Tesis Doctoral

Métodos combinatorios y algoritmos en topología de dimensiones bajas y la conjetura de Andrews-Curtis

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Métodos combinatorios y algoritmos en topología de dimensiones bajas y la conjetura de Andrews-Curtis.

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A mi abuela del alma

Métodos combinatorios y algoritmos en topología de dimensiones bajas y la conjetura de Andrews-Curtis

Resumen

En esta Tesis estudiamos la conjetura de Andrews-Curtis desde un nuevo enfoque combinatorio y computacional, utilizando como herramienta principal la teoría de espacios topológicos finitos.

La *conjetura de Andrews Curtis* (1965) es uno de los problemas abiertos más relevantes de la topología geométrica, con raíces en la teoría de homotopía simple de Whitehead y la teoría combinatoria de grupos, de íntima conexión con otros problemas de la topología algebraica, como la conjetura de asfericidad de Whitehead, la conjetura de Zeeman y la conjetura (ahora teorema) de Poincaré. Tiene formulaciones equivalentes en el contexto de la teoría combinatoria de grupos y en el de la topología algebraica en dimensiones bajas. Esencialmente apunta a reconocer, con métodos discretos, *grupos triviales* o *espacios contráctiles*, problemas frecuentes en topología algebraica y teoría de grupos, que en general no son decidibles computacionalmente. Si bien la conjetura se sabe verdadera para cierta clase de complejos (ej, los *standard spines*, o los *complejos cuasi-construibles*), ha resultado difícil realizar avances generales. Los abordajes computacionales se vieron limitados por la complejidad exponencial de los algoritmos. Como consecuencia de los resultados de este trabajo, obtenemos una serie de métodos combinatorios y algorítmicos que permiten reconocer complejos o presentaciones que satisfacen la conjetura, así como también ampliar las clases para las cuales la conjetura se sabe cierta.

Nuestro punto de partida fue la construcción de un nuevo modelo finito asociado a una presentación: un poset fácil de describir en términos de los generadores y relaciones. Desarrollamos una serie de métodos de reducción y métodos discretos de transformación de posets. Los últimas están inspirados en técnicas de *coloreo* de posets, y en teoría de Morse discreta y *matchings acíclicos*. Demostramos una versión más fuerte de la teoría de Morse, que precisa equivalencias *simples* y cotas en las dimensiones los complejos involucrados en la deformación, así como también detalles de las funciones de adjunción. Estas mejoras resultaron esenciales para que la teoría se torne aplicable a nuestro problema.

Estos métodos nos permitieron abordar desde un nuevo punto de vista algunos de los *potenciales contraejemplos*, que consisten de presentaciones balanceadas del grupo trivial para las cuales no se conoce si satisfacen o no la conjetura. Mostramos que nuestros métodos son útiles para estudiar presentaciones sin imponer cotas en la longitud de relaciones.

Dada la íntima conexión que existe entre presentaciones y CW complejos de dimensión 2, nuestros resultados se pudieron aplicar también a otros problemas de la topología algebraica en dimensiones bajas.

Los algoritmos que construimos de reconocimiento de espacios/presentaciones que satisfacen la conjetura se encuentran implementados en lenguaje Python, en el marco del software libre SAGE. El código se encuentra disponible en el siguiente repositorio de GitHub: `https://github.com/ximenafernandez/Finite-Spaces`.

Palabras clave: conjetura de Andrews-Curtis, espacios topológicos finitos, complejos celulares, homotopía simple, presentaciones de grupos, teoría de Morse.

Combinatorial methods and algorithms in low dimensional topology and the Andrews Curtis conjecture

Abstract

In this Thesis we study the Andrews-Curtis conjecture from a new combinatorial and computational approach, using as a main tool the theory of finite topological spaces.

The Andrews Curtis conjecture (1965) is one the most relevant open problems in geometric topology, with roots in Whitehead's simple homotopy theory and combinatorial group theory, and close relation with other important problems in algebraic topology, such as Whitehead asphericity conjecture, Zeeman conjecture and the Poincaré conjecture (now a theorem). It has equivalent statements in the combinatorial group theory context, as well as in the context of low dimensional topology. It basically aims to recognize, with discrete methods, *triviality of groups* or *contractibility of spaces*, frequent questions in algebraic topology and group theory, which are computationally undecidable in general. Although the conjecture is known to be true for some clases of complexes (such as the *standard spines* or the *quasi-constructible complexes*), it has been difficult to obtain general progress. The computational approaches have been shown to be limited by the exponential complexity on the amount of generators and total length of the relators. As a consequence of the results of this work, we obtain a collection of combinatorial and algorithmic methods to recognize complexes or presentations satisfying the conjecture and to increase the classes for which the conjecture is known to be true.

Our starting point is the construction of a new finite model associated to a presentation: a poset easy to describe in terms of the generators and relators. We developed a number of reduction methods and discrete transformation methods for posets. The last ones are inspired, on the one hand, in *coloring* techniques of posets and, on the other, in discrete Morse theory and *acyclic matchings*. We prove a stronger version of Morse theory, which specifies simple homotopies and bounds on the dimensions of the involved complexes, as well as details about the adjunction maps. These improvements were essential to make the theory applicable to our problem.

These methods allowed us to attack from a new point of view various examples from the classical list of *potential counterexamples*, consisting of balanced presentations of the trivial group for which it is not known whether the conjecture is satisfied. We show that our methods are useful to study presentations without imposing bounds on the total length of relators.

Given the close relationship between group presentations and CW complexes of dimension 2, our results could be applied to other problems in low dimensional topology.

The algorithms that we constructed for identifying spaces/presentations satisfying the conjecture, are implemented in Python, using the framework of the free software SAGE. The source code is available in `https://github.com/ximenafernandez/Finite-Spaces`.

Key words: Andrews-Curtis Conjecture, finite topological spaces, cellular complexes, simple homotopy, group presentations, Morse theory.

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Contents

Introducción

Uno de los principales propósitos de la Topología Algebraica es clasificar a los espacios según su tipo homotópico. Se sabe que este problema resulta algorítmicamente indecidible, incluso trabajando con modelos combinatorios como los complejos simpliciales y una noción más rígida de homotopía: la homotopía simple. Asimismo, el problema de decidir si dos grupos son isomorfos es un problema fundamental en teoría de grupos. Pese a los avances realizados por Tietze, quien generó una lista corta y efectiva de transformaciones para llevar una presentación de un grupo a cualquier otra de un grupo isomorfo, este problema también fue probado no resoluble computacionalmente incluso para grupos finitamente presentados.

Por supuesto, la teoría combinatoria de grupos y la topología crecieron juntas y su conexión vía el grupo fundamental es bien sabida. Como consecuencia de la interacción de estas teorías, podemos decir que los problemas que mencionamos recién son esencialmente el mismo.

Uno de los principales objetivos de este trabajo es atacar desde un nuevo punto de vista la conjetura de Andrews-Curtis, un problema abierto desde hace 50 años que está íntimamente relacionado con las preguntas anteriores. Introducimos nuevos métodos combinatorios y algoritmos para manipular CW-complejos y presentaciones de grupos preservando su tipo homotópico (simple) y -respectivamente- clase de equivalencia.

Orígenes del problema

En 1965, en su famoso artículo "*Free groups and handlebodies*" [AC65], James J. Andrews y Morton L. Curtis plantearon una pregunta sobre presentaciones de grupos, generalizando resultados previos de Nielsen [Nie18] sobre grupos libres y explicando algunas consecuencias relevantes de una posible respuesta afirmativa a su pregunta sobre la (en ese momento abierta) conjetura de Poincaré. El artículo gana relevancia con una observación del referí sobre una versión más débil de la conjetura y su conexión con un problema (a abierto) sobre deformaciones de complejos celulares.

Dado un grupo libre F finitamente generado en $x_1, \cdots, x_n$, el teorema de Nielsen afirma que toda otra base $y_1, \cdots, y_n$ de F se puede obtener a partir de $x_1, \cdots, x_n$ aplicando una sucesión finita de los siguientes movimientos elementales: invertir un elemento, intercambiar elementos, multiplicar un elemento por otro. Ahora, si F es libre en $x_1, \cdots, x_n$ y el conjunto $r_1, \cdots, r_n$ cumple que su clausura normal es F, entonces no necesariamente será posible transformar $r_1, \cdots, r_n$ en $x_1, \cdots, x_n$ mediante operaciones de Nielsen. Sin embargo, Andrews y Curtis conjeturaron que esto sería posible si se permite una cuarta operación: conjugar un elemento por cualquier otro elemento de F. Probaron que, si su conjetura fuera cierta, los entornos regulares de subcomplejos contráctiles de dimensión 2 de 5-variedades combinatorias serían 5-celdas. Notemos que la conjetura de Andrews-Curtis puede reformularse en términos de presentaciones de grupos. Si $\mathcal{P} = \langle x_1, \cdots, x_n : r_1, \cdots, r_n \rangle$ es una presentación balanceada del grupo trivial, definimos como operaciones válidas en la presentación al resultado de aplicar cualquiera de los movimientos definidos por Andrews y Curtis en el conjunto de relaciones. La conjetura afirma que $\mathcal{P}$ se puede transformar en $\langle x_1, \cdots, x_n : x_1, \cdots, x_n \rangle$ mediante una sucesión finita de operaciones válidas.

El referí sugirió incluir dos nuevas operaciones, que consisten en agregar un nuevo generador, digamos x, y una nueva relación x, y la operación inversa. Esto llevó a la siguiente conjetura actualmente abierta:

Conjetura (Andrews-Curtis)**.** *Si $\mathcal{P} = \langle x_1, \cdots, x_n | r_1, \cdots, r_n \rangle$ es una presentación balanceada del grupo trivial, entonces $\mathcal{P}$ se puede llevar a la presentación vacía $\langle | \rangle$ mediante una sucesión finita de las siguientes transformaciones:*

- *reemplazar r_i por r_i^{-1},*
- *reemplazar r_i por $r_i r_j$, $j \neq i$,*
- *reemplazar r_i por $w r_i w^{-1}$, donde w es una palabra en los generadores,*
- *agregar un nuevo generador y y la relación y, o el inverso de esta operación.*

También señaló que esta versión de la conjetura era equivalente a un problema aún abierto sobre deformaciones de complejos celulares y la teoría de homotopía simple de Whitehead: si K es un CW complejo de dimensión 2, ¿se puede expandir a un complejo L de dimensión 3 que colapse a un punto? La versión n-dimensional de esta pregunta ya había sido probada para $n \neq 2$ por C.T.C. Wall [Wal66b].

Conjetura (Andrews-Curtis, versión topológica)**.** *Todo complejo celular finito de dimensión 2 contráctil se 3-deforma a un punto.*

La generosa sugerencia del referí fue crucial para agregar un punto de vista topológico a la conjetura, que hoy tiene gran importancia. Las dos versiones apuntan a entender desde un enfoque discreto el difícil problema de detectar espacios contráctiles o grupos triviales.

Reseña histórica

Los complejos celulares de dimensión 2 no son tan inocentes como parecen a primera vista. Históricamente, muchos problemas pudieron ser probados (o refutados) en dimensiones altas, pero permanecen aún abiertos en las dimensiones más bajas. La conjetura de Poincaré generalizada es un ejemplo notable de este problema. Ésta fue probada por Stephen Smale para dimensiones mayores a 4 en los 60's, y por M. Freedman para dimensión 4 en 1982, pero fue recién en 2002 que Perelman obtuvo una demostración de su validez para dimensión 3.

Muchos problemas vinculados a la topología algebraica de complejos celulares permanecen aún abiertos para dimensión 2, como la conjetura de asfericidad de Whitehead [Whi41a], la conjetura de Zeeman [Zee64] y la conjetura de Andrews-Curtis [AC65]. La conjetura de

Zeeman implica la conjetura de Andrews-Curtis, y su validez proveería una demostración combinatoria de la conjetura de Poincaré 3-dimensional. Además, se puede deducir de la conjetura de Poincaré 3-dimensional que ambas conjeturas son ciertas para cierta clase de complejos llamados *standard spines*. Sin embargo, ambas conjeturas permanecen abiertas para complejos en general.

La complejidad de los complejos celulares de dimensión 2 se puede comprender mejor si tenemos en mente su correspondencia con las presentaciones de grupos. Dado un complejo celular conexo de dimensión 2, obtenemos una presentación de su grupo fundamental del siguiente modo. Se colapsa un árbol generador de su 1-esqueleto y se calcula una presentación del grupo fundamental del cociente que se obtiene, que resulta homotópicamente equivalente al original. Distintas elecciones del árbol, punto base u orientaciones de las celdas dan lugar a distintas presentaciones, que se pueden transformar una en otra mediante una sucesión finita de movimientos de Andrews-Curtis. Recíprocamente, una presentación de grupo puede modelarse como el grupo fundamental de un 2-complejo construido como sigue: comenzar con una única 0-celda, adjuntar una 1-celda por cada generador y darle una orientación, y finalmente adjuntar una 2-celda por cada relación, con la función de adjunción asociada a la palabra que se lee en la relación. Por el teorema de van Kampen, el grupo fundamental de este complejo tiene la presentación deseada. Diferentes elecciones de orientación dan lugar a complejos celulares 3-*deformables* (ver [Wri75]).

Esta correspondencia provee numerosas traducciones de problemas de teoría combinatoria de grupos y computabilidad en problemas de topología algebraica, y viceversa. Algunos ejemplos son el *problema de la palabra* y el *problema de isomorfismo*:

Sea $\mathcal{P} = \langle x_1, x_2, \cdots x_n : r_1, r_2, \cdots, r_m \rangle$ una presentación de un grupo finitamente presentado G. Si w es una palabra en los generadores, ¿es w trivial en G? ¿Es G el grupo trivial?

En 1911, Max Dehn cuestionó la existencia de algoritmos que respondan las preguntas anteriores para grupos en general, y produjo algoritmos que las responden para grupos fundamentales de 2-variedades cerradas orientables. Reconoció una característica crucial en estos grupos: están definidos por relaciones con propiedades de *cancelación pequeña*, dando lugar a la teoría de pequeña cancelación. Lo métodos de Dehn eran geométricos, hacía uso de teselaciones regulares del plano hiperbólico, y sus algoritmos se extendieron a una amplia clase de grupos [Deh11]. Sin embargo, él mismo conjeturó que el problema general era no decidible. En efecto, con la formalización de las nociones de *algoritmo* y *computabilidad* realizadas por Turing en los 30's [Tur37], Novikov [Nov55] y Boone [Boo54a, Boo54b, Boo55a, Boo55b] lograron hallar un grupo finitamente presentado G para el cual el problema de la palabra es no decidible. Más tarde, Adian y Rabin [Adi93, Rab58] probaron que, en general, no es decidible el problema de determinar si una presentación define el grupo trivial. Sin embargo, no se sabe aún si es decidible este problema para presentaciones *balanceadas*. No obstante, recientemente fue probado por Gadgil [Gad01] que si la conjetura de Andrews-Curtis fuera cierta, entonces efectivamente existiría un algoritmo para decidir si una presentación balanceada define el grupo trivial.

Vía la correspondencia de presentaciones de grupo con 2-complejos celulares, hay una traducción de los problemas anteriores en el marco topológico algebraico.

Sea K un complejo celular arcoconexo finito de dimensión 2. Si α es un lazo en K, ¿es α trivial? ¿Es K simplemente conexo? ¿Es K contráctil?

Es claro que las primeras dos preguntas son problemas topológicos no decidibles, y que la última pregunta sería decidible si la conjetura de Andrews-Curtis fuera cierta [Gad01].

A pesar de que determinar si un grupo es trivial o un espacio es contráctil son problemas no tratables computacionalmente, han surgido varios abordajes que enfrentan el problema desde un punto de vista discreto. Los principales ejemplos son la teoría de Tietze para grupos y la teoría de Whitehead para complejos celulares.

En 1908, Tietze [Tie08] mostró que dadas dos presentaciones finitas de un mismo grupo, se pueden obtener una de la otra mediante una secuencia finita de cierta lista de transformaciones elementales. Las *transformaciones de Tietze* permiten: invertir una relación, multiplicar una relación por otra, conjugar una relación por otro elemento en el grupo libre de generadores, agregar un nuevo generador y una nueva relación igual al generador agregado (y la operación inversa), y *agregar una nueva relación igual a un generador ya existente (y la operación inversa)*. Notar que cada una de las operaciones previas en la presentación de grupo preservan la clase de isomorfismo, pero no todas mantienen el tipo homotópico del complejo celular asociado. Las primeras cuatro son justamente los movimientos de Andrews-Curtis, y se pueden interpretar topológicamente como cambiar la función de adjunción de la celda asociada a la relación por una homotópica, lo cual no cambia el tipo homotópico *simple*. Agregar un nuevo generador y una relación igual al generador agregado es justamente la operación sugerida por el referí, y significa una *expansión elemental* en el complejo celular. Finalmente, la última transformación de Tietze se traduce en la adjunción de una esfera que cambia el tipo homotópico del complejo asociado pero no altera el grupo fundamental. La conjetura de Andrews-Curtis dice entonces que se puede prescindir del último movimiento de Tietze en presentaciones balanceadas del grupo trivial.

Como una simple consecuencia del teorema de Whitehead, se puede ver que las *presentaciones balanceadas del grupo trivial* se corresponden 1-1 con *complejos celulares de dimensión 2 contráctiles*. Más aún, se puede ver que los *movimientos de Andrews-Curtis* se corresponden con *3-deformaciones* de complejos celulares.

La teoría de homotopía simple de Whitehead es el análogo topológico de la teoría de Tietze para presentaciones de grupos, dando un enfoque combinatorio al clásico problema de clasificación de tipos homotópicos. Sus contribuciones [Whi41b, Whi50] resultaron ser fundamentales para el desarrollo de la topología lineal a trozos y la geometría combinatoria. Algunos de sus principales logros y aplicaciones son: el teorema de s-cobordismo, la conjetura de Zeeman [Zee64], la conjetura de Andrews-Curtis [AC65], las aplicaciones a la teoría de cirugía, el clásico artículo de Milnor sobre la torsión de Whitehead [Mil66], la invariancia topológica de la torsión.

De manera similar al caso de las transformaciones elementales de Tietze, Whitehead construyó su teoría sobre dos movimientos elementales de deformación sobre complejos celulares que preservan su tipo homotópico: las *expansiones* y los *colapsos*. Una sucesión finita de colapsos y expansiones es una deformación formal, que da lugar a la noción de homotopía simple. Uno podría preguntarse cuándo es cierto una suerte de análogo del teorema de Tietze, es decir, cuándo dos complejos homotópicamente equivalentes se pueden conectar mediante una cadena de expansiones y colapsos. Desafortunadamente, esto no se puede lograr en general. De hecho, existe una obstrucción algebraica, hoy llamada el *grupo de Whitehead*, que cuantifica la brecha. Si el grupo de Whitehead $\mathrm{Wh}(K)$ de un complejo celular K es trivial, entonces todo complejo homotópicamente equivalente a K es simplemente equivalente a K. En particular, como el grupo de Whitehead de los complejos contráctiles es trivial, entonces todo complejo contráctil se puede transformar en un punto mediante una sucesión finita de expansiones y colapsos. Más aún, si el complejo tiene dimensión $n \neq 2$, existe una deformación en la cual los complejos involucrados tienen dimensión a lo sumo $n+1$ [Wal66b]. La pregunta para $n = 2$ aún se encuentra abierta y es la conjetura de Andrews-Curtis.

Recientemente, el desarrollo de la teoría de homotopía de espacios finitos [Bar11a] proporcionó un nuevo enfoque combinatorio a muchos problemas de la topología algebraica. Los espacios topológicos finitos son una herramienta doble: por un lado, son modelos finitos de complejos celulares regulares; por el otro, se pueden pensar como posets finitos, un objeto manejable que permite hacer fácilmente los cálculos. Para cada complejo celular *regular* K, existe un espacio topológico finito $\mathcal{X}(K)$ con el mismo tipo homotópico débil. El tipo homotópico (fuerte) de los espacios finitos está completamente determinado por un algoritmo goloso de complejidad polinomial, que consiste en remover uno a uno puntos del *diagrama de Hasse* con cierta característica combinatoria (que el grado de entrada o el de salida sea igual a 1) [Sto66]. En cambio, clasificar el tipo homotópico débil de espacios finitos es tan difícil como clasificar el tipo homotópico de espacios topológicos en general, es decir, no es decidible computacionalmente. En [BM08b], Barmak y Minian desarrollaron un análogo de la teoría de homotopía simple para espacios finitos, de modo que K se $(n+1)$-deforma a L si y sólo si $\mathcal{X}(K)$ se $(n+1)$-deforma a $\mathcal{X}(L)$. Las deformaciones simples de espacios finitos también están también definidas a través de dos movimientos elementales: los *colapsos* y sus inversos, las *expansiones*. Pero esta vez, un colapso elemental es simplemente remover un vértice con cierta característica combinatoria: que el conjunto de puntos comparables distintos sea contráctil. Luego, se obtiene una versión combinatoria de la conjetura.

Conjetura (Andrews-Curtis, versión espacios finitos)**.** *[Bar11a] Si X es un espacio topológico finito homotópicamente trivial de altura 2, entonces X se 3-deforma a un punto.*

Además, los autores introducen las *qc-reducciones* y hallan una gran clase de complejos simpliciales, los *qc-construibles*, que satisfacen la conjetura.

A lo largo de los últimos 50 años, se ha construido una lista de ejemplos de presentaciones balanceadas del grupo trivial para los cuales no se conoce trivialización vía movimientos de Andrews-Curtis. Sirven como *potenciales contraejemplos* de la conjetura.

(i) $\langle x, y | xy^2x^{-1} = y^3, yx^2y^{-1} = x^3\rangle$, Crowell & Fox (1963) [CF77, p.41].

(ii) $\langle x, y, z | z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2\rangle$, Rapaport (1968) [Rap68a].

(iii) $\langle x, y | x = [x^m, y^n], y = [x^p, y^q]\rangle$, $m, n, p, q \in \mathbb{Z}$, Gordon (1984) [Bro84].

(iv) $\langle x, y | xyx = yxy, x^4 = y^5\rangle$, Akbulut & Kirby (1985) [AK85].

A continuación, hacemos un pequeño resumen y comentarios sobre sus orígenes y el trabajo previamente realizado en torno a estos ejemplos.

El ejemplo (i) está contenido en una serie más general de presentaciones balanceadas del grupo trivial que se encuentra en un artículo más reciente de Miller y Schupp [MS99]: $\langle x, y | yx^n y^{-1} = x^{n+1}, y = w\rangle$, donde $n \geq 1$, y w es una palabra en x e y de modo que los exponentes de las y suman 0.

El ejemplo (ii) pertenece a una serie de presentaciones con n generadores x_i, y relaciones cíclicamente indexadas $x_{i+1}^{-1} x_i x_{i+1} = x_i^2$. Para $n = 2$, la presentación se puede trivializar fácilmente con transformaciones de Andrews-Curtis. Para $n = 3$, se puede transformar en una presentación de 2 generadores y 2 relaciones, aunque esta "reducción" incrementa la longitud de las relaciones. Para $n > 4$, la presentación corresponde a grupos infinitos no triviales.

El ejemplo (iii) fue encontrado por Gordon y comunicado por Lickorish a Ronald Brown, quien lo incluyó en su artículo [Bro84] con el permiso del creador. En [BM06] se probó que las presentaciones de esta serie cuya longitud total de relaciones es igual a 14 satisfacen la conjetura de Andrews-Curtis.

El ejemplo (iv) es el más estudiado. Surge a partir de una descomposición en manijas de la 4-esfera de Akbulut-Kirby [AK85]. Todas las presentaciones de la familia $\langle x, y | xyx = yxy, x^n = y^{n+1}\rangle$ corresponden al grupo trivial. Para $n > 2$, no se sabe si son Andrews-Curtis trivializables. Para $n = 2$, Gersten mostró en 1982 que es trivializable con movimientos de Andrews-Curtis. Sin embargo, el artículo no se encuentra publicado. No fue hasta varios años después que Havas & Ramsay [HR03] y Miasnikov [Mia99, MM01] muestran trivializaciones concretas. Para hallarlas, Miasnikov creó nuevos *algoritmos genéticos* diseñados para testear la validez de la conjetura. Su principal resultado dice que todas las presentaciones balanceadas del grupo trivial cuya longitud total de las relaciones es a lo sumo 12 satisfacen la conjetura. Usaron el software computacional MAGNUS. Por su parte, Havas & Ramsay exhibieron un abordaje computacional basado en un *breadth-first search* sobre el árbol de presentaciones AC-equivalentes. Desarrollaron el software libre ACME. Para $n = 3$, no se lograron obtener resultados con ninguno de los algoritmos anteriores. En realidad, los enfoques anteriores tienen el mismo púnto débil: tratan de exhibir explícitamente la sucesión de movimientos que llevan a una presentación a la presentación trivial. Recientemente, en 2015, Bridson [Bri15] halló ejemplos de presentaciones balanceadas del grupo trivial, que son trivializables mediante movimientos de Andrews-Curtis, pero que la cantidad mínima de transformaciones requeridas para lograrlo crece más rápido que cualquier torre de exponenciales. Los ejemplos están construidos con la idea de codificar en presentaciones balanceadas la complejidad del problema de la palabra de ciertos grupos.

Contribuciones

Hasta ahora, no se conocen métodos para manejar las 3-deformaciones algorítmicamente. En este trabajo, proponemos una nueva estrategia para estudiarlas: introducimos nuevos *métodos de reducción* de espacios finitos, que codifican una 3-deformación de dos pasos. Esto es, $X \nearrow Y \searrow *$, donde la altura de Y es menor o igual que 3. Cada reducción decrece el número de puntos o aristas del diagrama de Hasse asociado al poset, y se puede describir fácilmente en términos de la combinatoria de este grafo. Algunos ejemplos de los nuevos métodos desarrollados son: las *reducciones de arista*, las *middle* reducciones, y varios otros tipos de reducciones de tipo cociente, incluyendo las *O-reducciones*, una generalización y mejora de los métodos de reducción de Osaki [Osa99] para tipos homotópicos débiles de espacios finitos. Estos métodos proveen una técnica algorítmica (no exhaustiva) para mostrar que un espacio se 3-deforma a un punto, dando lugar a la caracterización de nuevas clases de espacios que satisfacen la conjetura.

La principal herramienta que usamos para las demostraciones es una generalización a relaciones en general del cilindro no-Hausdorff de una función: construimos el *cilindro no-Hausdorff de una relación*, que denotamos $B_{\mathcal{R}}$.

Corolario 3.1.4. *Sea $\mathcal{R} \subseteq X \times Y$ una relación entre espacios topológicos finitos. Si $\underline{\mathcal{R}^{-1}(U_y)}$ y $\overline{\mathcal{R}(F_x)}$ son homotópicamente triviales para cada $x \in X$ e $y \in Y$, entonces $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$. Más aún, si $\overline{\mathcal{R}(F_x)}$, $\underline{\mathcal{R}^{-1}(U_y)}$ son colapsables, entonces $\mathcal{K}(X) \diagup\!\!\searrow^{n} \mathcal{K}(Y)$, donde n es la altura del cilindro.*

Los nuevos métodos de reducción que formulamos en esta tesis para estudiar la conjetura de Andrews-Curtis son descriptos en los siguientes resultados.

Las middle-reducciones son una variación de las qc-reducciones para puntos que no son maximales ni minimales, que satisfacen ciertas condiciones.

Proposición 3.2.7. *Sea X un espacio finito de altura a lo sumo 2. Si hay una middle-reducción de X a $X/_{\{a,b\}}$, entonces $X \diagup\!\!\searrow^{3} X/_{\{a,b\}}$.*

Las reducciones de arista son reducciones de cierto tipo de arcos del diagrama de Hasse de un espacio finito.

Proposición 3.2.12. *Sea X un espacio finito de altura a lo sumo 2. Sea $e = (a \prec b)$ una arista tal que existe una reducción de X a $X \setminus e$. Entonces $X \diagup\!\!\searrow^{3} X \setminus e$.*

Un resultado relevante fue la prueba de que las reducciones de Osaki [Osa99] en espacios de dimensión 2 son 3-deformaciones.

Teorema 3.2.15. *Sea X un espacio finito de altura n y sea $x_0 \in X$. Si $U_x \cap U_{x_0}$ es vacío o contráctil para todo $x \in X$, entonces $X \diagup\!\!\searrow^{n+1} X/U_{x_0}$.*

En general, un espacio finito X no tiene el mismo tipo homotópico débil que sus cocientes X/A, incluso si A es contráctil. Hallamos una condición general sobre A que asegura que si X es de altura n, entonces $X \diagup\!\!\searrow^{n+1} X/A$.

Teorema 3.2.19. *Sean X un espacio finito de altura n, $A \subseteq X$ un subespacio abierto conexo. Si $\overline{A}$ es homotópicamente trivial, entonces $X \diagup\!\!\searrow X/A$. Más aún, si $\overline{A}$ es colapsable, entonces $X \diagup\!\!\searrow^{n+1} X/A$.*

Mostramos también que el cilindro de una relación es útil para el estudio de varios otros problemas de la topología algebraica con métodos combinatorios, como clasificar el tipo homotópico de colímites homotópicos [FM16], generalizar el Teorema del Nervio [FM17].

El siguiente resultado es una generalización del clásico Teorema del Nervio (probado originalmente por Borsuk [Bor48], con versiones más generales de Björner [Bjö03] y Barmak [Bar11b]), en la cual permitimos al cubrimiento tener intersecciones no vacías no contráctiles.

Dado $\mathcal{U}$ un cubrimiento finito por abiertos X, el *nervio no-Hausdorff de $\mathcal{U}$* es el face poset del nervio clásico $\mathcal{N}(\mathcal{U})$. Lo denotamos por $N(\mathcal{U})$.

Teorema 4.1.4. *Sea X un espacio topológico finito y sea $\mathcal{U} = \{U_i\}_{i \in I}$ un cubrimiento abierto de X. Sea $N_0(\mathcal{U})$ el subespacio del nervio no-Hausdorff $N(\mathcal{U})$ de todas las intersecciones homotópicamente triviales. Si para todo $x \in X$, el subespacio $\mathcal{I}_x$ de $N_0(\mathcal{U})$ de las intersecciones que contienen a x, es homotópicamente trivial, entonces X tiene el mismo tipo homotópico simple que $N_0(\mathcal{U})$.*

Decimos que una familia $\mathcal{U}$ de subespacios de un espacio finito X es un *cubrimiento cuasi-bueno* si toda intersección no vacía de una subfamilia de $\mathcal{U}$ tiene sus componentes conexas homotópicamente triviales. A partir de cubrimientos cuasi-buenos, se pueden construir cubrimientos que satisfagan las hipótesis del Teorema 4.1.4. Denotamos por $M(\mathcal{U})$ al espacio finito cuyos elementos son las componentes conexas del espacio finito cuyos elementos son las componentes conexas de cada intersección de elementos de $\mathcal{U}$, con el orden dado por la inclusión.

Corolario 4.1.9. *Sea X un espacio finito y sea $\mathcal{U}$ un cubrimiento cuasi-bueno de X. Entonces, $X \diagup\!\!\searrow M(\mathcal{U})$.*

Por otra parte, introdujimos un modelo finito de colímites homotópicos y usamos la construcción del cilindro de una relación y un teorema clásico de McCord para probar una generalización del célebre Teorema de Thomason [Tho79] en el contexto de colímites homotópicos sobre posets. También obtuvimos análogos de resultados conocidos, como el Teorema Cofinal [BK72, Vog73] y una generalización de Teorema A de Quillen para posets [Qui73]. En particular, esto permitió caracterizar los colímites homotópicos de diagramas de complejos simpliciales en términos de la construcción de Grothendieck de diagramas de sus face posets. Estos resultados aparecieron en [FM16].

Dado $X : P \to \mathcal{P}_{<\infty}$ un diagrama de posets finitos indexado en un poset, definimos $\underline{\mathrm{hocolim}}\, X$, el *colímite homotópico no-Hausdorff* de X, como su construcción de Grothendieck $P \int X$, la cual resulta también un poset fácil de describir.

Tratamos a la construcción de Grothendieck de diagramas de posets como un espacio finito, y usamos análogos a los conocidos sobre colímites homotópicos, y para probar una generalización del teorema de Thomason. Esta generalización nos permitió aplicar *métodos de reducción* para estudiar el tipo homotópico de diagramas de poliedros (indexados en posets finitos).

El Teorema A de Quillen para posets se sigue inmediatamente de las siguientes proposiciones, aplicando los resultados al poset **1**, con elementos $0 < 1$.

Proposición 5.3.1. *Sea $X : P \to \mathcal{P}_{<\infty}$ un P diagrama de espacios finitos. Si $p \in P$ es un up beat point, entonces $\underline{\mathrm{hocolim}}\, X \searrow\!\!\searrow \underline{\mathrm{hocolim}}\, X|_{P \smallsetminus \{p\}}$.*

Proposición 5.3.5. *Sea $X : P \to \mathcal{P}_{<\infty}$ un P diagrama de espacios finitos. Si p un down beat point de P dominado por q y $f_{qp}^{-1}(U_x)$ es contráctil para cada $x \in X_p$, entonces*

$$\underline{\mathrm{hocolim}}\, X \searrow \underline{\mathrm{hocolim}}\, X|_{P \smallsetminus \{p\}}$$

El principal resultado de esta parte es la siguiente generalización del teorema de Thomason en el contexto de espacios finitos.

Teorema 5.4.1. *Sea P un poset finito. Sea $K : P \to \mathrm{Top}$ un diagrama de espacios y $X : P \to \mathcal{P}_{<\infty}$ un diagrama de posets finitos. Sea $\phi : K \to \mathrm{Top}$ un morfismo de diagramas (donde X es* *visto como diagrama de espacios topológicos finitos) tal que $\phi_p : K_p \to X_p$ es una equivalencia homotópica débil para cada $p \in P$. Entonces, existe una equivalencia homotópica débil*

$$\hat{\varphi} : \operatorname{hocolim} K \to \underline{\operatorname{hocolim}}\, X$$

entre el colímite homotópico de K al colímite homotópico no-Hausdorff de X (visto como espacio topológico finito).

Como consecuencia directa de este resultado, obtuvimos por un lado un caso particular del teorema de Thomason en el contexto de posets, y por el otro, una suerte de recíproca del teorema de Thomason, que relaciona el colímite homotópico de un diagrama de complejos simpliciales con el colímite homotópico no-Hausdorff del diagrama de sus face posets. Como corolario del Teorema 5.4.1, pudimos probar también la invariancia por subdivisión baricéntrica del tipo homotópico de colímites homotópicos de complejos simpliciales en el caso general (no necesariamente ordenado). La teoría desarrollada permite simplificar en general el cálculo de colímites homotópicos de espacios.

Por otra parte, también desarrollamos otro ataque combinatorio distinto de la conjetura de Andrews-Curtis, esta vez haciendo una nueva interpretación de su versión para presentaciones de grupos. Dada una presentación balanceada del grupo trivial, elaboramos dos técnicas para transformarla en una nueva presentación que puede obtenerse de la original vía una sucesión de movimientos de Andrews-Curtis, pero sin especificar la lista explícita de movimientos que llevan de una a la otra. Dada una presentación $\mathcal{P}$ del grupo G, denotamos $X_{\mathcal{P}}$ al face poset de la subdivisión baricéntrica del CW-complejo asociado a la presentación. Claramente, este poset tiene su grupo fundamental isomorfo a G, y su diagrama de Hasse puede ser descripto fácilmente en términos de $\mathcal{P}$.

Una de las técnicas desarrolladas está inspirada en la teoría de coloreo de posets y es, de hecho, una generalización e implementación discreta del hecho de que colapsar un árbol en el 1-esqueleto de un 2-complejo celular es una 3-deformación. Dado X un poset finito y A un subdiagrama colapsable de X que contiene todos sus puntos, construimos una presentación $\mathcal{P}_{X,A}$ del grupo fundamental de X, que resulta AC-equivalente a $\mathcal{P}_{\mathcal{K}(X)}$. Como consecuencia, obtuvimos el siguiente

Teorema 6.3.1. *Si $\mathcal{P}$ es una presentación de grupo y A es un subdiagrama generador colapsable de $\mathcal{H}(X_{\mathcal{P}})$, entonces $\mathcal{P} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$.*

La otra técnica que desarrollamos está inspirada en la teoría de Morse discreta, introducida por Robin Forman [For98] en los '90s como una versión combinatoria de la teoría clásica para variedades diferenciables. Obtuvimos una versión más fuerte del teorema principal de la teoría de Morse, haciendo la teoría ahora aplicable a nuestro contexto. Nuestro resultado tiene la ventaja de describir más precisamente la equivalencia homotópica. Mostramos que una función de Morse (o, equivalentemente, un matching M en el diagrama de Hasse del face poset del complejo) induce no sólo una equivalencia homotópica, sino también una $(n+1)$-deformación entre el complejo celular (de dimensión n) y el complejo de Morse K_M. También hicimos precisa la descripción de este complejo, describiendo las funciones de adjunción de sus celdas y no sólo la cantidad de celdas de cada dimensión.

Teorema 6.4.17. *Sea K un complejo celular regular de dimensión n y sea M un matching acíclico en $\mathcal{H}(\mathcal{X}(K))$. Entonces $K \diagup\!\!\searrow^{n+1} K_M$.*

Dimos una descripción explícita de la presentación estándar de K_M, que denotamos por $\mathcal{Q}_{\mathcal{X}(K),M}$.

Corolario 6.5.6. *Sea K un complejo celular regular de dimensión 2, y sea M un matching acíclico en $\mathcal{H}(\mathcal{X}(K))$ con una sola celda crítica de dimensión 0. Entonces $\mathcal{Q}_{\mathcal{X}(K),M}$ es una presentación balanceada de $\pi_1(K)$. Más aún, $\mathcal{Q}_{\mathcal{X}(K),M} \sim_{AC} \mathcal{P}_K$.*

A partir de las estrategias anteriores desarrollamos algoritmos concretos para obtener presentaciones AC-equivalentes, evitando tener que exhibir la lista de movimientos necesarios para lograr tal transformación, lo cual es un avance considerable a la luz de los resultados de [Bri15]. Bajo este enfoque, obtuvimos, entre otras, demostraciones simples de los siguientes hechos sobre la lista de potenciales contraejemplos.

- La presentación $\langle x, y | xyx = yxy, x^n = y^{n+1} \rangle$ [AK85] para $n = 2$, satisface la conjetura de Andrews-Curtis.
- La presentación $\langle x, y, z | z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2 \rangle$ [Rap68a], se puede trasformar en una presentación con 2 generadores y 2 relaciones.
- Hay una gran clase de enteros n, m, p, q para la cual la presentación $\langle x, y | x = [x^m, y^n], y = [x^p, y^q] \rangle$ [Bro84] satisface la conjetura.

Nuestros métodos se pueden manejar fácilmente con una computadora. Implementamos todos los algoritmos que existían sobre espacios finitos [Fer11] y los nuevos que desarrollamos en esta tesis usando el software libre SAGE [S^+17]. Creamos un módulo anexable a SAGE (ver Apéndices 2.A, 3.A y 6.A) que permite trabajar con espacios finitos y sus aplicaciones, incluyendo el estudio de presentaciones de grupos y complejos celulares. El código está disponible en `https://github.com/ximenafernandez/Finite-Spaces`.

Descripción de los capítulos

La tesis está organizada en seis capítulos. En el primer capítulo hacemos una reseña histórica del problema. Analizamos los distintos enfoques existentes y los estudiamos en un marco unificado. En el segundo capítulo, explicamos el punto de vista combinatorio bajo el cual trabajaremos en los siguientes capítulos. Describimos la conexión entre complejos celulares regulares, espacios topológicos finitos y posets. Explicamos la teoría de homotopía simple para espacios finitos y la reformulación de la conjetura de Andrews-Curtis desde esta perspectiva. El tercer capítulo introduce los nuevos métodos combinatorios en espacios finitos que desarrollamos para estudiar 3-deformaciones, y sus aplicaciones al estudio de la versión de la conjetura para espacios finitos y otros problemas de la topología algebraica. El cuarto capítulo muestra nuevas versiones del clásico teorema del Nervio, y aplicaciones al estudio de 3-deformaciones. El quinto capítulo trata de colímites homotópicos. En el sexto capítulo, desarrollamos nuevos métodos combinatorios para presentaciones de grupos, y sus aplicaciones al estudio de los potenciales contraejemplos. Casi todos los resultados de los capítulos 3, 4, 5 y 6 son nuevos y originales. Algunos de los resultados de los capítulos 3, 4 y 5 aparecieron en [FM16, FM17]. Los resultados del capítulo 6 no están publicados aún y serán objeto de un próximo artículo [Fer17].

Introduction

The classification of spaces up to homotopy equivalence is one of the primary objectives of Algebraic Topology. This problem is known to be algorithmically undecidable, even restricted to combinatorial models such as simplicial complexes with a more rigid notion of homotopy, the simple homotopy. Likewise, deciding whether two groups are isomorphic is a fundamental problem in group theory. Despite the efforts of Tietze to produce a short effective list of transformations, this problem was also proved to be unsolvable even for finitely presented groups. Of course, combinatorial group theory and topology grew up together and their connection via the fundamental group is well known. As a consequence of the interaction between these theories, one could say that the problems mentioned above are essentially the same.

One of the main goals of this work is to attack from a new point of view the Andrews-Curtis conjecture, a fifty-year old open problem which is closely related to the preceding problems. We introduce novel combinatorial methods and algorithmic procedures to manipulate CW-complexes and group presentations preserving their (simple) homotopy type and -respectively-equivalence class.

Origins of the problem

In 1965, in their famous article "*Free groups and handlebodies*" [AC65], J.J. Andrews and M. L. Curtis raised a question about presentations of groups, generalizing previous ideas of Nielsen [Nie18] about free groups and accounting for some relevant topological consequences of a possible affirmative answer of their question to the (at that time open) Poincaré conjecture. The article increased its relevance with the referee's remark on a weakening of the conjecture and its relationship with a (also open) homotopical problem concerning deformations of cellular complexes.

Given a finitely generated free group F on $x_1, \cdots, x_n$, Nielsen's theorem asserts that any other basis $y_1, \cdots, y_n$ of F can be obtained from $x_1, \cdots, x_n$ by applying a finite sequence of the following elementary moves: inverting one element, interchanging elements, multiplying an element by another one. Now, if F is free on $x_1, \cdots, x_n$, and the set $r_1, \cdots, r_n$ is such that its normal closure is F, then it may not be possible to change $r_1, \cdots, r_n$ to $x_1, \cdots, x_n$ by Nielsen operations. However, Andrews and Curtis conjectured that this should be possible if a fourth operation is allowed: conjugating an element by any element of F. They proved that if the conjecture were true, regular neighbourhoods of contractible 2-dimensional subcomplexes of combinatorial 5-manifolds would be 5-cells. Notice that the Andrews-Curtis conjecture can be reformulated in terms of group presentations. For $\mathcal{P} = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle$ a balanced presentation of the trivial group, define a valid operation on this presentation to be the result of applying any of the moves already defined by Andrews and Curtis on the set of relators. The conjecture states that $\mathcal{P}$ can be transformed in $\langle x_1, \cdots, x_n \mid x_1, \cdots, x_n \rangle$ by a finite sequence of valid operations.

The referee suggested to include two new operations, consisting of adding a new generator, say x, and the additional relator x; and also the inverse of the previous procedure; giving rise to the following open problem:

Conjecture (Andrews-Curtis)**.** *If $\mathcal{P} = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle$ is a balanced presentation of the trivial group, then $\mathcal{P}$ can be transformed in the empty presentation $\langle \mid \rangle$ by a finite sequence of the following transformations:*

- *replace some r_i by r_i^{-1},*
- *replace some r_i by $r_i r_j$, $j \neq i$,*
- *replace some r_i by $w r_i w^{-1}$, where w is any word in the generators,*
- *introduce a new generator y and the relator y; or the inverse of this operation.*

He also pointed out that this version of the conjecture was equivalent to a still open problem about deformations of cellular complexes of dimension 2 and Whitehead's simple homotopy theory: given K a contractible cellular complex of dimension 2 , is it possible to expand K to a 3-dimensional cellular complex L which collapses to a point? Note that the n-dimensional version of this statement had already been proved for $n \neq 2$ by C.T.C. Wall [Wal66b].

Conjecture (Andrews-Curtis, topological version)**.** *If K is a contractible finite cellular complex of dimension 2, then K can be 3-deformed to a point.*

The generous remark of the referee was crucial in adding a new topological point of view to the conjecture, which nowadays has a great importance. Both versions of the conjecture are an attempt to understand from a discrete point of view the hard problem of detecting contractible spaces or trivial groups.

Historical overview

Cellular complexes of dimension two are not as innocent as they seem at first glance. Historically, many problems have been proved (or disproved) in higher dimensions but they remain open in lower ones. The generalized Poincaré conjecture is a notable example of such a problem. Proved by Stephen Smale for dimensions greater than 4 in the sixties and by M. Freedman for dimension 4 in 1982, it was not until the late 2002 that Perelman obtained a proof of its validity for dimension 3.

Many problems related to the algebraic topology of the cellular complexes are still open for dimension 2, such as the Whitehead asphericity conjecture [Whi41a], the Zeeman conjecture [Zee64] and the Andrews-Curtis conjecture [AC65]. The Zeeman conjecture implies the Andrews-Curtis conjecture, and its validity would provide an alternative combinatorial proof of the 3-dimensional Poincaré conjecture. Furthermore, it can be deduced from the 3-dimensional

Poincaré conjecture that both Zeeman and Andrews-Curtis conjectures are true for some class of complexes called *standard spines*. However, both conjectures are still open for general complexes.

The complexity of the cellular complexes of dimension 2 can be better understood by keeping in mind their correspondence with group presentations. Given a connected cellular complex of dimension 2 it is possible to obtain a presentation of its fundamental group as follows. Collapse a spanning tree of the 1-skeleton and compute the presentation of the fundamental group of the resulting homotopy equivalent quotient space. Different choices of the tree, the base point or the orientation of the cells lead to different presentations, which can be transformed one in each other by a sequence of Andrews-Curtis moves. Reciprocally, a group presentation is modeled by a two-complex constructed as follows: start with one 0-cell, attach one 1-cell for every generator and give an orientation to each one, and finally attach one 2-cell for every relator, with attaching map spelling the word associated to the relator. By the van Kampen theorem, the fundamental group of this complex has the desired presentation. Different choices of orientations results in 3-*deformable* cellular complexes (see [Wri75]).

This correspondence leads to many translations of problems in combinatorial group theory and computability into the algebraic topology side, and vice-versa. Very first examples would be the *word problem* and the *isomorphism problem*:

Let $\mathcal{P} = \langle x_1, x_2, \cdots x_n \mid r_1, r_2, \cdots, r_m \rangle$ be a presentation of a finitely presented group G. If w is a word on the generators, is w trivial in G? Is G the trivial group?

In 1911, Max Dehn posed the existence of algorithms answering the previous questions for groups in general, and provided algorithms solving them for fundamental groups of closed orientable 2-dimensional manifolds. He recognized a crucial feature of these groups: they are defined by relations with *small cancellation* properties, giving rise to the small cancellation theory. Dehn's methods were geometric, making use of regular tessellations of the hyperbolic plane, and their algorithms have been extended to large classes of groups [Deh11]. However, he conjectured that the general problem was undecidable. Indeed, with the formalization of the notions of *algorithm* and *computability* due to Turing in the 1930s [Tur37], Novikov [Nov55] and Boone [Boo54a, Boo54b, Boo55a, Boo55b] found a finitely presented group G such that the word problem in G is undecidable. Later, Adian and Rabin [Adi93, Rab58] proved that it is undecidable whether a finite presentation defines the trivial group. However, the decidability of this problem for *balanced* presentations is not settled. It was recently proved by Gadgil [Gad01] that if the Andrews-Curtis conjecture were true, there would actually exist an algorithm to decide if a balanced presentation defines the trivial group.

Via the correspondence of group presentations with 2-dimensional cellular complexes, there is a translation of the previous problems in the algebraic topology setting.

Let K be a finite path-connected cellular complex of dimension 2. If α is a loop in K, is α trivial? Is K simply connected? Is K contractible?

It is clear that the fist two questions are undecidable topological problems, but it was proved that the latter would be decidable if the geometric Andrews Curtis conjecture were true [Gad01].

Although determining the triviality of groups or contractibility of spaces is intractable, different approaches to deal with these problems from a discrete point of view have been pursued. The main examples are the Tietze theory for groups and the Whitehead theory for cellular complexes.

In 1908, Tietze [Tie08] showed that any two finite presentations of the same group can be obtained from the other by a finite sequence of certain list of elementary transformations. *Tietze transformations* allow: to invert a relation, to multiply a relation by another one, to conjugate a relation by any element in the free group on the generators, to add an additional generator and a relation equal to the added generator (and the inverse operation), and *to add a new relation equal to an already existing generator (and the inverse operation).* Notice that each of the previous operations on the group presentation preserves the isomorphism type, but not all of them maintain the homotopy type of the associated cellular complex. The first four operations are precisely Andrews-Curtis moves, and they can be topologically seen as changing the attaching map of the cell associated to the relation involved by a homotopic one, which does not change the *simple* homotopy type. Adding a new generator and a relation equal to the added generator is the referee's suggested operation, which amounts to an *elementary expansion* on the cellular complexes. Finally the last transformation implies the attaching of a sphere, which changes the homotopy type but leaves the fundamental group unaltered. The Andrews-Curtis conjecture states that the last Tietze move is not necessary for balanced presentations of the trivial group.

As a simple consequence of Whitehead's theorem, one can see that *balanced presentations of the trivial group* are in one to one correspondence with *contractible cellular complexes of dimension 2*. Moreover, it can be proved that the *Andrews-Curtis moves* correspond to *3-deformations* of the cellular complexes.

Whitehead's simple homotopy theory is the topological analogue of Tietze's theory for group presentations, providing a combinatorial approach of the classical problem of classification of homotopy types. His contributions [Whi41b, Whi50] turned out to be fundamental for the development of piecewise-linear topology and combinatorial geometry. Some of the main achievements and applications are: the s-cobordism theorem, Zeeman's conjecture [Zee64], the Andrews-Curtis conjecture [AC65], the applications of the theory in surgery, Milnor's classical paper on Whitehead Torsion [Mil66] and the topological invariance of torsion.

Similarly as in the case of Tietze transformations, Whitehead built his theory on two elementary deformation moves on cellular complexes which preserve the homotopy type: expansions and collapses. A finite sequence of collapses and expansions is a formal deformation, which give rise to the notion of simple homotopy theory. One might wonder whether a sort of analogue of Tietze's theorem holds, that is, whether any two homotopy equivalent complexes can be connected by a chain of expansions and collapses. Unfortunately, this is not true in general. In fact, there exists an algebraic obstruction, now called *the Whitehead group*, that quantifies the gap. If the Whitehead group $\mathrm{Wh}(K)$ of the cellular complex K is trivial, then any complex homotopy equivalent to K is also simple homotopy equivalent to K. In particular, since the Whitehead group of contractible complexes is trivial, any contractible cellular complex can be transformed into a point by a finite sequence of expansions and collapses. Moreover, if the complex has dimension $n \neq 2$, there exists a deformation in which the complexes involved have dimension less than or equal to $n + 1$ [Wal66b]. The question for $n = 2$ is still open: that is the Andrews-Curtis conjecture.

Recently, the development of the homotopy theory of finite spaces [Bar11a] provided a new combinatorial approach to many problems in algebraic topology. Finite topological spaces are a two-side tool: on the one hand, they are finite models of regular cellular complexes; on the other, they can be thought of as finite posets, a powerful handy object that allows many computations. For each *regular* cellular complex K, there exists a finite topological space $\mathcal{X}(K)$ with the same weak homotopy type. The (strong) homotopy type of finite spaces is completely determined by a greedy algorithm with polynomial complexity, which consists of removing one by one points of the *Hasse diagram* with certain combinatorial characteristic (in-degree equal to one or out-degree equal to 1) [Sto66]. By contrast, classifying the weak homotopy type of finite spaces is as difficult as classifying the homotopy type of topological spaces, that is, it is not decidable computationally. In [BM08b], Barmak and Minian developed an analogue of the simple homotopy theory for finite spaces, so that K $(n+1)$-deforms to L if and only if $\mathcal{X}(K)$ $(n+1)$-deforms to $\mathcal{X}(L)$. Deformations of finite spaces were also defined by means of two elementary movements: *collapses* and their inverses, *expansions*. But this time, an elementary collapse is just removing a single point with certain combinatorial characteristic: the set of comparable (and non equal) points must be contractible (as a finite space). Thus, a combinatorial version of the conjecture is obtained.

Conjecture (Andrews-Curtis, finite spaces version)**.** *If X is a homotopically trivial finite topological space of height 2, then X 3-deforms to a point.*

They also introduced the *qc-reductions* and found a huge class of simplicial complexes, the *qc-constructible* ones, that satisfies the conjecture.

Over the last fifty years mathematicians have constructed a list of examples of balanced presentations of the trivial group which are not known to be trivializable via Andrews-Curtis transformations. They serve as *potential counterexamples* to disprove the conjecture.

(i) $\langle x, y \mid xy^2x^{-1} = y^3, yx^2y^{-1} = x^3\rangle$, Crowell & Fox (1963) [CF77, p.41].

(ii) $\langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2\rangle$, Rapaport (1968) [Rap68a].

(iii) $\langle x, y \mid x = [x^m, y^n], y = [x^p, y^q]\rangle$, $m, n, p, q \in \mathbb{Z}$, Gordon (1984) [Bro84].

(iv) $\langle x, y \mid xyx = yxy, x^4 = y^5\rangle$, Akbulut & Kirby (1985) [AK85].

A brief summary of facts and extra comments about their origins and the work already done on these examples is featured below.

Example (i) is contained in the most general series of balanced presentations of the trivial group presented in the recent paper of Miller and Schupp [MS99]: $\langle x, y \mid yx^ny^{-1} = x^{n+1}, y = w\rangle$, where $n \geq 1$, and w is a word in x and y with exponent sum 0 in y.

Example (ii) belongs to a series of presentations with n generators x_i, and cyclically indexed defining relations $x_{i+1}^{-1}x_ix_{i+1} = x_i^2$. For $n = 2$, the presentation can easily be trivialized by a Andrews-Curtis transformations. For $n = 3$, it can be transformed into a presentation with 2 generators and 2 relators, although this "reduction", increases the length of the relators. For $n > 4$, they present nontrivial infinite groups.

Example (iii) was found by Gordon, and was communicated by Lickorish to Ronald Brown, who included the example in his paper [Bro84] with Gordon's permission. The presentations of this series with length relator equal to 14 were proved to satisfy the Andrews-Curtis conjecture [BM06].

Example (iv) is the most widely studied. It corresponds to a handle decomposition of the Akbulut-Kirby 4-sphere [AK85]. All the presentations of the family $\langle x, y \mid xyx = yxy, x^n = y^{n+1}\rangle$ yield the trivial group. For $n > 2$ it is unknown whether they are Andrews-Curtis trivializable. For $n = 2$, Gersten (unpublished) showed in 1982 that the presentation to be trivializable via Andrews-Curtis transformations. It was not until several years later that Havas & Ramsay [HR03] and Miasnikov [Mia99, MM01] exhibited concrete trivializations. In order to find them, Miasnikov described new *genetic algorithms* designed to test the validity of the Andrews-Curtis conjecture. His main result states that all balanced presentations of the trivial group with total length of defining relators at most 12 satisfy the conjecture. He used the computational software package MAGNUS. On the other hand, Havas & Ramsay showed that a computational attack based on a *breadth-first search* of the tree of equivalent presentations is also viable. They developed the free software ACME. For $n = 3$, none of the algorithms could give any results. In fact, the previous approaches have a common weakness: they aim to exhibit the explicit sequence of moves transforming the original presentation into the trivial one. But recently, in 2015, Bridson [Bri15] found examples of balanced presentations of the trivial group, which are trivializable via Andrews-Curtis moves, while the number of the minimum amount of transformations required grows faster than any tower of exponentials. The examples are built with the aim of encoding into balanced presentations the complexity of the word problem in groups of a certain type.

Contributions

3-deformations are known not to be manageable algorithmically. We propose a new strategy to study them: the introduction of new *methods of reduction* of finite spaces, representing two-step 3-deformations. That is, $X \nearrow Y \searrow *$, where the height of Y is less than or equal to 3. Each reduction method decreases the number of points or edges of the Hasse diagram associated to the poset, and can be easily described in terms of the combinatoric of that graph. Some examples of the new methods that we developed are: the *edge reduction*, the *qc-middle-reductions* and several other types of quotient reductions, including the *O-reductions*, a generalization and improvement of Osaki's reduction method [Osa99] for weak homotopy types of finite spaces. These methods provide a (non exhaustive) algorithmic technique to show that a finite topological space 3-deforms to a point, leading to the characterization of new classes of spaces satisfying the Andrews-Curtis conjecture.

The main tool that we use to prove these methods is a generalization of the non-Hausdorff mapping cylinder for general relations: the *non-Hausdorff relation cylinder*, denoted by $B(\mathcal{R})$.

Corollary 3.1.4. *Let $\mathcal{R} \subseteq X \times Y$ be a relation between finite spaces. If $\underline{\mathcal{R}^{-1}(U_y)}$, $\overline{\mathcal{R}(F_x)}$ are homotopically trivial for every $x \in X$, $y \in Y$, then $X \diagup\!\!\searrow Y$. Moreover, if $\overline{\mathcal{R}(F_x)}$, $\underline{\mathcal{R}^{-1}(U_y)}$ are collapsible, then $\mathcal{K}(X) \diagup\!\!\searrow^{n} \mathcal{K}(Y)$, with $n = h(B(\mathcal{R}))$ the height of the cylinder.*

The new methods of reduction of finite spaces developed in this thesis to study the Andrews-Curtis conjecture are described in the following results.

Middle-reductions are a variation of qc-reductions for points which are not maximal nor minimal satisfying some conditions.

Proposition 3.2.7. *Let X be a finite space of height at most 2. If there is a middle-reduction from X to $X/_{\{a,b\}}$, then $X \nearrow\searrow^{3} X/_{\{a,b\}}$.*

Edge-reductions is a reduction of special kind of edges of the Hasse diagram of the finite space.

Proposition 3.2.12. *Let X be a finite space of height less than or equal to 2. Let $e = (a \prec b)$ be an edge such that there is an edge-reduction from X to $X \smallsetminus e$. Then $X \nearrow\searrow^{3} X \smallsetminus e$.*

A relevant result was the proof that Osaki's reductions [Osa99] in spaces of dimension 2 are actually 3-defomations.

Theorem 3.2.15. *Let X be a finite space of height n and let $x_0 \in X$. If $U_x \cap U_{x_0}$ is empty or contractible for every $x \in X$, then $X \nearrow\searrow^{n+1} X/U_{x_0}$.*

In general, a finite space X does not have the same weak homotopy type as its quotients X/A, even if A is contractible. We found a general condition on A that ensures that if height of X is n, then $X \nearrow\searrow^{n+1} X/A$.

Theorem 3.2.19. *Let X be a finite space of height n, $A \subsetneq X$ be a connected open subspace. If $\overline{A}$ is homotopically trivial, then $X \nearrow\searrow X/A$. Moreover, if $\overline{A}$ is collapsible, then $X \nearrow\searrow^{n+1} X/A$.*

We also show that the cylinder of a relation is very useful in the study of many other existing problems in algebraic topology via combinatorial methods, such as the homotopy type of homotopy colimits [FM16] and generalizations and improvements of the Nerve Theorem [FM17].

The following result is a generalization of the classic Nerve Theorem (proved by Borsuk [Bor48] and improved by Björner [Bjö03] and Barmak [Bar11b]), in which we do not require the covering to have all its intersections contractible.

Given $\mathcal{U}$ a finite open cover of a space X, the *non-Hausdorff nerve of $\mathcal{U}$* is the face poset of the classical nerve $\mathcal{N}(\mathcal{U})$. We denoted it by $N(\mathcal{U})$.

Theorem 4.1.4. *Let X be a finite topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Let $N_0(\mathcal{U})$ be the subspace of the non-Hausdorff nerve $N(\mathcal{U})$ consisting of all homotopically trivial intersections. If for every $x \in X$, the subspace $\mathcal{I}_x$ of $N_0(\mathcal{U})$ of the intersections which contain x, is homotopically trivial, then X has the same simple homotopy type as $N_0(\mathcal{U})$.*

We say that a family $\mathcal{U}$ of open subspaces of a finite space X is a *quasi-good cover* if every nonempty intersection of a subfamily of $\mathcal{U}$ has homotopically trivial connected components. From quasi-good covers we can construct coverings satisfying hypotheses of Theorem 4.1.4. Denote by $M(\mathcal{U})$ to the finite space whose elements are the connected components of the intersections of elements in $\mathcal{U}$, with the order given by the inclusion.

Corollary 4.1.9. *Let X be a finite space and let $\mathcal{U}$ be a quasi-good cover of X. Then, $X \nearrow\!\searrow M(\mathcal{U})$.*

We also introduced a finite model for homotopy colimits, and used the relation cylinder construction and a classical result of McCord to generalize a celebrated theorem of Thomason [Tho79], in the context of homotopy colimits over posets. We also derived analogues of well known results on homotopy colimits in the combinatorial setting, including a cofinality theorem [BK72, Vog73] and a generalization of Quillen's Theorem A [Qui73] for posets. In particular this allowed us to characterize the homotopy colimits of diagrams of simplicial complexes in terms of the Grothendieck construction on the diagrams of their face posets. These results appeared in our article [FM16].

Given $X : P \to \mathcal{P}_{<\infty}$ a diagram from a poset P to the category of finite posets, we define $\underline{\mathrm{hocolim}}\, X$, the *non-Hausdorff homotopy colimit* of X, as its Grothendieck construction $P \int X$, which is also a poset with an easy description.

We handle the Grothendieck construction on a diagram of finite posets as a finite topological space and use a local-to-global theorem of McCord to derive analogues of well known results on homotopy colimits in the combinatorial setting and to prove a generalization of Thomason's theorem. This generalization allows us to apply *reduction methods* to investigate homotopy colimits of diagrams of polyhedra (indexed by finite posets) Quillen's Theorem A for posets follows immediately from the propositions below, by applying the results to the poset **1**, with elements $0 < 1$.

Proposition 5.3.1. *Let $X : P \to \mathcal{P}_{<\infty}$ be a P-diagram of finite posets. If $p \in P$ is an up beat point, then $\underline{\mathrm{hocolim}}\, X \searrow\!\!\searrow \underline{\mathrm{hocolim}}\, X_{|_{P\smallsetminus\{p\}}}$. In particular, they are weak equivalent.*

Proposition 5.3.5. *Let $X : P \to \mathcal{P}_{<\infty}$ be a P-diagram of finite posets. If p is a down beat point of P dominated by an element q and $f_{qp}^{-1}(U_x)$ is contractible for every $x \in X_p$, then $\underline{\mathrm{hocolim}}(X) \searrow \underline{\mathrm{hocolim}}(X_{|P\smallsetminus\{p\}})$. In particular they are weak equivalent.*

The main result of this part is the following generalization of Thomason's theorem in the context of finite posets.

Theorem 5.4.1. *Let P be a finite poset. Let $K : P \to \mathrm{Top}$ be a diagram of spaces and $X : P \to \mathcal{P}_{<\infty}$ be a diagram of finite posets. Let $\phi : K \to X$ be a diagram morphism (where X is viewed as a diagram of finite topological spaces) such that $\phi_p : K_p \to X_p$ is a weak equivalence for every $p \in P$. Then there exists a weak equivalence*

$$\hat{\phi} : \mathrm{hocolim}\, K \to \underline{\mathrm{hocolim}}\, X.$$

As a direct consequence of this result we derive a particular case of Thomason's theorem in the context of posets, and also a kind of converse of Thomason's theorem, which relates the homotopy colimit of a diagram of simplicial complexes with the non-Hausdorff homotopy colimit of the diagram of their face posets. As a corollary of Theorem 5.4.1, we also prove invariance of homotopy type under barycentric subdivision for homotopy colimits of simplical complexes in the (general) unordered setting. The theory developed allows us to simplify the computation of homotopy colimits of diagrams of spaces in general.

On the other hand, we also made another different combinatorial approach to the Andrews-Curtis conjecture, this time making a new interpretation of its "group presentation" version. Given a balanced presentation of the trivial group, we develop two techniques to transform it into a new presentation which can be obtained from the original one through a sequence of Andrews-Curtis transformations, without specifying the list of movements to go from one to the other. Given a group presentation $\mathcal{P}$ of the group G we define $X_{\mathcal{P}}$, the associated poset, to be the face poset of a canonical subdivision of the cellular complex associated to $\mathcal{P}$. The fundamental group of this poset is clearly isomorphic to G and its Hasse diagram can be easily described by means of $\mathcal{P}$.

One of the developed techniques is inspired on colorings of posets and it is indeed a generalization and discrete implementation of the fact that the collapse of a tree into the 1-skeleton in a 2-dimensional cellular complex is a 3-deformation. Namely, given X a finite poset and A a collapsible subdiagram of X containing all its points, we define $\mathcal{P}_{X,A}$ a presentation of the fundamental group of X.

Theorem 6.3.1. *If $\mathcal{P}$ be a group presentation and A is a spanning collapsible subdiagram of $\mathcal{H}(X_{\mathcal{P}})$, then $\mathcal{P} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$.*

The other technique that we designed is inspired in discrete Morse theory, introduced by Robin Forman [For98] on the '90s as a combinatorial version of the classic theory for differentiable manifolds. We obtain a stronger version of the main theorem of Morse theory, making the theory more applicable to our setting.

Our result has the advantage of tracking down accurately the homotopy equivalence obtained. We show that a discrete Morse function (or equivalently, a matching M in the Hasse diagram of the face poset of the complex) induces an $(n+1)$-deformation between the original (n-dimensional) complex and the Morse complex K_M, not only a homotopy equivalence. We also describe precisely this Morse complex, indicating its attaching maps instead of just its cellular decomposition.

Theorem 6.4.17. *Let K be a regular cell complex of dimension n and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$. Then $K \overset{n+1}{\nearrow\searrow} K_M$.*

We give an explicit description of the standard presentation $\mathcal{Q}_{\mathcal{X}(K),M}$ of K_M.

Corollary 6.5.6. *Let K be a regular cell complex of dimension 2, and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$ with only one critic cell of dimension 0. Then, $\mathcal{Q}_{\mathcal{X}(K),M}$. is a balanced presentation of $\pi_1(K)$. Moreover, $\mathcal{Q}_{\mathcal{X}(K),M} \sim_{AC} \mathcal{P}_K$.*

As a consequence of the previous approaches, we obtain concrete algorithms to construct AC-equivalent presentations, in which we avoid the need of exhibiting the list of moves to assert an Andrews-Curtis transformation, which is a substantial advantage in the light of [Bri15].

Using this strategy we provide, for example, proofs to the following facts about the list of potential counterexamples.

- The presentation $\langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle$ [AK85] for $n = 2$, satisfies the Andrews-Curtis conjecture.

- The presentation $\langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2\rangle$ [Rap68a], can be transformed into a presentation with 2 generators and 2 relators.

- There is a large class of integers n, m, p, q for which the group presentations $\langle x, y \mid x = [x^m, y^n], y = [x^p, y^q]\rangle$ [Bro84] satisfy the conjecture.

Our methods are easily handled by a computer. We implemented the already existing algorithms about finite spaces [Fer11] and the new ones that we develop in this Thesis using the free software SAGE [S⁺17]. We created a module to be added to SAGE (see Appendices 2.A, 3.A and 6.A) which allow one to work with finite spaces and its applications, including the study of finite group presentations and cellular complexes. The source code is available in `https://github.com/ximenafernandez/Finite-Spaces`.

Outline of the chapters

The Dissertation is organized in six chapters. In the fist chapter we give an historical overview of the problem. We survey existing approaches, and propose a unified framework to see all of them. In the second chapter, we develop the combinatorial point of view on which we will work on over the next chapters. We review the connection between regular cell complexes, finite topological spaces and posets. We explain the simple homotopy theory for finite spaces and the reformulation of the Andrews-Curtis conjecture from this perspective. The third chapter introduces the novel combinatorial methods on finite spaces that we developed to study 3-deformations, and its applications to the finite space version of the conjecture and other problems in algebraic topology. The fourth chapter shows new versions of the classical Nerve theorem and some applications to $(n+1)$-deformations. The fifth chapter deals with homotopy colimits. In the sixth chapter we design new combinatorial methods for group presentations and their applications to study the list of potential counterexamples. Almost all the results of Chapters 3,4,5 and 6 are new and original. Some of the results of Chapter 3, 4 and 5 appeared in [FM16, FM17]. The results of Chapter 6 are still unpublished and subject to a future paper [Fer17].

Chapter 1

Group presentations, 3-deformations and the Andrews-Curtis conjecture

The Andrews-Curtis conjecture is an open problem formulated more than 50 years ago, with equivalent formulations in the context of combinatorial group theory and topological spaces. The question was raised in 1965 by J.J. Andrews and M.L. Curtis in their famous article "*Free groups and handlebodies*", extending an idea of Nielsen about free groups and addressing some topological consequences which would follow if the conjecture were true. One of their motivations was the Poincaré conjecture. It was the anonymous referee who noticed that a slight modification of his problem was equivalent to another important open problem in low dimensional topology. From then on, many mathematicians and computer scientists attacked the problem, with only partial success. In this chapter, we will make an overview of the origins of the problem and a review of the classic and more recent works that have been done on this topic.

1.1 Group presentations

Definition 1.1.1. A *presentation* $\mathcal{P} = \langle X \mid R\rangle$ of a group G consists of a set of *generators* X and a set of *relators* $R \subseteq F(X)$, that is a subset of the free group generated by X, such that $G = F(X)/N(R)$, the quotient of $F(X)$ by the normal subgroup generated by R. The group presented by $\mathcal{P}$ is also denoted by $G(\mathcal{P})$.

In this work we will consider only finite group presentations; i.e, presentations $\mathcal{P} = \langle X \mid R\rangle$ where X and R are finite sets.

Example 1.1.2. The following are presentations of the trivial group:

- $\mathcal{P} = \langle x \mid x, x\rangle$
- $\mathcal{P} = \langle x \mid xx^{-1}x \rangle$
- $\mathcal{P} = \langle x, y \mid xyx^{-1}y^{-2}, yxy^{-1}x^{-2} \rangle$
- $\mathcal{P} = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1}, x^3y^{-4}\rangle$

A group always admits many different presentations, so a natural problem is to decide whether two different presentations correspond to isomorphic groups. In 1911, Max Dehn formulated in [Deh11] three fundamental problems for an infinite group given by a finite presentation $\mathcal{P} = \langle X \mid R \rangle$, which in modern terms can be described as follows:

- **The word problem:** Given $w \in F(X)$, find an algorithm to decide whether or not this element equals the identity element.
- **The conjugacy problem:** Given $s, t \in F(X)$, find an algorithm to decide whether or not s and t are conjugated.
- **The isomorphism problem:** If $\mathcal{P}' = \langle X' \mid R' \rangle$ is another presentation, find an algorithm to decide whether or not $G(\mathcal{P})$ is isomorphic to $G(\mathcal{P}')$.

In 1912, he gave an algorithm that solved both the word and conjugacy problem for fundamental groups of closed orientable 2-dimensional manifolds [Deh12]. However, the problems remained unsolved in general. Dehn stated that "solving the word problem for all groups may be as impossible as solving every mathematical problem". It was as late as 1936 that the development of the theory of computability by Alan Turing [Tur37] allowed to give precise meaning to Dehn's premonition. A *decision problem* is a problem that can be formulated as a yes-no question on the input values. A decision problem is said to be *decidable* if there exists a Turing machine that halts on all possible inputs and returns the right answer. With this formal model, Turing provided the first example of undecidable problem: the famous *halting problem*. It can be described as the problem of deciding, given the description of a Turing machine and an input, whether the machine produces an answer or loops forever. In 1955, Novikov [Nov55] and Boone [Boo54a, Boo54b, Boo55a, Boo55b] constructed independently examples of finite presentations with unsolvable word problem.

As a counterpart, in 1908 Heinrich Tietze introduced a short list of transformations or elementary steps to perform on the group presentations that preserves the presented group [Tie08]. The power of his theory it that any two presentations of the same group can be transformed one in the other through a finite number of his elementary transformations.

Definition 1.1.3. Let $\mathcal{P} = \langle X \mid R \rangle$ be a group presentation. The *Tietze transformations* are:

(T1) add a new generator x and a new relator xw^{-1} with $w \in F(X)$;

(T2) if there exists a relator $r = x^{-1}w$ with $w \in F(X \smallsetminus \{x\})$ such that any other relator also belongs to $F(X \smallsetminus \{x\})$, then remove the generator x and the relator r;

(T3) add a new relator $r \in N(R)$;

(T4) if there is $r \in N(R \smallsetminus \{r\})$, then remove the relator r.

We say that $\mathcal{P}$ and $\mathcal{Q}$ are *Tietze-equivalent*, and we denote by $\mathcal{P} \sim_T \mathcal{Q}$ if we can transform $\mathcal{P}$ into $\mathcal{Q}$ by a finite sequence of Tietze transformations.

Example 1.1.4. The last presentation of the Example 1.1.2

$$\mathcal{P} = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1}, x^3y^{-4}\rangle$$

can be transformed into a presentation of the trivial group by the following sequence of Tietze transformations:

$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4}\rangle$	
$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4},\ x^4y^{-1}x^{-1}y^{-4}xy\rangle$	(Add $x^4y^{-1}x^{-1}y^{-4}xy$)
$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4},\ x^4y^{-1}x^{-1}x^{-3}xy\rangle$	(Add $x^4y^{-1}x^{-1}x^{-3}xy$ and remove $x^4y^{-1}x^{-1}y^{-4}xy$)
$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4},\ x^4y^{-1}x^{-3}y\rangle$	(Add $x^4y^{-1}x^{-3}y$ and remove $x^4y^{-1}x^{-1}x^{-3}xy$)
$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4},\ x^4y^{-1}y^{-4}y\rangle$	(Add $x^4y^{-1}y^{-4}y$ and remove $x^4y^{-1}x^{-3}y$)
$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4},\ x^4y^{-4}\rangle$	(Add x^4y^{-4} and remove $x^4y^{-1}y^{-4}y$)
$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4},\ x^4x^{-3}\rangle$	(Add x^4x^{-3} and remove x^4y^{-4})
$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1},\ x^3y^{-4},\ x\rangle$	(Add x and remove x^4x^{-3})
$\langle x, y \mid yy^{-1}y^{-1},\ x^3y^{-4},\ x\rangle$	(Add $yy^{-1}y^{-1}$ and remove $xyxy^{-1}x^{-1}y^{-1}$)
$\langle x, y \mid y,\ x^3y^{-4},\ x\rangle$	(Add y and remove $yy^{-1}y^{-1}$)
$\langle x, y \mid y,\ x\rangle$	(Remove x^3y^{-4})

The previous sequence of transformations can be easily understood if we think about the relators as "equations" and about Tietze transformations as "valid operations". In the previous case, the equations can be thought as:

(I) $xyx = yxy$,

(II) $x^3 = y^4$.

From the equation (I), $x = y^{-1}x^{-1}yxy$. By raising to the fourth power each side, we obtain $x^4 = y^{-1}x^{-1}y^4xy$. By equation (II), we can replace the latter y^4 by x^3. Then, $x^4 = y^{-1}x^{-1}x^3xy$. That is, $x^4 = y^{-1}x^3y$. Again, by (II) $x^4 = y^{-1}y^4y$, so that $x^4 = y^4$. Finally, $x^4 = x^3$. It follows that $x = 1$ and then $y = 1$.

Theorem 1.1.5. *(Tietze) Let $\mathcal{P}$, $\mathcal{Q}$ be finite group presentations of the groups $G(\mathcal{P})$ and $G(\mathcal{Q})$ respectively. Then $G(\mathcal{P})$ is isomorphic to $G(\mathcal{Q})$ if and only if $\mathcal{P} \sim_T \mathcal{Q}$.*

Proof. It is clear that the Tietze transformations preserve the presented group. For the converse, assume that $\mathcal{P} = \langle X_1 \mid R_1\rangle$ and $Q = \langle X_2 \mid R_2\rangle$ are two group presentations of isomorphic groups. Then there exist a group G and epimorphisms $\phi_i : F(X_i) \to G$ such that $\ker(\phi_i) = N(R_i)$ for $i = 1, 2$. Without loss of generality, we can suppose $X_1 \cap X_2 = \varnothing$. Let X be the disjoint union $X_1 \cup X_2$. Let F be the free group $F(X)$ and $\phi : F \to G$ be the map defined by:

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \in X_1 \\ \phi_2(x) & \text{if } x \in X_2 \end{cases}$$

For every $x \in X_1$, there exists an element $w_x \in F(X_2)$ such that $\phi(x) = \phi(w_x)$. Define $S_1 = \{xw_x^{-1} : x \in X_1\}$. Analogously, for every $x \in X_2$, there exists an element $w_x \in F(X_1)$ such that $\phi(x) = \phi(w_x)$. Define $S_2 = \{xw_x^{-1} : x \in X_2\}$. In a first step, transform the presentation $\mathcal{P} = \langle X_1 \mid R_1\rangle$ into $\langle X_1 \cup X_2 \mid R_1 \cup S_2\rangle$ using the Tietze transformations of type (T1). After that, since $\ker(\phi) = N(R_1 \cup S_2)$ and $R_2 \cup S_1 \subseteq \ker(\phi)$, the relators in $R_2 \cup S_1$ can be added to $\langle X_1 \cup X_2 \mid R_1 \cup S_2\rangle$ according to the move of type (T3). Thus, $\langle X_1 \mid R_1\rangle \sim_T \langle X \mid R_1 \cup R_2 \cup S_1 \cup S_2\rangle$. Similarly, it can be proven that $\langle X_2 \mid R_2\rangle \sim_T \langle X \mid R_1 \cup R_2 \cup S_1 \cup S_2\rangle$. Thus, $\mathcal{P} \sim_T \mathcal{Q}$. □

1.2 Simple homotopy theory of cellular complexes

This section is intended to briefly recall some of the basic constructions and results of Whitehead's simple homotopy theory, which will be used in the sequel. For a more detailed exposition of this subject, we refer the reader to [Coh67], [HAM93, Ch. 1].

Cellular complexes are combinatorial models of topological spaces. They are built as unions of cells attached with a less rigid structure than simplicial complexes. This class of spaces has many good properties. Some of them are: they serve as a weak homotopy model for any topological space, the notion of homotopy equivalence of cellular complexes coincides with the (easier to check) notion of weak equivalence, singular homology and cohomology of cellular complexes are readily computable via the cellular homology theory. The main goal of Whitehead's work [Whi39] was to understand the classic homotopy theory combinatorially. Simple homotopy theory was the attempt to formulate homotopy theory in a simpler and more concrete way, as an analogous of Tietze's theory to describe the isomorphism class of group presentations.

In 1939, Whitehead introduced the simple homotopy theory for simplicial complexes. Some years later, he extended the theory to a more general class of cell complexes, and finally, to the well known CW-complexes. In this thesis we will use *cell complexes or celular complexes* to mean CW-complexes. In many cases, we will work with simplicial complexes, regular cell complexes or combinatorial cell complexes (such as the complexes associated to presentations).

The aim will be to describe particular simple operations which do not change the homotopy type of a given complex L; namely, the attaching of an n-ball along a $(n-1)$-ball in its boundary. The pair (D^n, D^{n-1}) is assumed to form a standard ball pair, i.e., a pair homeomorphic to $(I^{n-1} \times I, I^{n-1} \times \{1\})$. In this work we will consider only finite connected cellular complexes.

Definition 1.2.1. Let K be a cellular complex and let L be a subcomplex. We say that K *elementary collapses* to L, and we denote it by $K \searrow^{e} L$, if $K = L \cup e^{n-1} \cup e^n$ with $e^{n-1}, e^n \notin L$ and there exists a map $\psi : D^n \to K$ such that ψ is the characteristic map of e^n, $\psi|_{\overline{\partial D^n \smallsetminus D^{n-1}}}$ is the characteristic map of e^{n-1} and $\psi(D^{n-1}) \subseteq L^{n-1}$.

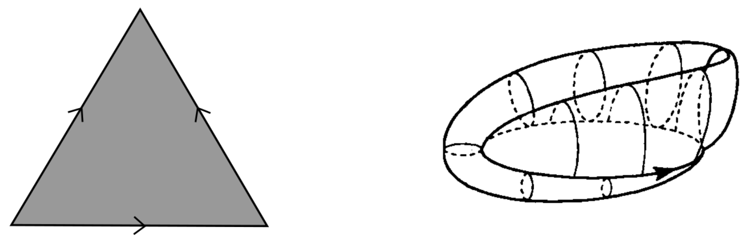
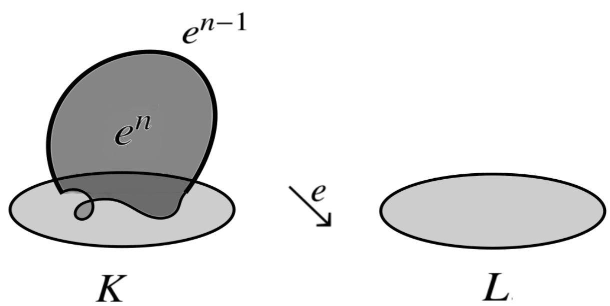


Figure 1.1: The Dunce Hat, figure adopted from [HAM93, p. 18]



Definition 1.2.2. Let K be a cellular complex and let L be a subcomplex. We say that K *collapses* to L (or L *expands* to K), and we denote it by $K \searrow L$ (resp. $L \nearrow K$), if there is a finite sequence of elementary collapses from K to L.

Remark 1.2.3. If $K \searrow L$, then $L \subseteq K$ is a strong deformation retract.

Many contractible complexes are collapsible; for instance, n-simplices are collapsible. But this fact does not hold in general. In some cases, it may be necessary to do some expansions before collapsing to a small complex. A famous example for this is the *Dunce Hat* studied by E. C. Zeeman is his article "*On The Dunce Hat*" [Zee64].

Example 1.2.4. The Dunce Hat H is a topological space constructed as the following identification of the sides of a triangle.

The Dunce Hat is the simplest example of a contractible but not collapsible polyhedron. A basic cellular structure is given by one 0-cell, one 1-cell x and one 2-cell attached with the map associated to the word xxx^{-1}. The contractibility of H results from the fact that the attaching map of the 2-cell is homotopic to the identity, and then $H \simeq D^2$, the 2-dimensional disk (see Lemma 1.4.6). On the other hand, any cellular structure of the Dunce Hat does not allow any collapse, as each 1-cell locally bounds at least two 2-cells. Nevertheless, Zeeman showed the following sequence of collapses and expansions:

$$H \nearrow H \times I \searrow * .$$

He conjectured that this procedure can be mimicked for any contractible 2-complex. This is known as the Zeeman's conjecture (see Conjecture 1.5.1).

Definition 1.2.5. Let K and L be cellular complexes. We say that K *deforms* into L (or that K and L have the same *simple homotopy type*), and we denote it by $K \nearrow\!\!\searrow L$, if there is a sequence of cellular complexes $K = K_0, K_1, \cdots K_r = L$ such that for each $0 \leq i \leq r-1$, $K_i \overset{e}{\searrow} K_{i+1}$ or $K_i \overset{e}{\nearrow} K_{i+1}$. For every $0 \leq i \leq r-1$, there is a homotopy equivalence $f_i : K_i \to K_{i+1}$ which is an inclusion or a retraction depending on whether $K_i \overset{e}{\nearrow} K_{i+1}$ or $K_{i+1} \overset{e}{\searrow} K_i$ respectively. In that case, K and L are then related by a *deformation* $f : K \to L$ defined to be the composition of the retractions and inclusions as above, i.e, $f = f_{r-1} \cdots f_1 f_0$.

Remark 1.2.6. If $K \nearrow\!\!\searrow L$, then $K \simeq L$, i.e., they are homotopically equivalent. The converse is false. Indeed, there is an obstruction called the Whitehead group $Wh(K)$ which quantifies the gap. It can be proven that the Whitehead group of simply connected complexes is trivial, and that if two cell complexes have trivial Whitehead group, then they are homotopy equivalent if and only if they are simple homotopy equivalent. In particular, a cell complex K is contractible if and only if it is possible to perform collapses and expansions from K to obtain the singleton $*$ (see **??** for a complete exposition).

Definition 1.2.7. Let K and L be cellular complexes. We say that K *n-deforms* to L, and we denote it by $K \nearrow\!\!\overset{n}{\searrow} L$, if K can be deformed to L through a sequence of collapses and expansions $K = K_0, K_1, \cdots, K_r = L$ with $\dim(K_i) \leq n$ for all $1 \leq i \leq r$.

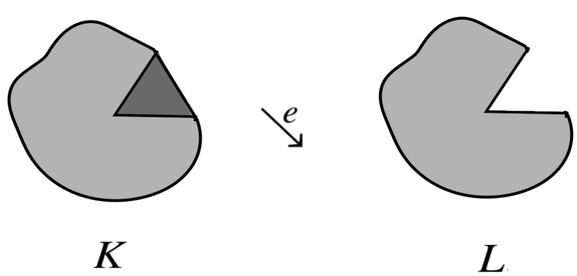
In dimension 1, using Nielsen's theorem [Nie18] on the decomposition of automorphisms of free groups, it can be proven that every map between finite connected 1-dimensional cell complexes which induces an isomorphism on the fundamental groups is homotopic to a 2-deformation.

Theorem 1.2.8. *(C.T.C.Wall) [Wal66b] Let K and L be cellular complexes of dimension $n \neq 2$. If $K \nearrow\!\!\searrow L$, then $K \nearrow\!\!\overset{n+1}{\searrow} L$.*

If K and L are cellular complexes of dimension $n = 2$, then Whitehead proved that if $K \nearrow\!\!\searrow L$, then $K \nearrow\!\!\overset{4}{\searrow} L$. It is still an open problem whether $K \nearrow\!\!\overset{3}{\searrow} L$. In particular, we have the following conjecture.

Conjecture 1.2.9 (Topological version of Andrews-Curtis conjecture)**.** *If K is a contractible 2-complex, then $K \nearrow\!\!\overset{3}{\searrow} *$.*

Remark 1.2.10. There is a more rigid version of simple homotopy theory for simplicial complexes. Whitehead introduced the notions of simplicial collapses and expansions in [Whi39]. The rigidity of simplicial complexes led him to extend these notions to a more general class of complexes [Whi49].



The following results relate both notions (the simplicial and the more general one) and allow us to work indistinctly in either category.

- If K and L are simplicial complexes, then there is a simplicial n-deformation $K \overset{n}{\nearrow\!\!\searrow} L$ if and only if there is a cellular n-deformation $K \overset{n}{\nearrow\!\!\searrow} L$ (see[RS72, Appendix B5], [Hud69, Chap. II], [HAM93, Prop. 2.3.]).

- If K is a cellular complex of dimension n, then there exists a simplicial complex L of dimension n such that $K \overset{n+1}{\nearrow\!\!\searrow} L$ (as cellular complexes) [Coh73, 7.2].

1.3 The connection between group presentations and 2-complexes

The complexity of homotopy theory of cellular complexes of dimension 2 (or *2-complexes*) can be best understood by considering its correspondence with group presentations. For a more complete exposition, see [HAM93, Chap.2 Section 2.3.] and [Wri75].

Definition 1.3.1. Let K be a 2-complex. A *standard group presentation* $\mathcal{P}_K$ associated to K is a group presentation defined as follows.

- Choose $e^0 \in K^0$ a 0-cell of K as basepoint.

- Choose $T \subseteq K^1$ a spanning tree (i.e., a contractible subcomplex of the 1-skeleton including all the vertices).

- Let $e^1_1, e^1_2 \cdots, e^1_n$ be the 1-cells in $K \smallsetminus T$. For every $1 \leq i \leq n$, choose an orientation for e^1_i and let x_i be the unique path in T from e_0 to the source of e^1_i followed by e^1_i and then the path from its target to e_0. Then, $x_i \in \pi_1(K^1, e_0)$ and, moreover, $\pi_1(K^1, e_0) = F(x_1, x_2, \cdots, x_n)$.

- Let $e^2_1, e^2_2, \cdots, e^2_m$ be the 2-cells of K. For every e^2_j, choose an attaching map φ_j, a basepoint s_0 and a orientation of ∂D^2. Thus, φ_j is a loop in $\varphi_j(s_0)$.

- For every $1 \leq j \leq m$, choose a path γ_j in K^1 from e_0 to $\varphi_j(s_0)$ and consider the loop $\gamma_j * \varphi_j * \bar{\gamma}_j$. Thus, $[\gamma_j * \varphi_j * \bar{\gamma}_j] \in \pi_1(K^1, e_0)$, and can be expressed as a product r_j of the elements $\{x_1, x_2, \cdots, x_n\}$ and its inverses.

- Define $\mathcal{P}_K = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_m \rangle$.

Example 1.3.2. Consider the following regular cell structure K for the projective plane, with three 0-cells, six 1-cells and four 2-cells as in Figure 1.2. The full line determines the chosen spanning tree in K^1.

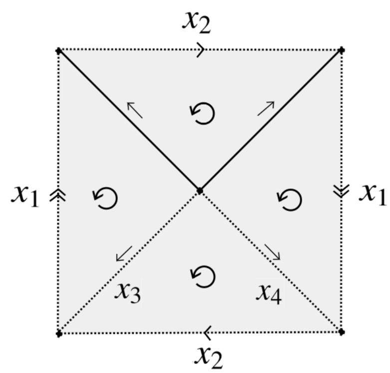


Figure 1.2: Regular cell structure of the projective plane.

Then,

$$\mathcal{P}_K = \langle x_1, x_2.x_3, x_4 \mid x_3^{-1}x_1^{-1}, x_3x_2^{-1}x_4^{-1}, x_4x_1^{-1}, x_2^{-1}\rangle.$$

It is easy to see that $G(\mathcal{P}_K) = \mathbb{Z}_2$.

Notice that the definition of $\mathcal{P}_K$ depends on many choices. Different choices will give rise to different presentations $\tilde{\mathcal{P}}_K$, which will be closely related with $\mathcal{P}_K$ (see Lemma 1.4.4).

Remark 1.3.3. By van Kampen's Theorem, $\pi_1(K, e_0) = F(x_1, x_2, \cdots, x_n)/N(r_1, r_2, \cdots, r_m)$. Note that $\mathcal{P}_K$ not only gives information about the fundamental group of K, but also encodes data about the homotopy type of K. If we consider the cellular decomposition of S^2 with one 0-cell, one 1-cell and two 2-cells, and one of D^2 consisting of one 0-cell, one 1-cell and one 2-cell, then $G(\mathcal{P}_{D^2}) = G(\mathcal{P}_{S^2})$ but $\mathcal{P}_{D^2} = \langle a \mid a\rangle \neq \langle a \mid a, a\rangle = \mathcal{P}_{S^2}$.

Many homotopy invariants can be derived from the presentation $\mathcal{P}_K = \langle X \mid R\rangle$, for instance, the Euler characteristic $\chi(K) = 1-|X|+|R|$. Indeed, since K is homotopy equivalent to K/T with T a spanning tree on K^1, $\chi(K) = \chi(K/T) = |\{\text{0-cells}\}| - |\{\text{1-cells}\}| + |\{\text{2-cells }\}| = 1 - |X| + |R|$.

We will see that $\mathcal{P}_K$ encodes the simple homotopy type of K (see Theorem 1.4.5).

Definition 1.3.4. Let $\mathcal{P} = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_m\rangle$ be a group presentation. The *deficiency* of $\mathcal{P}$ is $d(\mathcal{P}) = m - n$. We say that $\mathcal{P}$ is *balanced* if $d(\mathcal{P}) = 0$.

Remark 1.3.5. If K is a 2-complex and $T \subseteq K^1$ is a spanning tree, then, since T is contractible, $q : K \to K/T$ is a homotopy equivalence. Notice that $\mathcal{P}_K = \mathcal{P}_{K/T}$ and $\mathcal{X}(K) = \mathcal{X}(K/T) = 1 - n + m = 1 + d(\mathcal{P}_K)$.

Proposition 1.3.6. *Let K be a 2-complex and $\mathcal{P}_K$ the associated group presentation. If K is simply connected, i.e., $\mathcal{P}_K$ is a presentation of the trivial group, then K is contractible if and only if $\mathcal{P}_K$ is balanced.*

Proof. If K is contractible, $\chi(K) = 1$. Therefore, $d(\mathcal{P}_K) = \chi(K) - 1 = 0$. Conversely, if $\mathcal{P}_K$ is balanced, then

$$1 = \chi(K) = \text{rg}(H_0(K)) - \text{rg}(H_1(K)) + \text{rg}(H_2(K)).$$

Since $\text{rg}(H_0(K)) = 1$ and $H_1(K) = \pi_1(K)_{ab} = 0$, it follows that $\text{rg}(H_2(K)) = 0$. On the other hand, since $\dim(K) = 2$, $H_2(K)$ is free. Thus, $H_2(K) = 0$. In conclusion, $\pi_1(K) = 0$ and $H_n(K) = 0$ for all $n \geq 1$. By Whitehead Theorem's, K is contractible. □

As an inverse process of the above constructions, we may associate a finite cellular complex to every finite group presentation as follows.

Definition 1.3.7. Let $\mathcal{P} = \langle x_1, x_2 \cdots, x_n \mid r_1.x_2, \cdots, r_m\rangle$ be a finite group presentation. The *presentation complex* $K_{\mathcal{P}}$ is the cellular complex constructed as follows:

- its 0-skeleton is a singleton;
- for every generator x_i, attach to the 0-cell an oriented 1-cell e_i^1, building a bouquet of circles $\bigvee_{i=1}^n S^1$;
- every relator r_j determines a closed path φ_j in $\bigvee_{i=1}^n S^1$; for every $1 \leq j \leq m$ attach a 2-cell e_j^2 with attaching map φ_j.

For instance, the trivial presentation of the trivial group $\mathcal{P} = \langle \mid \rangle$ has as standard complex the singleton. However, the presentation $\mathcal{P} = \langle x \mid x\rangle$ of the trivial group yields a disc D^2 and $\mathcal{P} = \langle x \mid xxx^{-1}\rangle$ gives rise to the Dunce Hat (see 1.2.4).

Remark 1.3.8. As consequence of van Kampen's theorem, $\pi_1(K_{\mathcal{P}}) = G(\mathcal{P})$. Although Tietze modifications of $\mathcal{P}$ preserve the presented group, they may change the (simple) homotopy type of the associated standard complex. For instance, $\mathcal{P} = \langle x \mid x, x\rangle \sim_T \langle x \mid x\rangle = \mathcal{Q}$ but $K_{\mathcal{P}} \equiv S^2$ and $\mathcal{K}_{\mathcal{Q}} \equiv D^2$.

One can see that $\mathcal{P} \sim_T \mathcal{Q}$ if and only if $K_{\mathcal{P}} \vee \bigvee_{i=1}^k S_i^2 \diagup\!\!\searrow^3 K_{\mathcal{Q}} \vee \bigvee_{j=1}^l S_j^2$. In other words, two cellular 2-complexes K and L have isomorphic fundamental groups if and only if $K \vee \bigvee_{i=1}^k S_i^2 \nearrow\!\!\searrow^3 L \vee \bigvee_{j=1}^l S_j^2$.

1.4 The Andrews-Curtis conjecture

The following transformations of presentations preserve the group and also the deficiency of the presentation. We will see that, in fact, they preserve the simple homotopy type of the associated presentation complex.

Definition 1.4.1. Let $\mathcal{P} = \langle x_1, x_2, \cdots, x_n \mid r_2, r_2, \cdots, r_m\rangle$ be a group presentation. The *extended Nielsen transformations* or *Andrews-Curtis transformations* are:

(AC1) replace a relator r_j by r_j^{-1};

(AC2) replace a relator r_j by $r_j r_k$ or $r_k r_j$ with $k \neq j$;

(AC3) replace a relator r_j by $w r_j w^{-1}$ with $w \in F(x_1, x_2, \cdots, x_n)$;

(AC4) replace all the occurrences of the generator x_i in the relators by x_i^{-1}, $x_i x_j$ or $x_j x_i$ with $i \neq j$.

(AC5) add a new generator x and a new relator x;

(AC6) the inverse of the previous operation, if it is possible.

We say that $\mathcal{P}$ and $\mathcal{Q}$ are *AC-equivalent*, and denote by $\mathcal{P} \sim_{AC} \mathcal{Q}$, if we can transform $\mathcal{P}$ into $\mathcal{Q}$ through a finite sequence of Andrews-Curtis transformations.

Remark 1.4.2. If $\mathcal{P} \sim_{AC} \mathcal{Q}$, then $\mathcal{P} \sim_T \mathcal{Q}$ but the converse is not true. For instance, $\langle a \,|\, a, a\rangle \sim_T \langle a \,|\, a\rangle$ but $\langle a \,|\, a, a\rangle \nsim_{AC} \langle a \,|\, a\rangle$ since the Andrews-Curtis transformations preserve the deficiency of the presentation.

The Andrews-Curtis transformations are equivalent to the following moves:

(AC1') replace a generator r_j by $wr_j^{\epsilon_j}w^{-1}$ with $\epsilon = \pm 1$ and $w \in F(x_1, x_2, \cdots, x_n)$.

(AC2') replace a relator r_j by $r_j r_k$ or $r_k r_j$ with $k \neq j$

(AC3') add a new generator x and a new relator xw^{-1} with $w \in F(x_1, x_2, \cdots, x_n)$.

(AC4') the inverse of the previous operation, if it is possible.

On the other hand, Tietze operations are equivalent to Andrews-Curtis operations plus the operation of adding a new relation equal to the identity element 1 of $F(x_1, x_2, \cdots, x_n)$, and the inverse operation.

In 1965, Andrews and Curtis conjectured that any *balanced* presentation of the trivial group can be transformed into the trivial presentation without using all the Tietze transformations, but only the Andrews-Curtis transformations [AC65] .

Conjecture 1.4.3 (Andrews-Curtis)**.** *If $\mathcal{P}$ is a balanced presentation of the trivial group, then $\mathcal{P} \sim_{AC} \langle \,|\, \rangle$.*

Notice that the operations used to transform the presentation $\mathcal{P}$ of Example 1.1.4 into the trivial presentation are not all equivalent to Andrews-Curtis transformations. Indeed, it is not known any sequence of Andrews-Curtis operations to achieve that. The presentation $\mathcal{P}$ is one of the *potential counterexamples* to disprove the conjecture (see Section 1.6).

We will see that Conjecture 1.4.3 is equivalent to Conjecture 1.2.9.

Proposition 1.4.4. *Let K be a 2-complex. The definition of $\mathcal{P}_K$ involves the choice of a spanning tree, base points, orientations, connecting arcs and attaching maps. If $\tilde{\mathcal{P}}_K$ is constructed under another choices, then $\mathcal{P}_K \sim_{AC} \tilde{\mathcal{P}}_K$.*

Proof. If T and $\tilde{T}$ are two different spanning trees for K^1, then there is a sequence of elementary changes that convert T into $\tilde{T}$. The elementary moves consist of adding to T an edge e^1 in $\tilde{T} \smallsetminus T$ and removing one of the $T \smallsetminus \tilde{T}$ edges of the path which connects the distinct endpoints of e^1 in T. This transformation of T into $\tilde{T}$ yields concrete elementary Nielsen transformations which changes the basis of $\pi_1(K^1, e^0)$ with respect to T into a basis with respect to $\tilde{T}$. The same situation occurs when choosing different orientations of the edges in K^1. Any change of basis can be thought as the result of a sequence of Nielsen transformations applied to the relators, which can be achieved by (AC4).

Different choices of basepoints and paths γ_j amount to replacing a relator r_j by $wr_j^{\pm 1}w^{-1}$, accomplished by (AC1) and (AC3). Finally, if $\varphi_j, \tilde{\varphi}_j : S^1 \to K^1$ are different attaching maps of the same 2-cell e_j^2, then $\varphi_j \simeq \tilde{\varphi}_j$ or $\varphi_j \simeq \tilde{\varphi}_j\gamma$, where $\gamma : S^1 \to S^1$ has degree equal to -1. Then, $[\varphi_j]$ and $[\tilde{\varphi}_j]$ are conjugated in $\pi_1(K^1, e^0)$. Thus, the change of the attaching map of e_j^2 from φ_j to $\tilde{\varphi}_j$ is translated into the replacement of r_j by $wr_j^{\pm 1}w^{-1}$, achieved by (AC1) and (AC3). □

The relationship between both formulations of the conjecture was first noticed by the anonymous referee of the article [AC65]. In 1975, Wright [Wri75] gave an explicit formulation and proof of the correspondence between 3-deformation classes and AC-classes of presentations. Other complete reference is [HAM93, Thm. 2.4].

Theorem 1.4.5. *There is a bijection between 3-deformation classes of 2-complexes and AC-classes of group presentations. Namely,*

- *if $K \overset{3}{\nearrow\searrow} L$, then $\mathcal{P}_K \sim_{AC} \mathcal{P}_L$;*
- *if $\mathcal{P} \sim_{AC} \mathcal{P}'$, then $K_{\mathcal{P}} \overset{3}{\nearrow\searrow} K_{\mathcal{P}'}$;*
- *$K \overset{3}{\nearrow\searrow} K_{\mathcal{P}_K}$;*
- *$\mathcal{P} \sim_{AC} \mathcal{P}_{K_P}$.*

One of the main reasons why AC-transformations preserve the 3-deformation type of the associated complexes is the following

Lemma 1.4.6. *[Coh73, Prop. 7.1][HR03, Lemma 2.1] Let K be an n-dimensional cell complex and let $\varphi \simeq \psi : S^{n-1} \to K^{n-1}$ be attaching maps for the n-cells e^n_φ and e^n_ψ respectively. If $K_\varphi = K \cup e^n_\varphi$ and $K_\psi = K \cup e^n_\psi$, then $K_\varphi \overset{n+1}{\nearrow\searrow} K_\psi$.*

If K is an n-dimensional complex and an $(n+1)$-cell is expanded to a cell complex $M = K \cup e^n \cup e^{n+1}$ and collapsed immediately afterwards from a potentially different free face $K \cup e^n \cup e^{n+1} \searrow K \cup e^n \smallsetminus e'^{\,n}$, we say that there is an $(n+1)$-*transient move*.

The following lemmas will be relevant to understand deeply a 3-deformation and translate the geometrical changes into combinatorial transformations of the associated presentations.

Lemma 1.4.7. *If $K \searrow L$ then any given sequence of elementary collapses can be reordered to yield a sequence $K = K_0 \overset{e}{\searrow} K_1 \overset{e}{\searrow} \cdots \overset{e}{\searrow} K_r = L$ with $K_i = K_{i+1} \cup e^{n_i} \cup e^{n_i - 1}$ where $n_0 \geq n_1 \geq \cdots n_{r-1}$.*

Lemma 1.4.8. *If $K \nearrow\searrow L$ then there is a cellular complex M such that $K \nearrow M \searrow L$.*

Lemma 1.4.9. *[HR03, Lemma 2.5] If $K \nearrow\searrow L$ by expanding finitely many $(n+1)$-cells e_i^{n+1} and afterwards collapsing these cells e_i^{n+1} again, then there is a transformation from K to L by finitely many $(n+1)$-transient moves.*

Proof. (of Theorem 1.4.5) We will show that the maps

$$\begin{array}{ccc} \{\text{2-complexes}\}/\overset{3}{\nearrow\searrow} & \longleftrightarrow & \{\text{group presentations}\}/\sim_{AC} \\ K & \longrightarrow & \mathcal{P}_K \\ K_{\mathcal{P}} & \longleftarrow & \mathcal{P} \end{array}$$

are well defined and mutually inverse.

Assume $\mathcal{P} \sim_{AC} \mathcal{P}'$. In order to prove $K_{\mathcal{P}} \overset{3}{\nearrow\searrow} K_{\mathcal{P}'}$, it suffices to show that every elementary AC-move in the group presentations produces a 3-deformation between the associated 2-complexes. The replacement of r_j by r_j^{-1} yields attaching maps with inverse orientation. Thus, the respective presentation complexes $K_{\mathcal{P}}$ and $K_{\mathcal{P}'}$ are homeomorphic. Now, the replacement of r_j by $wr_jw^{-1} = r'_j$ is translated in homotopic attaching maps of the 2-cells e_j^2 and e'^2_j in $K_{\mathcal{P}}$ and $K_{\mathcal{P}'}$ respectively. By Lemma 1.4.6, $K_{\mathcal{P}} \overset{3}{\nearrow\searrow} K_{\mathcal{P}'}$. The replacement of r_j by $r_jr_k = r'_j$ yields a 3-deformation since $e_k^2 \in \mathcal{K}_{\mathcal{P}} \smallsetminus \{e_j^2\}$, and then, the attaching maps of e_j^2 and e'^2_j are homotopic. Finally, the addition of a new generator x and a new relator xw^{-1} with $w \in F(x_1, x_2, \cdots, x_n)$ corresponds to an elementary expansion in $K_{\mathcal{P}}$; the inverse operation, to an elementary collapse.

Now, assume that $K \overset{3}{\nearrow\searrow} L$. By Lemmas 1.4.7 and 1.4.8, there is a deformation which can be decomposed in the following way

$$K \nearrow \tilde{K} \nearrow M \searrow \tilde{L} \searrow L$$

where $\dim(\tilde{K}) = 2$, $\dim(\tilde{L}) = 2$, $\dim(M) = 3$ and $\tilde{K} \nearrow M \searrow \tilde{L}$ are 3-dimensional expansions and collapses.

Suppose $K = K_0 \overset{e}{\nearrow} K_1 \overset{e}{\nearrow} \cdots \overset{e}{\nearrow} K_r = \tilde{K}$. Every elementary expansion of dimension 1 from K_i to K_{i+1} increases in one edge the spanning tree used in the construction of the associated presentation, but $\mathcal{P}_{K_i} = \mathcal{P}_{K_{i+1}}$. If $K_i \overset{e}{\nearrow} K_{i+1} = K_i \cup e^1 \cup e^2$ is an elementary expansion of dimension 2, then $\mathcal{P}_{K_{i+1}}$ is obtained from $\mathcal{P}_{K_1}$ after adding one generator x associated to e^1 and one relator xw associated to e^2. Thus, $\mathcal{P}_{K_i} \sim_{AC} \mathcal{P}_{K_{i+1}}$. Therefore, $\mathcal{P}_K \sim_{AC} \mathcal{P}_{\tilde{K}}$. A similar reasoning shows that $\mathcal{P}_L \sim_{AC} \mathcal{P}_{\tilde{L}}$.

By Lemma 1.4.9, the deformation $\tilde{K} \nearrow M \searrow \tilde{L}$ can be decomposed into a sequence of transient moves. We proceed by induction. Suppose $\tilde{K} \overset{e}{\nearrow} M \overset{e}{\searrow} \tilde{L}$. That is, $M = \tilde{K} \cup e^2 \cup e^3 = \tilde{L} \cup e'^2 \cup e^3$. Notice that $e^2 \in \tilde{L}$ and $e'^2 \in \tilde{K}$. If r is the relator in $\mathcal{P}_{\tilde{K}\cup e^2}$ associated to e^2 and r' is the relator in $\mathcal{P}_{\tilde{L}\cup e'^2}$ associated to e'^2, then it can be shown that $r' = wr^{\pm 1}w^{-1}s$, where s is the product of the relators associated to the 2-cells in $\partial e^3 \smallsetminus (e^2 \cup e'^2)$ (see [HR03, Lemma 2.6]). Thus, $\mathcal{P}_{\tilde{K}} \sim_{AC} \mathcal{P}_{\tilde{L}}$. Therefore, we have proved that $K \overset{3}{\nearrow\searrow} L$, then $\mathcal{P}_K \sim_{AC} \mathcal{P}_L$.

We show now that $K \overset{3}{\nearrow\searrow} K_{\mathcal{P}_K}$. If $T \subseteq K^1 \subseteq K$ is a spanning tree, then it can be shown that $K \overset{3}{\nearrow\searrow} K/T$. Note that $(K/T)^1$ is homeomorphic to $K_{\mathcal{P}(K)}$. The 2-cells of each K/T and $K_{\mathcal{P}_K}$ are in correspondence, differing maybe in the orientation and in the attaching maps, which are homotopic. By Lemma 1.4.6, $K/T \overset{3}{\nearrow\searrow} K_{\mathcal{P}_K}$.

Finally, to see that $\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}}$, it suffices to note that the choice of orientations and attaching maps in $K_{\mathcal{P}}$ can be done in such a way that $\mathcal{P}_{K_{\mathcal{P}}} = \mathcal{P}$.

Hence, the assignments $K \mapsto \mathcal{P}_K$ and $\mathcal{P} \mapsto K_{\mathcal{P}}$ induce inverse maps between 3-deformation classes of 2-complexes and AC-classes of group presentations. $\square$

1.5 Relationship with other open problems in low dimensional topology

This is the statement of the original Poincaré conjecture, which was proved by Perelman in 2002 [Per02, Per03b, Per03a].

Poincaré Conjecture. *Let M be a compact 3-dimensional manifold without boundary. If M is homotopy equivalent to S^3, then M is homeomorphic to S^3.*

A question related with the Poincaré conjecture was formulated by Zeeman [Zee64].

Conjecture 1.5.1. *(Zeeman) If K is a 2-dimensional finite contractible simplicial complex, then $K \times I \searrow *$.*

Remark 1.5.2. Zeeman's conjecture implies the Andrews-Curtis conjecture. In fact, if K is a contractible 2-dimensional cell complex, then there exists a simplicial complex L of dimension 2 such that $K \nearrow\!\!\searrow^3 L$ (see 1.2.10). By Zeeman's conjecture $L \times I \searrow *$. Therefore, $K \nearrow\!\!\searrow^3 L \nearrow L \times I \searrow *$. Since $L \times I$ has dimension 3, we can conclude that $K \nearrow\!\!\searrow^3 *$.

Zeeman proved that his conjecture was relevant in the study of the Poincaré conjecture.

Theorem 1.5.3. *Zeeman's conjecture implies the Poincaré conjecture.*

Proof. Let M be a triangulated compact 3-dimensional manifold without boundary such that M is homotopy equivalent to S^3. Let $\sigma \in M$ be a 3-simplex. Consider $N = M \smallsetminus \mathring{\sigma}$, a 3-manifold with boundary $\partial N = \partial\sigma$ homeomorphic to S^2. On the one hand, $\chi(N) = \chi(M) + 1 = \chi(S^3) + 1 = 1$. On the other hand, since N is a 3-manifold with boundary, there exists a subcomplex $K \subseteq N$ of dimension 2 such that $N \searrow K$. In particular, K is homotopy equivalent to N and then $\pi_1(K) = \pi_1(N) = \pi_1(M) = \pi_1(S^3) = 0$. Thus, K is simply connected and $\chi(K) = \chi(N) = 1$. Therefore, K is contractible. By Zeeman's conjecture, $K \times I \searrow *$. Hence, $N \times I \searrow K \times I \searrow *$. By Whitehead's characterization of p.l. balls (see [Coh69, RS72]) $N \times I$ is homeomorphic to D^4 and $N = N \times \{0\}$ is contained in $\partial(N \times I) = N \times I \cup N \times \{0\} \cup N \times \{1\}$, which is homeomorphic to S^3. By the Alexander-Schoenflies Theorem (see [Moi77, Ch. I]), N is homeomorphic to D^3. Therefore $M = N \cup_{\partial\sigma} \sigma$, which is homeomorphic to $D^3 \cup_{\partial D^3} D^3 = S^3$. □

In [GR83], Gillman and Rolfsen proved that for a class of complexes called *standard spines*, the Poincaré conjecture and the Zeeman conjecture are equivalent. In particular, since the Poincaré conjecture is true, we know that the Andrews-Curtis conjecture holds for standard spines.

The Andrews-Curtis conjecture is also related with the 4-dimensional smooth Poincaré conjecture [AK85] and the Whitehead's asphericity conjecture [How83].

1.6 Potential counterexamples

An *AC-trivialization* of a finite presentation $\mathcal{P}$ is an AC-transformation of $\mathcal{P}$ into the presentation $\langle \mid \rangle$. No AC-trivialization is known for the following examples of balanced presentations of the trivial group. They are known as potential counterexamples to disprove the Andrews-Curtis conjecture.

(i) $\langle x, y \mid xy^2x^{-1} = y^3, yx^2y^{-1} = x^3\rangle$, Crowell & Fox (1963) [CF77, p.41].

(ii) $\langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2\rangle$, Rapaport (1968) [Rap68a].

(iii) $\langle x, y \mid xyx = yxy, x^4 = y^5\rangle$, Akbulut & Kirby (1985) [AK85].

(iv) $\langle x, y \mid x = [x^m, y^n], y = [x^p, y^q]\rangle$, $m, n, p, q \in \mathbb{Z}$, Gordon (1984) [Bro84].

(v) $\langle x, y \mid x^4y^3 = y^2x^2, x^6y^4 = y^3x^3\rangle$, M. Wicks (folklore).

Example (i) was introduced by R. H. Crowell and R. H. Fox. It belongs to a more general class of presentations of the trivial group proposed by Miller and Schupp in 1999, decribed as $\langle x, y \mid xy^nx^{-1} = y^{n+1}, x = w\rangle$, with w a word in x and y with exponent sum 0 on x and $n > 0$ [MS99].

Example (ii) belongs to a series of presentations with n generators x_i and relators $x_i^{-1}x_ix_{i+1} = x_i^2$. However, for $n \geq 4$, the presented group is not trivial, and for $n = 2$, it can be easily AC-trivialized.

Example (iii) corresponds to a handle decomposition of the Akbulut-Kirby 4-sphere. It belongs to the famous series $\langle x, y \mid xyx = yxy, x^n = y^{n+1}\rangle$ of presentations of the trivial group. For $n = 1$, it is trivially AC-trivializable. For $n = 2$, it was proved to be AC-trivializable in 1988 by Gersten in an unpublished work. It was not until the 2000's that Miasnikov [Mia99] and Havas and Ramsay [HR03] exhibited independently concrete AC-trivializations of this presentation by using different computational techniques. For $n > 2$ there are not known AC-trivializations, despite the efforts of many authors. The presentation $\langle x, y \mid xyx = yxy, x^3 = y^4\rangle$ is the smallest potential counterexample to the Andrews-Curtis conjecture.

Example (iv) was due to C. Gordon. Many of the presentations of this series are easily AC-trivializable; for instance, those presentations in this sequence whose total relator length is 10 or 12. In [BM06], Bowman and McCaul proved that all the presentations with total relator length equal to 14 are AC-trivializable.

1.7 Outline of the previous works on the conjecture

Classical works on the conjecture were made by Andrews and Curtis [AC66], Rapaport [Rap68b, Rap68a], Metzler and Hog-Angeloni [Met79, HAM90]. They contributed to the question of whether there exists some analogy between the problem of finding an AC-trivialization for a given balanced presentation of the trivial group and the pre-existing Nielsen's or Whitehead's algorithm for free groups (see [LS01, Ch. 1]). These works are focused on looking for strategies to "simplify" the presentation, where the simplification can be interpreted as a reduction of the number of relators, or the total length of the relators (the *relator length*, for short). They also found algebraic criteria to decide weather two presentations are AC-equivalent. For a detailed exposition of these results, see [HAM93, Ch. XII].

Modern advances in the study of the conjecture were developed by means of computational tools. In 1999, Miasnikov [Mia99] designed a genetic algorithm which proved that all balanced presentations of the trivial group in which the relator length is at most 12 satisfy the conjecture. In particular, he showed that the Andrews-Curtis conjecture holds for the presentation $\langle x, y \mid xyx = yxy, x^2 = y^3\rangle$, one of the potential counterexamples.

Miasnikov algorithm was non-deterministic, making a limited exploration on the search space. Note that an exhaustive search in the tree of AC-equivalent presentations is extremely expensive in memory and execution time. For instance, if the presentation has n relators then there are $(3n^2)^k$ presentations equivalent to the original in k steps if we do not allow to increase/decrease the number of relators.

In 2003, Havas and Ramsay [HR03] exhibited a computational approach based on a breadth-first constrained search of the tree of equivalent presentations with bounds on the relator length. They implemented a software, the *AC-move enumeration software* (also known as ACME), to prove that if the relator length is 13 and there are two generators, all presentations of the trivial group are AC-equivalent to the presentation $\langle x, y \mid x^3 = y^4, xyx = yxy \rangle$.

In 2006, Bowman and McCaul [BM06] made an optimized computer program for fast searching sequences of AC-transformations and they applied it to test the validity of the conjecture for the potential counterexamples, which are particularly hard to reduce. The program was again based in the breadth-first search technique, but they also added an effective use of disk memory and used the idea of "canonical form" of a presentation to reduce the search space. Despite the improvements in the implementation, they were not able to prove that $\langle x, y \mid x^3 = y^4, xyx = yxy \rangle$ satisfies the conjecture. However, they showed that all the presentations in the sequence due to Gordon whose total relator length is 14 are AC-trivializable.

In 2015, Bridson [Bri15] constructed explicit balanced presentations $\mathcal{P}_n$ of the trivial group with length relator at most $24(n+1)$ for which the number of AC-transformations required in any trivialization grows more quickly than any tower of exponentials. Thus, it is physically impossible to exhibit an explicit sequence of AC-transformations trivializing rather small balanced presentations of the trivial group. While various researchers were using computer experiments to prove potential counterexamples to the Andrews-Curtis conjecture, he warned to be when designing an experiment to find a trivializing sequence of AC-moves.

The construction of Bridson was based in groups G with presentations of deficiency 1 satisfying certain technical conditions, which were transformed into balanced group presentations $\mathcal{P}_w$ indexed by words w in the generators of G such that $w = 1$. The groups $\mathcal{P}_w$ result Andrews-Curtis trivializable and the number of Andrews-Curtis transformations required to trivialize them can be bounded above and below in terms of how hard it is to prove that $w = 1$ in G. In short, he encoded the complexity of the word problem in certain groups in terms of balanced presentations.

Summarizing, one of the main difficulties in working with finitely presented groups is the fact that a lot of problems about them are unsolvable. Even if algorithms exist, many of them are exponential or super exponential in nature, making it very unlikely for them to produce results in an acceptable period of time. The decidability of the problem of determining if a *balanced* presentation corresponds to the trivial group is not known. In this direction, Siddhartha Gadgil [Gad01] proved that if the Andrews-Curtis conjecture were true, then there would be an algorithm to recognize balanced presentations of the trivial group.

Resumen del capítulo 1: Presentaciones de grupo, 2-complejos y la conjetura de Andrews-Curtis

En este capítulo hacemos una reseña histórica del problema, desde su formulación original hasta los avances realizados al día de hoy. Para una explicación más detallada, ver [HAM93].

En 1965, en su famoso artículo "*Free groups and handlebodies*" [AC65], Andrews y Curtis plantearon una pregunta sobre presentaciones de grupos, generalizando el teorema de Nielsen sobre grupos libres [Nie18]. Su motivación en ese momento era el estudio de la (en ese momento abiera) conjetura de Poincaré, pues si su conjetura fuera cierta, se hubieran podido obtener consecuencias sobre posibles contraejemplos de la conjetura de Poincaré. El referí (anónimo) del artículo les observa que (una versión más débil de) la conjetura está en estrecha conexión con un problema abierto sobre deformaciones de CW-complejos.

Conjetura 1.4.3 (Andrews-Curtis)**.** *Si $\mathcal{P} = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle$ es una presentación balanceada del grupo trivial, entonces $\mathcal{P}$ se puede llevar a la presentación vacía $\langle \mid \rangle$ mediante una sucesión finita de las siguientes transformaciones:*

- *reemplazar r_i por r_i^{-1},*
- *reemplazar r_i por $r_i r_j$, $j \neq i$,*
- *reemplazar r_i por $w r_i w^{-1}$, donde w es una palabra en los generadores,*
- *agregar un nuevo generador x y la relación x, o el inverso de esta operación.*

La relación notada por el referí entre presentaciones grupos y CW-complejos de dimensión 2 fue recién formalizada y probada por Wright [Wri75].

Definición 1.3.1. Sea K un complejo celular de dimensión 2. La *presentación de grupo estándar* $\mathcal{P}_K$ asociada a K es la presentación definida como sigue.

- Elegir $e^0 \in K^0$ una 0-celda de K como punto base.
- Elegir $T \subseteq K^1$ un spanning tree (i.e., un subcomplejo contráctil de su 1-esqueleto que contenga todos los vértices).
- Sean $e_1^1, e_2^2 \cdots, e_n^1$ las 1-celdas de in $K \setminus T$. Para cada $1 \leq i \leq n$, elegir una orientación para e_i^1, y sea x_i el único camino en T desde e_0 al inicio de e_i^1 seguido por e_i^1 y luego por el camino desde su final a e_0. Entonces, $x_i \in \pi_1(K^1, e_0)$ y, más aún, $\pi_1(K^1, e_0) = F(x_1, x_2, \cdots, x_n)$.
- Sean $e_1^2, e_2^2, \cdots, e_m^2$ las 2-celdas de K. Para cada e_j^2, elegir una función de adjunción φ_j, un punto base s_0 y una orientación ∂D^2. Luego, φ_j es un ciclo en $\varphi_j(s_0)$.
- Para cada $1 \leq j \leq m$, elegir un camino γ_j in K^1 desde e_0 a $\varphi_j(s_0)$ y considerar el ciclo $\gamma_j * \varphi_j * \bar{\gamma}_j$. Luego, $[\gamma_j * \varphi_j * \bar{\gamma}_j] \in \pi_1(K^1, e_0)$, y puede expresarse como un producto r_j de los elementos $\{x_1, x_2, \cdots, x_n\}$ y sus inversos.

- Definir $\mathcal{P}_K = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_m \rangle$.

Definición 1.3.7. Sea $\mathcal{P} = \langle x_1, x_2 \cdots, x_n \mid r_1.x_2, \cdots, r_m \rangle$ una presentación de grupo finita. El *complejo de presentación* $K_{\mathcal{P}}$ es el complejo celular construido como sigue:

- su 0-esqueleto es un único punto;
- para cada generador x_i, adjuntar a la 0-celda una 1-celda orientada e_i^1, obteniendo un bouquet de círculos $\bigvee_{i=1}^n S^1$;
- cada relación r_j determina un camino cerrado φ_j en $\bigvee_{i=1}^n S^1$; para cada $1 \leq j \leq m$ adjuntar una 2-celda e_j^2 con función de adjunción φ_j.

Teorema 1.4.5. Hay una correspondencia entre clases de 3-deformación de 2-complejos y clases de Andrews-Curtis de presentaciones de grupos. Concretamente

- si $K \nearrow\!\!\!\searrow^3 L$, entonces $\mathcal{P}_K \sim_{AC} \mathcal{P}_L$;
- si $\mathcal{P} \sim_{AC} \mathcal{P}'$, entonces $K_{\mathcal{P}} \nearrow\!\!\!\searrow^3 K_{\mathcal{P}'}$;
- $K \nearrow\!\!\!\searrow^3 K_{\mathcal{P}_K}$;
- $\mathcal{P} \sim_{AC} \mathcal{P}_{K_P}$.

Luego, la conjetura de Andrews-Cutis se puede reformular en términos topológicos del siguiente modo.

Conjetura 1.2.9 (Conjetura de Andrews-Curtis, versión topológica)**.** *Si K es un 2-complejo contráctil, entonces $K \nearrow\!\!\!\searrow^3 *$.*

Ambas versiones de la conjetura fueron ampliamente estudiadas, obteniéndose hasta ahora resultados parciales. Por ejemplo, se sabe que la conjetura es cierta para ciertas clases de complejos, como los *standar spines* y los complejos *cuasi-construibles*.

Hay una lista de presentaciones balanceadas del grupo trivial ampliamente estudiada para las cuales no se conoce ninguna lista de movimientos de Andrews-Curtis que las lleven a la presentación vacía. Son *potenciales contraejemplos* para refutar la conjetura.

(i) $\langle x, y \mid xy^2x^{-1} = y^3, yx^2y^{-1} = x^3 \rangle$, Crowell & Fox (1963) [CF77, p.41].

(ii) $\langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2 \rangle$, Rapaport (1968) [Rap68a].

(iii) $\langle x, y \mid x = [x^m, y^n], y = [x^p, y^q] \rangle$, $m, n, p, q \in \mathbb{Z}$, Gordon (1984) [Bro84].

(iv) $\langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle$, $n \geq 3$, Akbulut & Kirby (1985) [AK85].

En los próximos capítulos desarrollaremos métodos para abordar ambas versiones de la conjetura así como también la lista de de potenciales contraejemplos, con técnicas de espacios topológicos finitos.

Chapter 2

The point of view of finite spaces

The theory of finite spaces can be used to study homotopy properties of more general spaces. In fact, finite spaces model the weak homotopy types of compact polyhedra. In this chapter we will recall the basic results and constructions about this theory.

2.1 Algebraic topology of finite spaces

The theory of finite spaces started in 1937 by P. S. Alexandroff [Ale37], who showed that they have an intrinsic combinatorial structure: they can be thought of as finite partially ordered sets (posets). In 1966, M. C. McCord [McC66] made a great advance in the theory establishing the correspondence of weak homotopy types between compact cell complexes and finite spaces. The same year, R. E. Stong [Sto66] proved that the homotopy type of finite spaces can be algorithmically decided by a simple reduction procedure of low complexity. More recently, J. A. Barmak and E. G. Minian studied different algebraic topology problems using the interaction between topology and combinatorics. Among other contributions, they developed the simple homotopy theory of finite spaces, and applied methods of finite spaces to the study of the Andrews-Curtis conjecture [Bar11a].

2.1.1 Finite spaces and posets

In 1937, Alexandroff [Ale37] proved that finite T_0-spaces and finite posets are basically the same object viewed from different perspectives. Given a finite topological space X, define a relation $\leq$ in X by $x \leq y$ if $x \in U_y$, where U_y is the minimal open set containing y. It is clear that $\leq$ is reflexive and transitive. If X is T_0, $\leq$ is also antisymmetric. Conversely, if $\leq$ is a reflexive and transitive relation on X the sets $\{U_x\}_{x \in X}$ where $U_x = \{y \in X : y \leq x\}$, constitute a basis for a topology on X. If $\leq$ is antisymetric, the topology defined on X will additionally satisfy the T_0-axiom. From now on, all finite spaces will be assumed to be T_0 and we will refer indistinctly to both structures, the topological one and the combinatorial one.

Note that a map $f : X \to Y$ between finite spaces is continuous if and only if it is an order-preserving map.

A poset X can be represented as a directed graph, its *Hasse diagram*, denoted by $\mathcal{H}(X)$. Its vertices are the elements of X, and its edges are the pairs (x, y) such that $x \prec y$, i.e., $x < y$ and there is no $z \in X$ satisfying $x < z < y$.

A chain in a finite poset X is a totally ordered subset $c = \{x_0 < x_1 < \cdots < x_n\}$ of X. The length of the chain c is n.

The *height* $h(X)$ of a finite poset X is the maximum length of a chain of X. The height of a point $x \in X$ is $h_X(x) = h(U_x)$.

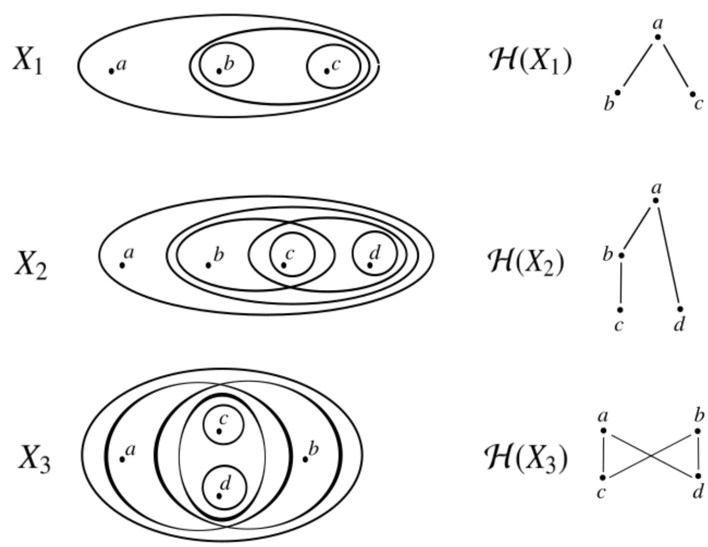


Figure 2.1: Hasse diagrams of some finite topological spaces.

2.1.2 Finite spaces and cellular complexes

Although general Hausdorff spaces do not have the homotopy type of a finite space, McCord proved in 1966 [McC66] that every finite cellular complex has an associated finite space with the same *weak homotopy type*.

Definition 2.1.1. There is a functor $\mathcal{K} : \mathrm{Top}_{<\infty} \to \mathcal{S}$ from the category of finite spaces to the category of finite complexes defined by:

- if X is a finite space, $\mathcal{K}(X)$ is the *order complex* of X, that is, the simplicial complex whose simplices are the nonempty chains of X;
- if $f : X \to Y$ is a continuous map between finite spaces, $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is the simplicial map defined by $\mathcal{K}(f)(x) = f(x)$.

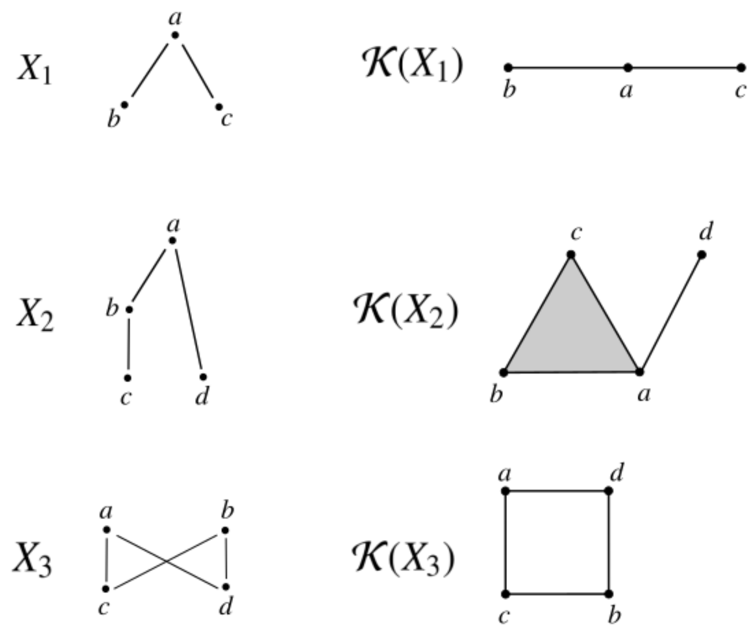


Figure 2.2: Order complexes of some finite topological spaces.

Notice that not every finite simplicial complex is the order complex of a finite space. Some necessary conditions are that K must be a clique complex and its 1-skeleton must be a comparability graph. Recall that a complex is a *clique complex* if its simplices are exactly the cliques [1] of the graph induced by its 1-skeleton. An undirected graph is a *comparability graph* if there exists a *transitive orientation* of its edges, that is, an assignment of directions to its edges such that for every pair (x, y), (y, z) of oriented edges, there exists a directed edge (x, z). This class of graphs is completely described by forbidden subgraphs: a graph is of comparability if and only if it does not contain cycles of odd length $n \geq 5$ as induced subgraphs. We prove that the previous conditions are sufficient to characterize order complexes.

Proposition 2.1.2. *A finite simplicial complex K is the order complex of some finite space if and only if its simplices are the cliques on its 1-skeleton, and its 1-skeleton does not contain odd cycles of length greater than 3.*

Proof. Assume $K = \mathcal{K}(X)$ for a finite space X. The vertices of K coincide with the underlying set of X and the simplices of K are the nonempty chains of X. Non empty chains of X corresponds with cliques in K^1. Thus, K is a clique complex. The order in X induces a transitive orientation of the edges of K^1. Thus, K^1 is a comparability graph.

Conversely, if K^1 is a comparability graph, then consider a transitive orientation on its edges. The orientation induces an acyclic order $\leq$ of its vertices K^0. Let X be the finite space associated with the poset $(K^0, \leq)$. Since K is clique complex, $K = \mathcal{K}(X)$. □

[1]A *clique* of an undirected graph is a subset C of its vertices such that every pair of elements in C are adjacent.

Recall that a CW-complex K is *regular* if for every open cell e^n, the attaching map $D^n \to K$ of its cells e^n is a homeomorphism.

Definition 2.1.3. There is a functor $\mathcal{X} : CW_{reg} \to \mathrm{Top}_{<\infty}$ from the category of finite regular cell complexes to the category of finite spaces, defined by:

- if K is a finite regular cell complex $\mathcal{X}(K)$ is the *face poset* of K, that is, the finite space whose underlying set is the set of all cells of K with the order-relation $e < e'$ if e is a face of e';
- if $f : K \to L$ is a cellular map between finite regular cell complexes, then $\mathcal{X}(f) : \mathcal{X}(K) \to \mathcal{X}(L)$ is the continuous map defined by $\mathcal{X}(f)(e) = f(e)$.

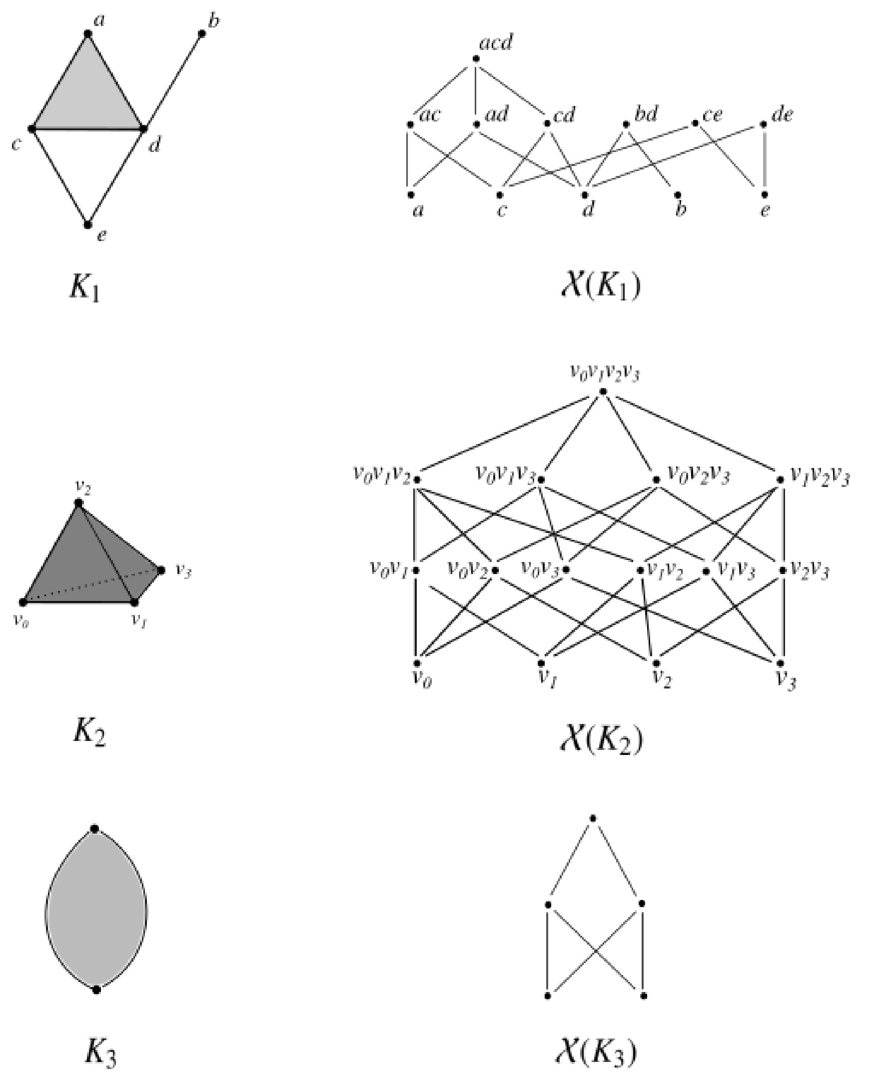


Figure 2.3: Face posets of some regular cell complexes.

Remark 2.1.4. The functors $\mathcal{K}$ and $\mathcal{X}$ are not mutually inverse. Given a finite regular cell complex K, the composition $\mathcal{K}(\mathcal{X}(K))$ is the *barycentric cell subdivision* K' of K, i.e. the simplicial complex whose vertices are the cells of K and $\{e_1, e_2, \cdots, e_{n+1}\}$ is an n-simplex if e_i is a face of e_{i+1} for each i. Conversely, if X is a finite space, then $\mathcal{X}(\mathcal{K}(X))$ is the so called *barycentric poset subdivision* X' of X, which is the poset of chains of X ordered by inclusion.

Definition 2.1.5. There are natural transformations:

- $\mu_X : |\mathcal{K}(X)| \to X$, defined by $\mu_X(\alpha) = \min(\text{support}(\alpha))$, where the support of a convex combination $\alpha = \sum_{i=1}^{r} t_i x_i$ of the vertices of $\mathcal{K}(X)$ is understood to be the set of vertices accompanied by a nonzero coefficient,
- $\mu_K : |K| \to \mathcal{X}(K)$, $\mu_K = \mu_{\mathcal{X}(K)} s_K^{-1}$, where $s_k : |K'| \to |K|$ is the linear homemorphism defined by $s_K(e) = b(e)$, the barycenter of e.

Recall that a continuous map $f : X \to Y$ between topological spaces is said to be a *weak homotopy equivalence* if it induces isomorphisms in all homotopy groups. We say that a topological space X is homotopically trivial if the map $X \to *$ is a weak equivalence.

McCord proved that the previous maps are weak equivalences.

Theorem 2.1.6. *[McC66] μ_X and μ_K are natural weak homotopy equivalences. In particular,*

- *a map $f : X \to Y$ between finite T_0-spaces is a weak equivalence if and only if $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is a homotopy equivalence.*
- *a map $f : K \to L$ between finite regular cell complexes is a homotopy equivalence if and only if $\mathcal{X}(f) : \mathcal{X}(K) \to \mathcal{X}(L)$ is a weak equivalence.*

Remark 2.1.7. In [Bar11a, Thm. 11.3.2.], Barmak gave an alternative proof of the fact that $\mathcal{X}(K)$ and K are weak equivalent. He defined the map

$$\nu_K : K \to \mathcal{X}(K)^{op}$$

as $\nu_K(x) = e$ if $x \in \mathring{e}$. He uses the same ideas of McCord to prove that ν_K is a weak equivalence.

Homotopy equivalences are weak homotopy equivalences and, in the category of cellular complexes weak homotopy equivalences are homotopy equivalences, by the Whitehead Theorem [Hat02, Thm. 4,5]. We will see in Section 2.1.3 that both notions do not coincide in the category of finite spaces.

We can deduce the following consequence of Theorem 2.1.6.

Corollary 2.1.8. *If K is a regular cell complex, then K is contractible if and only if $\mathcal{X}(K)$ is homotopically trivial. If X is a finite space, then X is homotopically trivial if and only if $\mathcal{K}(X)$ is contractible.*

McCord's result is supported on the following Theorem, which gives a criterion for recognizing weak homotopy equivalences between topological spaces.

An open cover $\mathcal{U}$ of a topological space X is called a *basis like open cover* if $\mathcal{U}$ is a basis for some topology in the underlying set of X (possibly different from the original one).

Theorem 2.1.9. *(McCord) [McC66, Thm. 6] Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. If there exists a basis like open cover $\mathcal{U}$ of Y such that each restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a weak equivalence for every $U \in \mathcal{U}$, then $f : X \to Y$ is a weak equivalence.*

In the same line, the celebrated Theorem A of Quillen [Qui73] provides a sufficient condition for a functor to induce a homotopy equivalence between the classifying spaces of the involved categories. Quillen's Theorem A applied to the case when both categories are finite posets coincides with McCord's Theorem applied to the minimal basis cover $\mathcal{U} = \{U_y : y \in Y\}$, and can be summarized in the following result.

Theorem 2.1.10. *(Quillen's Fiber Lemma)[Qui78, Prop. 1.6] Let $f : X \to Y$ be an order preserving map between finite posets such that $f^{-1}(U_y)$ is homotopically trivial for every $y \in Y$. Then $f : X \to Y$ is a weak equivalence.*

Barmak exposed a simpler proof of the previous statement using the notion of *non-Hausdorff mapping cylinder* (see Proposition 3.0.1 below and [Bar11b, Thm. 1.1, 1.2]).

2.1.3 Homotopy types

The homotopy type of finite spaces can be classified combinatorially by means of a method of reduction: the *beat points*. This description was discovered in 1966 by Stong [Sto66].

Recall that, given a finite space X and $x \in X$, we denote by U_x^X the minimal open set containing x, i.e., $U_x^X = \{y \in X : y \leq x\}$. Similarly, F_x^X denotes the closure of x in X, that is, $F_x^X = \{y \in X : x \leq y\}$. We omit the superscript when the poset is understood.

Definition 2.1.11. Let X be a finite space. An element $x \in X$ is called an *up beat point* if there is a unique element $y \in X$ such that $x \prec y$, i.e., the set $\hat{F}_x = F_x \smallsetminus \{x\}$ has a minimum. Analogously x is called a *down beat point* if there is a unique y such that $y \prec x$, i.e., the set $\hat{U}_x = U_x \smallsetminus \{x\}$ has maximum. In both cases we say that x is a beat point *dominated* by y.

If $x \in X$ is a beat point, then $X \smallsetminus \{x\}$ is a strong deformation retract of X. We denote it by $X \searrow\searrow X \smallsetminus \{x\}$. In general we denote $X \searrow\searrow Y$ if there is a sequence $X = X_0, \dots, X_n = Y$ such that X_i is obtained from X_{i-1} by removing a beat point. Note that in this case, $Y \subseteq X$ is a strong deformation retract.

A *core* of a finite T_0-space X is a strong deformation retract of X without beat points.

The following result was proved by Stong [Sto66] (see also [Bar11a]).

Theorem 2.1.12. *The core of a finite space is unique up to homeomorphism and two finite spaces are homotopy equivalent if and only if the have homeomorphic cores. In particular, a finite space is contractible if and only if its core is a singleton.*

As consequence, it can be deduced that finite spaces with maximum or minimum are contractible. Indeed, if X has a maximum x, then the maximal points of $X \smallsetminus \{x\}$ are up beat points, so we can remove them and iterate. Thus, the core of X is homeomorphic to the singleton. An analogous procedure works for spaces with minimum.

Example 2.1.13. The finite space of Figure 2.4 is contractible, the labels on the vertices indicate a sequence of beat points to be removed one by one.

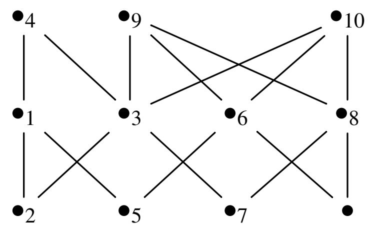


Figure 2.4: Contractible finite space.

Remark 2.1.14. In order to find the core of a finite space it is enough to greedily remove its beat points one by one. The complexity of this algorithms is quadratic in the number of points of the space. See Appendix 2.A for an implementation.

Note that there are finite spaces which are homotopically trivial (i.e. weak equivalent to the singleton) but not contractible. That is, Whitehead's theorem is not longer valid in this context. Throughout this dissertation, we will see many examples of such finite spaces.

2.1.4 Simple homotopy types

In 2012, Barmak and Minian [BM12b] introduced a simple homotopy theory for posets which is in correspondence with the classic theory developed by Whitehead by means of the McCord functors $\mathcal{K}$ and $\mathcal{X}$. In the same work, another method of reduction was presented: the *weak points*, which give rise to the analogues of elementary moves of collapse and expansion in this setting.

Definition 2.1.15. Let X be a finite T_0-space. A point $x \in X$ is a *weak beat point* (or simply *weak point*) if either $\hat{U}_x$ or $\hat{F}_x$ is contractible. More precisely, x is called *down weak point* and *up weak point* respectively.

Beat points are particular cases of weak points, since finite spaces with maximum or minimum element are contractible. If $x \in X$ is a weak point, then the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. There is a *collapse* from a finite space X to a subspace Y (or an *expansion* from Y to X) if the latter can be obtained from X by removing a sequence of weak points one by one. We denote it by $X \searrow Y$ (or $Y \nearrow X$). We say that X and Y have the same *simple homotopy type*, and we denote it by $X \diagup\!\!\searrow Y$, if there is a finite sequence of expansions and collapses connecting X with Y.

The following theorem is one of the main results of this theory (see [BM08b, Bar11a]).

Theorem 2.1.16.

i) Let X and Y be finite topological spaces. Then, $X \diagup\!\!\searrow Y$ if and only if $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.

ii) Let K and L be finite regular cell complexes. Then, $K \diagup\!\!\!\searrow L$ if and only if $\mathcal{X}(K) \diagup\!\!\!\searrow \mathcal{X}(L)$. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.

Example 2.1.17. The finite space of Figure 2.5 is collapsible but not contractible, since it does not have any beat point (compare with Figure 2.4). The label on the vertices indicates a sequence of reduction of weak points.

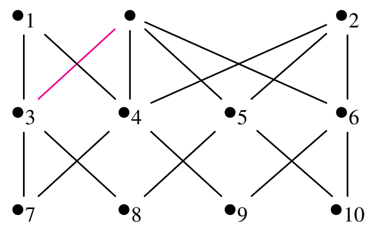


Figure 2.5: Collapsible but not contractible finite space.

Example 2.1.18. Consider the regular cell structure H of the Dunce Hat of Figure 2.6.

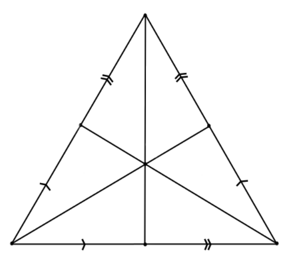


Figure 2.6: Regular cell structure of the Dunce Hat.

The face poset $\mathcal{X}(H)$ of H is the poset of Figure 2.7. $\mathcal{X}(H)$ is homotopically trivial since it is weak equivalent to H, but it does not have any weak point. Thus, it is not collapsible.

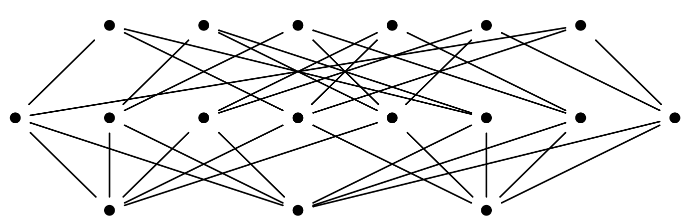


Figure 2.7: The face poset associated to a regular structure of the Dunce Hat.

Remark 2.1.19. Since the removal of different sequences of weak points may lead to different *weak cores* (i.e. finite spaces without weak points obtained from another after removing successively weak points), an exhaustive exploration of all possible sequences of reductions is necessary to find the weak core with minimum cardinality. A backtracking algorithm to explore all possible sequences of collapses is implemented in 2.A. Memorization of already explored spaces and pruning criteria is used to optimize the complexity, which is a priori the factorial of the number of points of the space.

Remark 2.1.20. If $\hat{U}_x$ or $\hat{F}_x$ is contractible, then $X \searrow X \smallsetminus \{x\}$. Thus, by Theorem 2.1.16, $\mathcal{K}(X) \searrow \mathcal{K}(X \smallsetminus \{x\})$. Less restrictive conditions can be considered to obtain similar results:

- If $\hat{U}_x$ or $\hat{F}_x$ is collapsible, then $\mathcal{K}(X) \searrow \mathcal{K}(X \smallsetminus \{x\})$. Such points x were studied in [BM12b] when investigating non-evasiveness and strong collapsibility from the point of view of finite spaces.

- If $\hat{U}_x$ or $\hat{F}_x$ is homotopically trivial, then $\mathcal{K}(X) \nearrow\!\searrow \mathcal{K}(X \smallsetminus \{x\})$. Such points x are called *γ-points* and were defined in [BM08a] when studying different one-point reduction methods.

- If $\hat{U}_x$ or $\hat{F}_x$ is acyclic, then $H_i(\mathcal{K}(X)) \cong H_i(\mathcal{K}(X \smallsetminus \{x\}))$ for all i. Such points x were also introduced in [BM08a]; we will refer to them as *acyclic points*.

- If $\hat{U}_x$ or $\hat{F}_x$ is n-connected (i.e. $\pi_i(\hat{U}_x) \cong 0$ for all $1 \leq i \leq n$), then $X \smallsetminus \{x\} \to X$ is an $(n + 1)$-equivalence (i.e. it induces an isomorphism $\pi_i(X \smallsetminus \{x\}, x_0) \to \pi_1(X, x_0)$ for $i < n + 1$ and an epimorphism for $i = n$). The proof of this fact can be found in [Bar11b, Lemma 6.2.].

2.2 The Andrews-Curtis conjecture in the context of finite spaces

The development of the simple homotopy theory of finite spaces allows one to reformulate the Andrews-Curtis conjecture in this context.

Definition 2.2.1. Let X and Y be two finite topological spaces. We say that X *n-deforms* to Y, and we denote it by $X \overset{n}{\nearrow\!\searrow} Y$, if Y can be obtained from X by performing a finite sequence of expansions and collapses in such a way that all the spaces involved are of height at most n.

The Andrews-Curtis Conjecture can be reformulated as follows. (See below for a proof of the equivalence).

Conjecture 2.2.2. *Let X be a homotopically trivial finite space of height 2. Then X 3-deforms to a point.*

The following result is well-known. See for example [Whi39, HAM93].

Lemma 2.2.3. *Let K be a regular n-dimensional cell complex and* $K' = \mathcal{K}(\mathcal{X}(K))$ *the barycentric subdivision of K. Then,* $K \overset{n+1}{\nearrow\!\searrow} K'$.

In the context of finite spaces we have the following similar result.

Lemma 2.2.4. *[Bar11a, Prop. 4.2.9.] Let X be a finite* T_0*-space of height n and let* $X' = \mathcal{X}(\mathcal{K}(X))$ *be the subdivision of X. Then,* $X \overset{n+1}{\nearrow\!\searrow} X'$.

Theorem 2.2.5. *Conjecture 2.2.2 is equivalent to the Andrews–Curtis conjecture.*

Proof. Assume Conjecture 2.2.2 holds. If K is a contractible regular 2-complex, then $\mathcal{X}(K)$ is a homotopically trivial finite space of height 2. Hence, $\mathcal{X}(K) \nearrow\!\!\searrow^{3} *$. Then, as a consequence of Theorem 2.1.16, $K' \nearrow\!\!\searrow^{3} *$. Finally, Lemma 2.2.3 implies that $K \nearrow\!\!\searrow^{3} *$. Conversely, suppose that the Andrews-Curtis conjecture is true and let X be a homotopically trivial finite space of height 2. Then $\mathcal{K}(X)$ is contractible and by the hypothesis $\mathcal{K}(X) \nearrow\!\!\searrow^{3} *$. This implies that $X' \nearrow\!\!\searrow^{3} *$. Since $X \nearrow\!\!\searrow^{3} X'$ by Lemma 2.2.4, the result follows. □

In [Bar11a, Ch.11] Barmak described a class of simplicial complexes, the *quasi-constructible* complexes, which satisfy the Andrews-Curtis conjecture. The definition of these complexes is supported in a class of homotopically trivial finite spaces of height 2: the *qc-reducible* spaces. Concretely, quasi-constructible complexes are the order complexes of qc-reducible spaces. The definition of the latter class is based in a method of reduction of finite spaces, called *qc-reduction*.

Definition 2.2.6. Let X be a finite space of height at most 2. Let $x, y \in X$ be two maximal elements such that $U_x \cup U_y$ is contractible. Then, there is a *qc-reduction* from X to the quotient space $X/\{x, y\}$. X is *qc-reducible* if after performing a sequence of qc-reductions starting from X, one can obtain a space with maximum.

Note that if there is a qc-reduction from X to $X/\{x, y\}$, then $X \nearrow\!\!\searrow^{3} X/\{x, y\}$. In fact, $X \nearrow X \cup \{z\}$ with $\hat{U}_z = U_x \cup U_y$, and $X \cup \{z\} \searrow X \cup \{z\} \smallsetminus \{x, y\}$ since x and y are up beat points. The latter space is isomorphic to $X/\{x, y\}$. In particular, qc-reducible spaces 3-deform to a point. The quasi constructible spaces are the contractible simplicial complexes whose face posets are qc-reducible.

For instance, in the Dunce Hat's Example 2.1.18, one can perform the following sequence of qc-reductions in $\mathcal{X}(H)$: $\mathcal{X}(H)$ qc-reduces to $\mathcal{X}(H)/\{x_1, y_1\} = Y$, and the last one qc-reduces to $Y/\{x_2, y_2\}$ (see Figure 2.8). The latter space is collapsible. Thus, $\mathcal{X}(H) \nearrow\!\!\searrow^{3} *$.

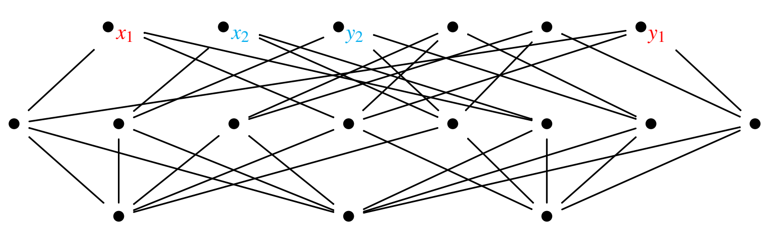


Figure 2.8: Qc-reductions in the face poset of the Dunce Hat.

In Chapter 3 we develop new methods of reduction and algorithms to decide (non-exhaustively) whether a finite space of height 2 satisfies Conjecture 2.2.2. In Chapter 6 we present two new methods to obtain presentations AC-equivalent to a given presentation of the trivial group and we implement both (non-deterministic) algorithms to decide if it satisfies Conjecture 1.2.9. We anticipate some examples to show the reach of the theory.

Example 2.2.7 (Triangle)**.** The cellular complex of Figure 2.9 is a contractible space. However, it does not have any free face.

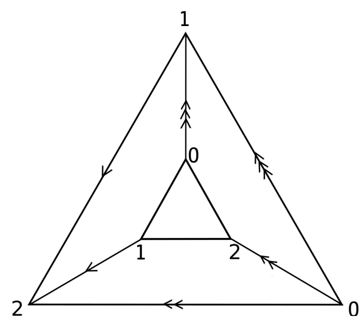


Figure 2.9: Triangle.

Any subdivision of the Triangle also has no free face to start collapsing. Consider the following regular subdivision T of the space.

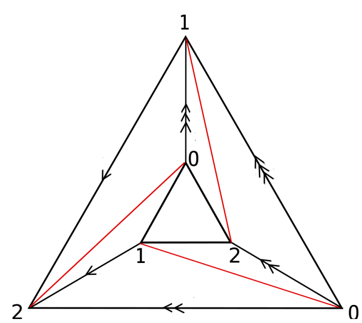


Figure 2.10: Regular cell structure of the Triangle.

The face poset $\mathcal{X}(T)$ of T is the poset of Figure 2.11. It is not collapsible since it has no weak points. In addition, T is not qc-reducible. However, using the methods of 3-deformation of finite spaces developed in Chapter 3, we will show that $\mathcal{X}(T) \nearrow\!\!\!\searrow^{3} *$. Thus, we actually prove that $T \nearrow\!\!\!\searrow^{3} *$, that is, T satisfies the Andrews-Curtis conjecture (see Figure 3.12).

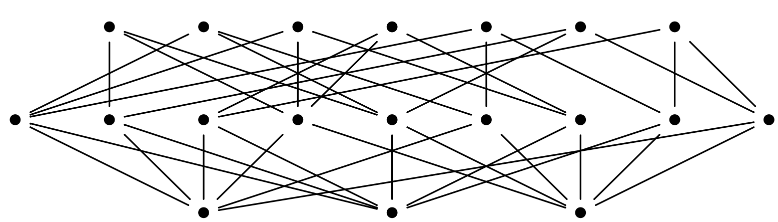


Figure 2.11: The face poset associated to a regular structure of the Triangle.

Example 2.2.8. Consider the following presentation of the trivial group (see the potential counterexample (iii) in Chapter 1, Section 1.6).

$$\mathcal{P} = \langle x, y \mid x^2y^{-3}, xyxy^{-1}x^{-1}y^{-1}\rangle.$$

The standard complex associated to $\mathcal{P}$ has one 0-cell, two 1-cells and two 2-cells attached as in Figure 2.12.

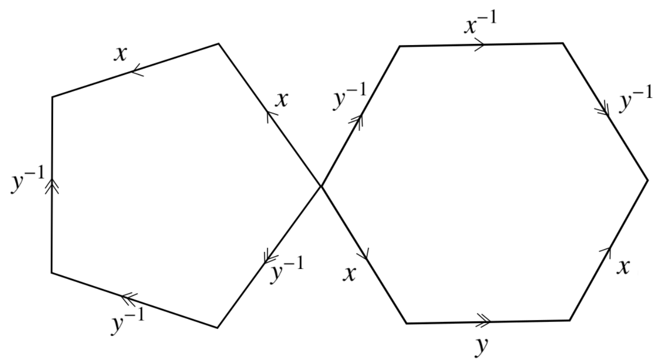


Figure 2.12: Standard complex $K_{\mathcal{P}}$ associated to the presentation $\mathcal{P}$.

A regular subdivision $K'_{\mathcal{P}}$ of $K_{\mathcal{P}}$ is shown in the Figure 2.13.

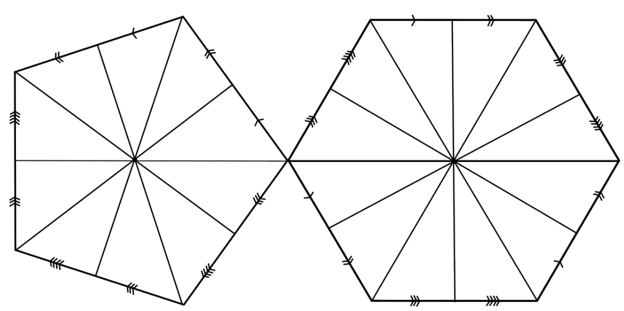


Figure 2.13: Barycentric subdivision of $K_{\mathcal{P}}$.

The face poset of $K'_{\mathcal{P}}$ is a finite space with 53 points, which is neither collapsible nor qc-reducible. However, with the coloring methods and Morse methods developed in Chapter 6, we will see that $K'_{\mathcal{P}} \nearrow\!\!\searrow^{3} *$ and then that we can deduce $\mathcal{P} \sim_{AC} \langle \mid \rangle$.

2.3 SAGE implementation

As we have seen in this chapter, the algebraic topology of finite spaces may often be combinatorially studied in terms of graphs, by viewing them as posets. For instance, elementary moves of finite spaces can be conveniently expressed in terms of algorithmic graph theory. This motivated us to undertake a computational approach of the previous methods. We employed the free software SAGE [S+17]. In the Appendix 2.A of this chapter we include the implementations of some basic operations on posets; most of them can be also found in [Fer11].

2.A Appendix: Finite spaces SAGE module

Generalities

```
#Opposite of a finite space
def op(X):
    elms = X.list()
    rels = [r[::-1 ] for r in X.cover_relations()]
    return Poset((elms,rels))

#Join
def join (X, Y):
    return X.ordinal_sum(Y)

#Cartesian Product
def cartesian_product (X, Y):
    return X.product(Y)

#Quotient
def quotient_poset(X, A): #X poset, A the list of elements of the
   subposet
    open_A = X.order_ideal(A)
    closed_A = X.order_filter(A)
    if set(open_A).intersection(closed_A) != set(A):
        return 'ERROR: X/A is not T_0'
    notinA = [x for x in X.list() if not x in A]
    elms = notinA + [A[0]]
    rels=[[x, y] for x in notinA for y in notinA if X.covers(x, y)]
   + [[A[0], x] for a in A for x in notinA if X.covers(a, x)] + [[x,
    A[0]] for a in A for x in notinA if X.covers(x, a)]
    return Poset((elms, rels))

#Wedge
def wedge(X, Y, x, y):
    return quotient_poset(X.disjoint_union(Y), [x, y])
```

Homology

```
#Homology of a finite space
def homology(X):
    return X.order_complex().homology()
```

Fundamental group

```
#Presentation of the fundamental group of a finite space
def fundamental_group(X):
    return X.order_complex().fundamental_group()
```

Closed sets, open sets, links and stars.

```
def F(X, x):
    return X.subposet(X.order_filter([x]))

def U(X, x):
    return X.subposet(X.order_ideal([x]))

def C(X, x):
    elms=[y for y in X.list() if X.is_gequal(x,y) or X.is_gequal(y,x)]
    return X.subposet(elms)

def F_hat(X, x):
    elms=[y for y in X.list() if X.is_greater_than(y, x)]
    return X.subposet(elms)

def U_hat(X, x):
    elms=[y for y in X.list() if X.is_greater_than(x, y)]
    return X.subposet(elms)

def C_hat(X,x):
    elms=[y for y in X.list() if X.is_greater_than(x, y) or X.is_greater_than(y, x)]
    return X.subposet(elms)
```

Reduction of posets

```
#remove the point x from the poset X
def remove_point(X, x):
    elms=[y for y in X.list() if not y == x]
    return X.subposet(elms)

#remove the edge x from the poset X
def remove_edge(X, e): #e=[e[0], e[1]]
    rels=[edge for edge in X.cover_relations() if not edge == e]
    return Poset((X.list(), rels))
```

Homotopy Theory

```
def is_beat_point(X, x):
    return len(X.upper_covers(x)) == 1 or len(X.lower_covers(x)) == 1

def beat_points(X):
    return [x for x in X.list() if is_beat_point(X,x)]

def core(X):
```

```
    for x in beat_points(X):
        X = remove_point(X, x)
        return core(X)
    return X

def is_contractible(X):
    if X.has_top() or X.has_bottom():
        return True
    return core(X).cardinality() == 1
```

Simple Homotopy Theory

```
def is_weak_point(X, x):
    return (is_contractible(U_hat(X, x)) or is_contractible(F_hat(X, x)))

def weak_points(X):
    return [x for x in X.list() if is_weak_point(X, x)]

def weak_core(X):
    for x in weak_points(X):
        X = remove_point(X, x)
        return weak_core(X)
    return X

def minwcoreaux(X):
    MX = X
    for x in weak_points(X):
        S = remove_point(X, x)
        K = tuple(S.list())
        if not visitados.has_key(K):
            visitados[K] = True
            Y = minwcoreaux(S)
            if Y.cardinality()< MX.cardinality(): MX = Y
  return MX

def min_weak_core(X):
    global visitados
    visitados = {}
    return minwcoreaux(X)

def is_collapsible_aux(X):
    if is_contractible(X):
        return True
    if non_collapsibles.has_key(tuple(X.list())):
        return False
    for x in weak_points(X):
        S = remove_point(X,x)
        if not non_collapsibles.has_key(tuple(S.list())):
```

```
            if is_collapsible(S):
                return True
            non_collapsibles[tuple(S.list())] = True
    return False

def is_collapsible(X):
    global non_collapsibles
    non_collapsibles = {}
    return is_collapsible_aux(X)
```

Qc-reductions

```
def is_qc_reduction(X, a, b):
    return is_contractible(X.subposet(X.order_ideal([a, b])))

def is_qc_op_reduction(X, a, b):
    return is_contractible(X.subposet(X.order_filter([a, b])))

#The qc-irreducible poset obtained from X by performing qc-
    reductions
def qc_core(X):
    if len(X.maximal_elements()) == 1:
  return X
    for a in X.maximal_elements():
        for b in X.maximal_elements():
            if a != b and is_qc_reduction(X, a, b):
                X=quotient_poset(X, [a, b])
                return qc_core(X)
    return X
```

Resumen del capítulo 2: El punto de vista de los espacios finitos

En este capítulo repasamos la teoría de espacios topológicos finitos, que será una herramienta fundamental en el presente trabajo. Mostramos también la reformulación de la conjetura desde este punto de vista. Para un desarrollo detallado de la teoría de espacios finitos, referimos al lector a [Bar11a].

Los espacios topológicos finitos T_0 y los conjuntos parcialmente ordenados son esencialmente el mismo objetos desde distintas perspectivas. Dado X un espacio topológico finito, definimos la relación $\leq$ en X por $x \leq y$ si x si x pertenece al abierto minimal que contiene a y. Es claro que $\leq$ es reflexiva y transitiva. Si X es T_0, entonces $\leq$ es además antisimétrica. Recíprocamente, si $\leq$ es una relación reflexiva y transitiva en X, los conjuntos $\{U_x\}_{x \in X}$, con $U_x = \{y \in X : y \leq x\}$, son una base para una topología en X. Si $\leq$ es antisimética, la topología definida en X satisfará además el axiona T_0.

De ahora en más, todos los espacios finitos se asumirán T_0, y nos referiremos indistintamente a ambas estructuras, la topológica y la combinatoria.

Los espacios finitos sirven como modelos topológicos simples para poliedros compactos.

Se define un funtor $\mathcal{K} : \text{Top}_{<\infty} \to \mathcal{S}$ de la categoría de espacios finitos a la categoría de complejos simpliciales. El *order complex* $\mathcal{K}(X)$ asociado a un espacio finito X es el complejo simplicial cuyos vértices son los elementos de X y cuyos símplices son las cadenas de elementos de X. Recíprocamente, se define el funtor $\mathcal{X} : CW_{reg} \to \text{Top}_{<\infty}$ desde la categoría de CW-complejos regulares. El *face poset* $\mathcal{X}(K)$ de un complejo regular K es el poset de celdas de K ordenadas por inclusión.

McCord probó en [McC66] que todo complejo celular regular es débilmente equivalente a $\mathcal{X}(K)$. Análogamente, dado un espacio finito X, hay una equivalencia débil entre X y $\mathcal{K}(X)$.

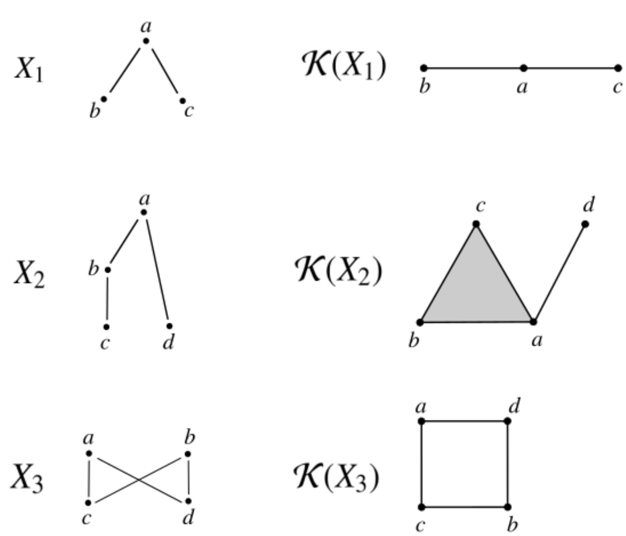


Figura 2.14: Order complex asociado a algunos espacios topológicos finitos.

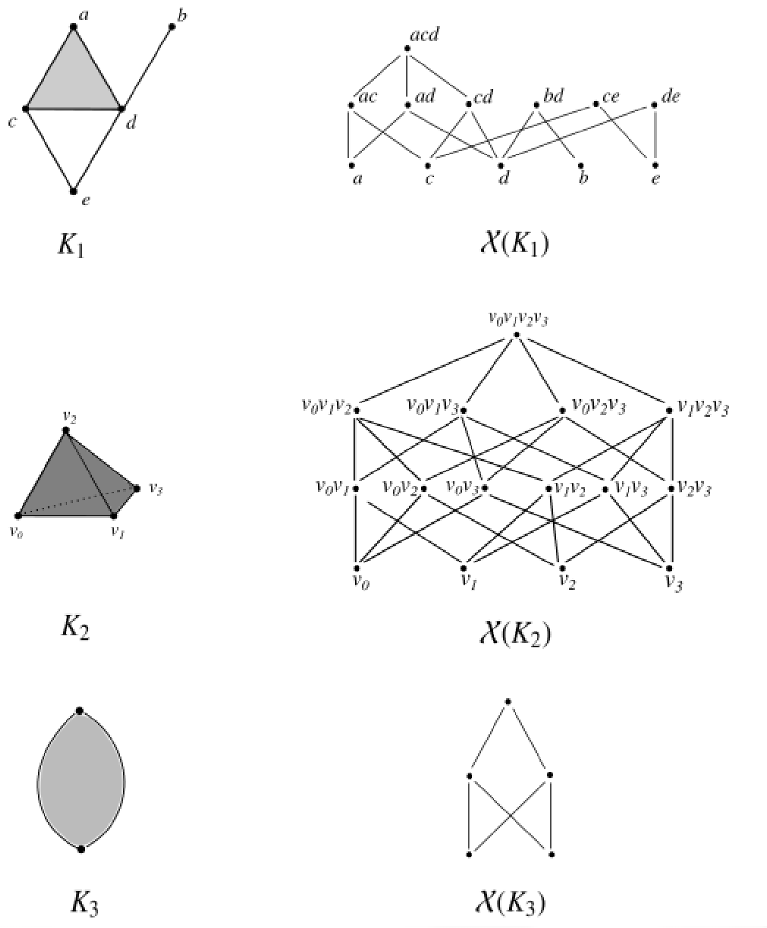


Figura 2.15: Face poset asociado a algunos complejos celulares regulares.

A diferencia de lo que ocurre en la categoría de CW-complejos, el tipo homotópico (fuerte) y el tipo homotópico débil difieren en la categoría de espacios finitos.

En [Sto66], Stong mostró que el tipo homotópico de espacios finitos puede describirse de manera muy simple a partir de un movimiento básico de reducción: remover *beat points*. Un punto $x \in X$ es un *beat point* si cubre o es cubierto por un único elemento de X. El *core* de un espacio finito es el espacio que se obtiene a partir de X luego de remover sucesivamente sus beat points hasta llegar a un espacio irreducible. Dos espacios finitos son homotópicamente equivalentes si y sólo si sus cores son isomorfos. En particular, un espacio finito es contráctil si su core es un punto.

En [BM08b], Barmak y Minian desarrollan la teoría de homotopía simple para espacios finitos. La construyen sobre la base de un movimiento elemental: quitar y agregar *weak points*. Decimos que dos espacios finitos tienen el mismo tipo homotópico simple si se puede llegar de uno a otro agregando y quitando una cantidad finita de weak points. El siguiente resultado muestra que la teoría de homotopía simple para espacios finitos modela la teoría homónima para CW-complejos.

Teorema 2.1.16. *i) Sean X e Y espacios topológicos finitos. Entonces, $X \diagup\!\!\searrow Y$ si y sólo si $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$. Más aún, si $X \searrow Y$ entonces $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.*

ii) Sean K y L complejos celulares regulares. Entonces, $K \diagup\!\!\searrow L$ si y sólo si $\mathcal{X}(K) \diagup\!\!\searrow \mathcal{X}(L)$. Más aún, si $K \searrow L$ entonces $\mathcal{X}(K) \searrow \mathcal{X}(L)$.

Decimos que un espacio finito X se *n-deforma* en Y, y lo denotamos por $X \diagup\!\!\overset{n}{\searrow} Y$, si Y se puede obtener a partir de X agregando y quitando weak points, de modo que todos los espacios involucrados tengan altura a lo sumo n.

La conjetura de Andrews-Curtis tiene una formulación equivalente en el contexto de espacios finitos.

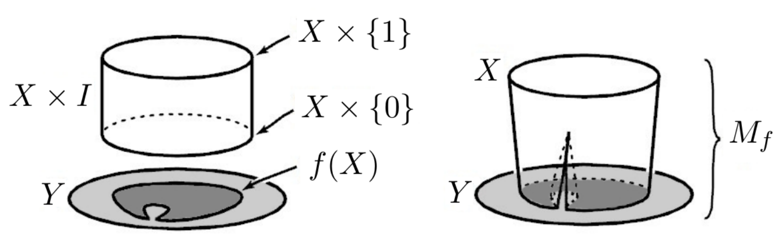
Conjetura 2.2.2 (Conjetura de Andrews-Curtis, versión para espacios finitos)**.** *Sea X un espacio topológico finito homotópicamente trivial de altura 2. Entonces X se 3-deforma a un punto.*

Chapter 3

3-deformation methods for finite spaces

The most natural way to relate the topology of two spaces is through continuous maps. Mapping cylinders embody this kind of connection in the shape of another topological space, a third one that encodes the homotopy type of the codomain and, under certain conditions over the map, also encodes the homotopy information of the domain.

Recall that the *mapping cylinder* of a map $f : X \to Y$ between topological spaces is the quotient space $M_f := (X \times I) \sqcup Y/_{(x,1)\sim f(x)}$.



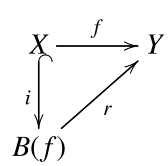
Both X and Y are subspaces of M_f, with $i : X \hookrightarrow M_f$ and $j : Y \hookrightarrow M_f$ the canonical inclusions. Moreover, Y is a strong deformation retract of M_f with a retraction $r : M_f \to Y$ which also satisfies $ri = f$. Then, f can be factorized as a composition of an inclusion (in fact, i is a cofibration) and a homotopy equivalence. In particular, f is a homotopy equivalence if and only if the inclusion i is.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & \nearrow_{r} & \\ M_f & & \end{array}$$

Mapping cylinders are very useful homotopical tools. They have shown to be highly suitable to reduce many proofs concerning homotopy classes of maps $f : X \to Y$ to the case of *good pairs* (Y, X) (a *cofibrant replacement*). For example, Wall's finiteness obstruction [Wal65, Wal66a] is an algebraic K-theory invariant which establishes if a *finitely dominated* space is homotopy equivalent to a finite CW complex. A finite domination of a space X is a CW complex K with maps $d : K \to X$, $s : X :\to K$ such that $ds \simeq id_X$. The very first step to deal with this problem is to assume that d is an inclusion by replacing X, if necessary, by M_d. This reduces the problem to the study of the relative homotopy groups associated to the pair (K, X). A construction due to Milnor [Mil59] allows to kill $\pi_n(K, X)$ for all $n \in \mathbb{N}$ by attaching cells to K and to extend r to a homotopy equivalence $\bar{r} : \bar{K} \to X$ from the new CW complex. Then $\bar{K}$ would be finite if all relative homotopy groups were finitely generated, which was proved to be equivalent to the vanishing of Wall's finiteness obstruction $w(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$.

Another good example of application of the mapping cylinder construction concerns formal deformations and Whitehead's simple homotopy theory (see [Coh73, Ch.2]). If $f : X \to Y$ is a cellular map, then $M_f \searrow Y$, and f is a homotopy equivalence if and only if X is a strong deformation retract of M_f. On the other hand, f is a simple homotopy equivalence if and only if $M_f \nearrow\!\!\!\searrow X$ rel X. So Whitehead's question about *when a homotopy equivalence $f : X \to Y$ is in fact a simple homotopy equivalence* can be reformulated as *when a CW pair (Y, X) such that X is a strong deformation retract of Y satisfies $Y \nearrow\!\!\!\searrow X$ rel X*. After identifying (Y, X) with (Y', X) if $Y \nearrow\!\!\!\searrow Y'$ rel X, Whitehead equipped this class of CW pairs with a group structure, obtaining as result a well defined abelian group $Wh(X)$, now known as the *Whitehead group* of X, which is in fact naturally isomorphic to $K_1(\mathbb{Z}[\pi_1(X)])/_{\langle \pm g : g \in \pi_1(X)\rangle}$, a quotient of the first K-theory group of the group ring $\mathbb{Z}[\pi_1(X)]$. This obstruction for a homotopy equivalence $f : X \to Y$ between finite cell complexes to be a simple homotopy equivalence is the *Whitehead torsion* $\tau(f)$, which is an element in $Wh(Y)$. Concretely, a cellular homotopy equivalence $f : X \to Y$ is a simple homotopy equivalence if and only if $\tau(f) = 0$. In particular, if X and Y are homotopy equivalent and $Wh(X) = Wh(Y) = 0$, then they are simple homotopy equivalent. It is well known that if $\pi_1(Y)$ is free, then $Wh(Y) = 0$. In particular, finite contractible cellular complexes are simple homotopy equivalent to a point.

In [BM08b, Bar11a], Barmak and Minian introduced a finite analogue of the classical mapping cylinder. The *non-Hausdorff mapping cylinder* of a map $f : X \to Y$ between finite topological spaces is the finite space $B(f)$ with underlying set $X \sqcup Y$, and order relation constructed by leaving unchanged the ordering in X and Y, and setting for each $x \in X$ and $y \in Y$, $x \leq y$ in $B(f)$ if $f(x) \leq y$ in Y. $B(f)$ has similar properties as the classical construction. Again X and Y turn out to be subspaces of $B(f)$. Denote by $i : X \hookrightarrow B(f)$ and $j : Y \hookrightarrow B(f)$ the canonical inclusions. Y is once again a strong deformation retract of $B(f)$ with a retraction which satisfies $ri = f$. In particular, $B(f) \searrow\!\!\!\searrow Y$.



The finite version of the mapping cylinder has been used to translate the simple homotopy theory of polyhedra into the setting of finite spaces.

Proposition 3.0.1. *(Barmak, Minian)[BM08b, Lemma 3.7] Let $f : X \to Y$ be a continuous map between finite spaces. If $f^{-1}(U_y)$ (resp. $f^{-1}(F_y)$) is contractible for every y in Y, then $B(f) \searrow X$. In particular, $X \diagup\!\!\searrow Y$. If $f^{-1}(U_y)$ (resp. $f^{-1}(F_y)$) is homotopically trivial for every y in Y, then $B(f) \overset{\gamma}{\searrow} X$. In particular, $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$.*

More recently, Barmak [Bar11b] used the non-Hausdorff mapping cylinder as the key tool to obtain a stronger version of Quillen's Fiber Lemma 2.1.10 (i.e. Quillen's Theorem A for posets).

In this chapter, we introduce a generalization of the non-Hausdorff mapping cylinder construction associated not only to functions between finite spaces, but also to *relations* in general. This construction will serve as a third space that links symmetrically the homotopy type of the related spaces. This new object will provide the suitable framework to investigate a variety of topological problems, including the studying of 3-deformations. More applications will appear in Chapter 4 when we prove a generalized Nerve Theorem, in Chapter 5 when investigating finite models for homotopy colimits and in the rest of this thesis to define some constructions in a more natural way.

3.1 The *relation* cylinder

Let $\mathcal{R} \subseteq X \times Y$ be a relation between the underlying sets of two finite posets. We will construct a finite topological space $B(\mathcal{R})$ associated to $\mathcal{R}$ which, under certain assumptions on $\mathcal{R}$, relates the homotopy properties of X with those of Y.

We write $x\mathcal{R}y$ to mean $(x, y) \in \mathcal{R}$.

Definition 3.1.1. Given $\mathcal{R} \subseteq X \times Y$, we define the *cylinder of the relation* $B(\mathcal{R})$ as the following finite poset. The underlying set is the disjoint union $X \sqcup Y$. We keep the given ordering in both X and Y, and for every $x \in X$ and $y \in Y$ we set $x \leq y$ in $B(\mathcal{R})$ if there points $x' \in X$ and $y' \in Y$ such that $x \leq x'$ in X, $y' \leq y$ in Y and $x'\mathcal{R}y'$ (i.e. we take the order relation generated by $x \leq y$ if $x\mathcal{R}y$).

The next results show how to use $B(\mathcal{R})$ to relate the homotopy properties of X with those of Y.

For $A \subseteq X$ and $B \subseteq Y$, we set

$$\mathcal{R}(A) = \{y \in Y : x\mathcal{R}y \text{ for some } x \in A\},$$
$$\mathcal{R}^{-1}(B) = \{x \in X : x\mathcal{R}y \text{ for some } y \in B\}.$$

If A is a subspace of a finite space, recall that

$$\overline{A} = \{x \in X : x \geq a \text{ for some } a \in A\} = \bigcup_{a \in A} F_a$$

is the *closure* of A, and

$$\underline{A} = \{x \in X : x \leq a \text{ for some } a \in A\} = \bigcup_{a \in A} U_a$$

is the *open hull* of A.

Proposition 3.1.2. *Let $\mathcal{R} \subseteq X \times Y$ be a relation between finite spaces. If $\underline{\mathcal{R}^{-1}(U_y)}$ is homotopically trivial for every $y \in Y$, then $B(\mathcal{R}) \diagup\!\!\searrow X$. In particular, $\mathcal{K}(B(\mathcal{R}))$ and $\mathcal{K}(X)$ are homotopically equivalent. Moreover, if for every $y \in Y$, $\underline{\mathcal{R}^{-1}(U_y)}$ is collapsible, then $\mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(X)$.*

Proof. Let $y_1, y_2, \cdots, y_n$ be a linear extension of Y; that is, if $y_i \leq y_j$ then $i \leq j$. We will show that $B(\mathcal{R}) \searrow^{\gamma} B(\mathcal{R}) \smallsetminus \{y_1\} \searrow^{\gamma} \cdots \searrow^{\gamma} B(\mathcal{R}) \smallsetminus \{y_1, y_2, \cdots, y_n\} = X$. Indeed, for $2 \leq i \leq n$,

$$\hat{U}_{y_i}^{B(\mathcal{R}) \smallsetminus \{y_1, y_2, \cdots, y_{i-1}\}} = \hat{U}_{y_i}^{B(\mathcal{R})} \smallsetminus Y = \underline{\mathcal{R}^{-1}(U_{y_i}^{Y})},$$

which is homotopically trivial by hypothesis. It follows that y_i is a γ-point of the subspace $B(\mathcal{R}) \smallsetminus \{y_1, y_2, \cdots, y_{i-1}\}$. Therefore, $B(\mathcal{R}) \diagup\!\!\searrow X$.

If for every $y \in Y$, $\underline{\mathcal{R}^{-1}(U_y)}$ is collapsible, then the previous collapses induce collapses between the associated simplicial complexes by Theorem 2.1.16. □

Analogously one can prove the following.

Proposition 3.1.3. *Let $\mathcal{R} \subseteq X \times Y$ be a relation between finite spaces. If $\overline{\mathcal{R}(F_x)}$ is homotopically trivial for every $x \in X$, then $B(\mathcal{R}) \diagup\!\!\searrow Y$. In particular, $\mathcal{K}(B(\mathcal{R}))$ and $\mathcal{K}(Y)$ are homotopically equivalent. Moreover, if for every $x \in X$, $\overline{\mathcal{R}(F_x)}$ is collapsible, then $\mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(Y)$.*

Corollary 3.1.4. *Let $\mathcal{R} \subseteq X \times Y$ be a relation between finite spaces. If $\underline{\mathcal{R}^{-1}(U_y)}$, $\overline{\mathcal{R}(F_x)}$ are homotopically trivial for every $x \in X$, $y \in Y$, then $X \diagup\!\!\searrow Y$. Moreover, if $\overline{\mathcal{R}(F_x)}$, $\underline{\mathcal{R}^{-1}(U_y)}$ are collapsible, then $\mathcal{K}(X) \diagup\!\!\searrow^{n} \mathcal{K}(Y)$, with $n = h(B(\mathcal{R}))$ the height of the cylinder.*

By relaxing the hypotheses on $\mathcal{R}$, we get weaker versions of Propositions 3.1.2 and 3.1.3.

Proposition 3.1.5. *Let $\mathcal{R} \subseteq X \times Y$ be a relation between finite spaces. If $\underline{\mathcal{R}^{-1}(U_y)}$, $\overline{\mathcal{R}(F_x)}$ are n-connected (resp. have trivial reduced homology groups for $k \leq n$) for every $x \in X$ and $y \in Y$, then $\pi_i(X) = \pi_i(Y)$ (resp. $H_i(X) = H_i(Y)$) for all $0 \leq i \leq n$.*

Proof. We follow the proof of Propositions 3.1.2 and 3.1.3. If $\underline{\mathcal{R}^{-1}(U_{y_i})}$ is n- connected, then the inclusion $i : B(\mathcal{R}) \smallsetminus \{y_1, \cdots, y_i\} \hookrightarrow B(\mathcal{R}) \smallsetminus \{y_1, \cdots, y_{i-1}\}$ is a $(n+1)$- equivalence by Remark 2.1.20. Then $i : X \hookrightarrow B(\mathcal{R})$ is a $(n+1)$- equivalence. Similarly, $j : Y \hookrightarrow B(\mathcal{R})$ is a $(n+1)$-equivalence.

The homology case is similar, using Remark 2.1.20. □

A map of posets $f : X \to Y$ induces a relation $\mathcal{R} \subset X \times Y$ via $x\mathcal{R}f(x)$. In this case, the relation cylinder $B(\mathcal{R})$ coincides with the non-Hausdorff mapping cylinder $B(f)$. Note that, when $\mathcal{R}$ is the relation associated to a map $f : X \to Y$, the hypotheses of Proposition 3.1.3 are automatically fulfilled and that $\underline{\mathcal{R}^{-1}(U_y)} = f^{-1}(U_y)$ for any $y \in Y$. Therefore, in this particular case, we obtain Proposition 3.0.1.

3.2 Combinatorial methods for 3-deformations

Simple, strong and weak homotopy can be thought of as a two-step procedure, as was noted in [Bar11a, Sect. 4.6]. Any two homotopy equivalent spaces can be embedded as deformation retracts into the same topological space. In simple homotopy context, we have a similar behavior. In the category of CW-complexes, any formal deformation $K_1 \nearrow\!\searrow K_2$ can be reformulated as an expansion $K_1 \nearrow L$ and a collapse $L \searrow K_2$ for some third CW complex L. That is, the formal deformation can be reordered in such a way all the expansions are made first, and all the collapses after (see Lemma 1.4.8). In the category of simplicial complexes or finite spaces, the increased rigidity of the model makes more challenging to achieve this result. If $K_1 \nearrow\!\searrow K_2$, with K_1 and K_2 finite simplicial complexes, there exists a third simplicial complex L such that $K_1 \nearrow L$ and $L \searrow \tilde{K}_2$, where $\tilde{K}_2$ is obtained from K_2 after performing stellar subdivisions [Whi39, Thm. 5]. If $X \nearrow\!\searrow Y$, with X, Y finite spaces, then there exists a finite space Z collapsing to both of them. Here, Z is obtained by considering an iterated construction of non-Hausdorff mapping cylinders [Bar11a, Sect. 4.6].

Motivated by these ideas, in this section we study formal deformations with bounded dimension. The purpose is to investigate new methods of reduction to perform 3-deformations between finite spaces of height 2 *in two steps*, by using the relation cylinder as the third linking space. That is, given X, Y finite spaces of height 2, we want to achieve a deformation of type $X \nearrow B(\mathcal{R}) \searrow Y$, with $h(B(\mathcal{R})) \leq 3$.

The following methods can also be applied to X^{op} (the *opposite poset* of X) obtaining a symmetric version of all of them. Since $\mathcal{K}(X) = \mathcal{K}(X^{op})$, $X' = (X^{op})'$. By Lemma 2.2.4, $X \nearrow\!\searrow^3 X^{op}$. Hence, any method of 3-deformation applied to X^{op} gives also information about the 3-deformation class of X.

3.2.1 Qc-reductions

Recall from Definition 2.2.6 that there is a *qc-reduction* from X to $X \cup \{c\} \smallsetminus \{a, b\} =: Y$, where $a \prec c \succ b$ in $X \cup \{c\}$, if $U_a \cup U_b$ is contractible. Notice that a qc-reduction involves a 3-deformation $X \overset{e}{\nearrow} X \cup \{c\} \searrow\!\searrow X \cup \{c\} \smallsetminus \{a, b\}$.

A *qc*-reduction can be stated in terms of a mapping cylinder. Given two maximal points $a, b \in X$, consider the quotient map $q : X \to X/_{\{a,b\}}$. Note that the height of the mapping cylinder of q is $h(B(q)) = h(X) + 1$. If $U_a \cup U_b$ is contractible, then $X \nearrow B(q) \searrow X/_{\{a,b\}}$ by Proposition 3.0.1.

Remark 3.2.1. The method of *qc*-reductions admits a generalization to an arbitrary number of points. If $\{x_1, \cdots, x_k\}$ is a subset of maximal elements of X such that $\bigcup_{i=1}^{k} U_{x_i}$ is collapsible, then via the mapping cylinder of the quotient map, we get a 3-deformation from X to $X/_{\{x_1,\cdots,x_k\}}$.

The following lemma gives a simple characterization of qc-reductions for finite spaces with height at most 2 and trivial second homology group.

Lemma 3.2.2. *[Bar11a, Prop. 11.2.3.] Let X be a finite space and let a, b be two maximal elements of X. Then $U_a \cup U_b$ is contractible if and only if $U_a \cap U_b$ is contractible. Moreover, if X is of height at most 2 and $H_2(X) = 0$, then the previous statements holds if and only if $U_a \cap U_b$ is nonempty and connected.*

Proposition 3.2.3. *Let X be a finite homotopically trivial space of height 2 which can be covered with a finite family $\mathcal{U} = \{U_1, U_2, \cdots, U_k\}$ of collapsible open subspaces without maximal elements in common such that $\bigcup_{i=1}^{j} U_i \cap U_{j+1}$ is non-empty and connected for every $1 \leq j \leq k-1$. Then $X \nearrow\!\searrow^3 *$.*

Proof. By Remark 3.2.1 X 3-deforms to the space Y obtained by identifying the set of maximal elements of U_i with a unique point y_i for all $1 \leq i \leq k$. The hypothesis on the intersections make it possible to perform inductively the following qc-reductions. There is a qc-reduction involving y_1, y_2, which will be identified with a unique y_2'. Inductively, there is a qc-reduction involving y_i', y_{i+1}. This proves that Y is qc-reducible. □

Example 3.2.4. The finite space of Figure 3.1 is not qc-reducible, but it is homotopically trivial.

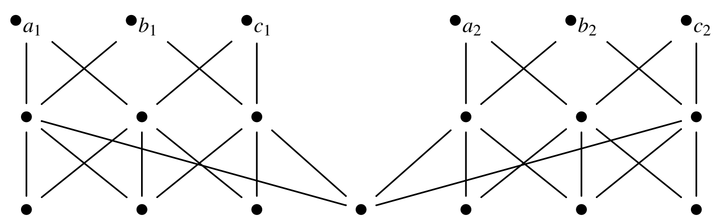
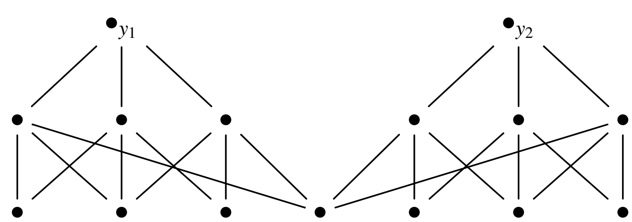


Figure 3.1: A homotopically trivial space which is not qc-reducible.

We can apply Proposition 3.2.3 using the cover by collapsibles $U_i = U_{a_i} \cup U_{b_i} \cup U_{c_i}$, $i = 1, 2$ and deduce that X 3-deforms to the space obtained by identifying a_1, b_1, c_1 with a point y_1, and a_2, b_2, c_2 with a point y_2, which is clearly qc-reducible.



3.2.2 Middle-reductions

One can define a variation of $qc-$reductions for pairs of points which are neither maximal nor minimal.

Definition 3.2.5. Let X be a finite space of height at most 2. Suppose $a, b \in X$ are neither maximal nor minimal points such that $U_a \cap U_b = \{*\}$. If for every $x \in F_a \setminus F_b$, $U_b \cap U_x = \{*\}$, and for every $x \in F_b \setminus F_a$, $U_a \cap U_x = \{*\}$, we say that there is a *middle-reduction* from X to the quotient $X/_{\{a,b\}}$.

Example 3.2.6. Figure 3.2 illustrates a middle-reduction involving the points a, b.

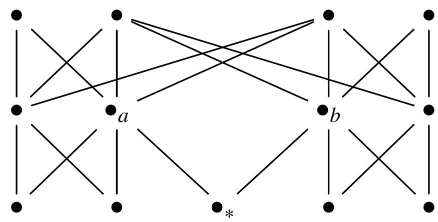


Figure 3.2: A space with a middle-reduction.

Proposition 3.2.7. *Let X be a finite space of height at most 2. If there is a middle-reduction from X to $X/_{\{a,b\}}$, then $X \nearrow\!\!\searrow^{3} X/_{\{a,b\}}$.*

Proof. Consider the mapping cylinder of the quotient map $q : X \to X/_{\{a,b\}}$. It is clear that $h(B(q)) \leq 3$.We only need to show that $q^{-1}(U_{q(x)})$ is contractible for every $x \in X$. Indeed,

$$q^{-1}(U_{q(x)}) = \begin{cases} U_x \cup U_b & \text{if } x \geq a, x \not\geq b \\ U_x \cup U_a & \text{if } x \geq b, x \not\geq a \\ U_x & \text{in other case.} \end{cases}$$

If $x \geq a, x \geq b$, then $U_x \cup U_b \searrow U_x \cup U_b \smallsetminus \{z \in X : z \leq x, z \not\leq b\}$. The latter is a finite space of height 1 with two maximal elements, x and b, and it is contractible if and only if $U_x \cap U_b$ is a singleton. A similar argument applies to $x \geq a, x \not\geq b$. □

We say that a finite space X of height 2 is *middle-reducible* if it can be transformed into a connected space with a unique middle point (i.e. a point of height one) by performing middle-reductions. In particular, if X is middle-reducible, then $X \nearrow\!\!\searrow^{3} *$.

Example 3.2.8. The space of Figure 3.3, 3-deforms to a point. Note that it does not have either weak points or qc-reductions. However, the points $a, b \in X$ allow to perform a middle-reduction from X to $X/_{\{a,b\}}$, and the last space is collapsible.

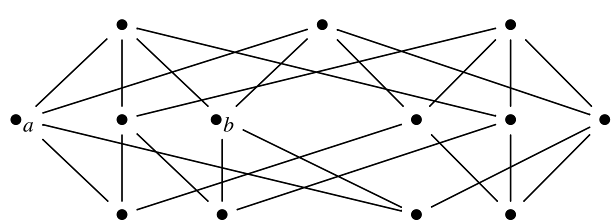


Figure 3.3: A model of a 2-complex which 3-deforms to a point via a middle-reduction.

The previous ideas could be applied to the general case of a quotient $q : X \to X/_{\{a,b\}}$ with a, b incomparable elements. But the main obstacle is that $h(B(q))$ could exceed 3 if $h(a) \neq h(b)$.

Remark 3.2.9. Middle-reductions can also be generalized to an arbitrary number of points. If $\{x_1, \cdots, x_k\}$ is a subset of non-maximal and non-minimal elements of X such that $\bigcup_{i=1}^k U_{x_i} \cup U_x$ is contractible for every $x \in \bigcup_{i=1}^k F_{x_i}$, applying Corollary 3.1.4 we get a 3-deformation from X to $X/_{\{x_1,\cdots,x_k\}}$.

3.2.3 Edge-reductions

We introduce now the method of edge reduction, which was motivated by the following example. Consider the contractible space P of Figure 3.4, which is obtained from a pentagon by identifying the edges as indicated in the figure. This example appeared in Barmak's book ([Bar11a], Example 11.3.7).

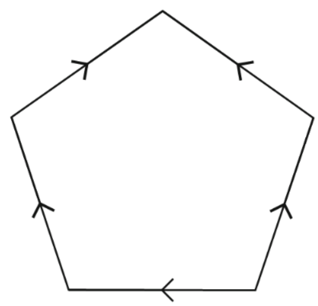


Figure 3.4: Pentagon.

Barmak observed that this space is contractible (because it can be obtained by attaching a 2-cell to the circle by means of a map which is homotopic to the identity) and that there was no reduction method to prove that the finite space $\mathcal{X}(K)$ associated to the h-regular structure[1] K of Figure 3.5 is homotopically trivial.

[1] h-regular complexes were introduced in [BM08a] as an extension of the class of regular complexes. A cellular complex is said to be *h-regular* if the attaching map of each cell is a homotopy equivalence with its image and the closed cells $\overline{e^n}$ are subcomplexes of K.

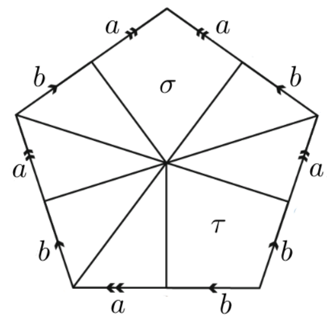


Figure 3.5: h-regular structure of the Pentagon.

We will show that an edge reduction in a finite space of height 2 induces a 3-deformation, and we will use edge reductions to prove that the finite space $\mathcal{X}(K)$ of Barmak's example can be 3-deformed to a point.

Given an edge e of the Hasse diagram of a finite space X, we denote by $X \setminus e$ to the finite space whose Hasse diagram is obtained by removing the edge e in $\mathcal{H}(X)$.

Definition 3.2.10. Let X be a finite space and let $e = (a \prec b)$ be an edge in the Hasse diagram $\mathcal{H}(X)$, with b a maximal element. If $U_b \setminus e$ is contractible, we say that there is an *edge-reduction* from X to $X \setminus e$.

Remark 3.2.11. To check if there is an edge-reduction involving $e = (a \prec b)$ in X, we have to verify the contractibility of $U_b \setminus e$. Nevertheless, we can see that $U_b \setminus e \searrow\!\!\searrow \hat{U}_b/_{\{x \in X : x \prec b, x \neq a\}}$, so the contractibility of the first one is equivalent to the same property in the latter. The previous strong collapse can be performed because the points covered by b in $U_b \setminus e$ are up beat points.

Proposition 3.2.12. *Let X be a finite space of height less than or equal to 2. Let $e = (a \prec b)$ be an edge such that there is an edge-reduction from X to $X \setminus e$. Then $X \nearrow\!\!\searrow^{3} X \setminus e$.*

Proof. Consider the mapping cylinder $B(\iota)$ of the inclusion $\iota : X \setminus e \to X$. To check that $B(\iota) \searrow X$, it suffices to verify that $i^{-1}(U_x)$ is contractible for every $x \in X$. Indeed,

$$i^{-1}(U_x) = \begin{cases} U_x & \text{if } x \neq b \\ U_b \setminus e & \text{if } x = b, \end{cases}$$

so the contractibility in both cases is evident. Finally, as $h(B(\iota)) = h(X) + 1$, it holds that $h(B(\iota)) \leq 3$. □

Example 3.2.13 (Pentagon)**.** Recall the pentagon P of Figure 3.4. Consider the *h-regular structure* K of Figure 3.5. It easy to check that the edges e_1 and e_2 of the Hasse diagram of $\mathcal{X}(K)$ satisfy the conditions of Definition 3.2.10 (see Figure 3.6).

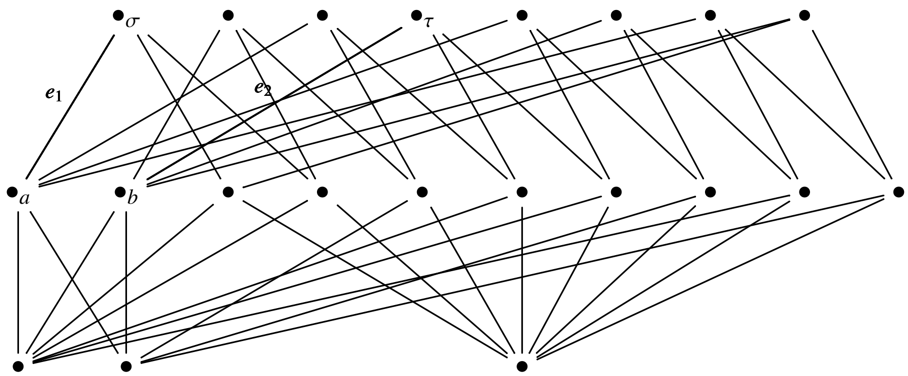


Figure 3.6: Face poset associated to the h-regular structure K of the Pentagon.

We can check that $U_\sigma \cup U_{\sigma'} \cup U_{\sigma''}$ and $U_\tau \cup U_{\tau'} \cup U_{\tau''}$ are both collapsibles, and, by Proposition 3.2.1, we can identify $\sigma, \sigma', \sigma''$ and τ, τ', τ'' with points Σ, $\mathcal{T}$ respectively (see Figure 3.7).

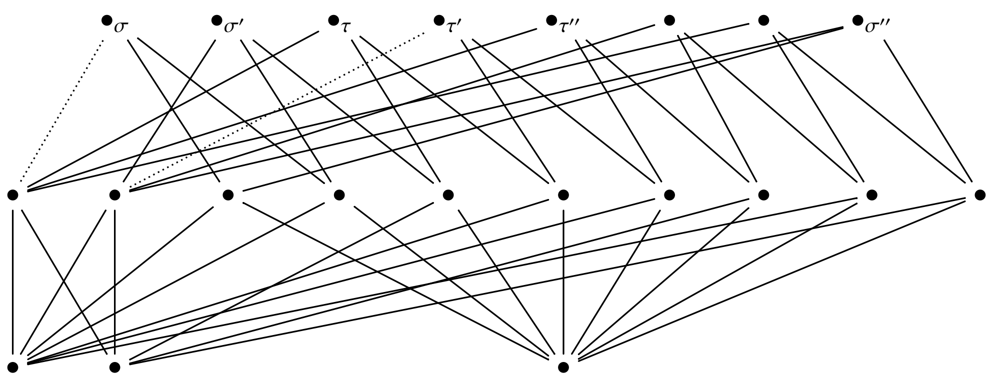


Figure 3.7: The finite space $\mathcal{X}(K) \smallsetminus \{e_1, e_2\}$.

The finite space obtained as result of the previous procedure is collapsible (see Figure 3.8).

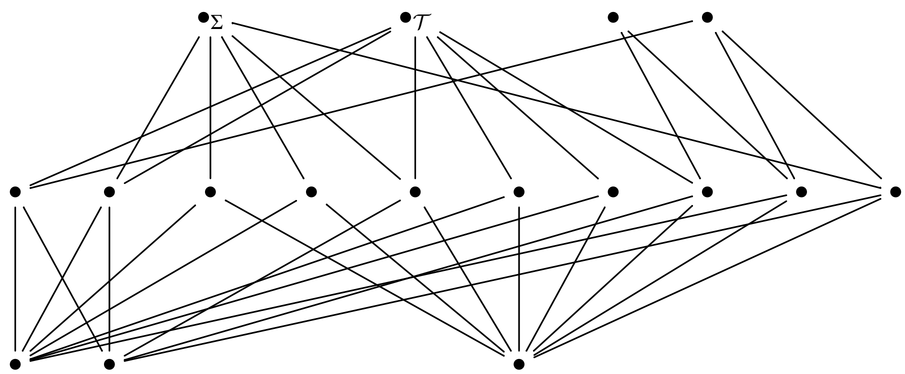


Figure 3.8: The collapsible finite space obtained from $\mathcal{X}(K)\smallsetminus\{e_1, e_2\}$ after applying generalized qc-reductions.

By Propositions 3.2.3 and 3.2.12, we can see that $\mathcal{X}(K) \nearrow\!\!\searrow^{3} *$.

Remark 3.2.14. Notice that in Example 3.2.13 we have not proved that $P \diagup\!\searrow^{3} *$. If fact, h-regular structures are not useful to prove simple deformations via finite spaces, since Theorem 2.1.16 cannot be generalized in this context.

However, we can prove $P \nearrow\!\!\searrow^{3} *$ as follows. Endow P with a regular structure L (see Figure 3.9).

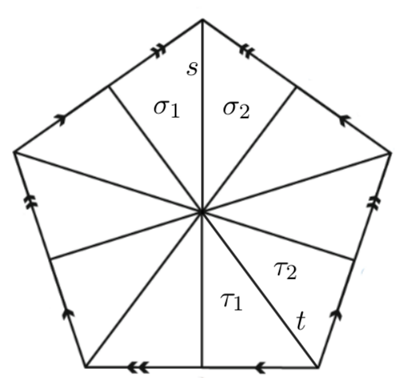


Figure 3.9: Regular cell structure of the Pentagon.

By Lemma 2.2.3, $P \diagup\!\searrow^{3} L$, and we show that $L \diagup\!\searrow^{3} *$. To begin with, we can perform in $\mathcal{X}(L)$ two qc-reductions involving the pairs of 2-cells σ_1, σ_2 and τ_1, τ_2, and then perform two point-reductions of the resultant up beat points s and t. As result, we obtain the face poset associated to the h-regular K structure showed in Figure 3.5. Now, we can continue as we outlined in Example 3.2.13.

Notice that a similar procedure can be done to transform many h-regular structures in regular ones and thus allowing to use Theorem 2.1.16.

3.2.4 Quotient reductions

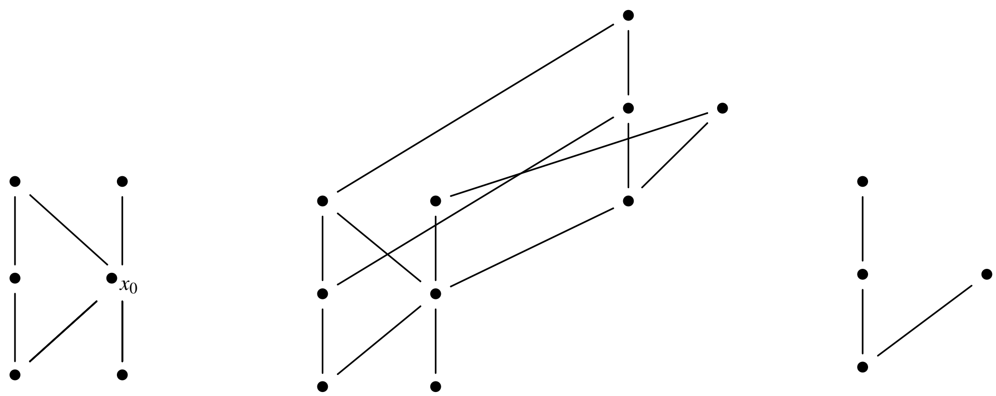
In [Osa99], Osaki introduced two methods of reduction which allow one to find a quotient of a given space with the same weak homotopy type. Namely, the quotient map $q : X \to X/U_{x_0}$ is a weak homotopy equivalence if $U_x \cap U_{x_0}$ is either empty or homotopically trivial for all $x \in X$. In this case, we say that there is a *down O-reduction* from X to X/U_{x_0} Similarly, $q : X \to X/F_{x_0}$ is a weak homotopy equivalence if $F_x \cap F_{x_0}$ is either empty or homotopically trivial. We say in this case that there is an *up O-reduction*. Barmak and Minian showed that, moreover, the quotient maps are simple homotopy equivalences [BM07]. To prove this they used the mapping cylinder of the quotient map. That is, $q : X \to X/U_{x_0}$ (resp. $q : X \to X/F_{x_0}$) satisfies the hypothesis of Proposition 3.0.1, so $X \overset{\gamma}{\nearrow\!\!\nearrow} B(q) \searrow\!\!\searrow X/U_{x_0}$ (resp. $X \overset{\gamma}{\nearrow\!\!\nearrow} B(q) \searrow\!\!\searrow X/F_{x_0}$). In fact,

$$q^{-1}(U_{q(x)}) = \begin{cases} U_x \cup U_{x_0} & \text{if } x \in \overline{U_{x_0}} \\ U_x & \text{if } x \notin \overline{U_{x_0}}. \end{cases}$$

It only remains to note that $U_x \cap U_{x_0}$ is nonempty if and only if $x \in \overline{U_{x_0}}$, and in that case, $U_x \cup U_{x_0} \searrow\!\!\searrow \mathbb{S}(U_x \cap U_{x_0})$, the *non-Hausdorff suspension* of $U_x \cap U_{x_0}$. [2] The latter is homotopically trivial if and only if $U_x \cap U_{x_0}$ so is.

If the height of X is 2, the previous argument proves that $X \nearrow B(q) \searrow\!\!\searrow X/U_{x_0}$ (resp. that $X \nearrow B(q) \searrow\!\!\searrow X/F_{x_0}$). In fact, as the homotopically trivial spaces of height at most one are actually contractible, the γ-points are now weak points. However, it does not necessarily show that there is a 3-deformation, because $B(q)$ not always has height at most 3 (see Figure 3.10).

[2] Given X a finite topological space, the *non-Hausdorff suspension* of X is the finite space whose underlying set is $X \sqcup S^0$ in which we keep the given ordering in X and we set $x \leq y$ for every $x \in X$ and $y \in S^0$. See [Bar11a, Def. 2.7.1., Prop 2.7.3] for a more detailed exposition.



X $\qquad$ $B(q)$ $\qquad$ X/U_{x_0}

Figure 3.10: A finite space X of height 2 with an Osaki's reduction, where $h(B(q)) = 4$.

A proof using the relation cylinder will show that if $h(X) = n$ and U_{x_0} satisfies Osaki's hypothesis, then $X \nearrow\!\!\searrow^{n+1} X/U_{x_0}$.

Theorem 3.2.15. *Let X be a finite space of height n and let $x_0 \in X$. If $U_x \cap U_{x_0}$ is empty or contractible for every $x \in X$, then $X \nearrow\!\!\searrow^{n+1} X/U_{x_0}$.*

Proof. Let $\mathcal{R} \subseteq X/U_{x_0} \times X$ be the relation defined by $q(x)\mathcal{R}x$ for every $x \in X/U_{x_0}$. If $x \in X$, then

$$\underline{\mathcal{R}^{-1}\left(U_x^X\right)} = U_{q(x)}^{X/U_{x_0}},$$

which is contractible since it has maximum. Hence, by Proposition 3.1.2, $\mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(X)$. On the other hand, for every $q(x) \in X/U_{x_0}$,

$$\overline{\mathcal{R}\left(F_{q(x)}^{X/U_{x_0}}\right)} = \begin{cases} F_x^X & \text{if } x \notin U_{x_0}, \\ \overline{U_{x_0}^X} & \text{if } x \in U_{x_0}. \end{cases}$$

In the first case, $\overline{\mathcal{R}\left(F_{q(x)}^{X/U_{x_0}}\right)}$ has minimum and then it is contractible. In the second case, we will see that $\overline{\mathcal{R}\left(F_{q(x)}^{X/U_{x_0}}\right)}$ is collapsible by proving that $\overline{U_{x_0}^X} \searrow U_{x_0}^X$. Let $\{x_1, x_2, \cdots, x_k\}$ be a linear extension of $\overline{U_{x_0}^X} \smallsetminus U_{x_0}^X$. For every $1 \leq i \leq k$,

$$\hat{U}_{x_i}^{\overline{U_{x_0}^X} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\}} = U_{x_i} \cap U_{x_0}^X.$$

Given that $U_{x_i} \cap U_{x_0}^X$ is contractible, $\overline{U_{x_0}^X} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\} \searrow \overline{U_{x_0}^X} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}, x_i\}$. Thus, we have proved that $\overline{U_{x_0}^X} \searrow U_{x_0}^X$. Since $U_{x_0}^X$ is collapsible, so is $\overline{U_{x_0}^X}$. By Proposition 3.1.3, $\mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(X)$.

It only remains to show that $h(B(\mathcal{R})) \leq n+1 = h(X)+1$. Since $h(X/U_{x_0}) \leq h(X)$, this follows from the fact that if $q(x)\mathcal{R}x$ then $h_{X/U_{x_0}}(q(x)) \leq h_X(x)$. Thus, $\mathcal{K}(X) \diagup\!\!\!\searrow^{n+1} \mathcal{K}(X/U_{x_0})$. By Lemma 2.2.4, $X \nearrow\!\!\!\searrow^{n+1} X/U_{x_0}$. □

Corollary 3.2.16. *If X is a finite space of height 2 and there is an up O-reduction from X to U/U_{x_0}, then $X \nearrow\!\!\!\searrow^{3} X/U_{x_0}$.*

Proof. If there is an up O-reduction from X to U/U_{x_0}, then $U_x \cap U_{x_0}$ is a homotopically trivial space of height one, and then, contractible. Hence, Theorem 3.2.15 applies. □

In general a finite space X and its quotients do not have the same weak homotopy type. We will make a detailed study of the problem of determining whether a finite space X of height n satisfies that it $(n+1)$-deforms to a quotient X/A.

Recall that if X is a finite space and $A \subseteq X$ is a subspace, then X/A is T_0 if and only if $\overline{A} \cap \underline{A} = A$. That is, if for every $a < b \in A$, $[a,b] \subseteq A$ (see [Bar11a, Prop. 2.7.8.]). Theorem 3.2.15 can be generalized to arbitrary quotients X/A, with a similar proof, as follows.

Proposition 3.2.17. *Let X be a finite space and $A \subseteq X$ be a connected subspace space such that $\overline{A} \cap \underline{A} = A$. If $F_x \cup \overline{A}$ is homotopically trivial or empty for every $x \in \underline{A}$, then $X \nearrow\!\!\!\searrow X/A$.*

Remark 3.2.18. Suppose that we are in the hypothesis of Proposition 3.2.17, but we also know that $F_x^X \cup \overline{A}$ is *collapsible* for every $x \in \underline{A}$. If $q : X \to X/A$ is the quotient map, define the relation $R \subseteq X/A \times X$ by $q(x)\mathcal{R}x$ for all $x \in X$. From Propositions 3.1.2 and 3.1.3 we can deduce $\mathcal{K}(X) \nearrow \mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(X/A)$.

Theorem 3.2.19. *Let X be a finite space and $A \subseteq X$ be a connected open subspace. If $\overline{A}$ is homotopically trivial, then $X \nearrow\!\!\!\searrow X/A$. Moreover, if $\overline{A}$ is collapsible, then $X \diagup\!\!\!\searrow^{n+1} X/A$ with $n = h(X)$.*

Proof. If A is open, then it is clear that $\overline{A} \cap \underline{A} = A$, so X/A is T_0. Moreover, $h(X/A) \leq h(X)$.

On the other hand, for every $x \in A$, $F_x \subseteq \overline{A}$. Since $\underline{A} = A$ and $\overline{A}$ is homotopically trivial, $F_x \cup \overline{A} = \overline{A}$ is homotopically trivial for every $x \in \underline{A}$. Thus, by Proposition 3.2.17, $X \nearrow\!\!\!\searrow X/A$. Moreover, if $\overline{A}$ is collapsible, by Remark 3.2.18, $\mathcal{K}(X) \nearrow \mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(X/A)$, where $\mathcal{R} \subseteq (X/A) \times X$ is the relation defined by $q(x)\mathcal{R}x$ for every $x \in X$. It only remains to see that $h(B(\mathcal{R})) \leq n+1 = h(X)+1$. The length of a chain in $B(\mathcal{R})$ involving $x \in X$ is $h_{X/A}(q(x)) + h_{X^{op}}(x) + 1$. But $h_{X/A}(q(x)) \leq h_X(x)$ and $h_X(x) + h_{X^{op}}(x) \leq h(X)$. Thus, $h(B(\mathcal{R})) \leq h(X)+1$. Finally, $\mathcal{K}(X) \nearrow\!\!\!\searrow^{n+1} \mathcal{K}(X/A)$ and by Lemma 2.2.4, $X \nearrow\!\!\!\searrow^{n+1} X/A$. □

The following definitions are motivated by Theorem 3.2.19, being generalizations of the Osaki methods of reduction.

Definition 3.2.20. Let X be a finite space and $A \subseteq X$ be a connected open subspace. If $\overline{A}$ is homotopically trivial, we say that there is a *generalized O-reduction* from X to X/A.

Definition 3.2.21. Let X be a finite topological space of height at most 2 and $A \subseteq X$ be a connected open subspace. If $\overline{A}$ is collapsible, we say that there is a *O-3-reduction* from X to X/A.

Remark 3.2.22. If X is a finite space and $A \subseteq X$ is an open subspace, then the fact that $U_x \cap A$ is either homotopically trivial or empty for every $x \in X$ implies that $\overline{A} \searrow\!\!\!\!\!\nearrow^{\gamma} A$, with a similar proof of the one of Theorem 3.2.15. Moreover, if $U_x \cap A$ is either contractible or empty for every $x \in X$, then $\overline{A} \searrow A$. Therefore, if there is an (up) O-reduction from X to X/U_{x_0} then $\overline{U_{x_0}}$ is homotopically trivial and there is a generalized O-reduction from X to X/U_{x_0}.

Remark 3.2.23. If X has height at most 2, then $A \cap U_x$ is homotopically trivial for every $x \in \overline{A}$ if and only if $\overline{A} \searrow A$ by a sequence of down weak points.

In fact, suppose $\overline{A} \searrow\!\!\!\!\!\nearrow^{e} \overline{A} \smallsetminus \{x_1\} \searrow\!\!\!\!\!\nearrow^{e} \cdots \searrow\!\!\!\!\!\nearrow^{e} \overline{A} \smallsetminus \{x_1, x_2 \cdots, x_n\} = A$ where x_i is a down weak point of $\overline{A} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\}$. Given $x \in X$, if $x \notin \overline{A} \smallsetminus A$, it is clear that $U_x \cap A$ is either homotopically trivial or empty. If $x \in \overline{A} \smallsetminus A$, then $x = x_i$ is a down weak point of $\overline{A} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\}$. But

$$\hat{U}_{x_i}^{\overline{A} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\}} = (U_{x_i}^X \cap A) \bigsqcup (U_{x_i}^X \cap (\overline{A} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\}) \cap A^c),$$

which is a contractible subspace of X of height at most 1. So, $U_x^X \cap A$ is contractible if and only if it is connected. If $U_{x_i}^X \cap A$ was non-connected, as $\hat{U}_y^{\overline{A}} \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\}$ is connected, there would exist a path connecting at least two components. So there exists $x_j \in U_{x_i} \cap \overline{A} \smallsetminus (A \cup \{x_1, x_2, \cdots, x_{i-1}\})$, with $i < j \leq n$, an element in the path. But $h(x_k) = 1$ and $\hat{U}_{x_k}^{\overline{A} \smallsetminus \{x_1, \cdots, x_{k-1}\}}$ has at least 2 elements. Therefore, x_k cannot be a down weak point of $\overline{A} \smallsetminus \{x_1, x_2, \cdots, x_{k-1}\}$, which is a contradiction.

Remark 3.2.24. The O-3-reduction method provides a way to "simplify" regular CW-complexes under 3-deformations. Let K be a regular CW-complex of dimension at most 2. Let e_0 be a cell in K such that $\bar{e}_0 \cap \bar{e}$ is either empty or contractible for every cell $e \in K$. By Theorem 3.2.19, $\mathcal{X}(K) \nearrow\!\!\!\!\!\searrow^{3} \mathcal{X}(K)/U_{e_0}$ But the hypothesis also implies that $K/\bar{e}$ is again a regular CW-complex, and $\mathcal{X}(K)/U_{e_0} = \mathcal{X}(K/\bar{e})$. By Lemma 2.2.3, it follows that $K \nearrow\!\!\!\!\!\searrow^{3} K/\bar{e}$.

Hence, we get a method of reduction for regular CW-complexes of dimension at most 2 that actually summarizes a 3-deformation.

Example 3.2.25. The finite space X of height 2 of Figure 3.11 is the wedge $Y_1 \vee_{y_0} Y_2$ of two copies of a collapsible space Y whose unique weak point is exactly y_0. Thus, X is a homotopically trivial space without weak points, and even without any O-reduction. However, X can be 3-deformed into a single point. The closure of the open subset $A = \bigcup_{i=1}^{5} U_{x_i}$ is trivially collapsible. Thus, there is a generalized O-reduction from X to X/A, where the latter collapses to Y_2. Thus, $X \nearrow\!\!\!\!\!\searrow^{3} *$.

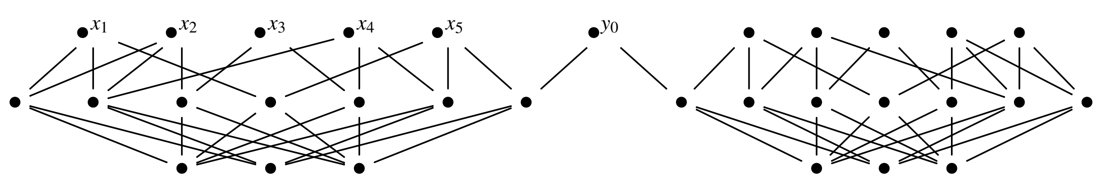


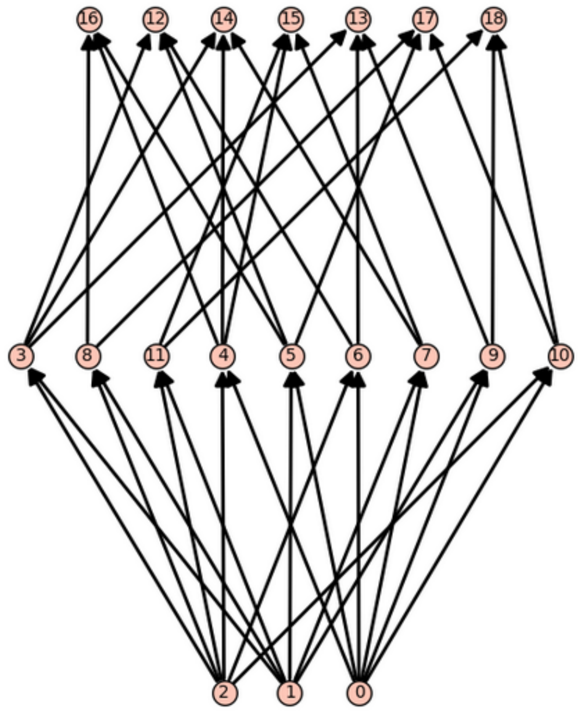
Figure 3.11: A finite space without weak points with no O-reductions, but with generalized O-reductions.

3.3 SAGE implementation.

We implemented each of the above described new methods of 3-deformation of finite spaces in the SAGE platform (see Appendix 3.A). The function `random_core(X)` makes use of all these methods. It takes as input a finite space X of height less or equal than 2 and returns a finite space which 3-deforms to the original one. The output is a non-reducible finite space obtained after applying a sequence of reductions to X, whose order is chosen at random. Each reduction corresponds to one of the methods previously developed (weak points, generalized qc-reductions, middle-reductions, edge reductions, O-3-reductions). Of course, if the output is a single point, then $X \diagup\!\!\searrow^{3} *$. However, the algorithm is not exhaustive.

In Figure 3.12 there is a sequence of random reductions allowing to 3-deform to a point the contractible space of Example 2.2.7.

```
Triangle=Poset({0:[4,5,6,7,9,10], 1:[3,5,7,8, 9,11], 2:[3,4,6,8,10,11], 3:[12,13,14],
4:[14,15,16], 5:[16,17,12], 6:[12,13], 7:[14,15], 8:[16,17], 9:[13,18], 10:[17,18], 11:[15,18]})
Triangle.show()
```




```
random_core(Triangle)
```

```
16 17 qc-reduction
12 13 qc-reduction
14 15 qc-reduction
7 weak point
6 up_o_reduction
[2, 8] edge-reduction
8 down_o_reduction
[5, 17] edge-reduction
17 18 qc-reduction
10 up_o_reduction
11 4 middle-reduction
4 up_o_reduction
4 6 qc-reduction
9 3 middle-reduction
[0, 5] edge-reduction
6 weak point
0 1 qc-op-reduction
1 up_o_reduction
1 weak point
se 3-deforma a un punto
```

Finite poset containing 1 elements

Figure 3.12: The program `random_core` applied to the Triangle.

3.A Appendix: 3-deformation SAGE module

Middle reduction

```
def is_middle_reduction(X, a, b):
    lower_intersection=Set(X.order_ideal([a])).intersection(Set(X.order_ideal([b])))
    if len(lower_intersection) != 1:
        return False
    Fa_minus_Fb=Set(X.order_filter([a])).difference (Set(X.order_filter([b])))
    for x in Fa_minus_Fb:
        if len(Set(X.order_ideal([b])).intersection(Set(X.order_ideal([x])))) != 1:
             return False
    Fb_minus_Fa=Set(X.order_filter([b])).difference (Set(X.order_filter([a])))
    for x in Fb_minus_Fa:
        if len(Set(X.order_ideal([a])).intersection(Set(X.order_ideal([x])))) !=  1:
            return False
    return True

def is_middle_op_reduction(X, a,  b):
    return is_middle_reduction(op(X),a, b)
```

Edge reduction

```
def is_down_edge(X, e): #e=[e[0], e[1]]
    if e[1] not in X.maximal_elements(): return False
    Y=remove_edge(U(X, e[1]), e)
    return is_contractible(Y)

def is_up_edge(X,e):
    if e[0] not in X.minimal_elements(): return False
    Y=remove_edge(F(X, e[0]), e)
    return is_contractible(Y)

def is_reducible_edge(X, e):
    return is_up_edge(X, e) or is_down_edge(X, e)
```

Osaki reductions

```
def is_down_O_reduction(X, x):
    if len(X.order_ideal([x])) == 1: return False
    for y in X:
        intersection = Set(X.order_ideal([x])).intersection(Set(X.order_ideal([y])))
```

```
        if not(is_contractible(X.subposet(list(intersection))) or len(intersection)==0):
            return False
    return True

def is_up_0_reduction(X,x):
    if len(X.order_filter([x]))==1: return False
    for y in X:
        intersection = Set(X.order_filter([x])).intersection(Set(X.order_filter([y])))
        if not(is_contractible(X.subposet(list(intersection))) or len(intersection) == 0):
            return False
    return True
```

Random Reduction

```
from random import shuffle

def random_core(X):
    if len(X.list())==1:
        print '3-deforms to a point'
        return X
    count=range(7)
    shuffle(count)
    for i in range(7):
        Y=random_reduction(X,count[i])
        if X!=Y:
            return random_core(Y)
    return X

def random_reduction(X,j):
    # Selects a reduction method according to parameter j, which is an integer that ranges from 0 to 6.
    # j = 0 -> weak_point
    # j = 1 -> qc-reduction
    # j = 2 -> qc-op-reduction
    # j = 3 -> middle-reduction
    # j = 4 -> edge-reduction
    # j = 5 -> down-0-reduction
    # j = 6 -> up-0-reduction

    elms = X.list()

    if j ==  0:
        shuffle(elms)
        for x in elms:
            if is_weak_point(X, x):
                print x, 'weak point'
```

```
                Y = remove_point(X, x)
                return Y
        return X

    if j == 1:
        M = X.maximal_elements()
        shuffle(M)
        for S in Set(M).subsets(2):
            l=list(S)
            if is_qc_reduction(X, l[0], l[1]):
                print l[0],l[1], 'qc-reduction'
                Y = quotient(X, [l[1], l[0]])
                return Y
        return X

    if j == 2:
        m = X.minimal_elements()
        shuffle(m)
        for S in Set(m).subsets(2):
            l = list(S)
            if is_qc_op_reduction(X, l[0], l[1]):
                print l[0], l[1], 'qc-op-reduction'
                Y = quotient_poset(X, [l[1], l[0]])
                return Y
        return X

    if j == 3:
        mid = [x for x in X.list() if not (x in X.maximal_elements()
    or x in X.minimal_elements())]
        shuffle(mid)
        for S in Set(mid).subsets(2):
            l = list(S)
            if is_middle_reduction(X, l[0], l[1]):
                print l[0], l[1], 'middle-reduction'
                Y=quotient_poset(X, [l[1], l[0]])
                return Y
        return X


    if j == 4:
        E = X.cover_relations()
        shuffle(E)
        for e in E:
            if is_reducible_edge(X, e):
                print e, 'edge-reduction'
                Y = remove_edge(X, e)
                return Y
        return X

```

```
    if j == 5:
        shuffle(elms)
        for x in elms:
            if is_down_O_reduction(X, x):
                print x, 'down_o_reduction'
                Y = quotient_poset(X, X.order_ideal([x]))
                return Y
        return X

    if j == 6:
        shuffle(elms)
        for x in elms:
            if is_up_O_reduction(X, x):
                print x, 'up_o_reduction'
                Y = quotient_poset(X, X.order_filter([x]))
                return Y
        return X
```

Resumen del capítulo 3: Métodos de 3-deformación de espacios finitos

En este capítulo estudiamos la formulación de la conjetura de Andrews-Curtis en el contexto de espacios finitos, es decir, el problema de decidir si un espacio finito homotópicamente trivial de altura 2 se 3-deforma a un punto. En general, no se conocen técnicas algorítmicas para resolver este problema. Por nuestra parte, desarrollamos nuevos *métodos de reducción* de espacios finitos, que representan una 3-deformación de dos pasos, es decir, una deformación de tipo $X \nearrow Y \searrow *$, donde la altura de Y es menor o igual que 3. Algunos ejemplos de los nuevos métodos desarrollados son: las *reducciones de arista*, las *middle* reducciones, y varios otros tipos de reducciones de tipo cociente, incluyendo las *O-reducciones*, una generalización y mejora de los métodos de reducción de Osaki [Osa99] para tipos homotópicos débiles de espacios finitos. Las reducciones disminuyen el número de puntos o aristas del diagrama de Hasse asociado al poset, y se pueden describir en términos de la combinatoria este grafo. Al final del capítulo, se encuentra la implementación de nuestros métodos algorítmicos en SAGE. En particular, implementamos una función llamada `random_core(X)`, que toma como input un espacio finito de dimensión 2 y devuelve como output el resultado de aplicarle al espacio una serie de reducciones al azar de las antes mencionadas hasta llegar a un espacio *irreducible*.

Para lograr las 3-deformaciones de dos pasos, creamos un modelo finito llamado el *cilindro no-Hausdorff de una relación* $\mathcal{R} \subseteq X \times Y$ entre espacios X e Y. Lo denotamos $B(\mathcal{R})$. Si $(x, y) \in \mathcal{R}$, lo denotamos por $x\mathcal{R}y$.

Definición 3.1.1. Dada $\mathcal{R} \subseteq X \times Y$ una relación, definimos el *cilindro de una relación* $B(\mathcal{R})$ como el siguiente poset finito. El conjunto subyacente es la unión disjunta $X \sqcup Y$. Preservamos el orden de X e Y y, para cada $x \in X$ e $y \in Y$, decimos que $x \leq y$ en $B(\mathcal{R})$ si existen puntos $x' \in X$ e $y' \in Y$ tales que $x \leq x'$ en X, $y' \leq y$ en Y y $x'\mathcal{R}y'$ (i.e. tomamos la relación generada por $x \leq y$ si $x\mathcal{R}y$).

Barmak y Minian prueban que se puede lograr una equivalencia simple entre dos espacios finitos X e Y mediante una función continua $f : X \to Y$ que cumpla ciertas características. Más aún, deformación se logra como $X \nearrow B(f) \searrow Y$, donde $B(f)$ es el cilindro no-Hausdorff de la función f.

Proposición 3.0.1. (Barmak, Minian)[BM08b, 3.7] [Bar11a, 4.2.7, 6.2.10] Sea $f : X \to Y$ una función continua entre espacios finitos. Si $f^{-1}(U_y)$ (resp. $f^{-1}(F_y)$) es contráctil para todo y in Y, entonces $B(f) \searrow X$.

Las siguientes proposiciones generalizan (y simetrizan) el resultado anterior para relaciones $\mathcal{R}$ entre dos espacios finitos X y Y. Nos sirvieron para formular 3-deformaciones del tipo $X \nearrow B(\mathcal{R}) \to Y$.

Proposición 3.1.2. Sea $\mathcal{R} \subseteq X \times Y$ una relación entre espacios finitos. Si $\underline{\mathcal{R}^{-1}(U_y)}$ es homotópicamente trivial para cada $y \in Y$, entonces $B(\mathcal{R}) \nearrow\searrow X$. Más aún, si para cada $y \in Y$, $\underline{\mathcal{R}^{-1}(U_y)}$ es colapsable, entonces $\mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(X)$.

Proposición 3.1.3. Sea $\mathcal{R} \subseteq X \times Y$ una relación entre espacios finitos. Si $\overline{\mathcal{R}(F_x)}$ es homotópicamente trivial para cada $x \in X$, entonces $B(\mathcal{R}) \nearrow\searrow Y$. Más aún, si para cada $x \in X$, $\overline{\mathcal{R}(F_x)}$ es colapsable, entonces $\mathcal{K}(B(\mathcal{R})) \searrow \mathcal{K}(Y)$.

Los nuevos métodos de reducción que formulamos en esta tesis para estudiar la conjetura de Andrews-Curtis son descriptos a continuación.

Las middle-reducciones son una variación de las qc-reducciones para puntos no maximales ni minimales.

Definición 3.2.5. Sea X un espacio finito de dimensión a lo sumo 2. Supongamos que $a, b \in X$ son puntos no maximales ni minimales tales que $U_a \cap U_b = \{*\}$. Si para cada $x \in F_a \smallsetminus F_b$, $U_b \cap U_x = \{*\}$, y para cada $x \in F_b \smallsetminus F_a$, $U_a \cap U_x = \{*\}$, decimos que hay una *middle-reducción* de X al cociente $X/_{\{a,b\}}$.

Proposición 3.2.7. Sea X un espacio finito de altura a lo sumo 2. Si hay una middle-reducción de X a $X/_{\{a,b\}}$, entonces $X \overset{3}{\nearrow\searrow} X/_{\{a,b\}}$.

Las reducciones de arista permiten eliminar cierto tipo de arcos del diagrama de Hasse de un espacio finito.

Definición 3.2.10. Sea X un espacio finito y sea $e = (a \prec b)$ una arista del diagrama de Hasse $\mathcal{H}(X)$, con b un elemento maximal. Si $U_b \smallsetminus e$ es contrácl, decimos que hay una reducción de arista de X a $X \smallsetminus e$.

Proposición 3.2.12. Sea X un espacio finito de altura a lo sumo 2. Sea $e = (a \prec b)$ una arista tal que existe una reducción de X a $X \smallsetminus e$. Entonces $X \overset{3}{\nearrow\searrow} X \smallsetminus e$.

Como consecuencia de este nuevo método, pudimos hallar una 3-deformación explícita con técnicas de espacios finitos del siguiente espacio que aparece al final del libro de Barmak [Bar11a], como un ejemplo de espacio homtópicamente trivial que no posee ninguna reducción posible.

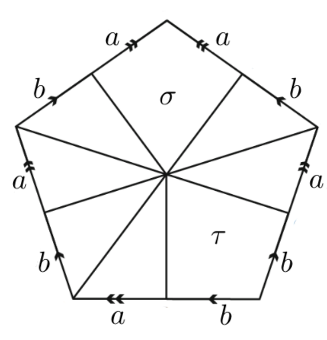


Figure 3.13: Estructura h-regular del Pentágono.

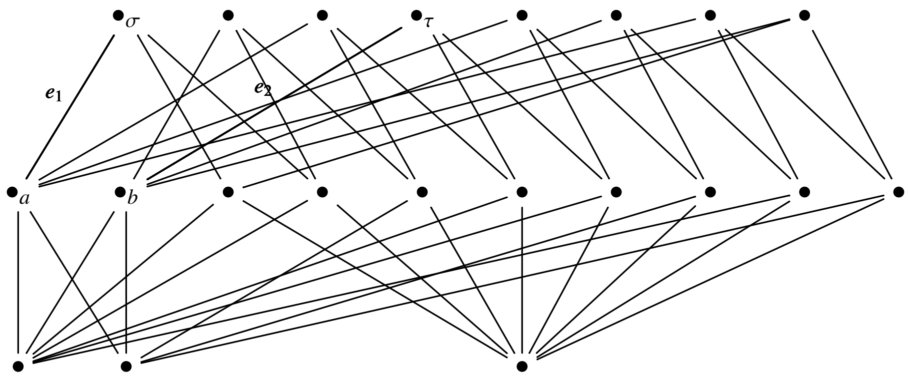


Figure 3.14: Face poset asociado a la estructura h-regular del Pentágono.

Un resultado relevante de este capítulo fue la prueba de que las reducciones de Osaki [Osa99] en espacios de dimensión son 3-deformaciones.

Teorema 3.2.15. Sea X un espacio finito de altura n y sea $x_0 \in X$. Si $U_x \cap U_{x_0}$ es vacío o contráctil para todo $x \in X$, entonces $X \nearrow\searrow^{n+1} X/U_{x_0}$.

Realizamos además un estudio general de reducciones dada por cociente. En general, un espacio finito X no tiene el mismo tipo homotópico débil que sus cocientes X/A, incluso si A es contráctil. Hallamos una condición general sobre A que asegura que si X es de altura n, entonces $X \nearrow\searrow^{n+1} X/A$.

Teorema 3.2.19. Sean X un espacio finito de altura n, $A \subseteq X$ un subespacio abierto conexo. Si $\overline{A}$ es homotópicamente trivial, entonces $X \nearrow\searrow X/A$. Más aún, si $\overline{A}$ es colapsable, entonces $X \nearrow\searrow^{n+1} X/A$.

Chapter 4

The Nerve Theorem

A very useful construction in algebraic topology and computational geometry is the notion of *nerve*. For instance, it is the main tool to prove the Crosscut Theorem, the Dowker Theorem and Helly type theorems for intersection patterns of covers. This kind of results was fruitfully applied in the area of topological data analysis, whose goal is to obtain information about the topology of a space from a discrete sample of the space.

The Nerve Theorem, due to Borsuk [Bor48] with alternative versions of Leray [Ler45], Weil (1952), Wu (1962), McCord (1967) and others, particularly guarantees that the nerve of a family of geometric objects has the same homotopy type as the union of the objects, if they form a *good cover*, that is, if the intersection of any subfamily is either empty or contractible. Nevertheless, this requirement is rather strong and not always adequate.

Many generalizations of the Nerve Theorem have been done since then (see [Bjö03, Bar11b]). We present a new result, obtained as a consequence of Propositions 3.1.2 and 3.1.3, which implies a new version of the Nerve Theorem.

4.1 A generalization of the classical Nerve Theorem

Recall that the *nerve* of a family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of a set X is the simplicial complex $\mathcal{N}(\mathcal{U})$ whose simplices are the finite subsets $J \subseteq I$ such that $\bigcap\limits_{i\in J} U_i \neq \varnothing$.

Several versions of the Nerve Theorem can be found, involving convex sets in $\mathbb{R}^d$ or good covers of a topological space. We mention here one version suitable for our purposes (see [Bar11b]).

Theorem 4.1.1 (Nerve Theorem)**.** *Let K be a finite simplicial complex (or a regular cell complex) and let $\mathcal{U} = \{L_i\}_{i\in I}$ be a finite family of subcomplexes of K such that $\bigcup_{i\in J} L_i = K$. If every intersection of elements of $\mathcal{U}$ is empty or contractible, then K has the same (simple) homotopy type as $\mathcal{N}(\mathcal{U})$.*

If $\mathcal{U}$ is a cover of a simplicial complex K satisfying the hypothesis of Theorem 4.1.1, then $\mathcal{U}$ is said to be a *good cover* of K.

Corollary 4.1.2. *If X is a finite topological space and $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X such that every intersection of elements of $\mathcal{U}$ is empty or homotopically trivial, then $X \diagup\!\!\!\!\searrow \mathcal{X}(\mathcal{N}(\mathcal{U}))$.*

Proof. The family of simplicial complexes $\mathcal{K}(\mathcal{U}) = \{\mathcal{K}(U_i)\}_{i\in I}$ is a cover by subcomplexes of $\mathcal{K}(X)$. The intersections are $\bigcap_{j\in J} \mathcal{K}(U_j) = \mathcal{K}(\bigcap_{j\in J} U_j)$ for every $J \subseteq I$. Since $\bigcap_{j\in J} U_j$ is either empty or homotopically trivial, then by Corollary 2.1.8 $\mathcal{K}(\mathcal{U})$ is a good cover of $\mathcal{K}(X)$. By the Nerve Theorem 4.1.1, $\mathcal{K}(X) \diagup\!\!\!\!\searrow \mathcal{N}(\mathcal{K}(\mathcal{U}))$. Notice that $\mathcal{N}(\mathcal{K}(\mathcal{U})) = \mathcal{N}(\mathcal{U})$. By Theorem 2.1.16, $X' \diagup\!\!\!\!\searrow \mathcal{X}(\mathcal{N}(\mathcal{U}))$. By Lemma 2.2.4, $X \diagup\!\!\!\!\searrow X'$. Thus, $X \diagup\!\!\!\!\searrow \mathcal{X}(N(\mathcal{U}))$. □

Remark 4.1.3. Corollary 4.1.2 is equivalent to the Nerve Theorem 4.1.1. In fact, if $\mathcal{U} = \{L_i\}_{i\in I}$ is a good cover of a simplicial complex K by a finite family of subcomplexes, then $\mathcal{X}(\mathcal{U}) = \{\mathcal{X}(L_i)\}_{i\in I}$ is a good open cover of $\mathcal{X}(K)$.

Now we deduce from Propositions 3.1.2 and 3.1.3 a more general statement in which we do not require the cover to be a good cover.

For simplicity, we will denote by $N(\mathcal{U})$ the face poset of $\mathcal{N}(\mathcal{U})$, and we will call it the *non-Hausdorff nerve* of $\mathcal{U}$.

Theorem 4.1.4. *Let X be a finite topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Let $N_0(\mathcal{U})$ be the subspace of the non-Hausdorff nerve $N(\mathcal{U})$ consisting of all homotopically trivial intersections. If for every $x \in X$, the subspace $\mathcal{I}_x$ of $N_0(\mathcal{U})$ of the intersections which contain x, is homotopically trivial, then X has the same simple homotopy type as $N_0(\mathcal{U})$.*

Proof. For every $J \subseteq I$, denote $\mathcal{I}_J = \bigcap_{i\in J} U_i$. So the elements of $N_0(\mathcal{U})$ are the subsets $J \subseteq I$ such that $\mathcal{I}_J$ is homotopically trivial.

Define the relation $\mathcal{R} \subseteq X \times N_0(\mathcal{U})^{op}$ as follows: for every $x \in X$ and $J \subseteq I$ such that $\mathcal{I}_J$ is homotopically trivial, set

$$x \,\mathcal{R}\, \mathcal{I}_J \text{ if } x \in \mathcal{I}_J.$$

On the one hand,

$$\underline{\mathcal{R}^{-1}(U_{\mathcal{I}_J})} = \underline{\mathcal{I}_J} = \mathcal{I}_J,$$

which is homotopically trivial by construction of $N_0(\mathcal{U})$.

On the other hand,

$$\overline{\mathcal{R}(F_x)} = \overline{\{\mathcal{I}_J : \mathcal{I}_J \ni y \text{ for some } y \geq x\}} = \overline{\{\mathcal{I}_J : \mathcal{I}_J \ni x\}} = \{\mathcal{I}_J : x \in \mathcal{I}_J\} = \mathcal{I}_x,$$

which is homotopically trivial by hypothesis.

Thus, by Propositions 3.1.2 and 3.1.3, $X \diagup\!\!\!\!\searrow N_0(\mathcal{U})^{op}$. Since $N_0(\mathcal{U})^{op} \diagup\!\!\!\!\searrow N_0(\mathcal{U})$, we deduce that $X \diagup\!\!\!\!\searrow N_0(\mathcal{U})$. □

Remark 4.1.5. If $\mathcal{U} = \{U_i\}_{i\in I}$ is a good cover of a finite space X then $N(\mathcal{U}) = N_0(\mathcal{U})$. Moreover, for every $x \in X$, the subspace of $N(\mathcal{U})$ of the intersections containing x has a maximum element, namely the intersection $\bigcap\limits_{\substack{U_i \in \mathcal{U} \\ U_i \ni x}} U_i$. Then, it is homotopically trivial. Thus, the Nerve Theorem for posets 4.1.2 can be deduced from Theorem 4.1.4.

Example 4.1.6. Consider the regular structure H of the Dunce Hat labeled as in Figure 4.1.

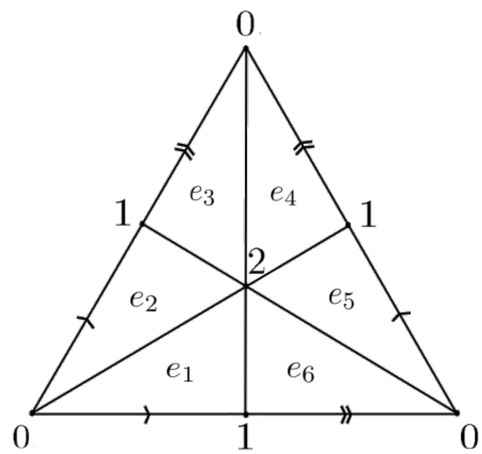


Figure 4.1: Regular cell structure of the Dunce Hat.

Attach "regularly" to H one 2-cell e_7, to the subcomplex of H isomorphic to S^1 determined by two of the 1-cells that joins the vertex of the triangle with the internal point.

It can be checked (we used SAGE) that the face poset of the regular cell complex $H \cup e_7$ has neither weak points nor any good cover (see Figure 4.2).

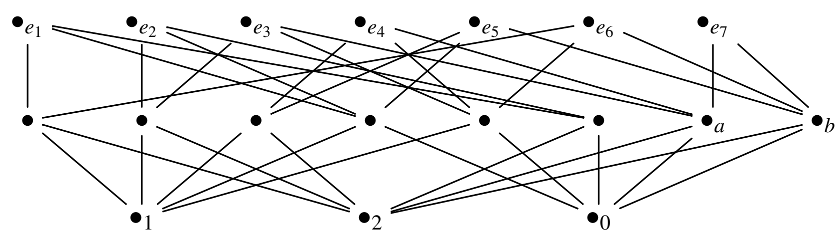


Figure 4.2: The face poset of $H \cup e_7$.

However, the open cover $\mathcal{U} = \{U_{e_1} \cup U_{e_2} \cup U_{e_3} \cup U_{e_4} \cup U_{e_5} \cup U_{e_6}, U_{e_7}, U_a, U_b, \{2\}, \{0\}\}$ satisfies the hypothesis of Theorem 4.1.4 and then, $X \diagup\!\!\searrow N_0(\mathcal{U})$.

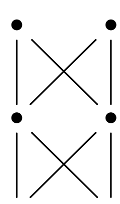


Figure 4.3: The finite space $N_0(\mathcal{U})$ associated to the covering $\mathcal{U}$ of $H \cup e$.

We investigate now some conditions on covers, which are easy to check, that imply the hypotheses of Theorem 4.1.4.

Definition 4.1.7. A family $\mathcal{U}$ of open subspaces of a finite space X is called a *quasi-good cover* if every nonempty intersection of a subfamily of $\mathcal{U}$ has homotopically trivial connected components.

Definition 4.1.8. Given a finite space X and a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, define $M(\mathcal{U})$ the finite space whose elements are the connected components C of $\bigcap_{i\in J} U_i$ for every $J \subseteq I$. If C is a connected component of $\bigcap_{i\in J} U_i$ and C' is a connected component of $\bigcap_{i\in J'} U_i$, then $c \leq C'$ if $C \supseteq C'$ and $J \subseteq J'$.

Corollary 4.1.9. *Let X be a finite space and let $\mathcal{U}$ be a quasi-good cover of X. Then, $X \diagup\!\!\searrow M(\mathcal{U})$.*

Proof. Define $\mathcal{U}' = \mathcal{U} \bigcup \{C : C \neq \emptyset$ connected component of $\bigcap_{i\in J} U_i\}_{J\subseteq I}$ a new cover of X. Notice that $M(\mathcal{U}) = N_0(\mathcal{U}')$. For every $x \in X$, the subspace I_x of $N_0(\mathcal{U}')$ of the intersections containing x has a minimum: the connected component containing x of the maximal subset $J \subseteq I$ such that $x \in \bigcap_{i\in J} U_i$, Thus, I_x is homotopically trivial. By Theorem 4.1.4, $X \diagup\!\!\searrow N_0(\mathcal{U}')$ and the result follows. □

To get an intuition, if $\mathcal{U}$ is a quasi-good cover, $N(\mathcal{U})$ not necessarily encodes the simple homotopy type of X, but it is enough to replace every element $\bigcap_{i\in J} U_i$ of $N(\mathcal{U})$ by its connected components, preserving the order given by the reversed inclusion of the subsets, and get a new finite space $M(\mathcal{U})$ satisfying $X \diagup\!\!\searrow M(\mathcal{U})$.

Example 4.1.10. Let X be the finite model of S^1 with 6 points of Figure 4.4. Although X is not a minimal finite model of S^1, it has not beat point nor weak points. Even more, there does not exists any cover of X with open subspaces satisfying the hypothesis of the Nerve Theorem 4.1.2.

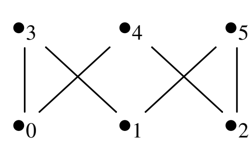


Figure 4.4: A finite model of S^1.

However, $\mathcal{U} = \{U_3 \cup U_4, U_5\}$ is a quasi-good cover of X, since $(U_3 \cup U_4) \cap U_5 = \{1\} \cup \{2\}$ has two contractible connected components. Despite the fact that $N(\mathcal{U})$ does not encodes the simple homotopy type of X, by of Corollary 4.1.9, we can deduce that X is simply equivalent to $M(\mathcal{U})$, which is actually the minimal model of S^1.

$N(\mathcal{U})$ $\qquad$ $M(\mathcal{U})$

Figure 4.5: The finite spaces $N(\mathcal{U})$ and $M(\mathcal{U})$ associated to a quasi-good cover $\mathcal{U}$ of X.

Theorem 4.1.4 and Corollary 4.1.9 have their counterparts in the context of complexes.

Theorem 4.1.11. *Let K be a finite simplicial complex (or a regular cell complex) and let $\mathcal{U} = \{L_i\}_{i\in I}$ be a finite family of subcomplexes of K such that $\bigcup_{i\in J} L_i = K$. Denote by $\mathcal{N}_0(\mathcal{U})$ the subcomplex of $\mathcal{N}(\mathcal{U})'$ whose simplices are the chains of contractible intersections. If for every $\sigma \in K$, the subcomplex S_σ of $\mathcal{N}_0(\mathcal{U})$ of chains with elements all containing σ is contractible, then K has the same simple homotopy type as $\mathcal{N}_0(\mathcal{U})$.*

Notice that if $\mathcal{U}$ is a covering of a simplicial complex K whose all nonempty intersections are contractible, then $\mathcal{N}_0(\mathcal{U}) = \mathcal{N}(\mathcal{U})' \diagup\!\!\searrow \mathcal{N}(\mathcal{U})$, and the contractibility of the complexes S_σ is trivially satisfied since they are cones with apex $\bigcap\limits_{\substack{L_i\in\mathcal{U}\\ L_i\ni\sigma}} L_i$. Thus, Nerve Theorem 4.1.1 is a particular case of Theorem 4.1.11.

Corollary 4.1.12. *Let K be a finite simplicial complex (or a regular cell complex) and let $\mathcal{U} = \{L_i\}_{i\in I}$ be a finite family of contractible subcomplexes of K such that $\bigcup_{i\in J} L_i = K$ and every nonempty intersection of a subfamily has contractible connected components. Then, $K \diagup\!\!\searrow \mathcal{M}(\mathcal{U})$, where $\mathcal{M}(\mathcal{U}) = \mathcal{K}(M(\mathcal{U}))$.*

4.2 Applications to $(n+1)$-deformations

The goal of this section is to find deformations with an upper bound on the dimensions of the involved spaces, by means of the Nerve Theorem.

A first preliminary remark is that if we replace the condition of "homotopically trivial" by "collapsible" in the statement of Theorem 4.1.4, we get that $X \nearrow B(\mathcal{R}) \searrow N_0(\mathcal{U})$, with $\mathcal{R}$ the relation defined in the proof of the theorem. However, the construction of $N_0(\mathcal{U})$ is not suitable to preserve low dimension.

We define the reduced version of the non-Hausdorff nerve, which fixes the redundancy of intersections $\mathcal{I}_J = \mathcal{I}_{J'}$ with $J \neq J'$.

Definition 4.2.1. Let X be a finite topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a finite cover of X by open subspaces. Define the *reduced non-Hausdorff nerve* $\tilde{N}(\mathcal{U})$ by the poset of all the non-empty finite spaces obtained as intersections of elements of the cover $\mathcal{U}$, ordered by inclusion.

Remark 4.2.2. Note that $N(\mathcal{U}) \searrow\!\!\searrow \tilde{N}(\mathcal{U})$. In fact, we can define an equivalence relation in the set of subsets J of I such that $I_J \neq \emptyset$. We say that $J \sim J'$ if $\mathcal{I}_J = \mathcal{I}_{J'}$. For every J, let J_0 be the union of all the elements in the equivalence class of J. In particular, $J_0 \sim J$. Thus, we can strong collapse

$$N(\mathcal{U}) \searrow\!\!\searrow N(\mathcal{U}) \smallsetminus \{\mathcal{I}_J : J \sim J_0, J \neq J_0\}$$

by removing successively the maximal elements of $\{\mathcal{I}_J : J \sim J_0, J \neq J_0\}$ since they are up beat points. The previous strong collapses can be done simultaneously for all the representatives J_0 of the equivalence classes.

Theorem 4.2.3. *Let X be a finite poset of height less or equal to n and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X such that $\tilde{N}_0(\mathcal{U})$, the subspace of $\tilde{N}(\mathcal{U})$ of collapsible intersections, is again of height less or equal to n. If for every $x \in X$, the subspace I_x of $\tilde{N}_0(\mathcal{U})$ of the intersections which contain x is collapsible and its height is less or equal than $n - h_X(x)$, then $X \diagup\!\!\searrow^{n+1} \tilde{N}_0(\mathcal{U})$.*

Proof. We follow the main idea of the proof of Theorem 4.1.4. Define the relation $\mathcal{R} \subseteq X \times \tilde{N}_0(\mathcal{U})^{op}$ as follows: for every $x \in X$ and $\mathcal{I} \in \tilde{N}_0(\mathcal{U})$, set

$$x \,\mathcal{R}\, \mathcal{I} \text{ if } x \in \mathcal{I}.$$

$\mathcal{R}$ satisfies hypothesis of Propositions 3.1.2 and 3.1.3. Thus, $X \nearrow B(\mathcal{R}) \searrow \tilde{N}_0(\mathcal{U})^{op}$. Note that $m = h(\tilde{N}_0(\mathcal{U})) \leq h(X) = n$ and $\tilde{N}_0(\mathcal{U})^{op} \diagup\!\!\searrow^{m+1} \tilde{N}_0(\mathcal{U})$. So we only have to see that $h(B(\mathcal{R})) \leq n+1$. A maximal chain in $B(\mathcal{R})$ is $c = x_1 < x_2 < \cdots < x_r < \mathcal{I}_1 < \mathcal{I}_2 < \cdots < \mathcal{I}_s$ with $x_1 < x_2 < \cdots < x_r$ a maximal chain in $U_{x_r} \subseteq X$, $\mathcal{I}_1$ a minimal element in the subspace of $\tilde{N}_0(\mathcal{U})^{op}$ of the intersections which contain x_r and $\mathcal{I}_1 < \mathcal{I}_2 < \cdots < \mathcal{I}_s$ a maximal chain in that subspace. Since, by hypothesis $s \leq n-r$, the length of c is less or equal than $r+1+n-r = n+1$. Therefore, $h(B(\mathcal{R})) \leq n+1$ and $X \diagup\!\!\searrow^{n+1} \tilde{N}_0(\mathcal{U})$. This completes the proof. □

Definition 4.2.4. A family $\mathcal{U}$ of open subsets of a finite space X is said a *collapsible quasi-good cover* if every nonempty intersection of a subfamily has collapsible connected components.

Given a finite space X and a quasi-good cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, denote by $\tilde{M}(\mathcal{U})$ the finite space whose elements are the topological spaces obtained as the connected components of $\bigcap_{i\in J} U_i$ for some $J \subseteq I$, ordered by inclusion.

Corollary 4.2.5. *Let X be a finite space of height n and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a collapsible quasi-good cover of X. If $h(\tilde{M}(\mathcal{U})) \leq n$ and the minimal connected component of the intersections of elements of $\mathcal{U}$ which contains an element $x \in X$ is of height less or equal than $h(x)$, then $X \diagup\!\!\searrow^{n+1} \tilde{M}(\mathcal{U})$.*

Example 4.2.6 (Dunce Hat)**.** Consider the regular structure H of the Dunce Hat described in Figure 4.1. Take the following cover $\mathcal{U} = \{U_{e_1} \cup U_{e_2}, U_{e_3} \cup U_{e_4}, U_{e_5}, U_{e_6}\}$ of $\mathcal{X}(H)$ by four collapsible open subspaces. Since $\mathcal{U}$ a collapsible quasi-good cover of $\mathcal{X}(H)$, by Corollary 4.1.9, $\mathcal{X}(H) \diagup\!\!\searrow M(\mathcal{U})$ (see Figure 4.6).

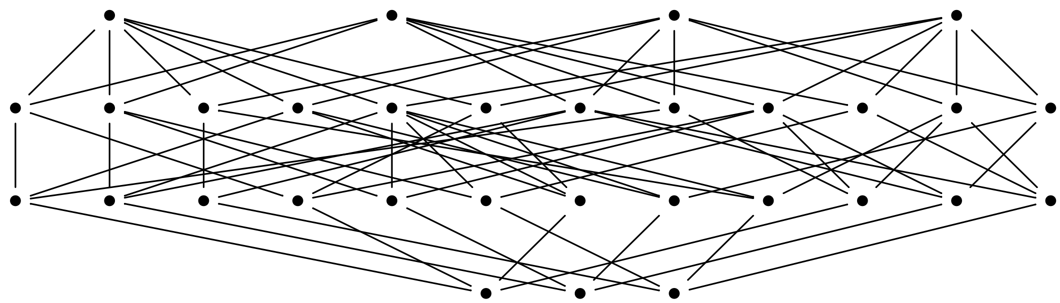


Figure 4.6: The finite space $M(\mathcal{U})$ associated to a quasi-good cover of the Dunce Hat.

Moreover, the cover $\mathcal{U}$ also satisfies the hypothesis on the height imposed in Corollary 4.2.5. Thus, $\mathcal{X}(H) \nearrow\searrow^{3} \tilde{M}(\mathcal{U})$, and the latter again collapsible. Finally, $H \nearrow\searrow^{3} H' \nearrow\searrow^{3} \mathcal{K}(\tilde{M}(\mathcal{U})) \searrow *$. This shows that H 3-deforms to a single point.

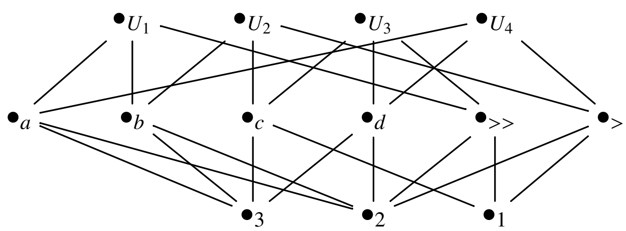


Figure 4.7: The finite space $\tilde{M}(\mathcal{U})$ associated to a collapsible quasi-good cover $\mathcal{U}$ of the Dunce Hat.

Resumen del capítulo 4: El Teorema del Nervio

Sea $\mathcal{U} = \{U_i\}_{i\in I}$ una familia de subconjuntos de un conjunto X. El *nervio* de $\mathcal{U}$ es el complejo simplicial $\mathcal{N}(\mathcal{U})$ cuyos símplices son los subconjuntos finitos $J \subseteq I$ tales que $\bigcap_{i\in J} U_i \neq \emptyset$.

El Teorema del Nervio, probado por Borsuk [Bor48], garantiza que el nervio de una familia de espacios tiene el mismo tipo homotópico que la unión de los espacios si éstos forman un *cubrimiento bueno*, es decir, si la intersección de toda subfamilia es vacía o contráctil.

Teorema 4.1.1 (Teorema del Nervio)**.** Sea K un complejo simplicial finito, (o complejo celular regular) y sea $\mathcal{U} = \{L_i\}_{i\in I}$ una familia finita de subcomplejos de K tales que $\bigcup_{i\in J} L_i = K$. Si toda intersección de elementos de $\mathcal{U}$ es vacía o contráctil, entonces K tiene el mismo tipo homotópico (simple) que $\mathcal{N}(\mathcal{U})$.

El Teorema del Nervio es equivalente a la siguiente formulación para espacios finitos. Dado $\mathcal{U}$ un cubrimiento finito por abiertos X, el *nervio no-Hausdorff de* $\mathcal{U}$ es el face poset del nervio clásico $\mathcal{N}(\mathcal{U})$. Lo denotamos por $N(\mathcal{U})$.

Corolario 4.1.2. Si X es un espacio topológico finito y $\mathcal{U} = \{U_i\}_{i\in I}$ es un cubrimiento abierto de X tal que toda intersección de elementos de $\mathcal{U}$ es vacía u homotópicamente trivial, entonces $X \diagup\!\!\searrow \mathcal{X}(\mathcal{N}(\mathcal{U}))$.

El siguiente resultado es una generalización del teorema anterior, en el cual permitimos al cubrimiento tener intersecciones no vacías no contráctiles.

Teorema 4.1.4. Sea X un espacio topológico finito y sea $\mathcal{U} = \{U_i\}_{i\in I}$ un cubrimiento abierto de X. Sea $N_0(\mathcal{U})$ el subespacio del nervio no-Hausdorff $N(\mathcal{U})$ de todas las intersecciones homotópicamente triviales. Si para todo $x \in X$, el subespacio $\mathcal{I}_x \subseteq N_0(\mathcal{U})$ de las intersecciones que contienen a x es homotópicamente trivial, entonces $X \diagup\!\!\searrow N_0(\mathcal{U})$.

Decimos que una familia $\mathcal{U}$ de subespacios de un espacio finito X es un *cubrimiento cuasi-bueno* si toda intersección no vacía de una subfamilia de $\mathcal{U}$ tiene sus componentes conexas homotópicamente triviales. El siguiente resultado que a partir de cubrimientos quasi-buenos también se pueden extraer modelos finitos de un espacio.

Denotamos por $M(\mathcal{U})$ al espacio finito cuyos elementos son las componentes conexas de cada intersección de elementos de $\mathcal{U}$, con el orden dado por la inclusión.

Corolario 4.1.9. Sea X un espacio finito y sea $\mathcal{U}$ un cubrimiento cuasi-bueno de X. Entonces, $X \diagup\!\!\searrow M(\mathcal{U})$.

Por otra parte, realizamos aplicaciones de los resultados anteriores al estudio de $(n+1)$-deformaciones. Para optimizar la altura del modelo finito que se puede obtener a partir de un cubrimiento $\mathcal{U}$, definimos $\tilde{N}(\mathcal{U})$, el *nervio no-Hausdorff reducido*. Se puede ver que $\tilde{N}(\mathcal{U}) \searrow\!\!\searrow N(\mathcal{U})$.

Definición 4.2.1. Sea X un espacio topológico finito y sea $\mathcal{U} = \{U_i\}_{i\in I}$ un cubrimiento finito de X por subespacios finitos. Definimos el *nervio no-Hausdorff reducido* $\tilde{N}(\mathcal{U})$ como el poset de todos los espacios finitos no vacíos obtenidos como intersección de elementos del cubrimiento $\mathcal{U}$, ordenados por inclusión.

Probamos los siguientes resultados.

Teorema 4.2.3. Sea X un espacio finito de altura menor o igual que n, y sea $\mathcal{U} = \{U_i\}_{i\in I}$ un cubrimiento abierto de X tal que $\tilde{N}_0(\mathcal{U})$, el subespacio de $\tilde{N}(\mathcal{U})$ de intersecciones colapsables, es también de altura menor o igual que n. Si para cada $x \in X$, el subespacio I_x de $\tilde{N}_0(\mathcal{U})$ de las intersecciones que contienen a x es colapsable y de altura menor o igual que $n - h_X(x)$, entonces $X \nearrow\searrow^{n+1} \tilde{N}_0(\mathcal{U})$.

Dados X un espacio finito y $\mathcal{U} = \{U_i\}_{i\in I}$ un cubrimiento cuasi-bueno de X, denotamos por $\tilde{M}(\mathcal{U})$ al espacio finito cuyos elementos son los espacios topológicos obtenidos como las componentes conexas de $\bigcap_{i\in J} U_i$ para algún $J \subseteq I$, ordenadas por inclusión.

Corolario 4.2.5. Sean X un espacio finito de altura n y $\mathcal{U} = \{U_i\}_{i\in I}$ un cubrimiento cuasi-bueno de X cuyas intersecciones no vacías son colapsables. Si $h(\tilde{M}(\mathcal{U})) \leq n$ y las componentes conexas minimales de intersecciones de elementos de $\mathcal{U}$ que contienen un elemento $x \in X$ tienen altura menor o igual que $h(x)$, entonces $X \nearrow\searrow^{n+1} \tilde{M}(\mathcal{U})$.

Ejemplo 4.2.6 (Dunce Hat)**.** Considerar la estructura regular H para el Dunce Hat descripta en la Figura 4.8.

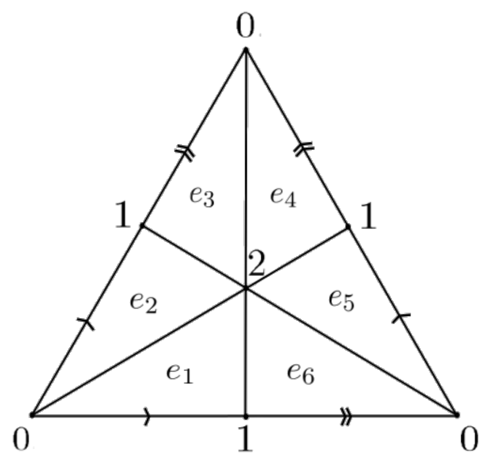


Figure 4.8: Estructura celular regular para el Dunce Hat.

Tomar el siguiente cubrimiento $\mathcal{U} = \{U_{e_1} \cup U_{e_2}, U_{e_3} \cup U_{e_4}, U_{e_5}, U_{e_6}\}$ de $\mathcal{X}(H)$ por cuatro subespacios abiertos colapsables. Dado que $\mathcal{U}$ es un cubrimiento cuasi-bueno colapsable, por Corolario 4.1.9, $\mathcal{X}(H) \nearrow\searrow M(\mathcal{U})$ (ver Figure 4.9).

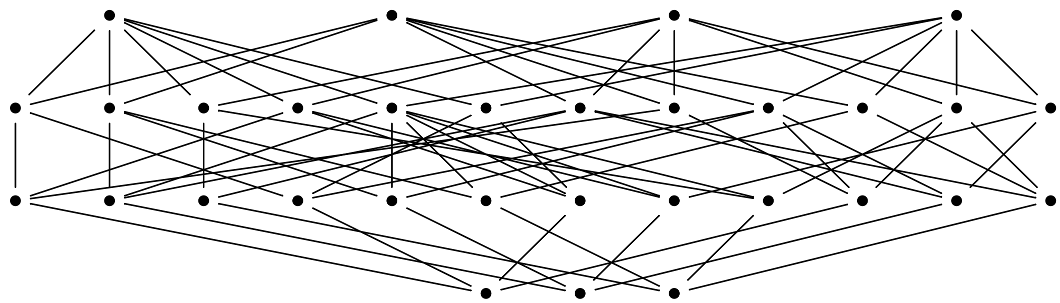


Figure 4.9: El espacio finito $M(\mathcal{U})$ asociado a un cubrimiento cuasi-bueno del Dunce Hat.

Más aún, el cubrimiento $\mathcal{U}$ también satisface las hipótesis de altura de del Corolario 4.2.5. Luego, $\mathcal{X}(H) \nearrow\searrow^{3} \tilde{M}(\mathcal{U})$. El último espacio es también colapsable. Finalmente, $H \nearrow\searrow^{3} H' \nearrow\searrow^{3} \mathcal{K}(\tilde{M}(\mathcal{U})) \searrow *$. Esto prueba que H se 3-deforma a un punto.

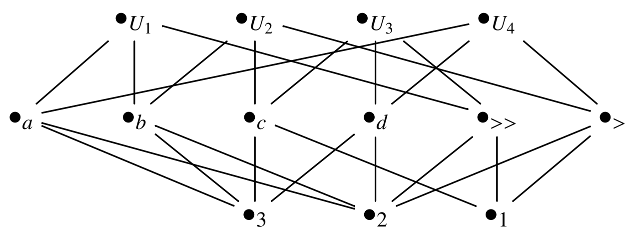


Figure 4.10: El espacio finito $\tilde{M}(\mathcal{U})$ asociado a un cubrimiento cuasi-bueno $\mathcal{U}$ del Dunce Hat.

Chapter 5

Homotopy colimits and deformations

5.1 Homotopy colimits

Limits and colimits are fundamental constructions in mathematics. They naturally arise as a generalization of well-known concepts as cartesian products, disjoint unions and direct sums. However, when the category in question has a "homotopy theory" limits and colimits are generally not homotopically well-behaved. For example, in general they are not invariant under homotopy equivalences. Thus, it is convenient to obtain a replacement with better properties, usually called a "homotopy limit" and "homotopy colimit".

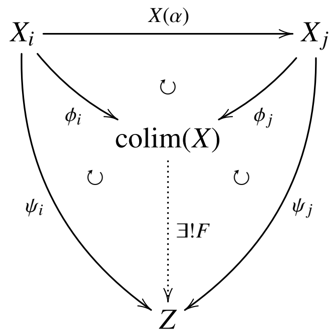
We are interested in studying *the homotopy type of the colimit of diagrams of topological spaces*.

Let I be a small category and $\mathcal{C}$ any category. Recall that an *I-diagram* in a category $\mathcal{C}$ is just a functor $X : I \to \mathcal{C}$. The category I is called the *indexing category* and gives the shape to the diagram. The actual objects and morphisms of I are irrelevant, but only the way they are interrelated is meaningful. Thus, a diagram X can be thought of as a collection $\{X(i)\}_{i\in I}$ of objects and morphisms $\{X(\alpha) : X(i) \to X(j)\}_{\alpha : i \to j}$ in $\mathcal{C}$ patterned on I. For simplicity, denote $X_i := X(i)$.

The *colimit* of a diagram $X : I \to \mathcal{C}$ (if it exists) is an object $\operatorname{colim} X$ in $\mathcal{C}$ equipped with morphisms $\phi_i : X_i \to \operatorname{colim} X$ for every $i \in I$ such that $\phi_j X(\alpha) = \phi_i$ for each $\alpha : i \to j$ morphism in I,

$$\begin{array}{ccc} X_i & \xrightarrow{X(\alpha)} & X_j \\ & \circlearrowleft & \\ {\scriptstyle\phi_i}\searrow & & \swarrow{\scriptstyle\phi_j} \\ & \operatorname{colim}(X) & \end{array}$$

and satisfies the following universal property: for every $Z \in \mathcal{C}$ together with morphisms $\psi_i : X_i \to Z$ for every $i \in I$ such that $\psi_j X(\alpha) = \psi_i$ for each $\alpha : i \to j$ morphism in I, there exists a unique morphism $F : \mathrm{colim}(X) \to Z$ such that $F\phi_i = \psi_i$ for every $i \in I$.

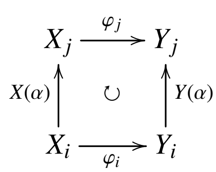


If $\mathcal{C}$ is the category of topological spaces, then $\mathrm{colim}\, X$ always exists and it is homeomorphic to

$$\bigsqcup_{j \in \mathcal{J}} X_j / \sim$$

where $\sim$ is the equivalence relation generated by $x \in X_i \sim X(\alpha)(x) \in X_j$ for every $\alpha : i \to j$ morphism in I. For example, if the indexing category is depicted by $0 \to 1$, then colim(X) of the diagram $X_0 \xrightarrow{f} X_1$ is isomorphic to X_1. If, instead, we index on the "push out category", with three objects and two non-identity maps: $1 \leftarrow 0 \to 2$, then colim(X) is precisely the push out of the diagram $X_1 \xleftarrow{f} X_0 \xrightarrow{g} X_2$, which is homeomorphic to $X_1 \sqcup X_2 / \sim$, where $f(x) \sim g(x)$ for every $x \in X_0$. In particular, if $X_2 = \{*\}$, then $\mathrm{colim}\, X$ is homeomorphic to $X_1 / f(X_0)$.

Given $X, Y : I \to \mathcal{C}$ two I-diagrams in $\mathcal{C}$, a *morphism of diagrams* $\varphi : X \to Y$ is just a natural transformation from X to Y. That is, for every $j \in I$, there is a map $\phi_j : X_j \to Y_j$ such that if $\alpha : i \to j$ is a morphism in I, then the following diagram commutes.



Notice that if X and Y are diagrams of topological spaces, a morphism $\varphi : X \to Y$ induces a continuous map $\hat{\varphi} : \mathrm{colim}\, X \to \mathrm{colim}\, Y$. But if $\varphi_i : X_i \to Y_i$ is a homotopy equivalence for every $i \in I$, unfortunately it does not follow in general that $\hat{\varphi}$ is a homotopy equivalence. Here is an example: let X, Y be the diagrams

$$\begin{aligned} D^n \leftarrow S^{n-1} \to D^n, \\ * \leftarrow S^{n-1} \to * \end{aligned} \tag{5.1}$$

respectively, indexed on $1 \leftarrow 0 \to 2$, with $n \geq 2$. Let $\varphi : X \to Y$ be the natural transformation which is the identity on S^{n-1} and collapses D^n to a point. Then $\mathrm{colim}\, X \simeq S^n$, but $\mathrm{colim}\, Y \simeq *$.

The problem here is that there are natural transformations which are objectwise homotopy equivalences, but the homotopy inverses of its components may not fit together into a natural transformation. So the colimit functor does not necessarily preserve homotopy equivalences, that is, it is not a *homotopy invariant*.

The homotopy colimits arise because of the previous basic difficulty, as a "correction" of the colimit to be homotopy invariant. There are simple examples of homotopy colimit constructions, which had been widely used, even before the development of the theory. For instance, the homotopy colimit of a diagram of topological spaces $X_0 \xrightarrow{f} X_1$ with indexing category $0 \to 1$ is precisely (homeomorphic to) the *mapping cylinder* of f, which deformation retracts to X_1. On the other hand, the homotopy colimit of a diagram indexed on the pushout category $1 \leftarrow 0 \to 2$ is the so called *homotopy pushout*, which is constructed by gluing together X_1 and X_2 "up to homotopy". Specifically, $\operatorname{hocolim} X \simeq X_1 \sqcup (X_0 \times I) \sqcup X_2 / \sim$, where $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for every $x \in X_0$. In particular, if $f : A \to X$ is a map and the diagram is $* \leftarrow A \to X$, then the homotopy pushout is homeomorphic to the mapping cone of f.

In the example of the diagrams X, Y described in (5.1) it is easy to check that $\operatorname{hocolim} X \simeq \operatorname{hocolim} Y \simeq S^n$.

There are two things to point out from the previous examples, which have been mimicked in the general theory. On the one hand, whereas the colimit of a diagram of spaces is constructed by merely gluing them, the homotopy colimit is obtained by gluing them *up to homotopy*, being a "fattened up" version of the classic colimit. On the other hand, note that there is a map $\operatorname{hocolim} X \to \operatorname{colim} X$ obtained by collapsing the homotopies. Under some conditions, this map is a homotopy equivalence.

The construction is due to Bousfield & Kan [BK72] and, independently, to Vogt [Vog73]. We will only work with homotopy colimit indexed by (finite) posets. In this case, the homotopy colimits are easier to define.

Let P be a finite a poset, viewed as a small category with a unique arrow $p \to q$ for each $p, q \in P$ such that $p \leq q$, and let $X : P \to \mathrm{Top}$ be a P-diagram of topological spaces. Recall that if $p \in P$, $F_p = \{q \in P : q \geq p\}$. The homotopy colimit of X, $\operatorname{hocolim} X$, is defined by

$$\bigsqcup_{p \in P} |\mathcal{K}(F_p)| \times X_p / \sim$$

where $\sim$ is the following equivalence relation. For every $p \leq q \in P$, we have a continuous map $f_{pq} : X_p \to X_q$ and an inclusion $i : |\mathcal{K}(F_q)| \to |\mathcal{K}(F_p)|$. The relation $\sim$ is generated by $(\alpha, f_{p,q}(x)) \in |\mathcal{K}(F_q)| \times X_q \sim (i(\alpha), x) \in |\mathcal{K}(F_p)| \times X_p$, for every $x \in X_p$, $\alpha \in |\mathcal{K}(F_q)|$.

Let I be a small category and let $X : I \to \mathrm{CAT}$ be a functor to the category CAT of small categories. Recall that the *Grothendieck construction* on X, which is usually denoted by $I \int X$, is the following category. The objects are the pairs (i, x), where i is an object of I and x is a object of $X(i)$, and the morphisms $(\alpha, \beta) : (i, x) \to (i', x')$ are given by morphisms $\alpha : i \to i'$ in I and $\beta : X(\alpha)(x) \to x'$ in $X(i')$.

Now, given any small category I, its nerve $\mathcal{N}(I)$ is a simplicial set whose n-simplices are the sequences of n composable morphisms in I. The geometric realization of this simplicial set is a topological space, called the classifying space of the category I.

Thomason's theorem [Tho79] establishes the existence of a natural homotopy equivalence between the nerve of the Grothendieck construction and the homotopy colimit of $\mathcal{N}(X)$ (here

$\mathcal{N}(X)$ is the diagram of simplicial sets given by the composition $I \xrightarrow{X} \mathrm{CAT} \xrightarrow{\mathcal{N}} \mathcal{SS}$ of X with the nerve functor).

Theorem 5.1.1 (Thomason)**.** *Let I be a small category and* CAT *the category of small categories. If $X : I \to$ CAT is an I-diagram of small categories, then there is a natural homotopy equivalence*

$$\operatorname{hocolim} \mathcal{N}(X) \to \mathcal{N}(I \int X).$$

In particular, this result says that the homotopy colimit of a diagram of topological spaces combinatorially described as the nerve of certain small categories has itself a combinatorial description as the nerve of a single category (namely, the Grothendieck construction of the diagram).

5.2 The Grothendieck construction on posets and non-Hausdorff homotopy colimits

Working with diagrams over finite posets will allow us to apply combinatorial methods to study their homotopy colimits. We also refer the reader to [Zv93] and [WZv99] for applications of homotopy colimits to combinatorial problems.

Let P be a finite poset and let X be a *P-diagram of finite posets*, i.e. a functor from P to the category $\mathcal{P}_{<\infty}$ of finite posets. In this case the Grothendieck construction on X can be described as a poset as follows.

Definition 5.2.1 (The non-Hausdorff homotopy colimit of finite posets)**.** Let $X : P \to \mathcal{P}_{<\infty}$ be a functor. The *non-Hausdorff homotopy colimit of X*, denoted by $\underline{\operatorname{hocolim}}\, X$, is the following poset. The underlying set is the disjoint union $\coprod_{p\in P} X_p$. We keep the given ordering within X_p for all $p \in P$, and for every $x \in X_p$ and $y \in X_q$ such that $p \leq q$, we set $x \leq y$ in $\underline{\operatorname{hocolim}}\, X$ if $f_{pq}(x) \leq y$ in X_q. Here $X_p = X(p)$ for each $p \in P$ and $f_{pq} = X(p \to q)$ for each $p \leq q$ in P.

By Thomason's theorem we have a homotopy equivalence

$$\mathcal{K}(\underline{\operatorname{hocolim}}\, X) \simeq \operatorname{hocolim}\ \mathcal{K}X,$$

since the nerve of a poset viewed as a small category is in fact its order complex.

In the context of finite posets, Thomason's theorem can be deduced from a more general result. This will be proved in Theorem 5.4.3.

Notice that the homotopy colimit of diagrams of finite posets can be thought as an iterated construction of the cylinder of a relation.

Lemma 5.2.2. *Let P be a finite poset and let X be a P-diagram of finite posets. Let $p_1, p_2, \cdots, p_n$ be a linear extension of P (i.e., an ordering of the elements of P such that if $p_i \leq p_j$, then $i \leq j$). Denote $X_i = X_{p_i}$ for all $1 \leq i \leq n$. Define $B_0 = X_0$ and, inductively, $B_i = B(R_i)$ for all $1 \leq i \leq n$, where $R_i \subseteq B_{i-1} \times X_i$ is the relation defined by $xR_i f_{pp_i}(x)$ for all $p < p_i \in P$, for each $x \in X_p$. Then, $\underline{\operatorname{hocolim}}\, X = B_n$.*

Remark 5.2.3. At this point it is worth noting the difference between $\underline{\mathrm{hocolim}}\, X$ and $\mathrm{hocolim}\, X$ for a given diagram $X : P \to \mathcal{P}_{<\infty}$. The first one is the Grothendieck construction on X and it is a finite poset (which can be viewed as a finite topological space). The second one is the classical construction of homotopy colimit of a diagram of topological spaces (applied, in this case, to a diagram of finite topological spaces) and it is not a finite space. However, as an immediate consequence of Theorem 5.4.3, we will see that they are weakly equivalent spaces (when we view $\underline{\mathrm{hocolim}}\, X$ as a finite topological space).

A very first and important example of non-Hausdorff homotopy colimit is the non-Hausdorff mapping cylinder.

Example 5.2.4. Any map $f : X_0 \to X_1$ between finite posets can be viewed as a diagram $X : \mathbf{1} \to \mathcal{P}_{<\infty}$ where $\mathbf{1}$ is the poset of two elements $0 < 1$. Similarly as in the topological context, we have $\underline{\mathrm{hocolim}}\, X = B_f$, the non-Hausdorff mapping cylinder of f.

Now we will reinterpret some well-known results about homotopy colimits reinterpreted in the non-Hausdorff setting (see [BK72, Seg68, tD71, Vog73]). We give direct and elementary proofs of their statements in this context.

Recall that given X a P-diagram, $\mathrm{colim}\, X = \bigcup_{p\in P} X_p/\sim$, where $x \sim f_{pp'(x)}$ for all $x \in X_p$ and $p \leq p'$ in P. Note that $\mathrm{colim}\, X$ is not always a poset. A necessary and sufficient condition is that if $x, y \in X_p$ are non-comparable and $f_{pp'}(x) < f_{pp'}(y)$ for some $p' > p$, then for all $p'' > p$, either $f_{pp''}(x) < f_{pp''}(y)$ or they are non-comparable.

The following lemma is a version of the Projection Lemma **??** for diagrams over posets.

Lemma 5.2.5 (Projection Lemma)**.** *Let P be a finite poset and $\mathcal{A}$ a finite collection of open subposets of P, ordered by inclusion. Let $X : (\mathcal{A}, \subseteq) \to \mathcal{P}_{<\infty}$ be the associated diagram, defined by $X(A) = A$ for all $A \in \mathcal{A}$, and $X(A \leq A')$ the inclusion map $\iota_{AA'}$. If for every $A, A' \in \mathcal{A}$, either $A \cap A' = \varnothing$ or $A \cap A'$ is union of elements of $\mathcal{A}$, then* $\underline{\mathrm{hocolim}}\, X = \bigcup_{A\in\mathcal{A}} A$ *(viewed as a subposet of P) and* $\underline{\mathrm{hocolim}}\, X \searrow\!\!\searrow \mathrm{colim}\, X$.

Proof. Let $A_1, A_2, \cdots, A_n$ be a linear extension of $(\mathcal{A}, \subseteq)$. For simplicity, if $A_j \subseteq A_i$, denote by ι_{ji} the inclusion map.

Define inductively $Y_0 = \underline{\mathrm{hocolim}}\, X$, $Y_i = Y_{i-1} \smallsetminus \left(\bigcup_{A_j\subseteq A_i} \iota_{ji}(A_j)\right)$ for all $1 \leq i \leq n$. Note that $\mathrm{colim}\, X$ (which is homeomorphic to Y_n) is a subposet of $\underline{\mathrm{hocolim}}\, X$ $(= Y_0)$. We will show inductively that $Y_i \searrow\!\!\searrow Y_{i+1}$ for all $0 \leq i \leq n-1$.

Suppose $\{A_j : j < i\}$ are disjoints, and iterate the following reasoning. Take $x \in A_i$ a minimal element of $\bigcup_{A_j\subseteq A_i} \iota_{ji}(A_j) \subseteq A_i$. Then, there exists a unique $j < i$ such that $x \in A_j$, and $\hat{U}_x^{Y_{i-1}} = U_{\iota_{ji}^{-1}(x)}^{Y_{i-1}}$ has maximum element. Hence $Y_{i-1} \searrow\!\!\searrow^{e} Y_{i-1} \smallsetminus \{x\}$. □

The Homotopy Lemma **??** is actually the statement that guarantees that homotopic colimits are invariant under homotopies.

Lemma 5.2.6 (Homotopy Lemma)**.** *Let P be a finite poset and let $X, Y : P \to \mathcal{P}_{<\infty}$ be two P-diagrams of finite posets. If $\alpha : X \to Y$ is a morphism of diagrams such that $\alpha_p : X_p \to Y_p$ is a weak equivalence for all $p \in P$, then α induces a weak equivalence*

$$\underline{\mathrm{hocolim}}\, X \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, Y.$$

The proof of the Homotopy Lemma can be deduced from the original Homotopy Lemma for diagrams of spaces and Thomason's theorem. However, we will show a direct proof in Remark 5.3.3 using combinatorial methods.

The following result is a reformulation of the Wedge Lemma [Zv93].

Lemma 5.2.7 (Wedge Lemma)**.** *Let $X : P \to \mathcal{P}_{<\infty}$ be P-diagram of finite posets. If for each $p \in P$, there exists $c_p \in X_p$ such that $f_{qp}(x) = c_p$ for all $x \in X_{p'}$ and $p' \leq p$ in P, then*

$$\underline{\operatorname{hocolim}}\, X \underset{we}{\simeq} P \vee \bigvee_{p\in P} X_p * \hat{F}_p,$$ [1]

where the wedge is formed by identifying $c_p \in X_p * \hat{F}_p$ with $p \in P$.

Björner, Wachs and Welker [BWW05] deduced a generalization of Quillen's Fiber Lemma 2.1.10 for maps not satisfying that the preimage of the minimal open sets is always homotopically trivial. We translate this result into the poset setting. The idea of the proof is the same as the original.

Theorem 5.2.8 (Björner, Wachs, Welker)**.** *Let $f : P \to Q$ be an order preserving map between finite posets such that for all $q \in Q$, the fiber $f^{-1}(U_q)$ is non-empty, and for all non-minimal $q \in Q$ the inclusion map $f^{-1}(\hat{U}_q) \hookrightarrow f^{-1}(U_q)$ is weak homotopic to a constant map which sends $f^{-1}(\hat{U}_q)$ to c_q for some $c_q \in f^{-1}(U_q)$. Then*

$$P \underset{we}{\simeq} Q \vee \bigvee_{q\in Q} f^{-1}(U_q) * \hat{F}_q,$$

where the wedge is formed by identifying each $q \in Q$ with c_q.

Proof. Let $\mathcal{A}$ be the collection of subposets of P given by $\{f^{-1}(U_q) : q \in Q\}$, and let $X : Q \to \mathcal{P}_{<\infty}$ be the associated diagram. By the Projection Lemma 5.2.5, $\underline{\operatorname{hocolim}}\, X \searrow\!\!\searrow \operatorname{colim} X \equiv P$. On the other hand, define $Y : Q \to \mathcal{P}_{<\infty}$ by $Y(q) = f^{-1}(U_q)$ with $Y(q' \leq q)$ the constant map c_q. By the Homotopy Lemma 5.2.6 for the diagrams $\mathcal{K}(X)$ and $\mathcal{K}(Y)$, $\mathcal{K}(X) \simeq \mathcal{K}(Y)$, and then $\underline{\operatorname{hocolim}}\, X \underset{we}{\simeq} \underline{\operatorname{hocolim}}\, Y$. Finally, by the Wedge Lemma 5.2.7,

$$\underline{\operatorname{hocolim}}\, Y \underset{we}{\simeq} Q \vee \bigvee_{q\in Q} f^{-1}(U_q) * \hat{F}_q.$$

We will show now that $P \simeq \operatorname{colim} X \nearrow\!\!\nearrow \underline{\operatorname{hocolim}}\, X$. The idea is similar to the proof of the Projection Lemma 5.2.5. Let $q_1, q_2, \cdots, q_n$ be a linear extension of Q. Define $Y_0 = \underline{\operatorname{hocolim}}\, X$, and inductively $Y_{i+1} = Y_i \smallsetminus f^{-1}(\hat{U}_{q_i})$, where $f^{-1}(\hat{U}_{q_i})$ is thought as a subspace of $f^{-1}(U_{q_i})$. Note that $P \simeq Y_n = \bigcup_{q\in Q} f^{-1}(\{q\}) \subseteq Y_0$.

To see that $Y_i \searrow\!\!\searrow Y_{i+1}$, take x a minimal element of $f^{-1}(\hat{U}_{q_i}) \subseteq f^{-1}(U_{q_i})$, and suppose $X_{q_j} = f^{-1}(q_j)$ for all $j \leq i$. Since $f^{-1}(\hat{U}_{q_i}) = \bigsqcup_{q_j \leq q_i} f^{-1}(q_j)$, $\hat{U}_x^{Y_i}$ has maximal element. □

[1] The *(non-Hausdorff) join* of two finite spaces X and Y is the disjoint union $X \sqcup Y$ keeping the given order within X and Y and setting $x \leq y$ for every $x \in X$ and $y \in Y$.

5.3 Methods of reduction for non-Hausdorff homotopy colimits

We will use reduction methods to derive old and new results on homotopy colimits, which will allow us to compute in a direct and easy way the homotopy type of homotopy colimits of diagrams over posets.

As we note in Chapter 3, if $f : X \to Y$ is a map of posets, the non-Hausdorff mapping cylinder B_f strong collapses to Y. This result will be viewed as a particular case of Corollary 5.3.2 below: if the indexing poset P has maximum p, the homotopy colimit of any P-diagram collapses to X_p.

On the other hand, we also note in Proposition 3.0.1 that if $f : X \to Y$ is a poset map such that $f^{-1}(U_y)$ is weakly contractible for every $y \in Y$, then f is a weak equivalence (i.e. it induces a homotopy equivalence between the classifying spaces). This is Quillen's Theorem A for posets [Qui73] and McCord's theorem [McC66] for finite topological spaces. In Proposition 5.3.5 we will generalize this result for homotopy colimits of P-diagrams. Moreover, Quillen's Theorem A for posets will follow immediately from Propositions 5.3.5 and 5.3.1 by applying the results to the poset **1**.

Let $X : P \to \mathcal{P}_{<\infty}$ be a P-diagram of finite posets. The following statement relates the homotopy type of P with the homotopy type of $\underline{\operatorname{hocolim}} X$. Notice that $P \searrow\!\!\searrow \{p\}$ does not imply that $\underline{\operatorname{hocolim}} X \searrow\!\!\searrow X_p$. For example, the homotopy pushout is indexed on the pushout category P defined by $1 \leftarrow 0 \to 2$, which collapses to 1 for example. Nevertheless, $\underline{\operatorname{hocolim}} X$ not necessarily is homotopic to X_1. See the example of the diagram $D^n \leftarrow S^{n-1} \to D^n$, whose homotopy colimit is homotopic to S^n. Now, if make a distinction between up and down beat points during the collapses, we can deduce collapses on the homotopy colimit.

Proposition 5.3.1. *Let $X : P \to \mathcal{P}_{<\infty}$ be a P-diagram of finite posets. If $p \in P$ is an up beat point, then $\underline{\operatorname{hocolim}} X \searrow\!\!\searrow \underline{\operatorname{hocolim}} X|_{P \smallsetminus \{p\}}$. In particular, they are weak equivalent.*

Proof. Let $x_0, x_1, \cdots, x_n$ be a linear extension of X_p^{op}, the opposite poset of X_p. Let $Y_0 = \underline{\operatorname{hocolim}} X$ and for each $0 \le i \le n$ define inductively $Y_{i+1} = Y_i \smallsetminus \{x_i\}$. Let q be the minimum element of $\hat{F}_p^P$. Note that $\hat{F}_{x_i}^{Y_i} = F_{f_{pq}(x_i)}^{Y_i}$. This implies that x_i is an up beat point of Y_i for all $1 \le i \le n$ and therefore

$$\underline{\operatorname{hocolim}} X = Y_0 \searrow\!\!\searrow Y_1 \searrow\!\!\searrow \cdots \searrow\!\!\searrow Y_{n+1} = (\underline{\operatorname{hocolim}} X) \smallsetminus X_p = \underline{\operatorname{hocolim}} X|_{P \smallsetminus \{p\}}.$$

□

Corollary 5.3.2. *Let $X : P \to \mathcal{P}_{<\infty}$ be P- diagram of finite posets. If P has a maximum element p, then $\underline{\operatorname{hocolim}}(X) \searrow\!\!\searrow X_p$. In particular, they are weak equivalent.*

Proof. Let $p = p_0, p_1, \cdots, p_n$ be a linear extension of P^{op}. Since P has a maximum element, there is a sequence of collapses

$$P \searrow\!\!\searrow P \smallsetminus \{p_1\} \searrow\!\!\searrow P \smallsetminus \{p_1, p_2\} \cdots \searrow\!\!\searrow P \smallsetminus \{p_1, p_2, \cdots, p_n\} = \{p\},$$

where p_i is an up beat point of $P \smallsetminus \{p_1, p_2, \cdots, p_{i-1}\}$. By applying recursively Proposition 5.3.1, we have

$$\underline{\mathrm{hocolim}}\, X \searrow\!\!\searrow (\underline{\mathrm{hocolim}}\, X) \smallsetminus X_{p_1} \searrow\!\!\searrow \cdots \searrow\!\!\searrow (\underline{\mathrm{hocolim}}\, X) \smallsetminus \bigcup_{i=1}^{n} X_{p_i} = X_p.$$

□

Remark 5.3.3. From the previous result and McCord's Theorem 2.1.9 we deduce an alternative proof of the Homotopy Lemma 5.2.6. For any $p \in P$ consider $\underline{\mathrm{hocolim}}\, X|_{U_p}$ the homotopy colimit of the diagram restricted to U_p. Since p is the maximum of U_p, by Proposition 5.3.2 $\underline{\mathrm{hocolim}}\, X|_{U_p} \searrow X_p$ and therefore α induces a weak equivalence $\underline{\mathrm{hocolim}}\, X|_{U_p} \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, Y|_{U_p}$. Now the result follows from McCord's Theorem 2.1.9 applied to the basis-like open cover $\{\underline{\mathrm{hocolim}}\, Y|_{U_p}\}_{p \in P}$ of $\underline{\mathrm{hocolim}}\, Y$.

Remark 5.3.4. Note that all the collapses in Proposition 5.3.1 and Corollary 5.3.2 are strong collapses, in the sense that all the points removed are beat points (not just weak points). This implies that if the finite space X_p in Corollary 5.3.2 is contractible then so is $\underline{\mathrm{hocolim}}\, X$.

Proposition 5.3.5. *Let $X : P \to \mathcal{P}_{<\infty}$ be a P- diagram of finite posets. If p is a down beat point of P dominated by an element q and $f_{qp}^{-1}(U_x)$ is contractible for every $x \in X_p$, then $\underline{\mathrm{hocolim}}(X) \searrow \underline{\mathrm{hocolim}}(X|_{P \smallsetminus \{p\}})$. In particular they are weak equivalent.*

Proof. Let $x_0, x_1, \cdots, x_n$ a linear extension of X_p. Define $Y_0 = \underline{\mathrm{hocolim}}\, X$, and inductively $Y_{i+1} = Y_i \smallsetminus \{x_i\}$ for every $0 \leq i \leq n$. We will show that $Y_0 \searrow Y_1 \searrow \cdots \searrow Y_{n+1} = (\underline{\mathrm{hocolim}}\, X) \smallsetminus X_p$.

Note that $\hat{U}_{x_i}^{Y_i} = \underline{\mathrm{hocolim}}\, \tilde{X}^i$, where $\tilde{X}^i : \hat{U}_p^P \to \mathcal{P}_{<\infty}$ is the functor defined by $\tilde{X}^i(p') = f_{p'p}^{-1}(U_{x_i})$, for all $p' < p$ in P (where the transition maps are induced by the original transition maps).

Since $\hat{U}_p^P$ has a maximum element q and $f_{qp}^{-1}(U_{x_i})$ is contractible, by Corollary 5.3.2 and Remark 5.3.4, $\hat{U}_{x_i}^{Y_i}$ is a contractible finite space. This proves that x_i is a weak point of Y_i. □

Given a poset map $\phi : P \to Q$ and a Q-diagram X, we denote by $\phi^* X$ the P-diagram obtained by pulling back X along ϕ. Concretely, $\phi^* X = X\phi$. There is a canonical map $\underline{\mathrm{hocolim}}\, \phi^* X \to \underline{\mathrm{hocolim}}\, X$ induced by the identities $(\phi^* X)_p = X_{\phi(p)}$. If $i : Q' \to Q$ is a subposet and X is a Q-diagram, the restriction $i^* X$ is what we have denoted by $X|_{Q'}$. Note that in this case $\underline{\mathrm{hocolim}}\, X|_{Q'}$ is a subposet of $\underline{\mathrm{hocolim}}\, X$.

Proposition 5.3.6. *Let $X : P \to \mathcal{P}_{<\infty}$ be a P- diagram of finite posets. If p is a down beat point of P dominated by q and f_{qp} is a weak equivalence, then*

$$\underline{\mathrm{hocolim}}\, X \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, X|_{P \smallsetminus \{p\}}.$$

Proof. Since p is dominated by q, there is a well-defined strong homotopy retraction $r : P \to P \smallsetminus \{p\}$ which is the identity for any $p' \neq p$ and $r(p) = q$. Denote by $i : P \smallsetminus \{p\} \to P$ the inclusion. There is a morphism of P-diagrams $\gamma : (ir)^* X \to X$ such that $\gamma_{p'}$ is the identity for every $p' \neq p$ and $\gamma_p = f_{qp} : ((ir)^* X)_p = X_q \to X_p$. By hypothesis and Corollary 5.2.6, γ induces a weak equivalence $\underline{\mathrm{hocolim}}(ir)^* X \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, X$, and by Proposition 5.3.5, $\underline{\mathrm{hocolim}}(ir)^* X \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, X|_{P \smallsetminus \{p\}}$. □

Example 5.3.7. By the previous proposition one can immediately deduce that the non-Hausdorff homotopy pushout of the poset diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\big\downarrow & & \\ X_2 & & \end{array}$$

is weak equivalent to X_0 provided the maps f_1 and f_2 are weak equivalences.

We prove now an analogue, in the context of posets, of a cofinality theorem of Bousfield and Kan. We need first a generalization of Proposition 5.3.1.

Proposition 5.3.8. *Let $X : P \to \mathcal{P}_{<\infty}$ be a P-diagram of finite posets. If p is a point of P such that $\hat{F}_p$ is homotopically trivial, then $\underline{\mathrm{hocolim}}\, X \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, X|_{P \smallsetminus \{p\}}$.*

Proof. Let $x_1, x_2, \cdots, x_n$ be a linear extension of X_p^{op}. For each $1 \leq i \leq n$, we define the $\hat{F}_p$-diagram $\tilde{X}^i : \hat{F}_p \to \mathcal{P}_{<\infty}$ as follows. Set $\tilde{X}^i(p') = F_{f_{pp'}(x_i)}$ and $\tilde{X}^i(p' \to p'') = f_{p'p''}|_{F_{f_{pp'}(x_i)}} : F_{f_{pp'}(x_i)} \to F_{f_{pp''}(x_i)}$.

Given that $\tilde{X}^i(p')$ is contractible for all $p' \in \hat{F}_p$, by Corollary 5.2.6 one can see that $\underline{\mathrm{hocolim}}\, \tilde{X}^i \underset{we}{\simeq} \hat{F}_p$, and by hypothesis, the last one is homotopically trivial. Therefore, since

$$\hat{F}_{x_i}^{(\underline{\mathrm{hocolim}}\, X) \smallsetminus \{x_1, x_2, \cdots, x_{i-1}\}} = \underline{\mathrm{hocolim}}\, \tilde{X}^i,$$

which is homotopically trivial, we can remove the points $x_1, \ldots, x_n$ of X_p one by one, similarly as in the proof of Proposition 5.3.1, and the result follows. □

We give now a direct and simple proof of a cofinality theorem for posets. This is a particular case of a known result for functors between small categories which satisfy the hypotheses of Quillens's Theorem A and it is an analogue of Bousfield-Kan's cofinality theorem in the combinatorial setting. The key tool of the proof is the construction of the cylinder of an appropriate relation.

Theorem 5.3.9 (Cofinality Theorem)**.** *Let $\varphi : P \to Q$ be an order preserving map between finite posets. Let $X : Q \to \mathcal{P}_{<\infty}$ be a Q-diagram. If $\varphi^{-1}(F_q)$ is homotopically trivial for all $q \in Q$, then the canonical map $\underline{\mathrm{hocolim}}\, \varphi^* X \to \underline{\mathrm{hocolim}}\, X$ is a weak equivalence.*

Proof. Let $\mathcal{R} \subseteq Q \times P$ the relation defined by $\phi(p)\, \mathcal{R}\, p$ for every $p \in P$ and construct the relation cylinder $B(\mathcal{R})$ (see Definition 3.1.1). Consider the following $B(\mathcal{R})$-diagram $\tilde{X} : B(\mathcal{R}) \to \mathcal{P}_{<\infty}$. For each $q \in Q$ take $\tilde{X}(q) = X_q$, set $\tilde{X}(p) = X_{\varphi(p)}$ for $p \in P$, $\tilde{X}(p \to p') = X(\varphi(p) \to \varphi(p'))$, $\tilde{X}(q \to q') = X(q \to q')$, and $\tilde{X}(\varphi(p) \to p) = \mathrm{id}_{X_{\varphi(p)}}$. Note that the restriction of $\tilde{X}$ to Q is the original diagram X and the restriction of $\tilde{X}$ to P is $\varphi^* X$.

Take a linear extension $q_1, q_2, \cdots, q_m$ of Q^{op}. For each $1 \leq j \leq m$ we have

$$\hat{F}_{q_j}^{B(\mathcal{R}) \smallsetminus \{q_1, q_2, \cdots, q_{j-1}\}} = \varphi^{-1}(F_{q_j})$$

which is homotopically trivial by hypothesis. By applying recursively Proposition 5.3.8, we get $\underline{\mathrm{hocolim}}\, \tilde{X} \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, \varphi^* X$.

Similarly, take a linear extension $p_1, p_2, \cdots, p_n$ of P. For each $1 \leq i \leq n$ we have

$$\hat{U}_{p_i}^{B(\mathcal{R})\smallsetminus\{p_1,p_2,\cdots,p_{i-1}\}} = \varphi(U_{p_i}) = U_{\varphi(p_i)}.$$

Therefore, p_i is a down beat point of $R \smallsetminus \{p_1, p_2, \cdots, p_{i-1}\}$ dominated by $\varphi(p_i)$, and $\tilde{X}(\varphi(p_i) \to p_i)$ is the identity. By Proposition 5.3.5 we have $\underline{\mathrm{hocolim}}\, \tilde{X} \underset{we}{\simeq} \underline{\mathrm{hocolim}}\, X$. □

If $X : P \to \mathcal{P}_{<\infty}$ is a P-diagram of finite posets, in some cases we can recover the homotopy type of P from the homotopy type of $\underline{\mathrm{hocolim}}\, X$.

Proposition 5.3.10. *Let $X : P \to \mathcal{P}_{<\infty}$ be a P-diagram of finite posets. If $\underline{\mathrm{hocolim}}\, X \searrow\!\!\searrow *$, then $P \searrow\!\!\searrow *$.*

Proof. Let $x_1, x_2, x_3, \cdots, x_m$ be a sequence of points of $\underline{\mathrm{hocolim}}\, X$ such that $\underline{\mathrm{hocolim}}\, X \searrow\!\!\searrow \underline{\mathrm{hocolim}}\, X \smallsetminus \{x_1\} \searrow\!\!\searrow \underline{\mathrm{hocolim}}\, X \smallsetminus \{x_1, x_2\} \searrow\!\!\searrow \cdots \searrow\!\!\searrow \underline{\mathrm{hocolim}}\, X \smallsetminus \{x_1, x_2 \cdots, x_m\} = \{*\}$. Let $\{x_{i_j}\}$ be the subsequence of $\{x_i\}$ of the last beat point removed of X_p for every $p \in P$. For every x_{i_j}, let X_{p_j} the poset of the diagram X such that $x_{i_j} \in X_{p_j}$. Finally, is easy to check that $P \searrow\!\!\searrow P \smallsetminus \{p_1\} \searrow\!\!\searrow P \smallsetminus \{p_1, p_2\} \searrow\!\!\searrow \cdots \searrow\!\!\searrow P \smallsetminus \{p_1, p_2, \cdots, p_{n-1}\} = \{p_n\}$, with $n = |P|$. □

5.4 Variations on Thomason's theorem and applications

We prove a result that relates the homotopy colimit of a diagram of spaces with the non-Hausdorff homotopy colimit of the diagram of their models. As a consequence we obtain an alternative and simple proof of Thomason's theorem in the context of posets. As another immediate consequence of the main result we deduce that the homotopy colimit of a diagram of finite simplicial complexes is weak equivalent to the non-Hausdorff homotopy colimit of the diagram of their face posets. This implies that all the techniques developed in the previous section for non-Hausdorff homotopy colimits can be used for diagrams of polyhedra (indexed by finite posets) by means of the face poset functor.

We denote by Top the category of topological spaces and continuous maps. Sometimes we require a diagram of spaces $D : P \to \mathrm{Top}$ to satisfy extra conditions (for instance D could be a diagram of simplicial complexes, finite topological spaces, etc), however in all these cases the homotopy colimit D is taken in the category Top.

Theorem 5.4.1. *Let P be a finite poset. Let $K : P \to \mathrm{Top}$ be a diagram of spaces and $X : P \to \mathcal{P}_{<\infty}$ be a diagram of finite posets. Let $\phi : K \to X$ be a diagram morphism (where X is viewed as a diagram of finite topological spaces) such that $\phi_p : K_p \to X_p$ is a weak equivalence for every $p \in P$. Then there exists a weak equivalence*

$$\hat{\phi} : \mathrm{hocolim}\, K \to \underline{\mathrm{hocolim}}\, X.$$

Proof. We define first the map $\hat{\phi} : \mathrm{hocolim}\, K \to \underline{\mathrm{hocolim}}\, X$. For every $p \leq p'$ denote by $f_{pp'} = K(p \to p')$ and $g_{pp'} = X(p \to p')$ the transition maps. Recall that $\mathrm{hocolim}\, K$ can be constructed from the disjoint union $\coprod_{p \in P} K_p \times \mathcal{K}(F_p)$ by identifying the pairs $(\alpha, \beta) \in K_p \times \mathcal{K}(F_p)$ with $(\alpha', \beta') \in K_{p'} \times \mathcal{K}(F_{p'})$ if $f_{pp'}(\alpha) = \alpha'$ and $\beta = \beta' \in \mathcal{K}(F_{p'})$. We denote by $\sim$ the equivalence relation generated by this identification.

For each $p \in P$, let $\mu_p : \mathcal{K}(F_p) \to F_p \subseteq X_p$ be the McCord map (see Definition 2.1.5). Given $(\alpha, \beta) \in K_p \times \mathcal{K}(F_p)$ we define

$$\hat{\phi}(\alpha, \beta) = \phi_{\mu_p(\beta)}(f_{p\mu_p(\beta)}(\alpha)) \in X_{\mu_p(\beta)} \subseteq \underline{\mathrm{hocolim}}\, X.$$

It is easy to verify that $\hat{\phi}$ is well defined. In order to see that it is a continuous map, it suffices to prove that for each $y \in \underline{\mathrm{hocolim}}\, X$, $\hat{\phi}^{-1}\big(U_y^{\underline{\mathrm{hocolim}}\, X}\big) \cap \big(K_q \times \mathcal{K}(F_q)\big)$ is open in $K_q \times \mathcal{K}(F_q)$ for every $q \in P$. Fix $y \in \underline{\mathrm{hocolim}}\, X$ and let $p \in P$ such that $y \in X_p$. Note that $\hat{\phi}^{-1}\big(U_y^{\underline{\mathrm{hocolim}}\, X}\big) \cap \big(K_q \times \mathcal{K}(F_q)\big)$ is empty if $q \nleq p$ and it is equal to $(g_{qp}\phi_q)^{-1}\big(U_y^{X_p}\big) \times \mu_q^{-1}\big(U_p^{F_q}\big)$ if $q \leq p$. This proves that $\hat{\phi}$ is continuous.

In order to prove that $\hat{\phi}$ is a weak homotopy equivalence, we use McCord's theorem 2.1.9 for the basis-like open cover $\big\{\underline{\mathrm{hocolim}}\, X|_{U_p}\big\}_{p \in P}$ of $\underline{\mathrm{hocolim}}\, X$. We have to see that

$$\hat{\phi} : \hat{\phi}^{-1}\left(\underline{\mathrm{hocolim}}\, X|_{U_p}\right) \to \underline{\mathrm{hocolim}}\, X|_{U_p}$$

is a weak equivalence for each p. Note that $\hat{\phi}^{-1}(\underline{\mathrm{hocolim}}\, X|_{U_p}) = \coprod_{q \leq p} K_q \times \mu_q^{-1}(U_p^{F_q})/\sim$ and that there is a commutative diagram

$$\begin{array}{ccc} K_p \times \{p\} & \xrightarrow{\ \phi_p\ } & X_p \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \left(\coprod_{q \leq p} K_q \times \mu_q^{-1}\left(U_p^{F_q}\right)/\sim\right) & \xrightarrow{\ \phi\ } & \underline{\mathrm{hocolim}}\, X|_{U_p} \end{array} .$$

The inclusion j is a weak equivalence by Corollary 5.3.2 and ϕ_p is a weak equivalence by hypothesis, thus we only need to check that the inclusion i is a homotopy equivalence.

Consider the retraction $r : \left(\coprod_{q \leq p} K_q \times \mu_q^{-1}\left(U_p^{F_q}\right)/\sim\right) \to K_p \times \{p\}$ defined by $r(\alpha, \beta) = (f_{qp}(\alpha), p)$ for $(\alpha, \beta) \in K_q \times \mu_q^{-1}\left(U_p^{F_q}\right)$. It is clear that $ri = 1$. We define a homotopy $H : ir \simeq 1$ as a composition of two linear homotopies. Any $\beta \in \mu_q^{-1}\left(U_p^{F_q}\right)$ can be written as $\beta = t\beta_1 + (1-t)\beta_2$ with $0 < t \leq 1$, $\beta_1 \in \mathcal{K}(U_p)$ and $\beta_2 \in \mathcal{K}(X_p \smallsetminus U_p)$. Take $H_1((\alpha, \beta), s) = (\alpha, (1-s)\beta + s\beta_1)$. Since $\beta_1 \in \mathcal{K}(U_p)$ which is a cone with apex p, we can define then $H_2((\alpha, \beta), s) = (\alpha, (1-s)\beta_1 + sp)$. □

Remark 5.4.2. Under the hypotheses of Theorem 5.4.1, there is an alternative way to prove that $\mathrm{hocolim}\, K$ and $\underline{\mathrm{hocolim}}\, X$ are weakly equivalent by means of a zigzag of weak equivalences. First one can see that it is enough to reduce to the case $K = X$ and $\phi = 1_X$. Then, in order to prove that $\mathrm{hocolim}\, X$ and $\underline{\mathrm{hocolim}}\, X$ are weakly equivalent one can use Thomason's theorem (to obtain a weak equivalence at the level of simplicial sets) and McCord's theorem (to come back to the context of topological spaces). Theorem 5.4.1 exhibits an explicit and direct weak equivalence $\mathrm{hocolim}\, K \to \underline{\mathrm{hocolim}}\, X$.

As a first immediate corollary we obtain the following particular case of Thomason's theorem for posets.

Corollary 5.4.3. *Given a diagram of finite posets $X : P \to \mathcal{P}_{<\infty}$, there is a homotopy equivalence*

$$\operatorname{hocolim} \mathcal{K}X \to \mathcal{K}(\underline{\operatorname{hocolim}}\, X).$$

Proof. Apply Theorem 5.4.1 to the diagram morphism $\mu : \mathcal{K}X \to X$, where $\mu_p : \mathcal{K}(X_p) \to X_p$ is the McCord map. □

Remark 5.4.4. From Theorem 5.4.1 one can easily deduce that if $X : P \to \mathrm{Top}$ is a diagram of finite topological spaces, although $\operatorname{hocolim} X$ and $\underline{\operatorname{hocolim}}\, X$ are very different, there is a weak homotopy equivalence $\operatorname{hocolim} X \to \underline{\operatorname{hocolim}}\, X$ induced by the identity map.

Now we prove a kind of converse of Thomason's result, which relates the homotopy colimit of a diagram of simplicial complexes with the non-Hausdorff homotopy colimit of their face posets.

Corollary 5.4.5. *Let $K : P \to \mathrm{Top}$ be a diagram of finite simplicial complexes. Then there is a weak equivalence*

$$\nu : \operatorname{hocolim} K \to \underline{\operatorname{hocolim}}(\mathcal{X}K)^{op}$$

from the homotopy colimit of K to the non-Hausdorff homotopy colimit of the diagram of the opposite of their face posets.

Proof. Apply Theorem 5.4.1 to the diagram morphism induced by the natural weak equivalences $\nu_p : K_p \to \mathcal{X}(K_p)^{op}$ defined in Remark 2.1.7. □

Since for every simplicial complex L, $\mathcal{K}(\mathcal{X}(L)^{op}) = \mathcal{K}(\mathcal{X}(L)) = L'$ (the barycentric subdivision of L), from Corollary 5.4.3 and Corollary 5.4.5 we prove invariance of homotopy type under barycentric subdivision for homotopy colimits in the (general) unordered setting.

Corollary 5.4.6. *Let $K : P \to \mathrm{Top}$ be a diagram of (unordered) finite simplicial complexes (and simplicial maps). Then* $\operatorname{hocolim} K$ *and* $\operatorname{hocolim} K'$ *are homotopy equivalent, where $K' : P \to \mathrm{Top}$ is the diagram of the barycentric subdivisions (of spaces and maps).*

Remark 5.4.7. It is well known that for any simplicial set T, there is a *natural* homotopy equivalence $sd\ T \to T$ from the (geometric realization of the) barycentric subdivision of T to T (see for example [May12, Thm. 12.2.5]). By Bousfield-Kan's homotopy lemma [BK72], this implies that the homotopy colimit of a diagram of simplicial sets is homotopy equivalent to the homotopy colimit of the diagram of their barycentric subdivisions. On the other hand, any ordered simplicial complex K (i.e. a simplicial complex together with a partial ordering of its vertices that restricts to a total ordering on the vertices of each simplex) can be seen as a simplicial set K_s. Moreover, the simplicial set associated to its (geometric) barycentric subdivision $(K')_s$ is naturally isomorphic to $sd\ K_s$, the subdivision of the simplicial set K_s (see [May12, Thm. 12.2.2]). This proves that the homotopy colimit of a diagram of ordered simplicial complexes (and ordered simplicial maps) is homotopy equivalent to the homotopy colimit of the diagram of their barycentric subdivisions. However, in a general geometric situation, one has to deal with diagrams of unordered simplicial complexes, and in the unordered context there is no natural homotopy equivalence between K and its barycentric subdivision K'. Concretely, the homotopy equivalence of the last corollary cannot be deduced directly from a diagram map between K and K' since, although the underlying topological spaces of K_p and K'_p are equal and the transition maps f_{qp} and f'_{qp} are (linearly) homotopic, in the context of unordered simplicial complexes there is no *natural* homotopy equivalence from the barycentric subdivision functor to the identity functor. Moreover, in general the homotopies $f_{qp} \simeq f'_{qp}$ cannot be taken coherently.

Corollary 5.4.5 allows one to apply the results of the previous section to homotopy colimits of diagrams of polyhedra, by means of the weak equivalence with the non-Hausdorff homotopy colimits of the opposite of the face posets. In particular, the following analogues of Propositions 5.3.8 and 5.3.6 are valid for diagrams of simplicial complexes.

Proposition 5.4.8. *Let* $K : P \to \mathrm{Top}$ *be a diagram of finite simplicial complexes (and simplicial maps). If* p *is a point of* P *such that* $\hat{F}_p$ *is homotopically trivial (in particular if* p *is an up beat point or an up weak point), then* $\mathrm{hocolim}\, K \simeq \mathrm{hocolim}\, K|_{P\smallsetminus\{p\}}$.

Proof. Consider the diagram $(\mathcal{X}K)^{op} : P \to \mathcal{P}_{<\infty}$, $(\mathcal{X}K)^{op}(p) = (\mathcal{X}(K_p))^{op}$, and apply Proposition 5.3.8 and Corollary 5.4.5. □

Proposition 5.4.9. *Let* $K : P \to \mathrm{Top}$ *be a diagram of finite simplicial complexes (and simplicial maps). If* p *is a down beat point of* P *dominated by* q *and* f_{qp} *is a homotopy equivalence, then*

$$\mathrm{hocolim}\, K \simeq \mathrm{hocolim}\, K|_{P\smallsetminus\{p\}}.$$

Proof. Consider the diagram $(\mathcal{X}K)^{op} : P \to \mathcal{P}_{<\infty}$ and apply Proposition 5.3.6 and Corollary 5.4.5. □

Suppose that $K : P \to \mathrm{Top}$ is a diagram of finite simplicial complexes such that all the transition maps $f_{qp} : K_q \to K_p$ are homotopy equivalences (in particular, if P is connected, all K_p have the same homotopy type). In general, although the simplicial maps $f_{qp} : K_q \to K_p$ are homotopy equivalences, the homotopy type of K_p and the topology of $\mathcal{K}(P)$ do not determine the homotopy type of $\mathrm{hocolim}\, K$. We will show that if the indexing poset P and the maps f_{qp} satisfy nice conditions, then $\mathrm{hocolim}\, K \simeq K_p$ (for any $p \in P$).

Corollary 5.4.10. *Let* $K : P \to \mathrm{Top}$ *be a diagram of finite simplicial complexes (and simplicial maps). If* P *is a contractible finite space and the transition maps* f_{qp} *are homotopy equivalences, then* $\mathrm{hocolim}\, K \simeq K_p$ *(for any* $p \in P$*).*

Proof. Since P is a contractible finite space, there is a sequence $p_1, \cdots, p_n$ such that p_i is a beat point (up or down) of $P \smallsetminus \{p_1, p_2, \cdots, p_{i-1}\}$ and $P \smallsetminus \{p_1, p_2, \cdots, p_n\} = \{p\}$. Now apply recursively Propositions 5.4.8 and 5.4.9. □

As we remarked in the second chapter, if P is contractible (as a finite space) then its classifying space $\mathcal{K}(P)$ is contractible but the converse does not hold. Suppose that the indexing poset P can be reduced to a single point by removing γ-points (recall that p is a γ-point if $\hat{F}_p$ or $\hat{U}_p$ is homotopically trivial). In that case the previous corollary is not longer valid since Proposition 5.4.9 works only for down beat points (i.e. when $\hat{U}_p$ has maximum, not just homotopically trivial). However, one can impose extra conditions on the maps $f_{qp} : K_q \to K_p$ in order to extend Proposition 5.4.9 to γ-points, and Corollary 5.4.10 to a more general class of homotopically trivial posets. To this end we replace homotopy equivalences by *contractible mappings.* This class of maps was introduced by Cohen in [Coh67]. A simplicial map $f : K \to L$ is called a *contractible mapping* if the preimage $f^{-1}(z)$ is contractible for every point z in the underlying space of L. In [Coh67, Thm. 11.1] Cohen proved that any contractible mapping $f : K \to L$ is a simple homotopy equivalence. In [Bar11b, Thm. 5.1] Barmak exhibited an alternative and simple proof of Cohen's result. From the proof of [Bar11b, Thm. 5.1] one can deduce the following.

Proposition 5.4.11 (Barmak)**.** *Let $f : K \to L$ be a contractible mapping and let $\mathcal{X}(f)^{op} : \mathcal{X}(K)^{op} \to \mathcal{X}(L)^{op}$ be the map induced in the opposite of their face posets. Then $(\mathcal{X}(f)^{op})^{-1}(U_\sigma)$ is homotopically trivial for every $\sigma \in \mathcal{X}(L)^{op}$.*

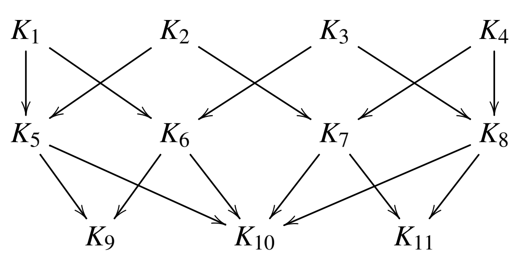
Corollary 5.4.12. *Let $K : P \to$ Top be a P-diagram of finite simplicial complexes (and simplicial maps). If p is a point of P such that $\hat{U}_p$ is homotopically trivial and the transition maps f_{qp} are contractible mappings for every $q \leq p$, then* hocolim $K \simeq$ hocolim $K|_{P \smallsetminus \{p\}}$.

Proof. Consider the diagram $(\mathcal{X}K)^{op} : P \to \mathcal{P}_{<\infty}$ and follow the proof of Proposition 5.3.5, using that $(\mathcal{X}(f_{qp})^{op})^{-1}(U_\sigma)$ are homotopically trivial by Proposition 5.4.11. □

Corollary 5.4.12 in combination with Proposition 5.4.8 allows us to extend Corollary 5.4.10 to a more general class of (homotopically trivial) indexing posets, under the stronger assumption that the transition maps are contractible mappings.

Corollary 5.4.13. *Let $K : P \to$ Top be a P-diagram of finite simplicial complexes. If the indexing poset P can be reduced to a point by removing γ-points (in particular, if P is collapsible), and the transition maps f_{qp} are contractible mappings, then* hocolim $K \simeq K_p$ *(for any $p \in P$).*

Example 5.4.14. If the transition maps of the following diagram of simplicial complexes are contractible mappings, its homotopy colimit is homotopy equivalent to any of the K_p. This is because the indexing poset is a collapsible (but non-contractible) finite space.



Resumen del capítulo 5: Colímites homotópicos y deformaciones

La teoría de colímites homotópicos fue intrducida independientemente por Bousfield y Kan [BK72] y Vogt [Vog73] como una "corrección" de la noción usual de colímite, para que, por un lado, cumpla la propiedad de ser invariante por homotopía, y por el otro, tenga el mismo tipo homotópico que el colímite usual bajo buenas propiedades en el diagrama.

En este capítulo, usamos un teorema clásico de McCord y métodos de reducción de espacios finitos para probar una generalización del teorema de Thomason [Tho79] sobre colímites homotópicos de diagramas sobre posets. En particular, esto permite caracterizar los colímites homotópicos de diagramas de complejos simpliciales en términos de la construcción de Grothendieck de los diagramas de sus face posets. También deducimos análogos de resultados conocidos sobre colímites homotópicos en el contexto combinatorio, incluyendo el teorema cofinal y una generalización del Teorema A de Quillen para posets.

Sea I una categoría pequeña, y sea $X : I \to CAT$ un funtor a la categoría de categorías pequeñas. La *construcción de Grothendieck* en X, denotada por $I \int X$, es la categoría cuyos objetos son los pares (i, x), con i un objeto de I y x un objeto de $X(i)$, y cuyos morfismos son de la forma $(\alpha,\beta) : (i, x) \to (i', x')$ con $\alpha : i \to i'$ un morfismo en I y $\beta : X(\alpha)(x) \to x'$ un morfismo en $X(i')$. El teorema de Thomason [Tho79] afirma que existe una equivalencia homotópica natural

$$\operatorname{hocolim} \mathcal{N}(X) \to \mathcal{N}(I \int X)$$

entre el colímite homotópico del nervio de X y el nervio de la construcción de Grothendieck en X.

En este capítulo, nos enfocamos en colímites homotópicos de diagramas de espacios indexados en posets. La idea es usar la interacción entre combinatoria y topología que logra la teoría de espacios topológicos finitos para estudiar colímites homotópicos de diagramas de poliedros. Si P es un poset finito y $X : P \to \mathcal{P}_{<\infty}$ es un diagrama a valores en la categoría de posets finitos, la construcción de Grothendieck $P \int X$ es también un poset y, de hecho, es fácil de caracterizar.

Definición 5.2.1. Sea $X : P \to \mathcal{P}_{<\infty}$ un funtor. El *colímite homotópico no Hausdorff de* X es el poset cuyo conjunto subyacente es la unión disjunta $\coprod_{p\in P} X_p$. El orden se obtiene preservando el orden de X_p para todo $p \in P$, y para cada $x \in X_p$ e $y \in X_q$ tal que $p \le q$, decimos que $x \le y$ en $\underline{\operatorname{hocolim}} X$ si $f_{pq}(x) \le y$ en X_q, donde $X_p = X(p)$ para cada $p \in P$ y $f_{pq} = X(p \to q)$ para cada $p \le q$ en P.

Por otra parte, del Capítulo 2, todo poset finito se puede pensar como un espacio topológico finito. Usando este hecho y el teorema de McCord 2.1.9, obtuvimos resultados análogos en el contexto combinatorio de resultados conocidos para colímites homotópicos en general, con demostraciones más simples. Citamos los más relevantes.

Lema 5.2.5. Sea P un poset finito y sea $\mathcal{A}$ una colección finita de subposets finitos de P, ordenados por inclusión. Sea $X : (\mathcal{A}, \subseteq) \to \mathcal{P}_{<\infty}$ el diagrama asociado, definido por $X(A) = A$ para todo $A \in \mathcal{A}$ y $X(A \le A')$ es la inclusión $\iota : A \to A'$. Si para todo par de elementos $A, A' \in$

$\mathcal{A}$, o bien $A\cap A' = \emptyset$ o bien $A\cap A'$ es union de elementos en $\mathcal{A}$, entonces $\underline{\text{hocolim}}(X) = \bigcup_{A\in\mathcal{A}} A$ (como poset) y $\underline{\text{hocolim}}(X) \searrow\!\!\!\searrow \text{colim}(X)$.

Lema 5.2.6. Sea P un poset finito, y sean $X, Y; P \to \mathcal{P}_{<\infty}$ P-diagramas de posets finitos y $\alpha : X \to Y$ un morfismo de diagramas. Si $\alpha_p : X_p \to Y_p$ es una equivalencia débil para todo $p \in P$, entonces α induce una equivalencia débil

$$\underline{\text{hocolim}}\, X \underset{we}{\simeq} \underline{\text{hocolim}}\, Y.$$

Probamos también una generalización del teorema de Thomason, la que nos permitió estudiar el tipo homotópico de diagramas de poliedros, indexados en posets. En [Bar11b], Barmak da una demostración simple del Teorema A de Quillen para posets 2.1.10 (o, equivalentemente, el teorema de MacCord para espacios topológicos finitos) usando B_f, el cilindro no Hausdorff de una función $f : X \to Y$. El cilindro no Hausdorff es el análogo finito del clásico cilindro de funciones continuas y, similarmente a lo de ocurre en el contexto clásico, B_f es el *colímite homotópico no Hausdorff* (i.e. la construcción de Grothendieck) del diagrama de posets $X \xrightarrow{f} Y$, indexado en el poset **1** de dos elementos $0 < 1$.

En la Sección 5.3 estudiamos a los colímites homotópicos desde el punto de vista de los espacios finitos. Usamos *métodos de reducción* para estudiar su tipo homotópico débil y simple.

Proposición 5.3.1. Sea $X : P \to \mathcal{P}_{<\infty}$ un P diagrama de espacios finitos. Si $p \in P$ es un up beat point, entonces $\underline{\text{hocolim}}\, X \searrow\!\!\!\searrow \underline{\text{hocolim}}\, X|_{P\smallsetminus\{p\}}$.

Corolario 5.3.2. Sea $X : P \to \mathcal{P}_{<\infty}$ un P diagrama de espacios finitos. Si P tiene máximo p, entonces $\underline{\text{hocolim}}\, X \searrow\!\!\!\searrow X_p$.

Proposición 5.3.5. Sea $X : P \to \mathcal{P}_{<\infty}$ un P diagrama de espacios finitos. Si p un down beat point de P dominado por q y $f_{qp}^{-1}(U_x)$ es contráctil para cada $x \in X_p$, entonces

$$\underline{\text{hocolim}}\, X \searrow \underline{\text{hocolim}}\, X|_{P\smallsetminus\{p\}}.$$

Proposición 5.3.10. Sea $X : P \to \mathcal{P}_{<infty}$ un P diagrama de espacios finitos. Si $\underline{\text{hocolim}}\, X \searrow\!\!\!\searrow *$, entonces $P \searrow\!\!\!\searrow *$.

Usando el *cilindro de una relación* apropiada probamos de manera más sencilla el

Teorema 5.3.9 (Cofinal)**.** Sea $\varphi : P \to Q$ una función de orden entre posets, y sea $X : Q \to \mathcal{P}_{<\infty}$ un Q-diagrama. Si $\varphi^{-1}(F_q)$ es homotópicamente trivial para todo $q \in Q$ entonces la función canónica $\underline{\text{hocolim}}\, \varphi^* X \to \underline{\text{hocolim}}\, X$ es una equivalencia débil.

El principal teorema de este capítulo es la siguiente generalización del teorema de Thomason en el contexto de espacios finitos.

Teorema 5.4.1. Sea P un poset finito. Sea $K : P \to \text{Top}$ un diagrama de espacios y $X : P \to \mathcal{P}_{<\infty}$ un diagrama de posets finitos. Sea $\phi : K \to \text{Top}$ un morfismo de diagramas (donde X es visto como diagrama de espacios topológicos finitos) tal que $\phi_p : K_p \to X_p$ es una equivalencia homotópica débil para cada $p \in P$. Entonces, existe una equivalencia homotópica débil

$$\hat{\varphi} : \text{hocolim}\, K \to \underline{\text{hocolim}}\, X.$$

entre el colímite homotópico de K al colímite homotópico no Hausdorff de X (visto como espacio topológico finito).

Una consecuencia inmediata de este resultado es el teorema de Thomason en el contexto de posets, y también una suerte de vuelta del teorema de Thomason, que relaciona el colímite homotópico de un diagrama de complejos simpliciales con el colímite homotópico no Hausdorff del diagrama de sus face posets. Junto con los métodos de reducción de la Sección 5.3, la teoría desarrollada permite el cálculo más simple del colímite homotópico de diagramas de espacios.

Corolario 5.4.3 (Thomason)**.** Dado un diagrama de posets $X : P \to \mathcal{P}_{<\infty}$, hay una equivalencia homotópica

$$\text{hocolim}\, \mathcal{K}X \to \mathcal{K}(\underline{\text{hocolim}}\, X).$$

Corolario 5.4.5. Dado un diagrama de complejos simpliciales $X : P \to \text{Top}$, hay una equivalencia homotópica débil

$$\nu : \underline{\text{hocolim}}\, K \to (\underline{\text{hocolim}}\, \mathcal{X}K)^{op}$$

entre el colímite homotópico de K al colímite homotópico no Hasdorff del diagrama de los opuestos de sus face posets.

Es conocido el hecho de que para todo conjunto simplicial T, hay una equivalencia homotópica *natural* $sdT \to T$ entre (la realización geométrica de) la subdivisión baricéntrica de T y T. Por el teorema de homotopía de Bousfield y Kan, esto implica que el colímite homotópico de un diagrama de conjuntos simpliciales es homotópicamente equivalente al colímite homotópico del diagrama de sus subdivisiones baricéntricas. Por otra parte, todo complejo simplicial ordenado K se puede ver como conjunto simplicial K_s. Más aún, el conjunto simplicial asociado a su subdivisión baricéntrica (geométrica) $(K')_s$ es naturalmente isomorfa a sdK_s, la subdivisión del conjunto simplicial. Esto prueba que el colímite homotópico de diagramas de complejos simpliciales ordenados (y funciones simpliciales ordenadas) es homotópicamente equivalente al colímite homotópico del diagrama de sus subdivisiones baricéntricas. Sin embargo, en el contexto geométrico general, los complejos simpliciales no están ordenados, aún cuando les diéramos un orden, no hay una manera coherente de hacerlo de modo que se obtenga una equivalencia natural entre K y su subdivisión baricéntrica K'. Nuestros métodos resultaron apropiados para evitar este problema. Como corolario del Teorema 5.4.1, probamos la invariancia por subdivisiones baricéntricas del tipo homotópico de colímites homotópicos de complejos simpliciales no ordenados.

Corolario 5.4.6. Sea $K : P \to \text{Top}$ un diagrama de complejos simpliciales (no ordenados) y funciones simpliciales. Entonces $\underline{\text{hocolim}}\, K$ y $\underline{\text{hocolim}}\, K'$ son homotópicamente equivalentes, donde $K' : P \to \text{Top}$ es el diagrama de las subdivisiones baricéntricas.

Chapter 6

New combinatorial methods for group presentations

Searching exhaustively for explicit sequences of Andrews-Curtis transformations is computationally prohibitive. Deterministic algorithms to explore all possible sequences up to a certain length are at least exponential in this number. We present some novel topological methods which allow the exploration of Andrews-Curtis equivalent presentations which are far (in the sense of number of transformations) from a given presentation.

6.1 The finite space associated to a group presentation

In this section, we show how to associate to each balanced group presentation $\mathcal{P}$ a finite space $X_{\mathcal{P}}$ of height 2 in such a way that two presentations are AC-equivalent if and only if their associated finite spaces are 3-deformable. This allows to translate the combinatorial group theory version of the problem into a combinatorial topology problem.

Recall from Theorem 1.4.5 that given $\mathcal{P}, \mathcal{Q}$ two balanced presentations, $\mathcal{P} \sim_{AC} \mathcal{Q}$ if and only if $K_{\mathcal{P}} \diagup\!\!\searrow^{3} K_{\mathcal{Q}}$. Conversely, if K, L are two cellular complexes of dimension 2, $K \diagup\!\!\searrow^{3} L$ if and only if $\mathcal{P}_K \sim_{AC} \mathcal{P}_L$. Moreover, $\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}}$ and $K \diagup\!\!\searrow^{3} K_{\mathcal{P}_K}$.

Definition 6.1.1. Let $\mathcal{P}$ be a finite group presentation. The barycentric subdivision of $K_{\mathcal{P}}$ is a regular cell complex which we denote by $K'_{\mathcal{P}}$. Define $X_{\mathcal{P}}$, the *presentation poset* of $\mathcal{P}$, as $\mathcal{X}(K'_{\mathcal{P}})$.

The poset $X_{\mathcal{P}}$ can be directly constructed from $\mathcal{P} = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_m \rangle$ as follows. For every generator x_i of $\mathcal{P}$ take X_i, a minimal finite model of S^1, as indicated in Figure 6.1.

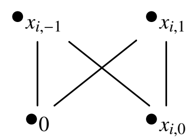


Figure 6.1: Minimal model of S^1 associated to the generator x_i of the presentation $\mathcal{P}$.

For every relator r_j of $\mathcal{P}$ take R_j, a finite model of D^2, as the cone on a finite model of S^1 whose number of vertices is four times the length of the relator. If the relator r_j is given by the word $x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_k}^{\epsilon_k}$, label the points as in Figure 6.1. Note that the word spelled by the cycle from any starting point is a (possibly inverted) cyclic permutation of r_j.

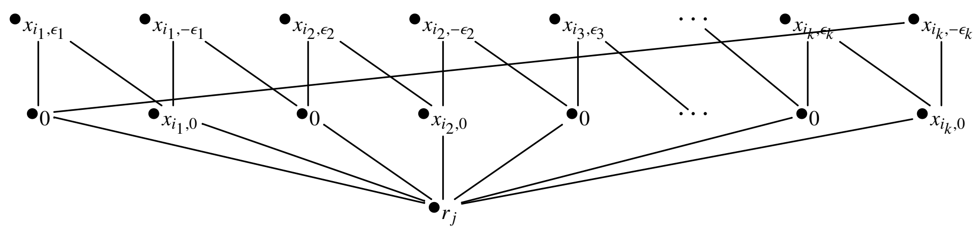


Figure 6.2: Finite model of D^2 associated to the relator r_j of the presentation $\mathcal{P}$.

Now define a relation $\mathcal{R}$ between the posets $\bigvee_i X_i$ (with wedge point 0) and $\bigsqcup_j R_j$, which relates the points with the same label and define $X_{\mathcal{P}} = B(\mathcal{R})$ (see Definition 3.1.1). Note that the Hasse diagram of $X_{\mathcal{P}}$ is obtained from $\bigvee_i X_i \sqcup \bigsqcup_j R_j$ by adding an edge between the elements of $\bigvee_i X_i$ and the elements of $\bigsqcup_j R_j$ with the same label.

Example 6.1.2. Let $\mathcal{P} = \langle x, y \mid x^2, xy^{-1} \rangle$. The cellular complex $K_{\mathcal{P}}$ associated to $\mathcal{P}$ has one 0-cell, two 1-cells (with an associated orientation), one for each generator, and two 2-cells, one for each relator. The attaching map of each 2-cell spells the word represented by the corresponding relator. Thus, $K'_{\mathcal{P}}$ is the following regular cell space.

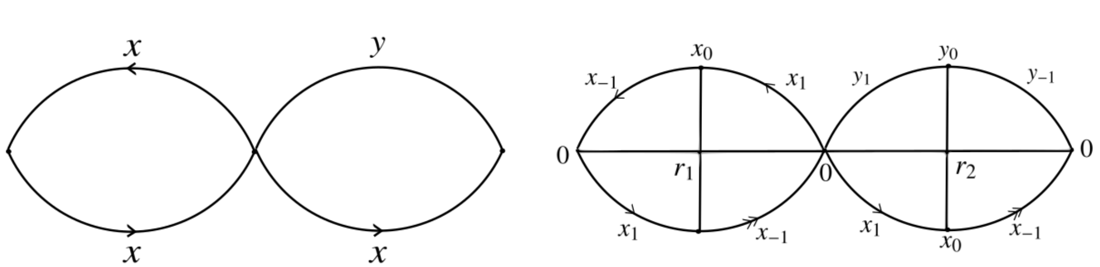


Figure 6.3: The standard complex associated to $\mathcal{P}$ and its barycentric subdivision.

Then, $X_{\mathcal{P}}$ is the finite poset of Figure 6.4.

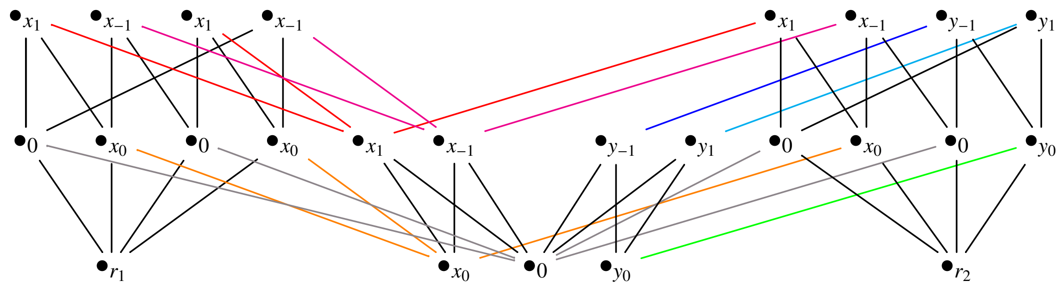


Figure 6.4: The presentation poset associated to $\mathcal{P} = \langle x, y \mid x^2, xy^{-1}\rangle$.

Remark 6.1.3. If $\mathcal{P}$ is a finite presentation of a group G, then $X_{\mathcal{P}}$ and $K_{\mathcal{P}}$ are weak homotopy equivalent. In particular $\pi_1(X_{\mathcal{P}}) = \pi_1(K_{\mathcal{P}}) = G$.

We will see that this assignment maps AC-equivalent presentations to 3-deformable finite spaces, and balanced presentations of the trivial group to homotopically trivial spaces.

Proposition 6.1.4. *Let $\mathcal{P}, \mathcal{Q}$ be presentations of a group G. Then,*

(i) $\mathcal{P} \sim_{AC} \mathcal{Q}$ if and only if $X_{\mathcal{P}} \diagup\!\!\searrow^{3} X_{\mathcal{Q}}$.

(ii) $\mathcal{P}$ is balanced and G is the trivial group if and only if $X_{\mathcal{P}}$ is homotopically trivial.

Proof. By Theorem 1.4.5, $\mathcal{P} \sim_{AC} \mathcal{Q}$ if and only if $K_{\mathcal{P}} \diagup\!\!\searrow^{3} K_{\mathcal{Q}}$. By Lemma 2.2.3 $K_{\mathcal{P}} \diagup\!\!\searrow^{3} K'_{\mathcal{P}}$ and $K_{\mathcal{Q}} \diagup\!\!\searrow^{3} K'_{\mathcal{Q}}$, hence $K'_{\mathcal{P}} \diagup\!\!\searrow^{3} K'_{\mathcal{Q}}$. Since $K'_{\mathcal{P}}$ and $K'_{\mathcal{Q}}$ are regular cell complexes, by 2.1.16, $X_{\mathcal{P}} = \mathcal{X}(K'_{\mathcal{P}}) \diagup\!\!\searrow^{3} \mathcal{X}(K'_{\mathcal{Q}}) = X_{\mathcal{Q}}$.

On the other hand, by Proposition 1.3.6, $\mathcal{P}$ is a balanced presentation of the trivial group if and only if $K_{\mathcal{P}}$ is contractible. The latter is equivalent to $K'_{\mathcal{P}}$ being contractible or $\mathcal{X}(K'_{\mathcal{P}}) = X_{\mathcal{P}}$ homotopically trivial. □

In the following sections we will develop two combinatorial methods to get new presentations of the same group which are AC-equivalent to the original one.

6.2 Colorings

A useful and well known combinatorial technique to solve problems is by coloring the involved objects and recognizing the existence of certain properties. For example, if we *color* a spanning simply connected subcomplex of a simplicial complex, then we can find a presentation of its edge-path group.

We will use colorings of the Hasse diagram of a poset to get associated group presentations. Good properties of colorings will lead to balanced presentations of the fundamental group of the finite space associated to the poset. As a matter of fact, different choices of good colorings will produce group presentations in the same AC-equivalence class. This construction was motivated by the techniques developed in [BM12a].

Definition 6.2.1. Let X be a finite space of height 2 and let C be a subgraph (also called subdiagram) of $\mathcal{H}(X)$. We will call C a *coloring* of $\mathcal{H}(X)$. We say that (x, y) is an *extremal pair* of X if $x < y$, $h(x) = 0$ and $h(y) = 2$. For every extremal pair (x, y) of X, if there exists any chain $c_{x,y} : x \prec z \prec y$ such that at least one of its edges is not colored by C, fix a *preferred* one. We associate to (X, C) a group presentation $\mathcal{P}_{X,C}$ whose generators are the edges of $\mathcal{H}(X)$ not belonging to C, and whose relators are induced by the *digons* (i.e. pairs of monotonic edge paths that only meet in the extremal points) containing a preferred chain. If the preferred chain $x < z < y$ and the chain $x < z' < y$ form a digon, then the relator induced is the induced by the identity between the words read off from this chains.

Example 6.2.2. Consider the finite space in Figure 6.5, colored by the dotted subdiagram C.

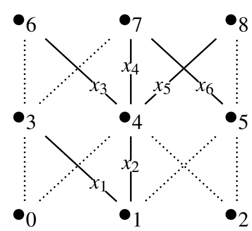


Figure 6.5: A simply connected finite space X colored by a spanning tree of $\mathcal{H}(X)$.

If we choose the following preferred chains:

$c_{0,6} : 0 < 4 < 6$, $c_{0,7} : 0 < 4 < 7$, $c_{0,8} : 0 < 4 < 8$, $c_{1,6} : 1 < 4 < 6$, $c_{1,7} : 1 < 4 < 7$,
$c_{1,8} : 1 < 4 < 8$, $c_{2,6} : 2 < 4 < 6$, $c_{2,7} : 2 < 4 < 7$, $c_{2,8} : 2 < 4 < 8$,

the associated presentation $\mathcal{P}_{X,C}$ is

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \,|\, x_3 = 1, x_4 = 1, x_1 = x_2x_3, x_1 = x_2x_4, x_6 = x_2x_4, x_2x_5 = 1, x_4 = x_6, x_5 = 1\rangle.$$

If we change the preferred chain $c_{1,7} : 1 < 4 < 7$ by $\tilde{c}_{1,7} : 1 < 3 < 7$ and preserve the rest of the choices, the associated presentation $\tilde{\mathcal{P}}_{X,C}$ is

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_3 = 1, x_4 = 1, x_1 = x_2x_3, x_1 = x_2x_4, x_6 = x_1, x_2x_5 = 1, x_4 = x_6, x_5 = 1\rangle.$$

Notice that both presentations are AC-equivalent balanced presentations of the trivial group. In Theorem 6.2.4 we will show that the presentation $\mathcal{P}_X$ associated to a poset X defined in this section is, under good choices of colorings, AC-equivalent to the classical presentation $\mathcal{P}_{\mathcal{K}(X)}$ of the associated complex.

Example 6.2.3. Let X be the finite space of Figure 6.6.

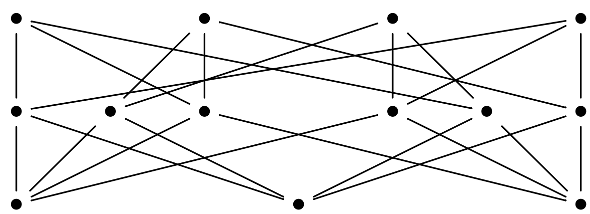


Figure 6.6: Finite model of the Projective Plane.

Notice that X is the face poset of the regular cell structure of the real projective plane of Figure 1.2. We consider the two different colorings in Figures 6.7 and 6.8 identified with doted lines. Since there are only two chains between every pair of points of heights 0 and 2 respectively, any choice of preferred chain will result in the same presentation.

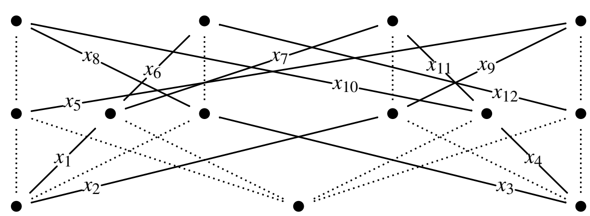


Figure 6.7: A coloring C_1 of X which is a spanning tree of $\mathcal{H}(X)$.

With the coloring C_1, a spanning tree of $\mathcal{H}(X)$, $\mathcal{P}_{X,C_1} = \langle x_1, x_2, \cdots, x_{12} \mid x_8 = 1,\ x_1x_6 = 1,\ x_1x_7x_2^{-1} = 1,\ x_2x_9x_5^{-1} = 1,\ x_{10} = 1,\ x_6x_{12}^{-1} = 1,\ x_7x_{11}^{-1} = 1,\ x_5 = 1, x_8x_{10}^{-1} = 1,\ x_{12} = 1,\ x_4x_{11}, = 1\ x_9 = 1\rangle$. It can be seen that $\mathcal{P}_{X,C_1}$ is a presentation of $\mathbb{Z}_2$.

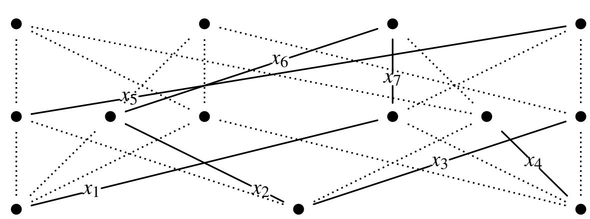


Figure 6.8: A coloring C_2 of X which is a spanning collapsible subdiagram of $\mathcal{H}(X)$.

The coloring C_2 is a collapsible subdiagram containing all the vertices of X. In this case, $\mathcal{P}_{X,C_2} = \langle x_1, x_2, \cdots, x_7 \mid x_1x_7 = x_6,\ x_1 = x_5,\ x_3 = 1,\ x_2 = x_3,\ x_2x_6 = 1,\ x_5 = x_3,\ x_4 = 1,\ x_4 = x_7\rangle$. It is easy to see that $\mathcal{P}_{X,C_2}$ is a presentation of $\mathbb{Z}_2$.

As we have seen in Chapter 1, it is possible to obtain a presentation of the fundamental group of a complex of dimension 2 from a spanning tree of its 1-skeleton, and different choices of spanning trees produce AC-equivalent presentations. The following theorem provides an analogue of this in our setting: if C is a spanning tree of $\mathcal{H}(X)$, the associated presentation $\mathcal{P}_{X,C}$ is in fact a presentation of the fundamental group of X AC-equivalent to $\mathcal{P}_{\mathcal{K}(X)}$.

Theorem 6.2.4. *Let X be a finite space and T be a spanning tree of $\mathcal{H}(X)$. Then, $\mathcal{P}_{X,T} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)}$.*

Proof. Let T be a spanning tree of $\mathcal{H}(X)$. Note that T can also be thought as a spanning tree of the 1-skeleton of $\mathcal{K}(X)$, since there is an inclusion $\mathcal{H}(X) \subseteq \mathcal{K}(X)^1$ as undirected graphs. Fix an orientation on the 1-cells inherited from the partial order in X. Let $\mathcal{P}_{\mathcal{K}(X),T}$ be the presentation of the fundamental group of $\mathcal{K}(X)/T$ obtained by previously selected orientation of the 1-cells. Let us see that $\mathcal{P}_{\mathcal{K}(X),T}$ can be transformed into $\mathcal{P}_{X,T}$ through AC-operations. For every extremal pair (x, y) of X, fix a preferred chain $c^0_{xy} : x < z < y$ such that at most one of the edges (x, z) and (z, y) is in T (note that such a chain exists for every extremal pair because T has no cycles). Label $e_1, \cdots, e_r$ the edges in $\mathcal{H}(X) \smallsetminus T$. Call w^0_{xy} the word in $F(e_1, \cdots, e_r)$ associated to c^0_{xy}. For every other chain c^i_{xy} between x, y, call w^i_{xy} the associated word. Thus, the generators of $\mathcal{P}_{X,T}$ are $e_1, \cdots, e_r$, and its relators are $(w^i_{xy})^{-1} w^0_{xy}$ for every extremal pair (x, y) of X and for every chain $c^i_{xy} \neq c^0_{xy}$. Summarizing,

$$\mathcal{P}_{X,T} = \langle e_1, \cdots, e_r \mid (w^i_{xy})^{-1} w^0_{xy} : i \rangle.$$

On the other hand, let e_{xy} be the edge in $\mathcal{K}(X)^1$ associated to the extremal pair (x, y). Then

$$\mathcal{P}_{\mathcal{K}(X),T} = \langle e_1, \cdots, e_r, e_{xy} \mid e^{-1}_{xy} w^0_{xy},\ (w^i_{xy})^{-1} e_{xy} : i \rangle.$$

For every extremal pair (x, y) call $r_i = (w^i_{xy})^{-1} e_{xy}$, $r_0 = e^{-1}_{xy} w^0_{xy}$. In $\mathcal{P}_{\mathcal{K}(X),T}$, replace r_i by $r_i r_0 = (w^i_{xy})^{-1} w^0_{xy}$, and finally eliminate the generator e_{xy} together with the relator $r_0 = w^0_{xy} e_{xy}$. □

Corollary 6.2.5. *Let X be a finite space of height 2. Then, different choices of a spanning tree in $\mathcal{H}(X)$ and preferred chains between related maximal and minimal elements of X give rise to AC-equivalent presentations of $\pi_1(X)$.*

Proof. The proof of Theorem 6.2.4 shows that for any choice of preferred chains, if we fix T a spanning tree in $\mathcal{H}(X)$, then $\mathcal{P}_{X,T} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)}$. Since the AC-equivalence class of $\mathcal{P}_{\mathcal{K}(X)}$ does not depend on the spanning tree T (see Lemma 1.4.4) the same holds for $\mathcal{P}_{X,T}$. □

We associate to each finite space X of height 2 a group presentation, which we will call $\mathcal{P}_X$ (without specifying the spanning tree chosen unless necessary). As a counterpart to Proposition 6.1.4, we derive a correspondence between AC-equivalent presentations and 3-deformable posets of height 2.

Theorem 6.2.6. *Let X, Y be finite spaces of height 2. Then,*

(i) $X \nearrow\!\!\searrow^{3} Y$ if and only if $\mathcal{P}_X \sim_{AC} \mathcal{P}_Y$;

(ii) X is homotopically trivial if and only if $\mathcal{P}_X$ is a balanced presentation of the trivial group.

Proof. By Theorem 2.1.16 and Lemma 2.2.4, $X \diagup\!\!\searrow^3 Y$ if and only if $\mathcal{K}(X) \diagup\!\!\searrow^3 \mathcal{K}(Y)$. By the correspondence proved in Theorem 1.4.5, $\mathcal{K}(X) \nearrow\!\!\searrow^3 \mathcal{K}(Y)$ if and only if $\mathcal{P}_{\mathcal{K}(X)} \sim_{AC} \mathcal{P}_{\mathcal{K}(Y)}$. By Theorem 6.2.4, $\mathcal{P}_{\mathcal{K}(X)} \sim_{AC} \mathcal{P}_X$ and $\mathcal{P}_{\mathcal{K}(Y)} \sim_{AC} \mathcal{P}_Y$. Hence, $X \diagup\!\!\searrow^3 Y$ is equivalent to $\mathcal{P}_X \sim_{AC} \mathcal{P}_Y$.

On the other hand, the Euler characteristic of X, denoted by $\chi(X)$, can be described in terms of the number of generator and relators of $\mathcal{P}_X$. Let T be the spanning tree on $\mathcal{H}(X)$ used to construct $\mathcal{P}_X$.

$$\begin{aligned}
\chi(X) &= \left|0\text{-chains}\right| - \left|1\text{-chains}\right| + \left|2\text{-chains}\right| && (6.1)\\
&= \left|X\right| - \Big(\left|E(\mathcal{H}(X))\right| + \left|\{x < y : h(x) = 0, h(y) = 2\}\right| \Big) + \sum_{\substack{x<y\\ h(x)=0\\ h(y)=2}} \left|\{z : x < z < y\}\right| && (6.2)\\
&= \left|X\right| - \Big(\left|E(T)\right| + \left|E(\mathcal{H}(X)) \smallsetminus E(T)\right| \Big) + \sum_{\substack{x<y\\ h(x)=0\\ h(y)=2}} \Big(\left|\{z : x < z < y\}\right| - 1 \Big) && (6.3)\\
&= 1 - \left|\text{generators of } \mathcal{P}_X\right| + \left|\text{relators of } \mathcal{P}_X\right|. && (6.4)
\end{aligned}$$

Equality (6.3) holds because

$$\left|E(\mathcal{H}(X))\right| = \left|E(T)\right| + \left|E(\mathcal{H}(X)) \smallsetminus E(T)\right|$$

and

$$-\left|\{x < y : h(x) = 0, h(y) = 2\}\right| + \sum_{\substack{x<y\\ h(x)=0\\ h(y)=2}} \left|\{z : x < z < y\}\right| = \sum_{\substack{x<y\\ h(x)=0\\ h(y)=2}} \Big(\left|\{z : x < z < y\}\right| - 1 \Big).$$

Equality (6.4) holds because T is a spanning tree, and thus $\left|E(T)\right| = \left|X\right| - 1$.

According to the formula above, if X is homotopically trivial, then $\chi(X) = 1$ and in consequence |generators of $\mathcal{P}_X$| = |relators of $\mathcal{P}_X$|. Conversely, if $\mathcal{P}_X$ is a balanced presentation of the trivial group, then $\pi_1(X) = 0$ and $\chi(X) = 1$. Hence $H_2(X) = 0$. Since X has height 2, $H_i(X) = 0$ for all $i > 2$. By Hurewicz Theorem, $\pi_i(\mathcal{K}(X)) = 0$ for all i and therefore X is homotopically trivial. □

We now extend the class of good choices of colorings of a poset to collapsible subdiagrams containing a spanning tree. We need some previous results.

Recall from Chapter 1 that if K is a finite cell complex of dimension 2 and T is a spanning tree of K^1, then $K \nearrow\!\!\searrow^3 K/T$ and $\mathcal{P}_K$ is defined as a presentation of the fundamental group of K/T. It is well known that if $C \leq K$ is a simply connected subcomplex, K and K/C have the same fundamental group. The fundamental group of K/C can be presented with generators the 1-cells of K not in C, and relators given by the words spelled in the boundary of the 2-cells of $K \setminus C$. In general, it is not known whether $K \nearrow\!\!\searrow^3 K/C$. We will see that if C is collapsible and contains all vertices of K, then they belong to the same AC-equivalence class.

Proposition 6.2.7. *Let K be a cellular complex of dimension 2. Let $A \leq K$ be a collapsible subcomplex containing all vertices of K. Then $\mathcal{P}_{K/A} \sim_{AC} \mathcal{P}_K$.*

Proof. Since A is a collapsible subcomplex of K, there exists a sequence of elemental collapses of decreasing dimension from A to a point. Let T be the maximal 1-dimensional subcomplex of K in that sequence. Thus,

$$A \searrow^{e} A \smallsetminus \{e_1, e_1'\} \searrow^{e} A \smallsetminus \{e_1, e_1', e_2, e_2'\} \searrow^{e} \cdots \searrow^{e} A \smallsetminus \{e_1, e_1', e_2, e_2', \cdots, e_k, e_k'\} = T \searrow *$$

and T is a spanning tree in K^1. We will see that $\mathcal{P}_{K/A} \sim_{AC} \mathcal{P}_K$, where the latter one is constructed using the spanning tree T. Call $A_0 := A$, $A_i := A_{i-1} \smallsetminus \{e_i, e_i'\}$ for $i \geq 1$. We will prove inductively that $\mathcal{P}_{K,A_i} \sim_{AC} \mathcal{P}_{K,A_{i+1}}$. Since $A_{i-1} \searrow^{e} A_i$, e_i is a free face of e_i' in A_{i-1}. Notice that if

$$\mathcal{P}_{K/A_{i-1}} = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_m \rangle$$

then,

$$\mathcal{P}_{K/A_i} = \langle x_1, x_2, \cdots, x_n, e_i \mid e_i^{\epsilon}, e_i^{\epsilon_1} r_1, e_i \epsilon_2 r_2, \cdots, e_i^{\epsilon_m} r_m \rangle,$$

where $\epsilon = \pm 1$, depending on the orientation of e_i', and

$$\epsilon_j = \begin{cases} \pm 1 & \text{if } e_i < e_j' \text{ (where the sign depends on the orientation of } e_i) \\ 0 & \text{if } e_i \not< e_i'. \end{cases}$$

for all $1 \leq j \leq m$. Note that the first relator corresponds to the word spelled on the boundary of e_i'. Now, for every j such that $\epsilon_j \neq 0$, multiply to the left by $e_i^{-\epsilon_j}$ (inverting the relation e_i^{ϵ} if necessary). Finally, invert the relation e_i^{ϵ} such that the exponent turn to be 1, and simplify the generator e_i with the relator e_i. As a result of the previous sequence of AC-transformations, we transform $\mathcal{P}_{K/A}$ into $\mathcal{P}_{K/T}$. □

Remark 6.2.8. The hypothesis that A is spanning may be dropped. It is enough to extend a spanning tree of A to a spanning tree of K. In Corollary 6.4.10 we will see that $K \nearrow\searrow^{3} K/A$, giving an alternative proof of the last proposition.

Theorem 6.2.9. *Let X be a finite space of height 2 and A be a spanning subdiagram of $\mathcal{H}(X)$. If A is collapsible (as finite space), then $\mathcal{P}_{X,A} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)}$.*

Proof. Notice that since A is collapsible, $\mathcal{K}(A)$ is collapsible. We will exhibit a sequence of AC-moves to transform $\mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)}$ into $\mathcal{P}_{X,A}$. The proof is similar to that of Theorem 6.2.4, and we adopt the notation therein. On the one hand,

$$\mathcal{P}_{X,A} = \langle e_1, \cdots, e_r \mid (w_{xy}^i)^{-1} w_{xy}^0 : i \rangle.$$

On the other hand, since $\mathcal{K}(A)$ is a clique complex, for every extremal pair x, y and every pair of 2-chains $(w_{xy}^i)^{-1}$ and w_{xy}^0, there are two 2-simplices

$$\mathcal{P}_{\mathcal{K}(X),\mathcal{K}(A)} = \langle e_1, \cdots, e_r, e_{xy} \mid e_{xy}^{-1} w_{xy}^0,\ (w_{xy}^i)^{-1} e_{xy} : i \rangle.$$

Call $r_i = (w_{xy}^i)^{-1} * e_{xy}$, $r_0 = e_{xy}^{-1} w_{xy}^0$ and replace in $\mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)}$ the relator r_i by $r_i r_0 = (w_{xy}^i)^{-1} w_{xy}^0$, and finally eliminate the generator e_{xy} together with the relator $r_0 = w_{xy}^0 e_{xy}$. □

Corollary 6.2.10. *Let X be a finite poset of height 2, A a spanning subdiagram of $\mathcal{H}(X)$. If A is collapsible as finite space, then $\mathcal{P}_{X,A} \sim_{AC} \mathcal{P}_X$.*

Proof. By Proposition 6.2.7 and Theorem 6.2.9, $\mathcal{P}_{X,A} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)/\mathcal{K}(A)} \sim_{AC} \mathcal{P}_{\mathcal{K}(X)}$. Finally, by Theorem 6.2.4, $\mathcal{P}_{\mathcal{K}(X)} \sim_{AC} \mathcal{P}_X$. □

6.3 Applications of colorings to the Andrews-Curtis conjecture

We relate the previous construction of $\mathcal{P}_{X,A}$ with the construction of $X_{\mathcal{P}}$ and deduce a method to obtain new presentations in the same AC-equivalence class of a given presentation $\mathcal{P}$.

Theorem 6.3.1. *Let $\mathcal{P}$ be a group presentation and A be a spanning collapsible subdiagram of $\mathcal{H}(X_{\mathcal{P}})$. Then $\mathcal{P} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$.*

Proof. It is well known that $\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}}$ (see Theorem 1.4.5). Since $\mathcal{K}(X_{\mathcal{P}})$ is the second barycentric subdivision of $K_{\mathcal{P}}$, by Lemma 2.2.3, $K_{\mathcal{P}} \diagup\!\!\searrow^{3} \mathcal{K}(X_{\mathcal{P}})$. Thus, by Theorem 1.4.5, $\mathcal{P}_{K_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})}$. Now, by Theorem 6.2.4 and Corollary 6.2.10, $\mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$. We conclude that $\mathcal{P} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$. □

Remark 6.3.2. If $\mathcal{P}$ is a group presentation and A a spanning collapsible in $\mathcal{H}(X_{\mathcal{P}})$, then $\mathcal{P} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$. We will estimate the (sufficient) number of AC-transformations to obtain $\mathcal{P}_{X_{\mathcal{P}},A}$ from $\mathcal{P}$. Consider the following chain of AC-equivalences (see the proof of Theorem 6.3.1):

$$\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})} \sim_{AC} \mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})/\mathcal{K}(A)} \sim \mathcal{P}_{X_{\mathcal{P}},A}.$$

Let n be the number of generators of $\mathcal{P}$ and m the number of relators. Let k be the total relator length. As we have seen in the proof of Theorem 1.4.5, we can choose the orientations and attaching maps in $K_{\mathcal{P}}$ in such a way that $\mathcal{P}_{K_{\mathcal{P}}} = \mathcal{P}$. Thus, the number of AC-transformations required to go from $\mathcal{P}$ to $\mathcal{P}_{K_{\mathcal{P}}}$ is proportional to $n+m$, or $O(n+m)$ using the big O notation. Now, since $K_{\mathcal{P}} \diagup\!\!\searrow^{3} \mathcal{K}(X_{\mathcal{P}}) = (K_{\mathcal{P}})''$, the quantity of AC-transformations needed to take $\mathcal{P}_{K_{\mathcal{P}}}$ to $\mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})}$ is at most a multiple of the number of elementary expansions and collapses performed to obtain $(K_{\mathcal{P}})''$ from $K_{\mathcal{P}}$. In general, the amount of elementary collapses and expansions employed to get from a regular cell complex K to its barycentric subdivision K' is a multiple of the number of cells of the latter. Hence, it is possible to transform $\mathcal{P}_{K_{\mathcal{P}}}$ into $\mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})}$ in $O(k)$ AC-transformations. On the other hand, the number of AC-transformations from $\mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})}$ to $\mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})/\mathcal{K}(A)}$ is of the order of the amount of elementary collapses from $\mathcal{K}(A)$ to a spanning tree in $K^1_{X_{\mathcal{P}}}$, which is at most $O(k)$. Lastly, the estimated number of AC-transformations from $\mathcal{P}_{\mathcal{K}(X_{\mathcal{P}})/\mathcal{K}(A)}$ to $\mathcal{P}_{X_{\mathcal{P}},A}$ is proportional to the number of extremal pairs (see Definition 6.2.1), that is $O(k(n+m))$.

We can generate AC-equivalent presentations of a given balanced presentation $\mathcal{P}$ by just choosing a spanning subdiagram A of $\mathcal{H}(X_{\mathcal{P}})$ and constructing the balanced presentation $\mathcal{P}_{X_{\mathcal{P}},A}$. This procedure could generate more tractable AC-equivalent presentations without specifying the actual AC-transformation, since it is "hidden" in the topological and combinatorial treatment of the presentation. Although the number of movements involved is estimated in Remark 6.3.2 it not necessary to explicitly compute them to obtain the desired AC-equivalence. We implemented the algorithm in the SAGE platform [S+17] (see Appendix 6.A). See also Section 6.7 for experimental results of the previous theory.

6.4 Matchings and Discrete Morse Theory

Discrete Morse theory was introduced in [For98]. It consists of a discrete approach to the ideas behind the original Morse theory used to analyze the topology of smooth manifolds. As such, discrete Morse theory involves a notion of Morse function. The discrete Morse functions assign a real number to each cell of the complex. They give information about the number of cells in each dimension of a new cell complex homotopy equivalent to the original one. Although discrete Morse theory has proved to be useful in practical computations of homology groups, it is generally not sufficient to recover the homotopy type of the original complex, since the construction of the new complex is ad-hoc and it does not provide a combinatorial description.

We will present a refinement of the theory, showing that it actually provides a method to simplify an n-dimensional complex through an $(n+1)$-deformation.

6.4.1 Forman's discrete Morse theory

We recall briefly the main results of Forman's version of the classic Morse theory for smooth manifolds in the context of regular cell complexes. We refer the reader to [For98, For02] for a more complete exposition.

Morse theory is built over the notion of *Morse function*, which can be thought of as a specific way of labeling the cells of a regular complex which is *almost increasing with respect the dimension*.

Definition 6.4.1. Let K be a regular cell complex. A *discrete Morse function* on K is a map $f : K \to \mathbb{R}$ such that for every cell e^n in K:

$$\left|\left\{e^{n+1} > e^n : f(e^{n+1}) \leq f(e^n)\right\}\right| \leq 1.$$

$$\left|\left\{e^{n-1} < e^n : f(e^{n-1}) \geq f(e^n)\right\}\right| \leq 1.$$

Definition 6.4.2. Let f be a discrete Morse function on K. An n-cell $e^n \in K^n$ is a *critical cell of index n* if

$$\left|\left\{e^{n+1} > e^n : f(e^{n+1}) \leq f(e^n)\right\}\right| = 0.$$

and

$$\left|\left\{e^{n-1} < e^n : f(e^{n-1}) \geq f(e^n)\right\}\right| = 0.$$

A Morse function induces an ordering in the cells, which determines *level subcomplexes* of K.

Definition 6.4.3. Let f be a discrete Morse function on a cellular complex K. For $c \in \mathbb{R}$ define the *level subcomplex of K* as

$$K(c) = \bigcup_{\substack{e \in K \\ f(e) \leq c}} \bar{e}$$

The following lemmas show how the homotopy type of the level subcomplexes evolves as the label increases, according to the type of cells which are successively added.

Lemma 6.4.4. *[For98, Thm. 3.3]. Let f be a discrete Morse function on a regular cell complex K. If $a < b$ are real numbers such that the cells e with $f(e) \in (a, b]$ are not critical, then $K(b) \searrow K(a)$.*

Lemma 6.4.5. *[For98, Thm. 3.4] Let f be a discrete Morse function on a regular cell complex K and suppose e_n is the only critical cell with $f(e_n) \in (a, b]$. Then there is a continuous map $\varphi : \partial D_n \to K(a)$ such that $K(b)$ is homotopy equivalent to $K(a) \cup_\varphi D^n$.*

The following is one of the central theorems of discrete Morse theory.

Theorem 6.4.6. *[For98, Thm. 10.2] Let K be a regular cell complex with a discrete Morse function $f : K \to \mathbb{R}$. Then K is homotopy equivalent to a cellular complex with exactly one cell of dimension k for every critical cell of index k.*

In 2000, Chari made a combinatorial description of a discrete Morse function f as a set M_f of pairings of cells, where

$$\{e, e'\} \in M_f \text{ if and only if } e \leq e' \text{ and } f(e) \geq f(e'). \tag{6.5}$$

As Forman had realized, the actual values of f on every cell are not essential, only the information of the critical cells is relevant. Thus, the problem of finding Morse functions can be reduced to simply deciding how to pair cells with their faces or cofaces, rather than assigning values over the entire complex such that the conditions of Definition 6.4.1 are simultaneously satisfied by every cell. The critical cells (see Definition 6.4.2) are in correspondence with those that do not appear in the pairing. Every Morse function has an associated pairing but the converse does not holds. Thus, he had to determine when a pairing M is equal to M_f for some f. Moreover, when such an f exists, it was not unique; each M_f actually corresponds to an equivalence class of discrete Morse functions, where f and f' are equivalent if they have the same critical cells and the same pairs of cells satisfying (6.5).

It is easy to deduce from the regularity of K that in a pairing M_f associated with a discrete Morse function f, each cell of K is involved in at most one pair, that is, M is actually a *matching*. This was a necessary but not sufficient condition for the existence of a Morse function associated to M. He said that a matching M is *acyclic* if the directed graph obtained by reversing the orientation of the matched pairs of cells in the Hasse diagram of $\mathcal{X}(K)$ is acyclic. We call it the *modified Hasse diagram of the poset with respect to M* and we denote it by $\mathcal{H}_M(\mathcal{X}(K))$. Chari proved that (classes of) Morse functions on K are in correspondence with acyclic matchings on $\mathcal{H}(X)$.

The following result summarizes the reformulation of Morse functions.

Theorem 6.4.7 (Forman-Chari)**.** *[Cha00] A pairing M on a regular cell complex is M_f for some Morse function f if and only if M is an acyclic matching.*

Proof. Every discrete Morse function f has an associated matching M_f defined as in (6.5). If the modified Hasse diagram of $\mathcal{X}(K)$ with respect to M_f had a cycle

$$e_1 \prec e'_1 \succ e_2 \prec e'_2 \succ \cdots e_n \prec e'_k \succ e_1,$$

with $(e_i, e'_i) \in M$ then we would have

$$f(e_1) > f(e'_1) > f(e_2) > f(e'_2) > \cdots > f(e_k) > f(e'_k) > f(e_1),$$

that is, $f(e_1) > f(e_1)$ which is a contradiction. Therefore, M_f is acyclic.

Conversely, M is acyclic if and only if the corresponding modified Hasse diagram with respect to M is acyclic. Thus, we can perform a *topological sort* of its vertices, i.e, a linear ordering $<_L$ of the cells such that if there is a directed edge (e, e'), then $e <_L e'$.[1] Let $\{e_1, e_2, \cdots, e_n\}$ be such total order. Then, assign values to the cells according to their labels: define $f(e_i) = i$. If there is an edge from e_i to e_j, then $e_i <_L e_j$, so $f(e_i) < f(e_j)$. By the construction of $\mathcal{H}_M(\mathcal{X}(K))$, f assigns higher values to higher-dimensional cells, except in cases where $\{e, e'\} \in M$. Since no cell appears in more than one pair of M, for every cell e, this function f can, at most, assign a higher value than $f(e)$ to one face of e, or assign a lower value to one coface of e. Hence, f is a discrete Morse function such that $M = M_f$. □

6.4.2 Formal deformations and internal collapses

Motivated by Kozlov's exposition of discrete Morse theory (see [Koz08, Ch 11]), we formalize the notion of *internal collapse*, a generalization of the usual collapses that can be thought of as a short way of performing a deformation. Internal collapses will be the key to a better understanding of discrete Morse theory and its connection with formal deformations.

The following inspiring result is a consequence of Lemma 1.4.6 and it is the basic idea behind the internal collapses.

Proposition 6.4.8. *Let K be a cellular complex of dimension less than or equal to n. Let $\varphi : \partial D^n \to K$ be the attaching map of a n-cell. If $K \searrow L$, then $K \cup_\varphi D^n \overset{n+1}{\nearrow\searrow} L \cup_{\tilde{\varphi}} D^n$, where $\tilde{\varphi} = r\varphi$ with r the canonical strong deformation retract $r : K \to L$.*

Proof. Let $j : L \to K$ be the inclusion. Then $jr\varphi \simeq_H \varphi$ with a homotopy $H : \partial D^n \times I \to K$. We can perform the following sequence of expansions and collapses

$$K \cup_\varphi D^n \nearrow \left(K \cup_\varphi D^n\right) \cup_{jr\varphi} D^n \cup_H D^n \times I \searrow K \cup_{jr\varphi} D^n \searrow L \cup_{r\varphi} D^n.$$

Since the dimension of $\left(K \cup_\varphi D^n\right) \cup_{jr\varphi} D^n \cup_H D^{n+1}$ is $n + 1$, we can conclude that

$$K \cup_\varphi D^n \overset{n+1}{\nearrow\searrow} L \cup_{\tilde{\varphi}} D^n.$$

□

The following result asserts that we can perform a more general procedure of deformation than the described in Proposition 6.4.8.

[1] Every acyclic directed graph G admits a topological sort. Start with an empty list $L = \varnothing$. Since G is acyclic, there must exist a vertex having no inward-oriented arrows. Add it to the queue of L and remove the vertex and its associated outward arrows from the graph. The remaining subgraph is still acyclic, so continue inductively until no vertices remain.

Theorem 6.4.9. *Let* $K \cup \bigcup_{i=1}^{N} e_i$ *be a cellular complex where* K *is a subcomplex of dimension at most* k *and such that* $k \leq \dim(e_i) \leq \dim(e_{i+1})$ *for all* i*. Denote by* $\varphi_j : \partial D_j \to K \cup \bigcup_{i<j} e_i$ *the attaching map of* e_j*. If* $K \searrow L$*, then*

$$K \cup \bigcup_{i=1}^{N} e_i \stackrel{n+1}{\nearrow\searrow} L \cup \bigcup_{i=1}^{N} \tilde{e}_i$$

with $n = \dim(e_N)$ *and* $\tilde{\varphi}_j : \partial D_j \to L \cup \bigcup_{i<j} \tilde{e}_i$ *the attaching map of* $\tilde{e}_j$ *defined inductively by* $\tilde{\varphi}_1 = r\varphi_1$ *and if* $j > 1$, $\tilde{\varphi}_j = f_j \varphi_j$*, where* $r : K \to L$ *is a strong deformation retraction and* $f_j : K \cup \bigcup_{i<j} e_i \to L \cup \bigcup_{i<j} \tilde{e}_i$ *is a deformation.*

Proof. We proceed by induction in N. If $N = 1$, the assertion follows from Proposition 6.4.8. Suppose that for every $j \leq N$, there exist deformations $f_j : K \cup \bigcup_{i<j} e_i \to L \cup \bigcup_{i<j} \tilde{e}_i$, defining $\tilde{\varphi}_j = f_j \varphi_j$. By inductive hypothesis, $K \cup \bigcup_{i=1}^{N} e_i \stackrel{n+1}{\nearrow\searrow} L \cup \bigcup_{i=1}^{N} \tilde{e}_i$ and then, there exists a deformation $f_N : K \cup \bigcup_{i=1}^{N} e_i \to L \cup \bigcup_{i=1}^{N} \tilde{e}_i$. Define $\tilde{\varphi}_{N+1} = f_N \varphi_{N+1}$ and take $L \cup \bigcup_{i=1}^{N+1} \tilde{e}_i$. We will prove that $K \cup \bigcup_{i=1}^{N+1} e_i \stackrel{n+2}{\nearrow\searrow} L \cup \bigcup_{i=1}^{N+1} \tilde{e}_i$. In fact,

$$K \cup \bigcup_{i=1}^{N+1} e_i \nearrow K \cup \bigcup_{i=1}^{N+1} e_i \cup \bigcup_{i=1}^{N+1} D_i \cup \bigcup_{i=1}^{N+1} D_i \times I$$

with D_i attached with the map $j_i f_i$ where j_i is a homotopical inverse of f_i, and $D_i \times I$ attached with the homotopy H_i between $j_i f_i$ and the identity. Now

$$K \cup \bigcup_{i=1}^{N+1} e_i \cup \bigcup_{i=1}^{N+1} D_i \cup \bigcup_{i=1}^{N+1} D_i \times I \searrow K \cup \bigcup_{i=1}^{N+1} D_i \searrow L \cup \bigcup_{i=1}^{N+1} \tilde{e}_i.$$

□

Corollary 6.4.10. *If* K *is a* n*-dimensional cell complex and* L *is a collapsible subcomplex of* K*, then* $K \stackrel{n+1}{\nearrow\searrow} K/L$*.*

Remark 6.4.11. As consequence of Corollary 6.4.10, we obtain another proof of the fact that if T is a spanning tree on the 1-skeleton of a n-dimensional cell complex K then $K \stackrel{n+1}{\nearrow\searrow} K/T$.

Following Kozlov's terminology, we have the following

Definition 6.4.12. If $K \searrow L$, we say that there is a *internal collapse* from $K \cup \bigcup_{i=1}^{n} e_i$ to $L \cup \bigcup_{i=1}^{n} \tilde{e}_i$, where the cells $\tilde{e}_i$ are attached as described in Theorem 6.4.9.

The next result shows that it is possible to compose successive internal collapses.

Corollary 6.4.13. *Let $L_1 \leq K_1 \leq L_2 \leq K_2 \leq \cdots \leq L_{N-1} \leq K_{N-1} \leq L_N$ be a chain of cellular subcomplexes of L_N such that $K_i \searrow L_i$ for all i. If $L_{i+1} = K_i \cup \bigcup_{j=1}^{k_i} e^i_j$, then*

$$L_N \overset{n+1}{\nearrow\searrow} L_1 \cup \bigcup_{i=1}^{N} \bigcup_{j=1}^{k_i} \tilde{e}^i_j,$$

with $n = \dim(L_N)$.

Proof. Just apply successively Theorem 6.4.9

$$L_N = K_{N-1} \cup \bigcup_{j=1}^{k_{N-1}} e^{N-1}_j \overset{n+1}{\nearrow\searrow} L_{N-1} \cup \bigcup_{j=1}^{k_{N-1}} \tilde{e}^{N-1}_j = K_{N-2} \cup \bigcup_{j=1}^{k_{N-2}} e^{N-2}_j \cup \bigcup_{j=1}^{k_{N-1}} \tilde{e}^{N-1}_j \overset{n+1}{\nearrow\searrow} \cdots$$

$$\overset{n+1}{\nearrow\searrow} L_2 \cup \bigcup_{i=2}^{N-1} \bigcup_{j=1}^{k_i} \tilde{e}^i_j = K_1 \cup \bigcup_{j=1}^{k_1} e^1_j \cup \bigcup_{i=2}^{N-1} \bigcup_{j=1}^{k_i} \tilde{e}^i_j \overset{n+1}{\nearrow\searrow} L_1 \cup \bigcup_{i=1}^{N-1} \bigcup_{j=1}^{k_i} \tilde{e}^i_j.$$

Here, the attaching maps of the cells e^i_j are successively modified according to the Theorem 6.4.9. By an abuse of notation, we still call them $\tilde{e}^i_j$. □

6.4.3 An $(n+1)$-deformation version of discrete Morse theory

We will see that discrete Morse theory gives a way to simplify the cell structure of a complex without changing its *simple* homotopy type. Concretely, any matching in the Hasse diagram of the face poset of a cell complex K can be interpreted as an ordered sequence of internal collapses in its cell structure.

Given X a poset and M an acyclic matching in $\mathcal{H}(X)$, it is possible to find a topological sort L of $\mathcal{H}_M(X)$ such that it respects the increasing height of critical points and whenever $(x, y) \in M$, the points x, y follow consecutively in L as follows (see [Koz08, Thm. 11.2.]). Define the ordered list L with the following inductive method. Start with $L = \emptyset$. In each step, denote by $\mathfrak{m}$ the set of minimal elements of $X \smallsetminus L$. Then, if any of the elements in $\mathfrak{m}$ is unmatched, add to the queue of L an unmatched element of $\mathfrak{m}$ with the lowest height in X. Else, all the elements in $\mathfrak{m}$ are matched. Consider the subposet $X_{\mathfrak{m}}$ of X with elements $\{(x, y) \in M : x \in \mathfrak{m}\}$. Since the matching M restricted to $X_{\mathfrak{m}}$ is acyclic, there must exist an element x_0 in $\mathfrak{m}$ such that $(x_0, y_0) \in M$ and $y_0 \not> x$ for every x in $\mathfrak{m} \smallsetminus \{x_0\}$. Add to the queue of L the elements y_0, x_0. The previous procedure generates a linear extension L of $\mathcal{H}_M(X)$ in which the critical cells are in increasing order of dimension, and every pair of matched cells is consecutive. We shall call it a *preferred topological sort* of $\mathcal{H}_M(X)$.

We can reformulate the notion of level subcomplex only in terms of the matching. If M is an acyclic matching in the Hasse diagram of a regular cell complex K and $L : e_1 \leq e_2 \leq \cdots \leq e_n$

is a preferred topological sort of $\mathcal{H}_M(X)$, then the Definition 6.4.3 of level subcomplex can be restated as

$$K(c) = \bigcup_{\substack{e_i \in K \\ i \leq c}} \bar{e}_i$$

that is, the subcomplex of K generated by the first i cells according to the total order L.

The following results are the combinatorial analogues of the Lemmas 6.4.4 and 6.4.5 and Theorem 6.4.6.

Lemma 6.4.14. *Let M be an acyclic matching in a regular cell complex K and let L be a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(K))$. If $a < b$ are real numbers such that the cells e_i with $i \in (a, b]$ are not critical, then $K(b) \searrow K(a)$.*

Lemma 6.4.15. *Let M be an acyclic matching on a regular cell complex K and let L be a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(K)$. If e_i is the only critical cell with $i \in (a, b]$, then there is a continuous map $\varphi : \partial D^n \to K(a)$ such that $K(b)$ is homotopy equivalent to $K(a) \cup_\varphi D^n$.*

Theorem 6.4.16. *Let K be a regular cell complex with an acyclic matching M on $\mathcal{H}(K)$. Then K is homotopy equivalent to a cellular complex with exactly one cell of dimension k for every critical cell of dimension k.*

We present next a simple homotopy version of Theorem 6.4.16, with an explicit construction of the cellular complex equivalent to the original one, and bounds on the deformation. This result will be useful for studying 3-deformations and the Andrews-Curtis conjecture.

Let K be a regular cell complex and M be an acyclic matching in $\mathcal{X}(K)$. If L is a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(K))$, then every pair of matched cells are consecutive in L. If (e_i, e_{i+1}) is a matched pair of cells, then by Lemma 6.4.14, there is a collapse of the level subcomplexes $K(i+1) \searrow K(i-1)$. Denote by K_M, *the reduced cellular complex associated to the matching M (and L)*, the cellular complex obtained after performing successively the internal collapses determined by the matched pairs of cells.

Theorem 6.4.17. *Let K be a regular cell complex of dimension n and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$. Then $K \overset{n+1}{\nearrow\searrow} K_M$.*

Proof. It is a direct consequence of Theorem 6.4.9, Corollary 6.4.13 and Theorem 6.4.16. □

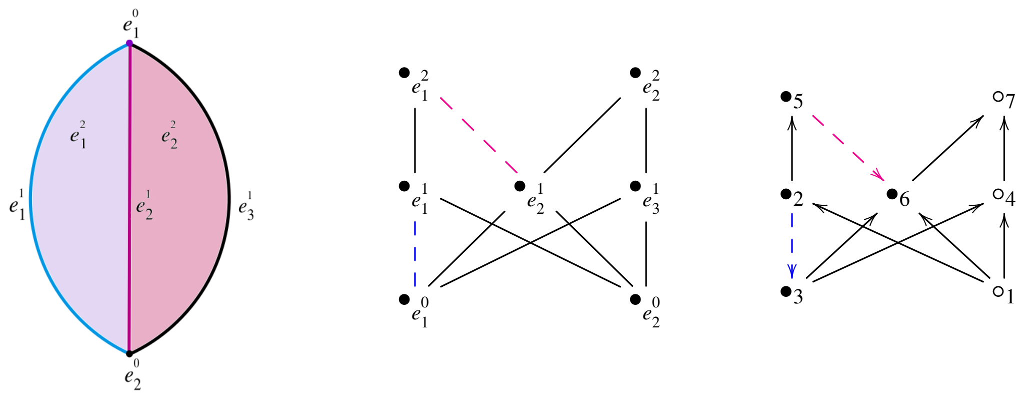


Figure 6.9: A regular 2-complex K, an acyclic matching M in $\mathcal{H}(\mathcal{X}(K))$ and a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(K))$.

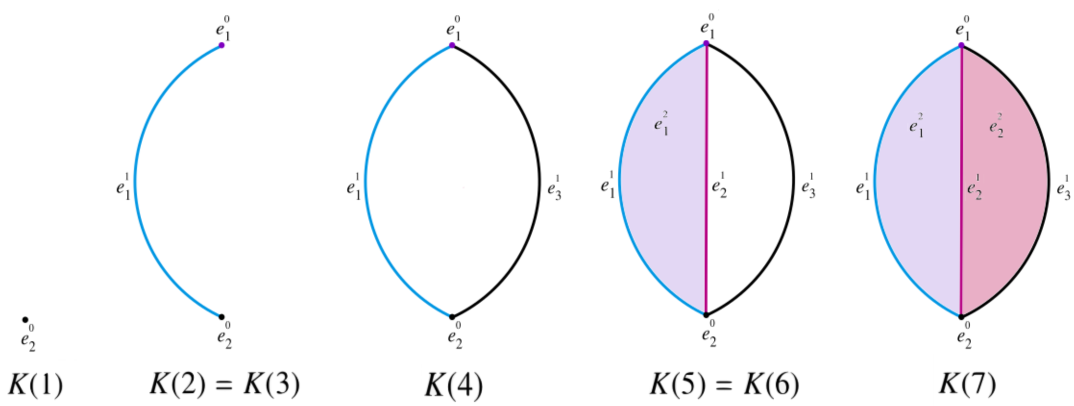


Figure 6.10: The level subcomplexes associated to the preferred topological of $\mathcal{H}_M(\mathcal{X}(K))$.

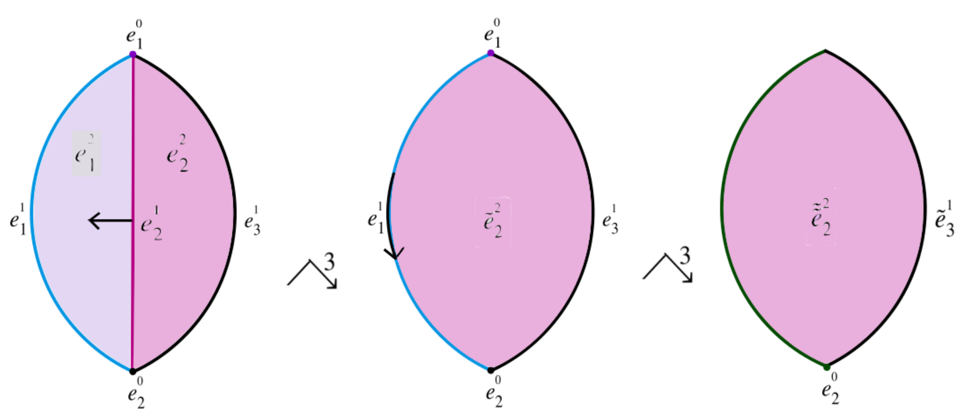


Figure 6.11: The 3-deformation from K to K_M.

As a consequence of this version of Discrete Morse Theory, we get a new proof in terms of acyclic matchings of the fact that an n-dimensional cell complex K $(n+1)$-deforms to K/T, where T is a spanning tree of K^1 (see Chapter 1). We state and prove a lemma first.

Lemma 6.4.18. *Let K be a regular cell complex and M be a matching in the subposet of $\mathcal{X}(K)$ of cells of dimension 0 and 1 with only one critical cell of dimension 0. Then M is acyclic if and only if the subcomplex $T = \bigcup_{e\in M} \bar{e}$ of matched cells is a spanning tree in the 1-skeleton of K.*

Proof. Observe that the subgraph of $\mathcal{H}(\mathcal{X}(K))$ formed by the vertices of height 0 and 1 is the barycentric subdivision of the 1-skeleton of K. A cycle

$$e_1 \prec e'_1 \succ e_2 \prec e'_2 \succ \cdots e_n \prec e'_k \succ e_1$$

in $\mathcal{H}_M(\mathcal{X}(K))$, with e_i of height 0 and e'_i of height 1, is in correspondence with a cycle

$$e_1, e_2, \cdots, e_k, e_1$$

in the 1-dimensional subcomplex of K of matched cells. The result follows easily from this fact. □

Corollary 6.4.19. *Let K be a regular cell complex of dimension n and M a matching in the subposet of $\mathcal{X}(K)$ of cells of dimension 0 and 1 with only one critical cell of dimension 0. Let T be the associated spanning tree in K^1. Then K_M is homeomorphic to K/T and $K \overset{n+1}{\diagup\!\!\searrow} K/T$.*

6.5 Applications of Morse theory to the Andrews-Curtis conjecture

We are particularly interested in the consequences of Theorem 6.4.17 when the dimension of the complex is 2.

Theorem 6.5.1. *Let K be a cellular regular cell complex of dimension 2 and let M be an acyclic matching in $\mathcal{H}(X(K))$. Then $K \overset{3}{\diagup\!\!\searrow} K_M$. In particular, $\mathcal{P}_K \sim_{AC} \mathcal{P}_{K_M}$.*

We will show a more concrete and algorithmically manageable version of Theorem 6.5.1 for group presentations.

Recall that if we are given a cell complex K of dimension 2, the group presentation $\mathcal{P}_K$ associated to K is a presentation of the fundamental group of K/T with T a spanning tree in K^1. By Lemma 6.4.18 and Corollary 6.4.19, we can describe $\mathcal{P}_K$ only in terms of matchings as the presentation of the fundamental group of K_M where M is an acyclic matching whose matched cells are of dimension 0 and 1, and it has only one critical cell of dimension 0.

We will provide an easy description of the presentation associated to K_M for a general matching M. Moreover, we present the previous theory in terms only of group presentations instead of cell complexes, which will be more tractable through the computer assistance.

Definition 6.5.2. Let $\mathcal{P}$ be a group presentation and r be a relator of $\mathcal{P}$ given by the word $w_1 x^\epsilon w_2$, where w_1 and w_2 are words on the generators, the generator x appear neither in w_1 nor in w_2, and $\epsilon = +1$ or -1. Then, the *equivalent expression of x inferred by r* is $(w_1^{-1} w_2^{-1})^\epsilon$.

Remark 6.5.3. If $\mathcal{P}$ is a group presentation and x is a generator of $\mathcal{P}$ such that it appears only once in a relator r, then $\mathcal{P}$ is AC-equivalent to the presentation obtained after eliminating the generator x and the relator r and replacing every occurrence of x in the other relators by its equivalent expression inferred by r. Indeed, suppose r' is another relator containing x. By cyclically permuting r' if necessary, we can assume r' reads as $x^\epsilon u$, with $\epsilon = 1$ or -1. We replace r' by the product sr' where s is a suitable cyclic permutation of r (or its inverse) to eliminate this occurrence of x. We iterate this procedure until no occurrence of x is left.

For instance, if $\mathcal{P} = \langle x, y, z \mid xyx^{-1}y^{-1},\ y^2z^3,\ zxz^{-1}y^{-1}\rangle$, then the equivalent expression of x inferred by $zxz^{-1}y^{-1}$ is $z^{-1}(z^{-1}y^{-1})^{-1}$, i.e., $z^{-1}yz$. Thus, $\mathcal{P}$ is AC-equivalent to the presentation $\tilde{\mathcal{P}} = \langle y, z \mid z^{-1}yzyz^{-1}y^{-1}zy^{-1},\ y^2z^3\rangle$.

At this point, a few words are in order about combinatorial complexes of dimension 2 and internal collapses. Recall that a cellular complex K of dimension 2 is called *combinatorial* if for each 2-cell e^2, its attaching map $\varphi : S^1 \to K^1$ sends each open 1-cell of some cellular structure on S^1 either homeomorphically onto an open 1-cell of K or collapses it to a 0-cell of K (see [HAM93]). Thus, we can think of the attaching map of a 2-cell in a combinatorial complex of dimension 2 simply as the ordered list of oriented 1-cells. Suppose that there is an internal collapse from the combinatorial 2-complex $K \cup \bigcup_{i=1}^n e_i$ to $L \cup \bigcup_{i=1}^n \tilde{e}_i$, where $K = L \cup \{a, e\} \searrow^{e} L$ and the attaching map of e is, say, $ax_1 \dots x_r$. We endow the cellular complex $L \cup \bigcup_{i=1}^n \tilde{e}_i$ with a combinatorial structure as follows. For every 2-cell containing edge a^ϵ (where, as usual, $\epsilon = 1$ or -1 and by a^{-1} we mean the 1-cell a traversed with the opposite orientation), modify its attaching map by replacing each occurrence of a^ϵ by $(x_1 \dots x_r)^{-\epsilon}$ and leaving the other cells unchanged. Therefore, after performing a sequence of internal collapses to a regular complex we obtain a complex with a natural combinatorial structure.

Definition 6.5.4. Let K be a regular cell complex of dimension 2. Let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$ such that there is only one critical cell of dimension 0. Denote by M_0 the subset of matched pairs of cells of dimension 0 and 1, and by $M_r = \{(x_1, y_1), (x_2, y_2), \cdots, (x_r, y_r)\}$ the subset of matched pairs of cells of dimension 1 and 2. Let L be a preferred topological sort in $\mathcal{H}_M(\mathcal{X}(K))$. Sort M_r respecting the total order L, that is, $M_r : y_1 < x_1 < y_2 < x_2 < \cdots < y_r < x_r$, with $(x_i, y_i) \in M$. The *group presentation* $\mathcal{Q}_{\mathcal{X}(K),M}$ *associated to the matching* M is the presentation $\mathcal{Q}_r$ defined by the following iterative procedure:

- $\mathcal{Q}_0$ is the standard presentation $\mathcal{P}_K$ constructed using the spanning tree T induced by M_0 (see Lemma 6.4.18). The generators of Q_0 are the unmatched 1-cells of K according to the matching M_0, and its relators are the words induced by the attaching maps of the 2-cells of K.

- For $0 \leq i < r$, let $\mathcal{Q}_{i+1}$ be the presentation obtained from $\mathcal{Q}_i$ after removing the relator associated to y_{r-i} and the generator x_{r-i}, and replacing every occurrence of the generator x_{r-i} in the remaining relators by the equivalent expression of x_{r-i} inferred by the relator associated to y_{r-i}.

It follows from the previous remarks that the group presentation $\mathcal{Q}_{\mathcal{X}(K),M}$ associated to a matching M in $\mathcal{H}(\mathcal{X}(K))$ is AC-equivalent to $\mathcal{P}_K$.

Theorem 6.5.5. *Let K be a regular cell complex of dimension 2, and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$ with only one critic cell of dimension 0. Then, $\mathcal{Q}_{\mathcal{X}(K),M} = \mathcal{P}_{K_M}$ for a suitable choice of orientations and basepoints in K_M.*

Proof. Let $r > 0$. Notice that the combinatorial movement accomplished to get $\mathcal{Q}_1$ from $\mathcal{Q}_0$ parallels exactly the geometric description of an internal collapse provided in the paragraph following Remark 6.5.3, where the collapse is the one indicated by the pair (x_r, y_r). Since the matching M is acyclic, for every $0 \leq i < r$ the relation corresponding to cell y_{r-i} in $\mathcal{Q}_i$ contains a unique occurrence of generator x_{r-i}. Therefore, presentations $\mathcal{Q}_i$ and $\mathcal{Q}_{i+1}$ are AC-equivalent and the sequence of AC-transformation employed to get from one to the other matches perfectly the corresponding internal collapse in complex K. Denote by K_{M_r} the complex obtained from K after performing the internal collapses induced by the pairs in M_r. Since $K_M = K_{M_r}/T$, it follows that $\mathcal{Q}_r = \mathcal{Q}_{\mathcal{X}(K),M}$ is the standard presentation associated to K_M for the right choice of orientations and basepoints. □

By Theorem 6.5.1, $K \nearrow\!\!\searrow^3 K_M$. Thus, $\mathcal{Q}_{\mathcal{X}(K),M}$ is another representative of the AC-class of $\mathcal{P}_K$.

Corollary 6.5.6. *Let K be a regular cell complex of dimension 2, and let M be an acyclic matching in $\mathcal{H}(\mathcal{X}(K))$ with only one critic cell of dimension 0. Then, $\mathcal{Q}_{\mathcal{X}(K),M} \sim_{AC} \mathcal{P}_K$.*

Example 6.5.7. Let T be the regular structure of the Triangle of Figure 6.12

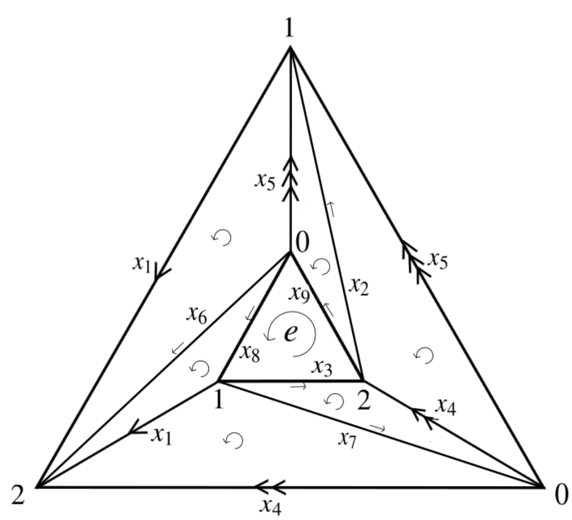


Figure 6.12: Regular cell structure of the Triangle T.

Let M be the acyclic matching in $\mathcal{H}(\mathcal{X}(T))$ of Figure 6.13.

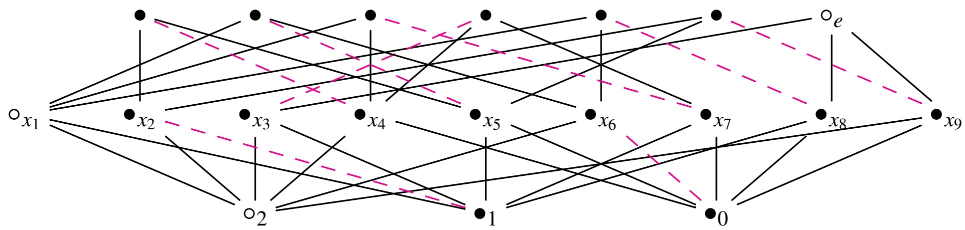


Figure 6.13: An acyclic matching M in $\mathcal{H}(\mathcal{X}(T))$.

The complex T_M has only one 0-cell, one 1-cell x_1 and one 2-cell $\tilde{e}$. However, we do not know beforehand what the attaching map of $\tilde{e}$ looks like. We will prove that $\tilde{e}$ has an attaching map homotopic to x_1^{-1} and so $T_M \nearrow\!\!\searrow^{3} D^2$. We take a preferred topological sort of $\mathcal{H}_M(\mathcal{X}(T))$ as in Figure 6.14.

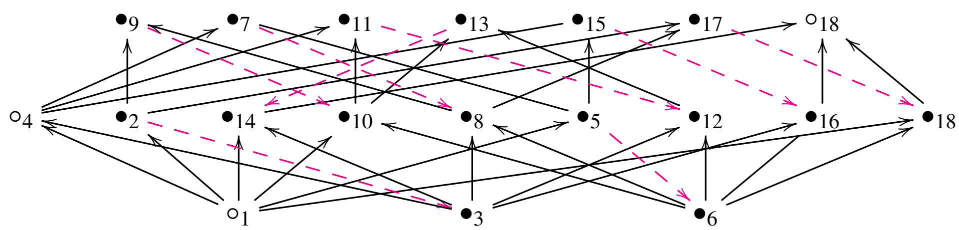


Figure 6.14: Preferred topological sort of $\mathcal{H}_M(T)$.

The attaching map of e in T can be described as the ordered list of 1-cells $x_9x_8x_3$. We carry out explicitly the recursive process described in Theorem 6.5.5 to obtain the attaching map of $\tilde{e}$ in T_M. By performing the internal collapses indicated by the matched pairs (17,18), (15,16) and (13,14) we get respectively

$$x_9 = x_2x_5^{-1},\ x_8 = x_6x_1^{-1},\ \text{and}\ x_3 = x_7x_4.$$

From the sequence of internal collapses induced by pairs (11,12), (9,10) and (7,8) we obtain respectively

$$x_7 = x_1x_4^{-1},\ x_4 = x_5x_2^{-1},\ \text{and}\ x_5 = x_6x_1^{-1},$$

so that $x_4 = x_6x_1^{-1}x_2^{-1}$ and $x_7 = x_1x_2x_1x_6^{-1}$. Thus, the attaching map of $\tilde{e}$ in the combinatorial complex which results from performing the internal collapses that correspond to cells of dimension 1 and 2, turns out to be $x_2x_6x_1^{-1}x_6x_1^{-1}x_1x_2x_1x_6^{-1}x_6x_1^{-1}x_2^{-1}$. The movements induced by the pairs (5,6) and (2,3) amount to the identities

$$x_6 = 1\ \text{and}\ x_2 = 1.$$

Finally the attaching map of $\tilde{e}$ in T_M is $x^{-1}x_1^{-1}x_1x_1x_1^{-1}$ which is easily seen to be homotopic to x_1^{-1}.

Theorem 6.5.8. *Let $\mathcal{P}$ be a balanced presentation of a group. Let M be an acyclic matching in $\mathcal{H}(X_{\mathcal{P}})$ with only one critic cell of dimension 0. Then $\mathcal{P} \sim_{AC} \mathcal{Q}_{X_{\mathcal{P}},M}$.*

Proof. On the one hand, Theorem 1.4.5 asserts that $\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}}$, and the latter is in the same AC-class as $\mathcal{P}_{K'_{\mathcal{P}}}$. On the other one, $X_{\mathcal{P}}$ is the face poset of $K'_{\mathcal{P}}$. Thus, by Corollary 6.5.6, $\mathcal{P}_{K'_{\mathcal{P}}} \sim_{AC} \mathcal{Q}_{\mathcal{X}(K'_{\mathcal{P}}),M} = \mathcal{Q}_{X_{\mathcal{P}},M}$. Therefore, $\mathcal{P} \sim_{AC} \mathcal{Q}_{X_{\mathcal{P}},M}$ and the proof is complete. □

Remark 6.5.9. Given $\mathcal{P}$ a presentation and M a matching in $\mathcal{H}(X_{\mathcal{P}})$ with only one critical 0-cell, we proved that $\mathcal{P} \sim_{AC} \mathcal{Q}_{X_{\mathcal{P}},M}$. We will estimate the (sufficient) number of AC-transformations to obtain $\mathcal{Q}_{X_{\mathcal{P}},M}$ from $\mathcal{P}$. According to the proof of Theorem 6.5.8, one can show the following chain of equivalences:

$$\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{K'_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{(K'_{\mathcal{P}})_M} = \mathcal{Q}_{X_{\mathcal{P}},M}.$$

Let n be the number of generators of $\mathcal{P}$. Let m be the number of relators and k the total relator length. We have already seen in Remark 6.3.2 that the equivalences $\mathcal{P} \sim_{AC} \mathcal{P}_{K_{\mathcal{P}}}$ and $\mathcal{P}_{K_{\mathcal{P}}} \sim_{AC} \mathcal{P}_{K'_{\mathcal{P}}}$ can be achieved in $O(n+m)$ and $O(k)$ AC-transformations respectively. Now, by Theorem 6.5.1, $K'_{\mathcal{P}} \nearrow\!\!\searrow^{3} (K'_{\mathcal{P}})_M$. Thus, the estimated number of AC-transformations required to obtain $\mathcal{P}_{(K'_{\mathcal{P}})_M}$ from $\mathcal{P}_{K'_{\mathcal{P}}}$ has the order of the number of elementary expansions and collapses needed to deform $K'_{\mathcal{P}}$ into $(K'_{\mathcal{P}})_M$. This is bounded by the square of the quantity of cells of $K'_{\mathcal{P}}$, which is proportional to k.

Independently, Brendel, Dlotko, Ellis, Juda and Mrozek presented in [BEJM15] an algorithm to describe a presentation of the fundamental group of a regular cell complex, using Forman's combinatorial version of Morse theory. They applied it to compute the fundamental group of point clouds.

6.6 SAGE implementation

We implemented the algorithms developed in the previous section in the SAGE platform [S+17] (see Appendix 6.A). Refer to Section 6.7 for some experimental results. We summarize below details of this implementation. The routine `coloring_presentation(X,A)` computes, given a poset `X` of height 2 and a subdiagram `A` of $\mathcal{H}$(`X`), the presentation $\mathcal{P}_{\texttt{X},\texttt{A}}$, which is AC-equivalent to $\mathcal{P}_{\mathcal{K}(\texttt{X})}$ whenever `A` is a subdiagram of $\mathcal{H}$(`X`) which contains all the vertices of X and is collapsible as finite space. The auxiliary functions `kruskal`(X) and `spanning_collapsible`(X) generate random spanning trees and spanning collapsible subdiagrams in $\mathcal{H}$(`X`), respectively. If `X` is the poset associated to a presentation $\mathcal{P}$, then `coloring_presentation(X,A)` is a presentation in the same AC-class as $\mathcal{P}$.

For the Morse theory part, the main function is `morse_presentation(gens,rels,M)` and it computes the presentation $\mathcal{Q}_{X_{\mathcal{P}},\texttt{M}}$ AC-equivalent to the presentation $\mathcal{P}$ where `gens` and `rels` are the list of generators and relators respectively of a presentation $\mathcal{P} = \langle$`gens` | `rels`$\rangle$. It employs the auxiliary routine `write(edge, isolate)`, which takes as arguments a matched 1-cell `edge` and a list `isolate` containing the expression by which `edge` is replaced when the corresponding internal collapse is performed and returns the attaching map of edge in the Morse complex. This is a recursive routine which mimics the process described in the proof of Theorem 6.5.5. It is also worth mentioning the function `greedy_acyclic_matching(X)`, which produces a valid matching on the face poset `X` of a regular CW complex. It implements a simple strategy, which as a first step randomly shuffles the edges of the Hasse diagram of X. After that, a matching is grown from the empty set by greedily adding an edge whenever is possible.

The routine `simplified`

The free software Sage [S+17] has a routine called `simplified`, which takes as argument a group presentation and produces a *simpler* presentation of the same group, that is, one with less generators and relators whose total length relator does not exceed the original by much. It essentially calls the Tietze reduction/elimination procedures of the free software GAP [GAP17] encoded in the function `SimplifyPresentation`.

GAP's method `SimplifyPresentation` modifies a presentation by Tietze transformations with a default strategy. It is supported on the *Tietze transformation program* designed by Havas, Kenne, Richardson and Robertson [HKRR84]. `SimplifyPresentation` is a procedure which consists in a loop of two phases. The *search phase* attempts to reduce the relator length by replacing long substrings of relators by shorter equivalent ones. Concretely, the algorithm looks for relators r_1 and r_2 such that

- a suitable cyclic permutation of r_1 reads uv and a cyclic permutation of r_2, or its inverse reads as wv,
- the length of v is greater than the length of w.

In this case, r_2 is replaced by wu^{-1}. The *elimination phase* tries to eliminate generators which occur only once in some relator.

We notice that the previous algorithm actually makes only AC-transformations if we start from a balanced presentation of the trivial group. In general, there is only one situation in which the procedure makes a transformation not allowed by Andrews-Curtis. Suppose a presentation $\mathcal{P}$ has a relator r_i which is equal to another relator r_j. Then, the `SimplifyPresentation` replaces relator r_i by a 1, and then eliminates the latter 1. This transformation changes the deficiency of the presentation and does not preserve the (simple) homotopy type of the associated complex $K_{\mathcal{P}}$. However, the previous situation is not possible if $\mathcal{P}$ is a balanced presentation of the trivial group, since if $\mathcal{P}$ has (after possibly a sequence of AC-transformations) one relator equal to another, then it is in the same AC-class as a presentation with a relator equal to 1. Thus, $K_{\mathcal{P}}$ has non-trivial second homology group, which is impossible.

Therefore, we have used the function `simplified` to obtain shorter presentations from those we have got as output of our functions `coloring_presentation` and `morse_presentation`.

6.7 Some experimental results

In this section, we expose some practical results we have obtained with the previous algorithms.

- Recall that the family of potential counterexamples proposed by Akbulut and Kirby [AK85] is defined as follows:

$$AK_n = \langle x, y \mid xyx = xyx, x^n = y^{n+1} \rangle.$$

By considering the collapsible subdiagram A detailed in Figure 6.15 we obtain that $\mathcal{P}_{K_{AK_2},A}$ is AC-equivalent to a balanced presentation with 42 generators and relators, which can be easily simplified to the trivial presentation $\langle \mid \rangle$. Since $AK_2 \sim_{AC} \mathcal{P}_{K_{AK_2},A}$, we proved that AK_2 satisfies the Andrews-Curtis conjecture.

- The following group presentation was posed by Rapaport [Rap68a]

$$R = \langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2 \rangle.$$

By considering the spanning collapsible subdiagram A of X_R detailed in Figure 6.16, R can be transformed into the following presentation with 2 generators and 2 relators

$$\tilde{R} = \langle x, y \mid x^{-1}yx^{-}2y^{-1}xyx^2y^{-1}x^{-1}yx^2y^{-1},\ y^{-1}x^{-1}yx^2y^{-}2x^{-1}yx^2yx^{-}2y^{-1}x \rangle.$$

- We have proved experimentally that there is a large class of integers n, m, p, q for which the group presentation

$$G_{n,m,p,q} = \langle x, y \mid x = [x^m, y^n], y = [x^p, y^q] \rangle$$

[Bro84] they satisfy the conjecture. [2]

It is not known in general if the presentations of this family satisfy the Andrews-Curtis conjecture. However, some of them can be shown to be AC-trivializable by means of the `simplified` routine.

For instance, we have verified that the presentations

$$G_{1,1,k,-1} = \langle x, y \mid x = [x, y], y = [x^p, y^{-1}] \rangle$$

for $1 \leq p \leq 100$ satisfies the Andrews-Curtis conjecture. If k is odd, the `simplified` routine is enough. Nevertheless, if k is even, it does not work. We have found for each k, an acyclic matching M_k in $\mathcal{H}(X_{G_{1,1,k,-1}})$ such that $Q_{X_{G_{1,1,k,-1}},M_k}$ can be reduced to the trivial presentation of the trivial group after applying the function `simplified`. See Figure 6.17 for an example of trivialization of $G_{1,1,15,-1} = \langle x, y \mid x = x^{-1}y^{-1}xy, y = x^{15}yx^{-15}y^{-1} \rangle$.

[2] Here, $[a, b] = a^{-1}b^{-1}ab$.

```
gens=['x','y']
rels=[[('x',2),('y',-3)], [('x',1),('y',1),('x',1),('y',-1),('x',-1),('y',-1)]]
#n=2

X=presentation_poset(gens, rels)

A=[['c2_y1_1', 'c2_y3_1'], ['c2_y1_1', 'c2_y2_1'], ['y3', 'c2_y3_1'], [0, 'y2'],
['c1_0_1', 'c1_y2_3'], ['x1', 'x2'], ['y1', 'y2'], ['y3', 'c1_y3_1'], ['c2_0_2',
'c2_x3_1'], ['y3', 'c1_y3_3'], ['y1', 'c1_y1_3'], ['c2_x1_2', 'c2_x2_2'], [2,
'c2_0_6'], ['y2', 'c2_y2_2'], ['y2', 'c1_y2_1'], ['c2_0_4', 'c2_x3_2'], ['y1',
'c1_y1_1'], ['c2_0_1', 'c2_x2_1'], ['c2_x1_1', 'c2_x3_1'], [1, 'c1_x1_2'], [2,
'c2_0_2'], ['c2_x1_3', 'c2_x3_3'], [0, 'c2_0_3'], [1, 'c1_x1_1'], [2, 'c2_y1_1'],
['y3', 'c2_y3_3'], ['y1', 'c1_y1_2'], ['x2', 'c2_x2_1'], ['x3', 'c1_x3_2'],
['c1_y1_2', 'c1_y2_2'], [1, 'c1_0_1'], [1, 'c1_y1_2'], ['y1', 'y3'], ['x1',
'c1_x1_2'], ['c1_0_3', 'c1_x3_2'], [2, 'c2_0_5'], ['y3', 'c2_y3_2'], ['x2',
'c2_x2_3'], ['x2', 'c1_x2_2'], ['x3', 'c2_x3_3'], ['x1', 'x3'], ['y1', 'c2_y1_2'],
['x3', 'c2_x3_2'], [0, 'c1_0_5'], ['c1_x1_1', 'c1_x2_1'], ['c1_0_2', 'c1_x2_2'],
['y3', 'c1_y3_2'], ['x3', 'c1_x3_1'], ['c1_0_4', 'c1_y2_1'], ['y2', 'c2_y2_3'],
['c2_y1_3', 'c2_y2_3'], [2, 'c2_x1_2'], ['y1', 'c2_y1_3'], ['y1', 'c2_y1_1'],
['x1', 'c2_x1_3'], [2, 'c2_x1_1'], ['c2_y1_2', 'c2_y2_2'], ['c2_y1_2', 'c2_y3_2'],
['c2_y1_3', 'c2_y3_3'], ['c2_x1_3', 'c2_x2_3'], ['c1_y1_3', 'c1_y3_3'],
['c1_y1_2', 'c1_y3_2'], ['c1_y1_1', 'c1_y2_1'], ['c1_y1_1', 'c1_y3_1'],
['c1_x1_2', 'c1_x2_2'], ['c1_x1_2', 'c1_x3_2'], [0, 'c1_0_4'], ['y2', 'c1_y2_2'],
['y2', 'c2_y2_1'], ['c1_0_5', 'c1_y2_2'], ['c2_0_2', 'c2_y2_1'], ['c1_0_1',
'c1_x2_1'], [1, 'c1_0_5'], [1, 'c1_0_2'], [1, 'c1_0_3'], [0, 'c2_0_2']]

G=coloring_presentation(X,A)
print is_collapsible(Poset((X.list(),A)))
print 'coloring presentation', G
print 'generators', len(G.generators()), 'relations', len(G.relations())
print G.simplified()
```

```
True
coloring presentation Finitely presented group < a0, a1, a2, a3, a4,
a5, a6, a7, a8, a9, a10, a11, a12, a13, a14, a15, a16, a17, a18, a19,
a20, a21, a22, a23, a24, a25, a26, a27, a28, a29, a30, a31, a32, a33,
a34, a35, a36, a37, a38, a39, a40, a41 | a41*a15^-1, a2*a38^-1, a0*a9,
a1*a39^-1, a2*a14, a40*a0^-1, a1*a10, a9*a6^-1, a7*a31, a3*a36^-1,
a10*a8^-1, a3*a34^-1, a4*a35^-1, a5*a30^-1*a6^-1, a7*a32,
a4*a33^-1*a8^-1, a5*a37^-1, a14*a26^-1, a11*a13^-1, a12*a15, a11*a27^-1,
a13*a28, a12*a29^-1, a16*a38^-1*a23^-1, a17*a23^-1, a19*a23^-1,
a31*a39^-1*a23^-1, a22*a36*a23^-1, a24*a26^-1*a17^-1, a24*a18^-1,
a24*a40, a24*a20^-1, a24*a34^-1*a21^-1, a41*a16^-1, a21*a35, a19*a30,
a25*a27^-1*a18^-1, a25*a28^-1, a25*a29^-1, a25*a32^-1,
a25*a33^-1*a20^-1, a25*a37^-1*a22^-1 >
generators 42 relations 42
Finitely presented group <  |  >
```

Figure 6.15: A proof using colorings that $\mathcal{P} = \langle x, y \mid xyx = xy, x^2 = y^3 \rangle$ satisfies the Andrews-Curtis conjecture

```
T=[[3, 'c3_a1_1'], [3, 'c3_0_1'], ['c2_0_4', 'c2_c3_2'], [0, 'c3'], ['c1', 'c1_c1_2'], ['a3',
'c2_a3_2'], [0, 'c1_0_3'], ['c1_0_1', 'c1_b2_3'], ['c3_a1_1', 'c3_a2_1'], [0, 'c1_0_2'], ['a1',
'c2_a1_2'], [0, 'a3'], ['c3', 'c2_c3_2'], [1, 'c1_0_4'], ['c1_0_1', 'c1_c3_1'], ['b3', 'c1_b3_1'],
[0, 'c3_0_2'], ['c2_0_2', 'c2_a2_1'], ['c2_0_5', 'c2_c2_2'], ['c3_a1_3', 'c3_a2_3'], [2, 'c2_c1_3'],
['c2_a1_2', 'c2_a3_2'], ['c2_0_3', 'c2_a2_2'], ['c1_0_5', 'c1_b3_3'], ['c3_0_1', 'c3_b3_1'], [3,
'c3_b1_2'], [2, 'c2_c1_1'], [0, 'c1_0_4'], ['c2', 'c2_c2_1'], ['c3_a1_2', 'c3_a2_2'], [2, 'c2_0_3'],
['b1', 'c3_b1_2'], ['a2', 'c3_a2_3'], [0, 'c3_0_5'], [0, 'c3_0_1'], [1, 'c1_c1_1'], [0, 'c2_0_5'],
['c2_0_1', 'c2_a3_1'], [2, 'c2_a1_1'], ['c2', 'c1_c2_1'], ['c3_0_2', 'c3_b2_1'], ['b1', 'c1_b1_2'],
['c1_b1_2', 'c1_b2_2'], ['c3_0_3', 'c3_b2_2'], ['c1_c1_1', 'c1_c3_1'], ['c3', 'c2_c3_1'], [0,
'c2_0_1'], ['c2_c1_1', 'c2_c3_1'], ['c3_0_4', 'c3_a3_2'], ['c2_c1_2', 'c2_c2_2'], [3, 'c3_a1_3'],
['c2', 'c2_c2_2'], ['c3_0_5', 'c3_a3_3'], ['c1', 'c2'], ['b1', 'c1_b1_1'], [1, 'c1_b1_3'], ['a3',
'c3_a3_2'], ['b1', 'b3'], ['c1_0_4', 'c1_c3_2'], ['c2', 'c1_c2_2'], ['c1_0_4', 'c1_b3_2'], [0,
'c3_0_3'], [0, 'c2_0_2'], [0, 'b2'], ['c3_b1_1', 'c3_b2_1'], [0, 'c1_0_5'], ['c3_0_5', 'c3_a2_2'],
['c3_0_3', 'c3_a3_1'], ['c2_c1_3', 'c2_c2_3'], ['c3_0_4', 'c3_b3_2'], ['c3', 'c2_c3_3'], ['c1_0_2',
'c1_b2_1']]

gens=['a', 'b', 'c']
rels=[[('c', -1), ('b', 1), ('c',1), ('b', -2)], [('a', -1), ('c', 1), ('a',1), ('c', -2)], [('b',
-1), ('a', 1), ('b',1), ('a', -2)] ]

X=presentation_poset(gens, rels)
P=coloring_presentation(X,T)
print P
G=P.simplified()
print G
```

```
Finitely presented group < a0, a1, a2, a3, a4, a5, a6, a7, a8, a9,
a10, a11, a12, a13, a14, a15, a16, a17, a18, a19, a20, a21, a22, a23,
a24, a25, a26, a27, a28, a29, a30, a31, a32, a33, a34, a35, a36, a37,
a38, a39, a40, a41, a42, a43, a44, a45, a46, a47, a48, a49, a50, a51,
a52, a53, a54, a55, a56, a57, a58, a59, a60, a61, a62, a63, a64, a65,
a66, a67, a68, a69, a70, a71, a72, a73, a74, a75, a76, a77, a78, a79,
a80, a81, a82, a83, a84, a85, a86, a87, a88, a89 | a3*a50, a51,
a80*a2^-1, a0*a35, a1, a3*a86^-1*a4^-1, a52*a87^-1*a4^-1, a0*a4^-1,
a1*a36*a4^-1, a2*a37*a4^-1, a6*a48*a67^-1*a7^-1, a45*a68^-1*a7^-1, a46,
a47*a72^-1, a7*a69, a6*a49*a73^-1, a5*a23*a74^-1, a5*a70^-1*a7^-1,
a24*a71^-1*a7^-1, a25*a75^-1, a11*a32*a64^-1*a13^-1,
a11*a33*a60^-1*a12^-1, a34*a61^-1*a12^-1, a13, a10*a12^-1,
a8*a62^-1*a12^-1, a8*a20*a65^-1*a13^-1, a9*a21*a13^-1, a9*a63^-1*a12^-1,
a10*a22*a66^-1*a13^-1, a88, a15*a23, a16*a89, a15*a16^-1, a17*a20^-1,
a17*a24^-1, a18*a21^-1*a14^-1, a18*a25^-1, a14*a19^-1, a22*a19^-1,
a28*a32^-1, a28*a81, a29*a33^-1, a29*a82*a35^-1, a26*a34,
a26*a84^-1*a30^-1, a27*a31^-1, a83, a27*a36*a30^-1, a37*a85^-1*a31^-1,
a48*a41^-1, a38*a45*a42^-1, a38*a46*a77^-1*a43^-1, a39*a47,
a44*a79*a39^-1, a44*a49^-1, a50*a76^-1*a42^-1, a40*a51*a78^-1*a43^-1,
a41, a40*a52, a67*a55^-1, a68, a54*a77^-1, a54*a72, a79*a69^-1, a54*a73,
a56*a76^-1, a56*a78^-1, a64, a56*a80*a81^-1, a53*a60, a56*a82^-1,
a57*a61^-1*a53^-1, a58*a84, a56, a86*a55^-1, a87, a57*a83, a58, a85,
a53*a88^-1, a54*a74, a89*a62^-1*a53^-1, a70, a65, a71, a59,
a59*a75^-1*a54^-1, a53*a63, a66 >
Finitely presented group < a0, a5 |
a0^-1*a5*a0^-2*a5^-1*a0*a5*a0^2*a5^-1*a0^-1*a5*a0^2*a5^-1,
a5^-1*a0^-1*a5*a0^2*a5^-2*a0^-1*a5*a0^2*a5*a0^-2*a5^-1*a0 >
```

Figure 6.16: A proof using colorings that $R = \langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2 \rangle$ is Andrews-Curtis equivalent to a presentation of 2 generators and 2 relators.

```
gens = ['x','y']
rels = [ [('x',-1), ('x', -1), ('y', -1), ('x', 1), ('y', 1)], [('y', -1), ('x', -15),
('y', 1), ('x', 15), ('y', -1)]]

Rels =  []
for r in rels:
    aux = F.one()
    for (g,e) in r:
        aux *= dict[g]^e
    Rels.append(aux)
Gor = F/Rels
print Gor
```

```
    Finitely presented group < a0, a1 | a0^-2*a1^-1*a0*a1,
    a1^-1*a0^-15*a1*a0^15*a1^-1 >
```

```
print Gor.simplified()
```

```
    Finitely presented group < a0, a1 | a0^-2*a1^-1*a0*a1,
    a1^-1*a0^-7*a1*a0^-1*a1^-1 >
```

```
X = pres_poset(gens,rels)
M  = [['c1_0_3', 'c1_x2_2'], ['c1_x1_3', 'c1_x2_3'], ['x2', 'c2_x2_12'], ['x3',
'c2_x3_18'], ['c2_0_26', 'c2_x3_23'], ['c2_x1_1', 'c2_x3_1'], ['x1', 'c2_x1_24'], [1,
'c1_x1_2'], ['c1_y1_1', 'c1_y2_1'], ['y2', 'c2_y2_1'], ['c2_x1_5', 'c2_x2_5'], [2,
'c2_0_13'], ['c2_0_30', 'c2_x3_27'], [0, 'y3'], ['c2_0_21', 'c2_x2_19'], ['c2_x1_2',
'c2_x3_2'], ['c2_0_15', 'c2_x3_14'], ['c2_0_25', 'c2_x3_22'], ['c2_x1_4', 'c2_x3_4'],
['c1_x1_1', 'c1_x2_1'], ['c2_x1_17', 'c2_x2_17'], ['c2_x1_7', 'c2_x2_7'], ['c2_x1_30',
'c2_x2_30'], ['c2_x1_15', 'c2_x3_15'], ['c2_0_28', 'c2_x3_25'], ['c2_0_1', 'c2_y3_1'],
['c2_0_8', 'c2_x3_7'], ['c2_0_31', 'c2_x2_29'], ['c2_0_18', 'c2_y3_2'], ['c1_y1_2',
'c1_y3_2'], ['c2_0_10', 'c2_x3_9'], ['c2_0_12', 'c2_x3_11'], ['c2_x1_22', 'c2_x2_22'],
['c2_0_33', 'c2_x3_30'], ['c2_x1_3', 'c2_x3_3'], ['c2_0_29', 'c2_x2_27'], ['c2_0_23',
'c2_x2_21'], ['c2_x1_26', 'c2_x2_26'], ['c2_y1_3', 'c2_y2_3'], ['c2_0_17', 'c2_x2_15'],
['c2_x1_18', 'c2_x2_18'], ['c2_x1_25', 'c2_x2_25'], ['c2_0_24', 'c2_x3_21'], ['c1_0_5',
'c1_x3_3'], ['c2_0_27', 'c2_x3_24'], ['c2_x1_13', 'c2_x2_13'], ['c2_x1_29', 'c2_x3_29'],
['c2_0_22', 'c2_x2_20'], ['c2_x1_16', 'c2_x2_16'], ['c2_0_3', 'c2_x2_1'], ['c2_0_6',
'c2_x3_5'], ['c2_x1_23', 'c2_x2_23'], ['c2_y1_2', 'c2_y2_2'], ['c2_x1_14', 'c2_x2_14'],
['c2_x1_20', 'c2_x3_20'], ['c2_0_19', 'c2_x3_16'], ['c2_x1_28', 'c2_x3_28'], ['c2_x1_6',
'c2_x3_6'], ['c2_x1_8', 'c2_x2_8'], ['c2_0_9', 'c2_x3_8'], ['c1_0_1', 'c1_x3_1'],
['c2_x1_9', 'c2_x2_9'], ['c2_x1_10', 'c2_x3_10'], ['c1_0_2', 'c1_x3_2'], ['c2_0_5',
'c2_x2_3'], ['c2_0_4', 'c2_x2_2'], ['c2_x1_11', 'c2_x2_11']]
P = morse_presentation(gens,rels,M)
print P
print P.simplified()
```

evaluate

```
    Finitely presented group < a0, a1, a2, a3, a4, a5, a6, a7, a8, a9,
    a10, a11, a12 | a10^-1*a1*a5*a6^-1*a9*a10^-1*a1*a6^-1*a9,
    a9^-1*a6*a1^-1*a10*a11^-1*(a9^-1*a6*a1^-1*a10)^2,
    (a1^-1*a10*a9^-1*a6)^2*a4^-1*a11*a9^-1*a6*a1^-1*a10*a9^-1*a6,
    a8^-1*a3*a0^-1*a9^-1*a6,
    a10^-1*a1*a2^-1*a3*(a6^-1*a9*a10^-1*a1)^4*a6^-1*a9,
    (a9^-1*a6*a1^-1*a10)^2*a12^-1*a10,
    (a10^-1*a1*a6^-1*a9)^2*a10^-1*a1*a5^-1*a2*a6^-1*a9*a10^-1*a1*a6^-1*a9,
    a3^-1*a8*(a6^-1*a9*a10^-1*a1)^2*a6^-1*a9*a0*a1^-1*a10*a9^-1*a6,
    a8^-1*a3*a8^-1*a4*a1^-1*a10*a9^-1*a6, (a6^-1*a9*a10^-1*a1)^2*a7^-1*a6,
    a6^-1*a9*a10^-1*a1*a6^-1,
    a9^-1*a6*a1^-1*a10*a9^-1*a6*a1^-1*a7*a6^-1*a9*a10^-1*a1*a3^-1*a8*a1^-1*a\
    10, a6^-1*a9*a12*(a9^-1*a6*a1^-1*a10)^3*a9^-1*a6 >
    Finitely presented group <  |  >
```

Figure 6.17: A proof using Morse theory that $\mathcal{P} = \langle x, y \mid x = x^{-1}y^{-1}xy, y = x^{15}yx^{-15}y^{-1}\rangle$ satisfies the Andrews-Curtis conjecture

6.A Appendix: Group presentations SAGE module

Presentation poset

```
#Poset associated to a group presentation

def aux_label(a, cycle, b, ind, c):
    return ('c' + str(a) + '_' + str(b) + ind + '_' + str(c))

def presentation_poset(gens,rels):

   #expanded relations
    Rels = []
    for i in range(len(rels)):
        Rels.append([])
        for l in rels[i]:
            if l[1] < 0:
                Rels[i] += [(l[0], -1) for k in range(abs(l[1]))]
            if l[1] > 0:
                Rels[i] += [(l[0], 1) for k in range(abs(l[1]))]

    #V is the set of elements of the presentation poset
    V = range(len(rels)+1) + [x + str(i) for x in gens for i in range(1,4)]

    #E is the list of cover relations of the presentaton poset
    E=[]
    #edges associated to the generators
    for x in gens:
        E += [[0, x +'2'], [0, x +'3'], [x +'1', x +'2'], [x +'1', x +'3']]

    for i in range(len(Rels)):
        letters = {}
        for x in gens: letters[x] = 0
        for j in range(len(Rels[i])):
            letters[Rels[i][j][0]] = letters[Rels[i][j][0]] + abs(Rels[i][j][1])
        total_letters = sum([letters[x] for x in gens])

        V = V + [aux_label(i+1, x, str(j), k) for x in gens if letters[x] != 0 for j in range(1, 4) for k in range(1, letters[x] + 1)] + [aux_label(i+1, 0, '', k) for k in range(1, total_letters + 1)]

        #the model of D^2 i associated to the cell corresponding to the relator i:

        #edges between the indicator i of the relator and the minimals of the cycle
```

```

    E = E + [[i+1, aux_label(i+1, 0, '', j+1)] for j in range(total_letters)] + [[i+1, aux_label(i+1, x, '1', j+1)] for x in gens for j in range(letters[x])]

    # edges between the models of S^1 associated to generators and relators

    for x in gens:
        for k in range(1,4):
            for j in range(letters[x]):
                E.append([x + str(k), aux_label(i+1, x, str(k), j+1)])
    for j in range(len(Rels[i])):
        E.append([0, aux_label(i+1, 0, '', j+1)])
    #edges of the cycle which do not start in 0
    for l in Rels[i]:
        for j in range(letters[l[0]]):
            E += [[aux_label(i+1, l[0], '1', j+1), aux_label(i+1, l[0],'2', j+1)],[aux_label(i+1, l[0], '1', j+1),aux_label(i+1, l[0], '3', j+1)]]
    #edges of the cycle starting in 0
    cont = {}
    for x in gens: cont[x] = 0

    for j in range(len(Rels[i])):
        cont[Rels[i][j][0]] += 1

        #edges to the 'right'
        if Rels[i][j][1] > 0:
            E.append([aux_label(i+1, 0, '', sum([cont[x] for x in gens])), aux_label(i+1, Rels[i][j][0], '2', cont[Rels[i][j][0]])])
        else:
            E.append([aux_label(i+1, 0, '', sum([cont[x] for x in gens])), aux_label(i+1, Rels[i][j][0], '3', cont[Rels[i][j][0]])])
        #edges to the 'left'
        if j != 0:
            if Rels[i][j-1][0] == Rels[i][j][0]:
                if Rels[i][j-1][1] > 0:
                    E.append([aux_label(i+1, 0, '', sum([cont[x] for x in gens])), aux_label(i+1, Rels[i][j-1][0], '3', cont[Rels[i][j-1][0]]-1)])
                else:
                    E.append([aux_label(i+1, 0, '', sum([cont[x] for x in gens])), aux_label(i+1,Rels[i][j-1][0], '2', cont[Rels[i][j-1][0]]-1) ])
            else:
```

```
                    if Rels[i][j-1][1] > 0:
                        E.append([aux_label(i+1, 0, '', sum([cont[x] for x in gens])), aux_label(i+1,Rels[i][j-1][0], '3', cont[Rels[i][j-1][0]])])
                    else:
                        E.append([aux_label(i+1, 0, '', sum([cont[x] for x in gens])), aux_label(i+1,Rels[i][j-1][0], '2', cont[Rels[i][j-1][0]]) ])

        #j=0
        n = len(Rels[i]) - 1
        if Rels[i][n][1] > 0:
            E.append([aux_label(i+1, 0, '', 1), aux_label(i+1, Rels[i][n][0], '3', cont[Rels[i][n][0]])])
        else:
            E.append([aux_label(i+1, 0, '', 1), aux_label(i+1, Rels[i][n][0], '2', cont[Rels[i][n][0]])])

    return Poset((V, E))
```

Colorings

```
#Random spanning tree
def find(C, u):
    if C[u] != u:
        C[u] = find(C, C[u])
    return C[u]

def union(C, u, v):
    u,v = find(C, u), find(C, v)
    C[u] = v

def kruskal(X):
    E = X.cover_relations()
    shuffle(E)
    T = []
    C = {u:u for u in X}
    for e in E:
        if find(C, e[0]) != find(C, e[1]):
            T.append(e)
            union(C, e[0], e[1])
    return T

#Random spanning collapsible
def expand(X,A):
    not_edges = [e for e in X.cover_relations() if not e in A]
    for e in not_edges:
        if is_collapsible(Poset((X.list(), A + [e]))):
            A.append(e)
```

```
    return A

def spanning_collapsible(X):
    A = kruskal(X)
    while len(expand(X,A)) != len(A):
        A = expand(X,A)
    return A


#Presentation associated to the coloring A
def coloring_presentation(X,A): #A is the list of edges of a spanning collapsible
    gens = [r for r in X.cover_relations() if not r in A]
    F = FreeGroup(len(gens), 'a')
    d = {}
    for i in range(len(gens)):
        d[tuple(gens[i])] = F.gens()[i]
    for j in range(len(A)):
        d[tuple(A[j])] = F.one()

    rels = []
    for x in X.minimal_elements():
        for w in X.maximal_elements():
            l = [u for u in X.upper_covers(x) if u in X.lower_covers(w)]
            ind = -1
            for j in range(len(l)):
                if(not [x,l[j]] in A or not [l[j],w] in A):
                    ind = j

            if(ind != -1):
                y = l[j]
                for z in l:
                    if(z != y):
                        s1 = [x,y]
                        s2 = [y,w]
                        s3 = [x,z]
                        s4 = [z,w]

                        rels.append(d[tuple(s1)] * d[tuple(s2)] * d[tuple(s4)]^-1 * d[tuple(s3)]^-1)
    return  (F / rels)
```

Morse Theory

```
def attaching(gens, rels):
    att = {} #dictionary of attaching maps

    #expanded relations of the presentation
```

```
    Rels=[]
    for i in range(len(rels)):
        Rels.append([])
        for l in rels[i]:
            if l[1] < 0:
                Rels[i] += [(l[0],-1) for k in range(abs(l[1]))]
            if l[1] > 0:
                Rels[i] += [(l[0],1) for k in range(abs(l[1]))]

    for i in range(len(Rels)):
        letters={}
        R = Rels[i]
        for (l,e) in R:
            letters[l] = 0
        for j in range(len(R)):
            (l,e) = R[j]
            letters[l] += 1
            if(e == 1):
                att[aux_label(i+1,l,'2',letters[l])] = [(l+'2',1),(aux_label(i+1,l,'1',letters[l]),-1),(aux_label(i+1,0,'',j+1),1)]
                att[aux_label(i+1,l,'3',letters[l])] = [(l+'3',1),(aux_label(i+1,0,'',(j+1)%(len(R))+1),-1), (aux_label(i+1),l,'1',letters[l]),1)]
            else:
                att[aux_label(i+1,l,'2',letters[l])] = [(l+'2',-1),(aux_label(i+1,0,'',(j+1)%(len(R))+1),-1),(aux_label(i+1,l,'1',letters[l]),1)]
                att[aux_label(i+1,l,'3',letters[l])] = [(l+'3',-1),(aux_label(i+1,l,'1',letters[l]),-1),(aux_label(i+1,0,'',j+1),1)]

    return att

def write(edge, isolate):
    global new_attaching
    new_attaching = []
    return (aux_write(edge, isolate))

def aux_write(edge, isolate):
    global new_attaching
    if edge in isolate.keys():
        if isolate[edge] == []:
            return new_attaching
        for edge in isolate[edge]:
            aux_write(edge, isolate)
        return new_attaching
    else:
        new_attaching += [edge]
        return new_attaching

```

```
def attaching_Morse(attaching, matching, critics_dim_2):
    dim2 = [] # cells of dimension 2 that collapse
    isolate = {}
    for (c1,c2) in matching:
        if c2 in attaching.keys(): # c2 of dimension 2
            dim2.append(c2)
            att = [] # new attaching map of c1
            orient = 1

            for (cell,e) in attaching[c2]:
                if cell != c1:
                    att.append((cell,-e))
                else:
                    orient = e
                    break
            att = att[::-1] # reverse list

            for (cell,e) in reversed(attaching[c2]):
                if cell != c1:
                    att.append((cell,-e))
                else:
                    break

            if orient == -1:
                att = att[::-1] # reverse list
                for i in range(len(att)): # inverse of every cell
                    (cell,e) = att[i]
                    att[i] = (cell,-e)
            isolate[(c1,1)] = att
            att_inv = []
            for i in range(len(att)): # inverse of every cell
                (cell,e) = att[i]
                att_inv.append((cell,-e))

            att_inv = att_inv[::-1]

            isolate[(c1,-1)] = att_inv
        else:
            isolate[(c2,1)] = []
            isolate[(c2,-1)] = []

    d = {}
    for c in critics_dim_2:
        rel = []
        for edge in attaching[c]:
            rel += write(edge, isolate)
        d[c] =  rel

    return d
```

```
def att_to_group(att, gens):
    F = FreeGroup(len(gens), 'a')
    d = {}
    for i in range(len(gens)):
       d[gens[i]] = F.gens()[i]
    Rels = []
    for c in att.keys():
        aux = F.one()
        for (cell,e) in att[c]:
            aux *= d[cell]^e
        Rels.append(aux)
    return F / Rels

def critical_by_level(X, M): #X the face poset of a regular CW, M an
     acyclic matching
    matched = [e[0] for e in M] + [e[1] for e in M]
    edges = X.cover_relations()
    G = DiGraph(edges)
    l = G.level_sets()
    C = []
    for i in range(len(l)):
        cr = []
        for x in l[i]:
            if x not in matched:
                cr.append(x)
        C.append(cr)
    return C

def morse_presentation(gens, rels, M):
    original_attaching = attaching(gens,rels)
    X = presentation_poset(gens,rels)
    criticals = critical_by_level(X, M)
    morse_attaching = attaching_Morse(original_attaching, M,
   criticals[2])
    return att_to_group(morse_attaching, criticals[1])

def total_relator_len_(G):
    return sum(sum(abs(e) for (l,e) in r.syllables()) for r in G.
   relations())

#Matchings

#Greedy algorithm that ouputs a random maximal matching
import random
def greedy_acyclic_matching(X): #X the face poset of regular CW.
    edges = X.cover_relations()
    in_match = {}
    for v in X.list():
```

```
        in_match[v] = False
    seed()
    shuffle(edges)
    M = []
    for e in available_edges:
        if(in_match[e[0]] or in_match[e[1]]):
            continue
        D = DiGraph(edges)
        D.reverse_edge(e)

        if D.is_directed_acyclic():
            edges.remove(e)
            edges.append([e[1], e[0]])
            M.append(e)
            in_match[e[0]] = True
            in_match[e[1]] = True
    return M

def spanning_matching(X):
    M = []
    n = 0
    while n != 1:
        M = greedy_acyclic_matching(X)
        n = len(critical_by_level(X, M)[0]) == 1
    return M
```

Resumen del capítulo 6: Nuevos métodos combinatorios para presentaciones de grupos

En este capítulo, realizamos un ataque combinatorio de la versión para presentaciones de grupos de la conjetura de Andrews-Curtis.

Dada una presentación $\mathcal{P}$ del grupo G, definimos $X_{\mathcal{P}}$, el *poset asociado a la presentación* $\mathcal{P}$, como el face poset de la subdivisión baricéntrica de $K_{\mathcal{P}}$. Este poset tiene claramente su grupo fundamental isomorfo a G, y su diagrama de Hasse puede ser descripto fácilmente en términos de $\mathcal{P}$. Sirve como modelo finito para estudiar su clase de Andrews-Curtis.

Proposición 6.1.4. Sean $\mathcal{P}, \mathcal{Q}$ dos presentaciones de un grupo. Entonces,

(i) $\mathcal{P} \sim_{AC} \mathcal{Q}$ si y sólo si $X_{\mathcal{P}} \nearrow\!\!\searrow^{3} X_{\mathcal{Q}}$.

(ii) $\mathcal{P}$ es balanceada y G es el grupo trivial si y sólo si $X_{\mathcal{P}}$ es homotópicamente trivial.

Dada una presentación balanceada del grupo trivial, elaboramos dos técnicas para transformarla en una nueva presentación en la misma clase de Andrews-Curtis, sin explicitar la lista de movimientos que llevan de una a la otra.

La primer técnica está inspirada en la teoría de coloreo de posets. Es una generalización e implementación discreta del hecho de que colapsar un árbol en el 1-esqueleto de un 2-complejo celular es una 3-deformación.

Dado X un espacio finito de altura 2 y C un coloreo de X, definimos una presentación $\mathcal{P}_{X,C}$.

Definición 6.2.1. Sea X un espacio finito de altura 2 y sea C un subdiagrama de $\mathcal{H}(X)$. Llamamos a C un *coloreo* de $\mathcal{H}(X)$. Decimos que (x, y) es un *par extremal* de X si $x < y$, $h(x) = 0, h(y) = 2$. Para cada par extremal (x, y) de X, fijar, si es posible, una *cadena preferencial* $c_{x,y} : x \prec z \prec y$ tal que al menos una de sus aristas no pertenece a C. Asociamos a (X, C) una presentación $\mathcal{P}_{X,C}$ cuyos generadores son las aristas de $\mathcal{H}(X)$ que no pertenecen a C, y cuyas relaciones están dadas por los *rombos* (i.e. pares de caminos monótonos que se intersecan en sus extremos) que contienen una cadena preferencial.

Dado un espacio finito X de altura 2, vimos que si $C \subseteq \mathcal{H}(X)$ es un árbol o un subdiagrama colapsable con todos los vértices de X, entonces las distintas presentaciones $\mathcal{P}_{X,C}$ pertenecen a la misma clase de equivalencia de Andrews-Curtis. La llamamos $\mathcal{P}_X$ cuando no queremos especificar el subdiagrama C.

Teorema 6.2.6. Sean X, Y espacios finitos de altura 2. Entonces,

(i) $X \nearrow\!\!\searrow^{3} Y$ si y sólo si $\mathcal{P}_X \sim_{AC} \mathcal{P}_Y$;

(ii) X es homotópicamente trivial si y sólo si $\mathcal{P}_X$ es una presentación balanceada del grupo trivial.

Teorema 6.3.1. Si $\mathcal{P}$ es una presentación de grupo y A es un subdiagrama generador colapsable de $\mathcal{H}(X_{\mathcal{P}})$, entonces $\mathcal{P} \sim_{AC} \mathcal{P}_{X_{\mathcal{P}},A}$.

La otra técnica está inspirada en la teoría de Morse discreta, introducida por Robin Forman [For98] en los '90s como una versión combinatoria de la teoría clásica para variedades diferenciables. Probamos una versión más fuerte del clásico teorema de Morse. Para lograrlo, hicimos una reinterpretación del complejo de Morse en términos de *colapsos internos.*

Teorema 6.4.9. Sea $K \cup \bigcup_{i=1}^{N} e_i$ un complejo celular, con K un subcomplejo de dimensión menor o igual que k, tal que $k \leq \dim(e_i) \leq \dim(e_{i+1})$ para todo i. Denotamos por $\varphi_j : \partial D_j \to K \cup \bigcup_{i<j} e_i$ a la función de adjunción de e_j. Si $K \searrow L$,

$$K \cup \bigcup_{i=1}^{N} e_i \nearrow\!\!\!\searrow^{n+1} L \cup \bigcup_{i=1}^{N} \tilde{e}_i$$

con $n = \dim(e_N)$ y $\tilde{\varphi}_j : \partial D_j \to L \cup \bigcup_{i<j} \tilde{e}_i$ la función de adjunción de $\tilde{e}_j$ definida inductivamente por $\tilde{\varphi}_1 = r\varphi_1$ y si $j > 1$, $\tilde{\varphi}_j = f_j \varphi_j$, donde $r : K \to L$ es un retracto por deformación fuerte y $f_j : K \cup \bigcup_{i<j} e_i \to L \cup \bigcup_{i<j} \tilde{e}_i$ es una deformación.

En las hipótesis del teorema anterior, decimos que hay un *colapso interno* de $K \cup \bigcup_{i=1}^{N} e_i$ a $L \cup \bigcup_{i=1}^{N} \tilde{e}_i$.

Dado K un complejo celular regular de dimensión n y M un matching acíclico en $\mathcal{H}(\mathcal{X}(K))$, sabemos por el teorema de Morse que K tiene el mismo tipo homotópico que K_M, un complejo con tantas celdas de dimensión k como celdas críticas en M de esa dimensión. Nosotros logramos describir explícitamente a K_M como el resultado de aplicarle a K una sucesión de colapsos internos determinados por M.

Teorema 6.4.17. Sea K un complejo celular regular de dimensión n sea M un matching acíclico en $\mathcal{H}(\mathcal{X}(K))$. Entonces $K \nearrow\!\!\!\searrow^{n+1} K_M$.

Dimos una descripción explícita de la presentación estándar asociada a K_M, que denotamos por $\mathcal{Q}_{\mathcal{X}(K),M}$.

Definición 6.5.4. Sea K un complejo celular regular de dimensión 2. Sea M un matching en $\mathcal{H}(\mathcal{X}(K))$ tal que hay una sola celda crítica de dimensión 0. Denotamos por M_0 al subconjunto de pares de puntos emparejados de alturas 0 y 1, y por $M_r = \{(x_1, y_1), (x_2, y_2), \cdots, (x_r, y_r)\}$ al subconjunto se pares emparejados a alturas 1 y 2. Sea L un orden topológico preferencial de $\mathcal{H}_M(\mathcal{X}(K))$. Ordenar M_k respecto de L, es decir, $M_k = y_1 < x_1 < y_2 < x_2 < \cdots < y_r < x_r$, con $(x_i, y_i) \in M$. La *presentación de grupo* $\mathcal{Q}_{\mathcal{X}(K),M}$ *asociada al matching* M es $\mathcal{Q}_r$ definida inductivamente por:

- $\mathcal{Q}_0$ es la presentación estándar $\mathcal{P}_K$ construida usando el spanning tree T inducido por M_0. Las relaciones de Q_0 con las 1-celdas de K no emparejadas según el matching M_0, y sus relaciones son las palabras inducidas por las funciones de adjunción de las 2-celdas de K.

- Para $0 \leq i < r$, definimos $\mathcal{Q}_{i+1}$ como la presentación que se obtiene de $\mathcal{Q}_i$ después de eliminar la relación asociada a y_{r-i} y el generador x_{r-i}, y reemplazar cada ocurrencia del generador x_{r-i} en los restantes generadores por la *expresión equivalente de x_{r-i} inferida por la relación asociada a y_{r-i}*.

Corolario 6.5.6. Sea K un complejo celular regular de dimensión 2, y sea M un matching acíclico en $\mathcal{H}(\mathcal{X}(K))$ con una sola celda crítica de dimensión 0. Entonces, $\mathcal{Q}_{\mathcal{X}(K),M}$ es una presentación balanceada de $\pi_1(K)$. Más aún, $\mathcal{Q}_{\mathcal{X}(K),M} \sim_{AC} \mathcal{P}_K$.

Usando los métodos anteriores, implementamos en SAGE algoritmos concretos para obtener presentaciones AC-equivalentes a una presentación dada, sin necesidad de explicitar la lista de movimientos de Andrews-Curtis que llevan de una a la otra.

Obtuvimos, entre otras, demostraciones experimentales simples de los siguientes hechos sobre la lista de potenciales contraejemplos.

- La presentación $\langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle$ [AK85] para $n = 2$, satisface la conjetura de Andrews-Curtis.

- La presentación $\langle x, y, z \mid z^{-1}yz = y^2, x^{-1}zx = z^2, y^{-1}xy = x^2 \rangle$ [Rap68a], se puede trasformar en una presentación con 2 generadores y 2 relaciones.

- Hay una gran clase de enteros n, m, p, q para la cual la presentación $\langle x, y \mid x = [x^m, y^n], y = [x^p, y^q] \rangle$ [Bro84] satisface la conjetura. [3]

  No se conocía si en general las presentaciones de esta familia satisfacen la conjetura de Andrews-Curtis. Sin embargo, algunas de ellas pueden probarse AC-tivializables usando la rutina `simplified`.

  Nosotros probamos experimentalmente que las presentaciones

  $$G_{1,1,k,-1} = \langle x, y \mid x = [x, y], y = [x^p, y^{-1}] \rangle$$

  para $1 \leq p \leq 100$ satisfacen la conjetura. Si k es impar, la rutina `simplified` alcanza, pero si k es par, no. Nosotros logramos hallar, para cada k, un matching acíclico M_k en $\mathcal{H}(X_{G_{1,1,k,-1}})$ tan que $Q_{X_{G_{1,1,k,-1}},M_k}$ se puede reducir a la presentación trivial del grupo trivial luego de aplicar la función `simplified`.

[3] Aquí, $[a, b] = a^{-1}b^{-1}ab$.

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