarged sequence $\{(\tilde{A}_1^{k_j}, \dots, \tilde{A}_m^{k_j})\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)^m$ such that

$$A_i^{k_j} \subset \tilde{A}_i^{k_j} \text{ for every } i = 1, \dots, m; \quad \text{and} \quad (\tilde{A}_1^{k_j}, \dots, \tilde{A}_m^{k_j}) \overset{\gamma_s}{\to} (A_1, \dots, A_m).$$

By Theorem 3.1.12, we know $|A_i| \le \liminf_{k\to\infty} |A_i|^k$, for $i = 1, \dots, m$. □

Remark 3.1.15. We want to emphasize that the γ_s-limit obtained in Theorem 3.1.14 is characterized by $A_i = \{u_i > 0\}$, where u_i is the $L^2(\Omega)$-limit of $\{u^s_{A_i^k}\}_{k\in\mathbb{N}}$, for each $i = 1, \dots, m$.

3.1.2 Existence of minimal shapes

Once we have proved a sort of compactness in, at least, two possible classes of admissible sets, we can stablish two existence results related to different shape optimization problems.

To solve a problem like (3.1.1), we need to introduce the class of admissible sets and a suitable cost functional.

Let $0 < s < 1$, $m \in \mathbb{N}$ and $F_s \colon \mathcal{A}_s(\Omega)^m \to [0, \infty]$ be such that

- F_s is γ_s-lower semicontinuous, that is,

$$F_s(A_1, \dots, A_m) \le \liminf_{k\to\infty} F_s(A_1^k, \dots, A_m^k),$$

for every sequence $\{(A_1^k, \dots, A_m^k)\}_{k\in\mathbb{N}}$ such that $(A_1^k, \dots, A_m^k) \overset{\gamma_s}{\to} (A_1, \dots, A_m)$.

- F_s is decreasing, that is, for every $(A_1, \dots, A_m), (B_1, \dots, B_m) \in \mathcal{A}_s(\Omega)^m$ such that $A_i \subset B_i$ for $i = 1, \dots, m$, we have

$$F_s(A_1, \dots, A_m) \ge F_s(B_1, \dots, B_m).$$

Before we start proving the existence of minimal shapes, we observe that the decreasing property of a functional F_s makes equivalent its weak and strong γ_s-lower semicontinuity, which is the content of next theorem.

Theorem 3.1.16. *Let $0 < s < 1$, $m \in \mathbb{N}$ and $F_s \colon \mathcal{A}_s(\Omega)^m \to [0, \infty]$ be a decreasing functional. Then, the following assertions are equivalent*

1. *F_s is weakly γ_s-lower semicontinuous.*

2. *F_s is γ_s-lower semicontinuous.*

Proof. It is enough to prove that the γ_s-lower semicontinuity implies the weak one. Indeed, consider $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ such that $A_k \xrightarrow{\gamma_s} A \in \mathcal{A}_s(\Omega)$. By Proposition 1.3.21, $A = \{u^s_A > 0\}$. Then, $A_k \xrightarrow{\gamma_s} A \in \mathcal{A}_s(\Omega)$. That proves (1) $\Rightarrow$ (2).

Assume F_s is γ_s-lower semicontinuous.

Let $\{A^k_i\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ such that $A^k_i \xrightarrow{\gamma_s} A_i \in \mathcal{A}_s(\Omega)$, for $i = 1, \dots, m$. That is, $u^s_{A^k_i} \to u_i$ strongly in $L^2(\Omega)$ and $A_i = \{u_i > 0\}$.

There exist enlarged sequences $\{\tilde{A}^k_1\}_{k\in\mathbb{N}}, \dots, \{\tilde{A}^k_m\}_{k\in\mathbb{N}}$ such that

$$A^k_i \subset \tilde{A}^k_i, \quad \text{and} \quad \tilde{A}^k_i \xrightarrow{\gamma_s} A_i,$$

for every $i = 1, \dots, m$, by Corollary 3.1.14 and Remark 3.1.15 .

Then, since F_s is γ_s-lower semicontinuous and decreasing, we obtain

$$F_s(A_1, \dots, A_m) \leq \liminf_{k\to\infty} F_s(\tilde{A}^k_1, \dots, \tilde{A}^k_m) \leq \liminf_{k\to\infty} F_s(A^k_1, \dots, A^k_m).$$

That means F_s is weak γ_s-lower semicontinuous, as we desired. □

A class of optimal shape problems: fixed measure

The first problem that we address in this second part of the thesis is the following:

$$\min\{F_s(A) \colon A \in \mathcal{A}_s(\Omega),\ |A| \leq c\}, \tag{3.1.7}$$

where F_s is a γ_s-lower semicontinuous and decreasing functional.

Remark 3.1.17. Observe that from the monotonicity assumption on F_s, this problem is equivalent to minimize in the class of s-quasi open sets A of **fixed measure** $|A| = c$. In fact, assume that a minimizer $A_0 \in \mathcal{A}_s$ for (3.1.7) verifies that $|A_0| < c$. Then, for any $\tilde{A}_0 \supset A_0$ such that $|\tilde{A}_0| = c$, since F_s is decreasing with respect of the set inclusion, we have

$$F_s(\tilde{A}_0) \leq F_s(A_0) = \inf_{A\in\mathcal{A}_s} F_s(A)$$

and so $\tilde{A}_0$ is a minimizer of F_s.

Following the same approach and ideas of [24], problem (3.1.7) can be analyzed and that is the content of next Theorem.

Theorem 3.1.18. *Let $0 < s < 1$ be fixed and $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $F_s : \mathcal{A}_s(\Omega) \to \mathbb{R}$ be a decreasing γ_s-lower semicontinuous functional.*

Then, for every $0 < c < |\Omega|$, problem (3.1.7) has a solution.

Proof. Take a minimizer sequence $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$, that is, $|A_k| \leq c$ and

$$\lim_{k\to\infty} F_s(A_k) = \inf\{F_s(A) \colon A \in \mathcal{A}_s(\Omega), |A| \leq c\} =: \alpha.$$

By Theorem 3.1.12, up to a subsequence, there exist an enlarged sequence $\{\tilde{A}_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ and a set $A \in \mathcal{A}_s(\Omega)$ such that

$$A_k \subset \tilde{A}_k, \quad \text{and} \quad \tilde{A}_k \overset{\gamma_s}{\to} A.$$

Then, since F_s is γ_s-lower semicontinuous and decreasing, we obtain

$$F_s(A) \leq \liminf_{k\to\infty} F_s(\tilde{A}_k) \leq \liminf_{k\to\infty} F_s(A_k) = \alpha.$$

To finish the proof, observe that that $|A| \leq \liminf_{k\to\infty} |A_k| \leq c$, also by Theorem 3.1.12. That means, A is an admissible domain for the minimization problem and so that A is a solution to (3.1.7). □

Thanks to Theorems 3.1.16 and 3.1.18, we easily obtain next Corollay.

Corollary 3.1.19. *Let $0 < s < 1$ be fixed and $\Omega \subset \mathbb{R}^n$ be open and bounded. Let F_s : $\mathcal{A}_s(\Omega) \to \mathbb{R}$ be a decreasing weak γ_s-lower semicontinuous functional.*

Then, for every $0 < c < |\Omega|$, problem (3.1.7) has a solution.

A class of optimal partition problems

Let $m \in \mathbb{N}$ be fixed and $0 < s < 1$. In the context of next problem, we understand by a **partition** of Ω to any collection of s-quasi open subset $A_1, \ldots, A_m$ such that $\Omega = \cup_{i=1}^m A_i$ and $\text{cap}_s(A_i \cap A_j, \Omega) = 0$ for $i \neq j$.

We are interested in considering the class of partitions of Ω as $\mathcal{A}$ in a type of problem (3.1.1).

We adapted the ideas from [22], where the authors consider the Laplacian operator, to recover their results for the fractional case. Rigorously speaking, under these assumptions, we have the following theorem.

Theorem 3.1.20. *Let $0 < s < 1$ be fixed and $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $F_s \colon \mathcal{A}_s(\Omega)^m \to [0, \infty]$ be a decreasing γ_s-lower semicontinuous functional. Then, there exists a solution to*

$$\min \{F_s(A_1, \ldots, A_m) \colon A_i \in \mathcal{A}_s(\Omega),\ \mathit{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\}. \tag{3.1.8}$$

Proof. Denote by

$$\alpha := \inf \{F_s(A_1, \ldots, A_m) \colon A_i \in \mathcal{A}_s(\Omega), \text{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\}.$$

Let $\{(A_1^k, \ldots, A_m^k)\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ be such that

$$\text{cap}_s(A_i^k \cap A_j^k, \Omega) = 0 \text{ for } i \neq j, \text{ and } \lim_{k\to\infty} F_s(A_1^k, \ldots, A_m^k) = \alpha.$$

By Corollary 3.1.14, there exist a subsequence $\{(A_1^k, \ldots, A_m^k)\}_{k\in\mathbb{N}}$ (still denoted by the same index), an enlarged sequence $\{(\tilde{A}_1^k, \ldots, \tilde{A}_m^k)\}_{k\in\mathbb{N}}$ and $(A_1, \ldots, A_m) \in \mathcal{A}_s(\Omega)^m$ such that

$$A_i^k \subset \tilde{A}_i^k \text{ for every } i = 1, \ldots, m, \quad \text{and} \quad (\tilde{A}_1^k, \ldots, \tilde{A}_m^k) \overset{\gamma_s}{\to} (A_1, \ldots, A_m).$$

Since F_s is γ_s-lower semicontinuous and decreasing in each coordinate, we obtain

$$F_s(A_1, \ldots, A_m) \leq \liminf_{k\to\infty} F_s(\tilde{A}_1^k, \ldots, \tilde{A}_m^k) \leq \liminf_{k\to\infty} F_s(A_1^k, \ldots, A_m^k) = \alpha. \tag{3.1.9}$$

To finish the proof, we should prove that $\operatorname{cap}_s(A_i \cap A_j, \Omega) = 0$ for $i \neq j$.

Let $i, j \in \{1, \ldots, m\}$ be such that $i \neq j$. Consider $u^s_{A_i^k}$ and $u^s_{A_j^k}$ defined in (1.3.4).

It becomes deduced by the proof of Theorem 3.1.12 that $A_l = \{u_l > 0\}$ for every $l = 1, \ldots, m$, where u_l is the $L^2(\Omega)$-limit of $\{u^s_{A_l^k}\}_{k\in\mathbb{N}}$. In addition, by Proposition 1.3.21, we know that $A_l^k = \{u^s_{A_l^k} > 0\}$ for every $l = 1, \ldots, m$.

Notice that this product $u^s_{A_i^k} \cdot u^s_{A_j^k}$ is an s-continuous function too, by Lemma 1.3.9, and $u^s_{A_i^k} \cdot u^s_{A_j^k} = 0$ s-q.e. in $\mathbb{R}^n \setminus (A_i^k \cap A_j^k)$. Moreover, since $\operatorname{cap}_s(A_i^k \cap A_j^k, \Omega) = 0$, we have $u^s_{A_i^k} \cdot u^s_{A_j^k} = 0$ s-q.e. in $\mathbb{R}^n$.

By Proposition 1.3.13, there exist subsequences $\{u^s_{A_i^k}\}_{k\in\mathbb{N}}$ and $\{u^s_{A_j^k}\}_{k\in\mathbb{N}}$, denoted with the same index, which converge s-q.e. to u_i and u_j respectively. Then, passing to the limit, we obtain $u_i \cdot u_j = 0$ s-q.e. in $\mathbb{R}^n$. That is $\operatorname{cap}_s(\{u_i \cdot u_j \neq 0\}, \Omega) = 0$. But, since $u_l \geq 0$ for every $l = 1, \ldots, m$, that means

$$\{u_i \cdot u_j \neq 0\} = \{u_i \neq 0\} \cap \{u_j \neq 0\} = \{u_i > 0\} \cap \{u_j > 0\} = A_i \cap A_j.$$

We have shown that $(A_1, \ldots, A_m)$ is admissible for the minimization problem (3.1.8) and recalling (3.1.9) the result is proved. □

Remark 3.1.21. Notice that it seems that we forgot to talk about **partition** in the class of domains where (3.1.8) was solved. The reason is the decreasing property of F_s. Indeed, take $(A_1, \ldots, A_m)$ a solution to (3.1.8) and suppose $\Omega \neq \cup_{i=1}^m A_i$. Denote by $B := \Omega \setminus \cup_{i=1}^m A_i$ and $\tilde{A}_1 := A_1 \cup B$. Then, $A_1 \subset \tilde{A}_1$, $\Omega = \tilde{A}_1 \cup \cup_{i=2}^m A_i$ and $\operatorname{cap}_s(\tilde{A}_1 \cap A_i, \Omega) = 0$ for every $i = 2, \ldots, m$.

By the decreasing property ef F_s, we obtain

$$F_s(\tilde{A}_1, A_2, \ldots, A_m) \leq F_s(A_1, A_2, \ldots, A_m).$$

We conclude $(\tilde{A}_1, A_2, \ldots, A_m)$ is also a solution to (3.1.8) and it is a **partition** of Ω.

Thanks to Theorems 3.1.16 and 3.1.20, we immediately obtain next Corollay.

Corollary 3.1.22. *Let $0 < s < 1$ be fixed and $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $F_s \colon \mathcal{A}_s(\Omega)^m \to [0, \infty]$ be a decreasing weak γ_s-lower semicontinuous functional. Then, there exists a solution to* (3.1.8).

3.2 Asymptotic behaviour of minimizers

We have proved the existence of solution to (3.1.7) and (3.1.8) for every $0 < s < 1$, inspired by works from Buttazzo-Dal Maso [24] and Bucur-Buttazzo-Henrot [22], where the authors considered shape optimization problems involving the Laplacian operator, which is the case $s = 1$ according to the notation used in this thesis. So, we want to answer the natural question about how probably the convergence from s-minimizers to the 1-minimizer is.

To this aim, we want to relate all the key elements involved. Given $0 < s_k \uparrow 1$, there exist a kind of *convergence* results between the $\|\cdot\|_{L^2(\Omega)}$-compact sets and, on the other hand, between the Sobolev spaces:

- $\mathcal{K}_{s_k} \to \mathcal{K}_1$, where $\mathcal{K}_{s_k}$ is defined by (3.1.2) and $\mathcal{K}_1$ is the analogous involving the Laplacian operator $-\Delta$ instead of the fractional operator.
- $H_0^{s_k}(A_k) \to H_0^1(\{u > 0\})$, where $A_k \in \mathcal{A}_{s_k}(\Omega)$ (see (1.3.4)), and u is the $L^2(\Omega)$-limit of the sequence of solutions $\{u_{A_k}^{s_k}\}_{k\in\mathbb{N}}$ defined by (1.3.4).

Such sense of *convergence* between those sets will be explained in the following.

3.2.1 Strong and weak γ-convergence

The goal in this subsection is to define certain notion of convergence from s-quasi open sets to 1-quasi open sets and obtain a compactness result. To this aim, first consider

$$\mathcal{K}_1 := \{w \in H_0^1(\Omega)\colon w \geq 0\,,\, -\Delta w \leq 1 \text{ in } \Omega\} \tag{3.2.1}$$

and

$$\mathcal{A}_1(\Omega) := \{A \subset \Omega\colon A \text{ is 1-quasi open}\}. \tag{3.2.2}$$

For $A \in \mathcal{A}_1(\Omega)$, we introduce the analogous notation $u_A^1 \in H_0^1(A)$ for the unique weak solution to

$$-\Delta u_A^1 = 1 \text{ in } A, \quad u_A^1 = 0 \text{ in } \mathbb{R}^n \setminus A. \tag{3.2.3}$$

With notations above, we are able to define a notion of set convergence.

Definition 3.2.1 (Strong γ-convergence)**.** Let $0 < s_k \uparrow 1$ and let $A_k \in \mathcal{A}_{s_k}(\Omega)$ and $A \in \mathcal{A}_1(\Omega)$. We say that $A_k \xrightarrow{\gamma} A$ if $u_{A_k}^{s_k} \to u_A^1$ strongly in $L^2(\Omega)$.

Let $m \in \mathbb{N}$, $(A_1^k, \dots, A_m^k) \in \mathcal{A}_{s_k}(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_1(\Omega)^m$.

We say that $(A_1^k, \dots, A_m^k) \xrightarrow{\gamma} (A_1, \dots, A_m)$ if $A_i^k \xrightarrow{\gamma} A_i$ strongly in $L^2(\Omega)$, for every $i = 1, \dots, m$.

Remark 3.2.2. Observe that the notion of $\gamma-$convergence of sets given in [24] is denoted in this thesis by γ_1-convergence. This should not cause any confusion.

Definition 3.2.3 (Weak γ-convergence)**.** Let $0 < s_k \uparrow 1$ and let $A_k \in \mathcal{A}_{s_k}(\Omega)$. We say that $A_k \xrightharpoonup{\gamma} A$ if $u_{A_k}^{s_k} \to u$ strongly in $L^2(\Omega)$ and $A := \{u > 0\}$.

Let $m \in \mathbb{N}$, $(A_1^k, \dots, A_m^k) \in \mathcal{A}_{s_k}(\Omega)^m$. We say that $(A_1^k, \dots, A_m^k) \xrightharpoonup{\gamma} (A_1, \dots, A_m)$ if $A_i^k \xrightharpoonup{\gamma} A_i$ strongly in $L^2(\Omega)$, for every $i = 1, \dots, m$.

We begin listing some important steps.

- Observe that $\mathcal{A}_1(\Omega) \subset \mathcal{A}_s(\Omega)$ for every $0 < s < 1$, by Remark 1.3.7.
- Let $0 < s_k \uparrow 1$ and $\mathcal{K}_{s_k}$ defined by (3.1.2) and $\mathcal{K}_1$ defined by (3.2.1). We prove a sort of *convergence* from $\mathcal{K}_{s_k}$ to $\mathcal{K}_1$.
- Let $0 < s_k \uparrow 1$ and $A_k \in \mathcal{A}_{s_k}(\Omega)$. Assume $u_{A_k}^{s_k} \to u$ in $L^2(\Omega)$, where $u_{A_k}^{s_k}$ is defined by (1.3.4). Then, we show a kind of *convergence* between the spaces $H_0^{s_k}(A_k)$ and $H_0^1(\{u > 0\})$.
- Given $A_k \in \mathcal{A}_{s_k}(\Omega)$, we apply a similar strategy to that used for the γ_s-convergence, to obtain an *enlarged γ-convergent sequence*. The techniques are more difficult since the domains are varying with s.

Our first goal is to show that a sequence $\{u_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$ such that $u_k \in \mathcal{K}_{s_k}$ is precompact and that every accumulation point belongs to $\mathcal{K}_1$.

Lemma 3.2.4. *Let $0 < s_k \uparrow 1$ and let $u_k \in \mathcal{K}_{s_k}$. Then, there exists $u \in H_0^1(\Omega)$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}} \subset \{u_k\}_{k\in\mathbb{N}}$ such that $u_{k_j} \to u$ strongly in $L^2(\Omega)$.*

Moreover, if $u_k \in \mathcal{K}_{s_k}$ is such that $u_k \to u$ strongly in $L^2(\Omega)$, then $u \in \mathcal{K}_1$.

Proof. From Remark 3.1.8, there exists a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}} (1 - s_k)[u_k]_{s_k}^2 \leq C.$$

Now the first claim follows from Theorem 1.1.11.

Now, assume that $u_k \to u$ in $L^2(\Omega)$. It is clear that $u \geq 0$. Since $(-\Delta)^{s_k} u_k \leq 1$ in Ω, for every nonnegative $\varphi \in C_c^\infty(\Omega)$ we have that

$$\int_\Omega (-\Delta)^{s_k} \varphi u_k \, dx = \langle (-\Delta)^{s_k} u_k, \varphi \rangle \leq \int_\Omega \varphi \, dx.$$

By the convergence assumption on u_k and the fact that the convergence (1.2.3) is also strong in $L^2(\Omega)$, we can take limit as $k \to \infty$ in the previous inequality to obtain that

$$\int_\Omega -\Delta \varphi u \, dx = \langle -\Delta u, \varphi \rangle \leq \int_\Omega \varphi \, dx,$$

and conclude that $-\Delta u \leq 1$ in Ω. Consequently, $u \in \mathcal{K}_1$ as required. □

Remark 3.2.5. Let $0 < s_k \uparrow 1$ and $A_k \in \mathcal{A}_{s_k}(\Omega)$. Then, by Corollary 3.1.6, $u_{A_k}^{s_k} \in \mathcal{K}_{s_k}$. Apply the previous Lemma 3.2.4 to conclude that there exist a subsequence (still denoted by the same index) and a function $u \in \mathcal{K}_1$ such that $u_{A_k}^{s_k} \to u$ in $L^2(\Omega)$. That means $A_k \xrightarrow{\gamma} A$.

Moreover, $u \leq u_A^1$ in $\mathbb{R}^n$, where $A := \{u > 0\}$, since u_A^1 is also a solution to

$$\max\left\{w \in H_0^1(\Omega) \colon w \leq 0 \text{ in } \mathbb{R}^n \setminus A, \, -\Delta w \leq 1 \text{ in } \Omega\right\}, \tag{3.2.4}$$

see [24, Section 3].

So, without loss of generality, given $0 < s_k \uparrow 1$ and $A_k \in \mathcal{A}_{s_k}(\Omega)$, we can assume that $A_k \xrightarrow{\gamma} A$, that is, $u^{s_k}_{A_k} \to u$ in $L^2(\Omega)$ and u is such that $u \in \mathcal{K}_1$ and, in addition, $u \leq u^1_A$ in $\mathbb{R}^n$, where $A = \{u > 0\}$.

Next lemma gives the continuity of u^s_A when $s \uparrow 1$, it means, when we fix de domain, the sequence of solutions to the fractional Laplacian converges to the solution to the Laplacian operetor.

Lemma 3.2.6. *For every $A \in \mathcal{A}_1(\Omega)$, $u^s_A \to u^1_A$ strongly in $L^2(\Omega)$, when $s \uparrow 1$.*

Proof. Let us remind that, from Proposition 3.1.5, u^s_A is also the solution to the minimization problem

$$I_s(u^s_A) = \min\{I_s(w) \colon w \in L^2(\Omega)\},$$

where

$$I_s(w) = \begin{cases} \frac{c(n,s)}{2}[w]^2_s - \int_\Omega w\,dx & \text{if } w \in H^s_0(A), \\ \infty & \text{otherwise.} \end{cases}$$

Notice that, as a consequence of Theorem 1.1.11, we have that $\frac{c(n,s)}{2}[w]^2_s \xrightarrow{\Gamma} \frac{1}{2}\|\nabla w\|^2_2$. Since the Γ-convergence is stable under continuous perturbations, we have that $I_s \xrightarrow{\Gamma} I_1$ in $L^2(\Omega)$, where

$$I_1(w) = \begin{cases} \frac{1}{2}\|\nabla w\|^2_2 - \int_\Omega w\,dx & \text{if } w \in H^1_0(A), \\ \infty & \text{otherwise.} \end{cases}$$

Thus, the minimizer of I_s converges to the minimizer of I_1. That is $u^s_A \to u^1_A$ strongly in $L^2(\Omega)$. □

Now we address the more difficult problem of understanding the limit behaviour of u^s_A when the domains also are varying with s.

Next lemma is key in understanding this limit behavior and the ideas are taken from [24].

Lemma 3.2.7. *Let $0 < s_k \uparrow 1$ and for every $k \in \mathbb{N}$ let $A_k \in \mathcal{A}_{s_k}(\Omega)$ be such that $u^{s_k}_{A_k} \to u$ strongly in $L^2(\Omega)$. Let $\{w_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$ be such that $w_k \in H^{s_k}_0(A_k)$ for every $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}}(1-s_k)[w_k]^2_{s_k} < \infty$. Assume, moreover that $w_k \to w$ strongly in $L^2(\Omega)$. Then, $w \in H^1_0(\{u > 0\})$.*

Proof. We need to show that $w = 0$ in $\mathbb{R}^n \setminus \{u > 0\}$, i.e., $w = 0$ in $\{u = 0\}$.

Let us define the functional

$$\Phi_k(v) = \begin{cases} \frac{c(n,s_k)}{2}[v]^2_{s_k} & \text{if } v \in H^{s_k}_0(A_k), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2.5}$$

defined in $L^2(\Omega)$. By the compactness of Γ-convergence, [31, Theorem 8.5], there exists a subsequence still denote by Φ_k such that

$$\Phi_k \xrightarrow{\Gamma} \Phi \quad \text{in } L^2(\Omega).$$

From Proposition 1.5.8, Φ is a quadratic form in $L^2(\Omega)$ with domain $D(\Phi) \subset L^2(\Omega)$.

Observe that $w \in D(\Phi)$, since

$$\Phi(w) \leq \liminf_{k\to+\infty} \Phi_k(w_k) \leq \sup_{k\in\mathbb{N}} \frac{c(n,s_k)}{2}[w_k]^2_{s_k} \leq C \sup_{k\in\mathbb{N}}(1-s_k)[w_k]^2_{s_k} < \infty.$$

Let $B\colon D(\Phi)\times D(\Phi) \to \mathbb{R}$ be the bilinear form associeted to Φ, which is defined by

$$B(v,\eta) = \frac{1}{4}(\Phi(v+\eta) - \Phi(v-\eta)).$$

Let us denote by V the closure of $D(\Phi)$ in $L^2(\Omega)$ and consider the linear operator $T\colon D(T) \subset L^2(\Omega) \to L^2(\Omega)$ defined as $Tv = f$ where

$$D(T) = \left\{ v \in D(\Phi) \colon \exists\, f \in V \text{ such that } B(v,\eta) = \int_\Omega f\eta\,dx,\ \forall \eta \in D(\Phi) \right\}.$$

By Proposition 1.5.11, $D(T)$ is dense in $D(\Phi)$ with respect to the norm

$$\|v\|_\Phi = (\|v\|_{L^2(\Omega)} + \Phi(v))^{\frac{1}{2}}.$$

Moreover, the following relation holds

$$\sqrt{2}\|\cdot\|_\Phi \geq \|\cdot\|_{H^1_0(\Omega)}. \tag{3.2.6}$$

Indeed, if $z \in D(\Phi)$, as $\Phi_k \xrightarrow{\Gamma} \Phi$ in $L^2(\Omega)$, there exists $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \to z$ in $L^2(\Omega)$ and

$$\infty > \Phi(z) = \lim_{k\to\infty} \Phi_k(z_k) = \begin{cases} \lim_{k\to\infty} \frac{c(n,s_k)}{2}[z_k]^2_{s_k} & \text{if } z_k \in H^{s_k}_0(A_k), \\ \infty & \text{otherwise.} \end{cases}$$

Thus, $z_k \in H^{s_k}_0(A_k)$ and then

$$\|z\|^2_{H^1_0(\Omega)} \leq \liminf_{k\to\infty} c(n,s_k)[z_k]^2_{s_k} = 2\lim_{k\to\infty} \Phi_k(z_k) = 2\Phi(z) \leq 2\|z\|^2_\Phi.$$

Since (3.2.6) holds, $D(T)$ is dense in $D(\Phi)$ with respect to the strong topology of $H^1_0(\Omega)$. Now to achieve the proof it is enough to prove that $v = 0$ in $\{u = 0\}$ for all $v \in D(T)$.

Let $v \in D(T)$ and let $f \in Tv$; then by [31, Proposition 12.12] v is a minimum point of the functional

$$\Psi(\eta) = \frac{1}{2}\Phi(\eta) - \int_\Omega f\eta\,dx.$$

Let v_k be the minimum point of functional

$$\Psi_k(\eta) := \frac{1}{2}\Phi_k(\eta) - \int_\Omega f\eta\,dx;$$

then v_k is the solution of the problem

$$(-\Delta)^{s_k} v_k = f, \qquad v \in H^{s_k}_0(A_k).$$

Since $\Phi_k \xrightarrow{\Gamma} \Phi$, then $\Psi_k \xrightarrow{\Gamma} \Psi$ and so we have that $v_k \to v$ strongly in $L^2(\Omega)$.

For $\varepsilon > 0$ we consider f^ε to be a bounded function with compact support such that $\|f^\varepsilon - f\|_2 < \varepsilon$ and v_k^ε is solution of

$$(-\Delta)^{s_k} v_k^\varepsilon = f^\varepsilon \text{ in } A_k, \qquad v_k^\varepsilon \in H_0^{s_k}(A_k).$$

By using the linearity of the operator together with Hölder's and Poincaré's inequalities we get

$$\begin{aligned} \frac{c(n,s_k)}{2}[v_k^\varepsilon - v_k]_{s_k}^2 &= \int_\Omega (f^\varepsilon - f)(v_k^\varepsilon - v_k)\,dx \\ &\le \|f_\varepsilon - f\|_2 \|v_k^\varepsilon - v_k\|_2. \end{aligned}$$

From Poincaré's inequality we obtain that

$$(1-s_k)[v_k^\varepsilon - v_k]_{s_k}^2 \le C\varepsilon^2,$$

where C is independent on k.

Then, from Theorem 1.1.11, up to a subsequence, $v_k^\varepsilon \to v^\varepsilon$ strongly in $L^2(\Omega)$ and $\|v^\varepsilon - v\|_{H_0^1(\Omega)} \le C\varepsilon$. At this point is enough to prove that $v^\varepsilon = 0$ in $\{u = 0\}$ for all $\varepsilon > 0$.

Since $f^\varepsilon \le c^\varepsilon := \|f^\varepsilon\|_\infty$ and

$$(-\Delta)^{s_k} v_k^\varepsilon = f^\varepsilon \le c^\varepsilon = (-\Delta)^{s_k}(c^\varepsilon u_{A_k}^{s_k}) \text{ in } A_k, \qquad v_k^\varepsilon = c^\varepsilon u_{A_k}^{s_k} = 0 \text{ in } \mathbb{R}^n \setminus A_k,$$

the comparison principle, Proposition 1.2.13, gives that $v_k^\varepsilon \le c^\varepsilon u_{A_k}^{s_k}$ in $\mathbb{R}^n$. Analogously, $-v_k^\varepsilon \le c^\varepsilon u_{A_k}^{s_k}$ in $\mathbb{R}^n$.

As $k \to \infty$, we obtain that $|v^\varepsilon| \le c^\varepsilon u$ in $\mathbb{R}^n$, which implies that $v^\varepsilon = 0$ in $\mathbb{R}^n$ in $\{u = 0\}$ for any $\varepsilon > 0$ and that completes the proof. □

Let $0 < s_k \uparrow 1$ and a sequence $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{s_k}(\Omega)$, by Remark 3.2.5, we can assume that $A_k \xrightarrow{\gamma} A$, that is, $u_{A_k}^{s_k} \to u$ in $L^2(\Omega)$ and $A := \{u > 0\}$. Moreover, $u \le u_A^1$ in $\mathbb{R}^n$. We want to enlarge the set sequence in such a way that its function $L^2(\Omega)$-limit associated to the γ-limit (set) is still less than u_A^1. That is the content of next lemma, which is the counterpart of Lemma 3.1.11.

Lemma 3.2.8. *Let $0 < s_k \uparrow 1$ and for every $k \in \mathbb{N}$, let $A_k \in \mathcal{A}_{s_k}(\Omega)$, $A \in \mathcal{A}_1(\Omega)$. Assume that $u_{A_k}^{s_k} \to u$ in $L^2(\Omega)$ and that $u \le u_A^1$ in $\mathbb{R}^n$.*

Then, if $u_{A_k \cup A^\varepsilon}^{s_k} \to u^\varepsilon$ strongly in $L^2(\Omega)$, where $A^\varepsilon := \{u_A^1 > \varepsilon\}$, it holds that $u^\varepsilon \le u_A^1$ in $\mathbb{R}^n$.

Proof. By Proposition 3.1.5 with $s = 1$, the inequality $u^\varepsilon \le u_A^1$ in $\mathbb{R}^n$ will follow if we prove that $u^\varepsilon \in H_0^1(\Omega)$, $u^\varepsilon \le 0$ in $\mathbb{R}^n$ in $\mathbb{R}^n \setminus A$ and $-\Delta u^\varepsilon \le 1$ in Ω.

Observe that by Lemma 3.2.4 we have that $u, u^\varepsilon \in \mathcal{K}_1$. Let us define

$$v^\varepsilon := 1 - \tfrac{1}{\varepsilon}\min\{u_A^1, \varepsilon\} = \tfrac{1}{\varepsilon}(\varepsilon - u_A^1)^+.$$

and observe that $0 \leq v^\varepsilon \leq 1$ and $v^\varepsilon = 0$ in A^ε since $0 \leq \min\{u_A^1, \varepsilon\} \leq \varepsilon$ and $\frac{1}{\varepsilon}\min\{u_A^1, \varepsilon\} = 1$ in A^ε. If we define

$$u_{k,\varepsilon} := u^{s_k}_{A_k \cup A^\varepsilon}, \qquad w_{k,\varepsilon} := \min\{v^\varepsilon, u_{k,\varepsilon}\},$$

it holds that $w_{k,\varepsilon} \geq 0$ since the comparison principle gives $u_{k,\varepsilon} \geq 0$, and also $v^\varepsilon \geq 0$.

Since $v^\varepsilon = 0$ in A^ε, it follows that $w_{k,\varepsilon} = 0$ in A^ε. Moreover, since $u_{k,\varepsilon} = 0$ in $\mathbb{R}^n \setminus (A_k \cup A^\varepsilon)$, it holds that $w_{k,\varepsilon} = 0$ in $\mathbb{R}^n \setminus (A_k \cup A^\varepsilon)$, and consequently, $w_{k,\varepsilon} \in H_0^{s_k}(A_k)$.

Notice that $w_{k,\varepsilon} \to w_\varepsilon := \min\{v^\varepsilon, u^\varepsilon\}$ strongly in $L^2(\Omega)$, and then, applying Lemma 3.2.7, we get $w_\varepsilon \in H_0^1(\{u > 0\})$, from where $w_\varepsilon = 0$ in $\{u = 0\}$. The relation $0 \leq u \leq u_A^1$ in $\mathbb{R}^n$ implies the inclusion $\{u_A^1 = 0\} \subset \{u = 0\}$ from where $w_\varepsilon \in H_0^1(\{u_A^1 > 0\})$. Moreover, since $\{u_A^1 > 0\} \subset A$, we have that $w_\varepsilon = 0$ in $\mathbb{R}^n \setminus A$. Now, being $v^\varepsilon = 1$ in $\mathbb{R}^n \setminus A$, we get $u^\varepsilon = 0$ in $\mathbb{R}^n \setminus A$, and in particular, $u^\varepsilon \leq 0$ in $\mathbb{R}^n \setminus A$.

Finally, it remains to see that $-\Delta u^\varepsilon \leq 1$ in Ω. Observe that $u_{k,\varepsilon} \in \mathcal{K}_{s_k}$ and $u_{k,\varepsilon} \to u^\varepsilon$ strongly in $L^2(\Omega)$. Then $u^\varepsilon \in \mathcal{K}_1$ by Lemma 3.2.4. Thus $-\Delta u^\varepsilon \leq 1$ in Ω and the proof is complete. □

With the help of these lemmas, we are now in position to prove the **compactness** result for the γ-convergence of sets.

Theorem 3.2.9. *Let $0 < s_k \uparrow 1$ and $\{A_k\}_{k\in\mathbb{N}} \in \mathcal{A}_{s_k}(\Omega)$, there exist a subsequence $\{A_{k_j}\}_{j\in\mathbb{N}} \subset \{A_k\}_{k\in\mathbb{N}}$, an enlarged sequence $\{\tilde{A}_{k_j}\}_{j\in\mathbb{N}}$ and $A \in \mathcal{A}_1(\Omega)$ such that*

$$A_{k_j} \subset \tilde{A}_{k_j}, \qquad \text{and} \qquad \tilde{A}_{k_j} \xrightarrow{\gamma} A.$$

Moreover, $|A| \leq \liminf_{k\to\infty} |A_k|$.

Proof. By Remark 3.2.5, we can suppose $A_k \xrightarrow{\gamma} A$, that is, $u^{s_k}_{A_k} \to u$ in $L^2(\Omega)$. In addition, $u \leq u_A^1$ holds, where $A := \{u > 0\}$ and u_A^1 is defined by (3.2.3).

Let $\varepsilon > 0$. Consider $A^\varepsilon := \{u_A^1 > \varepsilon\}$ and $u_{k,\varepsilon} := u^{s_k}_{A_k \cup A^\varepsilon} \in \mathcal{K}_{s_k}$. Then, by Lemma 3.2.4, there exist a subsequence (still denoted by the same index) and a function $u^\varepsilon \in \mathcal{K}_1$ such that $u_{k,\varepsilon} \to u^\varepsilon$ in $L^2(\Omega)$, when $k \to \infty$.

By Lemma 3.2.8, it holds that $u^\varepsilon \leq u_A^1$ in $\mathbb{R}^n$.

Since $A^\varepsilon \subset A_k \cup A^\varepsilon$, we obtain

$$u^{s_k}_{A^\varepsilon} \leq u^{s_k}_{A_k \cup A^\varepsilon}.$$

On the other hand, by Lemma 3.2.6, $u^{s_k}_{A^\varepsilon} \to u^1_{A^\varepsilon}$ strongly in $L^2(\Omega)$. Then, we can pass to the limit as $k \to \infty$ in the previous inequality to conclude that

$$u^1_{A^\varepsilon} \leq u^\varepsilon.$$

It can be easily checked that $u^1_{A^\varepsilon} = (u_A^1 - \varepsilon)_+$. Moreover, from Lemma 3.2.8,

$$(u_A^1 - \varepsilon)_+ \leq u^\varepsilon \leq u_A^1.$$

Observe that $\{u^\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}_1$. By [24], $\mathcal{K}_1$ is a compact set in $L^2(\Omega)$, so that there exists an $L^2(\Omega)$-convergent subsequence. So, the previous inequality tells that this $L^2(\Omega)$-limit function should be u^1_A.

Thus, there exists a sequence $0 < \varepsilon_k \downarrow 0$ such that

$$u^{s_k}_{A_k \cup A^{\varepsilon_k}} \to u^1_A \text{ strongly in } L^2(\Omega).$$

That is, the enlarged sequence $A_k \cup A^{\varepsilon_k} =: \tilde{A}_k$ γ-converges to A.

It is remained to prove that $|A|$ is bounded from above by $\liminf_{k\to\infty} |A_k|$. We omit its proof since it can be demonstrated with the same strategy from Theorem 3.1.12, using the convergence $u^{s_k}_{A_k} \to u$ in $L^2(\Omega)$ and Chebyshev's inequality. □

We extend the γ-compactness result for fixed $m \in \mathbb{N}$ coordinates.

Corollary 3.2.10. *Let $0 < s_k \uparrow 1$ and $(A^k_1, \ldots, A^k_m) \in \mathcal{A}_{s_k}(\Omega)^m$.*

Then, there exist a subsequence $\{(A^{k_j}_1, \ldots, A^{k_j}_m)\}_{j\in\mathbb{N}} \subset \{(A^k_1, \ldots, A^k_m)\}_{k\in\mathbb{N}}$, an enlarged sequence $\{(\tilde{A}^{k_j}_1, \ldots, \tilde{A}^{k_j}_m)\}_{j\in\mathbb{N}}$ and $(A_1, \ldots, A_m) \in \mathcal{A}_1(\Omega)^m$ such that

$$A^{k_j}_i \subset \tilde{A}^{k_j}_i \text{ for every } i = 1, \ldots, m, \quad \text{and} \quad (\tilde{A}^{k_j}_1, \ldots, \tilde{A}^{k_j}_m) \xrightarrow{\gamma} (A_1, \ldots, A_m).$$

Proof. By Theorem 3.2.9, there exist $A_1 \in \mathcal{A}_1(\Omega)$, a subsequence $\{A^{k_j}_1\}_{j\in\mathbb{N}} \subset \{A^k_1\}_{k\in\mathbb{N}}$ and an enlarged sequence $\{\tilde{A}^{k_j}_1\}_{j\in\mathbb{N}}$ such that

$$A^{k_j}_1 \subset \tilde{A}^{k_j}_1, \quad \text{and} \quad \tilde{A}^{k_j}_1 \xrightarrow{\gamma} A_1.$$

Now, consider $A^{k_j}_2 \in \mathcal{A}_{s_{k_j}}(\Omega)$ and apply again Theorem 3.2.9. Thus, there exist $A_2 \in \mathcal{A}_1(\Omega)$, a subsequence $\{A^{k_{j_l}}_2\}_{l\in\mathbb{N}} \subset \{A^{k_j}_2\}_{j\in\mathbb{N}}$ and an enlarged sequence $\{\tilde{A}^{k_{j_l}}_2\}_{l\in\mathbb{N}}$ such that

$$A^{k_{j_l}}_i \subset \tilde{A}^{k_{j_l}}_i, \quad \text{and} \quad A^{k_{j_l}}_i \xrightarrow{\gamma} A_i \text{ for } i = 1, 2.$$

Repeating this argument and renaming the final subsequence extracted, we obtain the enlarged sequence $(\tilde{A}^{k_j}_1, \ldots, \tilde{A}^{k_j}_m) \in \mathcal{A}_{s_{k_j}}(\Omega)^m$ and $(A_1, \ldots, A_m) \in \mathcal{A}_1(\Omega)^m$ such that

$$A^{k_j}_i \subset \tilde{A}^{k_j}_i \text{ for every } i = 1, \ldots, m; \quad \text{and} \quad (\tilde{A}^{k_j}_1, \ldots, \tilde{A}^{k_j}_m) \xrightarrow{\gamma} (A_1, \ldots, A_m).$$

□

3.2.2 Transition from nonlocal to local minimizers

Once we know the existence of an optimal shape for each $0 < s < 1$, we want to analyze the limit of these minimizers and its minimum values when $s \uparrow 1$.

Recall that the existence of solution to the first problem (3.1.7) in the case $s = 1$ was solved by Buttazzo-Dal Maso in [24] and the second problem (3.1.8) in the case $s = 1$ was proved by Bucur-Buttazzo-Henrot in [22]. Both works are related to shape optimization problems involving the Laplacian operator. We prove in this thesis the fractional version of both, and that is the motivation for the name **transition from nonlocal to local minimizers**.

In order to perform such analysis we need to assume some asymptotic behaviour on the cost functionals.

Let $0 < s \leq 1$, $m \in \mathbb{N}$ and $F_s \colon \mathcal{A}_s(\Omega)^m \to [0, \infty)$. Now, we give the assumptions:

(H_1) **Continuity**. For every $(A_1, \ldots, A_m) \in \mathcal{A}_1(\Omega)^m$,

$$F_1(A_1, \ldots, A_m) = \lim_{s \uparrow 1} F_s(A_1, \ldots, A_m).$$

(H_2) **Liminf inequality**. For every $0 < s_k \uparrow 1$, $(A_1^k, \ldots, A_m^k) \in \mathcal{A}_{s_k}(\Omega)^m$ and $(A_1, \ldots, A_m) \in \mathcal{A}_1(\Omega)^m$ such that $(A_1^k, \ldots, A_m^k) \xrightarrow{\gamma} (A_1, \ldots, A_m)$,

$$F_1(A_1, \ldots, A_m) \leq \liminf_{k \to \infty} F_{s_k}(A_1^k, \ldots, A_m^k).$$

For a class of shape optimization problems: *fixed measure*

First, we introduce the notation

$$m_s := \min \left\{ F_s(A) \colon A \in \mathcal{A}_s(\Omega),\ |A| \leq c \right\}, \tag{3.2.7}$$

for every $0 < s \leq 1$. The case $s = 1$ is due to Buttazzo-Dal Maso [24] and $0 < s < 1$ to Theorem 3.1.18.

Theorem 3.2.11. *For any $0 < s \leq 1$, let $F_s \colon \mathcal{A}_s(\Omega) \to \mathbb{R}$ be a decreasing γ_s-lower semicontinuous functional. Assume that (H_1) and (H_2) are satisfied, for $m = 1$.*

Then

$$m_1 = \lim_{s \uparrow 1} m_s.$$

Moreover, if $A_s \in \mathcal{A}_s(\Omega)$ is a minimizer for (3.1.7), then there exists a sequence $0 < s_k \uparrow 1$, sets $\tilde{A}_{s_k} \supset A_{s_k}$ and a set $A_1 \in \mathcal{A}_1(\Omega)$ such that $\tilde{A}_{s_k} \xrightarrow{\gamma} A_1$ and A_1 is a minimizer for (3.1.7) with $s = 1$.

Proof. By Theorem 3.1.18, there exists $A_k \in \mathcal{A}_{s_k}(\Omega)$ such that

$$F_{s_k}(A_k) = \min\{F_{s_k}(A) \colon A \in \mathcal{A}_{s_k}(\Omega), |A| \leq c\}.$$

Let $A \in \mathcal{A}_1(\Omega)$ be such that $|A| \leq c$. Observe that $\mathcal{A}_1(\Omega) \subset \mathcal{A}_{s_k}(\Omega)$ for every $k \in \mathbb{N}$, see Remark 1.3.7. Since A_k is the minimizer, we know that

$$\limsup_{k \to \infty} F_{s_k}(A_k) \leq \lim_{k \to \infty} F_{s_k}(A) = F_1(A),$$

where we use condition (H_1) to obtain the last identity. It follows that

$$\limsup_{k\to\infty} m_{s_k} \le m_1. \tag{3.2.8}$$

Now, we use the compactness result for the γ-convergence. By Theorem 3.2.9, there exist a subsequence (still denoted by the same index) $A_k \in \mathcal{A}_{s_k}(\Omega)$, an enlarged sequence $\tilde{A}_k \in \mathcal{A}_{s_k}(\Omega)$ and a set $A_1 \in \mathcal{A}_1(\Omega)$ such that

$$A_k \subset \tilde{A}_k, \quad \tilde{A}_k \xrightarrow{\gamma} A_1 \quad \text{and} \quad |A| \le \liminf_{k\to\infty} |A_k| \le c.$$

Finally, from condition (H_2) (**Liminf**) and the fact that each functional is decreasing with respect to the set inclusion, we conclude that

$$F_1(A) \le \liminf_{k\to\infty} F_{s_k}(\tilde{A}_k) \le \liminf_{k\to\infty} F_{s_k}(A_k),$$

from where it follows that

$$m_1 \le \liminf_{k\to\infty} m_s. \tag{3.2.9}$$

Putting together (3.2.8) and (3.2.9) the result follows. □

For a class of optimal partition problems

Let $m \in \mathbb{N}$ and $0 < s \le 1$. Let $F_s\colon \mathcal{A}_s(\Omega)^m \to [0,\infty]$ be a decreasing weak γ_s-lower semicontinuous functional. Then, by Theorem 3.1.20($0 < s < 1$) and [22, Theorem 3.2](the case $s = 1$) there exists a solution $(A_1^s, \dots, A_m^s)$ to

$$m_s := \min\left\{F_s(B_1,\dots,B_m)\colon B_i \in \mathcal{A}_s(\Omega),\, \operatorname{cap}_s(B_i \cap B_j, \Omega) = 0 \text{ for } i \ne j\right\}. \tag{3.2.10}$$

Theorem 3.2.12. *Let $m \in \mathbb{N}$ be fixed and $0 < s \le 1$. Let $F_s\colon \mathcal{A}_s(\Omega)^m \to [0,\infty]$ be a decreasing weak γ_s-lower semicontinuous functional. Assume that (H_1)-(H_2) are verified. Then,*

$$m_1 = \lim_{s\uparrow 1} m_s, \tag{3.2.11}$$

where m_s is defined in (3.2.10).

Moreover, if $(A_1^s, \dots, A_m^s)$ is a minimizer of (3.2.10), then, there exist a subsequence $0 < s_k \uparrow 1$, $(\tilde{A}_1^{s_k}, \dots, \tilde{A}_m^{s_k}) \in \mathcal{A}_{s_k}(\Omega)^m$ and $(A_1^1, \dots, A_m^1) \in \mathcal{A}_1(\Omega)^m$ such that

$$A_i^{s_k} \subset \tilde{A}_i^{s_k} \quad \textit{and} \quad (\tilde{A}_1^{s_k}, \dots, \tilde{A}_m^{s_k}) \xrightarrow{\gamma} (A_1^1, \dots, A_m^1),$$

where $(A_1^1, \dots, A_m^1)$ is a minimizer of (3.2.10) with $s = 1$.

Proof. First, notifce that m_1 is achieved by [22, Theorem 3.2].

Let $0 < s_k \uparrow 1$. By Theorem 3.1.20, there exists $(A_1^k, \dots, A_m^k) \in \mathcal{A}_{s_k}(\Omega)^m$ such that

$$\operatorname{cap}_{s_k}(A_i^k \cap A_j^k, \Omega) = 0 \text{ for } i \ne j \text{ and } F_{s_k}(A_1^k, \dots, A_m^k) = m_k, \tag{3.2.12}$$

where $m_k = m_{s_k}$ defined in (3.1.8).

Let $(A_1, \dots, A_m) \in \mathcal{A}_1(\Omega)^m$ be such that $\text{cap}_1(A_i \cap A_j, \Omega) = 0$ for $i \neq j$. Since $0 < s_k \uparrow 1$, we can assume $0 < \varepsilon_0 < s_k \uparrow 1$, for some fixed ε_0.

Now, recalling Corollary 1.3.6 and Remark 1.3.7, we know that $(A_1, \dots, A_m)$ belongs to

$$\{(B_1, \dots, B_m) \colon B_i \in \mathcal{A}_{s_k}(\Omega),\ \text{cap}_{s_k}(B_i \cap B_j, \Omega) = 0 \text{ for } i \neq j\},$$

for every $k \in \mathbb{N}$. This fact and condition (H_1) imply that

$$\limsup_{k\to\infty} F_{s_k}(A_1^k, \dots, A_m^k) \leq \lim_{k\to\infty} F_{s_k}(A_1, \dots, A_m) = F_1(A_1, \dots, A_m).$$

It follows that

$$\limsup_{k\to\infty} m_k \leq m_1. \tag{3.2.13}$$

To see the remaining inequality, let us denote $u_i^k := u_{A_i^k}^{s_k} \in \mathcal{K}_{s_k}$. By Lemma 3.2.4, there is $u_i \in \mathcal{K}_1$ such that, up to a subsequence, $u_i^k \to u_i$ strongly in $L^2(\Omega)$ and a.e. in Ω.

Denote by $A_i := \{u_i > 0\} \in \mathcal{A}_1(\Omega)$ for every $i = 1, \dots, m$. We claim that $\text{cap}_1(A_i \cap A_j, \Omega) = 0$ for $i \neq j$.

Indeed, let $i \neq j$ be fixed. For each $k \in \mathbb{N}$, due to Lemma 1.3.3 and (3.2.12), we know that

$$|\{u_i^k \cdot u_j^k \neq 0\}| = |A_i^k \cap A_j^k| \leq C(n, s_k)\, \text{cap}_{s_k}(A_i^k \cap A_j^k, \Omega) = 0.$$

Then, $u_i^k \cdot u_j^k = 0$ a.e. in $\mathbb{R}^n$. Since $u_l^k \to u_l$ a.e. in Ω for $l = 1, 2$, we conclude $u_i \cdot u_j = 0$ a.e in Ω, it is still true in $\mathbb{R}^n \setminus \Omega$ considering that they belong to $H_0^s(\Omega)$. So, $u_i \cdot u_j = 0$ a.e. in $\mathbb{R}^n$.

Reminding that we are working with 1-quasi continuous representative functions in $H_0^1(\Omega)$, the previous identity $u_i \cdot u_j = 0$ a.e. in $\mathbb{R}^n$ and [56, Lemma 3.3.30] tells that $u_i \cdot u_j = 0$ 1-q.e. in $\mathbb{R}^n$. That means, $\text{cap}_1(A_i \cap A_j, \Omega) = 0$.

Consequently, $(A_1, \dots, A_m)$ is admissible to the problem (3.1.8) with $s = 1$ and we obtain $m_1 \leq F_1(A_1, \dots, A_m)$.

Moreover, by Theorem 3.2.9, there exists an enlarged sequence $\tilde{A}_i^k \in \mathcal{A}_{s_k}(\Omega)$ such that $A_i^k \subset \tilde{A}_i^k$ and $(\tilde{A}_1^k, \dots, \tilde{A}_m^k)$ $\gamma-$converges to $(A_1, \dots, A_m)$, occasionally taking a subsequence.

Finally, from condition (H_2) and the decreasing property of F_{s_k}, we conclude that

$$\begin{aligned} m_1 \leq F_1(A_1, \dots, A_m) &\leq \liminf_{k\to\infty} F_{s_k}(\tilde{A}_1^k, \dots, \tilde{A}_m^k) \\ &\leq \liminf_{k\to\infty} F_{s_k}(A_1^k, \dots, A_m^k) = \liminf_{k\to\infty} m_k. \end{aligned}$$

Therefore, from the previous conclusion and (3.2.13) we have the identity (3.2.11) and the results follow. □

3.3 Examples

Let first establish some notations. Given a bounded domain $A \in \mathcal{A}_s(\Omega)$, consider the problem

$$(-\Delta)^s u = \lambda^s u \quad \text{in } A, \qquad u \in H_0^s(A) \tag{3.3.1}$$

where $\lambda^s \in \mathbb{R}$ is the eigenvalue parameter. It is well-known that there exists a discrete sequence $\{\lambda_k^s(A)\}_{k\in\mathbb{N}}$ of positive eigenvalues of (3.3.1) approaching $+\infty$ whose corresponding eigenfunctions $\{u_k^s\}_{k\in\mathbb{N}}$ form an orthogonal basis in $L^2(A)$. These facts follows directly from the spectral theorem for compact and self adjoints operators, see [20]. Moreover, the following variational characterization holds for the eigenvalues

$$\lambda_k^s(A) = \min_{u \perp W_{k-1}} \frac{c(n,s)}{2} \frac{[u]_s^2}{\|u\|_2^2}, \tag{3.3.2}$$

where W_k is the space spanned by the first k eigenfunctions $u_1^s, \dots, u_k^s$.

Functions F_s being decreasing γ_s-lower semicontonuous include a large family of examples.

Consider the application $A \mapsto \lambda_k^s(A)$. As in the local case, one can prove its γ_s-lower semicontinuity.

Fixed measure examples

For instance, if we consider the application $A \mapsto \lambda_k^s(A)$, Theorem 3.1.18 and Remark 3.1.17 claim that for every $k \in \mathbb{N}$ and $0 < c < |\Omega|$, the minimum

$$\min\{\lambda_k^s(A) \colon A \in \mathcal{A}_s(\Omega),\, |A| = c\}$$

is achieved. More generally, the minimum

$$\min\{\Phi_s(\lambda_{k_1}^s(A), \dots, \lambda_{k_N}^s(A)) \colon A \in \mathcal{A}_s(\Omega),\, |A| = c\}$$

is achieved, where $\Phi_s \colon \mathbb{R}^N \to \bar{\mathbb{R}}$ is lower semicontinuous and increasing in each coordinate.

Moreover, if $\Phi_s(t_1, \dots, t_N) \to \Phi_1(t_1, \dots, t_N)$ for every $(t_1, \dots, t_N) \in \mathbb{R}^N$ and

$$\Phi_1(t_1, \dots, t_N) \leq \liminf_{k\to\infty} \Phi_{s_k}(t_1^k, \dots, t_N^k),$$

for every $(t_1^k, \dots, t_N^k) \to (t_1, \dots, t_N)$, then Theorem 3.2.11, Remark 3.1.17 together with the result of [19] imply that

$$\begin{aligned}\min\{\Phi_1(\lambda_{k_1}(A), \dots, \lambda_{k_N}(A)) \colon A \in \mathcal{A}_1(\Omega),\, |A| = c\} \\ = \lim_{s\uparrow 1} \min\{\Phi_s(\lambda_{k_1}^s(A), \dots, \lambda_{k_N}^s(A)) \colon A \in \mathcal{A}_s(\Omega),\, |A| = c\}.\end{aligned}$$

Optimal partition examples

Consider functionals $F_s(A_1, \dots, A_m) = \Phi_s(\lambda_{k_1}^s(A_1), \dots, \lambda_{k_m}^s(A_m))$. Theorem 3.1.20 and Remark 3.1.21 claim that for every $(k_1, \dots, k_m) \in \mathbb{N}^m$, the minimum

$$\min\{\Phi_s(\lambda_{k_1}^s(A_1), \dots, \lambda_{k_m}^s(A_m)) \colon A_i \in \mathcal{A}_s(\Omega),\, \operatorname{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\}$$

is achieved, where $\Phi_s\colon \mathbb{R}^m \to \bar{\mathbb{R}}$, is increasing in each coordinate and lower semicontinuous.

Moreover, if $\Phi_s(t_1,\dots,t_m) \to \Phi_1(t_1,\dots,t_m)$ for every $(t_1,\dots,t_m) \in \mathbb{R}^m$ and

$$\Phi_1(t_1,\dots,t_m) \le \liminf_{k\to\infty} \Phi_{s_k}(t_1^k,\dots,t_m^k),$$

for every $(t_1^k,\dots,t_m^k) \to (t_1,\dots,t_m)$, then Theorem 3.2.12, Remark 3.1.21 together with the existence result of [22] imply that

$$\begin{aligned}&\min\{\Phi_1(\lambda_{k_1}(A_1),\dots,\lambda_{k_m}(A_m))\colon A_i \in \mathcal{A}_1(\Omega),\ \mathrm{cap}_1(A_i\cap A_j,\Omega)=0 \text{ for } i\neq j\}\\ &= \lim_{s\uparrow 1} \min\{\Phi_s(\lambda^s_{k_1}(A_1),\dots,\lambda^s_{k_m}(A_m))\colon A_i\in\mathcal{A}_s(\Omega),\ \mathrm{cap}_s(A_i\cap A_j,\Omega)=0 \text{ for } i\neq j\}.\end{aligned}$$

Resumen del capítulo

En este capítulo, contamos el aporte de esta tesis en problemas de diseño óptimo donde se ve involucrado el laplaciano fraccionario. Además, se estudia el comportamiento asintótico de dichos problemas, obteniendo un resultado de convergencia a los valores mínimos y las formas óptimas para el caso $s = 1$, estudiado en [22, 24]. Nuestro resultados pueden ser encontrados en [47, 78].

Se introduce una noción de convergencia para s-quasi abiertos, γ_s-convergencia, que resulta precompacta. Gracias a este resultado de compacidad, se logra probar existencia de solución para los siguientes problemas:

$$\min\{F_s(A)\colon A\in\mathcal{A}_s(\Omega),\ |A| = c\}, \qquad \text{para } 0 < c < |\Omega| \text{ fija},$$

y en segundo lugar,

$$\min\{F_s(A_1,\dots,A_m)\colon A_i\in\mathcal{A}_s(\Omega),\ A_i\cap A_j=\emptyset \text{ para } i\neq j\},\ \text{ para } m\in\mathbb{N} \text{ fija},$$

donde $\mathcal{A}_s(\Omega)$ es la clase de dominios admisibles, y los funcionales de costo son decrecientes y semi continuos inferiores respecto a la γ_s-convergencia.

Para lidiar con el comportamiento asintótico de los problemas anteriores, se introduce una segunda noción de convergencia de conjuntos: la γ-convergencia.

Además, se asume que para cada $0 < s \le 1$, tenemos la siguiente relación entre los funcionales de costo,

Continuidad. Para todo $(A_1,\dots,A_m)\in\mathcal{A}_1(\Omega)^m$, se tiene que

$$F_1(A_1,\dots,A_m) = \lim_{s\uparrow 1} F_s(A_1,\dots,A_m).$$

Desigualdad de liminf. Para toda $0 < s_k \uparrow 1$, $(A_1^k,\dots,A_m^k)\in\mathcal{A}_{s_k}(\Omega)^m$ y $(A_1,\dots,A_m)\in\mathcal{A}_1(\Omega)^m$ tales que $(A_1^k,\dots,A_m^k) \xrightarrow{\gamma} (A_1,\dots,A_m)$, se tiene que

$$F_1(A_1,\dots,A_m) \le \liminf_{k\to\infty} F_{s_k}(A_1^k,\dots,A_m^k).$$

Obteniendo, en primer lugar,

$$m_1 = \lim_{s \uparrow 1} m_s, \qquad \text{donde} \qquad m_s := \min \left\{ F_s(A) \colon A \in \mathcal{A}_s(\Omega),\ |A| \le c \right\},$$

para $0 < s \le 1$. Más aún, si $A_s \in \mathcal{A}_s(\Omega)$ es un minimizante del s-*problema*, entonces existe una sucesión $0 < s_k \uparrow 1$, conjuntos $\tilde{A}_{s_k} \supset A_{s_k}$ y un $A_1 \in \mathcal{A}_1(\Omega)$ tales que $\tilde{A}_{s_k} \xrightarrow{\gamma} A_1$ y A_1 es un minimizante para el 1-*problema*.

En segundo lugar, con la notación

$$m_s := \min \left\{ F_s(B_1, \dots, B_m) \colon B_i \in \mathcal{A}_s(\Omega),\, \mathrm{cap}_s(B_i \cap B_j, \Omega) = 0 \text{ for } i \ne j \right\},$$

se prueba que $m_1 = \lim_{s \uparrow 1} m_s$. Más aún, si $(A_1^s, \dots, A_m^s)$ es un minimizante para el s-*problema*, entonces existe una sucesión $0 < s_k \uparrow 1$, $(\tilde{A}_1^{s_k}, \dots, \tilde{A}_m^{s_k}) \in \mathcal{A}_{s_k}(\Omega)^m$ y $(A_1^1, \dots, A_m^1) \in \mathcal{A}_1(\Omega)^m$ tales que

$$A_i^{s_k} \subset \tilde{A}_i^{s_k} \quad \text{and} \quad (\tilde{A}_1^{s_k}, \dots, \tilde{A}_m^{s_k}) \xrightarrow{\gamma} (A_1^1, \dots, A_m^1),$$

donde $(A_1^1, \dots, A_m^1)$ es un minimizante del 1-*problema*.

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