Laplaciano fraccionario: regularidad de soluciones y aproximaciones por elementos finitos

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2017-06-15

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Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Buenos Aires, 2017

Resumen

El objetivo de esta tesis es estudiar aproximaciones para el laplaciano fraccionario por medio del método de elementos finitos. Dado $s \in (0,1)$, este operador esta definido por

$$(-\Delta)^s u(x) = C(n,s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \, dy,$$

donde $C(n,s)$ es una constante de normalización. El laplaciano fraccionario de orden s es el operador pseudo-diferencial con símbolo $|\xi|^{2s}$ y es el generador infinitesimal del proceso estocástico $2s$-estable llamdo vuelo de Lévy.

En primer lugar se estudia la regularidad de soluciones de ecuaciones que involucran al laplaciano fraccionario sobre dominios acotados. Se muestran caracterizaciones precisas de propiedades de mapeo de este operador sobre dominios unidimensionales y se obtienen estimaciones de regularidad sobre dominios n-dimensionales en término tanto de la regularidad Sobolev como de la regularidad Hölder de los datos. En este último caso, los resultados en espacios estándar son extendidos a espacios con peso, que permiten obtener caracterizaciones finas del comportamiento de las soluciones cerca de la frontera del dominio en cuestión.

Luego se aborda el tratamiento numérico del problema de Dirichlet con condiciones de borde homogéneas. El método de elementos finitos con funciones de base continuas y lineales a trozos es implementado en una y dos dimensiones. Asimismo, se desarrolla la teoría tanto para mallas uniformes como adecuadamente graduadas.

Posteriormente, se trata el problema de autovalores con condiciones de borde Dirichlet homogéneas para el laplaciano fraccionario. Este problema es un modelo simplificado de la ecuación de Schrödinger no local con potencial infinito, por lo que su abordaje numérico ha sido tratado entre la comunidad física; sin embargo, los esquemas preexistentes solo son aplicables o bien a dominios unidimensionales o a dominios con simetría. Las aproximaciones por elementos finitos desarrolladas en esta tesis, al ser conformes (el espacio discreto es un subespacio del continuo) permiten obtener con facilidad y de forma relativamente eficiente cotas superiores para los autovalores, con versatilidad respecto al dominio.

Finalmente, se estudian aproximaciones para el problema de Dirichlet con condiciones de borde no homogéneas. Debido al carácter no local del laplaciano fraccionario, a pesar de resolver una ecuación en un dominio acotado es necesario prescribir los datos de contorno sobre todo el complemento del mismo. Aquí se utiliza una fórmula de integración por partes no local que permite introducir una *derivada normal no local.* El enfoque planteado consiste en incluir esta derivada normal no local en forma de multiplicador de Lagrange, obteniendo así un problema en forma mixta.

Para los tres problemas analizados se obtienen estimaciones de error de los esquemas propuestos, y se realizan experimentos numéricos que muestran resultados en concordancia con las predicciones teóricas.

Palabras clave: Laplaciano fraccionario, método de elementos finitos, espacios de Sobolev fraccionarios.

Abstract

The goal of this thesis is to study finite element approximations for the fractional Laplacian. Given $s \in (0,1)$, this operator is defined by

$$(-\Delta)^s u(x) = C(n,s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,$$

where $C(n,s)$ is a normalizing constant. The fractional Laplacian of order s is a pseudo-differential operator with symbol $|\xi|^{2s}$, and it is the infinitesimal generator of a $2s$-stable stochastic process, called Lévy flight.

In first place we study regularity of solutions of equations involving the fractional Laplacian in bounded domains. Precise mapping properties of this operator over one-dimensional domains are shown, and general regularity estimates over n-dimensional domains are obtained in terms both of the Sobolev and the Hölder regularity of the data. In the latter case, results in standard spaces are extended to weighted spaces, and these allow to have a sharp characterization of the behavior of solutions near the boundary of the domain under consideration.

Then we deal with the numerical treatment of the homogeneous Dirichlet problem for the fractional Laplacian. The finite element method with continuous, piecewise linear basis functions is implemented in one and two dimensions. Moreover, theory is developed for uniform and suitably graded meshes.

Afterwards, the thesis treats the fractional eigenvalue problem with homogeneous Dirichlet conditions. Such problem serves as a simplified model of the nonlocal Schrödinger equation with an infinite well, and thus has been considered in the physics' community; however, the preexisting schemes are only applicable either to one-dimensional or radial domains. The conforming nature of the finite element approximations developed in this thesis allow to obtain upper bound for eigenvalues easily and efficiently, with versatility with respect to the domain.

Finally, we study discrete approximations to the nonhomogeneous Dirichlet problem. Due to the nonlocal character of the fractional Laplacian, even though we are solving an equation over a bounded domain it is necessary to enforce the Dirichlet condition on the whole complement. Here we utilize a nonlocal integration by parts formula that allows to define a *nonlocal normal derivative*. The approach we propose consists on including this nonlocal normal derivative as a Lagrange multiplier, which leads to a mixed formulation of the problem.

For the discrete schemes we propose for three problems described above we obtain error estimates and we carry out numerical experiments. The results of these are in good agreement with the theoretical predictions.

Keywords: Fractional Laplacian, finite element method, fractional Sobolev spaces.

Contents

Introduction

Diffusion is the net movement of particles from regions with higher concentration towards regions with lower concentration. By now it is clear that Fick's first law, which is a constitutive relation for diffusive fluxes, is a questionable model for numerous phenomena (see, for example, [84, 85]). Equivalently, whenever the associated underlying stochastic process is not given by Brownian motion, the diffusion is regarded as *anomalous*. In particular, anomalous superdiffusion refers to situations that can be modeled using fractional spatial derivatives or fractional spatial differential operators.

Integer-order differentiation operators are local, because the derivative of a function at a given point depends only on the values of the function in a neighborhood of it. In contrast, fractional-order derivatives are nonlocal, integro-differential operators.

In recent years, nonlocal models have increasingly impacted upon a number of important fields in science and technology. The study of nonlocal operators has been an active area of research in different branches of mathematics, and these operators have been employed to model situations in which different length scales are involved. Evidence of anomalous diffusion processes has been found in several physical and social environments [70, 85], and corresponding transport models have been proposed in various areas such as electrodiffusion in nerve cells [77], electromagnetic fluids [83], and ground-water solute transport [38]. Nonlocal models have also been proposed in fields as diverse as finance [29, 36], machine learning [94], peridynamics [100] and image processing [25, 52, 54, 80].

Naturally, the operators that appear in these applications can vary, and a number of discrete approximation strategies have been proposed for these. Accordingly, offering a unified discretization of all these operators is a too ambitious goal. This dissertation focuses on finite element approximation schemes for one of the most striking examples of nonlocal operators, the fractional Laplacian of order s ($s \in (0,1)$), which we will denote by $(-\Delta)^s$. In the theory of stochastic processes, this operator appears as the infinitesimal generator of a $2s$-stable Lévy process [14, 103].

We remark the ubiquity of fractional Laplacians in the modeling of physical and social phenomena with two examples. Firstly, in porous media flow, particles may experience very large transitions arising both from high heterogeneity and very long spatial autocorrelation. In [11] the authors have performed experiments and theoretical studies into contaminant transport in aquifers and have provided evidence of diffusion governed by α-stable processes with orders $\alpha = 1.55$, 1.65 and 1.8 for various aquifer locations. Thus, the corresponding fractional advection-dispersion equation modeling these phenomena included fractional Laplacians of orders 0.775, 0.825 and 0.9, respectively.

Secondly, Lévy flights have been utilized to model human locomotion in relation to crime diffusion [31]. Starting with an agent-based cellular automata model, the authors derive its continuum limit, that consists of two equations and involves the fractional Laplacian operator, which allows for the superdiffusion of criminals.

There are several characterizations of the fractional Laplacian of order s if the domain under consideration is the whole space $\mathbb{R}^n$. A first definition, in order to understand why it is called a *Laplacian*, is given by deeming it as a pseudo-differential operator with symbol $|\xi|^{2s}$. Indeed, for a function u in the Schwartz class $\mathcal{S}$, let

$$(-\Delta)^s u := \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}u\right),$$

where $\mathcal{F}$ denotes the Fourier transform. Upon inverting this transform, this operator can equivalently be defined by means of the identity

$$(-\Delta)^s u(x) = C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy, \tag{$*$}$$

where $C(n,s)$ is a suitable normalizing constant. The last identity illustrates the *non-local* character of the fractional Laplacian: modifying a function u in an open set $\mathcal{O}$ may alter the value of $(-\Delta)^s u$ at points x arbitrarily away from $\mathcal{O}$.

There are two different approaches to the definition of the fractional Laplacian on an open bounded set Ω. On the one hand, to consider powers of the Laplacian in a spectral sense: given a function u, to consider its spectral decomposition in terms of the eigenfunctions of the Laplacian with homogeneous Dirichlet boundary condition, and to take the operator that acts by raising to the power s the corresponding eigenvalue. Namely, if $\{\psi_k, \lambda_k\}_{k\in\mathbb{N}} \subset H^1_0(\Omega)\times\mathbb{R}_+$ denotes the set of normalized eigenfunctions and eigenvalues, then this operator is defined as

$$(-\Delta)^s_S\, u(x) = \sum_{k=1}^{\infty} \lambda_k^s (u,\psi_k)_{L^2(\Omega)}\psi_k(x). \tag{$**$}$$

On the other hand, there is the possibility to keep the motivation coming from the stochastic process leading to the definition of $(-\Delta)^s$ in $\mathbb{R}^n$. This option leads to two different types of operators: one in which the stochastic process is restricted to Ω and one in which particles are allowed to jump anywhere in the space. The first of these two is the infinitesimal generator of a censored stable Lévy process in Ω. We refer to it as *regional* fractional Laplacian, and it is given by

$$(-\Delta)^s_\Omega u(x) = C(n,s,\Omega) \text{ P.V.} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy.$$

The second of the two operators motivated by Lévy processes leads to considering the integral formulation $(*)$. Observe that, unlike the aforementioned fractional Laplacians, the definition of this operator does not depend on the domain Ω. In this thesis we deal with this last operator, which we denote by $(-\Delta)^s$ and simply call the fractional Laplacian. The possibility of having arbitrarily long jumps in the corresponding stochastic process explains why, when considering problems involving the fractional Laplacian on a bounded domain Ω, the usual boundary conditions need to be substituted by analogous volume constraints on $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$.

In this dissertation we shall be concerned with some problems involving the fractional Laplacian on bounded domains, such as the following. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth enough boundary, $f\colon \Omega \to \mathbb{R}$ be a smooth enough function and $s \in (0,1)$, we seek u such that

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{$***$}$$

Our main purpose is to convey a complete finite element analysis of problems like this one. More precisely, besides $(***)$, this thesis is concerned with a problem with nonhomogeneous Dirichlet condition and with an homogeneous fractional eigenvalue problem. As well as including proof of convergence of the algorithms we propose for these problems, we seek to estimate their computational efficiency. For the purpose of obtaining a priori error estimates, it is fundamental to know how regular solutions are; this dissertation develops regularity estimates for solutions of the aforementioned problems in the Sobolev scale. We are also interested in giving numerical evidence of the accuracy of our theoretical predictions, and this in turn requires a feasible finite element implementation.

The finite element method is one of the preferred numerical tools in scientific and engineering communities. It counts with a solid and long established theoretical foundation, mainly in the linear case of second order *elliptic* partial differential equations. The fractional Laplace equation in the form $(***)$ shares some key analytical features of the classical Laplacian making it amenable, in principle, to a direct finite element treatment. Some numerical difficulties –such as the need to cope with integration on unbounded domains and handling the singularity of the kernel appearing in $(*)$– seem to be the main disadvantages that have discouraged a direct finite element approach. Concerning the latter, applying rather standard techniques (actually borrowed from the theory of the boundary element method [95]) together with an appropriate treatment of the integrals involving the unbounded domain Ω^c, accurate finite element solutions can be delivered for one and two-dimensional problems.

State of the art

Before proceeding with the thesis outline and its specific contributions, we review the state of the art regarding regularity theory for the fractional Laplacian and different numerical methods for fractional differential equations advocated in the literature.

Regularity theory

The study of mapping properties of the fractional Laplacian and the regularity of solutions of problems such as $(***)$ dates back to the 1960s, with the works by Višik and Èskin (for example, [104]). These authors utilized a factorization property of pseudo-differential symbols to derive important mapping properties of the fractional Laplacian. Furthermore, their results yield a Sobolev regularity estimate provided $s \in (0, 1/2)$. Later, this pseudo-differential approach was also analyzed by Grubb [56], who obtained regularity estimates for problems posed on smooth domains and $s \in (0, 1)$.

Preceding the work by Grubb, the study of regularity of solutions for boundary value problems involving the fractional Laplacian had already regained popularity after the seminal work [27] by Caffarelli and Silvestre. Therein, the authors prove that the fractional Laplacian in $\mathbb{R}^n$ can be regarded as a Dirichlet-to-Neumann map for a certain local elliptic equation posed in $\mathbb{R}^{n+1}_+$. This allows for an analysis via differential operators.

Resorting to potential theoretic and integral operator methods, Hölder regularity up to the boundary of solutions to the Dirichlet problem for the fractional Laplacian was proved by Ros-Oton and Serra [92]. Their estimates are valid for Lipschitz domains satisfying the exterior ball condition, which makes them more appropriate for a finite element analysis than the ones developed via pseudo-differential methods. Moreover, these estimates measure in a precise way the singular behavior of solutions near the boundary.

Numerical methods

In contrast to elliptic PDEs, numerical developments for problems involving the fractional Laplacian in the form $(*)$, even in simplified contexts, are seldom found in the literature. The reason for this is related to two major challenging tasks usually involved in its numerical treatment: the handling of highly singular kernels and the need to cope with an unbounded region of integration. This is precisely the case of $(***)$, for which just a few numerical methods have been proposed.

Huang and Oberman [62] developed a one-dimensional scheme that combines finite differences with a quadrature rule in an unbounded domain; their analysis proves the algorithm to be convergent under the condition that the solution is of class C^4. The numerical evidence provided in that paper for smooth right-hand sides indicates convergence with an order s, in the infinity norm, as the mesh size tends to zero. Orders as high as $3-2s$ are demonstrated in that contribution for singular right-hand sides that make the solution smoother.

D'Elia and Gunzburger [39], in turn, analyzed a nonlocal diffusion operator that approximates the one-dimensional fractional Laplacian. This work demonstrates that, under certain conditions, the solution of the nonlocal equation converges to the solution of ($***$) as the nonlocal interactions become infinite. The continuous Galerkin finite element discretization showed convergence of order 1/2 in the energy norm. These authors also suggested that an improved solution algorithm, with increased convergence order, might require explicit consideration of the solution's boundary singularities.

Motivated by applications to fractional quantum mechanics, some schemes oriented to the resolution of eigenvalue problems for the fractional Laplacian in one-dimensional domains have been introduced in recent years. On the one hand, Zoia, Rosso and Kardar [109] provided a discretized version of this operator; different types of boundary conditions are justified by appealing to two physical models: hopping particles and elastic springs. On the other hand, in [44] the authors find numerically the ground and first excited states for linear and nonlinear fractional Schrödinger equation. The technique they employ is based on the introduction of a fractional gradient flow with discrete normalization, which is then discretized by using a trapezoidal quadrature rule in space and the semi-implicit Euler method in time. This work also reports on the emergence of boundary layers in the ground states of the fractional nonlinear Schrödinger equation.

There have been other approaches to fractional diffusion, more specifically, related to the discretization of the spectral fractional Laplacian ($**$). Nochetto, Otárola and Salgado [87] exploited the localization technique from [23, 27, 102] and analyzed a local problem posed on a certain cylindrical extension. Since the variable of interest is a conormal derivative of the extended function, they considered tensor product finite elements on anisotropic meshes. The numerical algorithm proposed and analyzed by the authors takes advantage of the rapid decay of the solution in the extended variable, that enables truncation to a bounded domain of modest size. Error estimates are derived in a weighted H^1-norm.

Another approach based on the spectral decomposition of the Laplacian has been proposed in [65, 66]. In those works, the authors introduced a matrix transference technique for problems posed on an interval. This method consists in utilizing as discrete operator the s-th power of a matrix approximation of the Laplacian –typically obtained from a finite difference scheme–.

The discretization of ($**$) proposed by Bonito and Pasciak [19] is based on the integral formulation of fractional powers of self-adjoint operators, representing solutions via Dunford-Taylor integrals. This method leads to the discretization of a sequence of uncoupled elliptic PDEs, and delivers quasi-optimal error estimates in the L^2-norm.

Thesis outline

This dissertation analyzes finite element approximations for some problems involving the fractional Laplacian ($*$) on bounded domains. In order to do so, Chapter 1 sets the notation and provides preliminary information regarding the fractional Laplacian, fractional-order spaces and variational formulations of problems like ($***$). We also compare our approach to fractional differentiation on bounded domains with other operators considered by other authors.

Regularity of solutions

An important component in the finite element analysis of problems such as ($***$) is the regularity of its solutions. Basically, the more regular a function is, the better it may be approximated by a discrete scheme. As we shall discuss in Chapter 2, independently of the smoothness of the domain, solutions to ($***$) have reduced Sobolev regularity. For one-dimensional and radial problems, the discussion of regularity of solutions is powerfully solved by utilizing the explicit eigendecomposition of a weighted operator closely related to the fractional Laplacian. However, the situation in arbitrary multi-dimensional domains is more delicate. We describe Sobolev regularity estimates based on Sobolev regularity of the right hand side function as well as Hölder-Hölder estimates. Then, we utilize the latter to develop a Hölder-Sobolev regularity theory; this analysis leads also to a fine characterization of the behavior of solutions near the boundary of the domain. This characterization leads naturally to the consideration of certain weighted fractional spaces.

Homogeneous Dirichlet problem

Having at our disposal sharp regularity estimates, in Chapter 3 we address a direct finite element analysis of ($***$) resorting to piecewise linear Lagrangian elements. The nonlocal nature of the problem under consideration is reflected in the fact that fractional Sobolev norms are not additive respect to the domains. Therefore, after defining a suitable interpolation operator and obtaining adequate local interpolation estimates, some cautions must be taken in order to bound the global interpolation error. In order to deal with graded meshes we extend well-known error estimates for the Scott-Zhang interpolation operator to weighted fractional Sobolev spaces. These estimates are derived by introducing Poincaré inequalities in the weighted fractional setting. We present error bounds in the energy and the L^2-norm and numerical experiments in one and two-dimensional domains, utilizing both uniform and graded meshes.

Fractional eigenvalue problem

Chapter 4 deals with finite element approximations to the eigenvalue problem for the fractional Laplacian on bounded domains with homogeneous Dirichlet conditions. We prove the convergence of a conforming finite element scheme in gap distance with minimal assumptions on the domain. Moreover, under certain smoothness of the boundary of the domain, we show that the Babuška-Osborn theory applies to give convergence rates; we also provide a self-contained proof of these orders of convergence. We perform numerical computations and compare our results with others found in the literature. Besides showing good agreement with the theory, these results deliver eigenvalue estimates that are consistent with those given by other authors. The conforming nature of the finite element approximations developed in this thesis allows to obtain upper bound for eigenvalues easily and efficiently, with versatility with respect to the domain.

Nonhomogeneous Dirichlet conditions and flux density approximation

In some applications, when dealing with problems such as $(***)$, the quantity of interest may not only be the solution but also the flux density between the domain and its complement. In Chapter 5 we propose a mixed finite element method to approximate both of these quantities, that is valid also when non-zero volume constraints are imposed. We propose a weak imposition of the Dirichlet condition and the incorporation of a nonlocal analogue of the normal derivative –the flux density in the complement of the domain– as a Lagrange multiplier in the formulation of the problem. Since the domains we discretize must be bounded, we estimate the error associated to discarding the tail of the volume constraint. Piecewise linear discrete functions are utilized, and the scheme we propose requires that, as meshes are refined, the discrete domains need to grow in diameter. We display evidence of convergence of the algorithm independently of whether the volume constraint is boundedly supported or not.

Contributions

The first part of Chapter 2 –more precisely, subsections 2.1.1 and 2.1.2– collects results from the work:

[4] G. Acosta, J. P. Borthagaray, O. Bruno and M. Maas. *Regularity theory and high order numerical methods for the (1d)-Fractional Laplacian.* To appear in Mathematics of Computation.

Section 2.3 and the discussion in Chapter 3 are mainly based on the reference:

[3] G. Acosta and J. P. Borthagaray, *A fractional Laplace equation: regularity of solutions and finite element approximations.* SIAM Journal on Numerical Analysis, 55(2):472–495, 2017.

The exposition in Chapter 4 follows the one in the paper:

[20] J. P. Borthagaray, L. M. Del Pezzo and S. Martínez, *Finite element approximation for the fractional eigenvalue problem*. Submitted. Preprint, arXiv.

The analysis and the scheme presented in Chapter 5 is mainly taken from the work:

[5] G. Acosta, J. P. Borthagaray and N. Heuer. *Finite element approximations for the nonhomogeneous fractional Dirichlet problem*. In preparation.

The two-dimensional finite element experiments carried out in this thesis have been performed with the aid of the code described in:

[2] G. Acosta, F. M. Bersetche and J. P. Borthagaray. *A short FEM implementation for a 2D homogeneous Dirichlet problem of a fractional Laplacian*. Submitted. Preprint, arXiv.

Chapter 1

Preliminaries

This chapter collects the preliminary material to develop a full finite element study of problems involving the fractional Laplacian: we motivate and define this operator and study the variational spaces involved, resorting to several viewpoints. To the best of the author's knowledge, there are essentially four approaches to the fractional Laplacian: as a singular integral, as the infinitesimal generator of a stable Lévy process, from pseudo-differential calculus, and from nonlocal calculus. Even though we mainly work with the first approach, each of these sheds light on different aspects of the operator.

Section 1.1 gives the definition of the fractional Laplacian that we use the most, as a singular integral. Also, a motivation for this operator by a random walk with jumps of arbitrary length is provided, illustrating the *nonlocal* nature of the fractional Laplacian.

Since the operator we study gives raise to a formulation set in fractional-order Sobolev spaces, in Section 1.2 we discuss several aspects of these spaces. Further, we examine the weak form of the fractional Laplacian and a nonlocal integration by parts formula. One issue raised by the nonlocal nature of fractional Sobolev spaces is that fractional seminorms are not additive with respect to domains. We discuss how to localize these norms as well.

Fractional Sobolev spaces with exponent $p = 2$ may equivalently be characterized by means of the decay of the Fourier coefficients. This characterization as Bessel potential spaces, developed in Section 1.3, leads to an alternative formulation of the fractional Laplacian that enables to show that this operator is in turn pseudo-differential.

Even though there is a well-defined unique characterization of fractional diffusion in $\mathbb{R}^n$, there are different possibilities on bounded domains. The aim of Section 1.4 is to discuss some of these variants and to comment on other nonlocal operators on bounded domains that share features of the fractional Laplacian.

Finally, in Section 1.5 we introduce and discuss some theoretical properties of a class of weighted fractional Sobolev spaces. In this section we make a distinction between one-dimensional and multi-dimensional domains. For the former, our spaces are based on expansions on bases of special functions and lead to a precise characterization of the mapping properties of the fractional Laplacian on one-dimensional domains. As for multi-dimensional domains, the spaces we introduce are of interest for finite element approximations because, as we show in Chapter 3, they allow to increase the order of convergence of the numerical schemes.

1.1 The fractional Laplacian in $\mathbb{R}^n$

We begin our exposition by defining the main object of study of this thesis, the fractional Laplace operator, which is an elliptic linear integro-differential operator. We first provide a definition as a singular integral and afterwards we show that it arises naturally as the infinitesimal generator of a random walk with jumps of arbitrary length.

1.1.1 Definition as a singular integral

In order to give a definition of the fractional Laplacian we consider its action over a family of regular enough functions. Consider the Schwartz class, consisting of smooth and rapidly decaying functions over $\mathbb{R}^n$,

$$\mathcal{S} := \left\{ u \in C^\infty(\mathbb{R}^n) \colon \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty \ \forall \alpha, \beta \in \mathbb{N}^n \right\}.$$

Definition 1.1.1. Let $s \in (0,1)$. The fractional Laplacian of order s of a function $u \in \mathcal{S}$, which we will denote by $(-\Delta)^s u$, is defined as

$$(-\Delta)^s u(x) := C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy, \tag{1.1.1}$$

where

$$C(n,s) := \frac{2^{2s} s \Gamma(s + \frac{n}{2})}{\pi^{n/2} \Gamma(1-s)} \tag{1.1.2}$$

is a normalizing constant.

Observe that the fractional Laplacian is defined by a principal value integral. However, if $s \in (0, 1/2)$ it is not necessary to include the principal value in the definition because if $u \in \mathcal{S}$, then

$$\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} \, dy \\ \leq C \int_{B(x,R)} |x-y|^{-n-2s+1} dy \ + \|u\|_{L^\infty(\mathbb{R}^n)} \int_{B(x,R)^c} |x-y|^{-n-2s} dy < \infty.$$

In order to get rid of the principal value in the definition of the fractional Laplacian, we may write the operator as the integral of a weighted second order differential quotient.

Proposition 1.1.2 (see [40, Lemma 3.2])**.** *Let $s \in (0,1)$, then for any $u \in \mathcal{S}$,*

$$(-\Delta)^s u(x) = -\frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{n+2s}} \, dy.$$

Remark 1.1.3. Definition (1.1.1) is not the only possible way to characterize the fractional Laplacian; rather, it is the one that we utilize for the finite element approximations we develop in this thesis. In Section 1.3 we introduce a definition based on the Fourier transform. Other characterizations of the fractional Laplacian include the one based on harmonic extensions and Bochner's subordination. See [75] for a proof of the equivalence of these definitions.

The constant $C(n,s)$, given by (1.1.2), ensures the consistence between Definition 1.1.1 and the definition of the fractional Laplacian motivated by Fourier analysis (cf. Proposition 1.3.4). It may look unimportant if s is fixed, but it provides the adequate scaling in the limits $s \to 0^+$ and $s \to 1^-$:

$$C(n,s) \sim s(1-s) \quad \text{for } s \to \{0^+, 1^-\}.$$

Actually, another important property of the fractional Laplacian is that, in some sense, it lies between the identity and the classical Laplacian.

Proposition 1.1.4 (see [40, Proposition 4.4])**.** *For any $u \in \mathcal{S}$, the following limits hold for every $x \in \mathbb{R}^n$:*

$$\lim_{s \to 0^+} (-\Delta)^s u(x) = u(x),$$
$$\lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x).$$

To conclude this section, we mention an useful characterization of the constant appearing in the definition of the fractional Laplacian.

Lemma 1.1.5 (see [26, Lemma 3.1.3])**.** *Let $C(n,s)$ be given according to* (1.1.2)*. Then, it holds that*

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(z_1)}{|z|^{n+2s}} \, dz \right)^{-1}.$$

1.1.2 A random walk approach

Next, we motivate the fractional Laplacian by probabilistic considerations. Namely, we show that the fractional heat equation naturally arises from a process in which a particle moves randomly in the space subject to a probability that allows long jumps with a polynomial tail. This subsection is based on the construction by Bucur and Valdinoci [26]. A similar construction, consisting of a random walk over a lattice, can be found in the expository article [103].

We begin with a probability defined over the set of natural numbers, having density p. Namely, let $p : \mathbb{N} \to [0,1]$ be such that $\sum_{\rho\in\mathbb{N}} p(\rho) = 1$. We assume that p is an homogeneous function of order $-(1+2s)$ for a certain $s \in (0,1)$:

$$p(\rho) = \begin{cases} c(s)\rho^{-(1+2s)}, & \rho \neq 0, \\ 0, & \rho = 0. \end{cases}$$

We consider a particle having a discrete motion in space and in time as follows. Let h denote a spatial length and τ be a time step. We link these quantities by $\tau = h^{2s}$. At each time step τ, the particle selects a natural number ρ according to the probability distribution with density p, a random direction v according to a uniform distribution in the $n-1$ dimensional sphere ∂B_1, and moves $v\rho h$.

Observe that, unlike the classical random walk, in this construction the particle is allowed to take jumps of arbitrary length, although with a small probability.

We define $u(x,t)$ as the probability density of the particle location, so that for any measurable set $E \subset \mathbb{R}^n$ and $t \in \tau\mathbb{N}$, it holds that

$$\mathbb{P}\left(\text{particle is in } E \text{ at time } t\right) = \int_E u(x,t)\,dx.$$

Then, $u(x,t+\tau)$ is given by the sum of the probabilities of x to be somewhere else, say $x+v\rho h$, at time t times the probability of jumping from $x+v\rho h$ to x,

$$u(x,t+\tau) = \frac{c(s)}{\sigma_{n-1}} \int_{\partial B_1} \sum_{\rho\in\mathbb{N}} \frac{u(x+v\rho h)}{|\rho|^{1+2s}}\,d\sigma(v),$$

where σ_{n-1} denotes the measure of the $(n-1)$-dimensional sphere ∂B_1. Therefore,

$$u(x,t+\tau) - u(x,t) = \frac{c(s)}{\sigma_{n-1}} \int_{\partial B_1} \sum_{\rho\in\mathbb{N}} \frac{u(x+v\rho h,t) - u(x,t)}{|\rho|^{1+2s}}\,d\sigma(v). \tag{1.1.3}$$

Given a direction $v \in \partial B_1$, let us define

$$\psi_v(y,x,t) = \frac{u(x+vy,t) - u(x,t)}{|y|^{1+2s}},$$

then, recalling that $\tau = h^{2s}$, we may write (1.1.3) as

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{c(s)}{\sigma_{n-1}} \int_{\partial B_1} \sum_{\rho\in\mathbb{N}} h\,\psi_v(\rho h,x,t)\,d\sigma(v).$$

The latter is a Riemann sum approximating $\int_0^\infty \psi_v(\rho, x, t)\, d\rho$, and thus, taking formally the limit $h \to 0$ above we obtain

$$\partial_t u(x,t) = \frac{c(s)}{\sigma_{n-1}} \int_{\partial B_1} \int_0^\infty \psi_v(\rho, x, t)\, d\rho\, d\sigma(v).$$

Equivalently, resorting to integration in polar coordinates,

$$\partial_t u(x,t) = \frac{c(s)}{\sigma_{n-1}} \int_{\mathbb{R}^n} \frac{u(x+y,t) - u(x,t)}{|y|^{n+2s}}\, dy = c(n,s)\, (-\Delta)^s u(x,t),$$

for some constant $c(n,s) > 0$. Therefore, we have shown that, in the limit, this random walk with arbitrarily long jumps gives raise to a fractional heat equation.

Remark 1.1.6. There are several variants of this random process that motivate related nonlocal operators (cf. subsections 1.4.2 and 1.4.3) or other problems for the fractional Laplacian (1.1.1).

For example, assume that the time step is $\tau = 1$, that the particle moves within a bounded domain Ω and that whenever the particle reaches a point $y \in \Omega^c$ the process ends. Denote the expected time of finishing the process starting at a point $x \in \Omega$ by $u(x)$.Then, a similar argument to the one we have just presented allows to show that u satisfies $(-\Delta)^s u = 1$ in Ω. A closed formula for the solution of this equation was obtained by Getoor [53], that gives an explicit computation of the expected finishing time of this process in case Ω is a ball.

Remark 1.1.7. Natural phenomena that may be successfully modeled by processes of this type have been extensively reported to occur. For example, see the biological observations in [63, 90, 101]. A "hit-and-run" hunting strategy is related to the random walk we have just considered: a predator chooses a random direction, moves randomly in that direction, stops to eat the prey in its surroundings and then goes on. For non-destructive foraging, that is, under the hypothesis of preys being located in patches and being only temporarily depleted, in [105] it is shown that whenever no a priori information about the surroundings is available, the value $s = 1/2$ delivers an optimal searching strategy. On the other hand, for destructive foraging, optimal search patterns correspond to the limit $s \to 0$. This is consistent with data gathered from in-situ observations; for example, in [101, Figure 1] the behavior of diverse marine vertebrates is compared, and the least-squares fitting shown there for those species yields values of s between 0.35 and 0.7.

From the mathematical viewpoint, [67] studies a model of two competing species that have the same population dynamics but two different dispersal strategies: the movement of one species is purely by random walk while the other species adopts a nonlocal dispersal strategy. Supported both by their local stability analysis and the results of their numerical simulations, the authors conjecture that nonlocal dispersal is always preferred over random dispersal.

1.2 Fractional Sobolev spaces

The fractional Laplacian gives raise to variational formulations set in fractional-order Sobolev spaces. Here we set the notation and review some properties of the spaces involved in the rest of the thesis. Nonlocality is reflected in the fact that fractional Sobolev norms are not additive respect to the domains.

1.2.1 Definition and properties

Definition 1.2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $s \in (0,1)$ and $p \in [1,\infty)$. The Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \left\{ v \in L^p(\Omega) \colon |v|_{W^{s,p}(\Omega)} < \infty \right\},$$

where

$$|v|_{W^{s,p}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}} dx\, dy \right)^{\frac{1}{p}}$$

denotes the Aronszajn-Slobodeckij seminorm. The space $W^{s,p}(\Omega)$ is furnished with the norm

$$\|v\|_{W^{s,p}(\Omega)} := \|v\|_{L^p(\Omega)} + |v|_{W^{s,p}(\Omega)}.$$

Fractional Sobolev spaces of order greater than 1 are defined in the following way. Let $s = k + \sigma$, with $k \in \mathbb{N}$ and $\sigma \in (0,1)$. Then,

$$W^{s,p}(\Omega) := \left\{ v \in W^{k,p}(\Omega) \colon |\partial^\alpha v| \in W^{\sigma,p}(\Omega) \, \forall \alpha \in \mathbb{N}^n \text{ such that } |\alpha| = k \right\},$$

and we endow this set with the norm

$$\|v\|_{W^{s,p}(\Omega)} := \|v\|_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} |\partial^\alpha v|_{W^{\sigma,p}(\Omega)}.$$

Completeness of fractional Sobolev spaces is well-known (see, for example, [6, Section 7.32]).

Proposition 1.2.2. *Let $s > 0$ and $p \in [1,\infty)$, then $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)})$ is a Banach space.*

There is more than one possible interpretation that can be made to functions "vanishing at the boundary" of Ω. We state them next.

Definition 1.2.3. We denote by $W^{s,p}_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the $W^{s,p}(\Omega)$ norm.

On the other hand, we define $\widetilde{W}^{s,p}(\Omega)$ as the set of functions in $W^{s,p}(\Omega)$ whose extension by zero over Ω^c belongs to $W^{s,p}(\mathbb{R}^n)$. The norm for functions in this set is given by

$$\|v\|_{\widetilde{W}^{s,p}(\Omega)} := \|\tilde{v}\|_{W^{s,p}(\mathbb{R}^n)},$$

where $\tilde{v}$ is the extension of v by zero outside Ω. For simplicity of notation, whenever we refer to a function in $W^{s,p}(\Omega)$, we assume that it is extended by zero onto Ω^c.

Notice that, unless $s \in \mathbb{N}$, the norm in $\widetilde{W}^{s,p}(\Omega)$ is not the same as the one in $W^{s,p}(\Omega)$, as the former includes integration over the set $\Omega \times \Omega^c$. However, smooth functions in Ω are dense in $\widetilde{W}^{s,p}(\Omega)$.

Proposition 1.2.4 (see [55, Theorem 1.4.2.2])**.** *Let Ω be a set with continuous boundary, then $C_0^\infty(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ for all $s > 0$.*

Moreover, whenever the domain Ω is regular enough but trace operators are unavailable, smooth functions are dense in $W^{s,p}(\Omega)$.

Proposition 1.2.5 (see [55, Theorem 1.4.2.4])**.** *Let Ω be an open, bounded set with continuous boundary. Then, for all $s \in (0, 1/p]$, $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$.*

Having defined fractional Sobolev spaces, we recall some of their basic properties that are needed in the following. We begin stating a fractional version of the Poincaré-Wirtinger inequality.

Proposition 1.2.6 (Poincaré inequality I)**.** *Let Ω be an open bounded set, $s \in (0,1)$ and $p \in [1,\infty)$. For any $v \in W^{s,p}(\Omega)$, we write $\bar{v} := \frac{1}{|\Omega|}\int_\Omega v$. Then it holds that*

$$\|v - \bar{v}\|_{L^p(\Omega)} \leq c d_\Omega^s |v|_{W^{s,p}(\Omega)}, \tag{1.2.1}$$

with c bounded in terms of $\frac{d_\Omega}{d_B}$, where $d_\Omega =$diam(Ω), $d_B =$diam(B) and B is the largest ball contained in Ω.

Proof. Applying Hölder's inequality if $p > 1$, we write

$$\int_\Omega |v - \bar{v}|^p dx = \frac{1}{|\Omega|^p}\int_\Omega \left|\int_\Omega (v(x) - v(y))dy\right|^p dx \leq \frac{1}{|\Omega|}\int_\Omega\int_\Omega |v(x) - v(y)|^p\, dydx.$$

Therefore,

$$\int_\Omega |v - \bar{v}|^p dx \leq \frac{d_\Omega^{n+sp}}{|\Omega|}\int_\Omega\int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+sp}} dydx.$$

Taking into account that $\frac{\sigma_{n-1}}{n2^n}\, d_B^n = |B| \leq |\Omega|$, the claimed identity (1.2.1) follows with $c = \left(\frac{2n^{1/n}}{\sigma_{n-1}^{1/n}}\,\frac{d_\Omega}{d_B}\right)^{n/p}$. □

Remark 1.2.7. In the same spirit as [24, Equation (4.2.17)] we call $\frac{d_\Omega}{d_B}$ the chunkiness parameter of Ω.

Another well-known result is the following.

Proposition 1.2.8 (Poincaré inequality II)**.** *Given an open, bounded set Ω, $s \in (0,1)$ and $p \in [1, \infty)$, there exists a constant $c = c(\Omega, n, s, p)$ such that*

$$\|v\|_{L^p(\Omega)} \leq c|v|_{W^{s,p}(\mathbb{R}^n)} \quad \forall v \in \widetilde{W}^{s,p}(\Omega).$$

Proof. By Lemma 6.1 of [40], there exists some constant $c(n, s, p) > 0$ such that for all $x \in \Omega$,

$$c(n, s, p)|\Omega|^{-\frac{sp}{n}} \leq \int_{\Omega^c} \frac{1}{|x-y|^{n+sp}} dy.$$

On the other hand, since $v \equiv 0$ on Ω^c we know that $|v(x)|^p = |v(x) - v(y)|^p$ for all $x \in \Omega$, $y \in \Omega^c$. So, we obtain

$$c(n, s, p)|\Omega|^{-\frac{sp}{n}} \int_\Omega |v(x)|^p dx \leq \iint_{\Omega \times \Omega^c} \frac{|v(x) - v(y)|^p}{|x-y|^{n+sp}} dx\, dy,$$

and the Poincaré inequality follows straightforwardly. □

An immediate consequence of the previous proposition is that the $W^{s,p}(\mathbb{R}^n)$ seminorm is equivalent to the $W^{s,p}(\mathbb{R}^n)$-norm on $\widetilde{W}^{s,p}(\Omega)$.

Definition 1.2.9. Given $\Omega \subset \mathbb{R}^n$, we denote by δ the function $\delta : \mathbb{R}^n \to [0, \infty)$,

$$\delta(x) = \delta(x, \partial\Omega). \tag{1.2.2}$$

Proposition 1.2.10 (Hardy inequality, see [45] and [55, Theorem 1.4.4.4])**.** *Let Ω be a bounded domain with Lipschitz boundary. Then, if $s - 1/p$ is not an integer, there exists $c = c(\Omega, n, s, p) > 0$ such*

$$\int_\Omega \frac{|\partial^\alpha v(x)|^p}{\delta(x)^{(s-|\alpha|)p}} dx \leq c\|v\|^p_{W^{s,p}(\Omega)} \; \forall\, v \in W^{s,p}_0(\Omega), \alpha \in \mathbb{N}^n, |\alpha| \leq s. \tag{1.2.3}$$

As a consequence of the previous proposition we deduce two important properties.

Corollary 1.2.11. *If $s \in (0,1)$ and $s \neq \frac{1}{p}$, there exists a constant $c = c(\Omega, n, s, p) > 0$ such that*

$$|v|_{W^{s,p}(\mathbb{R}^n)} \leq c\|v\|_{W^{s,p}(\Omega)} \quad \forall v \in W^{s,p}_0(\Omega).$$

Proof. Let $v \in C_0^\infty(\Omega)$, then, since $v = 0$ in Ω^c,

$$
\begin{aligned}
|v|^p_{W^{s,p}(\mathbb{R}^n)} &= \iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}} dxdy + 2\iint_{\Omega\times\Omega^c} \frac{|v(x)|^p}{|x-y|^{n+sp}} dxdy \\
&\le c(n,s,p)\left[|v|^p_{W^{s,p}(\Omega)} + \int_\Omega |v(x)|^p \int_{B(x,d(x))^c} \frac{1}{|x-y|^{n+sp}}\, dy\, dx \right] \\
&= c(n,s,p)\left[|v|^2_{W^{s,p}(\Omega)} + \int_\Omega \frac{|v(x)|^p}{\delta(x)^{sp}}\, dx \right].
\end{aligned}
$$

Applying the Hardy inequality (1.2.3), the estimate follows. □

Corollary 1.2.12. *Let Ω be an open, bounded set with Lipschitz boundary. If $s - 1/p \notin \mathbb{N}$, then*

$$\widetilde{W}^{s,p}(\Omega) = W_0^{s,p}(\Omega).$$

Furthermore, if $0 < s < 1/p$, then

$$\widetilde{W}^{s,p}(\Omega) = W_0^{s,p}(\Omega) = W^{s,p}(\Omega).$$

Proof. From the previous corollary, we know that the norms in $W_0^{s,p}(\Omega)$ and $\widetilde{W}^{s,p}(\Omega)$ are equivalent if $s - 1/p \notin \mathbb{N}$. Since $C_0^\infty(\Omega)$ is dense in both spaces (recall Definition 1.2.3 and Proposition 1.2.4), the first claim follows.

The second part is easily proved by combining the first assertion with Proposition 1.2.5. □

Remark 1.2.13. In view of Proposition 1.2.8, if $s \in (1/p, 1)$, then in the conclusion of Corollary 1.2.11 we may substitute the $W^{s,p}(\Omega)$ norm in the right hand side by the corresponding seminorm.

Remark 1.2.14. The case $s = 1/p$ has been excluded from most of the preceding discussion. From Proposition 1.2.5, we know that $W_0^{1/p,p}(\Omega) = W^{1/p,p}(\Omega)$. In turn, this space contains strictly the set $\widetilde{W}^{1/p,p}(\Omega)$. The latter may be characterized as

$$\widetilde{W}^{1/p,p}(\Omega) = \left\{ v \in W^{1/p,p}(\Omega) \colon \frac{v}{\delta^{1/p}} \in L^p(\Omega) \right\},$$

where δ is defined according to (1.2.2) (see [55, Corollary 1.4.4.10] or [79, Theorem 1.11.7]). In case $p = 2$, this set is usually called *Lions-Magenes space*, and is denoted by $H_{00}^{1/2}(\Omega)$. See the discussion in Section 1.2.2.

Sobolev spaces of negative order are defined by duality.

Definition 1.2.15. Let $s < 0$, $p \in (1, \infty)$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. We denote by $W^{s,p}(\Omega)$ the dual space of $\widetilde{W}^{-s,q}(\Omega)$, and by $\widetilde{W}^{s,p}(\Omega)$ the one of $W^{-s,q}(\Omega)$.

To finish this section, we comment about the important question of whether the fractional Sobolev spaces we have defined are consistent with the usual definition of Sobolev spaces of integer order. Up to normalizing constants, this is the case. Observe that the factors multiplying the fractional order norms below have the same scaling as the constant $C(n,s)$ defined by (1.1.2).

Proposition 1.2.16. *Let $v \in L^p(\Omega)$, $1 < p < \infty$. Then (see [21]),*

$$\lim_{s \to 1^-} (1-s) \|v\|^p_{W^{s,p}(\Omega)} = C(n,p) \|v\|^p_{W^{1,p}(\Omega)}.$$

On the other hand, if there exists $s_0 > 0$ such that $v \in W^{s_0,p}(\mathbb{R}^n)$, then (see [82])

$$\lim_{s \to 0^+} s \|v\|^p_{W^{s,p}(\mathbb{R}^n)} = C(n,p) \|v\|^p_{L^p(\mathbb{R}^n)}.$$

Remark 1.2.17. Fractional Sobolev spaces may equivalently be defined as real interpolation spaces. Indeed, if $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in (0,1)$, and $p \in [1, \infty)$, either for $\Omega = \mathbb{R}^n$ or Ω a domain with a Lipschitz boundary, then (for example, [79, Chapter 1])

$$\begin{aligned} W^{s,p}(\Omega) &= \left[W^{k,p}(\Omega), W^{k+1,p}(\Omega) \right]_{s,p} \\ W^{s,p}_0(\Omega) &= \left[W^{k,p}(\Omega), W^{k+1,p}_0(\Omega) \right]_{s,p}. \end{aligned}$$

Interpolation above may be taken either by the $K-$ or the $J-$method (see [12, Chapter 3]). Thus, estimates in these norms may be obtained by interpolation between estimates in integer order Sobolev spaces [12, Theorems 3.1.2 and 3.2.2].

Bearing in mind applications to the finite element method, Heuer [61] proved that the Aronzajn-Slobodeckij and interpolation seminorms are uniformly equivalent under affine mappings that ensure shape regularity of the domains under consideration.

1.2.2 The spaces $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$

The variational space associated to the fractional Laplacian is the Sobolev space of order s and integrability exponent $p = 2$; it is customary to use the notation

$$H^s(\Omega) := W^{s,2}(\Omega), \quad \widetilde{H}^s(\Omega) := \widetilde{W}^{s,2}(\Omega), \quad H^s_0(\Omega) := W^{s,2}_0(\Omega).$$

This notation is justified by the characterization of the fractional Sobolev spaces as Bessel potential spaces when the integrability exponent equals to 2 (cf. Subsection 1.3.1).

Proposition 1.2.18. *Let Ω be an open set. Define the bilinear forms $(\cdot,\cdot)_{L^2(\Omega)}$ and $\langle \cdot,\cdot \rangle_{H^s(\Omega)}$,*

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(x)\, v(x)\, dx, \tag{1.2.4}$$

$$\langle u,v\rangle_{H^s(\Omega)} = \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx. \tag{1.2.5}$$

Then, the space $H^s(\Omega)$, furnished with the inner product $(u,v)\mapsto (u,v)_{L^2(\Omega)}+\langle u,v\rangle_{H^s(\Omega)}$ is a Hilbert space.

The previous proposition also applies to the zero-extension space $\widetilde{H}^s(\Omega)$. Recall that the norm in this space is the one from $H^s(\mathbb{R}^n)$, so that over this set the bilinear form $\langle u,v\rangle_{H^s(\mathbb{R}^n)}$ takes the form

$$\langle u,v\rangle_{H^s(\mathbb{R}^n)} = \langle u,v\rangle_{H^s(\Omega)} + 2\int_\Omega u(x)\, v(x) \int_{\Omega^c} \frac{1}{|x-y|^{n+2s}} dydx, \quad u,v\in \widetilde{H}^s(\Omega).$$

Thus, the inner product in $\widetilde{H}^s(\Omega)$ is the sum of the one from $H^s(\Omega)$ plus a weighted $L^2(\Omega)$ inner product. For convenience, we set the following notation regarding this weight.

Definition 1.2.19. Given an (not necessarily bounded) open set Ω and $s\in(0,1)$, we denote by $\omega^s_\Omega : \Omega\to(0,\infty)$ the function given by

$$\omega^s_\Omega(x) = \int_{\Omega^c} \frac{1}{|x-y|^{n+2s}} dy. \tag{1.2.6}$$

Recall the function δ provided by Definition 1.2.9. An upper bound for ω^s_Ω is easily obtained by integration in polar coordinates. Furthermore, if $\partial\Omega$ is Lipschitz continuous, then the order of such a bound (with respect to δ) is accurate,

$$0 < \frac{C}{\delta(x)^{2s}} \le \omega^s_\Omega(x) \le \frac{\sigma_{n-1}}{2s\,\delta(x)^{2s}} \quad \forall x\in\Omega. \tag{1.2.7}$$

Above, σ_{n-1} denotes the measure of the $n-1$ dimensional sphere and $C>0$ is a constant that depends on Ω. For the lower bound for ω^s_Ω we refer to [55, formula (1.3.2.12)].

Next, we state a nonlocal version of the integration by parts formula involving the fractional Laplacian. For that purpose, we first define a *nonlocal normal derivative* introduced in [41].

Definition 1.2.20. Let $\Omega\subset\mathbb{R}^n$ be an open set and v be a smooth enough function defined over $\mathbb{R}^n$, then the nonlocal normal derivative of v with respect to Ω is the operator $\mathcal{N}_s v\colon \Omega^c\to\mathbb{R}$ given by

$$\mathcal{N}_s v(x) = C(n,s)\int_\Omega \frac{v(x)-v(y)}{|x-y|^{n+2s}}\, dy.$$

Notice the dependence of the nonlocal normal derivative with respect to the domain Ω. The integration by parts formula for the fractional Laplacian reads as follows.

Proposition 1.2.21 (see [41, 43]). *Let $u, v\colon \mathbb{R}^n \to \mathbb{R}$ be smooth enough functions, then*

$$\begin{aligned}\frac{C(n,s)}{2}\iint_Q &\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\\ &= \int_\Omega v(x)(-\Delta)^s u(x)\,dx + \int_{\Omega^c} v(x)\,\mathcal{N}_s u(x)\,dx,\end{aligned} \tag{1.2.8}$$

where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$.

In the previous proposition we assumed that u, v are smooth functions. However, as $C_0^\infty(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$ and the double integral in (1.2.8) is a constant times $\langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^n)}$ in $\widetilde{H}^s(\Omega)$, we may extend (1.2.8) to the latter space:

$$\frac{C(n,s)}{2}\iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy = \int_\Omega v(-\Delta)^s u,\ \ u,v \in \widetilde{H}^s(\Omega). \tag{1.2.9}$$

This integration by parts formula is used for defining the nonlocal normal derivative of a function in $H^s(\mathbb{R}^n)$ in the following fashion. Let $u \in C^\infty(\mathbb{R}^n)$ and $v \in \widetilde{H}^s(\Omega^c)$, so that upon extending v by zero on Ω, formula (1.2.8) gives

$$\int_{\Omega^c} \mathcal{N}_s u(x)\,v(x)\,dx = \frac{C(n,s)}{2}\iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy,$$

and this in turn equals to $\frac{C(n,s)}{2}(\langle u,v\rangle_{H^s(\mathbb{R}^n)} - \langle u,v\rangle_{H^s(\Omega^c)})$. Thus, we can extend the domain of definition of the operator $\mathcal{N}_s$. If $u \in H^s(\mathbb{R}^n)$ we set $\mathcal{N}_s u \in H^{-s}(\Omega^c)$ as

$$(\mathcal{N}_s u, v) := \frac{C(n,s)}{2}\left(\langle u,v\rangle_{H^s(\mathbb{R}^n)} - \langle u,v\rangle_{H^s(\Omega^c)}\right).$$

More generally, if Ω is a domain such that there exists a continuous extension operator $H^s(\Omega^c) \to H^s(\mathbb{R}^n)$[1], then the integration by parts formula gives that the nonlocal normal derivative induces a bounded map $H^s(\mathbb{R}^n) \to \widetilde{H}^{-s}(\Omega^c)$. Indeed, if $u \in H^s(\mathbb{R}^n)$ and $v \in H^s(\Omega^c)$, considering an extension $Ev \in H^s(\mathbb{R}^n)$ we set

$$(\mathcal{N}_s u, v) := \frac{C(n,s)}{2}\left(\langle u,Ev\rangle_{H^s(\mathbb{R}^n)} - \langle u,v\rangle_{H^s(\Omega^c)}\right) - \int_\Omega Ev\,(-\Delta)^s u.$$

This definition is readily seen to be independent of the extension of v considered.

Remark 1.2.22. It is possible to give the nonlocal derivative operator $\mathcal{N}_s$ an interpretation in the context of the probabilistic motivation for the fractional Laplacian from Section 1.1.2. Indeed, assume that the random walk with jumps takes place inside a

[1] This is true, for example if Ω is a Lipschitz domain. See Remark 4.3.2 for a characterization of fractional extension domains.

set Ω and, as before, let $u(x,t)$ be the probability distribution of the position of the particle. Modify the process in the following way: whenever the particle exits Ω, it immediately comes back into Ω; if the particle reaches a point $x \in \Omega^c$, it may return to a point $y \in \Omega$, with probability proportional to $|x-y|^{-n-2s}$. These modifications lead, in the same fashion as in Section 1.1.2 to the fact that $u(x,t)$ solves problem

$$\begin{cases} u_t + (-\Delta)^s u = 0 & \text{in } \Omega \times (0,\infty), \\ \mathcal{N}_s u = 0 & \text{in } \Omega^c \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases}$$

where u_0 denotes the initial probability distribution of the position of the particle. See [41] for details.

1.2.3 Weak formulation

We illustrate how to write weak formulations for problems involving the fractional Laplacian. From now on we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Whenever regularity on the boundary of Ω is required, we mention it explicitly. For simplicity, let us focus on the homogeneous Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \qquad \text{(Homogeneous)}$$

where $(-\Delta)^s u$ denotes the operator defined in (1.1.1) and $f \in H^{-s}(\Omega)$ (recall Definition 1.2.15 and the comments at the beginning of Subsection 1.2.2). Observe that we seek a function that vanishes on Ω^c. Recall that, due to the Poincaré inequality (Proposition 1.2.8), the $H^s(\mathbb{R}^n)$ seminorm is an equivalent norm to the $H^s(\mathbb{R}^n)$ one over $\widetilde{H}^s(\Omega)$. In order to simplify the notation, we multiply this seminorm by a factor $\sqrt{\frac{C(n,s)}{2}}$, with $C(n,s)$ defined according to (1.1.2). So, we consider the variational space

$$(\mathbb{V}, \|\cdot\|_{\mathbb{V}}) := \left(\widetilde{H}^s(\Omega), \sqrt{\frac{C(n,s)}{2}}\, |\cdot|_{H^s(\mathbb{R}^n)} \right),$$

which is a Hilbert space endowed with the inner product

$$\langle u, v\rangle_{\mathbb{V}} := \langle u, v\rangle_{H^s(\mathbb{R}^n)} = \frac{C(n,s)}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy, \;\; u,v \in \mathbb{V}. \tag{1.2.10}$$

Recall that the integration above is just carried over $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$.

Weak solutions of (Homogeneous) are defined by multiplying the equation by a test function and applying the integration by parts formula (1.2.9). Thus, using the notation we have just set, the weak formulation of this model problem is:

$$\text{find } u \in \mathbb{V} \text{ such that } \;\langle u, v\rangle_{\mathbb{V}} = \int_\Omega f\, v \quad \forall v \in \mathbb{V}. \tag{1.2.11}$$

Application of the Lax-Milgram Theorem allows to prove well-posedness of the weak problem. Coercivity is a consequence of the considerations we have made, whereas continuity follows from the Cauchy-Schwarz inequality.

Proposition 1.2.23. *Problem* (1.2.11) *is well-posed: there exists a unique solution $u \in \mathbb{V}$, and the solution map $f \mapsto u$ is continuous (with modulus of continuity equal to* 1*):*

$$\|u\|_{\mathbb{V}} \leq \|f\|_{H^{-s}(\Omega)}.$$

1.2.4 Localization of fractional norms

The problems we study in this thesis involve the fractional Sobolev spaces described previously. For our purposes, there are two immediate consequences of nonlocality we need to take into account.

In first place, recall that $\widetilde{H}^s(\Omega)$ is a Hilbert space. As shown in Chapter 3, when computing discrete solutions, we need to calculate the inner product between every pair of basis functions and assemble this information in a matrix. Nonlocality implies that, independently of the distance between the supports of basis functions, this inner product may be nonzero. Indeed, assume φ_i and φ_j are two nonnegative functions such that $\text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) = \emptyset$, then

$$\langle \varphi_i, \varphi_j \rangle_{\mathbb{V}} = -2\, C(n,s) \iint_{\text{supp}(\varphi_i)\times\text{supp}(\varphi_j)} \frac{\varphi_i(x)\varphi_j(y)}{|x-y|^{n+2s}} dy\, dx < 0.$$

This means that the stiffness matrices involved are full, and it also affects the efficiency of algorithms, as it is necessary to run a double loop in the elements to compute the entries of such matrices.

The second issue raised by nonlocality is that fractional seminorms are not additive with respect to domain decompositions. Namely, if we decompose $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1 \cap \Omega_2 = \emptyset$, then

$$|v|^2_{H^s(\Omega)} = |v|^2_{H^s(\Omega_1)} + |v|^2_{H^s(\Omega_2)} + 2 \iint_{\Omega_1\times\Omega_2} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dydx.$$

Unless some additional assumptions are taken on the function v, it is not possible to bound the integral over $\Omega_1 \times \Omega_2$ in terms of the norms on Ω_1, Ω_2. This implies that, whenever calculating error (or interpolation) estimates, it is not possible to sum elementwise. Nevertheless, Faermann [49, 50] showed that certain localization is possible by adding some overlapping. As we are stating the result in a slightly different way than Faermann, we include a proof here.

Proposition 1.2.24. *Let $s \in (0,1)$ and Ω be a bounded domain. Assume there is a decomposition $\overline{\Omega} = \overline{\cup_i \Omega_i}$, where the subdomains Ω_i are open and pairwise disjoint. Then, for any $v \in H^s(\Omega)$ it holds that*

$$|v|^2_{H^s(\Omega)} \leq \sum_i \left[\iint_{\Omega_i \times S_i} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} \, dy \, dx + \frac{2\sigma_{n-1}}{s\delta_i^{2s}} \, \|v\|^2_{L^2(\Omega_i)} \right], \tag{1.2.12}$$

where

$$S_i := \bigcup_{j \, : \, \overline{\Omega_j} \cap \overline{\Omega_i} \neq \emptyset} \Omega_j,$$

$\delta_i = d(\Omega_i, \Omega \setminus S_i)$ and σ_{n-1} denotes the measure of the $(n-1)$-dimensional sphere.

Proof. Given an element Ω_i of the partition, we define $D_i = \Omega \setminus S_i$. Then,

$$|v|^2_{H^s(\Omega)} = \sum_i \left[\iint_{\Omega_i \times S_i} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} \, dy \, dx + \iint_{\Omega_i \times D_i} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} \, dy \, dx \right].$$

We may bound the integrals in the right hand side as

$$\begin{aligned} \iint_{\Omega_i \times D_i} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} \, dy \, dx \leq 2 \Bigg[& \int_{\Omega_i} |v(x)|^2 \int_{D_i} |x-y|^{-n-2s} dy \, dx \\ & + \int_{D_i} |v(y)|^2 \int_{\Omega_i} |x-y|^{-n-2s} dx \, dy \Bigg] \\ =: {} & J_{i,1} + J_{i,2}. \end{aligned}$$

Let us show that $\sum_i J_{i,1} = \sum_i J_{i,2}$. Indeed, we write

$$\begin{aligned} \sum_i J_{i,2} &= \sum_i \int_\Omega \chi_{D_i}(y) |v(y)|^2 \int_{\Omega_i} |x-y|^{-n-2s} dx \, dy \\ &= \int_\Omega |v(y)|^2 \left(\sum_i \chi_{D_i}(y) \int_{\Omega_i} |x-y|^{-n-2s} dx \right) dy = \int_\Omega |v(y)|^2 f(y) \, dy, \end{aligned}$$

where $f(y) = \sum_i \chi_{D_i}(y) \int_{\Omega_i} |x-y|^{-n-2s} dx$. Next, we write the integral over Ω as a sum of integrals over subdomains Ω_j. Observe that, if $y \in \Omega_j$, then

$$\chi_{D_i}(y) = \begin{cases} 1 & \text{if } \overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset \text{ (i.e., if } \Omega_i \subset D_j), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $y \in \Omega_j$,

$$f(y) = \sum_i \chi_{D_i}(y) \int_{\Omega_i} |x-y|^{-n-2s} dx = \int_{D_j} |x-y|^{-n-2s} dx,$$

and then

$$\sum_i J_{i,2} = \sum_j \int_{\Omega_j} |v(y)|^2 \int_{D_j} |x-y|^{-n-2s} dx\, dy = \sum_j J_{j,1}.$$

Therefore, we have shown that

$$|v|^2_{H^s(\Omega)} \leq \sum_i \left[\iint_{\Omega_i \times S_i} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dy\, dx + 4J_{i,1} \right].$$

Finally, $J_{i,1}$ is easily bounded by noticing that $D_i \subset \Omega \setminus B(x, \delta_i)$ for all $x \in \Omega_i$ and integrating in polar coordinates:

$$J_{i,1} \leq \sigma_{n-1} \int_{\Omega_i} |v(x)|^2 \int_{\delta_i}^{\infty} \rho^{-1-2s} d\rho\, dx = \frac{\sigma_{n-1}}{2s\delta_i^{2s}} \|v\|^2_{L^2(\Omega_i)}.$$

This concludes the proof. □

Remark 1.2.25. As stated, the previous lemma does not assume any type of shape regularity on the elements of the partition of Ω. Actually, this dependence is hidden in the variable δ_i: in order to apply this result in the context of finite element approximations, it is necessary to link δ_i with the diameter of Ω_i. Thus, local quasi-uniformity should be assumed.

Remark 1.2.26. Another subject -not addressed in this thesis- where nonlocality raises difficulties is in the development of a posteriori error indicators. Actually, Faermann developed the localization estimate with the purpose of applying it to adaptive boundary element methods. The technique from Proposition 1.2.24 is not the only possible way to localize fractional order norms. Nochetto, von Petersdorff and Zhang [88], bearing in mind applications to residual error indicators for integral equations, take advantage of the partition of unity given by the nodal basis functions, leading them to local H^{-s}-norms over element patches. Estimating these dual norms by local L^p-norms raises restrictions on the range of s, n within which the method is applicable.

1.3 The Fourier approach

In this section we utilize the Fourier transform to define the fractional Laplacian. First, Subsection 1.3.1 reviews basic properties of the Fourier transform and defines Bessel potential spaces. Afterwards, pseudo-differential operators are considered in Subsection 1.3.2 and the link between the Bessel potential spaces and the fractional Sobolev spaces from Section 1.2 is established. The fractional Laplacian plays a pivotal role in this regards.

1.3.1 Fourier transform and Bessel potential spaces

We denote the Fourier transform $\mathcal{F}$ of a function $u \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u(x)\, dx.$$

It is well-known that the Fourier transform maps the Schwartz class $\mathcal{S}$ continuously into itself. Basic properties of the Fourier transform can be found, for example, in [51, Section 8.3]. For the sake of completeness, we state here some of them that will be useful in the sequel.

Proposition 1.3.1. *Let $u \in L^1(\mathbb{R}^n)$. Given $z \in \mathbb{R}^n$, define $\tau_z u(x) := u(x - z)$. Then,*

$$\mathcal{F}(\tau_z u)(\xi) = e^{-2\pi i \xi \cdot z} \hat{u}(\xi). \tag{1.3.1}$$

If $u \in C^k(\mathbb{R}^n)$, $\partial^\alpha u \in L^1(\mathbb{R}^n)$ for $|\alpha| \leq k$ and $\partial^\alpha u \in C_0(\mathbb{R}^n)$ for $|\alpha| \leq k-1$, then

$$\mathcal{F}(\partial^\alpha u)(\xi) = (2\pi i \xi)^\alpha \hat{u}(\xi), \quad \forall \xi \in \mathbb{R}^n,\ |\alpha| \leq k. \tag{1.3.2}$$

If $\hat{u} \in L^1(\mathbb{R}^n)$, then the following Fourier inversion formula holds,

$$u(x) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} \hat{u}(\xi)\, d\xi. \tag{1.3.3}$$

The Fourier transform can be (uniquely) extended to an unitary isomorphism on $L^2(\mathbb{R}^n)$: if $u \in L^2(\mathbb{R}^n)$, then $\hat{u} \in L^2(\mathbb{R}^n)$ and

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{1.3.4}$$

The Fourier transform allows to define Bessel potential spaces as follows.

Definition 1.3.2. Let $s \geq 0$ and $p \in (1, \infty)$, then the space

$$H^{s,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) \colon \mathcal{F}^{-1}\left((1 + 4\pi^2 |\cdot|^2)^{s/2} \hat{u} \right) \in L^p(\mathbb{R}^n) \right\},$$

endowed with the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}\left((1 + 4\pi^2 |\cdot|^2)^{s/2} \hat{u} \right)\|_{L^p(\mathbb{R}^n)}.$$

From Proposition 1.3.1, it is evident that the previous definition leads to the identity $W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, with equivalence of norms. Interestingly, this identity also holds for non integer orders; this justifies the use of the notation introduced in Subsection 1.2.2. We postpone a proof of this equivalence to next subsection (cf. Proposition 1.3.6).

1.3.2 The fractional Laplacian as a pseudo-differential operator

Recall Proposition 1.3.1: the Fourier transform maps derivatives into polynomials in the frequency space. Furthermore, from the definition of the fractional Laplacian (Definition 1.1.1), we know that this operator leads naturally to fractional order Sobolev spaces, and in turn, these spaces may be characterized in terms of the decay of the Fourier transform of its elements. Thus, it seems natural to wonder how to put together the link between the fractional Laplacian and the Fourier transform. This is attained by introducing the concept of pseudo-differential operators.

In particular, from (1.3.2) we observe that

$$\mathcal{F}(-\Delta u)(\xi) = 4\pi^2|\xi|^2\hat{u}(\xi),$$

and upon applying the inversion formula (1.3.3), we obtain the expression

$$-\Delta u(x) = \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x} 4\pi^2|\xi|^2\hat{u}(\xi)\,d\xi.$$

Therefore, the Laplacian can be represented by an integral operator in the frequency space, multiplying the Fourier transform by the function $\sigma(\xi) = 4\pi^2|\xi|^2$. Such function σ is called a symbol[2].

Definition 1.3.3. A symbol is a function σ defined over $\mathbb{R}^n$ such that there exist $C > 0$ and $\alpha \in \mathbb{R}$ such that

$$|\sigma(\xi)| \leq C(1 + |\xi|)^\alpha \quad \forall \xi \in \mathbb{R}^n.$$

Given a symbol σ, the pseudo-differential operator T_σ associated to it is defined by

$$T_\sigma u(x) := \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\sigma(\xi)\hat{u}(\xi)\,d\xi, \quad u \in \mathcal{S}.$$

Pseudo-differential operators generalize differential operators; the symbols of the latter are just polynomials in the ξ-variable. The fractional Laplacian $(-\Delta)^s$ is one of the simplest examples of a pseudo-differential operator that is not a differential operator, because its symbol is an homogeneous function in ξ of degree $2s$.

Proposition 1.3.4. *Let $s \in (0, 1)$ and $(-\Delta)^s : \mathcal{S} \to \mathbb{R}^n$ denote the fractional Laplacian, given by Definition 1.1.1. Then, for any $u \in \mathcal{S}$, the following identity holds:*

$$\mathcal{F}\left((-\Delta)^s u\right)(\xi) = (2\pi|\xi|)^{2s}\hat{u}(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{1.3.5}$$

This means that the fractional Laplacian is the pseudo-differential operator T_σ associated to the symbol $\sigma(\xi) = (2\pi|\xi|)^{2s}$.

[2] Actually, the class of symbols that give raise to pseudo-differential operators may be taken much larger than the one considered here. For example, symbols may be allowed to depend on the variable x as well. See [48, Chapter V] for a discussion on this topic.

Proof. We will take advantage of the definition of the fractional Laplacian given by Proposition 1.1.2. First, observe that

$$\begin{aligned}
&\left|\frac{u(x+y)-2u(x)+u(x-y)}{|y|^{n+2s}}\right| \\
&\le \chi_{B_1}(y)|y|^{-n-2s+2}\sup_{z\in B_1(x)}|D^2u(z)| + 4\chi_{B_1^c}(y)|y|^{-n-2s}\sup_{z\in\mathbb{R}^n}|u(z)| \\
&\le C\left(\chi_{B_1}(y)|y|^{-n-2s+2}(1+|x|^{n+1})^{-1}+\chi_{B_1^c}(y)|y|^{-n-2s}\right)\in L^1(\mathbb{R}^n\times\mathbb{R}^n).
\end{aligned}$$

Thus, we may apply Fubini's theorem to exchange the integral in y from the definition of the fractional Laplacian with the integral in x from the Fourier transform. Using identity (1.3.1) we obtain

$$\begin{aligned}
\mathcal{F}\left((-\Delta)^s u\right)(\xi) &= -\frac{C(n,s)}{2}\int_{\mathbb{R}^n}\frac{\mathcal{F}(\tau_y u-2u+\tau_{-y}u)(\xi)}{|y|^{n+2s}}\,dy \\
&= -\frac{C(n,s)}{2}\int_{\mathbb{R}^n}\frac{e^{-2\pi i\xi\cdot y}+e^{2\pi i\xi\cdot y}-2}{|y|^{n+2s}}\,\hat{u}(\xi)\,dy \\
&= C(n,s)\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy\ \hat{u}(\xi).
\end{aligned}$$

So, we just need to show that

$$C(n,s)\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy = (2\pi|\xi|)^{2s}. \tag{1.3.6}$$

To do this, consider the function $\Psi\colon\mathbb{R}^n\to\mathbb{R}$, $\Psi(\xi)=\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy$. It is simple to check that Ψ is invariant under rotations centered at the origin, so that $\Psi(\xi)=\Psi(|\xi|e_1)$. Substituting $z=2\pi|\xi|y$ in the definition of Ψ, we obtain

$$\Psi(\xi)=(2\pi|\xi|)^{2s}\int_{\mathbb{R}^n}\frac{1-\cos(z_1)}{|z|^{n+2s}}\,dz=\frac{(2\pi|\xi|)^{2s}}{C(n,s)},$$

according to Lemma 1.1.5. This concludes the proof. □

Combining the previous proposition with the definition of the L^2-based Bessel potential spaces (Definition 1.3.2), we deduce that $(-\Delta)^s$ is an operator of order $2s$ on this class of spaces.

Corollary 1.3.5. *For any $s\in\mathbb{R}$, the operator $(-\Delta)^s$ is of order $2s$, that is, $(-\Delta)^s: H^\ell(\mathbb{R}^n)\to H^{\ell-2s}(\mathbb{R}^n)$ is continuous for any $\ell\in\mathbb{R}$.*

Moreover, if $u\in L^2(\mathbb{R}^n)$ satisfies $(-\Delta)^s u=f$ in $\mathbb{R}^n$ for some $f\in H^\ell(\mathbb{R}^n)$, then $u\in H^{\ell+2s}(\mathbb{R}^n)$.

From the latter part of the previous corollary, it seems expectable to have a gain of $2s$ derivatives for problems involving $(-\Delta)^s$ on bounded domains. However, this is not true: if $u \in \widetilde{H}^s(\Omega)$ satisfies $(-\Delta)^s u = f$ for some $f \in H^\ell(\mathbb{R}^n)$ it does not imply that $u \in H^{\ell+2s}(\Omega)$. The characterization of the fractional Laplacian as a pseudo-differential operator is of key importance in order to obtain Sobolev regularity results of solutions of problems involving it on bounded domains. We address this subject in Chapter 2.

Finally, the previous construction allows us to easily link L^2-based fractional Sobolev spaces with Bessel potential spaces, using the fractional Laplacian.

Proposition 1.3.6. *Let $s \in (0,1)$, then $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$. Furthermore,*

$$|v|^2_{W^{s,2}(\mathbb{R}^n)} = \frac{2(2\pi)^{2s}}{C(n,s)} \| |\cdot|^s \hat{v} \|^2_{L^2(\mathbb{R}^n)} = \frac{2}{C(n,s)} \|(-\Delta)^{s/2} v\|^2_{L^2(\mathbb{R}^n)}, \tag{1.3.7}$$

where $C(n,s)$, given by (1.1.2)*, is the constant appearing in the definition of the fractional Laplacian.*

Proof. Let $v \in W^{s,2}(\mathbb{R}^n)$. Given $z \in \mathbb{R}^n$, recall that we denote $\tau_z v(\cdot) = v(z - \cdot)$. Then, using the definition of the Aronzajn-Slobodeckij seminorm and Plancherel's formula (1.3.4),

$$\begin{aligned} |v|^2_{W^{s,2}(\mathbb{R}^n)} &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left| \frac{v(x) - \tau_z v(x)}{|z|^{n/2+s}} \right|^2 dxdz \\ &= \int_{\mathbb{R}^n} \left\| \frac{v - \tau_z v}{|z|^{n/2+s}} \right\|^2_{L^2(\mathbb{R}^n)} dz = \int_{\mathbb{R}^n} \left\| \mathcal{F}\left(\frac{v - \tau_z v}{|z|^{n/2+s}} \right) \right\|^2_{L^2(\mathbb{R}^n)} dz. \end{aligned}$$

Recalling identities (1.3.1) and (1.3.6), we deduce

$$\begin{aligned} |v|^2_{W^{s,2}(\mathbb{R}^n)} &= 2 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1 - \cos(2\pi\xi \cdot z)}{|z|^{n+2s}} |\hat{v}(\xi)|^2 d\xi dz \\ &= \frac{2}{C(n,s)} \int_{\mathbb{R}^n} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 d\xi. \end{aligned}$$

This proves the equivalence of spaces and the first equality in (1.3.7). The second one follows by recalling identity (1.3.5) and applying Plancherel's formula. □

Remark 1.3.7. The equivalence between Sobolev spaces and Bessel potential spaces stated in Proposition 1.3.6 relies on Plancherel's Formula. As it is well-known, the Fourier transform is not an invertible map between $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ unless $p = q = 2$. This implies that, in general, the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ do not coincide with the Sobolev spaces $W^{s,p}(\mathbb{R}^n)$.

1.4 Other fractional order operators over bounded sets

In this section we recapitulate some operators that are closely related to the fractional Laplacian. As before, let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth enough boundary, and suppose we want to study a problem such as

$$\mathcal{L}u = f \text{ in } \Omega,$$

with adequate boundary conditions, where $\mathcal{L}$ is a nonlocal, fractional order operator. For the sake of clarity, let us say that we want to analyze the case of homogeneous Dirichlet conditions. It is not obvious what the definition of the operator $\mathcal{L}$ should be, nor how boundary conditions should be imposed. Even though throughout this thesis we make use of the fractional Laplacian (1.1.1), other operators could be taken as well.

The three operators analyzed here share important properties with the fractional Laplacian, and the three of them allow to obtain $(-\Delta)^s$ as a certain limit. Nevertheless, there are also crucial differences between them and our object of study.

The *spectral fractional Laplacian* corresponds to a non-integer power of the Laplace operator in the spectral sense, and its connection with the fractional Laplacian is given by an extension technique based on an identification of the latter as a Dirichlet-to-Neumann map for a certain elliptic operator. However, Dirichlet conditions for the spectral fractional Laplacian just need to be imposed on the boundary of the domain.

Restricting the random process described in Subsection 1.1.2 to the domain Ω gives raise to a *regional fractional Laplacian.* This operator is related to a Neumann problem for the fractional Laplacian. Also, for the regional fractional Laplacian, imposing a homogeneous Dirichlet condition in the boundary of the domain leads to a well-posed problem.

Finally, restricting the maximum jump length in the random process from Subsection 1.1.2 to be finite, in the limit it leads to a nonlocal operator with finite interaction radius. We refer to it as a *nonlocal diffusion operator.* Boundary conditions for this operator need to be replaced by adequate volume constraints.

In the following subsections, we comment in more detail some properties of these three operators.

1.4.1 Spectral fractional Laplacian

One of the most striking properties of the fractional Laplacian is its nonlocality; in order to localize it, Caffarelli and Silvestre [27] showed that it can be realized as a Dirichlet-to-Neumann operator by means of an extension problem in the half-space $\mathbb{R}^{n+1}_+$. Namely, given a function $u \colon \mathbb{R}^n \to \mathbb{R}$, they considered an extension $U : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ that satisfies the equation

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(\cdot,0) = u(\cdot) & \text{in } \mathbb{R}^n, \end{cases}$$

and showed that the so-called conormal exterior derivative of U coincides (up to a multiplicative constant) with the fractional Laplacian of u, namely,

$$\lim_{y\to 0^+} \frac{\partial U}{\partial y^{1-2s}}(\cdot,y) = \lim_{y\to 0^+} y^{1-2s}\partial_y U(\cdot,y) = c(n,s)(-\Delta)^s u(\cdot).$$

This extension can be adapted in two different ways to convey a definition of a fractional order operator over bounded domains. On one hand, restricting the domain of the extension problem (1.4.1) to functions supported in Ω leads to the fractional Laplacian (1.1.1).

On the other hand, denote by $\mathcal{C}$ the cylinder $\Omega \times (0,\infty)$ and by $\partial_L\mathcal{C}$ its lateral boundary, $\partial_L\mathcal{C} = \partial\Omega \times (0,\infty)$. Given $u\colon \Omega \to \mathbb{R}$, consider the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}, \\ U = 0 & \text{in } \partial_L\mathcal{C}, \\ U(\cdot,0) = u(\cdot) & \text{in } \Omega. \end{cases} \tag{1.4.1}$$

Upon consideration of the conormal exterior derivative, the solution of (1.4.1) gives a fractional power of the Dirichlet Laplace operator in the sense of spectral theory. Indeed, in [23, 102], a Caffarelli-Silvestre result was proved for this operator: let U be the solution to the extension problem (1.4.1), then

$$\lim_{y\to 0^+} \frac{\partial U}{\partial y^{1-2s}}(\cdot,y) = (-\Delta)^s_S u(\cdot).$$

The operator $(-\Delta)^s_S$ is called the spectral fractional Laplacian, and is defined as follows. Let $\{\psi_k,\lambda_k\}_{k\in\mathbb{N}} \subset H^1_0(\Omega)\times\mathbb{R}_+$ be the set of normalized eigenfunctions and eigenvalues for the Laplace operator in Ω with homogeneous Dirichlet boundary conditions, so that $\{\psi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and

$$\begin{cases} -\Delta\psi_k = \lambda_k\psi_k & \text{in } \Omega, \\ \psi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, the spectral fractional Laplacian $(-\Delta)^s_S$ is defined for $u \in C_0^\infty(\Omega)$ by

$$(-\Delta)^s_S u := \sum_{k=1}^{\infty} \langle u,\psi_k\rangle \lambda_k^s u_k,$$

and can be subsequently extended by density to the Hilbert space $H^s(\Omega)$.

This spectral operator is different from the fractional Laplacian; for example, their difference is positive definite and positivity preserving [86]. See also [32, 99], where the spectra of these operators are compared.

From the numerical point of view, the localization technique [23, 102] was exploited by Nochetto, Otárola and Salgado [87]. As problem (1.4.1) is local and posed on the semi-infinite cylinder $\mathcal{C}$, but the variable of interest is the conormal exterior derivative, the authors study the numerical approximation of the spectral fractional Laplacian by considering graded meshes in the extended variable. See also [89] for further details.

Remark 1.4.1. There is a remarkable difference between the extension problems for $(-\Delta)^s$ and $(-\Delta)^s_S$. Recall that in these extension problems, the operators are recovered as a conormal derivative of the extension. However, the extension problem for the spectral fractional Laplacian is set on a cylinder whose only unbounded component is the extended variable, while the corresponding problem for the fractional Laplacian is set in an unbounded domain in $n+1$ dimensions. The decay in the extended variable is exponential, but in the first n variables it is polynomial. This precludes the use of extension techniques for the fractional Laplacian (1.1.1).

1.4.2 Regional fractional Laplacian

It is also possible to restrict integration in (1.1.1) to Ω. This leads to a regional fractional Laplacian

$$(-\Delta)^s_\Omega u(x) = C(n,s,\Omega) \text{ P.V.} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy. \tag{1.4.2}$$

This operator is known to be the infinitesimal generator of the so-called censored stable Lévy processes [18, 59]. Observe that, for a function $u \in \widetilde{H}^s(\Omega)$,

$$(-\Delta)^s_\Omega u(x) = (-\Delta)^s u(x) - C(n,s)\omega^s_\Omega(x)u(x),$$

where ω^s_Ω is defined according to (1.2.6). Clearly, the regional fractional Laplacian of a smooth function u depends on the domain Ω.

This operator is related to fractional diffusion with homogeneous Neumann conditions, as there is no exchange of mass between the domain Ω and its complement. See [7, Chapter 3] for an account on such type of problems. Dirichlet data for the regional fractional Laplacian need to be imposed on the boundary of Ω, although non-homogeneous boundary conditions may lead to an ill-posed problem for $s \in (0,1/2]$ [58, 106].

We do not consider finite element discretizations for the regional fractional Laplacian in this thesis. Nevertheless, it is noteworthy that the same algorithm developed for the fractional Laplacian serves to approximate this operator. The only change to perform on the scheme described in Appendix A is to omit integration over Ω^c.

1.4.3 Nonlocal diffusion operator

Nonlocal models differ from the classical partial differential equation models in the fact that in the latter interactions between two domains occur only due to contact, whereas in the former interactions can occur at a positive distance. In this subsection we comment on an operator that arises as the infinitesimal generator of a process where particles may jump to Ω^c, but the maximum length of jumps is finite. Namely, given an interaction radius λ, and $s \in (0,1)$, we consider the nonlocal diffusion operator

$$\mathcal{L}_\lambda^s u(x) = C(n,s) \int_{B(x,\lambda)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

The fractional Laplacian is the limit of this nonlocal operator as the interactions' horizon λ becomes infinite. This fact has important consequences in the treatment of problems involving the fractional Laplacian equations on bounded domains. It is possible to generalize the usual differential operators to nonlocal counterparts, see [42, 43]. Exploiting this nonlocal vector calculus, D'Elia and Gunzburger [39] performed a finite element analysis of volume constrained problems over one-dimensional domains. Further, they studied convergence of the approximate nonlocal solutions to the solution of the fractional Laplacian equation in the limit where the interaction radius tends to infinity and the mesh size to zero.

Moreover, from the analytical perspective, if λ is large then the nonlocal diffusion operator $\mathcal{L}_\lambda^s$ has the same properties as the fractional Laplacian $(-\Delta)^s$. Indeed, let $u \in \widetilde{H}^s(\Omega)$ and assume that $\lambda > \text{diam}(\Omega)$. It is simple to check that

$$\mathcal{L}_\lambda^s u(x) = (-\Delta)^s u(x) - \frac{C(n,s)\sigma_{n-1}}{2s\lambda^{2s}} u(x), \quad x \in \Omega.$$

This means, for example, that under homogeneous Dirichlet conditions, eigenspaces of the fractional Laplacian coincide with the ones of the fractional diffusion operator, and eigenvalues are shifted by a factor $\frac{C(n,s)\sigma_{n-1}}{2s\lambda^{2s}}$.

As a consequence of this remark, we obtain a lower bound for the first eigenvalue of the fractional Laplacian. Indeed, as the operator $\mathcal{L}_\lambda^s$ is definite positive [43] its eigenvalues are non negative. Thus, upon using (1.1.2) and simplifying, we deduce that the first eigenvalue of the fractional Laplacian with homogeneous Dirichlet conditions satisfies

$$\lambda^{(s)} \geq \frac{2^{2s}\Gamma(s+n/2)}{\Gamma(1-s)\Gamma(n/2)\text{diam}(\Omega)^{2s}}.$$

This bound is valid for open domains, without any condition on boundary regularity.

1.5 Weighted fractional Sobolev spaces

Weighted Sobolev spaces are a customary tool for dealing with singular solutions. Characterizing the regularity of a function in term of a weighted space leads to more precise knowledge of its behavior, specially wherever the weight tends to 0. The weights we consider are powers of the distance to the boundary of Ω, and the spaces they induce give a precise characterization of the behavior of solutions to (Homogeneous).

1.5.1 Spaces on an interval

We begin our discussion of weighted spaces of non integer order by analyzing the simple case of one-dimensional domains. Namely, throughout this subsection we set $\Omega = (-1, 1)$; results for arbitrary intervals follow by applying affine transformations. The theory developed here allows to characterize regularity of solutions of the Dirichlet problem for the fractional Laplacian on one-dimensional (not necessarily connected) domains. We address such topic in Section 2.1.

On the interval $(-1, 1)$, the distance to the boundary function (1.2.2) takes the simple form $\delta(x) = \min\{x + 1, 1 - x\}$. Here we make use of the equivalent weight

$$\omega(x) = (1 - x^2), \tag{1.5.1}$$

and as a first step, we consider a weighted L^2 space,

$$L^2_s(-1, 1) := \left\{ \phi : (-1, 1) \to \mathbb{R} \, : \, \int_{-1}^{1} |\phi(x)|^2 \omega^s(x) \, dx < \infty \right\}, \tag{1.5.2}$$

which, together with the inner product

$$(\phi, \psi)^s_{-1,1} := \int_{-1}^{1} \phi(x) \, \psi(x) \, \omega^s(x) \, dx$$

and associated norm is a Hilbert space. Moreover, let us introduce a family of special functions.

Definition 1.5.1. Let $\alpha \in \mathbb{R}$, the family of Gegenbauer polynomials $\left\{ C_n^{(\alpha)} \right\}_{n \in \mathbb{N}}$ is defined by the recurrence

$$\begin{aligned}
C_0^{(\alpha)}(x) &= 1, \\
C_1^{(\alpha)}(x) &= 2\alpha x, \\
C_n^{(\alpha)}(x) &= \frac{1}{n} \left[2x(n + \alpha - 1) C_{n-1}^{(\alpha)}(x) - (n + 2\alpha - 2) C_{n-2}^{(\alpha)}(x) \right].
\end{aligned}$$

Lemma 1.5.2 ([1, Chapter 22])**.** *Given $s \in (0,1)$, the set of Gegenbauer polynomials $\left\{C_n^{(s+1/2)}\right\}_{n\in\mathbb{N}}$ constitutes an orthogonal basis of $L_s^2(-1,1)$.*

For simplicity of notation, we write

$$\tilde{C}_j^{(s+1/2)}(x) := \frac{C_j^{(s+1/2)}(x)}{\left\|C_j^{(s+1/2)}\right\|_{L_s^2(-1,1)}}$$

for the normalized polynomials. In view of the previous lemma, given a function $v \in L_s^2(-1,1)$ we may consider the expansion

$$v(x) = \sum_{j=0}^{\infty} v_j \tilde{C}_j^{(s+1/2)}(x), \tag{1.5.3}$$

where the Gegenbauer coefficients are

$$v_j := \int_{-1}^{1} v(x)\tilde{C}_j^{(s+1/2)}(x)\omega^s(x)\,dx. \tag{1.5.4}$$

The next proposition relates the smoothness of a function and the decay of these coefficients. We refer the reader to [4, Section 4] for a proof.

Proposition 1.5.3. *Let $k \in \mathbb{N}$ and let $v \in C^k[-1,1]$ such that for a certain decomposition $[-1,1] = \bigcup_{i=1}^{n}[\alpha_i,\alpha_{i+1}]$ $(-1 = \alpha_1 < \alpha_i < \alpha_{i+1} < \alpha_n = 1)$ and for certain functions $\tilde{v}_i \in C^{k+2}[\alpha_i,\alpha_{i+1}]$ we have $v(x) = \tilde{v}_i(x)$ for all $x \in (\alpha_i,\alpha_{i+1})$ and $1 \le i \le n$.. Then the Gegenbauer coefficients v_j in equation* (1.5.4) *are quantities of order $O(j^{-(k+2)})$ as $j \to \infty$:*

$$|v_j| < Cj^{-(k+2)}$$

for a constant C that depends on v and k.

Thus, the decay of the coefficients in the Gegenbauer expansion leads naturally to a definition of a class of weighted Sobolev spaces.

Definition 1.5.4. Let $r, s \in \mathbb{R}$, $r \ge 0$, $s > -1/2$ and, for $v \in L_s^2(-1,1)$ let v_j be the corresponding Gegenbauer coefficient (1.5.4). We define the s-weighted Sobolev space of order r,

$$H_s^r(-1,1) := \left\{ v \in L_s^2(-1,1) \colon \sum_{j=0}^{\infty}(1+j^2)^r|v_j|^2 < \infty \right\}.$$

The proof of the next lemma is completely analogous to that of [71, Theorem 8.2].

Lemma 1.5.5. *Let $r, s \in \mathbb{R}$, $r \geq 0$, $s > -1/2$. Then the space $H_s^r(-1,1)$ endowed with the inner product $\langle v, w\rangle_s^r = \sum_{j=0}^{\infty} v_j w_j \left(1+j^2\right)^r$ and associated norm*

$$\|v\|_{H_s^r(-1,1)} \coloneqq \sum_{j=0}^{\infty} \left(1+j^2\right)^r |v_j|^2$$

is a Hilbert space.

Remark 1.5.6. By definition it can be immediately checked that for every function $v \in H_r^s(-1,1)$ the Gegenbauer expansion (1.5.3) with expansion coefficients (1.5.4) is convergent in $H_s^r(-1,1)$.

Remark 1.5.7. In view of the Parseval identity $\|v\|_{L_s^2(-1,1)}^2 = \sum_{n=0}^{\infty} |v_n|^2$ it follows that the Hilbert spaces $H_s^0(-1,1)$ and $L_s^2(-1,1)$ coincide. Further, we have the dense compact embedding $H_s^t(-1,1) \subset H_s^r(-1,1)$ whenever $r < t$. (The density of the embedding follows directly from Remark 1.5.6 since all polynomials are contained in $H_s^r(-1,1)$ for every r.) Finally, by proceeding as in [71, Theorem 8.13] it follows that for any $r > 0$, $H_s^r(-1,1)$ constitutes an interpolation space between $H_s^{\lfloor r\rfloor}(-1,1)$ and $H_s^{\lceil r\rceil}(-1,1)$ in the sense defined in [12, Chapter 2].

Closely related "Jacobi-weighted Sobolev spaces" $\mathcal{H}_s^k$ (see Definition 1.5.8 below) were introduced by Babuška and Guo [9] in connection with Jacobi approximation problems in the p-version of the finite element method.

Definition 1.5.8. Let $k \in \mathbb{N}$ and $r > 0$. The k-th order non-uniformly weighted Sobolev space $\mathcal{H}_s^k(-1,1)$ is defined as the completion of the set $C^\infty(-1,1)$ under the norm

$$\|v\|_{\mathcal{H}_s^k(-1,1)} = \left(\sum_{j=0}^{k} \int_{-1}^{1} |v^{(j)}(x)|^2 \omega^{s+j}(x)dx\right)^{1/2} = \left(\sum_{j=0}^{k} \|v^{(j)}\|_{L_{s+j}^2(-1,1)}^2\right)^{1/2}.$$

The r-th order space $\mathcal{H}_s^r(a,b)$, in turn, is defined by interpolation of the spaces $\mathcal{H}_s^k(a,b)$ ($k \in \mathbb{N}$) by the K-method (see [12, Section 3.1]).

Finally, the proposition below shows that, in fact, the spaces $\mathcal{H}_s^k$ coincide with the spaces H_s^k defined above, and the respective norms are equivalent. This is an important tool in the study of regularity of solutions in one-dimensional domains.

Proposition 1.5.9 ([9, Theorem 2.1 and Remark 2.3])**.** *Let $r > 0$. The spaces $H_s^r(-1,1)$ and $\mathcal{H}_s^r(-1,1)$ coincide, and their corresponding norms $\|\cdot\|_{H_s^r(-1,1)}$ and $\|\cdot\|_{\mathcal{H}_s^r(-1,1)}$ are equivalent.*

1.5.2 Spaces in multi-dimensional domains

The spaces considered in the previous subsection lead to a precise characterization of the mapping properties on one-dimensional domains. This is caused mainly because they are based on Gegenbauer expansions, and in next Chapter we show that these special polynomials are the eigenfunction of a certain weighted fractional Laplacian. However, the drawback of these Gegenbauer-based approach is that it lacks the flexibility to cope with more general domains. Thus, for arbitrary n-dimensional problems it is necessary to consider spaces that do not depend on special functions.

In analogy with Definition 1.2.9, we introduce the notation

$$\delta(x, y) := \min\{\delta(x), \delta(y)\}. \tag{1.5.5}$$

Definition 1.5.10. Let Ω be an open bounded set with Lipschitz boundary, $s > 0$ and $\alpha \in \mathbb{R}$. Writing $s = k + \sigma$, with $k \in \mathbb{N}$ and $\sigma \in (0, 1]$, we define the weighted Sobolev space

$$H^s_\alpha(\Omega) := \left\{ v \in H^k(\Omega) \colon |\partial^\beta v|_{H^\sigma_\alpha(\Omega)} < \infty \ \forall \beta \in \mathbb{N}^n \text{ s.t. } |\beta| = k \right\},$$

where

$$|w|_{H^\sigma_\alpha(\Omega)} := \iint_{\Omega\times\Omega} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2\sigma}} \, \delta(x, y)^{2\alpha} dx\, dy.$$

We furnish this space with the norm

$$\|v\|^2_{H^s_\alpha(\Omega)} := \|v\|^2_{H^k(\Omega)} + \sum_{|\beta|=k} |\partial^\beta v|_{H^\sigma_\alpha(\Omega)}.$$

We also need to define spaces over $\mathbb{R}^n$.

Definition 1.5.11. Let Ω, s and α be as in the previous definition. The global weighted Sobolev space $H^s_{\alpha,\Omega}(\mathbb{R}^n)$ is

$$H^s_{\alpha,\Omega} := \left\{ v \in H^k(\mathbb{R}^n) \colon |D^\beta v|_{H^\sigma_{\alpha,\Omega}(\mathbb{R}^n)} < \infty \ \forall \beta \in \mathbb{N}^n \text{ s.t. } |\beta| = k \right\},$$

where

$$|w|_{H^\sigma_{\alpha,\Omega}(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2\sigma}} \, \delta(x, y)^{2\alpha} dx\, dy.$$

The norm on this space is given by

$$\|v\|^2_{H^s_{\alpha,\Omega}(\mathbb{R}^n)} := \|v\|^2_{H^k(\mathbb{R}^n)} + \sum_{|\beta|=k} |D^\beta v|_{H^\sigma_{\alpha,\Omega}(\mathbb{R}^n)}.$$

Whenever the set Ω is clear from the context, we drop the reference to it in the global case and simply write $H^\ell_\alpha(\mathbb{R}^n)$.

Remark 1.5.12. Although we are interested in the case $\alpha \geq 0$, we recall that in the definition of weighted Sobolev spaces $H^k_\alpha(\Omega)$, with k being a nonnegative integer, arbitary powers of $\delta(x)$ can be considered [72, Theorem 3.6]. On the other hand, for general weights some restrictions must be taken into account in order to get an adequate definition of the spaces, namely, to ensure their completeness. A classical family of weights is that of the Muckenhoupt A_2 class [69]. In the global version $H^\ell_\alpha(\mathbb{R}^n)$ we need to restrict the range of α to $|\alpha| < 1/2$ in order to have $\delta^{2\alpha} \in A_2$.

Poincaré inequalities play a key role in the analysis of finite element methods. Thus, it is of interest in our applications to obtain an analogue to Proposition 1.2.8 valid for these weighted spaces. The term *improved* in this context usually involves weights which are powers of the distance to the boundary.

Our starting point is the following fractional improved Sobolev-Poincaré inequality for functions with zero average.

Proposition 1.5.13 ([64, Theorem 4.10])**.** *Let Ω be an open bounded star-shaped domain with respect to a ball, $\delta, \tau \in (0,1)$, $1 < p \leq q \leq \frac{np}{n-\delta p}$ and $p < n/\sigma$. Then, the following inequality holds*

$$\left(\int_\Omega |v(x) - \bar{v}|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_\Omega \int_{\Omega \cap B(x,\tau\delta(x))} \frac{|v(x) - v(y)|^p}{|x-y|^{n+\sigma p}} \, dy \, dx \right)^{\frac{1}{p}}. \tag{1.5.6}$$

Remark 1.5.14. In [64] the domain Ω is assumed to belong to the class of John domains (for a definition and properties of this class see for instance [81]); this class is much larger than the one considered here.

Furnished with the previous proposition, we aim to prove a weighted analogue of (1.2.1).

Proposition 1.5.15 (Weighted fractional Poincaré inequality)**.** *Let $s \in (0,1)$, $\alpha < s$ and Ω a domain which is star-shaped with respect to a ball B. Then, there exists a constant C such that for every $v \in L^2(\Omega)$, it holds*

$$\|v - \bar{v}\|_{L^2(\Omega)} \leq C d_\Omega^{s-\alpha} |v|_{H^s_\alpha(\Omega)}, \tag{1.5.7}$$

with a constant C depending on the chunkiness parameter of Ω (cf. Remark 1.2.7).

Proof. Set $\tau = 1/2$ in (1.5.6). Without loss of generality, we may assume $\bar{v} = 0$; moreover, we begin considering Ω such that $d_\Omega = 1$. For σ to be chosen, we consider p such that $\frac{np}{n-\sigma p} = 2 = q$. Observe that this choice obviously implies that $p < 2$, and therefore for all $\alpha \in \mathbb{R}$, applying Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$,

$$\|v\|_{L^2(\Omega)} \leq C I_1^{\frac{1}{2}} I_2^{\frac{2-p}{2p}},$$

where

$$I_1 = \int_\Omega \int_{\Omega\cap B(x,\frac{\delta(x)}{2})} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, \delta(x,y)^{\frac{2\alpha}{p}}\, dy\, dx,$$

and

$$I_2 = \int_\Omega \int_{\Omega\cap B(x,\frac{\delta(x)}{2})} |x-y|^{-n+\frac{2p(s-\sigma)}{2-p}}\, \delta(x,y)^{-\frac{2\alpha}{2-p}}\, dy\, dx.$$

Since for every $x\in\Omega$ and $y\in B(x,\frac{\delta(x)}{2})$ it holds that $\delta(x,y)\in\left[\frac{\delta(x)}{2},\delta(x)\right]$, assuming that $\sigma<s$ the second integral I_2 can be estimated as follows:

$$I_2 \le C\int_\Omega \left(\int_0^{\frac{\delta(x)}{2}} \rho^{-1+\frac{2p(s-\sigma)}{2-p}}\, d\rho\right) \delta(x)^{-\frac{2\alpha}{2-p}}\, dx \le C\int_\Omega \delta(x)^{\frac{2p(s-\sigma)-2\alpha}{2-p}}\, dx.$$

This integral is finite if and only if $\frac{2p(s-\sigma)-2\alpha}{2-p} > -1$, and recalling the choice of p we made, it is enough to consider

$$\alpha < \frac{2n(s-\sigma)+2\sigma}{n+2\sigma}.$$

Choosing α according to this restriction, we obtain that the weight in the term I_1 must satisfy

$$\frac{2\alpha}{p} < 2s - 2\sigma\left(1-\frac{1}{n}\right).$$

Therefore, taking $\sigma = \frac{\varepsilon n}{2(n-1)}$ for $\varepsilon\in(0,2s)$, we obtain

$$\|v\|_{L^2(\Omega)} \le C\left(\int_\Omega \int_{\Omega\cap B(x,\frac{\delta(x)}{2})} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, \delta(x,y)^{2s-\varepsilon} dy\, dx\right)^{\frac{1}{2}},$$

where the constant C above depends on n, s, ε and the one appearing in (1.5.6). The latter, in turn, depends on the constants associated to the John domain Ω. In the case of a star-shaped domain the John constants are easily bounded in terms of the chunkiness parameter.

For domains of arbitrary diameter, a scaling argument leads straightforwardly to the final dependence on the d_Ω. □

Resumen del capítulo

Este capítulo recolecta material preliminar necesario para desarrollar el análisis por elementos finitos de problemas que involucran al laplaciano fraccionario sobre dominios acotados. Motivamos y definimos este operador y estudiamos los espacios involucrados en las formas variacionales asociadas al mismo.

La **Sección 1.1** provee la definición del laplaciano fraccionario que más utilizamos a lo largo de la tesis, como una integral singular. Además, motivamos este operador a partir de un paseo al azar con saltos de largo arbitrario, lo que ilustra el carácter no local del laplaciano fraccionario.

Como el operador que estudiamos da lugar a formulaciones planteadas en espacios de Sobolev fraccionarios, en la **Sección 1.2** discutimos varios aspectos de los mismos. Más aún, examinamos la forma débil del laplaciano fraccionario y presentamos una fórmula de integración por partes no local. Una dificultad asociada a la naturaleza no local de estos espacios fraccionarios es que sus seminormas no son aditivas respecto a dominios; también discutimos una estrategia para localizarlas.

Los espacios de Sobolev fraccionarios con exponente $p = 2$ pueden ser caracterizados por medio del decaimiento de los coeficientes de Fourier de sus funciones. Esta caracterización como espacios potenciales de Bessel, desarrollada en la **Sección 1.3**, conduce a una formulación alternativa del laplaciano fraccionario que permite mostrar su carácter como operador pseudo-diferencial.

A pesar de que existe una única definición bien definida para difusión fraccionaria en $\mathbb{R}^n$, hay varias alternativas en el caso de dominios acotados. El objetivo de la **Sección 1.4** es discutir algunas de estas posibilidades, y comentar aspectos de algunos de estos operadores no locales sobre dominios acotados que comparten ciertas características del laplaciano fraccionario.

Finalmente, en la **Sección 1.5** introducimos y analizamos propiedades de ciertos espacios de Sobolev fraccionarios con peso. En esta sección distinguimos entre dominios uni- y multidimensionales. Para los primeros, nuestros espacios están basados en expansiones en ciertas bases de funciones especiales, y conducen a una caracterización precisa de las propiedades de mapeo del laplaciano fraccionario sobre dominios unidimensionales. Respecto a dominios multidimensionales, los espacios que introducimos son de interés para realizar aproximaciones por elementos finitos porque, como mostramos en el Capítulo 3, permiten aumentar el orden de convergencia de nuestros esquemas numéricos.

Chapter 2

Regularity theory

Given a function $f \in H^r(\Omega)$ $(r \geq -s)$, recall the homogeneous Dirichlet problem for the fractional Laplacian,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{Homogeneous}$$

and the variational framework from Section 1.2.3. In particular, existence and uniqueness of a weak solution $u \in \widetilde{H}^s(\Omega)$ has been established in Proposition 1.2.23. A natural question that arises subsequently is whether such weak solution is more regular than $\widetilde{H}^s(\Omega)$, and what assumptions on f are needed to ensure that. Further, taking into account our goal of performing finite element analysis on problems involving the fractional Laplacian, we aim these regularity estimates to bound higher-order Sobolev norms of the solution.

A more general question is to provide a characterization of the mapping properties of the fractional Laplacian. More precisely, whether it is possible to invert the operator in the Sobolev scale. Actually, as we show in Section 2.1 this is possible in one-dimensional domains. We prove that the a certain variant of the fractional Laplacian in fact induces a *bijection* between the weighted Sobolev spaces discussed in Subsection 1.5.1. Thus, based on a factorization of solutions as a product of a certain edge-singular weight times a regular unknown, a weighted-Sobolev characterization of the regularity of solutions is obtained in terms of the smoothness of the corresponding right-hand sides. An explicit eigendecomposition for problems set in n-dimensional balls shows that this construction also carries for radial domains.

Unfortunately, this powerful construction does not hold for more general geometries. The characterization of the fractional Laplacian as a pseudo-differential operator allows to state regularity of solutions in smooth domains in terms of the so-called Hörmander μ-spaces. These combine certain pseudo-differential operators with zero-extensions and restriction operators. In Section 2.2 we explore the connection between these spaces and Sobolev spaces, and this leads to Sobolev-Sobolev regularity estimates for (Homogeneous).

Estimates valid for a broader class of domains are discussed in Section 2.3. The price we have to pay is that we bound Sobolev norms of solutions of (Homogeneous) in terms of Hölder norms of the data. However, in that section we prove Sobolev regularity in standard and weighted fractional spaces. The latter measures in a precise way the behavior of solutions near the boundary of the domain Ω.

2.1 One-dimensional and radial problems

Our analysis begins with the consideration of one-dimensional and radial problems. The contents of this section serve as a guide of what type of results should be expected to hold in general domains. We study the regularity of solutions of problem (Homogeneous) under various smoothness assumptions on the right-hand side f, including treatments in both Sobolev and analytic function spaces, and for multi-interval domains Ω.

The main result in Subsection 2.1.1 establishes that for right-hand sides f in the space $H_s^r(\Omega)$ (cf. Defintion 1.5.4) with $r \geq 0$ the solution u of equation (Homogeneous) can be expressed in the form $u(x) = \omega^s(x)\phi(x)$, where ϕ belongs to $H_s^{r+2s}(\Omega)$. This section considers the single-interval case; generalizations of all results to the multi-interval context are presented in Subsection 2.1.2. Related results for n-dimensional balls are displayed in Subsection 2.1.3, where explicit examples are provided in such domains.

The theoretical background presented in this section is based on [4] and, in that work, has been exploited to develop and analyze a class of high-order algorithms for the numerical solution of equation (Homogeneous) in one-dimensional domains.

2.1.1 Sobolev Regularity, single interval case

In this subsection we recast the fractional Laplacian as an integral operator in a bounded domain. This motivates naturally to utilize the weighted Sobolev spaces from Subsection 1.5.1, which provide a sharp regularity result for a weighted fractional Laplacian $(-\Delta)^s_\omega(\cdot) = (-\Delta)^s(\omega^s\cdot)$: we show that it induces a *bijection* between these weighted Sobolev spaces. Using an appropriate version of the Sobolev lemma, these results are seen to imply, in particular, that the regular factor of the fractional Laplacian solutions admit k continuous derivatives for a certain value of k that depends on the regularity of the right-hand side. Additionally, we establish the operator regularity in spaces of analytic functions. For notational convenience, we restrict our attention to the domain $\Omega = (-1, 1)$; the corresponding definition for general multi-interval domains then follows easily.

The following lemma provides a useful expression for the fractional Laplacian operator in terms of a certain integro-differential operator. For a proof, we refer the reader to [4, Lemma 2.3]

Lemma 2.1.1. *Let* $s \in (0,1)$*, let* $u \in C_0^2(-1,1)$ *such that* $|u'|$ *is integrable in* $(-1,1)$*, let* $x \in \mathbb{R}, x \notin \partial\Omega = \{-1,1\}$*, and recalling* (1.1.2)*, define*

$$C_s = \frac{C(1,s)}{2s(1-2s)} = -\Gamma(2s-1)\sin(\pi s)/\pi \quad (s \neq 1/2). \tag{2.1.1}$$

If $s \neq \frac{1}{2}$*, then*

$$(-\Delta)^s u(x) = C_s \frac{d}{dx} \int_{-1}^{1} |x-y|^{1-2s} \frac{d}{dy} u(y) dy,$$

whereas if $s = \frac{1}{2}$*, then*

$$(-\Delta)^{1/2} u(x) = \frac{1}{\pi} \frac{d}{dx} \int_{-1}^{1} \ln|x-y| \frac{d}{dy} u(y) dy.$$

Using the weight function $\omega(x) = 1 - x^2$ (cf. (1.5.1)), for $\phi \in C^2(-1,1) \cap C^1[-1,1]$ (that is, ϕ smooth up to the boundary but it does not necessarily vanish on the boundary) we introduce the weighted version

$$(-\Delta)^s_\omega \phi(x) = \begin{cases} C_s \frac{d}{dx} \int_{-1}^{1} |x-y|^{1-2s} \frac{d}{dy} \left(\omega^s \phi(y)\right) dy & (s \neq 1/2), \\ \frac{1}{\pi} \frac{d}{dx} \int_{-1}^{1} \ln|x-y| \frac{d}{dy} \left(\omega^{1/2} \phi(y)\right) dy & (s = 1/2). \end{cases} \tag{2.1.2}$$

Remark 2.1.2. Clearly, given a solution ϕ of the equation

$$(-\Delta)^s_\omega \phi = f$$

in the domain $\Omega = (-1,1)$, the function $u = \omega^s \phi$ extended by zero outside Ω solves the Dirichlet problem for the fractional Laplacian (Homogeneous) (cf. Lemma 2.1.1).

Recall that the set of normalized Gegenbauer polynomials $\left\{C_n^{(s+1/2)}\right\}_{n \in \mathbb{N}}$ constitutes an orthogonal basis of $L^2_s(-1,1)$ (cf. (1.5.2)). The key result to link this family of polynomials with the fractional Laplacian in one-dimensional domains is that the Gegenbauer polynomials are actually eigenfunctions of the weighted operator (2.1.2). We refer the reader to [4, Theorem 3.14 and Corollary 3.15] for a proof.

Theorem 2.1.3. *The weighted operator* $(-\Delta)^s_\omega$ *in the interval* $(-1,1)$ *satisfies the identity*

$$(-\Delta)^s_\omega (C_n^{(s+1/2)}) = \lambda_n^s \, C_n^{(s+1/2)},$$

where

$$\lambda_n^s = \frac{\Gamma(2s+n+1)}{n!}. \tag{2.1.3}$$

Remark 2.1.4. It is useful to note that, in view of the formula $\lim_{n\to\infty} n^{\beta-\alpha}\Gamma(n+\alpha)/\Gamma(n+\beta)=1$ (see e.g. [1, 6.1.46]) we have the asymptotic relation $\lambda_n^s \approx O(n^{2s})$ for the eigenvalues (2.1.3).

Recalling Definition 1.5.4, it is now apparent that the Sobolev spaces $H_s^r(-1,1)$ completely characterize the Sobolev regularity of the weighted fractional Laplacian operator $(-\Delta)_\omega^s$.

Theorem 2.1.5. *Let $r \geq 0$. Then, the weighted fractional Laplacian operator* (2.1.2) *can be extended uniquely to a continuous linear map $(-\Delta)_\omega^s$ from $H_s^{r+2s}(-1,1)$ into $H_s^r(-1,1)$. The extended operator is bijective and bicontinuous.*

Proof. Let $\phi \in H_s^{r+2s}(-1,1)$, and let $\phi^n = \sum_{j=0}^n \phi_j \tilde{C}_j^{(s+1/2)}$ where ϕ_j denotes the Gegenbauer coefficient of ϕ as given by equation (1.5.4) with $v=\phi$. According to Theorem 2.1.3 we have $(-\Delta)_\omega^s \phi^n = \sum_{j=0}^n \lambda_j^s \phi_j \tilde{C}_j^{(s+1/2)}$. In view of Remarks 1.5.6 and 2.1.4 it is clear that $(-\Delta)_\omega^s\phi^n$ is a Cauchy sequence (and thus a convergent sequence) in $H_s^r(-1,1)$. We may thus define

$$(-\Delta)_\omega^s \phi = \lim_{n\to\infty} (-\Delta)_\omega^s \phi^n = \sum_{j=0}^\infty \lambda_j^s \phi_j \tilde{C}_j^{(s+1/2)} \in H_s^r(-1,1).$$

The bijectivity and bicontinuity of the extended mapping follows easily, in view of Remark 2.1.4, as does the uniqueness of continuous extension. The proof is complete. □

Corollary 2.1.6. *The solution u of* (Homogeneous) *with right-hand side $f \in H_s^r(-1,1)$ ($r \geq 0$) can be expressed in the form $u = \omega^s\phi$ for some $\phi \in H_s^{r+2s}(-1,1)$.*

Proof. Follows from Theorem 2.1.5 and Remark 2.1.2. □

The classical smoothness of solutions of equation (Homogeneous) for sufficiently smooth right-hand sides results from the following version of the Sobolev embedding theorem.

Theorem 2.1.7 (Sobolev's Lemma for weighted spaces [4, Theorem 4.14])**.** *Let $s \geq 0$, $k \in \mathbb{N}$ and $r > 2k+s+1$. Then we have a continuous embedding $H_s^r(-1,1) \subset C^k[-1,1]$ of $H_s^r(-1,1)$ into the Banach space $C^k[-1,1]$ of k-continuously differentiable functions in $[-1,1]$ with the usual norm $\|v\|_k$ (given by the sum of the L^∞-norms of the function and the k-th derivative): $\|v\|_k := \|v\|_\infty + \|v^{(k)}\|_\infty$.*

Remark 2.1.8. The previous result is sharp as can be checked by the following example in the case $k=0$. The function $v(x) = |\log(x)|^\beta$ with $0<\beta<1/2$ is not bounded, but a straightforward computation shows that, for $s \in \mathbb{N}$, $v \in \mathcal{H}_s^{s+1}(0,1)$, or equivalently (see Lemma 1.5.9), $v \in H_s^{s+1}(0,1)$.

Corollary 2.1.9. *The weighted fractional Laplacian operator* (2.1.2) *maps bijectively the space* $C^\infty[-1,1]$ *into itself.*

Proof. Follows directly from Proposition 1.5.3 together with theorems 2.1.5 and 2.1.7. □

Remark 2.1.10. Analytic regularity estimates for the weighted fractional Laplacian (2.1.2) are also attainable by a technique similar to the one shown above and by consideration of extensions of analytic functions defined on $[-1,1]$ to relevant neighborhoods of such interval in the complex plane. We consider the *Bernstein ellipse* $\mathcal{E}_\rho$, that is, the ellipse with foci ± 1 whose minor and major semiaxial lengths add up to $\rho \geq 1$. Clearly, any analytic function f over the interval $[-1,1]$ can be extended analytically to $\overline{\mathcal{E}_\rho}$ for some $\rho > 1$. We thus consider the set $A_\rho = \{f : f \text{ is analytic on } \overline{\mathcal{E}_\rho}\}$ endowed with the L^∞-norm $\|\cdot\|_{L^\infty(\overline{\mathcal{E}_\rho})}$. Then, taking advantage of sharp bounds on the decay of the Gegenbauer coefficients of functions in A_ρ [107] it is possible to prove that, for each $f \in A_\rho$, it holds that $((-\Delta)^s_\omega)^{-1} f \in A_\rho$; furthermore, the mapping $((-\Delta)^s_\omega)^{-1} : A_\rho \to A_\rho$ is continuous. A proof of these facts can be found in [4, Theorem 4.18]

2.1.2 Regularity on multi-interval domains

This subsection concerns multi-interval domains Ω of the form

$$\Omega = \bigcup_{i=1}^{M} (a_i, b_i), \tag{2.1.4}$$

where the intervals (a_i, b_i) have disjoint closures. Having at hand regularity estimates for the fractional Laplacian on intervals, analysis of the homogeneous Dirichlet problem (Homogeneous) on domains such as (2.1.4) is not a difficult task. The technique we describe is based on the idea of splitting the weighted fractional Laplacian as a singular self-interaction component plus a smooth remainder. The self-interaction component is a block-diagonal operator that shares the mapping properties of the fractional Laplacian over intervals (cf. Subsection 2.1.1), whereas the remainder is a sum of convolutions with respect to smooth kernels.

In first place, a similar result to Lemma 2.1.1 holds for multi-interval domains.

Lemma 2.1.11. *Given a domain* Ω *according to* (2.1.4)*, and with reference to equation* (2.1.1)*, for* $u \in C_0^2(\Omega)$ *we have*

- *Case* $s \neq \frac{1}{2}$*:*

$$(-\Delta)^s u(x) = C_s \frac{d}{dx} \sum_{i=1}^{M} \int_{a_i}^{b_i} |x-y|^{1-2s} \frac{d}{dy} u(y) dy$$

- *Case* $s = \frac{1}{2}$*:*

$$(-\Delta)^{1/2}u(x) = \frac{1}{\pi}\frac{d}{dx}\sum_{i=1}^{M}\int_{a_i}^{b_i}\ln|x-y|\frac{d}{dy}u(y)dy$$

for all $x \in \mathbb{R} \setminus \partial\Omega = \cup_i^M\{a_i, b_i\}$.

Using the characteristic functions $\chi_{(a_i,b_i)}$ of the individual component interval, letting $\omega^s(x) = \sum_{i=1}^M (x-a_i)^s(b_i-x)^s\chi_{(a_i,b_i)}(x)$ and relying on the previous lemma, we define the multi-interval weighted fractional Laplacian operator on Ω by $(-\Delta)^s_\omega\phi = (-\Delta)^s(\omega^s\phi)$, where $\phi : \mathbb{R} \to \mathbb{R}$. We decompose this operator as $(-\Delta)^s_\omega = \mathcal{K}_s + \mathcal{R}_s$, where

$$\mathcal{K}_s(\phi)(x) = C_s\sum_{i=1}^{M}\chi_{(a_i,b_i)}(x)\frac{d}{dx}\int_{a_i}^{b_i}|x-y|^{1-2s}\frac{d}{dy}(\omega^s\phi)(y)dy$$

is a block-diagonal operator and where $\mathcal{R}_s$ is the associated off-diagonal remainder. Using integration by parts it is easy to check that

$$\mathcal{R}_s\phi(x) = C(1,s)\int_{\Omega\setminus(a_j,b_j)}|x-y|^{-1-2s}\omega^s(y)\phi(y)dy \quad \text{for} \quad x \in (a_j, b_j).$$

The block-diagonal operator resembles the fractional Laplacian of functions defined on a single interval. As the kernel involved in $\mathcal{R}_s$ is smooth, writing the equation $(-\Delta)^s_\omega\phi = f$ as $\mathcal{K}_s\phi = f - \mathcal{R}_s\phi$, upon proving existence and uniqueness of solutions, in a similar fashion to the single-interval case it is possible to prove their regularity in weighted Sobolev spaces as well as in spaces of analytic functions.

Theorem 2.1.12 ([4, Theorem 4.21])**.** *Let* Ω *be given according to* (2.1.4)*. Then, given* $f \in L^2_s(\Omega)$*, there exists a unique* $\phi \in L^2_s(\Omega)$ *such that* $(-\Delta)^s_\omega\phi = f$*. Moreover, for* $f \in H^r_s(\Omega)$ *(resp.* $f \in A_\rho(\Omega)$*) we have* $\phi \in H^{r+2s}_s(\Omega)$ *(resp.* $\phi \in A_\nu(\Omega)$ *for some* $\nu > 1$*).*

2.1.3 The fractional Laplacian in balls

In this subsection we provide a family of explicit solutions in n-dimensional balls. Even though the results presented here do not address the issue of regularity of solutions directly, they are closely related to the ones we have developed in the one-dimensional setting. Further, they are also a rich source of examples that allow to illustrate the sharpness of our regularity estimates and the order of convergence of the numerical methods we propose.

Independently to [4], a related diagonal form for the fractional Laplacian was obtained by Dyda, Kuznetsov and Kwaśnicki [46] by employing arguments based on Mellin transforms. The diagonal form developed in [46] provides, in particular, a family of explicit solutions in the n-dimensional unit ball in $\mathbb{R}^n$, which are given by products of a singular term and general Meijer G-functions.

Here we mention the simplest construction that stems from that work. Consider the Jacobi polynomials $P_k^{(\alpha,\beta)}\colon [-1,1] \to \mathbb{R}$, given by

$$P_k^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+k+1)}{k!\,\Gamma(\alpha+\beta+k+1)} \sum_{m=0}^{k} \binom{k}{m} \frac{\Gamma(\alpha+\beta+k+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{x-1}{2}\right)^m,$$

and the weight function $\omega : \mathbb{R}^n \to \mathbb{R}$, $\omega(x) = 1 - |x|^2$.

In [46, Theorem 3] it is shown how to construct explicit eigenfunctions for a weighted fractional Laplacian by using $P_k^{(s,n/2-1)}$. To be more precise, the authors prove the following result.

Theorem 2.1.13. *Let $B(0,1) \subset \mathbb{R}^n$ the unitary ball. For $s \in (0,1)$ and $k \in \mathbb{N}$, define*

$$\lambda_{k,s} = \frac{2^{2s}\,\Gamma(1+s+k)\Gamma\left(\frac{n}{2}+s+k\right)}{k!\,\Gamma\left(\frac{n}{2}+k\right)} \tag{2.1.5}$$

and $p_k^{(s)}\colon \mathbb{R}^n \to \mathbb{R}$,

$$p_k^{(s)}(x) = P_k^{(s,\,n/2-1)}(2|x|^2-1)\chi_{B(0,1)}(x).$$

Then the following equation holds

$$(-\Delta)^s\left(\omega^s p_k^{(s)}\right)(x) = \lambda_{k,s}\, p_k^{(s)}(x) \ \text{ in } B(0,1).$$

The connection between the previous theorem and Theorem 2.1.3 is given by the relation between Jacobi and Gegenbauer polynomials [1, equation 22.5.22]

$$P_k^{(s,-1/2)}(x) = c(s,k)C_{2k}^{(s+1/2)}\left(\sqrt{\frac{1+x}{2}}\right).$$

Simple manipulations lead to linking the corresponding eigenvalues. Namely, that the identity $\lambda_{2k}^s = \lambda_{k,s}$ holds (cf. (2.1.3) and (2.1.5)).

Remark 2.1.14. The solutions expressed in Theorem 2.1.13 neither constitute an orthogonal basis of a weighted L^2 space in the ball, nor are the only ones obtained in [46]. In that work, besides utilizing Jacobi polynomials, the main results are stated in terms of other special functions such as Meijer G-functions and hypergeometric functions. See Section 1.6 therein for further examples and the construction of a complete orthogonal system of eigenfunctions of the weighted fractional Laplacian in the n-dimensional ball.

2.2 Sobolev-Sobolev regularity

We now turn our attention to Sobolev regularity estimates valid in all dimensions. In this section, we address estimates in terms of Sobolev norms of the right hand side function. Even though this Sobolev-Sobolev estimates are desirable when bearing in mind applications to the finite element method, the technique employed to derive them requires the domain Ω to be smooth. Thus, they are not entirely satisfactory for our finite element purposes, where discrete domains are polygons.

We follow the recent work by Grubb [56] to show Sobolev regularity results for the solution. Nevertheless, the idea of applying pseudo-differential theory to analyze problem (Homogeneous) is not new. Other estimates built on Fourier-based spaces have been known for a long time in the Russian school. In particular, certain regularity estimates for follow from the work by Višik and Èskin in the 60's (for example, [104]). See [48, Chapter VI] for a more detailed account of this theory.

In that paper, the author deals with Hörmander $\mu-$spaces $H^{\mu(\ell),p}$, where $\mu \in \mathbb{C}$, $1 < p < \infty$, $\ell > \text{Re}(\mu) + 1/p - 1$. These mix the features of supported and restricted Sobolev spaces by means of combining certain pseudo-differential operators with zero-extensions and restriction operators. In order to define these spaces, consider the operator e^+ that corresponds to extending a function by zero on Ω^c. Then, Hörmander spaces can be characterized for the half-space by

$$H^{\mu(\ell),p}(\mathbb{R}^n_+) = \Xi_+^{-\mu} e^+ W^{\ell - \text{Re}(\mu),p}(\mathbb{R}^n_+),$$

where $\Xi_+^{-\mu}$ is the pseudo-differential operator with symbol $((1 + |\xi'|^2)^{1/2} + i\xi_n)^{-\mu}$, with $\xi = (\xi', \xi_n)$. This operator is shown to be an homeomorphism $W^{\ell,p}(\mathbb{R}^n_+) \to W^{\ell+\text{Re}(\mu),p}(\mathbb{R}^n_+)$. See [56] for further details. The above characterization generalizes to any smooth domain Ω by means of employing local coordinates.

Hörmander and Sobolev spaces are related in the following way.

Proposition 2.2.1 ([56, Theorem 5.4])**.** *If $Re(\mu) > -1$, $\ell > Re(\mu) - 1/p'$ and $M \in \mathbb{N}$, then*

$$H^{\mu(\ell),p}(\Omega) \begin{cases} = \widetilde{W}^{\ell,p}(\Omega), & \text{if } \ell - Re(\mu) \in (-1/p', 1/p), \\ \subset W^{\ell-\varepsilon,p}(\Omega), & \text{if } \ell - Re(\mu) = 1/p, \ \forall \varepsilon > 0, \end{cases}$$

and

$$H^{\mu(\ell),p}(\Omega) \subset e^+ \delta(x)^\mu W^{\ell - Re(\mu),p}(\Omega) + \begin{cases} \widetilde{W}^{\ell,p}(\Omega), & \text{if } \ell - Re(\mu) \in (M - 1/p', M + 1/p), \\ W^{\ell-\varepsilon,p}(\Omega), & \text{if } \ell - Re(\mu) = M + 1/p, \ \forall \varepsilon > 0. \end{cases}$$

The precise regularity estimate by Grubb reads as follows [56, Theorem 7.1].

Theorem 2.2.2. *Assume that Ω is a C^∞ domain and $s \in (0,1)$. Let $1 < p < \infty$ and $\ell > s - 1 + 1/p$. Assume that the solution u of* (Homogeneous) *belongs to $W_0^{\sigma,p}(\Omega)$ for some $\sigma > s - 1 + 1/p$ and consider a right hand side function $f \in W^{\ell - 2s,p}(\Omega)$. Then, it holds that $u \in H^{s(\ell),p}(\Omega)$.*

In particular, considering $\ell = r + 2s$ and $p = 2$ in the previous theorem and taking into account, from Proposition 2.2.1, that

$$H^{s(r+2s),2}(\overline{\Omega}) \begin{cases} = \widetilde{H}^{2s+r}(\Omega) & \text{if } 0 < s + r < 1/2, \\ \subset \widetilde{H}^{s+1/2-\varepsilon}(\Omega)\ \forall \varepsilon > 0, & \text{if } 1/2 \leq s + r < 1, \end{cases}$$

we conclude the desired Sobolev regularity result.

Proposition 2.2.3. *Let $f \in H^r(\Omega)$ for $r \geq -s$, $u \in \widetilde{H}^s(\Omega)$ be the solution of the Dirichlet problem* (Homogeneous) *and let $\alpha = s + r$ if $s + r < 1/2$ or $\alpha = 1/2 - \varepsilon$ if $s + r \geq 1/2$, with $\varepsilon > 0$ arbitrarily small. Then, $u \in \widetilde{H}^{s+\alpha}(\Omega)$ and it holds that*

$$|u|_{H^{s+\alpha}(\mathbb{R}^n)} \leq C(n, s, \Omega, \alpha) \|f\|_{H^r(\Omega)}.$$

Remark 2.2.4. Assuming further Sobolev regularity in the right hand side function does not imply that the solution will be any smoother than what is given by the previous proposition. Indeed, if $f \in H^r(\Omega)$, then Theorem 2.2.2 gives $u \in H^{s(r+2s)}(\Omega)$, which can not be embedded in any space smoother than $H^{s+1/2-\varepsilon}(\Omega)$ if $r + s \geq 1/2$. The sharpness of the previous proposition can also be seen from Remark 2.3.12 below, where an example is given.

2.3 Hölder-Sobolev regularity

Here we address Sobolev regularity results for (Homogeneous) valid for a more general class of domains compared to previous section. Furthermore, we are also able to deliver these estimates in the weighted fractional spaces defined in Subsection 1.5.2. However, the drawback of this procedure is that the estimates we obtain are expressed in terms of the Hölder regularity of the data.

The proof we present follows [3] and relies on Hölder regularity results by Ros-Oton and Serra [92]. In particular, some of these estimates measure in a precise way the singular behavior of solutions near the boundary. We show in Chapter 3 how to take advantage of the increased regularity in weighted spaces to enhance the order of convergence of finite element approximations to the homogeneous Dirichlet fractional problem.

We start by reviewing some key results given in [92].

Theorem 2.3.1 (See Prop. 1.1 in [92])**.** *If Ω is a bounded, Lipschitz domain satisfying the exterior ball condition and $f \in L^\infty(\Omega)$, then any solution u of* (Homogeneous) *belongs to $C^s(\mathbb{R}^n)$ and*

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C(\Omega, s) \|f\|_{L^\infty(\Omega)}. \tag{2.3.1}$$

Moreover, if f is Hölder continuous, then higher order Hölder estimates for u are also obtained in [92]; these are expressed in terms of certain weighted norms. For $0 < \beta$, we denote by $|\cdot|_{C^\beta(\Omega)}$ the $C^\beta(\Omega)$ seminorm. For $\theta \geq -\beta$, we write $\beta = k + \beta'$ with k integer and $\beta' \in (0, 1]$. Recalling definition (1.5.5), we define the seminorm

$$|w|_\beta^{(\theta)} = \sup_{x,y\in\Omega} \delta(x,y)^{\beta+\theta} \frac{|D^k w(x) - D^k w(y)|}{|x-y|^{\beta'}},$$

and the associated norm $\|\cdot\|_\beta^{(\theta)}$ in the following way: for $\theta \geq 0$,

$$\|w\|_\beta^{(\theta)} = \sum_{\ell=0}^{k} \left(\sup_{x\in\Omega} \delta(x)^{\ell+\theta} |D^\ell w(x)| \right) + |w|_\beta^{(\theta)},$$

while for $-\beta < \theta < 0$,

$$\|w\|_\beta^{(\theta)} = \|w\|_{C^{-\theta}(\Omega)} + \sum_{\ell=1}^{k} \left(\sup_{x\in\Omega} \delta(x)^{\ell+\theta} |D^\ell w(x)| \right) + |w|_\beta^{(\theta)}.$$

The higher order Hölder estimate for solutions reads as follows.

Theorem 2.3.2 (See Prop. 1.4 in [92])**.** *Let Ω be a bounded domain and $\beta > 0$ be such that neither β nor $\beta + 2s$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(s)} < \infty$, and $u \in C^s(\mathbb{R}^n)$ be a solution of* (Homogeneous)*. Then, $u \in C^{\beta+2s}(\Omega)$ and*

$$\|u\|_{\beta+2s}^{(-s)} \leq C(\Omega, s, \beta) \left(\|u\|_{C^s(\mathbb{R}^n)} + \|f\|_\beta^{(s)} \right).$$

In the next remarks we explore some consequences of the previous theorems written in a way useful in the sequel.

Remark 2.3.3 (Case $s \in (0, 1/2)$). Taking $\beta \in (0, 1-2s)$ in Theorem 2.3.2, we obtain that there exists a constant $C(\Omega, s, \beta)$ such that

$$\sup_{x,y\in\Omega} \delta(x,y)^{\beta+s} \frac{|u(x) - u(y)|}{|x-y|^{\beta+2s}} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|f\|_\beta^{(s)} \right). \tag{2.3.2}$$

Moreover, since $\beta < 1$, for $f \in C^\beta(\Omega)$ it is simple to prove that

$$\|f\|_\beta^{(s)} \leq C(\Omega, s) \|f\|_{C^\beta(\Omega)}.$$

Remark 2.3.4 (Case $s \in (1/2, 1)$). Considering $\beta \in (0, 2-2s)$, Theorem 2.3.2 implies that

$$\sup_{x,y\in\Omega} \delta(x,y)^{\beta+s} \frac{|Du(x) - Du(y)|}{|x-y|^{\beta+2s-1}} \leq C \left(\Omega, s, \beta, \|f\|_\beta^{(s)} \right),$$

and

$$\sup_{x\in\Omega} \delta(x)^{1-s} |Du(x)| \leq C \left(\Omega, s, \beta, \|f\|_\beta^{(s)} \right).$$

In the remainder of this section we show how to use these results to bound Sobolev norms of u. First we focus on regularity within Ω both in standard and weighted spaces; afterwards we extend our argument to the space $\mathbb{R}^n$.

For our purposes, it is useful to divide $\Omega \times \Omega$ into a set in which the distance between x and y is bounded below by $\delta(x, y)$ and a set in which $|x - y|$ is smaller than that. Roughly, for the first set, Hölder regularity of the solution is enough to control the integrand involved in fractional seminorms of u, as this region is away from the diagonal $\{x = y\}$. As for the second one, since the weight involving $|x - y|$ is singular at $y = x$, some extra term is required in order to control its growth; this is obtained by means of Theorem 2.3.2.

It is convenient to observe that, given a function $v : \Omega \times \Omega \to \mathbb{R}$ such that $v(x, y) = v(y, x)$ for all $x, y \in \Omega$, the integral of v over $\Omega \times \Omega$ equals 2 times its integral over the set

$$A = \{(x, y) \in \Omega \times \Omega : \ \delta(x, y) = \delta(x)\}. \tag{2.3.3}$$

We make use of the decomposition mentioned in the previous paragraph by defining

$$B = \{(x, y) \in A : \ |x - y| \geq \delta(x)\}. \tag{2.3.4}$$

Remark 2.3.5. We recall an useful identity regarding integrability of powers of the distance to the boundary function. The following holds whenever $\alpha < 1$:

$$\int_\Omega \delta(x)^{-\alpha} dx = \mathcal{O}\left(\frac{1}{1-\alpha}\right) \tag{2.3.5}$$

See, for example, the proof of Lemma 2.14 in [28].

After these preliminary considerations, we split the argument into two: first we address regularity within Ω in standard spaces, and afterwards we focus on weighted spaces. Then, we extend easily our argument to $\mathbb{R}^n$ and focus on the case $s = 1/2$.

2.3.1 The case $s \in (0, 1/2)$: regularity in standard fractional spaces

We are now in position to prove that, if the right hand side is smooth enough, then solutions gain almost half a derivative in the Sobolev sense.

Theorem 2.3.6. *Let $s \in (0, 1/2)$ and $f \in C^{\frac{1}{2}-s}(\Omega)$. Then, for every $\varepsilon > 0$, the solution u of* (1.2.11) *belongs to $H^{s+\frac{1}{2}-\varepsilon}(\Omega)$, with*

$$|u|_{H^{s+\frac{1}{2}-\varepsilon}(\Omega)} \leq \frac{C(\Omega, s, n)}{\varepsilon} \|f\|_{C^{\frac{1}{2}-s}(\Omega)}.$$

Proof. Take $\theta \in (s, 1)$ and consider the splitting of A mentioned before. Then, applying estimate (2.3.1),

$$\iint_B \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\theta}} dx\, dy \leq$$

$$\leq C(\Omega, s) \|f\|^2_{L^\infty(\Omega)} \int_\Omega \int_{B(x,\delta(x))^c} |x - y|^{-n-2\theta+2s} dy\, dx$$

$$\leq \frac{C(\Omega, s, n) \|f\|^2_{L^\infty(\Omega)}}{\theta - s} \int_\Omega \delta(x)^{2(s-\theta)} dx.$$

A necessary and sufficient condition for the finiteness of the right hand side in the previous inequality is that $\theta < s + \frac{1}{2}$.

On the other hand, assume $f \in C^\beta(\Omega)$ for some $\beta > 0$. In a similar fashion the application of inequality (2.3.2) yields

$$\iint_{A \setminus B} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\theta}} dx\, dy \leq$$

$$\leq C \int_\Omega \delta(x)^{-2(\beta+s)} \left(\int_{B(x,\delta(x))} |x - y|^{-n-2\theta+2\beta+4s} dy \right) dx.$$

Now, the integral over $B(x, \delta(x))$ is finite if and only if $\beta + 2s > \theta$. So, in this case we obtain

$$\iint_{A \setminus B} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\theta}} dx\, dy \leq C \int_\Omega \delta(x)^{2(s-\theta)} dx, \tag{2.3.6}$$

where in the end the constant is of the form

$$C = \frac{C(\Omega, s, n, \beta)}{\beta + 2s - \theta} \|f\|^2_{C^\beta(\Omega)}.$$

Once again, the integral in the right hand side of (2.3.6) is finite if and only if $\theta < s + \frac{1}{2}$. If $\beta = \frac{1}{2} - s$, choosing $\theta = s + \frac{1}{2} - \varepsilon$, we find

$$\iint_{A \setminus B} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\theta}} dx\, dy \leq \frac{C(\Omega, s, n)}{\varepsilon} \int_\Omega \delta(x)^{-1+2\varepsilon} dx.$$

Since the integral in the right hand side is $\mathcal{O}(\varepsilon^{-1})$ (recall identity (2.3.5)), the proof is concluded. □

Remark 2.3.7. If f is more regular than $C^{\frac{1}{2}-s}(\Omega)$, then no further gain of regularity from estimate (2.3.2) is possible by means of the technique of the previous proof. This is indeed sharp, see Remark 2.3.12. The matter is that the parameter β disappears in the estimate over $A \setminus B$: in inequality (2.3.6), the dependence on the regularity of the data is hidden on the constant appearing in the right hand side, but not in the exponent in the integrand.

2.3.2 The case $s \in (1/2, 1)$: regularity in standard and weighted spaces

Next we show that an analogue of Theorem 2.3.6 is possible for $s \in (1/2, 1)$ and hence almost half a derivative is also gained in the a priori estimate. Moreover, along the proof of this result it becomes clear that if the right hand side is smooth enough the singular behavior of the solution is localized near the boundary. Therefore, by introducing appropriate weights we find alternative regularity results that afterwards are used to build a priori adapted meshes.

Before proceeding, we remark that the expected gain of half a derivative would imply that the solution belongs at least to $H^1(\Omega)$. Thus, one important tool to make use of is Proposition 1.2.16, that characterizes the behavior of the fractional seminorms $|\cdot|_{H^{1-\varepsilon}(\Omega)}$ as $\varepsilon \to 0$.

In first place, we want to prove that for the solution u of (1.2.11), the product $\varepsilon^{1/2}|u|_{H^{1-\varepsilon}(\Omega)}$ remains bounded as $\varepsilon \to 0$, so that u belongs to $H^1(\Omega)$. For that purpose, we require the following local Hölder regularity estimate (see [92, Lemma 2.9]).

Lemma 2.3.8. *If $f \in L^\infty(\Omega)$ and $\gamma \in (0, 2s)$, then u verifies*

$$|u|_{C^\gamma(\overline{B_R(x)})} \leq C R^{s-\gamma} \|f\|_{L^\infty(\Omega)} \ \forall x \in \Omega, \tag{2.3.7}$$

where $R = \frac{\delta(x)}{2}$ and the constant C depends only on Ω, s and γ, and blows up only when $\gamma \to 2s$.

The aforementioned H^1 regularity follows using this lemma and by a similar argument to the one of Theorem 2.3.6.

Lemma 2.3.9. *If $s \in (1/2, 1)$ and $f \in L^\infty(\Omega)$, then the solution u of* (1.2.11) *belongs to $H^1(\Omega)$ and therefore to $H^1(\mathbb{R}^n)$. Moreover, it satisfies*

$$|u|_{H^1(\Omega)} \leq \frac{C(\Omega, s, n)\|f\|_{L^\infty(\Omega)}}{2s-1},$$

where the constant $C(\Omega, s, n)$ is uniformly bounded for all $s \in (1/2, 1)$.

Proof. Take $\varepsilon \in (0, 1-s)$ and in the same fashion as before consider the sets A and B, with the slight difference of a $\frac{\delta(x)}{2}$ instead of a $\delta(x)$ in the definition of the latter. Taking $\gamma = 1 - C(\varepsilon)$ for some $0 < C(\varepsilon) < \varepsilon$ to be chosen, applying estimate (2.3.7) and proceeding as in the proof of Theorem 2.3.6, it follows

$$\iint_{A \setminus B} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2(1-\varepsilon)}} \, dy \, dx \leq \frac{C(\Omega, s, n)\|f\|^2_{L^\infty(\Omega)}}{\varepsilon - C(\varepsilon)} \int_\Omega \delta(x)^{2(s-1+\varepsilon)} dx.$$

Observe that the constant C in the previous inequality remains bounded for $s \in (1/2, 1)$, and that the integral is $\mathcal{O}\left((2s-1+2\varepsilon)^{-1}\right)$.

On the other hand, taking into account the global Hölder regularity of u (cf. equation (2.3.1)) it is immediate to obtain

$$\iint_B \frac{|u(x)-u(y)|^2}{|x-y|^{n+2(1-\varepsilon)}}\, dy\, dx \leq C(\Omega, s, n)\|f\|^2_{L^\infty(\Omega)} \int_\Omega \delta(x)^{2(s-1+\varepsilon)} dx.$$

Combining the previous estimates, we obtain

$$|u|^2_{H^{1-\varepsilon}(\Omega)} \leq \frac{C(\Omega, s, n)\|f\|^2_{L^\infty(\Omega)}}{(\varepsilon - C(\varepsilon))(2s-1+\varepsilon)},$$

where the constant $C(\Omega, s, n)$ remains bounded for $s \in (1/2, 1)$. Taking $C(\varepsilon)$ such that $\varepsilon - C(\varepsilon) = \mathcal{O}(\varepsilon)$, the desired conclusion follows thanks to Proposition 1.2.16. □

Next, we require some regularity on Du. Let $\beta \in (0, 2-2s)$ and assume that $f \in C^\beta(\Omega)$. Consider the subsets A and B of $\Omega \times \Omega$ as before (cf. (2.3.3) and (2.3.4)) and introduce the weighted integral

$$I := \iint_{A \setminus B} \frac{|Du(x)-Du(y)|^2}{|x-y|^{n+2(\ell-1)}}\, \delta(x,y)^{2\alpha} dx\, dy.$$

Using the first inequality of Remark 2.3.4 we explore how to take the involved parameters ℓ and α in order to keep I bounded. On one hand,

$$\begin{aligned} I &\leq C \int_\Omega \left(\int_{B(x,\delta(x))} |x-y|^{2(\beta+2s-1)-n-2(\ell-1)}\, dy \right) \delta(x)^{2(\alpha-\beta-s)} dx \\ &\leq \frac{C}{\beta+\ell-2s} \int_\Omega \delta(x)^{2(\alpha+s-\ell)} dx \leq \frac{C}{(\beta+\ell-2s)(1+2(\alpha-s-\ell))}, \end{aligned}$$

where, in order to ensure the convergence of the integrals involved, we must require

$$\ell - \beta < 2s \quad \text{and} \quad \ell < \alpha + s + 1/2. \tag{2.3.8}$$

On the other hand, for

$$II := \iint_B \frac{|Du(x)-Du(y)|^2}{|x-y|^{n+2(\ell-1)}}\, \delta(x,y)^{2\alpha} dx\, dy,$$

again due to Remark 2.3.4,

$$\begin{aligned} II &\leq C \int_\Omega \left(\int_{B(x,\delta(x))^c} |x-y|^{-n-2(\ell-1)} dy \right) \delta(x)^{2(\alpha+s-1)} dx \\ &\leq C \int_\Omega \delta(x)^{2(\alpha+s-\ell)} dx \leq \frac{C}{1+2(\alpha-s-\ell)}, \end{aligned}$$

where the condition for the finiteness of II is guaranteed if we restrict our attention to (2.3.8). Under these conditions, we have proved that

$$|Du|_{H^{\ell-1}_\alpha(\Omega)} \leq \frac{C}{(\beta+\ell-2s)(1+2(\alpha-s-\ell))}. \tag{2.3.9}$$

Within the range provided in (2.3.8) we can highlight some cases of interest. In the same spirit of Theorem 2.3.6, we have, considering $\alpha = 0$ and $\ell = s+1/2-\varepsilon$ in (2.3.9):

Theorem 2.3.10. *If $s \in (1/2,1)$ and $f \in C^\beta(\Omega)$ for some $\beta > 0$, then the solution u of* (1.2.11) *belongs to $H^{s+\frac{1}{2}-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, with*

$$|Du|_{H^{s-\frac{1}{2}-\varepsilon}(\Omega)} \leq \frac{C(\Omega,s,n,\beta)}{\sqrt{\varepsilon}(2s-1)} \|f\|_{C^\beta(\Omega)}.$$

Next, we turn our attention to weighted spaces. If we restrict the weight to the Muckenhoupt A_2 class (see Remark 1.5.12), which can be relevant for extending these considerations to the global case treated later, we need to choose $\alpha < 1/2$. This restriction is also of importance in the optimality of the graded meshes proposed later (see Section 3.4). Accordingly, assume $\alpha = 1/2-\varepsilon$ for $\varepsilon > 0$ small enough and take $\ell = 1+s-2\varepsilon$ and $\beta = 1-s$. From (2.3.9) we obtain the following weighted version, where the gaining of regularity is of almost one derivative.

Theorem 2.3.11. *Let $s \in (1/2,1)$, $f \in C^{1-s}(\Omega)$ and u be the solution of* (1.2.11)*. Then, given $\varepsilon > 0$ it holds that $u \in H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ and*

$$\|u\|_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C(\Omega,s,\|f\|_{1-s})}{\varepsilon}.$$

Remark 2.3.12. The regularity estimates given in this section are sharp, in the sense that if we consider the problem

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } B(x_0,r), \\ u = 0 & \text{in } B(0,r)^c, \end{cases} \tag{2.3.10}$$

for $x_0 \in \mathbb{R}^n$ and $r > 0$, then its solution is given by (cf. Theorem 2.1.13)

$$u(x) = \frac{2^{-2s}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)} \left(r^2 - |x-x_0|^2\right)^s \text{ in } B(x_0,r).$$

It is straightforward to check that this function belongs to $H^{s+\frac{1}{2}-\varepsilon}(\Omega)$ for all $s \in (0,1)$, to $H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ if $s \in (1/2,1)$ and that the parameter ε can not be removed. This explicit solution was first obtained by Getoor [53], in connection with the probability density function of the first exit time of the symmetric $2s$-stable Lévy process from Ω.

Remark 2.3.13. The technique we employed to obtain higher regularity in weighted spaces for $s \in (1/2, 1)$ does not carry immediately to the case $s \leq 1/2$. Indeed, from the considerations we have made we know that solutions cannot be expected to behave more regularly than $H^{s+1/2-\varepsilon}(\Omega)$. If $s \leq 1/2$, this exponent is less than 1. Thus, it would not be possible to apply Proposition 1.2.16 as it is stated; a weight should be included in the space $H^{1-\varepsilon}(\Omega)$. However, to the best of the author's knowledge, such a weighted version of this proposition has not been proved yet.

2.3.3 Global Regularity

A direct derivation of global regularity is a simple task in the present context. First we present the following lemma.

Lemma 2.3.14. *For $\frac{1}{2} < s < 1$, $\varepsilon > 0$ and $u \in H^{s+\frac{1}{2}-\varepsilon}(\Omega)$, it holds*

$$\int_\Omega \int_{\Omega^c} \frac{|Du(x)|^2}{|x-y|^{n+2(s-\frac{1}{2}-\varepsilon)}} \, dy \, dx \leq \frac{C(\Omega, s, n)}{2s-1-2\varepsilon} \|Du\|^2_{H^{s-\frac{1}{2}-\varepsilon}(\Omega)}.$$

Proof. This is a simple consequence of the inclusion $\Omega^c \subset B(x, \delta(x))^c$ for all $x \in \Omega$ and the Hardy inequality (1.2.3). □

Combining Lemmas 2.3.9, 2.3.14, and Theorem 2.3.10 we have proved:

Proposition 2.3.15. *If $1/2 < s < 1$ and $f \in C^\beta(\Omega)$ for some $\beta > 0$, then the solution u of* (1.2.11) *belongs to $\widetilde{H}^{s+\frac{1}{2}-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ and*

$$|u|_{H^{s+\frac{1}{2}-\varepsilon}(\mathbb{R}^n)} \leq \frac{C(\Omega, s, n, \beta)}{\sqrt{\varepsilon}(2s-1)} \|f\|_{C^\beta(\Omega)}.$$

In a similar fashion we prove:

Proposition 2.3.16. *Let $1/2 < s < 1$, $f \in C^{1-s}(\Omega)$ and u be the solution of our problem. Then, given $\varepsilon > 0$, $u \in H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\mathbb{R}^n)$ and*

$$\|u\|_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\mathbb{R}^n)} \leq \frac{C(\Omega, s, \|f\|_{1-s})}{\varepsilon}.$$

2.3.4 The case $s = 1/2$

Up to now, the possibility of s being equal to $1/2$ has been excluded from our analysis. In order to obtain a regularity estimate, the arguments to be carried are in the same spirit as before; the only issue to overcome is the need for $\beta > 0$ in Theorem 2.3.2. In this case, the argument demands less regularity of the function f. Indeed, the same technique as in the proof of Lemma 2.3.9 gives $u \in H^{1-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, with a bound of the type

$$|u|_{H^{1-\varepsilon}(\Omega)} \leq \frac{C(\Omega, n)}{\varepsilon} \|f\|_{L^\infty(\Omega)}. \tag{2.3.11}$$

Observe that we cannot assure $u \in H^1(\Omega)$ by taking $\varepsilon \to 0$ in the previous inequality, which is coherent with example (2.3.10).

Moreover, in this case the variational space $\mathbb{V}$ coincides with the Lions-Magenes space $H^{1/2}_{00}(\Omega)$ considered in Remark 1.2.14. Given $\varepsilon > 0$, the energy norm in this case can be bounded by resorting to (1.2.7) and applying the Hardy inequality (1.2.3) as follows,

$$\begin{aligned} \|u\|^2_{\mathbb{V}} &= |u|^2_{H^{1/2}(\Omega)} + \int_\Omega u^2 \omega^s_\Omega \\ &\leq |u|^2_{H^{1/2+\varepsilon}(\Omega)} + C \int_\Omega \frac{u^2}{\delta^{1+2\varepsilon}} \leq C\|u\|^2_{H^{1/2+\varepsilon}(\Omega)}. \end{aligned} \tag{2.3.12}$$

As a consequence, finite element error estimates for uniform meshes in this case follow from the theory developed in Chapter 3 for $s \neq 1/2$.

Resumen del capítulo

Dada una función $f \in H^r(\Omega)$ $(r \geq -s)$, consideramos el problema con condiciones de tipo Dirichlet homogéneas para el laplaciano fraccionario,

$$\left\{ \begin{aligned} (-\Delta)^s u &= f \quad \text{en } \Omega, \\ u &= 0 \quad \text{en } \Omega^c. \end{aligned} \right.$$

Recordemos que, en el contexto variacional descrito en la Subsección 1.2.3, la existencia y unicidad de una solución débil $u \in \widetilde{H}^s(\Omega)$ fueron demostradas en la Proposición 1.2.23. Un pregunta natural subsiguiente es si dicha solución débil es más regular que $\widetilde{H}^s(\Omega)$, y qué hipótesis sobre f son suficientes para asegurar esto. Más aún, teniendo en cuenta nuestro objetivo final de realizar análisis de elementos finitos sobre problemas que involucran el laplaciano fraccionario, es deseable que estas estimaciones de regularidad permitan acotar normas Sobolev de mayor orden de la solución.

Una pregunta más general es caracterizar las propiedades de mapeo del laplaciano fraccionario. Más precisamente, si es posible invertirlo como operador actuando entre espacios de Sobolev. En la **Sección 2.1** mostramos que esto es posible en dominios unidimensionales. Allí demostramos que cierta variante del laplaciano fraccionario induce una biyección entre los espacios de Sobolev con pesos introducidos en la Subsección 1.5.1. Luego, basados en la factorización de soluciones como el producto de un cierto peso explícito por una incógnita regular, obtenemos una caracterización de la regularidad de soluciones en esta escala de Sobolev con pesos. Una decomposición explícita en autoespacios para problemas planteados sobre bolas n-dimensionales muestra que esta construcción también es válida para dominios radiales.

Lamentablemente, esta construcción no es válida para dominios más generales. La caracterización del laplaciano fraccionario como un operador pseudo-diferencial permite escribir la regularidad de soluciones para dominios suaves en términos de los llamados μ-espacios de Hörmander. Estos combinan ciertos operadores pseudo-diferenciales con operadores de extensión por cero y de restricción. En la **Sección 2.2** describimos la conexión entre estos espacios y los espacios de Sobolev, lo que conduce a obtener estimaciones de regularidad Sobolev para soluciones en términos de normas Sobolev del dato f.

Estimaciones válidas para una clase más general de dominios son estudiadas en la **Sección 2.3**. El costo que debemos pagar allí es que podemos acotar las normas Sobolev de las soluciones en términos de normas Hölder del dato. Sin embargo, en esta sección demostramos la regularidad de soluciones tanto en espacios estándar como con peso. Este último resultado permite medir de un modo preciso el comportamiento de las soluciones cerca del borde de Ω.

Chapter 3

Finite element approximations for the homogeneous problem

In this chapter we carry out a complete finite element study of the homogeneous Dirichlet problem for the fractional Laplacian,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

Numerical approximations to problem (Homogeneous) have been addressed in the last years. D'Elia and Gunzburger [39] exploited the nonlocal vector calculus introduced in [42, 43] in order to perform a study of convergence of certain the approximations to the fractional Laplacian given by the nonlocal diffusion operators presented in Section 1.4.3 as the nonlocal interactions become infinite.

Also, Huang and Oberman [62] proposed a method which combines finite differences with numerical quadrature, obtained a discrete convolution operator and studied numerically the convergence and order of their method in the $L^\infty(\Omega)$-norm. The evidence provided in that paper indicates convergence with an order s with respect to the meshsize h, although orders as high as $3-2s$ are demonstrated if the solution is smooth.

Further, in [4], based on the theoretical considerations from Section 2.1, a high-order Nyström method was implemented. The algorithm developed in that work is spectrally accurate, with convergence rates that only depend on the smoothness of the right-hand side. In particular, convergence is exponentially fast (resp. faster than any power of the mesh-size) for analytic (resp. infinitely smooth) right-hand sides.

However, these methods were implemented only in one-dimensional domains. It is also noteworthy that, in [62], convergence of the algorithm is proved assuming that solutions are of class C^4, whereas in [39] regularity of solutions is assumed as part of the hypotheses. We have already developed regularity theory for the fractional Laplacian, and in turn, borrowing techniques from the boundary element method, in [3] it was found that the singular kernel arising in this problem in two-dimensional domains can be accurately handled.

Section 3.1 introduces the discrete spaces we use, sets the discrete formulation of our problem and recalls some properties necessary for the sequel. The approach we shall follow to obtain error estimates is to consider an adequate interpolator in a finite element space $\mathbb{V}_h$, and estimate the interpolation error. Section 3.2 analyzes stability and approximability properties of the Scott-Zhang operator in fractional Sobolev spaces. These properties are used to prove optimal order of convergence of the finite element approximations both in the standard and weighted context.

In first place, in Section 3.3 we deduce the orders of convergence of the discrete scheme in the energy and the $L^2(\Omega)$-norms. Afterwards, in Section 3.4 we adapt the theory we have developed for weighted fractional spaces, and take advantage of the new estimates by introducing approximations on a family of tailored graded meshes.

The finite element method is implemented in one and two dimensions, where uniform as well as graded meshes are proposed. Numerical experiments are presented in Section 3.5, showing orders of convergence in full agreement with our theoretical predictions.

3.1 Discrete problem and discrete spaces

We assume that $\cup_{T\in\mathcal{T}_h} T = \overline{\Omega}$, where $\mathcal{T}_h$ is an admissible triangulation of Ω made up of elements T of diameter h_T and with ρ_T equal to the diameter of the largest ball contained in T. We require that the family of triangulations under consideration satisfies:

$$\exists \sigma > 0 \text{ s.t. } h_T \leq \sigma \rho_T \ \forall T \in \mathcal{T}_h, \qquad \text{(Regularity)}$$

$$\exists \kappa > 0 \text{ s.t. } h_T \leq \kappa h_{T'} \ \forall T, T' \in \mathcal{T}_h \colon \overline{T} \cap \overline{T'} \neq \emptyset. \qquad \text{(Local quasi-uniformity)}$$

Naturally the second condition is a consequence of the first one. In this way κ can be expressed in terms of σ.

We consider Lagrangian finite elements (see, for example [24, Section 3.2]) of degree 1 and we set discrete functions to vanish on $\partial\Omega$. Namely,

$$\mathbb{V}_h = \{ v \in C_0(\Omega) : \ v\big|_T \in \mathcal{P}_1 \, \forall T \in \mathcal{T}_h \}. \tag{3.1.1}$$

The simpler case of piecewise constants, which provides a conforming method for $s \in (0, 1/2)$, is not addressed in the thesis in order to present an unified approach for the whole range $s \in (0, 1)$.

Throughout the remainder of the thesis, we denote by $\left\{X^{(i)}\right\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ the nodes and the nodal basis of $\mathbb{V}_h$, respectively. Recall (Section 1.2.3) that the variational space for problem (Homogeneous) is

$$(\mathbb{V}, \|\cdot\|_{\mathbb{V}}) = \left(\widetilde{H}^s(\mathbb{R}^n), \ \sqrt{\frac{C(n,s)}{2}}, |\cdot|_{H^s(\mathbb{R}^n)} \right).$$

Naturally, the space $\mathbb{V}_h$ is contained in $\mathbb{V}$, so that we are dealing with a conforming method. Further, it is immediate to check that there exists a unique solution to the discrete problem

$$\text{find } u_h \in \mathbb{V}_h \text{ such that } \langle u_h, v_h \rangle_{\mathbb{V}} = \int_\Omega f v_h \ \forall v_h \in \mathbb{V}_h, \tag{3.1.2}$$

where $\langle u_h, v_h \rangle_{\mathbb{V}}$ is the bilinear form defined by (1.2.10) and that Céa's Lemma holds in this context. Namely, the finite element solution is the best approximation in $\mathbb{V}_h$ to the solution of problem (Homogeneous):

$$\|u - u_h\|_{\mathbb{V}} = \min_{v_h \in \mathbb{V}_h} \|u - v_h\|_{\mathbb{V}}. \tag{3.1.3}$$

Thus, the question of convergence of the finite element approximations is equivalent to the question of how do the discrete spaces $\mathbb{V}_h$ approximate solutions of (1.2.11) in the energy norm.

We end this section by setting some further notations and recalling basic properties. Given a subdomain $\Lambda = \Lambda_i \subset \overline{\Omega}$, consider an affine mapping $F_\Lambda : \hat{\Lambda} \to \Lambda$, $F_\Lambda(\hat{x}) = B_\Lambda \hat{x} + x_0$, where $\hat{\Lambda}$ is a reference set. Then, it is straightforward to check that (see, for example, [34, Theorem 3.1.3])

$$\begin{aligned} |\det B_\Lambda| &\le C h_\Lambda^{\dim(\Lambda)}, & \|B_\Lambda\| &\le C h_\Lambda, \\ |\det B_\Lambda^{-1}| &\le C h_\Lambda^{-\dim(\Lambda)}, & \|B_\Lambda^{-1}\| &\le C h_\Lambda^{-1}, \end{aligned} \tag{3.1.4}$$

where the constants depend on the chunkiness of either $\hat{\Lambda}$ or Λ. In case Λ corresponds to an element of the triangulation $\mathcal{T}_h$, we denote the reference set by $\hat{T}$, and define it by

$$\hat{T} \coloneqq \left\{ \hat{x} = (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2 \colon 0 \le \hat{x}_1 \le 1, \ 0 \le \hat{x}_2 \le \hat{x}_1 \right\}. \tag{3.1.5}$$

Given a function $v \colon T \to \mathbb{R}$, define $\hat{v} = v \circ F_T$. By changing variables and using (3.1.4), it is immediate to verify that (for example, [34, Theorem 3.1.2])

$$\begin{aligned} &\|\hat{v}\|_{L^2(\hat{T})} \le C(\sigma) h_T^{-n/2} \|v\|_{L^2(T)}, \\ &\|D^\alpha \hat{v}\|_{L^2(\hat{T})} \le C(\sigma, k) h_T^{k-n/2} \|D^\alpha v\|_{L^2(T)} \qquad \forall \alpha \text{ s.t. } |\alpha| = k, \\ &|D^\alpha \hat{v}|_{H^{\ell-k}(\hat{T})} \le C(\sigma, k, \ell) h_T^{\ell-n/2} |D^\alpha v|_{H^{\ell-k}(T)} \quad \forall \alpha \text{ s.t. } |\alpha| = k, \ \ell \ge k. \end{aligned} \tag{3.1.6}$$

Finally, observe that the basis functions we are considering are Lipschitz continuous, with modulus of continuity $\text{Lip}(\varphi_i) \le \frac{C(\sigma)}{h_T}$ for all i, where T is any element having $X^{(i)}$ as a vertex.

3.2 Quasi-interpolation estimates

Identity (3.1.3) allows us to estimate the finite element solution error by choosing an adequate discrete function v_h in the right hand side. Typically, this is achieved by using an interpolation operator. However, as we cannot assure that pointwise values of solutions are well-defined[1] we need to resort to operators that, instead of using pointwise values of functions, take as input certain integrals. These are called quasi-interpolation operators and use integrals of the functions to be interpolated either over elements or over element boundaries, whenever traces are well-defined. So, estimates involving these quasi-interpolants typically bound norms in an element in terms of norms in a patch surrounding it.

One difficult aspect dealing with fractional seminorms is that they are not additive with respect to the decomposition of domains. Summing up bounds over all the elements of a triangulation does not lead to a global bound. Thus, we resort to the localization technique explained in Section 1.2.4. This requires obtaining estimates not only over elements, but to include a certain overlapping. Namely, we bound functions over sets of the form $T \times S_T$, where

$$S_T := \bigcup_{T' \colon \bar{T}' \cap \bar{T} \neq \emptyset} T'.$$

Observe that, due to (Local quasi-uniformity), the factor δ_i in (1.2.12) behaves like h_T. Thus, in this context, the conlcusion of Proposition 1.2.24 reads:

$$|v|^2_{H^s(\Omega)} \leq \sum_{T \in \mathcal{T}_h} \left[\iint_{T \times S_T} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dy \, dx + \frac{C(n,\sigma)}{s \, h_T^{2s}} \, \|v\|^2_{L^2(T)} \right]. \tag{3.2.1}$$

Our next step is to define an adequate quasi-interpolation operator. We work with the Scott-Zhang interpolator [96]. Other choices that would deliver similar approximation results are the Clément operator [35], the Bernardi-Girault operator [13] or the more recent construction by Guermond and Ern [60].

For the purpose of the following definition and only for this section, we assume that the finite element space $\mathbb{V}_h$ is constructed by Lagrange elements of an arbitrary degree k.

Definition 3.2.1 (Scott-Zhang interpolator)**.** Given an index $i \in I$, define a number a_i as follows: for $X^{(i)} \in \overline{T}$, pick $\Lambda_i \subset \partial T$ s.t. $X^{(i)} \in \Lambda_i$ (if $X^{(i)} \in \partial\Omega$, then Λ_i must lie in $\partial\Omega$). Take a projection $P_{\Lambda_i} : L^1(\Lambda_i) \to \mathcal{P}_1(\Lambda_i)$, let $\left\{\varphi_j^{(i)}\right\}_{j \in J}$ be the basis functions that do not vanish on Λ_i and $\left\{\psi_j^{(i)}\right\}$ be its dual basis,

$$\int_{\Lambda_i} \psi_j^{(i)} \varphi_k^{(i)} \; = \; \delta_{j,k} \; \forall k \in J.$$

[1]For example, in dimension $n=2$, the embedding $H^\ell(\Omega) \subset C(\Omega)$ is valid only if $\ell > 1$.

Then, we consider $a_i = P_{\Lambda_i} v(X^{(i)})$, and given a function $v \in H^\ell(\Omega)$ ($\ell > 1/2$), we define the Scott-Zhang interpolator of v as

$$I_h v(x) = \sum_{i \in I} a_i \varphi_i(x),$$

or equivalently,

$$I_h v(x) = \sum_{i \in I} \left(\int_{\Lambda_i} v \psi_i^{(i)} \right) \varphi_i(x).$$

We recall some of the basic properties of the operator I_h [96].

Theorem 3.2.2. *Let $\ell > 1/2$, then $I_h : H^\ell(\Omega) \to \mathbb{V}_h$ satisfies that $I_h(v_h) = v_h$ for all $v_h \in \mathbb{V}_h$ and I_h preserves boundary conditions, in the sense that $H_0^\ell(\Omega)$ is mapped to $\mathbb{V}_{h\,0} := \{v_h \in \mathbb{V}_h : \, v_h\big|_{\partial\Omega} = 0\}$.*

Stability and approximability results for the Scott-Zhang interpolation in fractional spaces were studied in [33], where estimates are developed elementwise. Here we follow the technique from that work and, in view of (3.2.1), adapt it to sets of the form $T \times S_T$.

Proposition 3.2.3. *Let $T \in \mathcal{T}_h$. If $s \in (0,1)$ and $\ell > \max\{1/2, s\}$, write $\ell = k + \ell'$, with $k \in \mathbb{N}$ and $\ell' \in (0,1]$. If $v \in H^\ell(\Omega)$, then*

$$\int_T \int_{S_T} \frac{|I_h v(x) - I_h v(y)|^2}{|x-y|^{n+2s}} \, dy \, dx \le \frac{C(n,\sigma,\ell)}{1-s} \left[\sum_{j=0}^{k} h_T^{2j-2s} \|D^j v\|^2_{L^2(S_T)} + h_T^{2\ell-2s} |v|^2_{H^\ell(S_T)} \right]. \tag{3.2.2}$$

Proof. Given $T \in \mathcal{T}_h$, $x \in T$ and $y \in S_T$, from the definition of the Scott-Zhang interpolator it is clear that

$$I_h v(x) - I_h v(y) = \sum_{i:\, X^{(i)} \in S_T} \left(\int_{\Lambda_i} v \psi_i \right) (\varphi_i(x) - \varphi_i(y)).$$

Therefore, raising to the square, applying Hölder's inequality and recalling that the number of terms in the sum is bounded by some constant depending on the mesh regularity, we obtain

$$\begin{aligned}\int_T \int_{S_T} &\frac{|I_h v(x) - I_h v(y)|^2}{|x-y|^{n+2s}} dy \, dx \le \\ &C(\sigma) \sum_{i / X^{(i)} \in S_T} \|\psi_i\|^2_{L^\infty(\Lambda_i)} \|v\|^2_{L^1(\Lambda_i)} \int_T \int_{S_T} \frac{|\varphi_i(x) - \varphi_i(y)|^2}{|x-y|^{n+2s}} dy \, dx.\end{aligned} \tag{3.2.3}$$

We estimate the terms in the right hand side of the previous inequality separately. With the notation from Definition 3.2.1, let T' be an element such that $T' = \Lambda_i$ (in this case, it is just $T' = T$) or $\Lambda_i \subset \partial T'$ (in this case, T' is either T or one of the neighbouring elements contained in S_T).

By [96, Lemma 3.1], it holds that

$$\|\psi_i\|_{L^\infty(\Lambda_i)} \leq C(\sigma) h_{T'}^{-\dim(\Lambda_i)}. \tag{3.2.4}$$

Next, we would like to estimate $\|v\|_{L^1(\Lambda_i)}$ in terms of $\|v\|_{L^2(T')}$. On one hand, if Λ_i is $n-$dimensional, then $\Lambda_i = T'$ and by Hölder's inequality it follows that

$$\|v\|_{L^1(\Lambda_i)} \leq h_{T'}^{n/2} \|v\|_{L^2(T')}. \tag{3.2.5}$$

On the other hand, if $\Lambda_i \subset \partial T'$ is $(n-1)-$dimensional, let us consider an affine mapping $F_{\Lambda_i} : \hat{\Lambda}_i \to \Lambda_i$, where $\hat{\Lambda}_i \subset \partial \hat{T}$, and $\hat{T}$ is the reference element (3.1.5). Then, applying the trace theorem $H^\ell(\hat{T}) \hookrightarrow L^1(\hat{\Lambda}_i)$, we obtain

$$\|v\|_{L^1(\Lambda_i)} \leq C(\ell) h_{T'}^{n-1} \|\hat{v}\|_{H^\ell(\hat{T})},$$

where $\hat{v} = v \circ F$. Combining this inequality with (3.1.6), we deduce

$$\|v\|_{L^1(\Lambda_i)} \leq C(\sigma, \ell) h_{T'}^{n/2-1} \left[\sum_{j=0}^{k} h_{T'}^j \|D^j v\|_{L^2(T')} + h_{T'}^\ell |v|_{H^\ell(T')} \right]. \tag{3.2.6}$$

As for the term involving the basis functions, assume T is an element such that $X^{(i)} \in S_T$. Recall that mesh regularity implies that $\mathrm{Lip}(\varphi_i) \leq \frac{C(\sigma)}{h_T}$, and that there exists a constant $C(\sigma)$ such that for all $x \in T$ it holds that

$$\alpha(x) \coloneqq \max_{z \in \partial S_T} d(x, z) \leq C(\sigma) h_T.$$

Therefore,

$$\begin{aligned} \int_T \int_{S_T} \frac{|\varphi_i(x) - \varphi_i(y)|^2}{|x-y|^{n+2s}} dy\, dx &\leq \frac{C(\sigma)}{h^2} \int_T \int_{S_T} |x-y|^{2-n-2s} dy\, dx \leq \\ &\leq \frac{C(n, \sigma)}{h^2} \int_T \int_0^{\alpha(x)} \rho^{1-2s} d\rho\, dx \leq \frac{C(n, \sigma)}{1-s} h_T^{n-2s}. \end{aligned}$$

Combining this estimate with (3.2.4) and either (3.2.5) or (3.2.6), and taking into account that $h_{T'}$ is comparable with h_T for all $T' \subset S_T$, we deduce that

- if $\Lambda_i = T$, then

$$\|\psi_i\|_{L^\infty(\Lambda_i)}^2 \|v\|_{L^1(\Lambda_i)}^2 \int_T \int_{S_T} \frac{|\varphi_i(x) - \varphi_i(y)|^2}{|x-y|^{n+2s}} dy\, dx \leq \frac{C(n, \sigma)}{1-s} h_T^{-2s} \|v\|_{L^2(T)}^2;$$

- if $\Lambda_i \subset \partial T$, then

$$\|\psi_i\|^2_{L^\infty(\Lambda_i)} \|v\|^2_{L^1(\Lambda_i)} \int_T \int_{S_T} \frac{|\varphi_i(x) - \varphi_i(y)|^2}{|x-y|^{n+2s}} dy\, dx \leq \frac{C(n,\sigma,\ell)}{1-s} h_T^{-2s} \left[\sum_{j=0}^{k} h^j_{T'} \|D^j v\|_{L^2(T')} + h^\ell_{T'} |v|_{H^\ell(T')} \right]^2 .$$

As $\#\{i/X^{(i)} \in \bar{T}\} \leq C(\sigma)$, inequality (3.2.2) follows upon combination of the last two estimates with (3.2.3). □

Before obtaining approximability estimates for the Scott-Zhang operator, we recall some facts about a well known key tool. Let S be an star-shaped domain with respect to a ball B. Introduce the polynomial $P_k u$ of degree k with the property

$$\int_S \partial^\alpha (v - P_k v) = 0,$$

for all multi-index α of order $|\alpha| \leq k$. In our context we need to focus on the cases $k = 0, 1$. For instance, Proposition 1.2.6 gives at once

$$\|v - P_0 v\|_{L^2(S_T)} \leq C h^\ell |v|_{H^\ell(S_T)},$$

for $0 < \ell < 1$ and with a constant depending on the chunkiness parameter of S_T. In this context, due to the mesh properties (Regularity) and (Local quasi-uniformity), such constant may be expressed in terms of σ.

Since $|v - P_0 v|_{H^\ell(S_T)} = |v|_{H^\ell(S_T)}$, by means of the L^2 estimate and interpolation (Remark 1.2.17) we obtain

$$|v - P_0 v|_{H^s(S_T)} \leq C h^{\ell - s} |v|_{H^\ell(S_T)}, \tag{3.2.7}$$

for any $0 \leq s \leq \ell < 1$, with a constant $C = C(\sigma)$.

Similarly, using the standard Poincaré inequality for functions with zero average together with Proposition 1.2.6 , we obtain for any $1 < \ell < 2$

$$\|v - P_1 v\|_{L^2(S_T)} + h_T |v - P_1 v|_{H^1(S_T)} \leq C h_T^\ell |v|_{H^\ell(S_T)}, \tag{3.2.8}$$

with C uniformly bounded in terms of σ.

Moreover, Remark 1.2.17 and (3.2.8) give for $0 < s < 1$ and $1 < \ell < 2$

$$|v - P_1 v|_{H^s(S_T)} \leq C h_T^{\ell - s} |v|_{H^\ell(S_T)}, \tag{3.2.9}$$

with C bounded again in terms of σ.

We are now in position to prove, by means of a standard argument, an approximation identity for the Scott-Zhang operator over $T \times S_T$.

Proposition 3.2.4. *Let either* $0 < s < \ell < 1$ *and* $\ell > 1/2$ *or* $1/2 < s < 1$ *and* $1 < \ell < 2$, *and let* I_h *be the Scott-Zhang operator. Then, for all* $T \in \mathcal{T}_h$,

$$\int_T \int_{S_T} \frac{|(v - I_h v)(x) - (v - I_h v)(y)|^2}{|x-y|^{n+2s}} \, dy \, dx \leq \frac{C(n,\sigma,\ell)}{1-s} h_T^{2\ell-2s} |v|^2_{H^\ell(S_T)}.$$

Proof. Assume $0 < s < \ell < 1$ and $\ell > 1/2$. Since I_h is a projection over $\mathbb{V}_h$, it holds that $v - I_h v = v - P_0 v + I_h(P_0 v - v)$.

Furthermore, combining the stability Proposition 3.2.3 with estimate (3.2.7), we deduce

$$\int_T \int_{S_T} \frac{|I_h(P_0 v - v)(x) - I_h(P_0 v - v)(y)|^2}{|x-y|^{n+2s}} \, dy \, dx \leq$$
$$\frac{C(n,\sigma,\ell)}{1-s} \left[h_T^{-2s} \|P_0 v - v\|^2_{L^2(S_T)} + h_T^{2\ell-2s} |P_0 v - v|^2_{H^\ell(S_T)} \right] \leq \frac{C(n,\sigma,\ell)}{1-s} h_T^{2\ell-2s} |v|^2_{H^\ell(S_T)}.$$

The proof for the case $1/2 < s < 1$ and $1 < \ell < 2$ follows in the same way, using the polynomial $P_1 v$ and estimate (3.2.9), respectively. □

Remark 3.2.5. Following the same lines as in the proofs of propositions 3.2.3 and 3.2.4 it is possible to obtain L^2 stability and approximability estimates valid for all $T \in \mathcal{T}_h$, $v \in H^\ell(\Omega)$, $\ell \in (1/2, 2)$:

$$\|I_h v\|^2_{L^2(T)} \leq C(n,\sigma,\ell) \left[\sum_{j=0}^{k} h_T^{2j} \|D^j v\|^2_{L^2(S_T)} + h_T^{2\ell} |v|^2_{H^\ell(S_T)} \right],$$
$$\|v - I_h v\|^2_{L^2(T)} \leq C(n,\sigma,\ell) h_T^{2\ell} |v|^2_{H^\ell(S_T)}.$$

3.3 Uniform meshes

Having at hand quasi-interpolation estimates, we are now in position to estimate the order of convergence of the finite element scheme for (Homogeneous). Up to now, we have only assumed that the family of triangulations $\{\mathcal{T}_h\}$ is quasi-uniform: there exists some constant C such that

$$h_T \leq C h_{T'} \quad \forall T, T' \in \mathcal{T}_h.$$

In this section we obtain a priori convergence estimates in the energy and in the $L^2(\Omega)$-norms without assuming any further condition on the family of meshes.

3.3.1 Convergence in the energy norm

To obtain finite element convergence estimates in the energy norm, from identity (3.1.3), we just need to bound $\|u-I_h u\|_{\mathbb{V}}$ adequately, where u is the solution of (Homogeneous). In first place, calling $h = \max_{T\in\mathcal{T}_h} h_T$ the mesh size parameter, combining the approximability estimate from Proposition 3.2.4 with the localization property (3.2.1) we obtain

$$|u - I_h u|^2_{H^s(\Omega)} \leq \sum_{T\in\mathcal{T}_h} \left[\frac{C(n,\sigma,\ell)}{1-s} h^{2\ell-2s} |u|^2_{H^\ell(S_T)} + \frac{C(n,\sigma)}{s\, h^{2s}} \|u - I_h u\|^2_{L^2(T)} \right].$$

Further, by Remark 3.2.5, the L^2-norm $\|u - I_h u\|_{L^2(T)}$ is bounded in terms of $|u|_{H^\ell(S_T)}$. Noticing that mesh regularity implies that, given an element T the number of stars $S_{T'}$ such that $T \subset S_{T'}$ is bounded by $C(\sigma)$, we deduce

$$|u - I_h u|_{H^s(\Omega)} \leq \frac{C(n,\sigma,\ell)}{\sqrt{s(1-s)}} h^{\ell-s} |u|_{H^\ell(\Omega)}. \tag{3.3.1}$$

It is noteworthy that the constant above has the 'correct' scaling for $s \to 0$ and $s \to 1$, i.e., $C \sim \sqrt{s(1-s)}^{\,-1}$. Thus, recalling that $\|\cdot\|_{\mathbb{V}} = \sqrt{\frac{C(n,s)}{2}} |\cdot|_{H^s(\mathbb{R}^n)}$, where the constant $C(n,s)$ is given by (1.1.2), and in turn, by Corollary 1.2.11 the $H^s(\mathbb{R}^n)$-seminorm is equivalent to the $H^s(\Omega)$-norm if $s < 1/2$ (or just to the $H^s(\Omega)$-seminorm if $s > 1/2$), we conclude

$$\|u - I_h u\|_{\mathbb{V}} \leq C(n,s,\sigma,\ell) h^{\ell-s} |u|_{H^\ell(\Omega)}. \tag{3.3.2}$$

The constant above remains uniformly bounded for all $s \in (0,1)$. Invoking identity (3.1.3), and combining it respectively with Theorem 2.3.6, estimate (2.3.11) and Theorem 2.3.10, we have proved the order of convergence of the finite element approximation in the energy norm.

Theorem 3.3.1. *Let Ω be a bounded, Lipschitz domain satisfying the exterior ball condition. For the solution u of* (1.2.11) *and its finite element approximation u_h given by* (3.1.2) *we have the a priori estimates*

$$\|u - u_h\|_{\mathbb{V}} \leq \frac{C(n,s,\sigma)}{\varepsilon} h^{\frac12-\varepsilon} \|f\|_{C^{\frac12-s}(\Omega)} \qquad \forall \varepsilon > 0, \quad \text{if } s < 1/2,$$

$$\|u - u_h\|_{\mathbb{V}} \leq \frac{C(n,\sigma)}{\varepsilon} h^{\frac12-\varepsilon} \|f\|_{L^\infty(\Omega)} \qquad \forall \varepsilon > 0, \quad \text{if } s = 1/2,$$

$$\|u - u_h\|_{\mathbb{V}} \leq \frac{C(n,s,\beta,\sigma)}{\sqrt{\varepsilon}(2s-1)} h^{\frac12-\varepsilon} \|f\|_{C^\beta(\Omega)} \qquad \forall \varepsilon > 0, \quad \text{if } s > 1/2.$$

So, if h is sufficiently small, taking $\varepsilon = |\ln h|^{-1}$ we obtain the quasi-optimal estimates

$$\|u - u_h\|_{\mathbb{V}} \leq C(n,s,\sigma) h^{\frac12} |\ln h| \|f\|_{C^{\frac12-s}(\Omega)}, \qquad \text{if } s < 1/2,$$

$$\|u-u_h\|_{\mathbb{V}} \le C(n,\sigma)h^{\frac{1}{2}}|\ln h|\|f\|_{L^\infty(\Omega)}, \qquad \text{if } s=1/2,$$
$$\|u-u_h\|_{\mathbb{V}} \le \frac{C(n,s,\beta,\sigma)}{2s-1}h^{\frac{1}{2}}\sqrt{|\ln h|}\|f\|_{C^\beta(\Omega)}, \qquad \text{if } s>1/2.$$

3.3.2 Convergence in $L^2(\Omega)$

After obtaining error estimates in the energy norm, it is natural to ask whether the order of convergence is improved if the norm in which the error is measured is weaker. Here we follow the well-known Aubin-Nitsche duality argument to obtain error estimates in the $L^2(\Omega)$-norm.

Since we require Sobolev norms in the convergence estimate to perform this trick, we assume that Ω is a smooth domain and resort to the results from Section 2.2. Error estimates in terms of Sobolev norms of the right hand side function follow from (3.3.2) and Proposition 2.2.3. Namely, if $f \in H^r(\Omega)$ for some $r \ge -s$, then

$$\|u-u_h\|_{\mathbb{V}} \le C(n,s,\sigma,\alpha)h^\alpha\|f\|_{H^r(\Omega)}, \tag{3.3.3}$$

where $\alpha = \min\{s+r, 1/2-\varepsilon\}$.

Proposition 3.3.2. *Let Ω be a smooth domain, $s \in (0,1)$, $f \in H^r(\Omega)$ for some $r \ge -s$ and u be the solution of* (Homogeneous). *Given a uniform mesh $\mathcal{T}_h$ with mesh size h, and the space $\mathbb{V}_h$ defined as in* (3.1.1)*, let u_h be the finite element solution of the discrete problem* (3.1.2)*. Then, it holds that*

$$\|u-u_h\|_{L^2(\Omega)} \le C(n,s,\sigma,\alpha)h^{\alpha+\beta}\|f\|_{H^r(\Omega)}, \tag{3.3.4}$$

where $\alpha = \min\{s+r, 1/2-\varepsilon\}$ and $\beta = \min\{s, 1/2-\varepsilon\}$.

Proof. Let $w \in \mathbb{V}$ be the weak solution of the boundary value problem

$$\begin{cases} (-\Delta)^s w = u-u_h & \text{in } \Omega, \\ w = 0 & \text{in } \Omega^c. \end{cases}$$

Then, resorting to Galerkin orthogonality, we obtain

$$\|u-u_h\|^2_{L^2(\Omega)} = \langle w, u-u_h\rangle_{\mathbb{V}} \le \|w-I_hw\|_{\mathbb{V}}\ \|u-u_h\|_{\mathbb{V}},$$

where $I_hw \in \mathbb{V}_h$ is the Scott-Zhang interpolator of w. Taking into account the regularity given by Proposition 2.2.3 with $r=0$, interpolation estimate (3.3.2) gives

$$\|w-I_hw\|_{\mathbb{V}} \le C(n,s,\sigma)h^\beta|w|_{H^{s+\beta}(\Omega)} \le C(n,s,\sigma,\beta)h^\beta\|u-u_h\|_{L^2(\Omega)},$$

where $\beta = \min\{s, 1/2-\varepsilon\}$. Finally, using the error estimate (3.3.3) we obtain

$$\|u-u_h\|^2_{L^2(\Omega)} \le C(n,s,\sigma,\alpha,\beta)h^{\alpha+\beta}\|u-u_h\|_{L^2(\Omega)}\|f\|_{H^r(\Omega)},$$

and then estimate (3.3.4) follows. □

3.4 Graded meshes

The approximability property for the Scott-Zhang operator we obtained in Section 3.2 is enough to deal with standard fractional spaces. Nevertheless, for $s \in (1/2, 1)$, in view of Proposition 2.3.16, further information about the behavior of solutions near the boundary of the domain is available. The procedure we propose here is standard, for example, in problems with corner singularities or to cope with boundary layers arising in convection-dominated problems. An increased rate of convergence is achieved by resorting to a priori adapted meshes. This approach requires dealing with the weights already introduced in Subsection 2.3.2 and suitably graded meshes. In order to obtain appropriate bounds in weighted fractional spaces we should replace the classical Poincaré inequality of Proposition 1.2.6 by the weighted counterpart from Proposition 1.5.15.

Now we want to exploit such proposition together with the a priori estimate of Theorem 2.3.11. Since the weights under consideration vanish only on the boundary of the domain we need to rely on (1.5.7) just for patches S_T touching $\partial\Omega$. Actually, for them we obtain the following improved version of (3.2.8), derived using Proposition 1.5.15 instead of Proposition 1.2.6,

$$\|v - P_1 v\|_{L^2(S_T)} + h_T |v - P_1 v|_{H^1(S_T)} \leq C h_T^{\ell-\alpha} |v|_{H^\ell_\alpha(S_T)},$$

where $1 < \ell < 2$ and $\alpha < \ell - 1$. Taking $s \in (1/2, 1)$, $\ell = 1 + s - 2\varepsilon$, and $\alpha = 1/2 - \varepsilon$ we obtain the analogue of (3.2.9)

$$|v - P_1 v|_{H^s(S_T)} \leq C h_T^{1/2-\varepsilon} |v|_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_T)}.$$

In particular, this property of P_1 and the stability estimate from Proposition 3.2.3 yield, following the same steps as in the proof of Proposition 3.2.4,

$$\int_T \int_{S_T} \frac{|(v - I_h v)(x) - (v - I_h v)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma)}{1 - s} h_T^{1-2\varepsilon} |v|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_T)}. \tag{3.4.1}$$

This approximability property is particularly useful for patches S_T touching the boundary of Ω. For these it must be recalled that $d(x, \partial S_T) \leq d(x, \partial\Omega)$ for $x \in S_T$. Naturally, as stated in Remark 3.2.5 for standard spaces, stability and approximability results in the L^2-norm in terms of weighted H^ℓ-norm hold as well,

$$\|I_h v\|^2_{L^2(T)} \leq C(n, \sigma, \ell) \left[\sum_{j=0}^{k} h_T^{2s} \|D^j v\|^2_{L^2(T)} + h_T^{2(\ell-s)} |v|^2_{H^\ell_\alpha(S_T)} \right],$$
$$\|v - I_h v\|^2_{L^2(T)} \leq C(n, \sigma, \ell) h_T^{2(\ell-\alpha)} |v|^2_{H^\ell_\alpha(S_T)}.$$

The following construction of graded meshes is based on [55, Section 8.4]. We assume that, in addition to (Regularity) and (Local quasi-uniformity) our sequence of meshes enjoys some extra properties, denoted below with (H). First, we pick an arbitrary mesh size parameter $h > 0$ and define, for ε small enough, a number $\mu \geq 1$. We assume that for any $T \in \mathcal{T}_h$,

$$\begin{array}{ll} \text{if } T \cap \partial\Omega \neq \emptyset, & \text{then } h_T \leq C(\sigma) h^{\mu}; \\ \text{otherwise}, & h_T \leq C(\sigma) h \, d(T, \partial\Omega)^{(\mu-1)/\mu}. \end{array} \tag{H}$$

In first place, we prove a convergence result stated in terms of the mesh size parameter h.

Proposition 3.4.1. *Let $s \in (1/2, 1)$ and assume that the triangulation $\mathcal{T}_h$ satisfies conditions* (Regularity)*,* (Local quasi-uniformity) *as well as the grading hypotheses* (H) *with $\mu = \frac{2}{1+2\varepsilon}$, where $\varepsilon > 0$. Then, if $f \in C^{1-s}(\Omega)$, it holds that*

$$\|u - u_h\|_{\mathbb{V}} \leq \frac{C(n, s, \beta, \sigma)}{2s - 1} h \sqrt{|\ln h|} \|f\|_{C^{1-s}(\Omega)}.$$

Proof. In first place, recall estimate (3.2.1):

$$|u - I_h u|^2_{H^s(\Omega)} \leq \sum_{T \in \mathcal{T}_h} \left[\iint_{T \times S_T} \frac{|(u - I_h u)(x) - (u - I_h u)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx + \frac{C(n, \sigma)}{s \, h_T^{2s}} \|u - I_h u\|^2_{L^2(T)} \right]. \tag{3.4.2}$$

Since we have bounds for $\|u - I_h u\|_{L^2(\Omega)}$ both in standard and weighted spaces, we just need to take care of the integrals over $T \times S_T$.

We split the sum in the right hand side of (3.4.2) according to whether $S_T \cap \partial\Omega$ is empty or not. For elements away from the boundary it is enough to utilize Proposition 3.2.4 and observe that the weight function $\delta(x, y)$ may be safely taken out of the integral. Indeed, if T is an element such that $S_T \cap \partial\Omega = \emptyset$,

$$\iint_{T \times S_T} \frac{|(u - I_h u)(x) - (u - I_h u)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma, \ell)}{1 - s} h_T^{2\ell - 2s} d(T, \partial\Omega)^{-1+2\varepsilon} |u|^2_{H^{\ell}_{1/2-\varepsilon}(S_T)}. \tag{3.4.3}$$

Using that $h_T \leq C(\sigma) h \, d(T, \partial\Omega)^{1/2-\varepsilon}$, and choosing $\ell = 1 + s - 2\varepsilon$ above, we obtain that, for every element T such that $S_T \cap \partial\Omega = \emptyset$,

$$\iint_{T \times S_T} \frac{|(u - I_h u)(x) - (u - I_h u)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma)}{1 - s} h^{2-4\varepsilon} |u|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_T)}.$$

On the other hand, if T is an element such that $S_T \cap \partial\Omega \neq \emptyset$, identity (3.4.1) together with the hypothesis $h_T \leq C(\sigma) h^{2/(1+2\varepsilon)}$ give immediately

$$\iint_{T \times S_T} \frac{|(u - I_h u)(x) - (u - I_h u)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \frac{C(n, \sigma)}{1 - s} h^{2-4\varepsilon} |u|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_T)}. \tag{3.4.4}$$

Putting the estimates over $T \times S_T$ all together and recalling the presence of the scaling factor (1.1.2) in the $\mathbb{V}$-norm, we deduce

$$\|u - I_h u\|_{\mathbb{V}}^2 \leq C(n,s,\sigma) h^{2-4\varepsilon} \sum_{T \in \mathcal{T}_h} |u|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_T)},$$

with a constant uniformly bounded with respect to s. Mesh regularity implies that the sum in the right hand side above is bounded by $C(\sigma)|u|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)}$, and an application of Theorem 2.3.11 allows to conclude the proof. □

Finally, if the mesh size parameter is appropriately related to the number of nodes of the mesh, then it is possible to obtain quasi-optimal order of convergence.

Theorem 3.4.2. *Let $s \in (1/2, 1)$ and assume that the triangulation $\mathcal{T}_h$ satisfies conditions* (Regularity)*,* (Local quasi-uniformity) *as well as the grading hypotheses* (H)*. If the mesh parameter h behaves like $h \sim \frac{1}{N^{1/n}}$, N being the number of mesh nodes, then for the solution u of* (1.2.11) *and its finite element approximation u_h given by* (3.1.2) *we have the following a quasi-optimal a priori estimate*

$$\|u - u_h\|_{\mathbb{V}} \leq \frac{C(n,s,\beta,\sigma)}{2s-1} N^{-1/n} \sqrt{|\ln N|} \|f\|_{C^{1-s}(\Omega)}.$$

Remark 3.4.3. Both Proposition 3.4.1 and Theorem 3.4.2 are stated for $s > 1/2$. Recall Remark 2.3.13: in case $s \leq 1/2$, we cannot assure that the solution u belongs to $H^1(\Omega)$, and so it would be necessary to add a weight function in order to make at least $u \in H^{1-\epsilon}_\alpha(\Omega)$ for some $\alpha > 0$. Therefore, in order to obtain the same order of convergence as in Theorem 3.4.2, an analogue result to Proposition 1.2.16 would be necessary in the context of weighted Sobolev spaces. However, the numerical evidence we present in Table 3.4 for graded meshes and $s \in (0,1)$ indicates that the restriction $s > 1/2$ might be unnecessary.

Remark 3.4.4. Taking into account the restrictions (2.3.8) necessary to prove the regularity theorem in weighted spaces (Theorem 2.3.11), it is possible to achieve differentiability orders between $1/2 + s < \ell < 2$ by choosing adequate weights; our choices delivered $\ell = 1 + s - 2\varepsilon$. It seems natural to ask whether the order of convergence (with respect to N) could be improved by considering a different value of ℓ and following the grading approach presented at the beginning of this section. This is not the case; actually the choice we made yields the best possible order with respect to the number of nodes with minimum grading requirements on the mesh.

To show this, we revisit the proof of Proposition 3.4.1. If we assume that $u \in H^\ell_\alpha(\Omega)$ for arbitrary ℓ, α, and we leave the parameter μ free as well, then estimates (3.4.3) and

(3.4.4) read

$$\iint_{T\times S_T} \frac{|(u-I_hu)(x)-(u-I_hu)(y)|^2}{|x-y|^{n+2s}}\,dy\,dx \leq \begin{cases} Ch^{2\ell-2s}d(T,\partial\Omega)^{-2\alpha+2\frac{(\mu-1)}{\mu}(\ell-s)}|u|^2_{H^\ell_\alpha(S_T)} & \text{if } S_T\cap\partial\Omega=\emptyset, \\ Ch^{2\mu(\ell-s-\alpha)}|u|^2_{H^\ell_\alpha(S_T)} & \text{if } S_T\cap\partial\Omega\neq\emptyset. \end{cases} \tag{3.4.5}$$

So, making the exponent in $d(S_T,\partial\Omega)$ to be zero, we obtain $\alpha=(\ell-s)\frac{(\mu-1)}{\mu}$, and thus, considering restriction (2.3.8), it must hold that $\mu>2(\ell-s)$ and the error is of order $h^{\ell-s}$.

For simplicity, we assume $n=2$. It is not a difficult task to verify that the total number of nodes is related with the mesh parameter h by (see Remark 3.5.5 below for the case that Ω is a ball)

$$N\sim\begin{cases} h^{-2} & \text{if } \mu\leq 2, \\ h^{-\mu} & \text{if } \mu>2. \end{cases}$$

On one hand, if $\mu\leq 2$, as ℓ increases there is a gain of order without an increment in the total number of nodes and the error behaves like $N^{-(\ell-s)/2}$. Within this range, the choice $\ell=1+s-\varepsilon$ (leading to $\mu>2-2\varepsilon$) is optimal.

On the other hand, if $\mu>2$, the order of convergence in N would be $-\frac{\ell-s}{\mu}$, which is poorer than $N^{-1/2}$. Here the expected gain of order due to the increase in differentiability is compensated by the cost of having to increase the weight power, which implies a growth in the number of nodes.

3.5 Numerical experiments

Numerical computation of solutions of (Homogeneous) has as main difficulties the fact that a singular kernel is involved, and that integrals over the whole $\mathbb{R}^n$ must be calculated. Algorithmic aspects and the quadrature rules utilized are described in Appendix A, where there are also further comments about implementation. More details about the code employed can be found in [2].

In this section we display the outcome of several experiments regarding problem (Homogeneous) carried out with our code, in one and two-dimensional domains. These results show good agreement with our theoretical predictions.

3.5.1 Uniform meshes

Our estimates bound the error both in the L^2 and the energy norms (cf. Proposition 3.3.2 and Theorem 3.3.1). If the exact solution of our problem is available, the error in the L^2-norm is easily computed by standard quadratures. However, as the energy norm is of fractional nature, computing it for an arbitrary function is a delicate task. The following lemma shows that in fact the energy norm of the error equals a quantity that does not involve singular kernels.

Lemma 3.5.1. *It holds that*

$$\|u - u_h\|_{\mathbb{V}} = \left(\int_{\Omega} f(x)(u(x) - u_h(x)) \, dx \right)^{\frac{1}{2}}.$$

Proof. It is an immediate consequence of the orthogonality condition

$$\langle v_h, u - u_h \rangle_{\mathbb{V}} = 0 \quad \forall v_h \in \mathbb{V}_h.$$

Indeed, from it we obtain

$$\|u - u_h\|_{\mathbb{V}}^2 = \langle u - u_h, u - u_h \rangle_{\mathbb{V}} = \langle u, u - u_h \rangle_{\mathbb{V}},$$

and the equality follows by (1.2.11). □

Next, we display the output of our numerical computations. Since explicit solutions to (Homogeneous) are only available if Ω is a ball, we limit the examples presented here to this type of domains. In the following chapters some numerical results in different domains are exhibited.

Example 3.5.2. As a first example, we analyze the problem given in Remark 2.3.12 with $n = 2$, $x_0 = 0$ and $r = 1$, and for several values of s. Namely, consider

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } B(0,1) \subset \mathbb{R}^2, \\ u = 0 & \text{in } B(0,1)^c. \end{cases} \tag{3.5.1}$$

Several orders are shown in Table 3.1; these results are in accordance with the estimates in Theorem 3.3.1 and Proposition 3.3.2. Although the computation Lemma 3.5.1 is subtle in general, in this particular case it can be carried out exactly since $f \equiv 1$ on Ω and a closed formula for $\int_\Omega u$ is easy to get while the exact value of $\int_\Omega u_h$ can be numerically evaluated.

Example 3.5.3. More generally, utilizing Theorem 2.1.13 a family of explicit solutions for (Homogeneous) is available. We take $k = 2$ in such theorem; we set $f(x) = \lambda_{2,s}\, p_2^{(s)}(x)$, for $s \in \{0.25, 0.75\}$. The exact solution is $u(x) = (1 - |x|^2)_+^s p_2^{(s)}(x)$. We compute the order of convergence in $L^2(\Omega)$ for these two values of s; according to Proposition 3.3.2, it is expected to have order of convergence 0.75 for $s = 0.25$ and 1 for $s = 0.75$ with respect to the mesh size h.

We summarize our numerical results in Table 3.2. These are in accordance with the predicted rates of convergence. Finally, in Figure 3.1 a finite element solution, corresponding to $s = 0.75$ and computed with a mesh of about 16000 triangles, is displayed.

Value of s	Order in L^2	Order in $\mathbb{V}$
0.1	0.621	0.500
0.2	0.721	0.496
0.3	0.804	0.492
0.4	0.880	0.491
0.5	0.947	0.492
0.6	1.003	0.496
0.7	1.046	0.501
0.8	1.059	0.494
0.9	0.999	0.467

Table 3.1: (Uniform Meshes) Computational rates of convergence for problem (3.5.1) with respect to the mesh size, measured in the L^2 and energy norms.

Mesh size h	$\|u-u_h\|_{L^2(\Omega)}$ $(s=0.25)$	$\|u-u_h\|_{L^2(\Omega)}$ $(s=0.75)$
0.0383	0.0801	0.01740
0.0331	0.0698	0.01388
0.0267	0.0605	0.01104
0.0239	0.0556	0.00965
0.0218	0.0513	0.00849

Table 3.2: Errors in the L^2-norm for $s = 0.25$ and $s = 0.75$ in Example 3.5.3. The estimated orders of convergence with respect to the mesh size are, respectively, 0.7669 and 1.2337. This is in good accordance with estimate (3.3.4).

Example 3.5.4. Even though the energy norm, that coincides with a multiple of the $H^s(\mathbb{R}^n)$ seminorm, involves integration over the whole space, our theoretical convergence estimates involve norms of the solution just in Ω (cf. (3.3.2)). The aim of this example is to illustrate this fact.

In general, there is no reason to expect solutions to (Homogeneous) to be smooth. Therefore, we need to enforce such regularity by applying the forward map to a smooth enough function. However, computing the fractional Laplacian of a function by hand is an extremely difficult task. Thus, the example we present corresponds to a one-dimensional domain, where this computation can be at least partially carried out. Let $s > 1/2$ and consider problem (Homogeneous) posed on the interval $\Omega = (-1, 1)$, with exact solution $u(x) = \sin(\pi x)\chi_{(-1,1)}(x)$, namely:

$$\begin{cases} (-\Delta)^s u = (-\Delta)^s \sin(\pi\cdot) & \text{in } (-1,1) \\ u = 0 & \text{in } (-\infty,-1)\cup(1,\infty). \end{cases} \tag{3.5.2}$$

The solution for this problem is smooth in $(-1, 1)$ but belongs to $H^{3/2-\varepsilon}(\mathbb{R})$. According to (3.3.2), the convergence in the energy norm is expected to be of order $2 - s$. Some results are shown in Table 3.3, where it can be seen that these orders are indeed achieved.

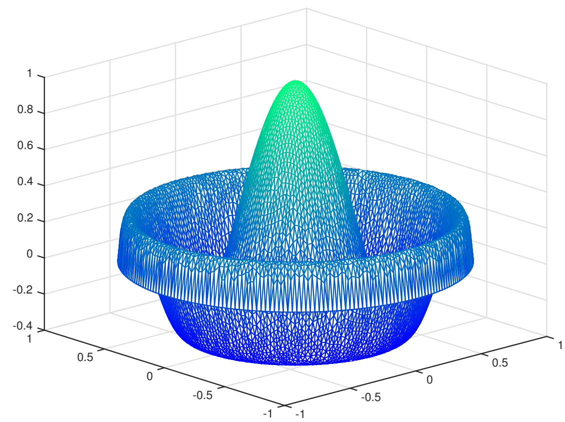

Figure 3.1: Finite element solution to Example 3.5.3 with $s = 0.75$, computed on a mesh containing about 16000 triangles.

Value of s	Order (in h)
0.6	1.4028
0.7	1.2993
0.8	1.2002
0.9	1.1002

Table 3.3: Rates of convergence for uniform meshes in the norm $\|\cdot\|_{\mathbb{V}}$ for problem (3.5.2) and $s \in (1/2, 1)$.

3.5.2 Graded meshes

Here we explain how to build appropriate graded meshes and display the output of computations involving them for the first two examples described in the previous subsection. These show good agreement with our predictions about convergence in the energy norm, and an enhanced order of convergence (with respect to uniform meshes) in the L^2-norm.

Since the examples we are going to consider only involve the unit ball, we show how to build graded meshes in this case. We pick a positive integer M and define an increasing sequence of radii $r_i := 1 - \left(1 - \frac{i}{M}\right)^{\mu}$ for $1 \leq i \leq M$. We can mesh the complete disk Ω by meshing each subdomain $\Omega_i = \{x \in \Omega : r_{i-1} < |x| < r_i\}$ with uniform elements of size $h_T = h_i = r_i - r_{i-1}$ (see Figure 3.2). The previous construction ensures that conditions (Regularity), (Local quasi-uniformity) and hypotheses (H) hold, taking $h = 1/M$.

Remark 3.5.5. It is simple to verify that the total number of nodes is $N \sim \sum_{i=1}^{M} r_i/h_i$, because essentially, for every $i \in \{1, \ldots, M\}$ we are dividing a circumference of ra-

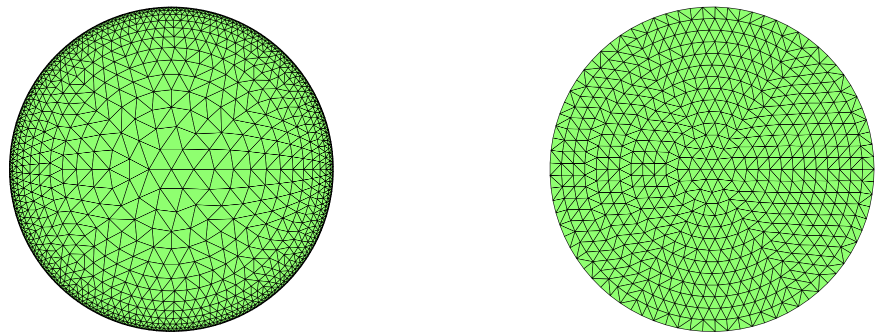

Figure 3.2: Graded mesh with $M = 15$ and $\mu = 2$ (left panel) and uniform mesh with $M = 15$ and $\mu = 1$ (right panel).

dius r_i with nodes within a distance h_i. Moreover, for M large enough we may write asymptotically

$$h_i \sim \frac{1}{M}\left(1 - \frac{i-1}{M}\right)^{\mu-1},$$

and thus, writing $j = M - i + 1$, the number of nodes is of order of

$$N \sim M \sum_{j=1}^{M} \frac{1 - \left(\frac{j}{M}\right)^{\mu}}{\left(\frac{j}{M}\right)^{\mu-1}}. \tag{3.5.3}$$

In first place, if $\mu < 2$, then we write (3.5.3) as

$$N \sim M^2 \sum_{j=1}^{M} \frac{1 - \left(\frac{j}{M}\right)^{\mu}}{\left(\frac{j}{M}\right)^{\mu-1}} \frac{1}{M},$$

and the sum in the right hand side above is just a Riemann sum of the function $x \mapsto \frac{1-x^{\mu}}{x^{\mu-1}}$ in the interval $(0, 1)$. Since $\mu < 2$, this integral is convergent and therefore the number of nodes is $N \sim M^2 \sim h^{-2}$.

Secondly, if $\mu \geq 2$, we rewrite (3.5.3) as

$$N \sim M^{\mu} \left(\sum_{j=1}^{M} j^{1-\mu}\right) - \sum_{j=1}^{M} j.$$

The sum in parentheses is convergent, and the second is of order M^2. So, in this case the number of nodes behaves like $N \sim M^{\mu} \sim h^{-\mu}$.

In view of remarks 3.4.4 and 3.5.5, we set the parameter μ equal to 2 when constructing graded meshes.

Example 3.5.6. Consider the same problem as in Example 3.5.2, and utilize finite element approximations with graded meshes as described above, with $\mu = 2$. Table 3.4 shows numerical results for this case. The accuracy is in full agreement with that predicted in Theorem 3.4.2 for $s \in (1/2, 1)$, and the same order is observed for $s \in (0, 1/2]$.

Value of s	Order (in h)
0.1	1.066
0.2	1.040
0.3	1.019
0.4	1.002
0.5	1.066
0.6	1.051
0.7	0.990
0.8	0.985
0.9	0.977

Table 3.4: (Graded Meshes) Rates of convergence in the energy norm for (2.3.10) and $s \in (0, 1)$. The mesh parameter h behaves like $N^{-1/2}$, N being the number of nodes.

Example 3.5.7. Consider the same problem as in Example 3.5.3. We compute finite element solutions utilizing graded meshes with $\mu = 2$. Errors in the L^2-norm are displayed in Table 3.5. Although our convergence estimates for graded meshes do not involve convergence in this norm, an enhanced order of convergence is observed.

Mesh parameter h	$\|u - u_h\|_{L^2(\Omega)}$ $(s = 0.25)$	$\|u - u_h\|_{L^2(\Omega)}$ $(s = 0.75)$
0.0769	0.0323	0.01875
0.0667	0.0251	0.01373
0.0556	0.0183	0.00920
0.0476	0.0141	0.00662
0.0417	0.0112	0.00503

Table 3.5: (Graded meshes) Errors in the L^2-norm for $s = 0.25$ and $s = 0.75$ in Example 3.5.7. The estimated orders of convergence with respect to the mesh size are, respectively, 1.7222 and 2.1501. The mesh parameter h behaves like $N^{-1/2}$, N being the number of nodes.

Resumen del capítulo

En este capítulo realizamos un análisis de elementos finitos completo para el problema con condiciones de tipo Dirichlet homogéneas para el laplaciano fraccionario,

$$\begin{cases} (-\Delta)^s u = f & \text{en } \Omega, \\ u = 0 & \text{en } \Omega^c. \end{cases}$$

En la **Sección 3.1** introducimos los espacios discretos que utilizamos, establecemos la formulación del problema discreto y recordamos algunas propiedades básicas y necesarias para nuestro análisis posterior.

Para obtener estimaciones de error, nuestro enfoque consiste en considerar un operador de interpolación adecuado en un espacio de elementos finitos $\mathbb{V}_h$ y estimar el error de interpolación. La **Sección 3.2** analiza la estabilidad y propiedades de aproximación del operador de Scott-Zhang en el marco de espacios de Sobolev fraccionarios. Estas propiedades son utilizadas para demostrar el orden de convergencia óptimo de las aproximaciones de elementos finitos que realizamos, en el contexto de tanto espacios estándar como con peso.

En primer lugar, en la **Sección 3.3** deducimos órdenes de convergencia del esquema discreto propuesto tanto en la norma de energía como en la de $L^2(\Omega)$. Lo primero es expresado en términos de normas Hölder del dato f. Lo segundo es realizado mediante un argumento de dualidad de Aubin-Nitsche, por lo que es necesario asumir la suavidad del dominio, de modo de que valgan las estimaciones de regularidad descritas en la Sección 2.2.

Posteriormente, en la **Sección 3.4** adaptamos la teoría que hemos desarrollado a los espacios fraccionarios con pesos. Aprovechamos nuestras nuevas estimaciones introduciendo aproximaciones en una familia de mallas apropiadamente graduadas, que nos permiten duplicar el orden de convergencia en la norma de energía.

Implementamos el método de elementos finitos en una y en dos dimensiones, donde empleamos mallas tanto uniformes como graduadas. Presentamos experimentos numéricos en la **Sección 3.5**, mostrando órdenes de convergencia en total acuerdo con nuestras predicciones teóricas.

Chapter 4

Fractional eigenvalue problem

Since the development of fractional quantum mechanics by Laskin [78], the study of eigenproblems involving the fractional Laplacian has been thoroughly pursued by the physics' community. The consideration of Lévy-like quantum mechanical paths, instead of Brownian paths, leads to a Schrödinger equation where the Laplace operator is substituted by the fractional Laplacian.

Letting $\Omega \subset \mathbb{R}^n$ be a bounded domain, in this chapter we analyze the fractional eigenvalue problem with homogeneous Dirichlet conditions. Namely, we seek a positive number λ (eigenvalue) and a function $u \not\equiv 0$ (eigenfunction) such that

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{Eigenproblem}$$

where $s \in (0,1)$ and $(-\Delta)^s$ is the fractional Laplacian (1.1.1). Recall that, due to the fact that pointwise values of $(-\Delta)^s u$ depend on the value of u over the whole space, Dirichlet data need to be imposed on the complement of Ω.

In fractional quantum mechanics, (Eigenproblem) corresponds to the steady-state, linear fractional Schrödinger equation with an infinite-potential well. The eigenfunctions are usually called the stationary states; in particular, the eigenfunction with the smallest eigenvalue is called the *ground* state, and those with the larger eigenvalues are called the *excited* states. The singular behavior of eigenfunctions near $\partial\Omega$ has been mentioned in many physics papers (see, for example, [44] and the references therein), but only very recently has been addressed mathematically [57, 93].

Even if Ω is an interval, it is very challenging to obtain closed analytical expressions for the eigenvalues and eigenfunctions of the fractional Laplacian. This motivates the utilization of discrete approximations for this problem (see, for example, [44, 74, 109]); here we consider a finite element method. Unlike preexisting schemes, the method we propose is flexible enough to deal with a variety of domains, and enables us to provide estimates and sharp upper bounds for eigenvalues even for non convex domains.

Moreover, as a consequence of our numerical experiments in the L-shaped domain $\Omega = [-1,1]^2 \setminus [0,1]^2$, we conjecture that the first eigenfunction for this domain is as regular as the first one in any smooth domain.

The discussion in this chapter begins by describing some basic theoretical aspects of the problem under consideration, setting the notation and utilizing the theory developed in Section 2.3 to study the Sobolev regularity of eigenfunctions. Afterwards, Section 4.2 describes the discrete setting we utilize and recalls some properties that are useful for our analysis.

The study of convergence of solutions begins in Section 4.3. There, it is proved that the scheme we consider leads to good approximation of all eigenspaces and does not generate spurious modes. Then, we show the order of convergence for eigenvalues and eigenfunctions, both in the energy and the L^2-norm. The first part of Section 4.4 is devoted to the Babuška-Osborn theory, that allows to directly derive these convergence rates, while the second provides an elementary and direct proof.

Finally, we perform several numerical experiments and compare our results with previous work by other authors. The results from Section 4.5 are in good agreement with the theory and the eigenvalue estimates we obtain are consistent with those available in the literature. Furthermore, we are able to sharpen preexisting upper bounds for some domains.

4.1 Theoretical aspects

In this section we review some basic features of (Eigenproblem). Besides setting the weak formulation of the problem, we recall some useful characterizations of the eigenvalues and analyze regularity of eigenfunctions.

Similarly to the homogeneous Dirichlet equation, we set the variational space

$$(\mathbb{V}, \|\cdot\|_{\mathbb{V}}) = \left(\widetilde{H}^s(\Omega), \sqrt{\frac{C(n,s)}{2}}\, |\cdot|_{H^s(\mathbb{R}^n)} \right).$$

Recall the bilinear forms (1.2.4) and (1.2.5). Multiplying (Eigenproblem) by a test function v and applying the integration by parts formula (1.2.9), we obtain the following variational formulation of the fractional eigenvalue problem:

$$\text{find } (u,\lambda) \in (\mathbb{V} \setminus \{0\}) \times (0,+\infty) \text{ such that } \langle u, v\rangle_{\mathbb{V}} = \lambda (u,v)_{L^2(\Omega)} \quad \text{for all } v \in \mathbb{V}. \tag{4.1.1}$$

It is well-known (see, for example, [98]) that there is an infinite sequence of eigenvalues $\{\lambda^{(k)}\}_{k\in\mathbb{N}}$,

$$0 < \lambda^{(1)} < \lambda^{(2)} \le \cdots \le \lambda^{(k)} \le \cdots, \quad \lambda^{(k)} \to \infty \text{ as } k \to \infty,$$

where the same eigenvalue can be repeated several times according to its multiplicity. Further, the first eigenvalue $\lambda^{(1)}$ is simple (see, for example, [98]). The corresponding eigenfunctions $u^{(k)}$, normalized by $\|u^{(k)}\|_{L^2(\Omega)} = 1$, form a complete orthonormal set in $L^2(\Omega)$. Also in [98], the authors prove that for any $k \in \mathbb{N}$ the eigenvalues of (4.1.1) can be characterized as:

$$\lambda^{(k)} = \min \left\{ \frac{\|u\|_{\mathbb{V}}^2}{\|u\|_{L^2(\Omega)}^2} : u \in \mathbb{V}^{(k)} \setminus \{0\} \right\},$$

where $\mathbb{V}^{(1)} = \mathbb{V}$ and

$$\mathbb{V}^{(k)} = \left\{ u \in \mathbb{V} \colon \langle u, u^{(j)} \rangle_{\mathbb{V}} = 0 \quad \forall j = 1, \ldots, k-1 \right\}$$

for all $k \geq 2$. Therefore, by the min-max theorem,

$$\lambda^{(k)} = \min_{E \in S^{(k)}} \max_{u \in E} \frac{\|u\|_{\mathbb{V}}^2}{\|u\|_{L^2(\Omega)}^2},$$

where $S^{(k)}$ denotes the set of all $k-$dimensional subspaces of $\mathbb{V}$.

Relying on the theory from Section 2.3 we immediately deduce Sobolev regularity of eigenfunctions; for this purpose we require the domain Ω to be Lipschitz and satisfying the exterior ball condition. Observe however that, independently of the smoothness of Ω, eigenfunctions are not expected to be any smoother than $H^{s+1/2-\varepsilon}(\Omega)$. In first place, we show that these are smooth in the interior of Ω.

Proposition 4.1.1. *If Ω is a Lipschitz domain satisfying the exterior ball condition then any solution of* (Eigenproblem) *is in $C^\infty(\Omega)$.*

Proof. By [97, Proposition 4] since Ω is a Lipschitz domain satisfying the exterior ball condition, then $u^{(k)} \in L^\infty(\Omega)$ for all $k \in \mathbb{N}$ and thus, by Theorem 2.3.1, $u^{(k)} \in C^s(\mathbb{R}^n)$ for all $k \in \mathbb{N}$.

Let $\beta > 0$ be such that neither β nor $\beta + ks$ is an integer for any $k \in \mathbb{N}$. Then, applying Theorem 2.3.2 and using that

$$\|u\|_{\beta+s}^{(s)} \leq C \|u\|_{\beta+2s}^{(-s)}$$

we deduce that $u \in C^{\beta+2s}(\Omega)$ and $\|u\|_{\beta+s}^{(s)} < \infty$. Subsequent applications of this argument give that $u \in C^{\beta+sk}(\Omega)$ for any $k \in \mathbb{N}$ and thus $u \in C^\infty(\Omega)$. □

Sobolev regularity of eigenfunctions follows easily from the interior Hölder regularity and the theory developed in Section 2.3.

Proposition 4.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain satisfying the exterior ball condition and let u be an eigenfunction of $(-\Delta)^s$ in Ω with homogeneous Dirichlet boundary conditions. Then, $u \in \widetilde{H}^{s+1/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.*

Moreover, considering the weighted Sobolev scale (cf. Definition 1.5.11) it also holds that $u \in H^{1+s-\varepsilon}_{1/2-2\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$.

Proof. Since by Proposition 4.1.1 it holds that $f = \lambda u \in C^\infty(\Omega)$, the claim follows easily applying either Theorem 2.3.6, 2.3.10 or 2.3.11. □

Remark 4.1.3. Hölder boundary regularity of eigenfunctions is a more sticky question. For smooth domains, estimates of this type are derived in [57, 93]. Letting d be a smooth function that behaves like $d(x, \partial\Omega)$ near the boundary of Ω, it is proved that any eigenfunction u lies in the space $d^s C^{2s(-\varepsilon)}(\overline{\Omega})$, where the ε is active only if $s = 1/2$ and that $\frac{u}{d^s}$ does not vanish near $\partial\Omega$. This also shows that no further regularity than $H^{s+1/2-\varepsilon}(\Omega)$ should be expected for eigenfunctions in smooth domains.

4.2 Discrete problem

The discrete scheme we utilize to approximate (Eigenproblem) is a variant of the one we analyzed in Chapter 3. Indeed, we consider a conforming finite element method, with continuous, piecewise linear functions. In this section we recall this discrete framework and some useful facts from the previous chapter. We also establish certain basic properties of the discrete eigenvalue problems we deal with.

As in Section 3.1, let $\mathcal{T}_h$ be a family of triangulations of Ω satisfying

$$\exists \sigma > 0 \text{ such that } h_T \leq \sigma \rho_T,$$

for any element $T \in \mathcal{T}_h$, where h_T is the diameter of T and ρ_T is the radius of the largest ball contained in T. This is the only requirement we need to impose to our family of triangulations. When considering graded meshes, we also utilize hypothesis (H): there is a constant $\mu \geq 1$ such that for every $T \in \mathcal{T}_h$,

$$\begin{array}{ll} \text{if } T \cap \partial\Omega \neq \emptyset, & \text{then } h_T \leq C(\sigma) h^\mu; \\ \text{otherwise,} & h_T \leq C(\sigma) h \, d(T, \partial\Omega)^{(\mu-1)/\mu}. \end{array} \tag{H}$$

Further, we restrict the analysis to the optimal grading choice $\mu = 2 - \varepsilon$, that ensures that the mesh size parameter h scales as $N^{-1/2}$, where N is the number of nodes, whilst being associated to maximal regularity in weighted Sobolev spaces (cf. Remark 3.4.4).

We consider continuous piecewise linear functions on $\mathcal{T}_h$, namely

$$\mathbb{V}_h = \left\{ v \in \mathbb{V} \colon v\big|_T \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}_h \right\}.$$

Then, the Galerkin approximation we utilize consists in looking for $\lambda_h \in \mathbb{R}$ and $u_h \in \mathbb{V}_h$ such that $u_h \not\equiv 0$ and

$$\langle u_h, v\rangle_{\mathbb{V}} = \lambda_h (u_h, v)_{L^2(\Omega)} \quad \forall v \in \mathbb{V}_h. \tag{4.2.1}$$

We can order the discrete eigenvalues of (4.1.1) as follows

$$0 < \lambda_h^{(1)} \le \lambda_h^{(2)} \le \cdots \le \lambda_h^{(k)} \le \cdots \le \lambda_h^{(\dim \mathbb{V}_h)},$$

where the same eigenvalue is repeated according to its multiplicity. The corresponding eigenfunctions $u_h^{(k)}$ (normalized by $\|u_h^{(k)}\|_{L^2(\Omega)} = 1$) form an orthonormal set in $L^2(\Omega)$. Moreover, a min-max characterization for eigenvalues of the discrete problem holds,

$$\lambda_h^{(k)} = \min_{E_h \in S_h^{(k)}} \max_{u_h \in E_h} \frac{\|u_h\|_{\mathbb{V}}^2}{\|u_h\|_{L^2(\Omega)}^2}.$$

Above, $S_h^{(k)}$ denotes the set of all k dimensional subspaces of $\mathbb{V}_h$.

Remark 4.2.1. Since for all $h > 0$ the discrete space $\mathbb{V}_h$ is a subset of the continuous space $\mathbb{V}$, the min-max characterization for both continuous and discrete eigenvalues implies that for every $k \in \{1, \ldots, \dim \mathbb{V}_h\}$,

$$\lambda^{(k)} \le \lambda_h^{(k)}.$$

As we show below, an important tool in the analysis of finite element approximations to (Eigenproblem) is the discrete solution of (Homogeneous) in the same discrete setting. Namely, we make use of the approximation properties of the operator $\Pi_h \colon \mathbb{V} \to \mathbb{V}_h$, such that $\Pi_h v$ is the energy-norm projection of the solution of (Homogeneous) with right hand side $(-\Delta)^s v$ over the discrete space $\mathbb{V}_h$. Explicitly, $\Pi_h v$ is the only element of $\mathbb{V}_h$ satisfying

$$\langle \Pi_h v, v_h\rangle_{\mathbb{V}} = ((-\Delta)^s v, v_h)_{L^2(\Omega)} \quad \forall v_h \in \mathbb{V}_h.$$

In particular, for every $v \in \mathbb{V}$ it holds that

$$\|v - \Pi_h v\|_{\mathbb{V}} = \inf_{v_h \in \mathbb{V}_h} \|v - v_h\|_{\mathbb{V}}. \tag{4.2.2}$$

We also require a quasi-interpolator like the Scott-Zhang operator analyzed in Section 3.2. Denoting as before by I_h such an operator, it holds that (cf. (3.3.1))

$$\|v - I_h v\|_{\mathbb{V}} \le C(n, s, \sigma, \ell) h^{\ell - s} |v|_{H^\ell(\Omega)} \; \forall v \in H^\ell(\Omega) \; (\ell > \max\{s, 1/2\}), \tag{4.2.3}$$

and when the family of meshes satisfies (H) with grading parameter $\mu = 2$ (cf. (3.4.5)),

$$\|v - I_h v\|_{\mathbb{V}} \le C(n, s, \sigma, \ell) h^{\ell - s - \varepsilon} |v|_{H^\ell_{1/2-\varepsilon}(\Omega)} \; \forall v \in H^\ell(\Omega) \; (\ell > s > 1/2). \tag{4.2.4}$$

4.3 Convergence in gap distance

In this section we show that the discrete approximations to (Eigenproblem) that we are considering are convergent under rather weak assumptions on the domain Ω. Indeed, we prove that all eigenspaces are well approximated and that there are no spurious solutions. This idea of convergence is made precise by the notion of gap between Hilbert spaces, that we define below. A discussion on the order of convergence, that requires further assumptions on the domain boundary, is postponed to next section.

We are concerned about working with the minimum smoothness on the domains as possible. For that purpose, we introduce the following class.

Definition 4.3.1. A domain Ω is called a fractional extension domain if, for every $\sigma \in (0,1)$ there is a continuous extension map $E : H^\sigma(\Omega) \to H^\sigma(\mathbb{R}^n)$.

Remark 4.3.2. A characterization of fractional extension domains was given in [108]. A set Ω is an extension domain if and only if there is a constant $C > 0$ such that for all $x \in \Omega$ and all $r \in (0,1]$,

$$|\Omega \cap B(x,r)| \geq C r^n \quad \forall x \in \Omega.$$

Naturally, the class of extension domains is larger than the class of Lipschitz domains. A direct proof of the existence of an extension map for Lipschitz domains can be found in [40, Theorem 5.4].

Our interest in extension domains is given by the following auxiliary result, that ensures the compactness of the embedding of the energy space in $L^2(\mathbb{R}^n)$.

Lemma 4.3.3 ([40, Theorem 7.1])**.** *Let Ω be a bounded fractional extension domain, then the embedding $\mathbb{V} \hookrightarrow L^2(\mathbb{R}^n)$ is compact.*

We also consider the solution operators of the continuous and discrete problems, $T : L^2(\Omega) \to \mathbb{V}$ and $T_h : L^2(\Omega) \to \mathbb{V}_h$, defined as follows. Given $f \in L^2(\Omega)$, we set $Tf \in \mathbb{V}$ as the unique solution of

$$\langle Tf, v\rangle_{\mathbb{V}} = (f, v)_{L^2(\Omega)} \quad \forall v \in \mathbb{V}, \tag{4.3.1}$$

and $T_h f \in \mathbb{V}_h$ as the unique solution of

$$\langle T_h f, v_h\rangle_{\mathbb{V}} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in \mathbb{V}_h.$$

Observe that if (u, λ) is an eigenpair, then $T(\lambda u) = u$ and, in connection to the operator Π_h introduced in the previous section, it also holds that $T_h(\lambda u) = \Pi_h u$.

Next, we make precise the notion of convergence we use in this section.

Definition 4.3.4. Let E, F be two subspaces of a certain Hilbert space H. Define the quantity $\delta(E, F)$ by

$$\delta(E,F) \coloneqq \sup_{u\in E, \|u\|_H=1} \inf_{v\in F} \|u-v\|_H.$$

Then, the gap between E and F is

$$\hat{\delta}(E,F) \coloneqq \max\{\delta(E,F), \delta(F,E)\}.$$

Taking $H = \mathbb{V}$ in the definition above, we are in position to give a notion of convergence. Essentially, the definition we give below, introduced in [17], states that all solutions are well approximated and no spurious eigenvalues pollute the spectrum.

Definition 4.3.5. We say that the discrete eigenvalue problem (4.2.1) converges to the continuous (4.1.1) in gap distance if, for any $\varepsilon > 0$ and $k > 0$, there is $h_0 > 0$ such that

$$\max_{1\le i\le m(k)} |\lambda^{(i)} - \lambda_h^{(i)}| \le \varepsilon, \qquad \hat{\delta}\left(\bigoplus_{i=1}^{m(k)} E^{(i)}, \bigoplus_{i=1}^{m(k)} E_h^{(i)}\right) \le \varepsilon,$$

for all $h < h_0$. Here, $m(k)$ is the dimension of the space spanned by the first distinct k eigenspaces and $E^{(i)}$ and $E_h^{(i)}$ are the eigenspace and the discrete eigenspace associated to $\lambda^{(i)}$ and $\lambda_h^{(i)}$, respectively.

The main result of this section reads as follows.

Theorem 4.3.6. *Let Ω be a bounded fractional extension domain and $s \in (0,1)$, then the discrete eigenvalue problem* (4.2.1) *converges to the continuous* (4.1.1) *in gap distance.*

The next proposition yields a fundamental characterization of gap convergence.

Proposition 4.3.7 (See [15, Proposition 7.4 and Remark 7.5] and [68, Chapter IV])**.** *If the operator T is compact, then* (4.2.1) *converges to* (4.1.1) *in gap distance if and only if the norm convergence*

$$\|T - T_h\|_{\mathcal{L}(L^2(\Omega),\mathbb{V})} \to 0 \text{ as } h \to 0.$$

holds true.

Hence, in order to prove Theorem 4.3.6, it suffices to show the compactness of the solution operator T and the norm convergence of T_h towards T as operators from $L^2(\Omega)$ to $\mathbb{V}$. The next two lemmas are devoted to this purpose.

Lemma 4.3.8. *The operator $T : L^2(\Omega) \to \mathbb{V}$ is compact.*

Proof. Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^2(\Omega)$. Then, there exists a subsequence (that we still denote by $\{f_k\}$) and $f \in L^2(\Omega)$ such that $f_k \rightharpoonup f$ weakly in $L^2(\Omega)$. Taking $v = Tf_k$ in (4.3.1), we obtain

$$\|Tf_k\|_{\mathbb{V}}^2 = (f_k, Tf_k)_{L^2(\Omega)} \leq C\|Tf_k\|_{L^2(\Omega)} \quad \forall k \in \mathbb{N}.$$

Therefore, by the Poincaré inequality (cf. Proposition 1.2.8), it holds that $\{Tf_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{V}$. Thus, there exists a subsequence of $\{f_k\}_{k\in\mathbb{N}}$ (still denoted by $\{f_k\}_{k\in\mathbb{N}}$) and $u \in \mathbb{V}$ such that $Tf_k \rightharpoonup u$ weakly in $\mathbb{V}$. Hence

$$\langle u, v\rangle_{\mathbb{V}} = \lim_{k\to\infty}\langle Tf_k, v\rangle_{\mathbb{V}} = \lim_{k\to\infty}(f_k, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in \mathbb{V},$$

that is, $u = Tf$.

On the other hand, since by Lemma 4.3.3 the embedding $\mathbb{V} \hookrightarrow L^2(\mathbb{R}^n)$ is compact, passing if necessary to a subsequence we may assume

$$\begin{aligned} &Tf_k \rightharpoonup Tf \text{ weakly in } \mathbb{V},\\ &Tf_k \to Tf \text{ strongly in } L^2(\mathbb{R}^n).\end{aligned}$$

Then,

$$\|Tf_k\|_{\mathbb{V}}^2 = (f_k, Tf_k)_{L^2(\Omega)} \to (f, Tf)_{L^2(\Omega)} = \|Tf\|_{\mathbb{V}}^2.$$

Since the space $\mathbb{V}$ is uniformly convex, it follows that

$$Tf_k \to Tf \text{ strongly in } \mathbb{V},$$

and thus T is a compact operator. □

Lemma 4.3.9. *The following norm convergence holds true:*

$$\|T - T_h\|_{\mathcal{L}(L^2(\Omega),\mathbb{V})} \to 0 \textit{ as } h \to 0.$$

Proof. For each h, take $f_h \in L^2(\Omega)$ such that $\|f_h\|_{L^2(\Omega)} = 1$ and

$$\sup_{\|f\|_{L^2(\Omega)}=1} \|Tf - T_hf\|_{\mathbb{V}} = \|Tf_h - T_hf_h\|_{\mathbb{V}}.$$

Then, to prove the result, it is enough to show that for any sequence $h_k \to 0$ there is a subsequence $\{h_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\|Tf_{h_{k_j}} - T_{h_{k_j}}f_{h_{k_j}}\|_{\mathbb{V}} \to 0 \quad \text{as } j \to \infty.$$

Let $\{h_k\}_{k\in\mathbb{N}}$ be a sequence such that $h_k \to 0$. It follows from $\|f_{h_k}\|_{L^2(\Omega)} = 1$ for all $k \in \mathbb{N}$ that there exist a subsequence $\{f_{h_{k_j}}\}_{j\in\mathbb{N}}$ and $f \in L^2(\Omega)$ such that $f_{h_{k_j}} \rightharpoonup f$

weakly in $L^2(\Omega)$. Being finite-rank operators, the discrete solution maps T_h are compact. Thus, passing if necessary to a subsequence, we may assume

$$T_{h_{k_j}} f_{h_{k_j}} \rightharpoonup v \text{ weakly in } \mathbb{V},$$
$$T_{h_{k_j}} f_{h_{k_j}} \to v \text{ strongly in } L^2(\mathbb{R}^n).$$

On the other hand, it follows from either (4.2.3) or (4.2.4) that

$$I_h \varphi \to \varphi \text{ strongly in } \mathbb{V},$$
$$I_h \varphi \to \varphi \text{ strongly in } L^2(\mathbb{R}^n),$$

for any $\varphi \in C_0^\infty(\Omega)$. Therefore,

$$\langle v, \varphi \rangle_{\mathbb{V}} = \lim_{j \to \infty} \left\langle T_{h_{k_j}} f_{h_{k_j}}, I_{h_{k_j}} \varphi \right\rangle_{\mathbb{V}} = \lim_{j \to \infty} (f_{h_{k_j}}, I_{h_{k_j}} \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in C_0^\infty(\Omega),$$

which means that $v = Tf$. Then,

$$\|T f_{h_{k_j}} - T_{h_{k_j}} f_{h_{k_j}}\|_{\mathbb{V}}^2 = (f_{h_{k_j}}, T f_{h_{k_j}} - T_{h_{k_j}} f_{h_{k_j}})_{L^2(\Omega)} \to 0.$$

□

Combining Proposition 4.3.7 and lemmas 4.3.8 and 4.3.9, the proof of Theorem 4.3.6 is concluded.

4.4 Order of convergence

The results from the previous section ensure the convergence in gap distance of the discrete problem towards (Eigenproblem), but do not give any information about the speed of such a convergence. Assuming certain regularity on the domain Ω, we are able to deduce orders of convergence of the finite element approximations. In first place, we set our problem in the framework of the Babuška-Osborn theory; this enables us to estimate the convergence of eigenvalues and eigenfunctions in the energy norm. Afterwards, we give an alternative, self-contained proof of the same results, and we also bound eigenfunction errors in the L^2-norm.

4.4.1 A proof via Babuška-Osborn theory

The Babuška-Osborn theory [10, Chapter II] allows to deduce the order of convergence for spectral approximation of variationally formulated eigenvalue problems. Recall that the norm convergence $T_h \to T$ and the compactness of the operator T imply convergence in gap distance. The Babuška-Osborn theory gives an estimate in the order of convergence of eigenvalues and eigenfunctions (in the energy norm). We start giving an overview of the main results of this theory.

Let H_1 and H_2 be two Hilbert spaces, with inner products and norms $(\cdot,\cdot)_1$, $\|\cdot\|_1$ and $(\cdot,\cdot)_2$, $\|\cdot\|_2$, respectively. Let a be a bilinear form on $H_1 \times H_2$ such that

$$\begin{aligned} & a(u,v) \le C_1 \|u\|_1 \|v\|_2, \quad \forall u \in H_1,\ v \in H_2, \\ & \inf_{\|u\|_1=1} \sup_{\|v\|_2=1} |a(u,v)| = \alpha > 0, \\ & \sup_u |a(u,v)| > 0, \quad \forall v \in H_2 \setminus \{0\}. \end{aligned} \tag{4.4.1}$$

Further, let $\|\cdot\|_1'$ be a norm on H_1 such that every bounded sequence with respect to $\|\cdot\|_1$ has a Cauchy subsequence in $\|\cdot\|_1'$, and let b be a bilinear form on $H_1 \times H_2$ satisfying

$$|b(u,v)| \le \|u\|_1' \|v\|_2 \quad \forall u \in H_1,\ v \in H_2. \tag{4.4.2}$$

Then, [10] develops an abstract theory regarding approximations of the problem

$$\text{find } (u,\lambda) \in (H_1 \setminus \{0\}) \times \mathbb{C} \text{ such that } a(u,v) = \lambda b(u,v) \quad \forall v \in H_2.$$

Specifically, let $S_{1,h} \subset H_1$ and $S_{2,h} \subset H_2$ be finite dimensional subspaces that satisfy

$$\begin{aligned} & \inf_{u \in S_{1,h},\ \|u\|_1=1} \sup_{v \in S_{2,h},\ \|v\|_2=1} |a(u,v)| = \beta(h) > 0, \\ & \sup_{u \in S_{1,h}} |a(u,v)| > 0, \quad \forall v \in S_{2,h} \setminus \{0\}, \end{aligned} \tag{4.4.3}$$

and consider the discrete eigenproblem: find $(u_h, \lambda_h) \in (S_{1,h} \setminus \{0\}) \times \mathbb{C}$ such that

$$a(u_h, v_h) = \lambda_h b(u_h, v_h) \quad \forall v_h \in S_{2,h}. \tag{4.4.4}$$

We also need to make use of the adjoint eigenvalue problem:

$$\text{find } (v,\lambda) \in (H_2 \setminus \{0\}) \times \mathbb{C} \text{ such that } a(u,v) = \lambda b(u,v) \quad \forall u \in H_1. \tag{4.4.5}$$

Finally, we assume that the convergence of the solution operators $T_h \to T$ in norm, or equivalently (cf. Proposition 4.3.7), that convergence in gap distance holds. Then, theorems 8.1 and 8.2 from [10] state the following.

Theorem 4.4.1. *In the setting described above, let $\lambda^{(k)}$ be an eigenvalue of multiplicity m (that is, $\lambda^{(k)} = \lambda^{(k+1)} = \cdots = \lambda^{(k+m-1)}$ and $\lambda^{(i)} \neq \lambda^{(k)}$ for $i \neq k, \ldots, k+m-1$), with associated eigenspace E. Also, let E^* be the eigenspace corresponding to $\lambda^{(k)}$ in the adjoint problem* (4.4.5)*. Consider the Galerkin approximations given by* (4.4.4)*, and let $\hat{\lambda}$ be the arithmetic mean of the m discrete eigenvalues that converge towards $\lambda^{(k)}$.*

If $u^{(k)}$ is an eigenfunction associated to $\lambda^{(k)}$, there is

$$\left\{w_h^{(k)}\right\} \subset E_h^{(k)} \oplus \ldots \oplus E_h^{(k+m-1)}$$

such that

$$\|u^{(k)} - w_h^{(k)}\|_1 \leq C \sup_{u \in E, \|u\|_1 = 1} \inf_{u_h \in S_{1,h}} \|u - u_h\|_1.$$

Moreover, it holds that

$$0 \leq \hat{\lambda} - \lambda^{(k)} \leq C\beta(h)^{-1} \sup_{u \in E, \|u\|_1 = 1} \inf_{u_h \in S_{1,h}} \|u - u_h\|_1 \sup_{v \in E^*, \|v\|_2 = 1} \inf_{v_h \in S_{2,h}} \|v - v_h\|_2.$$

With the notation of the theorem above, we consider $H_1 = H_2 = \mathbb{V}$ and the bilinear forms $a(\cdot,\cdot) = \langle \cdot,\cdot \rangle_{\mathbb{V}}$, $b(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$. Then, the continuity and coercivity of the form a imply that conditions (4.4.1) are satisfied, and taking $\|\cdot\|_1' = \|\cdot\|_{L^2(\mathbb{R}^n)}$, the continuity of b (4.4.2) holds as well. Also, the coercivity of a ensures that (4.4.3) holds with β independent of h, and the symmetry of a implies that the adjoint problem (4.4.5) equals the original eigenproblem.

Next, we assume Ω to be a Lipschitz domain satisfying the exterior ball condition and the family of meshes to be shape-regular. We deduce from Proposition 4.1.2 and (4.2.3) that, given $\varepsilon > 0$, there exists a constant $C > 0$ such that for every eigenfunction u it holds that

$$\inf_{u_h \in \mathbb{V}_h} \|u - u_h\|_{\mathbb{V}} \leq C h^{1/2-\varepsilon}.$$

Further, if $s \in (1/2, 1)$ and the meshes are graded according to (H) with $\mu = 2$, due to Proposition 4.1.2 and (4.2.4) it holds that

$$\inf_{u_h \in \mathbb{V}_h} \|u - u_h\|_{\mathbb{V}} \leq C h^{1-2\varepsilon}.$$

We conclude the order of convergence for our discrete scheme for the fractional eigenvalue problem with Dirichlet boundary conditions.

Theorem 4.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain satisfying the exterior ball condition and let $\lambda^{(k)}$ be an eigenvalue of multiplicity m (that is, $\lambda^{(k)} = \lambda^{(k+1)} = \cdots = \lambda^{(k+m-1)}$ and $\lambda^{(i)} \neq \lambda^{(k)}$ for $i \neq k, \ldots, k+m-1$). Consider the Galerkin approximations given by (4.2.1) on a shape-regular familiy of meshes.*

1. *For any $\varepsilon > 0$, there exists a positive constant C independent of h such that if $u^{(k)}$ is an eigenfunction associated to $\lambda^{(k)}$, there is*

$$\left\{w_h^{(k)}\right\} \subset E_h^{(k)} \oplus \ldots \oplus E_h^{(k+m-1)}$$

satisfying

$$\|u^{(k)} - w_h^{(k)}\|_{\mathbb{V}} \leq Ch^{1/2-\varepsilon}.$$

Moreover, it holds that

$$0 \leq \lambda_h^{(j)} - \lambda^{(k)} \leq Ch^{1-\varepsilon} \quad \forall k \leq j \leq k+m-1.$$

2. *On the other hand, if $s \in (1/2, 1)$ and the meshes are graded according to* (H) *with $\mu = 2$, then the estimates above can be refined to be*

$$\|u^{(k)} - w_h^{(k)}\|_{\mathbb{V}} \leq Ch^{1-\varepsilon}.$$

and

$$0 \leq \lambda_h^{(j)} - \lambda^{(k)} \leq Ch^{2-\varepsilon} \quad \forall k \leq j \leq k+m-1,$$

respectively.

4.4.2 Direct proof of order of convergence

Here we provide an alternative proof of Theorem 4.4.2, without resorting to the Babuška-Osborn theory, and we also study convergence of discrete eigenfunctions in the L^2 norm. Even though the statements and arguments we give are for simple eigenspaces, they may be straightforwardly extended to multiple eigenspaces.

We begin proving the L^2 convergence of the energy-norm projection over the discrete space. Notice that smoothness of the domain is required in order to apply Proposition 3.3.2. The proof of this theorem follows the ideas from [91, Lemma 6.4-3].

Theorem 4.4.3. *Assume Ω is a smooth domain, let $\alpha = \min\{s, 1/2 - \varepsilon\}$ for any $\varepsilon > 0$, and let $\left(u^{(k)}, \lambda^{(k)}\right)$ be a simple eigenpair. Then, there is a positive constant C independent of h such that*

$$\|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \leq Ch^{\alpha+1/2-\varepsilon}, \tag{4.4.6}$$

where $u_h^{(k)}$ is chosen in such a way that $\left(\Pi_h u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)} \geq 0$, and $\Pi_h u^{(k)}$ is defined by (4.2.2).

Proof. In first place, we define the L^2-projection of $\Pi_h u^{(k)}$ over $E_h^{(k)} = \text{span}\left\{u_h^{(k)}\right\}$,

$$v_h^{(k)} = \left(\Pi_h u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)} u_h^{(k)},$$

and the quantity

$$\rho_h^{(k)} = \max_{i \neq k} \frac{\lambda^{(k)}}{|\lambda^{(k)} - \lambda_h^{(i)}|}.$$

Then, it holds that

$$\|u^{(k)}-u_h^{(k)}\|_{L^2(\Omega)} \leq \|u^{(k)}-\Pi_h u^{(k)}\|_{L^2(\Omega)}+\|\Pi_h u^{(k)}-v_h^{(k)}\|_{L^2(\Omega)}+\|v_h^{(k)}-u_h^{(k)}\|_{L^2(\Omega)}. \quad (4.4.7)$$

We are going to estimate the terms in the right hand side separately.

Given $\varepsilon > 0$, it follows from propositions 3.3.2 and 4.1.2 that there exists $C > 0$ independent of h such that

$$\|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)} \leq C h^{\alpha+1/2-\varepsilon}, \quad (4.4.8)$$

where $\alpha = \min\{s, 1/2 - \varepsilon\}$. Moreover, since

$$\left(\Pi_h u^{(k)}, u_h^{(i)}\right)_{L^2(\Omega)} = \frac{1}{\lambda_h^{(i)}} \left\langle \Pi_h u^{(k)}, u_h^{(i)} \right\rangle_{\mathbb{V}} = \frac{1}{\lambda_h^{(i)}} \left\langle u^{(k)}, u_h^{(i)} \right\rangle_{\mathbb{V}} = \frac{\lambda^{(k)}}{\lambda_h^{(i)}} \left(u^{(k)}, u_h^{(i)}\right)_{L^2(\Omega)},$$

we deduce

$$\left|\left(\Pi_h u^{(k)}, u_h^{(i)}\right)_{L^2(\Omega)}\right| \leq \rho_h^{(k)} \left|\left(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)}\right)_{L^2(\Omega)}\right|.$$

Therefore,

$$\begin{aligned}\|\Pi_h u^{(k)} - v_h^{(k)}\|^2_{L^2(\Omega)} &= \sum_{i\neq k} \left(\Pi_h u^{(k)}, u_h^{(i)}\right)^2_{L^2(\Omega)} \leq \left[\rho_h^{(k)}\right]^2 \sum_{i\neq k} \left(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)}\right)^2_{L^2(\Omega)} \\ &\leq \left[\rho_h^{(k)}\right]^2 \|u^{(k)} - \Pi_h u^{(k)}\|^2_{L^2(\Omega)} \leq C h^{\alpha+1/2-\varepsilon}.\end{aligned} \quad (4.4.9)$$

Finally, let us show that

$$\|v_h^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \leq \|v_h^{(k)} - u^{(k)}\|_{L^2(\Omega)}, \quad (4.4.10)$$

so that

$$\|v_h^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \leq \|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)} + \|\Pi_h u^{(k)} - v_h^{(k)}\|_{L^2(\Omega)}.$$

Indeed, on the one hand we have

$$u_h^{(k)} - v_h^{(k)} = \left[1 - \left(\Pi_h u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)}\right] u_h^{(k)}.$$

On the other hand, due to the normalizations $\|u^{(k)}\|_{L^2(\Omega)} = \|u_h^{(k)}\|_{L^2(\Omega)} = 1$, we have

$$\left|1 - \|v_h^{(k)}\|_{L^2(\Omega)}\right| \leq \|u^{(k)} - v_h^{(k)}\|_{L^2(\Omega)}$$

and

$$\|v_h^{(k)}\|_{L^2(\Omega)} = \left|\left(\Pi_h u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)}\right|.$$

So, choosing the sign of $u_h^{(k)}$ in such a way that $\left(\Pi_h u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)} \geq 0$, we deduce

$$\|u_h^{(k)} - v_h^{(k)}\|_{L^2(\Omega)} = \left|1 - \left(\Pi_h u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)}\right| = \left|1 - \left|\left(\Pi_h u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)}\right|\right| \leq \|u^{(k)} - v_h^{(k)}\|_{L^2(\Omega)},$$

as stated in (4.4.10).

Hence, estimate (4.4.6) is obtained by combining (4.4.7), (4.4.8), (4.4.9) and (4.4.10). □

Next, we utilize the L^2 estimate to produce a direct proof of the rate of convergence for the eigenvalues.

Proposition 4.4.4. *For any $\varepsilon > 0$ there is a positive constant C independent of h such that*

$$0 \leq \lambda_h^{(k)} - \lambda^{(k)} \leq C h^{1-\varepsilon}.$$

Proof. By Remark 4.2.1 we only have to prove the second inequality. Developing the inner product and taking an eigenfunction $u^{(k)}$ and $u_h^{(k)}$ its corresponding discrete eigenfunction, we obtain

$$\begin{aligned}
\|u^{(k)} - u_h^{(k)}\|_{\mathbb{V}}^2 &= \left\langle u^{(k)} - u_h^{(k)},\, u^{(k)} - u_h^{(k)} \right\rangle_{\mathbb{V}} \\
&= \lambda^{(k)} \left(u^{(k)}, u^{(k)} - u_h^{(k)}\right)_{L^2(\Omega)} - \lambda^{(k)} \left(u^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)} + \lambda_h^{(k)} \left(u_h^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)} \\
&= \lambda^{(k)} \left(u^{(k)} - u_h^{(k)}, u^{(k)} - u_h^{(k)}\right)_{L^2(\Omega)} + \left(\lambda_h^{(k)} - \lambda^{(k)}\right) \left(u_h^{(k)}, u_h^{(k)}\right)_{L^2(\Omega)}.
\end{aligned}$$

That is,

$$\left(\lambda_h^{(k)} - \lambda^{(k)}\right) \|u_h^{(k)}\|_{L^2(\Omega)}^2 = \|u^{(k)} - u_h^{(k)}\|_{\mathbb{V}}^2 - \lambda^{(k)} \|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)}^2. \tag{4.4.11}$$

Additionally,

$$\begin{aligned}
\|u_h^{(k)} - \Pi_h u^{(k)}\|_{\mathbb{V}}^2 &= \left(\lambda_h^{(k)} u_h^{(k)} - \lambda^{(k)} u^{(k)}, u_h^{(k)} - \Pi_h u^{(k)}\right)_{L^2(\Omega)} \\
&\leq \|\lambda_h^{(k)} u_h^{(k)} - \lambda^{(k)} u^{(k)}\|_{L^2(\Omega)} \|u_h^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)} \\
&\leq \left(\lambda_h^{(k)} \|u_h^{(k)} - u^{(k)}\|_{L^2(\Omega)} + |\lambda_h^{(k)} - \lambda^{(k)}| \|u^{(k)}\|_{L^2(\Omega)}\right) \|u_h^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}.
\end{aligned}$$

Dividing by $\|u_h^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}$ and using the fact that

$$\|u_h^{(k)} - \Pi_h u^{(k)}\|_{\mathbb{V}}^2 \geq \lambda^{(1)} \|u_h^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}^2,$$

we obtain

$$\|u_h^{(k)} - \Pi_h u^{(k)}\|_{\mathbb{V}} \leq C \left(\lambda_h^{(k)} \|u_h^{(k)} - u^{(k)}\|_{L^2(\Omega)} + |\lambda_h^{(k)} - \lambda^{(k)}| \|u^{(k)}\|_{L^2(\Omega)} \right). \qquad (4.4.12)$$

Therefore, by (4.4.11), (4.4.12), (3.3.3) and normalizing $\|u_h^{(k)}\|_{L^2(\Omega)} = \|u^{(k)}\|_{L^2(\Omega)} = 1$, we arrive at

$$\begin{aligned}
\lambda_h^{(k)} - \lambda^{(k)} &\leq C \left(\|u^{(k)} - \Pi_h u^{(k)}\|_{\mathbb{V}}^2 + \|u_h^{(k)} - \Pi_h u^{(k)}\|_{\mathbb{V}}^2 \right) - \lambda^{(k)} \|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)}^2 \\
&\leq C \left(h^{1-\varepsilon} + \left[\lambda_h^{(k)} \right]^2 \|u_h^{(k)} - u^{(k)}\|_{L^2(\Omega)}^2 + \left| \lambda_h^{(k)} - \lambda^{(k)} \right|^2 \right) - \lambda^{(k)} \|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)}^2 \\
&= C \left(h^{1-\varepsilon} + \left| \lambda_h^{(k)} - \lambda^{(k)} \right|^2 \right) + \left(C \left[\lambda_h^{(k)} \right]^2 - \lambda^{(k)} \right) \|u_h^{(k)} - u^{(k)}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Finally, using that $\lambda_h^{(k)} \to \lambda^{(k)}$ and that $\|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)}^2 \leq C h^{2\alpha+1}$, we conclude

$$\lambda_h^{(k)} - \lambda^{(k)} \leq C h^{1-\varepsilon} + O(h^{2\alpha+1}) \leq C h^{1-\varepsilon}.$$

□

The error in the energy norm may be bounded directly from the eigenvalue and L^2-error estimates.

Proposition 4.4.5. *Let Ω be a smooth domain and let $\left(u^{(k)}, \lambda^{(k)}\right)$ be a simple eigenpair. For any $\varepsilon > 0$ there is a positive constant C independent of h such that*

$$\|u^{(k)} - u_h^{(k)}\|_{\mathbb{V}} \leq C h^{1/2-\varepsilon},$$

where $u_h^{(k)}$ is chosen in such a way that $(\Pi_h u^{(k)}, u_h^{(k)}) \geq 0$.

Proof. By (4.4.11) and normalizing $\|u_h^{(k)}\|_{L^2(\Omega)} = \|u^{(k)}\|_{L^2(\Omega)} = 1$, we have that

$$\|u^{(k)} - u_h^{(k)}\|_{\mathbb{V}}^2 = \lambda^{(k)} \|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)}^2 + \lambda_h^{(k)} - \lambda^{(k)}.$$

This implies that

$$\|u^{(k)} - u_h^{(k)}\|_{\mathbb{V}} \leq C \left(\|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} + \sqrt{\lambda_h^{(k)} - \lambda^{(k)}} \right) \leq C h^{1/2-\varepsilon}.$$

□

4.5 Examples and applications

In order to illustrate the convergence estimates obtained in Section 4.4, we present the results of numerical tests for finite element discretizations of one and two-dimensional eigenvalue problems. Moreover, in the latter case, some examples in a domain that does not satisfy the hypotheses of Theorem 4.4.2 are displayed. These examples provide numerical evidence that the assertions of this theorem still hold true under weaker assumptions about the domain.

Since in general no closed formula for the eigenvalues of the fractional Laplacian is available, we estimate the order of convergence by means of a least-squares fitting of the model

$$\lambda_h^{(k)} = \lambda^{(k)} + Ch^{\alpha}.$$

This allows to extrapolate approximations of the eigenvalues as well (in the tables below we denote this extrapolated value of $\lambda^{(k)}$ as $\lambda_{ext}^{(k)}$).

Throughout this section, our results are compared with those available in the literature. In first place we consider one-dimensional problems, which have been widely studied both theoretically and from the numerical point of view. Next, we show some examples in two-dimensional domains: the unit ball, a square and an L-shaped domain. As for the ball, the deep results of [47] allow to obtain sharp estimates on the eigenvalues, and thus provide a point of comparison for the validity of the finite element implementation. Regarding the square, some estimates for the eigenvalues are found in [74]. The main interest of the L-shaped domain is that, although it does not satisfy the "standard" requirements to regularity of eigenfunctions to hold, the numerical order of convergence is the same as in problems posed on smooth, convex domains.

General estimates for eigenvalues, valid for a class of domains, have been derived by Chen and Song [32]. In that paper, the authors state upper and lower bounds for eigenvalues of the fractional Laplacian on domains satisfying the exterior cone condition. Calling $\mu^{(k)}$ the k-th eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions in a domain Ω, they prove that there exists a constant $C = C(\Omega)$ such that

$$C\left(\mu^{(k)}\right)^s \leq \lambda^{(k)} \leq \left(\mu^{(k)}\right)^s. \tag{4.5.1}$$

If Ω is a bounded convex domain, then C can be taken as $1/2$. It is noteworthy that, due to the scaling property of the fractional Laplacian, eigenvalues for the dilations of a domain Ω are obtained by means of $\lambda^{(k)}(\gamma\Omega) = \gamma^{-2s}\lambda^{(k)}(\Omega)$.

The main advantage of employing the finite element method is that it is flexible enough to cope with a variety of domains. Moreover, as we are working with conforming approximations, sharp upper bounds for the eigenvalues may be obtained by considering discrete solutions on refined meshes. This is of special interest for applications in domains in which theoretical estimates are not sharp, or the non-symmetry of the domain precludes the possibility of developing arguments such as the ones in [47].

4.5.1 One-dimensional intervals

Eigenproblems for the fractional Laplacian in intervals have been widely studied in recent years. In [109], a discretized model of the fractional Laplacian is developed, and a numerical analysis of eigenfunctions and eigenvalues is implemented for different boundary conditions. In [73], the authors deal with one dimensional problems for $s = 1/2$, and provide asymptotic expansion for eigenvalues. Later, Kwaśnicki [74] extended this work to the whole range $s \in (0,1)$. Namely, he showed the following identity for the k-th eigenvalue in the interval $(-1,1)$:

$$\lambda^{(k)} = \left(\frac{k\pi}{2} - \frac{(1-s)\pi}{4} \right)^{2s} + \frac{1-s}{\sqrt{s}} O\left(k^{-1}\right). \tag{4.5.2}$$

Moreover, in that work a method to obtain lower bounds in arbitrary bounded domains is developed, and it is proved that, on one spacial dimension, all eigenvalues are simple if $s \geq 1/2$. As eigenvalues are simple and we are working in one dimension, it is not difficult to numerically estimate the order of convergence of eigenfunctions in the L^2-norm. Indeed, normalizing the discrete eigenfunctions so that $\|u_h^{(k)}\|_{L^2(-1,1)} = 1$ and choosing their sign adequately (recall Subsection 4.4.2), these are then compared with a solution on a very fine grid.

On the other hand, [44] performs a numerical study of the fractional Schrödinger equation in an infinite potential well in one spacial dimension. The authors find numerically the ground and first excited states and their corresponding eigenvalues for the stationary linear problem, which corresponds to the first two eigenpairs of (4.1.1).

In Table 4.5.1, our results for the first two eigenpairs are displayed. The extrapolated numerical values are compared with the estimates from [44, 74]; the orders of convergence are in good agreement with those predicted correspondingly by theorems 4.4.2 and 4.4.3.

	Numerical values						Orders			
s	$\lambda^{(1)}_{ext}$	$\lambda^{(1)}$ [44]	$\lambda^{(1)}$ [74]	$\lambda^{(2)}_{ext}$	$\lambda^{(2)}$ [44]	$\lambda^{(2)}$ [74]	$\lambda^{(1)}$	$\lambda^{(2)}$	$u^{(1)}$	$u^{(2)}$
0.05	0.9726	0.9726	0.9809	1.0922	1.0922	1.0913	1.108	1.149	0.551	0.568
0.1	0.9575	0.9575	0.9712	1.1965	1.1966	1.1948	1.071	1.102	0.612	0.625
0.25	0.9702	0.9702	0.9908	1.6015	1.6016	1.5977	1.021	1.038	0.762	0.782
0.5	1.1577	1.1578	1.1781	2.7548	2.7549	2.7488	1.001	0.979	0.961	0.969
0.75	1.5975	1.5976	1.6114	5.0598	5.0600	5.0545	0.998	0.999	0.998	0.998
0.9	2.0487	—	2.0555	7.5031	—	7.5003	1.004	1.021	0.999	0.999
0.95	2.2481	2.2441	2.2477	8.5958	8.5959	8.5942	1.035	1.142	0.999	0.999

Table 4.1: First two eigenpairs in the interval $(-1,1)$. On the left, the extrapolated numerical values are compared with the results from [44] and with approximation (4.5.2), obtained in [74]. On the right, orders of convergence for eigenvalues and eigenfunctions in the L^2-norm (obtained by a least-square fitting) are displayed.

To illustrate the sharpness of Proposition 4.1.2, we display in Figure 4.1 the first two L^2-normalized eigenfunctions for $s = 0.1$ and $s = 0.9$. It is observed that these functions are smooth within the interval, but as they are extended by zero in the complement of the interval, they are not globally smooth.

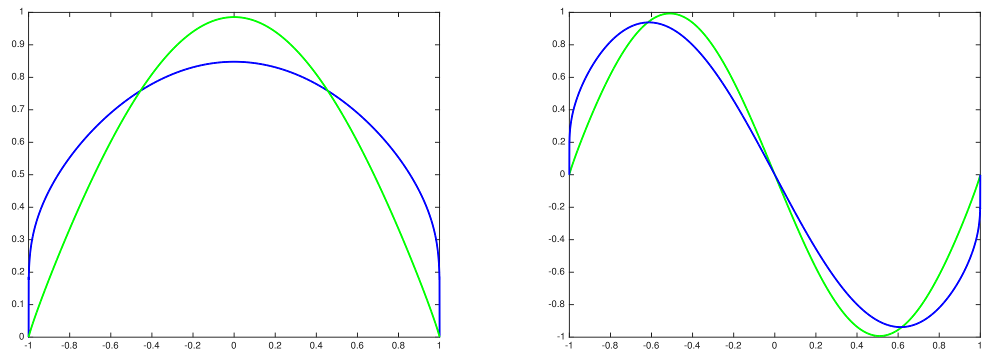

Figure 4.1: First (left panel) and second (right panel) L^2-normalized discrete eigenfunctions in the interval $(-1, 1)$. Color blue corresponds to $s = 0.1$, while green corresponds to $s = 0.9$.

4.5.2 Two-dimensional experiments

The theoretical order of convergence for eigenvalues is also attained in the following examples in $\mathbb{R}^2$. The implementation for these experiments is based on the code from [2], and the stiffness matrix can be computed following the details provided in Appendix A. We exhibit examples posed in a smooth domain (unit ball), and in Lipschitz domains satisfying and not satisfying the exterior ball condition (square and L-shaped domain, respectively). Also, for the unit ball we show that, in agreement with Theorem 4.4.2, the order of convergence is increased by utilizing graded meshes.

Unit ball

Consider the fractional eigenvalue problem on the two-dimensional unit ball. As we discussed in Subsection 2.1.3, in [46] explicit formulas for eigenvalues and eigenfunctions of the weighted operator $u \mapsto (-\Delta)^s(\omega^s u)$ are established, where $\omega(x) = 1 - |x|^2$. Furthermore, in [47] the same authors exploit these expressions to obtain two-sided bounds for the eigenvalues of the fractional Laplacian in the unit ball in any dimension. This method provides sharp estimates; however, it depends on the decomposition of the fractional Laplacian as a weighted operator, and the weight ω that delivers an explicit eigendecomposition of this operator is only known for balls.

In Table 4.2, our results for the first eigenvalue are compared with those from [47] for different values of s. This comparison serves as a test for the validity of the code we are employing. As well as the extrapolated value of $\lambda^{(1)}$ and the numerical order of convergence, for every s considered we exhibit an upper bound for the first eigenvalue. These outcomes are consistent with those from [47] and the theoretical order of convergence given by Theorem 4.4.2.

s	$\lambda^{(1)}$	$\lambda^{(1)}_{ext}$	$\lambda^{(1)}_h$ (UB)	Order
0.005	1.00475	1.00475	1.00480	0.9462
0.05	1.05095	1.05094	1.05145	0.9455
0.25	1.34373	1.34367	1.34626	0.9497
0.5	2.00612	2.00607	2.01060	0.9686
0.75	3.27594	3.27632	3.28043	1.0092

Table 4.2: First eigenvalue in the unit ball in $\mathbb{R}^2$. Estimate from [47]; extrapolated value of $\lambda^{(1)}$; upper bound obtained by the finite element method with a meshsize $h \sim 0.02$; numerical order of convergence.

Computations with graded meshes were carried out for this domain as well. The grading parameter μ in (H) was set to be equal to 2. See the discussion in Subsection 3.5.2 about the construction of these meshes, and remarks 3.4.4 and 3.5.5 for a justification of the choice of this grading parameter. In Table 4.3 we summarize our findings. For $s \in [1/2, 1)$, we estimated the order of convergence towards the first eigenvalue both with uniform and graded meshes, and also compared the extrapolated value of this eigenvalue. An increase in the convergence rate, in agreement with Theorem 4.4.2, is observed, while the extrapolated eigenvalues remain almost unchanged.

s	Order (unif.)	Order (graded)	$\lambda^{(1)}_{ext}$ (unif.)	$\lambda^{(1)}_{ext}$ (graded)
0.5	0.9686	2.1528	2.0061	2.0061
0.6	0.9808	2.1720	2.4165	2.4165
0.7	0.9969	2.1066	2.9506	2.9507
0.8	1.0348	2.0497	3.6494	3.6498
0.9	1.1654	2.0943	4.5691	4.5695

Table 4.3: Computational results in the unit ball in $\mathbb{R}^2$, for uniform and graded meshes. Orders of convergence are stated in terms of the mesh parameter h; this behaves like $N^{-1/2}$, N being the number of nodes.

Square

Eigenvalue estimates for the case in which the domain Ω is a square are also addressed in [74]. However, in order to obtain upper bounds, the method proposed in that work depends on having pointwise bounds of the Green function for the fractional Laplacian on Ω. The estimates from [32, 74] are compared with our results in Table 4.4, where numerical orders of convergence are also displayed. As in the previous example, the observed order of convergence is in agreement with Theorem 4.4.2. Moreover, the proposed scheme with uniform meshes with parameter $h \sim 0.04$ delivers a more accurate upper bound for the first eigenvalue than [32, 74].

s	$\lambda^{(1)}$ (LB)	$\lambda^{(1)}$ (UB)	$\lambda_h^{(1)}$ (UB)	$\lambda_{ext}^{(1)}$	Order
0.05	1.0308[b]	1.0831[a]	1.0412	1.0405	0.9229
0.1	1.0506[b]	1.1731[a]	1.0895	1.0882	0.9230
0.25	1.1587[b]	1.4905[a]	1.2844	1.2813	0.9283
0.5	1.3844[b]	2.2214[a]	1.8395	1.8344	0.9622
0.75	1.6555[a]	3.3109[a]	2.8921	2.8872	0.9940
0.9	2.1034[a]	4.2067[a]	3.9492	3.9467	1.0654
0.95	2.2781[a]	4.5562[a]	4.4083	4.4062	1.1496

[a]See [32]. [b]See [74].

Table 4.4: First eigenvalue in the square $[-1,1]^2$. Best lower (LB) and upper (UB) bounds known before; upper bound obtained by the finite element method with a meshsize $h \sim 0.04$; extrapolated value of $\lambda^{(1)}$; numerical order of convergence.

L-shaped domain

To the best of the author's knowledge, there is no *efficient* method to estimate eigenvalues of the fractional Laplacian if the domain Ω lacks symmetry. The bound (4.5.1) remains valid as long as Ω satisfies the assumptions required, but the range that estimate provides is quite wide, especially if Ω is not convex.

In Proposition 4.1.2, which states that eigenfunctions belong to $\widetilde{H}^{s+1/2-\varepsilon}(\Omega)$, it is assumed that the domain Ω satisfies the exterior ball condition. For the Laplacian, in order to prove regularity of solutions, it is customary to assume that Ω is either smooth or at least convex. In those cases, it is well known that if $f \in H^r(\Omega)$ for some r, then the solutions of the Dirichlet problem with right hand side f belong to $H^{r+2}(\Omega)$. However, if the domain has a re-entrant corner, solutions are less regular. This also applies to eigenfunctions: in the L-shaped domain $\Omega = [-1,1]^2 \setminus [0,1]^2$, the first eigenvalue of the Laplacian is known not to belong to $H^{3/2}(\Omega)$.

Surprisingly, numerical evidence indicates that eigenvalues of the fractional Laplacian on this L-shaped domain converge with the same order as in the previous examples. Our findings are summarized in Table 4.5, and motivate the conjecture that eigenfunctions and solutions to the Dirichlet equation (Homogeneous) have the same Sobolev regularity than in smooth domains. In figure 4.2 we exhibit the first discrete eigenfunction in this domain for $s = 0.1$.

s	$\lambda_h^{(1)}$ (UB)	$\lambda_{ext}^{(1)}$	Order $\lambda^{(1)}$
0.1	1.1434	1.1413	0.9085
0.2	1.3386	1.3342	0.9103
0.3	1.6025	1.5956	0.9160
0.4	1.9593	1.9499	0.9267
0.5	2.4440	2.4322	0.9459
0.6	3.1072	3.0936	0.9812
0.7	4.0228	4.0069	0.9822
0.8	5.2994	5.2831	1.0609
0.9	7.0975	7.0790	1.1891

Table 4.5: First eigenvalues in the $L-$shaped domain $[-1,1]^2 \setminus [0,1]^2$. Upper bound obtained by the finite element method with a meshsize $h \sim 0.04$; extrapolated value of $\lambda^{(1)}$; numerical order of convergence.

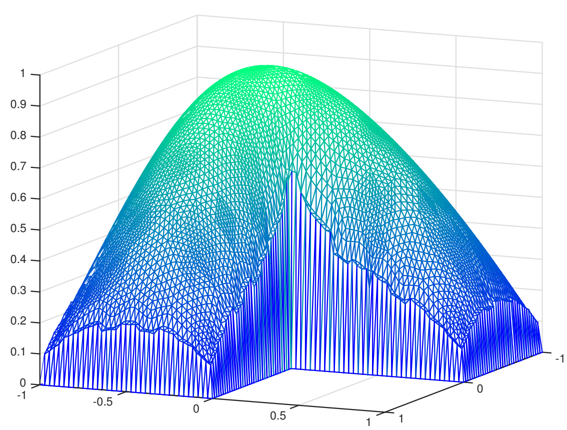

Figure 4.2: First discrete eigenfunction in the L-shaped domain with $s = 0.1$, computed on a mesh containing about 11000 elements within Ω.

Resumen del capítulo

Sea $\Omega \subset \mathbb{R}^n$ un dominio acotado. En este capítulo estudiamos el problema de autovalores fraccionario con condiciones de Dirichlet homogéneas. Esto es, buscamos un número positivo λ (autovalor) y una función $u \not\equiv 0$ (autofunción), tales que

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{en } \Omega, \\ u = 0 & \text{en } \Omega^c. \end{cases}$$

Incluso si Ω es un intervalo, obtener expresiones analíticas cerradas para autovalores y autofunciones del laplaciano fraccionario es muy dificultoso. Esto motiva al empleo de aproximaciones discretas para este problema; aquí consideramos el método de elementos finitos conforme. A diferencia de otros esquemas preexistentes, el que proponemos es suficientemente flexible como para tratar con una variedad de dominios, y nos permite obtener estimaciones y cotas superiores finas para autovalores, incluso en dominios no convexos. Más aún, como consecuencia de los experimentos que realizamos en el dominio $\Omega = [-1,1]^2 \setminus [0,1]^2$, conjeturamos que la primera autofunción en este dominio es tan regular como la primera autofunción de cualquier dominio suave.

En la **Sección 4.1** describimos algunos aspectos teóricos básicos del problema en consideración, y utilizamos la notación y teoría desarrolladas en la Sección 2.3 para estudiar la regularidad Sobolev de autofunciones. Posteriormente, la **Sección 4.2** describe el contexto discreto que utilizamos y recuerda ciertas propiedades útiles para nuestro análisis.

El estudio de convergencia de soluciones comienza en la **Sección 4.3**. Allí se demuestra que el esquema que consideramos conduce a una buena aproximación de todos los autoespacios y no genera modos espúreos. Luego, mostramos el orden de convergencia para autovalores y autofunciones, tanto en la norma de energía como en la de $L^2(\Omega)$. La primera parte de la **Sección 4.4** está dedicada a la teoría de Babuška-Osborn, que permite deducir estos órdenes de convergencia de forma directa, mientras que la segunda ofrece una prueba directa y elemental.

Finalmente, realizamos varios experimentos numéricos y comparamos nuestros resultados con trabajos anteriores por otros autores. Los resultados de la **Sección 4.5** muestran un buen acuerdo con la teoría y las estimaciones de autovalores que obtenemos son consistentes con las encontradas en la literatura.

Chapter 5

A mixed method for the nonhomogeneous Dirichlet problem

In some applications, it may be of interest not only to approximate the solutions of problems involving the fractional Laplacian but also to estimate the flux between the domain where the problem is posed and its complement.

In the local framework, the flux of a vector-valued quantity between two domains is identified with the integral of the normal component of this quantity across the interface between the domains. So, the integral of the normal derivative of a scalar function u over the boundary of an open set Ω is interpreted as the flux of u between Ω and its complement. The normal derivative, supported on $\partial\Omega$, is understood as a flux density.

The nonlocal nature of the fractional Laplacian implies that interactions occur at any positive distance, and this in turn suggests that the flux density between a domain and its complement must be supported in the complement of the domain. The nonlocal normal derivative (cf. Definition 1.2.20) plays exactly the same role as does the normal derivative in the local context. We refer the reader to [43, Section 2] for a detailed comparison between local and nonlocal fluxes.

In this chapter we study finite element approximations to problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \Omega^c, \end{cases} \qquad \text{(Nonhomogeneous)}$$

where the functions f and g are data belonging to suitable spaces. We perform approximations of both the solution u of (Nonhomogeneous) as well as of its nonlocal derivative $\mathcal{N}_s u$. Finite element methods in which two different unknowns are treated as primary variables receive the general denomination of mixed methods. Upon utilizing the integration by parts formula (1.2.8), we introduce the nonlocal derivative as a Lagrange multiplier. As usual with mixed formulations, we are led to solving a saddle-point problem.

One of the main difficulties to be overcome is that the supports of the volume constraint g and of the nonlocal derivative of u are not necessarily bounded. Our strategy includes truncating the Dirichlet condition and approximating the nonlocal derivative on a sequence of bounded but growing discrete domains. Besides developing a finite element scheme, we need to estimate how solutions are affected by the truncation of the volume constraint and the decay of the flux density at infinity.

The chapter is organized as follows. Section 5.1 introduces some notation, states the mixed formulation of the problem under consideration and proves its well-posedness. Regularity of solutions and of its nonlocal normal derivatives is addressed in Section 5.2, where these estimates are shown to be sharp.

Afterwards, the chapter is devoted to the mixed finite element analysis of the problem with piecewise linear functions to approximate both the solution and its nonlocal normal derivative. The discrete spaces and problem are introduced in Section 5.3. Moreover, in that section, assuming the support of the volume constraint to be bounded and contained in a certain auxiliary domain, it is proved that the discrete problem is well-posed and approximation estimates are derived for the solution and its flux density in $H^s(\mathbb{R}^n)$ and $\widetilde{H}^{-s}(\Omega^c)$, respectively.

The method we propose to deal with unboundedly supported volume constraints consists of simply truncating the Dirichlet condition. So, Section 5.4 analyzes the error associated to this truncation process, comparing the solution of the continuous problems. When solving the corresponding discrete problems, the diameter of the truncated domain has to be suitably associated to the mesh size in order to preserve the accuracy of the scheme. Finally, in Section 5.5, computational results are displayed; these provide evidence of convergence of the algorithm.

5.1 Statement of the problem

Let Ω be a bounded Lipschitz domain satisfying the exterior ball condition and let $f \in \widetilde{H}^{-s}(\Omega)$ and $g \in H^s(\Omega^c)$. As mentioned at the beginning of this chapter, we study the nonhomogeneous Dirichlet problem for the fractional Laplacian (Nonhomogeneous). Besides approximating the solutions inside Ω, we aim to approximate the flux density of solutions between Ω^c and Ω. That is, we seek to approximate the nonlocal normal derivative (cf. Definition 1.2.20) of the solution as well.

The approach we propose is to weakly impose the Dirichlet condition and to incorporate the normal derivative as a Lagrange multiplier in the formulation of the problem. For this purpose, an important theoretical tool is the fractional integration by parts formula (1.2.8): if $u, v \in H^s(\mathbb{R}^n)$, then

$$\frac{C(n,s)}{2}\iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy \\ = \int_\Omega v(x)(-\Delta)^s u(x)\,dx + \int_{\Omega^c} v(x)\,\mathcal{N}_s u(x)\,dx.$$

Recall that the nonlocal normal derivative is given by

$$\mathcal{N}_s u(x) := C(n,s)\int_\Omega \frac{v(x)-v(y)}{|x-y|^{n+2s}}\,dy, \quad x \in \Omega^c,$$

and that this definition induces an operator $\mathcal{N}_s \colon H^s(\mathbb{R}^n) \to H^{-s}(\Omega^c)$. Moreover, if u is the solution to (Nonhomogeneous), since $((-\Delta)^s u)\big|_\Omega = f \in \widetilde{H}^{-s}(\Omega)$, it simple to verify that it also holds that $\mathcal{N}_s u \in \widetilde{H}^{-s}(\Omega^c) := (H^s(\Omega^c))'$.

Multiplying the first equation in (Nonhomogeneous) by a test function v and applying (1.2.8), we obtain

$$\frac{C(n,s)}{2}\iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy - \int_{\Omega^c} v(x)\,\mathcal{N}_s u(x)\,dx \\ = \int_\Omega f(x)v(x)\,dx.$$

Throughout this chapter we consider the spaces $\mathbb{V} = H^s(\mathbb{R}^n)$ and $\Lambda = \widetilde{H}^{-s}(\Omega^c)$, furnished with their usual norms, and the bilinear forms $a\colon \mathbb{V}\times\mathbb{V}\to\mathbb{R}$, $b\colon \mathbb{V}\times\Lambda\to\mathbb{R}$,

$$\begin{aligned} a(u,v) &= \frac{C(n,s)}{2}\iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy, \\ b(u,\mu) &= \int_{\Omega^c} u(x)\,\mu(x)\,dx, \end{aligned} \tag{5.1.1}$$

with $Q = (\Omega\times\mathbb{R}^n)\cup(\mathbb{R}^n\times\Omega)$.

Remark 5.1.1. The form a satisfies the identity

$$a(u,v) = \frac{C(n,s)}{2}\left(\langle u,v\rangle_{H^s(\mathbb{R}^n)} - \langle u,v\rangle_{H^s(\Omega^c)}\right)\ \forall u,v\in\mathbb{V}.$$

In particular, over the set $\widetilde{H}^s(\Omega)$, $a(v,v)$ coincides with $\frac{C(n,s)}{2}|v|^2_{H^s(\mathbb{R}^n)}$.

It is trivial to verify that b is continuous, while the previous remark implies that $|a(u,v)| \le C(n,s)|u|_{H^s(\mathbb{R}^n)}|v|_{H^s(\mathbb{R}^n)}$ for all $u,v\in\mathbb{V}$. Moreover, let us consider the continuous functionals

$$F:\mathbb{V}\to\mathbb{R}, \quad F(v) = \int_\Omega f(x)\,v(x)\,dx,$$

$$G : \Lambda \to \mathbb{R}, \quad G(\lambda) = \int_{\Omega^c} g(x)\, \lambda(x)\, dx.$$

We are now in position to give the precise formulation of the problem we deal with in this chapter: find $(u, \lambda) \in \mathbb{V} \times \Lambda$ such that

$$\begin{cases} a(u, v) - b(v, \lambda) = F(v) & \forall v \in \mathbb{V}, \\ \qquad\qquad b(u, \mu) = G(\mu) & \forall \mu \in \Lambda. \end{cases} \tag{5.1.2}$$

Remark 5.1.2. As can be seen from the above considerations, the Lagrange multiplier λ, which is associated to the restriction $u = g$ in Ω^c, coincides with the nonlocal derivative $\mathcal{N}_s u$ in that set. In order to simplify the notation, in the following we refer to it as λ.

The main result of this section is the well-posedness of (5.1.2). In order to prove it, we resort to the Babuška-Brezzi theory. We are going to make use of the following pivotal theorem.

Theorem 5.1.3 ([16, Theorem 4.2.3])**.** *Let $\mathbb{V}$, Λ be two Hilbert spaces and $a\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$, $b\colon \mathbb{V} \times \Lambda \to \mathbb{R}$ be two continuous bilinear forms, and let $B\colon \mathbb{V} \to \Lambda'$ the linear operator*

$$\langle Bv, \mu \rangle_{\Lambda' \times \Lambda} := b(v, \mu).$$

Assume that the restriction of a on $\ker B$ satisfies the double inf-sup condition

$$\begin{aligned} \inf_{u \in \ker B} \sup_{v \in \ker B} \frac{a(u, v)}{\|u\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} \geq \alpha > 0, \\ \inf_{v \in \ker B} \sup_{u \in \ker B} \frac{a(u, v)}{\|u\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} \geq \alpha > 0, \end{aligned} \tag{5.1.3}$$

and that b satisfies the inf-sup condition

$$\inf_{\mu \in \Lambda} \sup_{v \in \mathbb{V}} \frac{b(v, \mu)}{\|v\|_{\mathbb{V}} \|\mu\|_{\Lambda}} \geq \beta > 0.$$

Then, for every $(f, g) \in \mathbb{V}' \times \Lambda'$, the problem

$$\text{find } (u, \lambda) \in \mathbb{V} \times \Lambda \text{ such that } \begin{cases} a(u, v) - b(v, \lambda) = F(v) & \forall v \in \mathbb{V}, \\ \qquad\qquad b(u, \mu) = G(\mu) & \forall \mu \in \Lambda, \end{cases}$$

has a unique solution (u, λ) that is bounded by

$$\|u\|_{\mathbb{V}} + \|\lambda\|_{\Lambda} \leq C \left(\|f\|_{\mathbb{V}'} + \|\lambda\|_{\Lambda'} \right)$$

for a constant C depending only on α, β and the modulus of continuity of a.

Remark 5.1.4. Naturally, if a is symmetric, then the two conditions (5.1.3) are the same. Moreover, these hold if a is coercive on the kernel of B, that is, if there exists $\alpha' > 0$ such that

$$a(v,v) \geq \alpha' \|v\|_{\mathbb{V}}^2, \ \forall v \in \ker B.$$

Now, we set our problem in the framework of Theorem 5.1.3. We take the bilinear forms a, b as in (5.1.1) and consider

$$K = \{v \in \mathbb{V} \colon b(v,\mu) = 0 \ \forall \mu \in \Lambda\} = \widetilde{H}^s(\Omega). \tag{5.1.4}$$

Recalling Poincaré inequality (cf. Proposition 1.2.8) and Remark 5.1.1, it follows that

$$\|v\|_{\mathbb{V}}^2 \leq C|v|_{H^s(\mathbb{R}^n)}^2 = Ca(v,v), \quad \forall v \in K. \tag{5.1.5}$$

According to Remark 5.1.4, this implies that (5.1.3) holds.

Another important ingredient in our analysis is the extension operator $E : H^\sigma(\Omega^c) \to H^\sigma(\mathbb{R}^n)$ (recall Definition 4.3.1 and Remark 4.3.2). Namely, that for every $\sigma \in (0,1)$, there exists a constant $C = C(n,\sigma,\Omega)$ such that, for all $u \in H^\sigma(\Omega^c)$,

$$\|Eu\|_{H^\sigma(\mathbb{R}^n)} \leq C\|u\|_{H^\sigma(\Omega^c)}. \tag{5.1.6}$$

This extension operator allows us to prove the inf-sup condition for the form b.

Lemma 5.1.5. *For all $\mu \in \Lambda$, it holds that*

$$\sup_{u \in \mathbb{V}} \frac{b(u,\mu)}{\|u\|_{\mathbb{V}}} \geq \frac{1}{C}\|\mu\|_\Lambda, \tag{5.1.7}$$

where $C > 0$ is the constant from (5.1.6).

Proof. Let $\mu \in \Lambda$. Recalling that $\Lambda = (H^s(\Omega^c))'$ and taking into account the extension operator given by (5.1.6), it holds that

$$\|\mu\|_\Lambda = \sup_{v \in H^s(\Omega^c)} \frac{b(v,\mu)}{\|v\|_{H^s(\Omega^c)}} \leq C \sup_{v \in H^s(\Omega^c)} \frac{b(Ev,\mu)}{\|Ev\|_{\mathbb{V}}} \leq C \sup_{u \in \mathbb{V}} \frac{b(u,\mu)}{\|u\|_{\mathbb{V}}}.$$

□

Due to the ellipticity of a on K (5.1.5) and the inf-sup condition (5.1.7), we deduce the well-posedness of our continuous problem by means of Theorem 5.1.3.

Proposition 5.1.6. *Problem* (5.1.2) *admits a unique solution* $(u,\lambda) \in \mathbb{V} \times \Lambda$, *and there exists $C > 0$, such that the bound*

$$\|u\|_{\mathbb{V}} + \|\lambda\|_\Lambda \leq C \left(\|f\|_{\widetilde{H}^{-s}(\Omega)} + \|g\|_{H^s(\Omega^c)} \right)$$

is satisfied.

5.2 Regularity of solutions

This section studies regularity properties of the solution (u, λ) to (5.1.2). For the sake of simplicity, we just state the maximal regularity possible in terms of Sobolev norms of the data. For this purpose we assume Ω to be a smooth domain and recall that, according to Proposition 2.2.3, if $f \in H^r(\Omega)$ for some $r \geq -s$ and $g \equiv 0$, then the solution belongs to $\widetilde{H}^{s+\alpha}(\Omega)$ with $\alpha = \min\{s + r, 1/2 - \varepsilon\}$.

Remark 5.2.1. Since the maximum gain of regularity for solutions of the homogeneous problem is "almost" half a derivative, from this point on we assume $f \in \widetilde{H}^{-s}(\Omega) \cap H^{-s+1/2-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Moreover, we require the Dirichlet condition g to belong to $H^{s+1/2-\varepsilon}(\Omega^c)$ and to be such that $(-\Delta)^s_{\Omega^c} g \in H^{-s+1/2-\varepsilon}(\Omega^c)$ $\forall \varepsilon > 0$, where $(-\Delta)^s_{\Omega^c}$ denotes the regional fractional Laplacian operator (1.4.2) in Ω^c. In order not to overload notation, we will avoid writing the ε when referring to data regularity.

The main objective of this section is to prove the following.

Theorem 5.2.2. *Let Ω be a smooth domain, $f \in \widetilde{H}^{-s}(\Omega) \cap H^{-s+1/2}(\Omega)$ and let $g \in H^{s+1/2}(\Omega^c)$ be such that $(-\Delta)^s_{\Omega^c} g \in H^{-s+1/2}(\Omega^c)$. Let $u \in H^s(\mathbb{R}^n)$ be the solution of* (Nonhomogeneous) *and λ be its nonlocal normal derivative. Then, for all $\varepsilon > 0$, $u \in H^{s+1/2-\varepsilon}(\mathbb{R}^n)$, $\lambda \in H^{1/2-s-\varepsilon}(\Omega^c)$, and there exists $C > 0$ such that*

$$\|u\|_{H^{s+1/2-\varepsilon}(\mathbb{R}^n)} + \|\lambda\|_{H^{-s+1/2-\varepsilon}(\Omega^c)} \leq C\, \Sigma_{f,g},$$

where

$$\Sigma_{f,g} := \left(\|f\|_{H^{-s+1/2}(\Omega)} + \|g\|_{H^{s+1/2}(\Omega^c)} + \|(-\Delta)^s_{\Omega^c} g\|_{H^{-s+1/2}(\Omega^c)} \right). \tag{5.2.1}$$

Proof. According to (5.1.6), we take an extension of the Dirichlet condition g. We denote this extension by $G \in H^{s+1/2}(\mathbb{R}^n)$ and consider problem (Homogeneous) with right hand side equal to $f - (-\Delta)^s G$:

$$\begin{cases} (-\Delta)^s v &= f - (-\Delta)^s G \quad \text{in } \Omega, \\ v &= 0 \quad \text{in } \Omega^c. \end{cases}$$

Due to Proposition 1.3.5, we know that $(-\Delta)^s G \in H^{-s+1/2}(\mathbb{R}^n)$, with

$$\|(-\Delta)^s G\|_{H^{-s+1/2}(\mathbb{R}^n)} \leq C \|G\|_{H^{s+1/2}(\mathbb{R}^n)} \leq C \|g\|_{H^{s+1/2}(\Omega^c)},$$

so that the right hand side function $f - (-\Delta)^s G$ belongs to $H^{-s+1/2}(\Omega)$. Applying Proposition 2.2.3, we obtain that the solution $v \in H^{s+1/2-\varepsilon}(\mathbb{R}^n)$, with

$$\|v\|_{H^{s+1/2-\varepsilon}(\mathbb{R}^n)} \leq C \left(\|f\|_{H^{-s+1/2}(\Omega)} + \|(-\Delta)^s G\|_{H^{-s+1/2}(\Omega)} \right).$$

Furthermore, as the solution of (Nonhomogeneous) is given by $u = v + G$, we deduce that $u \in H^{s+1/2-\varepsilon}(\mathbb{R}^n)$, and

$$\|u\|_{H^{s+1/2-\varepsilon}(\mathbb{R}^n)} \leq C\left(\|f\|_{H^{-s+1/2}(\Omega)} + \|g\|_{H^{s+1/2}(\Omega^c)}\right). \tag{5.2.2}$$

Next, we want to prove that $\lambda \in H^{-s+1/2-\varepsilon}(\Omega^c)$. Since $u = g$ in Ω^c, given $x \in \Omega^c$ it holds that

$$\lambda(x) = C(n,s)\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = (-\Delta)^s u(x) - C(-\Delta)^s_{\Omega^c} g(x).$$

So, given a function $v \in \widetilde{H}^{s-1/2+\varepsilon}(\Omega^c)$, we write

$$\left|\int_{\Omega^c} \lambda v\right| \leq \left(\|(-\Delta)^s u\|_{H^{-s+1/2-\varepsilon}(\Omega^c)} + C\|(-\Delta)^s_{\Omega^c} g\|_{H^{-s+1/2-\varepsilon}(\Omega^c)}\right)\|v\|_{\widetilde{H}^{s-1/2+\varepsilon}(\Omega^c)}.$$

Using Proposition 1.3.5, we deduce

$$\left|\int_{\Omega^c} \lambda v\right| \leq C\left(\|u\|_{H^{s+1/2-\varepsilon}(\mathbb{R}^n)} + \|(-\Delta)^s_{\Omega^c} g\|_{H^{-s+1/2-\varepsilon}(\Omega^c)}\right)\|v\|_{\widetilde{H}^{s-1/2+\varepsilon}(\Omega^c)}$$

and taking supremum in v we conclude that $\lambda \in H^{-s+1/2-\varepsilon}(\Omega^c)$, with

$$\|\lambda\|_{H^{-s+1/2-\varepsilon}(\Omega^c)} \leq C\,\Sigma_{f,g},$$

where we have used (5.2.2) in the last inequality and the notation (5.2.1). □

Remark 5.2.3. In view of Proposition 1.3.5, it might seem true that for every $\ell \in \mathbb{R}$ and $g \in H^\ell(\Omega^c)$ it holds that $(-\Delta)^s_{\Omega^c} g \in H^{\ell-2s}(\Omega^c)$, which in turn would imply that the hypothesis $(-\Delta)^s_{\Omega^c} g \in H^{-s+1/2}(\Omega^c)$ is superfluous. However, we have not been able neither to prove nor to disprove this claim. As a reference on what type of additional hypotheses are utilized to ensure this type of behavior of the restricted fractional Laplacian, we refer the reader to [106, Lemma 5.6].

Naturally, the homogeneous case $g \equiv 0$ satisfies the assumptions of Theorem 5.2.2.

Corollary 5.2.4. *Let Ω be a smooth domain and $f \in H^{-s+1/2}(\Omega)$. Let $u \in \widetilde{H}^s(\Omega)$ be the solution of* (Homogeneous) *and λ be its nonlocal normal derivative. Then, for all $\varepsilon > 0$, it holds that $\lambda \in H^{1/2-s-\varepsilon}(\Omega^c)$ and*

$$\|\lambda\|_{H^{-s+1/2-\varepsilon}(\Omega^c)} \leq C(n,s,\Omega,\varepsilon)\|f\|_{H^{-s+1/2}(\Omega)}.$$

Remark 5.2.5. We illustrate the sharpness of the regularity estimate for the nonlocal derivative from Theorem 5.2.2 (or from Corollary 5.2.4) with the following simple example. Let $\Omega = (-1,1)$ and consider the problem from Remark 2.3.12 with $x_0 = 0$:

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } (-1,1), \\ u = 0 & \text{in } \mathbb{R}\setminus(-1,1), \end{cases}$$

whose solution is given by $u(x) = c(s)(1-x^2)_+^s$ for some constant $c(s) > 0$. Our goal is to characterize the behavior of the nonlocal normal derivative of u near $\partial\Omega$, say, for $x \sim 1$. Let $x \in (1,2)$, then

$$\mathcal{N}_s u(x) = -C(s) \int_{-1}^{1} \frac{(1-y^2)^s}{(x-y)^{1+2s}}\, dy.$$

Observe that the integrand above is strictly positive, so that by restricting the integral to a region "close" to the boundary of Ω, we may bound

$$|\mathcal{N}_s u(x)| > C(s) \sum_{k=0}^{\infty} \int_{I_k} \frac{(1-y)^s}{(x-y)^{1+2s}}\, dy,$$

where

$$I_k := \left\{ y \in \Omega \colon \delta(y) \in \left(\frac{\delta(x)}{2^{k+1}}, \frac{\delta(x)}{2^k} \right] \right\} = \left[1 - \frac{\delta(x)}{2^k}, 1 - \frac{\delta(x)}{2^{k+1}} \right).$$

Since $\delta(y) = (1-y) > \frac{\delta(x)}{2^{k+1}}$ and $x - y = \delta(x) + \delta(y) \le \delta(x)\left(1 + \frac{1}{2^k}\right)$ for every $y \in I_k$, we deduce

$$|\mathcal{N}_s u(x)| > C(s) \sum_{k=0}^{\infty} \frac{|I_k|}{\delta(x)^{1+s}\, 2^s (2^k+1)^s}.$$

Finally, as $|I_k| = \frac{\delta(x)}{2^{k+1}}$ we conclude

$$|\mathcal{N}_s u(x)| > \frac{C(s)}{\delta(x)^s}. \tag{5.2.3}$$

Next, observe that $\delta(x)^\alpha \in H^\ell(1,2)$ if and only if $\ell < \alpha + 1/2$. Thus, by duality, we conclude from (5.2.3) that $\mathcal{N}_s u \notin H^{s-1/2}(1,2)$.

More generally, let $f : (-1,1) \to \mathbb{R}$ be a function such that its Gegenbauer coefficients f_j (in the expansion with respect to the basis $\{C_j^{(s+1/2)}\}$) satisfy

$$\sum_{j=0}^{\infty} \frac{f_j}{\lambda_j} C_j^{(s+1/2)}(1) \neq 0.$$

Recall from Theorem 2.1.3 that the solution to problem (Homogeneous) may be written as $u = \omega^s \phi$, where ϕ is a smooth, non-vanishing function as $x \to 1$. Therefore, the same argument as above applies: the nonlocal derivative of the solution of the homogeneous Dirichlet problem belongs to $H^{-s+1/2-\varepsilon}(\mathbb{R} \setminus (-1,1))$, and the $\varepsilon > 0$ cannot be removed.

5.3 Finite Element approximations

In this section we begin the study of finite element approximations to problem (5.1.2); we assume the volume constraint g to have bounded support. This assumption allows to simplify the error analysis of the numerical method we propose, but it is not necessary: in the next section, estimates for data not satisfying such hypothesis are deduced.

5.3.1 Domains and spaces

Given $H > 0$ big enough, we denote by Ω_H a domain containing Ω and such that

$$cH \leq \min_{x\in\partial\Omega,\, y\in\partial\Omega_H} d(x,y) \leq \max_{x\in\partial\Omega,\, y\in\partial\Omega_H} d(x,y) \leq CH, \tag{5.3.1}$$

where c, C are constants independent of H (see Figure 5.1). We set admissible trian-

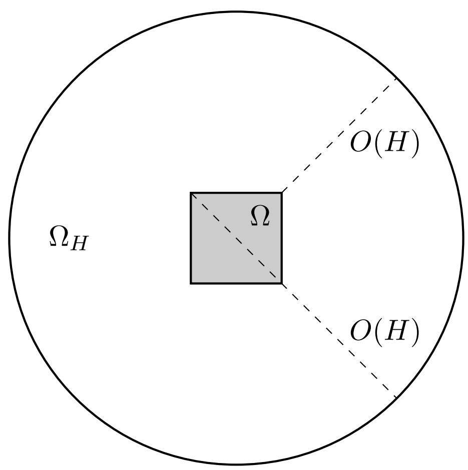

Figure 5.1: Auxiliary domain Ω_H, as defined by (5.3.1). The distance between any pair of points $(x,y) \in \partial\Omega \times \partial\Omega_H$ is of order of H.

gulations $\{\mathcal{T}_h\}$ defined on Ω_H in such a way that every triangulation meshes exactly Ω and $\Omega_H \setminus \Omega$. Moreover, to simplify our analysis, we assume the family of meshes to be globally quasi-uniform.

Remark 5.3.1. The parameter H depends on the mesh size h in such a way that as h goes to zero, H tends to infinity. The purpose of Ω_H is twofold: in first place, to provide a domain in which to implement the finite element approximations. In second place, the behavior of solutions may be controlled in the complement of Ω_H. Assuming g to have bounded support implies that, for h small enough, the domain Ω_H contains the support of the Dirichlet condition g. Moreover, since there is no reason to expect λ to be compactly supported, taking H depending adequately on h ensures that the decay of the nonlocal derivative in Ω_H^c is of the same order as the approximation error of u and λ in Ω_H.

We consider nodal basis functions

$$\varphi_1, \ldots, \varphi_{N_{int}}, \varphi_{N_{int}+1}, \ldots, \varphi_{N_{int}+N_{ext}},$$

where the first N_{int} nodes belong to the interior of Ω and the last N_{ext} to $\Omega_H \setminus \Omega$. The discrete spaces we consider consist of continuous, piecewise linear functions. We set

$$\begin{aligned}
\mathbb{V}_h &= \text{span}\,\{\varphi_1, \ldots, \varphi_{N_{int}+N_{ext}}\}, \\
K_h &= \text{span}\,\{\varphi_1, \ldots, \varphi_{N_{int}}\}, \\
\Lambda_h &= \text{span}\,\{\varphi_{N_{int}+1}, \ldots, \varphi_{N_{int}+N_{ext}}\}.
\end{aligned}$$

The spaces $\mathbb{V}_h$ and Λ_h are endowed with the $\|\cdot\|_{\mathbb{V}}$ and $\|\cdot\|_{\Lambda}$ norms, respectively. We set all the discrete functions to vanish on $\partial\Omega_H$, so that $\mathbb{V}_h \subset \widetilde{H}^s(\Omega_H)$.

5.3.2 Discrete problem

With the notation previously introduced, the discrete formulation of (5.1.2) reads: find $(u_h, \lambda_h) \in \mathbb{V}_h \times \Lambda_h$ such that

$$\left\{ \begin{aligned} a(u_h, v_h) - b(v_h, \lambda_h) &= F(v_h) \quad \forall v_h \in \mathbb{V}_h, \\ b(u_h, \mu_h) &= G(\mu_h) \quad \forall \mu_h \in \Lambda_h. \end{aligned} \right. \tag{5.3.2}$$

Unlike the problems from chapters 3 and 4, the well-posedness of the discrete problem is not an immediate consequence of the well-posedness of the continuous one. This is caused because coercivity is a property that is carried into subspaces, but the inf-sup conditions are not. Thus, as the schemes proposed for the Dirichlet homogeneous and the eigenvalue problem for the fractional Laplacian were based on the coercivity of the bilinear form involved, discrete well-posedness was not an issue. Here, we have to check that the formulation (5.3.2) satisfies the hypotheses of Theorem 5.1.3.

First, observe that the discrete kernel of the restriction of b to $\mathbb{V}_h \times \Lambda_h$ coincides with the space K_h, that is, it consists of piecewise linear functions over the triangulation of Ω that vanish on $\partial\Omega$. The coercivity of a on K_h is deduced immediately.

Lemma 5.3.2. *There exists a constant $C > 0$, independent of h, such that for all $v_h \in K_h$,*

$$a(v_h, v_h) \geq C \|v_h\|_{\mathbb{V}}^2.$$

Proof. Observe that K_h is a subspace of the continuous kernel K given by (5.1.4). The lemma follows by the coercivity of a on K. □

In order to prove the discrete inf-sup condition, we utilize a projection over the discrete space. Since $\mathbb{V}_h \subset \widetilde{H}^{3/2-\varepsilon}(\Omega_H)$ for all $\varepsilon > 0$, it is possible to define the the L^2-projection of functions in the dual space of $\widetilde{H}^{3/2-\varepsilon}(\Omega_H)$. Namely, we consider $P_h : H^{-\sigma}(\Omega_H) \to \mathbb{V}_h$ for $0 \leq \sigma \leq 1$, the operator characterized by

$$\int_{\Omega_H} (w - P_h w)\, v_h = 0 \quad \forall v_h \in \mathbb{V}_h. \tag{5.3.3}$$

Under certain regularity assumption on the meshes, this L^2-projection turns out to be H^σ-stable.

Lemma 5.3.3. *Let $0 < \sigma < 1$, and assume the triangulation to be quasi-uniform. Then, there exists a constant C, independent of h, such that*

$$\|P_h w\|_{H^\sigma(\Omega_H)} \leq C\|w\|_{H^\sigma(\Omega_H)}$$

for all $w \in H^\sigma(\Omega_H)$.

Proof. The proof will follow by interpolation. On the one hand, the L^2-stability estimate

$$\|P_h w\|_{L^2(\Omega_H)} \leq \|w\|_{L^2(\Omega_H)}$$

is obvious. On the other hand, the H^1 bound

$$\|P_h w\|_{H^1(\Omega_H)} \leq C\|w\|_{H^1(\Omega_H)}$$

is a consequence of a global inverse inequality (see, for example [8]). □

Remark 5.3.4. The global quasi-uniformity hypothesis could actually be weakened and substituted by the ones from [22, 30, 37]. In these works, meshes are required to be just locally quasi-uniform, but some extra control on the change in measures of neighboring elements is requested as well.

By duality, it is possible to obtain stability estimates in negative-order norms.

Lemma 5.3.5. *Let $0 \leq \sigma \leq 1$, and assume the triangulation to be quasi-uniform. There exists a constant C, independent of h, such that*

$$\|P_h w\|_{\widetilde{H}^{-\sigma}(\Omega_H)} \leq C\|w\|_{\widetilde{H}^{-\sigma}(\Omega_H)}$$

for all $w \in \widetilde{H}^{-\sigma}(\Omega_H)$.

Proof. Consider $v \in H^\sigma(\Omega_H)$. Since P_h is self-adjoint, it holds that

$$\int_{\Omega_H} P_h w\, v = \int_{\Omega_H} w\, P_h v \leq \|w\|_{\widetilde{H}^{-\sigma}(\Omega_H)} \|P_h v\|_{H^\sigma(\Omega_H)}.$$

The proof follows by the H^σ-stability of P_h. □

Remark 5.3.6. For simplicity, the previous lemma was stated for functions defined in Ω_H, but clearly it is also valid over $\Omega_H \setminus \Omega$:

$$\|P_h w\|_{\widetilde{H}^{-\sigma}(\Omega_H\setminus\Omega)} \leq C\|w\|_{\widetilde{H}^{-\sigma}(\Omega_H\setminus\Omega)} \quad \forall w \in \widetilde{H}^{-\sigma}(\Omega_H \setminus \Omega). \tag{5.3.4}$$

The discrete inf-sup condition is a consequence of the fact that the L^2-projection is a Fortin operator for (5.3.2).

Proposition 5.3.7. *There exists a constant C, independent of h, such that the following discrete inf-sup condition holds:*

$$\sup_{v_h\in\mathbb{V}_h} \frac{b(v_h,\mu_h)}{\|v_h\|_{\mathbb{V}}} \geq C\|\mu_h\|_\Lambda \quad \forall \mu_h \in \Lambda_h. \tag{5.3.5}$$

Proof. In first place, let $E\colon H^s(\Omega^c) \to H^s(\mathbb{R}^n)$ be the extension operator (5.1.6) and P_h be the L^2-projection considered in this section. We denote, for $v \in H^s(\Omega^c)$, $P_h(Ev) := P_h\left((Ev)\big|_{\Omega_H}\right)$. Taking into account the fact that $P_h(Ev) \in \widetilde{H}^s(\Omega_H)$ (so that we may apply Corollary 1.2.11) and the continuity of these operators, it is clear that

$$\|P_h(Ev)\|_{\mathbb{V}} = \|P_h(Ev)\|_{\widetilde{H}^s(\Omega_H)} \leq C\|v\|_{H^s(\Omega^c)} \quad \forall v \in H^s(\Omega^c).$$

The conclusion follows easily. Indeed, let $\mu_h \in \Lambda_h$, $v \in H^s(\Omega^c)$ and write

$$\sup_{v_h\in\mathbb{V}_h} \frac{b(v_h,\mu_h)}{\|v_h\|_{\mathbb{V}}} \geq \frac{b(P_h(Ev),\mu_h)}{\|P_h(Ev)\|_{\mathbb{V}}} \geq C\frac{b(v,\mu_h)}{\|v\|_{H^s(\Omega^c)}}.$$

Using the fact that v is arbitrary together with (5.1.7), we deduce (5.3.5). □

Due to the standard theory of finite element approximations of saddle point problems (see, for example, [16, Theorem 5.2.5]), we deduce the following estimate.

Proposition 5.3.8. *Let $(u,\lambda) \in \mathbb{V}\times\Lambda$ and $(u_h,\lambda_h) \in \mathbb{V}_h\times\Lambda_h$ be the respective solutions of problems* (5.1.2) *and* (5.3.2)*. Then there exists a constant C, independent of h, such that*

$$\|u-u_h\|_{\mathbb{V}} + \|\lambda-\lambda_h\|_\Lambda \leq C\left(\inf_{v_h\in\mathbb{V}_h}\|u-v_h\|_{\mathbb{V}} + \inf_{\mu_h\in\Lambda_h}\|\lambda-\mu_h\|_\Lambda\right). \tag{5.3.6}$$

5.3.3 Approximation properties

In order to obtain convergence order estimates for the finite element approximations under consideration, it remains to estimate the infima on the right hand side of (5.3.6). Within Ω_H, this is achieved by means of a quasi-interpolation operator, as described in

Section 3.2. We denote such an operator by I_h; depending on whether discrete functions are required to have zero trace or not, I_h could be either the Clément or the Scott-Zhang operator. Recall identity (3.3.1):

$$\|v - I_h v\|_{H^\sigma(\Omega)} \le C h^{\ell-\sigma} \|v\|_{H^\ell(\Omega)} \quad \forall v \in H^\ell(\Omega),\ 0 \le \sigma \le \ell. \tag{5.3.7}$$

In first place, based on a simple duality argument, we bound the L^2-projection error in negative-order norms.

Lemma 5.3.9. *Given $v \in L^2(\Omega_H \setminus \Omega)$ and $0 \le \sigma \le 1$, the following estimate holds:*

$$\|v - P_h v\|_{\widetilde{H}^{-\sigma}(\Omega_H\setminus\Omega)} \le C h^\sigma \|v\|_{L^2(\Omega_H\setminus\Omega)}. \tag{5.3.8}$$

Proof. Let $v \in L^2(\Omega_h \setminus \Omega)$. Given $\varphi \in H^\sigma(\Omega_H \setminus \Omega)$, considering the quasi-interpolation operator I_h and taking into account (5.3.3),

$$\begin{aligned}\frac{\int_{\Omega_H\setminus\Omega}(v - P_h v)\varphi}{\|\varphi\|_{H^\sigma(\Omega_H\setminus\Omega)}} &= \frac{\int_{\Omega_H\setminus\Omega}(v - P_h v)(\varphi - I_h\varphi)}{\|\varphi\|_{H^\sigma(\Omega_H\setminus\Omega)}} \\ &\le \|v - P_h v\|_{L^2(\Omega_H\setminus\Omega)} \frac{\|\varphi - I_h\varphi\|_{L^2(\Omega_H\setminus\Omega)}}{\|\varphi\|_{H^\sigma(\Omega_H\setminus\Omega)}}.\end{aligned}$$

Combining (5.3.7) with the trivial estimate $\|v - P_h v\|_{L^2(\Omega_H\setminus\Omega)} \le \|v\|_{L^2(\Omega_H\setminus\Omega)}$, we conclude the proof. □

Next, we estimate the approximation errors within the meshed domain.

Proposition 5.3.10. *If supp$(g) \subset \Omega_H$, then the following estimates hold:*

$$\inf_{v_h\in\mathbb{V}_h} \|u - v_h\|_{H^s(\Omega_H)} \le C h^{1/2-\varepsilon}\, \Sigma_{f,g}, \tag{5.3.9}$$

$$\inf_{\mu_h\in\Lambda_h} \|\lambda - \mu_h\|_{\widetilde{H}^{-s}(\Omega_H\setminus\Omega)} \le C h^{1/2-\varepsilon}\, \Sigma_{f,g}, \tag{5.3.10}$$

where $\Sigma_{f,g}$ is given by (5.2.1).

Proof. Estimate (5.3.9) is easily attained by taking into account that u vanishes on $\partial\Omega_H$, and applying the regularity estimate (5.2.2) jointly with (3.3.1).

In order to prove (5.3.10), in first place we assume $s \le 1/2$, so that $\lambda \in L^2(\Omega_H \setminus \Omega)$. Since $I_h\lambda = P_h(I_h\lambda)$, it follows that

$$\begin{aligned}\|\lambda - P_h\lambda\|_{L^2(\Omega_H\setminus\Omega)} &\le \|\lambda - I_h\lambda\|_{L^2(\Omega_H\setminus\Omega)} + \|P_h(I_h\lambda - \lambda)\|_{L^2(\Omega_H\setminus\Omega)} \\ &\le 2\|\lambda - I_h\lambda\|_{L^2(\Omega_H\setminus\Omega)}.\end{aligned}$$

Therefore, writing $\lambda - P_h\lambda = \lambda - P_h\lambda - P_h(\lambda - P_h\lambda)$, applying (5.3.8) and (5.3.7), we obtain (5.3.10) immediately:

$$\|\lambda - P_h\lambda\|_{\widetilde{H}^{-s}(\Omega_H\setminus\Omega)} \le Ch^s\|\lambda - P_h\lambda\|_{L^2(\Omega_H\setminus\Omega)} \le Ch^s\|\lambda - I_h\lambda\|_{L^2(\Omega_H\setminus\Omega)} \le Ch^{1/2-\varepsilon}\Sigma_{f,g}.$$

Meanwhile, if $s > 1/2$, considering $\sigma = s$ in (5.3.4) and (5.3.8), we obtain:

$$\|w - P_h w\|_{\widetilde{H}^{-s}(\Omega_H\setminus\Omega)} \le C\|w\|_{\widetilde{H}^{-s}(\Omega_H\setminus\Omega)},$$
$$\|w - P_h w\|_{\widetilde{H}^{-s}(\Omega_H\setminus\Omega)} \le Ch^s\|w\|_{L^2(\Omega_H\setminus\Omega)}.$$

Interpolating these two identities, recalling the regularity of λ given by Theorem 5.2.2 and observing that $\widetilde{H}^{-s+1/2-\varepsilon}(\Omega_H \setminus \Omega) = H^{-s+1/2-\varepsilon}(\Omega_H \setminus \Omega)$ (cf. Corollary 1.2.12), we deduce that

$$\|\lambda - P_h\lambda\|_{\widetilde{H}^{-s}(\Omega_H\setminus\Omega)} \le Ch^{1/2-\varepsilon}\|\lambda\|_{H^{-s+1/2-\varepsilon}(\Omega_H\setminus\Omega)} \le Ch^{1/2-\varepsilon}\,\Sigma_{f,g}.$$

□

As the norms in both $\mathbb{V}$ and Λ involve integration on unbounded domains and the discrete functions vanish outside Ω_H, to estimate the infima in (5.3.6) we also need to rely on identities that do not depend on the discrete approximation but on the behavior of u and λ. For the part corresponding to the norm of u the Poincaré inequality suffices, while for the nonlocal derivative contribution it is necessary to formulate an explicit decay estimate.

Proposition 5.3.11. *If supp$(g) \subset \Omega_H$, then there exists a constant C, independent of f and g, such that the estimate*

$$\|\lambda\|_{\widetilde{H}^{-s}(\Omega_H^c)} \le \|\lambda\|_{L^2(\Omega_H^c)} \le CH^{-(n/2+2s)}\,\Sigma_{f,g}$$

holds.

Proof. It is evident that $\|\lambda\|_{\widetilde{H}^{-s}(\Omega_H^c)} \le \|\lambda\|_{L^2(\Omega_H^c)}$. Since supp$(g) \subset \Omega_H$, for every $x \in \Omega_H^c$ it holds that

$$|\lambda(x)| \le C(n,s)\int_\Omega \frac{|u(y)|}{|x-y|^{n+2s}}\,dy.$$

Consider the auxiliary function $\omega : \Omega \to \mathbb{R}$,

$$\omega(y) = \left(\int_{\Omega_H^c} \frac{1}{|x-y|^{2(n+2s)}}\right)^{1/2}.$$

Integrating in polar coordinates, we straightforwardly deduce

$$|\omega(y)| \le CH^{-(n/2+2s)} \quad \forall y \in \Omega.$$

As a consequence, integrating the pointwise estimate for λ, applying Minkowski's integral inequality and resorting to the pointwise estimate for ω,

$$\|\lambda\|_{L^2(\Omega_H^c)} \leq C H^{-(n/2+2s)} \int_\Omega |u(y)|\, dy.$$

Since Ω is bounded, the L^1-norm of u may be bounded by its L^2-norm, and this in turn is controlled in terms of the data (see, for example, (5.2.2)). □

Remark 5.3.12. As the finite element approximation u_h to u in Ω_H has an H^s-error of order $h^{1/2-\varepsilon}$, we need the previous estimate for the nonlocal derivative to be at least of the same order. Thus, we require $H^{-(n/2+2s)} \leq Ch^{1/2}$, that is, $H \geq Ch^{-1/(n+4s)}$.

Collecting the estimates we have developed so far, we are in position to prove the following.

Theorem 5.3.13. *Let Ω be a bounded, smooth domain, $f \in H^{-s+1/2-\varepsilon}(\Omega)$ and $g \in H^{s+1/2-\varepsilon}(\Omega^c)$ for some $\varepsilon > 0$. Moreover, assume that g has bounded support and consider Ω_H according to* (5.3.1)*, with $H \geq Ch^{-1/(n+4s)}$. For the finite element approximations considered in this work and h small enough, recalling* (5.2.1)*, the following a priori estimates hold:*

$$\|u - u_h\|_{\mathbb{V}} \leq Ch^{1/2-\varepsilon}\, \Sigma_{f,g}, \tag{5.3.11}$$

$$\|\lambda - \lambda_h\|_{\Lambda} \leq Ch^{1/2-\varepsilon}\, \Sigma_{f,g}. \tag{5.3.12}$$

Proof. In order to obtain the above two inequalities, it is enough to estimate the infima in (5.3.6). Since g is boundedly supported and $H \to \infty$ as $h \to 0$, if h is small enough then $\operatorname{supp}(g) \subset \Omega_H$. So, $u - v_h \in \widetilde{H}^s(\Omega_H)$ for all $v_h \in \mathbb{V}_h$ and thus we may apply the Hardy inequality (cf. Corollary 1.2.11) together with (5.3.9):

$$\inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{\mathbb{V}} \leq C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{H^s(\Omega_H)} \leq Ch^{1/2-\varepsilon}\, \Sigma_{f,g}.$$

The infimum involving the nonlocal derivative is estimated as follows. Because functions in Λ_h are supported in Ω_H, it holds that

$$\inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_{\Lambda} \leq \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_{\widetilde{H}^{-s}(\Omega_H \setminus \Omega)} + \|\lambda\|_{\widetilde{H}^{-s}(\Omega_H^c)}.$$

The first term on the right hand side is bounded by means of equation (5.3.10), whereas for the second one we apply Proposition 5.3.11 and notice that the choice of H implies that $H^{-(n/2+2s)} \leq Ch^{1/2}$. It follows that

$$\inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_{\Lambda} \leq Ch^{1/2-\varepsilon}\, \Sigma_{f,g},$$

and the proof is completed. □

5.4 Volume constraint truncation error

The finite element approximations performed in the previous section refer to a problem in which the Dirichlet condition g has bounded support. Here, we develop error estimates without this restriction on the volume constraints. However, as it is not possible to mesh the whole support of g, we are going to take into account the Dirichlet condition in the set Ω_H considered in the previous section. So, we need to compare u, the solution to (Nonhomogeneous) to $\tilde{u}$, the solution to

$$\begin{cases} (-\Delta)^s \tilde{u} = f & \text{in } \Omega, \\ \tilde{u} = \tilde{g} & \text{in } \Omega^c, \end{cases} \tag{5.4.1}$$

where $\tilde{g}$ is a smooth truncation of g: given a function $\phi \in C^\infty(\mathbb{R}^n)$ such that $0 \le \phi \le 1$, $\phi = 1$ on Ω_{H-1}, $\phi = 0$ on Ω_H^c, we set $\tilde{g} := g\phi$. This allows to apply the finite element estimates developed in Section 5.3 to problem (5.4.1), because $\text{supp}(\tilde{g}) \subset \overline{\Omega_H}$. The objective of this section is to show that choosing H in the same fashion as there, namely $H \ge Ch^{-1/(n+4s)}$, leads to the same order of error between the continuous truncated problem and the original one.

Since the problems under consideration are linear, without loss of generality we may assume that $g \ge 0$ (otherwise, split $g = g_+ - g_-$ and work with the two problems separately).

Proposition 5.4.1. *The following estimate holds:*

$$|u - \tilde{u}|_{H^s(\Omega)} \le CH^{-(n/2+2s)} \|g\|_{L^2(\Omega_H^c)}. \tag{5.4.2}$$

Proof. We denote $\varphi = u - \tilde{u}$ the difference between the solutions to equations (Nonhomogeneous) and (5.4.1); it is noteworthy that φ is nonnegative, $s-$harmonic in Ω and vanishes in $\Omega_{H-1} \setminus \Omega$.

Moreover, let us consider $\tilde{\varphi} = \varphi \chi_\Omega$. As $\varphi \in H^{s+1/2-\varepsilon}(\mathbb{R}^n)$ vanishes in $\Omega_{H-1} \setminus \Omega$, it is clear that $\tilde{\varphi} \in \widetilde{H}^{s+1/2-\varepsilon}(\Omega)$, and applying the integration by parts formula (1.2.8):

$$a(\varphi, \tilde{\varphi}) = \int_\Omega \tilde{\varphi} (-\Delta)^s \varphi = 0.$$

The nonlocal normal derivative term in last equation is null because $\tilde{\varphi}$ vanishes in Ω^c. Splitting the integrand appearing in the form a and recalling the definition of ω_Ω^s (1.2.6), we obtain

$$\begin{aligned} |\varphi|^2_{H^s(\Omega)} &= -2 \int_\Omega \varphi^2(x)\, \omega^s_\Omega(x)\, dx + 2 \int_\Omega \varphi(x) \left(\int_{\Omega^c_{H-1}} \frac{g(y) - \tilde{g}(y)}{|x-y|^{n+2s}}\, dy \right) dx \\ &\le 2 \int_\Omega \varphi(x) \left(\int_{\Omega^c_{H-1}} \frac{g(y) - \tilde{g}(y)}{|x-y|^{n+2s}}\, dy \right) dx. \end{aligned} \tag{5.4.3}$$

Applying the Cauchy-Schwarz inequality in the integral over Ω^c_{H-1} and taking into account that $g - \tilde{g} \leq g$ and that $(H-1)^{-(n/2+2s)} \simeq H^{-(n/2+2s)}$, it follows immediately that

$$|\varphi|^2_{H^s(\Omega)} \leq C(n,s) H^{-(n/2+2s)} \|\varphi\|_{L^1(\Omega)} \|g\|_{L^2(\Omega^c_{H-1})}. \tag{5.4.4}$$

We need to bound $\|\varphi\|_{L^1(\Omega)}$ adequately. Let $\psi \in H^s(\mathbb{R}^n)$ be a function that equals 1 over Ω. Multiplying $(-\Delta)^s \varphi$ by ψ, integrating on Ω and applying (1.2.8), since φ is s-harmonic in Ω, we obtain

$$0 = a(\varphi, \psi) - \int_{\Omega^c} \mathcal{N}_s \phi(y)\, \psi(y)\, dy,$$

or equivalently,

$$\begin{aligned} 0 = C(n,s) &\int_\Omega \int_{\Omega^c} \frac{(\varphi(x) - \varphi(y))(1 - \psi(y))}{|x-y|^{n+2s}}\, dy\, dx \\ - C(n,s) &\int_{\Omega^c} \left(\int_\Omega \frac{\varphi(y) - \varphi(x)}{|x-y|^{n+2s}}\, dx \right) \psi(y)\, dy. \end{aligned}$$

This implies that

$$\int_\Omega \int_{\Omega^c} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2s}}\, dy\, dx = 0.$$

Recalling that φ is zero in $\Omega_{H-1} \setminus \Omega$ and that $g - \tilde{g} \leq g$, we have

$$\int_\Omega \varphi(x)\, \omega^s_\Omega(x)\, dx = \int_\Omega \int_{\Omega^c_{H-1}} \frac{g(y) - \tilde{g}(y)}{|x-y|^{n+2s}} dy dx \leq C H^{-(n/2+2s)} \|g\|_{L^2(\Omega^c_{H-1})}.$$

Recall that the function ω^s_Ω is uniformly bounded below in Ω and that $\varphi \geq 0$. We deduce

$$\|\varphi\|_{L^1(\Omega)} \leq C H^{-(n/2+2s)} \|g\|_{L^2(\Omega^c_{H-1})}, \tag{5.4.5}$$

and combining this bound with (5.4.4) yields (5.4.2). □

As a byproduct of the proof of the previous proposition, we obtain the following.

Lemma 5.4.2. *There is a constant C such that the bound*

$$\|u - \tilde{u}\|_{L^2(\Omega)} \leq C H^{-(n/2+2s)} \|g\|_{L^2(\Omega^c_{H-1})} \tag{5.4.6}$$

holds.

Proof. As before, we write $\varphi = u - \tilde{u}$. From the first line in (5.4.3),

$$2 \int_\Omega \varphi^2(x)\, \omega^s_\Omega(x)\, dx \leq \int_\Omega \varphi(x) \left(\int_{\Omega^c_{H-1}} \frac{g(y) - \tilde{g}(y)}{|x-y|^{n+2s}}\, dy \right) dx \leq$$

$$\leq CH^{-(n/2+2s)} \|\varphi\|_{L^1(\Omega)} \|g\|_{L^2(\Omega^c_{H-1})}.$$

Combining this estimate with (5.4.5), we deduce

$$\int_\Omega \varphi^2(x)\omega^s_\Omega(x)\,dx \leq CH^{-(n+4s)}\|g\|^2_{L^2(\Omega^c_{H-1})},$$

where the function ω^s_Ω is given by Definition 1.2.19. The lower uniform boundedness of ω^s_Ω implies (5.4.6) immediately. □

Given $\tilde{u}$, the solution to (5.4.1), let us denote $\tilde{\lambda} := \mathcal{N}_s \tilde{u}$ its nonlocal normal derivative.

Proposition 5.4.3. *There is a constant C such that*

$$\|\lambda - \tilde{\lambda}\|_\Lambda \leq CH^{-(n/2+2s)}\|g\|_{L^2(\Omega^c_{H-1})}. \tag{5.4.7}$$

Proof. Let $\phi \in H^s(\Omega^c)$. We consider an extension $E\phi \in H^s(\mathbb{R}^n)$ such that $\|E\phi\|_{H^s(\mathbb{R}^n)} \leq C\|\phi\|_{H^s(\Omega^c)}$. By linearity, it is clear that $\lambda - \tilde{\lambda} = \mathcal{N}_s\varphi$, where $\varphi = u - \tilde{u}$. Applying the integration by parts formula (1.2.8) and recalling that φ is $s-$harmonic in Ω,

$$\int_{\Omega^c} (\lambda - \tilde{\lambda})\phi = \frac{C(n,s)}{2} \iint_Q \frac{(\varphi(x)-\varphi(y))(E\phi(x)-E\phi(y))}{|x-y|^{n+2s}}\,dx\,dy.$$

Since φ vanishes in $\Omega_{H-1}\setminus\Omega$, it is simple to bound

$$\begin{aligned}&\int_{\Omega^c} (\lambda - \tilde{\lambda})\,\phi \\ &\qquad \leq C\left(\left|\langle \varphi, E\phi\rangle_{H^s(\Omega)}\right| + \left| \int_\Omega \int_{\Omega^c_{H-1}} \frac{(\varphi(x)-\varphi(y))(E\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy \right| \right).\end{aligned}$$

The first term in the right hand side above is bounded by $C|\varphi|_{H^s(\Omega)}\|\phi\|_{H^s(\Omega^c)}$, and Proposition 5.4.1 allows to bound $|\varphi|_{H^s(\Omega)} \leq CH^{-(n/2+2s)}\|g\|_{L^2(\Omega^c_{H-1})}$. For the second term, splitting the integrand it is simple to obtain the estimates:

$$\begin{aligned}
\left| \int_\Omega \varphi(x)E\phi(x) \left(\int_{\Omega^c_{H-1}} \frac{1}{|x-y|^{n+2s}}\,dy \right) dx \right| &\leq C\|\varphi\|_{L^2(\Omega)}\|E\phi\|_{L^2(\Omega)}, \\
\left| \int_\Omega \varphi(x) \left(\int_{\Omega^c_{H-1}} \frac{\phi(y)}{|x-y|^{n+2s}}\,dy \right) dx \right| &\leq CH^{-(n/2+2s)}\|\varphi\|_{L^1(\Omega)}\|\phi\|_{L^2(\Omega^c_{H-1})}, \\
\left| \int_\Omega E\phi(x) \left(\int_{\Omega^c_{H-1}} \frac{\varphi(y)}{|x-y|^{n+2s}}\,dy \right) dx \right| &\leq CH^{-(n/2+2s)}\|E\phi\|_{L^1(\Omega)}\|\varphi\|_{L^2(\Omega^c_{H-1})},
\end{aligned}$$

$$\left| \int_\Omega \left(\int_{\Omega^c_{H-1}} \frac{\varphi(y)\phi(y)}{|x-y|^{n+2s}} \, dy \right) dx \right| \le C H^{-(n+2s)} \|\varphi\|_{L^2(\Omega^c_H)} \|\phi\|_{L^2(\Omega^c_{H-1})}.$$

The terms in the right hand sides of the inequalities above are estimated applying Lemma 5.4.2 and Proposition 5.4.1, as well as recalling the continuity of the extension operator and of the inclusion $L^2(\Omega) \subset L^1(\Omega)$. We obtain

$$\frac{\int_{\Omega^c} (\lambda - \tilde{\lambda})\phi}{\|\phi\|_{H^s(\Omega^c)}} \le C H^{-(n/2+2s)} \|g\|_{L^2(\Omega^c_{H-1})} \quad \forall \phi \in H^s(\Omega^c).$$

Taking supremum in ϕ, estimate (5.4.7) follows. □

Combining the estimates obtained in this section, we immediately prove the following result.

Theorem 5.4.4. *Let* (u, λ) *be the solution of problem* (1.2.11)*, and consider* $\tilde{g}$ *as in the beginning of this section. Moreover, let* (u_h, λ_h) *be the finite element approximations of the truncated problem* (5.4.1)*, defined on* Ω_H*, where* H *behaves as* $h^{-1/(n+4s)}$*. Then,*

$$\|u - u_h\|_{H^s(\Omega_{H-1})} \le C h^{1/2-\varepsilon} \, \Sigma_{f,g}$$

and

$$\|\lambda - \lambda_h\|_\Lambda \le C h^{1/2-\varepsilon} \, \Sigma_{f,g}. \tag{5.4.8}$$

Proof. Applying the triangle inequality, we write

$$\|u - u_h\|_{H^s(\Omega_{H-1})} \le \|u - \tilde{u}\|_{H^s(\Omega_{H-1})} + \|\tilde{u} - u_h\|_{H^s(\Omega_{H-1})}.$$

The second term above is bounded by $\|\tilde{u} - u_h\|_{\mathbb{V}}$, which is controlled by (5.3.11). As for the first one, recall that $u = \tilde{u}$ in $\Omega_{H-1} \setminus \Omega$, so that

$$\begin{aligned} &\|u - \tilde{u}\|^2_{H^s(\Omega_{H-1})} \\ &\qquad = \|u - \tilde{u}\|^2_{H^s(\Omega)} + 2 \int_\Omega |u(x) - \tilde{u}(x)|^2 \left(\int_{\Omega_{H-1} \setminus \Omega} \frac{1}{|x-y|^{n+2s}} \, dy \right) dx. \end{aligned}$$

The integral above is bounded, if $s \neq 1/2$, by means of Hardy-type inequalities from Proposition 1.2.10, because $(u - \tilde{u})\chi_\Omega$ belongs to $\widetilde{H}^s(\Omega)$. If $s = 1/2$ proceed as in Subsection 2.3.4 (more specifically, as in (2.3.12)). So, resorting to Proposition 5.4.1 and Lemma 5.4.2,

$$\|u - \tilde{u}\|_{H^s(\Omega_{H-1})} \le C \|u - \tilde{u}\|_{H^s(\Omega)} \le C H^{-(n/2+2s)},$$

which –taking into account the behavior of H– is just (5.3.11).

Estimate (5.4.8) is an immediate consequence of the triangle inequality, the dependence of H on h and equations (5.4.7) and (5.3.12). Indeed,

$$\begin{aligned}\|\lambda - \lambda_h\|_\Lambda &\le \|\lambda - \tilde{\lambda}\|_\Lambda + \|\tilde{\lambda} - \lambda_h\|_\Lambda \\ &\le C h^{1/2-\varepsilon}\, \Sigma_{f,g}.\end{aligned}$$

□

Remark 5.4.5. Since it is only possible to mesh a bounded domain, there is no hope in general to obtain convergence estimates for $\|u - u_h\|_{\mathbb{V}}$, unless some extra hypothesis on the decay of the volume constraint is included.

5.5 Numerical experiments

We display the results of the computational experiments performed for the mixed formulation of (Nonhomogeneous). The scheme utilized for these two-dimensional examples is based on [2]. Appendix A includes some details about the computation of the matrix having entries $a(\varphi_i, \varphi_j)$, while in Appendix B some auxiliary computations can be found.

Besides the scarcity of explicit solutions, we face another challenge in these experiments: our convergence estimates (theorems 5.3.13 and 5.4.4) are expressed in terms of fractional-order norms, and it is not possible to carry out a computation such as Lemma 3.5.1. The examples we provide give evidence of the convergence of the scheme towards the solution u both for Dirichlet data with bounded and unbounded support. We compute orders of convergence in L^2-norms.

Example 5.5.1. Our first example is the same as in Remark 2.3.12 and Example 3.5.2; however, we shrink the domain Ω so that we produce a nonhomogeneous boundary condition with bounded support. Namely, for $\Omega = B(0, 1/2) \subset \mathbb{R}^2$ we study

$$\begin{cases} (-\Delta)^s u &= 1 \quad \text{in } \Omega, \\ \qquad u &= \frac{1}{2^{2s}\Gamma(1+s)^2}(1 - |\cdot|^2)_+^s \quad \text{in } \Omega^c. \end{cases}$$

The exact solution of this problem is $u(x) = \frac{1}{2^{2s}\Gamma(1+s)^2}(1 - |x|^2)_+^s$.

We estimate the L^2 norms of the finite element errors inside the domain Ω and in the whole space. In every case, the auxiliary domains were taken in such a way that $\text{supp}(g) \subset \Omega_H$. So, errors in $L^2(\Omega_H)$ coincide with errors in $L^2(\mathbb{R}^n)$.

Our results are summarized in Table 5.1. Since the exact solution is smooth in $\overline{\Omega}$ (see Figure 5.2), a fast convergence is observed in that domain; an order of convergence 2 is expected. The situation in $\mathbb{R}^n$ is quite different, because the solution belongs to $H^{s+1/2-\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$; in general, we observe convergence with order approximately $s + 1/2$ in the $L^2(\mathbb{R}^n)$-norm.

s	Order u_h ($L^2(\Omega)$)	Order u_h ($L^2(\mathbb{R}^n)$)
0.1	1.9212	0.6999
0.2	2.2540	0.8303
0.3	2.4332	0.8495
0.4	2.0137	0.8766
0.5	2.0518	1.0236
0.6	2.0849	1.0974
0.7	2.2134	1.2408
0.8	1.9691	1.3334
0.9	1.8033	1.4392

Table 5.1: Computational results for Example 5.5.1. The support of the volume constraint was contained in every auxiliary domain Ω_H.

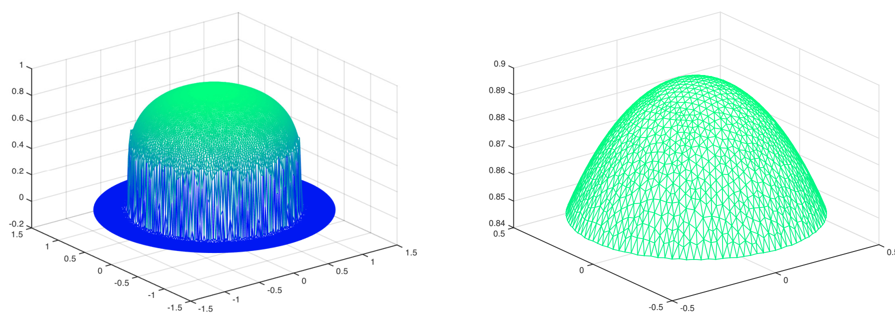

Figure 5.2: Discrete solutions to Example 5.5.1 with $s = 0.2$, computed on a mesh with size $h = 0.03$ in an auxiliary domain $\Omega_H = B(0, H)$, $H = 1.39$. Left panel displays the solution in Ω_H, while the right panel exhibits the solution in Ω. Although the solution is not smooth in $\mathbb{R}^n$, it is smooth in $\overline{\Omega}$.

We next display two examples where the Dirichlet condition has unbounded support, posed in the two-dimensional unit ball. The Poisson kernel for this domain is known, and thus it is simple to obtain an explicit expression for the solutions of problems as the two we analyze next (see (B.0.2)). Evaluating those expressions is not so simple; see Appendix B for a description of the method utilized for this purpose.

Example 5.5.2. Consider problem

$$\begin{cases} (-\Delta)^s u &= 0 \qquad \text{in } B(0,1) \subset \mathbb{R}^2, \\ u &= \exp(-|\cdot|^2) \qquad \text{in } \Omega^c. \end{cases}$$

In Table 5.2 we exhibit some computations. We display orders of convergence in $L^2(\Omega)$, $L^2(\Omega_H)$ and $L^2(\mathbb{R}^n)$. Unlike the previous example, solutions are not smooth up to the boundary of Ω. So, the observed convergence with orders approximately $s + 1/2$ is expected.

s	Order in $L^2(\Omega)$	Order in $L^2(\Omega_H)$	Order in $L^2(\mathbb{R}^n)$
0.1	0.6355	0.9328	0.8996
0.2	0.7774	1.0555	0.8132
0.3	0.8559	1.0441	0.7387
0.4	0.9052	0.9574	0.6214
0.5	0.9720	0.9180	0.5723
0.6	1.1535	0.8727	0.5163
0.7	1.2742	0.8241	0.4667
0.8	1.3152	0.7885	0.4338
0.9	1.3742	0.7800	0.4042

Table 5.2: Computational results for Example 5.5.2. The auxiliary domains Ω_H where considered in such a way that $H = Ch^{-1/(2+4s)}$ for constants ranging between $C = 0.19$ for $s = 0.1$ and $C = 0.34$ for $s = 0.9$.

Moreover, since we cannot mesh the support of the volume constraint, as h decreases the actual region where we measure the error is expanded. According to Remark 5.3.12, in these experiments we have considered $H = Ch^{-1/(4+2s)}$, with a constant depending on s. Nevertheless, the computational cost of solving (5.3.2) for H large is extremely high. In practice, we have worked with small values of the constant C that relates H with h, especially for s small.

In this example, the Dirichlet condition decays exponentially, and it is easy to calculate its L^2-norm in the complement of a ball centered at the origin of radius R,

$$\|\exp(-|\cdot|^2)\|_{L^2(B(0,R)^c)} = \sqrt{\frac{\pi}{2}}\exp(-R^2).$$

So, recalling (5.3.1), we deduce that the L^2-error in Ω_H^c is exponential in h:

$$\|\exp(-|\cdot|^2)\|_{L^2(\Omega_H^c)} \leq C_1 \exp\left(-C_2 h^{\frac{1}{1+2s}}\right).$$

However, the rather small diameter of the auxiliary domain Ω_H explains why the exponential regimen is still not observed in our experiments, and the rather poor order of convergence in $L^2(\mathbb{R}^n)$.

In order to illustrate the singular character of the nonlocal derivative, in Figure 5.3 we plot the computed λ_h for two different values of s. A behavior similar to the one described in Remark 5.2.5 is observed. As s increases, the computed nonlocal derivatives become more singular near $\partial\Omega$.

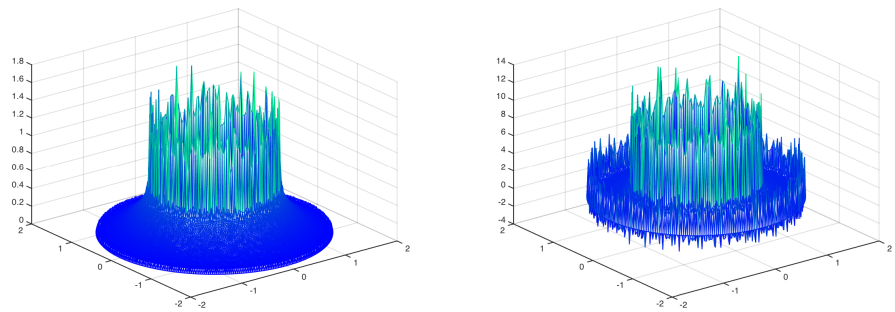

Figure 5.3: Computed nonlocal normal derivatives from Example 5.5.2 for $s = 0.1$ (left panel) and $s = 0.5$ (right panel). Both blow up near the boundary of the domain, and the greater the s, the more singular these nonlocal derivatives are observed to be.

Example 5.5.3. Our final example deals with a volume constraint that decays polynomially at infinity; we consider the problem

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } B(0,1) \subset \mathbb{R}^2, \\ u &= \frac{1}{|\cdot|^4} \quad \text{in } \Omega^c. \end{cases}$$

Table 5.3 shows the computed orders of convergence in $L^2(\Omega)$, $L^2(\Omega_H)$ and $L^2(\mathbb{R}^n)$. Regarding the first one, we observe convergence of order approximately $s + 1/2$, which is consistent with the fact that solutions belong to $H^{s+1/2-\varepsilon}(\Omega)$.

s	Order u_h ($L^2(\Omega)$)	Order u_h ($L^2(\Omega_H)$)	Order u_h ($L^2(\mathbb{R}^n)$)
0.1	0.5505	0.5927	0.6400
0.2	0.6417	0.7360	0.6306
0.3	0.7423	0.8673	0.5940
0.4	0.8929	0.9619	0.5555
0.5	1.0320	0.9961	0.5179
0.6	1.1367	0.9910	0.4841
0.7	1.1553	0.9292	0.4365
0.8	1.2571	0.8699	0.4035
0.9	1.4017	0.8590	0.3681

Table 5.3: Computational results for Example 5.5.3.

As in the previous example, the results in $L^2(\Omega_H)$ are explained by the change of diameter of these auxiliary domains. Concerning convergence in $L^2(\mathbb{R}^n)$, observe that

$$\| |\cdot|^{-4} \|_{L^2(B(0,R)^c)} = \sqrt{\frac{\pi}{3}}\, R^{-3}.$$

This implies that the decay of the error is polynomial in H,

$$\| | \cdot |^{-4} \|_{L^2(\Omega_H)^c} \leq C h^{\frac{3}{2+4s}}.$$

So, if we utilize a sequence of domains $\{\Omega_H\}$ with H not large enough, the tail of the L^2-norm of the volume constraint has a large impact on the $L^2(\mathbb{R}^n)$-error. In Figure 5.4 we compare the effect of increasing the constant in the identity $H = Ch^{-1/(2+4s)}$. Errors are observed to diminish considerably, and there is a slight improvement in the orders of convergence as well. Notice also that the errors in $L^2(\mathbb{R}^n)$ are one order of magnitude larger than errors in $L^2(\Omega_H)$.

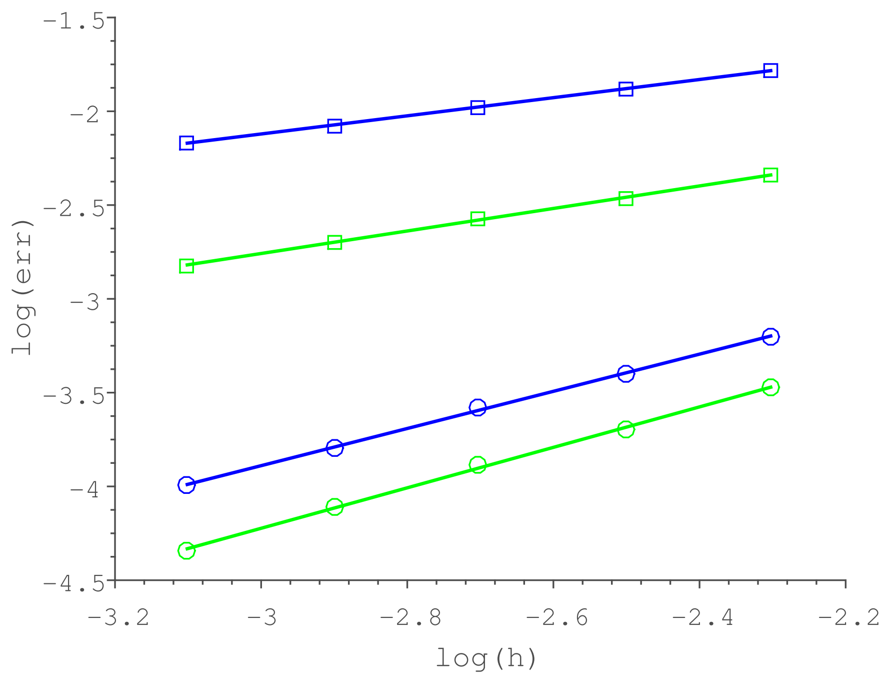

Figure 5.4: Convergence in $L^2(\Omega_H)$ (circles) and in $L^2(\mathbb{R}^n)$ squares for Example 5.5.3 with $s = 0.6$. In blue, we have plotted the results collected to build Table 5.3; the slopes of the best fitting lines are 0.9910 and 0.4841, respectively. In green, we display the results with a sequence of larger auxiliary domains. The slopes of the green lines are 1.0785 and 0.6002.

Resumen del capítulo

En este capítulo estudiamos aproximaciones por elementos finitos al problema

$$\begin{cases} (-\Delta)^s u = f & \text{en } \Omega, \\ u = g & \text{en } \Omega^c, \end{cases}$$

donde las funciones f y g son datos pertenecientes a espacios adecuados. Desarrollamos un esquema mixto que permite aproximar tanto a la solución u como a su derivada no local $\mathcal{N}_s u$. Luego de aplicar la fórmula de integración por partes no local (1.2.8), introducimos la derivada no local en la formulación del problema como un multiplicador de Lagrange.

Una de las mayores dificultades a sortear es que los soportes del dato de Dirichlet g y de la derivada no local de u no tienen por qué ser acotados. Nuestra estrategia incluye el truncamiento de g y la aproximación de la derivada no local en una sucesión de dominios discretos acotados, aunque con diámetros crecientes. Además de desarrollar un algoritmo de elementos finitos, necesitamos estimar cómo afecta el truncamiento del dato a la solución y cómo es el decaimiento de $\mathcal{N}_s u$ en el infinito.

En la **Sección 5.1** introducimos la notación, planteamos la formulación mixta del problema y probamos que ésta está bien planteada. La regularidad de soluciones y de sus derivadas no locales es tratada en la **Sección 5.2**, y se muestra que nuestras estimaciones son precisas.

Posteriormente, el capítulo trata sobre el análisis de elementos finitos del problema utilizando funciones continuas y lineales a trozos para aproximar tanto a la solución como a su derivada no local. Los espacios y el problema discreto son introducidos en la **Sección 5.3**. Además, en dicha sección, asumiendo que el soporte del dato de Dirichlet está contenido en un cierto dominio auxiliar, demostramos que el problema discreto está bien planteado y ofrecemos estimaciones de aproximación para las incógnitas de nuestro problema mixto.

El método que proponemos para tratar con datos de Dirichlet con soporte no acotado consiste simplemente en truncarlos. En la **Sección 5.4** analizamos el error asociado a este procedimiento, comparando las soluciones de los respectivos problemas continuos. Para resolver los problemas discretos correspondientes, el diámetro de los dominios truncados tiene que estar adecuadamente vinculado con el tamaño de la malla de modo de no deteriorar el orden de convergencia. Finalmente, en la **Sección 5.5**, mostramos algunos resultados computacionales que proveen evidencia de la convergencia del algoritmo propuesto.

Appendix A

Implementation of the finite element method

The aim of this appendix is to comment some algorithmic aspects of finite element approximations of the fractional Laplacian. Emphasis is put on the computation and assembly of the stiffness matrices associated to this operator. For further details on the implementation we refer the reader to [2], where a comprehensive and simple 2D *MATLAB*® finite element code for problem (Homogeneous) was presented.

The emphasis in our presentation is on the computation of the nonlocal bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{V}}$ (cf. (1.2.5)) between basis functions. Thus, for the sake of clarity, we discuss implementation details just for the homogeneous Dirichlet problem (Homogeneous). Recall, from Section 3.5, that our discrete space consists of continuous, piecewise linear functions defined over a shape-regular mesh,

$$\mathbb{V}_h = \{ v \in C_0(\Omega) : \, v\big|_T \in \mathcal{P}_1 \, \forall T \in \mathcal{T}_h \},$$

and that writing the discrete solution as $u_h = \sum_j u_j \varphi_j$, the discrete problem to be solved is equivalent to solving the linear system

$$KU = b, \tag{A.0.1}$$

where the stiffness matrix $K = (K_{ij}) \in \mathbb{R}^{N \times N}$ and the right hand side $b = (b_j) \in \mathbb{R}^N$ are given by

$$K_{ij} = \langle \varphi_i, \varphi_j \rangle_{\mathbb{V}}, \quad b_j = \int_\Omega f \varphi_j$$

and the unknown is $U = (u_j)$ in $\mathbb{R}^N$.

The fractional stiffness matrix K is symmetric and positive definite, so that (A.0.1) has a unique solution. Notice that the integrals in the inner product involved in computation of K_{ij} should be carried over $\mathbb{R}^n$. For this reason we find it useful to consider a ball B containing Ω and such that the distance from $\overline{\Omega}$ to B^c is an arbitrary positive number. This is needed in order to avoid difficulties caused by lack of symmetry when dealing with the integral over Ω^c when Ω is not a ball. Together with B, we introduce an auxiliary triangulation $\mathcal{T}_A$ on $B \setminus \Omega$ such that the complete triangulation $\tilde{\mathcal{T}}$ over B (that is $\tilde{\mathcal{T}} = \mathcal{T} \cup \mathcal{T}_A$) is admissible (see Figure A.1). As we are considering an homogeneous Dirichlet problem, the discrete solution is simply set to vanish on Ω^c by defining its value to be zero on the nodes in $B \setminus \Omega$.

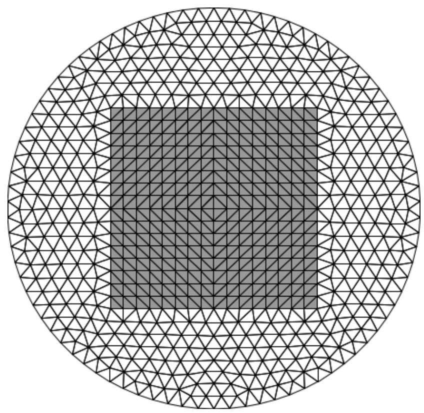

Figure A.1: A square domain Ω (gray) and an auxiliary ball containing it. Regular triangulations $\mathcal{T}$ and $\mathcal{T}_A$ for Ω and $B \setminus \Omega$ are shown. The final symmetry of the admissible triangulation $\tilde{\mathcal{T}} = \mathcal{T} \cup \mathcal{T}_A$, exhibited in the example, is not relevant.

We denote by $N_{\tilde{\mathcal{T}}}$ the number of elements on the triangulation of B. Then, defining for $1 \leq \ell, m \leq N_{\tilde{\mathcal{T}}}$ and $1 \leq \ell \leq N_{\tilde{\mathcal{T}}}$

$$\begin{aligned} I^{i,j}_{\ell,m} &= \int_{T_\ell} \int_{T_m} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{2+2s}} \, dxdy, \\ J^{i,j}_{\ell} &= \int_{T_\ell} \int_{B^c} \frac{\varphi_i(x)\varphi_j(x)}{|x-y|^{2+2s}} \, dydx, \end{aligned} \tag{A.0.2}$$

we may write

$$K_{ij} = \frac{C(n,s)}{2} \sum_{\ell=1}^{N_{\tilde{\mathcal{T}}}} \left(\sum_{m=1}^{N_{\tilde{\mathcal{T}}}} I^{i,j}_{\ell,m} + 2 J^{i,j}_{\ell} \right).$$

Remark A.0.1. Let $1 \leq \ell, m \leq \mathcal{N}_{\tilde{\mathcal{T}}}$. When computing $I^{i,j}_{\ell,m}$ or $J^{i,j}_{\ell}$, the basis function indices i and j do not refer to a global numbering but to a local. This means, for example, that if $\overline{T_\ell} \cap \overline{T_m} = \emptyset$, then $1 \leq i, j \leq 6$. See Remark A.2.1 for details on this convention.

As mentioned above, the computation of each integral $I^{i,j}_{\ell,m}$ and $J^{i,j}_{\ell}$ is challenging for different reasons: the former involves a singular integrand if $\overline{T_\ell} \cap \overline{T_m} \neq \emptyset$, while the latter needs to be calculated on an unbounded domain. Further, although for disjoint elements T_ℓ and T_m the integrands in $I^{i,j}_{\ell,m}$ are bounded, these integrals should be calculated efficiently.

The integrals in (A.0.2) are computed by running a double loop over the elements of the triangulation. First, in Section A.1 we describe the finite element algorithm we utilized to solve problem (Homogeneous), assuming that the integrals $I^{i,j}_{\ell,m}$ and $J^{i,j}_{\ell}$ are accurately computed. That section illustrates the double loop structure of the scheme, and the challenges in efficiency caused by nonlocality. Afterwards, Section A.2 treats the quadrature rules employed for computing the integrals over two elements T and T' (with the possibility that $T = T'$). These are related to the ones presented in Chapter 5 of [95]; the advantage of applying the transformations from that book for this problem is that they convert an integral over the product of two elements into an integral over $[0,1]^4$, in which variables can be separated and the singular part can be solved analytically.

A.1 Algorithm

One of the main challenges to build up a finite element implementation to problem (Homogeneous) is to assemble the stiffness matrix in an efficient mode. Independently of whether the supports of two given basis functions φ_i and φ_j are disjoint, the interaction $\langle \varphi_i, \varphi_j \rangle_{H^s(\mathbb{R}^n)}$ is not null. This yields a paramount difference between finite element implementations for the classical and the fractional Laplace operators; in the former the stiffness matrix is sparse, while in the latter it is full. Therefore, unless some care is taken, the amount of memory required and the number of operations needed to assembly the stiffness matrix increases quadratically with the number of nodes. In this section we briefly describe a simple algorithm to compute the stiffness matrix and solve the discrete problem (A.0.1).

Since the computation of the entries of the stiffness matrix requires calculating integrals on *pairs* of elements, it is required to perform a double loop. It is simple to check the identity $I^{i,j}_{\ell,m} = I^{i,j}_{m,\ell}$ for all i, j, ℓ, m, and therefore it is enough to carry the computations out only for the pairs of elements T_ℓ and T_m with $\ell \leq m$.

The main loop goes through all the elements T_ℓ of the mesh of Ω. Observe that auxiliary elements are excluded from it. Fixed ℓ, the first task is to classify all the mesh elements T_m ($\ell \leq m \leq N_{\tilde{\mathcal{T}}}$) according to whether $\overline{T_\ell} \cap \overline{T_m}$ is empty, a vertex, an edge or the element $\overline{T_\ell}$ itself. For this purpose, after initializing the problem variables, we create a patches' structure. This contains the information about elements' intersection. Namely, for every pair of different elements T_ℓ and T_m, it allows to recognize efficiently whether their intersection is empty or if it is a vertex or an edge.

Then, the main loop goes through elements T_ℓ, and begins by computing the local contribution to the right hand side and the local matrices $(I^{i,j}_{\ell,\ell})$ and $(J^{i,j}_{\ell})$. Afterwards, interactions are computed for pairs of elements that are within a positive distance from T_ℓ, and then for neighboring elements, and the stiffness matrix is updated after every computation of $(I^{i,j}_{\ell,m})$. Once the matrix K and right hand side b are assembled, system (A.0.1) is solved. A pseudocode is exhibited in Algorithm 1.

```
Data: s, f, mesh data (including auxiliary domain)
Result: discrete solution u_h
initialize K, u_h and b;
build patches' structure;
for ℓ = 1 : nt-nt_aux do
    create vectors edge, vertex and empty;
    compute local vector b and assemble;
    compute the 3 × 3 local matrices (I^{i,j}_{ℓ,ℓ}) and (J^{i,j}_ℓ);
    update K = K + (I^{i,j}_{ℓ,ℓ}) + 2(J^{i,j}_ℓ);
    for m ∈ empty do
        compute the 6 × 6 matrix (I^{i,j}_{ℓ,m});
        update K = K + (I^{i,j}_{ℓ,m});
    end
    for m ∈ vertex do
        compute the 5 × 5 matrix (I^{i,j}_{ℓ,m});
        update K = K + (I^{i,j}_{ℓ,m});
    end
    for m ∈ edge do
        compute the 4 × 4 matrix (I^{i,j}_{ℓ,m});
        update K = K + (I^{i,j}_{ℓ,m});
    end
end
solve u_h = K \ b
```

Algorithm 1: Pseudocode for finite element implementation of problem (Homogeneous).

A.2 Quadrature rules

Here we give details about how to compute the integrals $I^{i,j}_{\ell,m}$ and $J^{i,j}_{\ell}$ defined in (A.0.2). In order to cope with $I^{i,j}_{\ell,m}$, we proceed according to whether $\overline{T_\ell} \cap \overline{T_m}$ is empty, a vertex, an edge or an element. Recall that $I^{i,j}_{\ell,m} = I^{i,j}_{m,\ell}$, so that we may assume $\ell \leq m$.

Consider two elements T_ℓ and T_m such that $\operatorname{supp}(\varphi_i), \operatorname{supp}(\varphi_j) \cap (T_\ell \cup T_m) \neq \emptyset$. Observe that if one of this intersections is empty, then $I^{i,j}_{\ell,m} = 0$. Moreover, it could be possible that one of the elements is disjoint with the support of both φ_i and φ_j, provided the other element intersects both supports and $I^{i,j}_{\ell,m} \neq 0$.

We are going to consider the reference element as in (3.1.5),

$$\hat{T} = \{\hat{x} = (\hat{x}_1, \hat{x}_2)\colon 0 \leq \hat{x}_1 \leq 1,\, 0 \leq \hat{x}_2 \leq \hat{x}_1\},$$

whose vertices are

$$\hat{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \hat{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{x}^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The basis functions on $\hat{T}$ are, obviously,

$$\hat{\varphi}_1(\hat{x}) = 1 - \hat{x}_1, \quad \hat{\varphi}_2(\hat{x}) = \hat{x}_1 - \hat{x}_2, \quad \hat{\varphi}_3(\hat{x}) = \hat{x}_2.$$

Remark A.2.1. Given two elements T_ℓ and T_m, we provide a local numbering in the following way. If T_ℓ and T_m are disjoint, we set the first three nodes to be the nodes of T_ℓ and the following three nodes to be the ones of T_m. Else, we set the first node(s) to be the ones in the intersection, then we insert the remaining node(s) of T_ℓ and finally the one(s) of T_m (see Figure A.2). For simplicity of notation, when computing $I^{i,j}_{\ell,m}$ and $J^{i,j}_\ell$, we assume that i, j denote the local numbering of the basis functions involved; for example, if T_ℓ and T_m share only a vertex, then $1 \leq i, j \leq 5$.

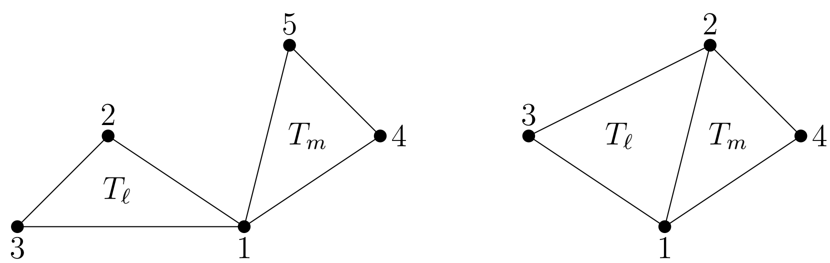

Figure A.2: Local numbering for elements with a vertex and an edge in common.

Consider the affine mappings

$$\begin{aligned} \chi_\ell &: \hat{T} \to T_\ell, \qquad & \chi_\ell(\hat{x}) &= B_\ell \hat{x} + x^{(1)}_\ell, \\ \chi_m &: \hat{T} \to T_m, \qquad & \chi_m(\hat{x}) &= B_m \hat{x} + x^{(1)}_m, \end{aligned} \tag{A.2.1}$$

where the matrices B_ℓ and B_m are such that $\hat{x}^{(2)}$ (resp. $\hat{x}^{(3)}$) is mapped respectively to the second (resp. third) node of T_ℓ and T_m in the local numbering defined above.

Then, it is clear that

$$
\begin{aligned}
I_{\ell,m}^{i,j} &= 4|T_\ell||T_m| \int_{\hat{T}} \int_{\hat{T}} \frac{(\varphi_i(\chi_\ell(\hat{x})) - \varphi_i(\chi_m(\hat{y})))(\varphi_j(\chi_\ell(\hat{x})) - \varphi_j(\chi_m(\hat{y})))}{|\chi_\ell(\hat{x}) - \chi_m(\hat{y})|^{2+2s}} \, d\hat{x} \, d\hat{y} \\
&= 4|T_\ell||T_m| \iiiint_{\hat{T}\times\hat{T}} F_{ij}(\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2) \, d\hat{x}_1 \, d\hat{x}_2 \, d\hat{y}_1 \, d\hat{y}_2.
\end{aligned} \tag{A.2.2}
$$

We discuss how to compute $I_{\ell,m}^{i,j}$ depending on the relative position of T_ℓ and T_m, and afterwards we tackle the computation of $J_\ell^{i,j}$.

A.2.1 Non-touching elements

This is the simplest case, since the integrand F_{ij} in (A.2.2) is not singular. Recall that

$$
I_{\ell,m}^{i,j} = \int_{T_\ell} \int_{T_m} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{2+2s}} \, dxdy, \quad 1 \le \ell, m \le N_{\tilde{\mathcal{T}}}
$$

Splitting the numerator above, we obtain

$$
\begin{aligned}
I_{\ell,m}^{i,j} =& \int_{T_\ell} \int_{T_m} \frac{\varphi_i(x)\varphi_j(x)}{|x-y|^{2+2s}} \, dxdy + \int_{T_\ell} \int_{T_m} \frac{\varphi_i(y)\varphi_j(y)}{|x-y|^{2+2s}} \, dxdy \\
&- \int_{T_\ell} \int_{T_m} \frac{\varphi_i(x)\varphi_j(y)}{|x-y|^{2+2s}} \, dxdy - \int_{T_\ell} \int_{T_m} \frac{\varphi_i(y)\varphi_j(x)}{|x-y|^{2+2s}} \, dxdy.
\end{aligned}
$$

Note that all the integrands depend on ℓ and m only through their denominators. Since $\varphi_i(x) = 0$ if $i \in \{1, 2, 3\}$ and $x \in T_m$ or if $i \in \{4, 5, 6\}$ and $x \in T_\ell$, given two indices i, j, only one of the four integrals above is not null. Thus, we may divide the 36 interactions between the 6 basis functions involved into four 3 by 3 blocks, and write the local matrix $\texttt{ML} = (I_{\ell,m}^{i,j})$ as:

$$
\texttt{ML} = \begin{pmatrix} A_{\ell,m} & B_{\ell,m} \\ C_{\ell,m} & D_{\ell,m} \end{pmatrix},
$$

where

$$
\begin{aligned}
A_{\ell,m}^{i,j} &= \int_{T_\ell} \int_{T_m} \frac{\varphi_i(x)\varphi_j(x)}{|x-y|^{2+2s}} \, dxdy, & B_{\ell,m}^{i,j} &= -\int_{T_\ell} \int_{T_m} \frac{\varphi_i(x)\varphi_{j+3}(y)}{|x-y|^{2+2s}} \, dxdy \\
C_{\ell,m}^{i,j} &= -\int_{T_\ell} \int_{T_m} \frac{\varphi_{i+3}(y)\varphi_j(x)}{|x-y|^{2+2s}} \, dxdy, & D_{\ell,m}^{i,j} &= \int_{T_\ell} \int_{T_m} \frac{\varphi_{i+3}(y)\varphi_{j+3}(y)}{|x-y|^{2+2s}} \, dxdy.
\end{aligned}
$$

If we use two nested Gaussian quadrature rules to estimate these integrals, with n_G quadrature nodes each, it amounts for a total of n_G^2 quadrature points in $\hat{T} \times \hat{T}$. Let us denote by p_k and w_k ($k \in \{1, \ldots, n_G\}$) the quadrature nodes and weights in $\hat{T}$, respectively. Changing variables we obtain

$$A^{i,j}_{\ell,m} = 4|T_\ell||T_m| \int_{\hat{T}} \int_{\hat{T}} \frac{\hat{\varphi}_i(x)\hat{\varphi}_j(x)}{|\chi_\ell(x) - \chi_m(y)|^{2+2s}} \, dxdy,$$

and applying the quadrature rule twice, we derive:

$$A^{i,j}_{\ell,m} \approx 4|T_\ell||T_m| \sum_{q=1}^{n_G} \sum_{k=1}^{n_G} \frac{w_q \, w_k \, \hat{\varphi}_i(p_k)\hat{\varphi}_j(p_k)}{|\chi_\ell(p_k) - \chi_m(p_q)|^{2+2s}}.$$

Note that the right hand side summands only depend on i and j through their numerators, and on ℓ and m through their denominators. This is an important remark to take advantage of to develop an efficient code, because it allows to compute $I^{i,j}_{\ell,m}$ at once for all the elements T_m such that $\overline{T}_\ell \cap \overline{T}_m = \emptyset$. This approach is explained in detail in [2, Section 5.2].

A.2.2 Vertex-touching elements

In case $\overline{T_\ell} \cap \overline{T_m}$ consists of a vertex, define $\hat{z} = (\hat{x}, \hat{y})$, identify $\hat{z}$ with a vector in $\mathbb{R}^4$, and split the domain of integration in (A.2.2) into two components D_1 and D_2, where

$$D_1 = \{\hat{z} \colon 0 \le \hat{z}_1 \le 1, \, 0 \le \hat{z}_2 \le \hat{z}_1, \, 0 \le \hat{z}_3 \le \hat{z}_1, \, 0 \le \hat{z}_4 \le \hat{z}_3\},$$
$$D_2 = \{\hat{z} \colon 0 \le \hat{z}_3 \le 1, \, 0 \le \hat{z}_4 \le \hat{z}_3, \, 0 \le \hat{z}_1 \le \hat{z}_3, \, 0 \le \hat{z}_2 \le \hat{z}_1\}.$$

Let $\xi \in [0, 1]$ and $\eta = (\eta_1, \eta_2, \eta_3) \in [0, 1]^3$. We consider the mappings $T_h \colon [0, 1] \times [0, 1]^3 \to D_h, \; h = 1, 2,$

$$T_1(\xi, \eta) = \begin{pmatrix} \xi \\ \xi\eta_1 \\ \xi\eta_2 \\ \xi\eta_2\eta_3 \end{pmatrix}, \quad T_2(\xi, \eta) = \begin{pmatrix} \xi\eta_2 \\ \xi\eta_2\eta_3 \\ \xi \\ \xi\eta_1 \end{pmatrix},$$

having Jacobian determinants $|JT_1| = \xi^3\eta_2 = |JT_2|$.

We perform the calculations in detail only for D_1. Observe that if $i = 1$, which corresponds to the vertex in common between T_ℓ and T_m, then

$$\varphi_i(\chi_\ell(\xi, \xi\eta_1)) - \varphi_i(\chi_m(\xi\eta_2, \xi\eta_2\eta_3)) = -\xi(1 - \eta_2).$$

Meanwhile, if the subindex i equals 2 or 3, it corresponds to one of the other two vertices of T_ℓ. Therefore, in those cases $\varphi_i(\chi_m(\xi\eta_2, \xi\eta_2\eta_3)) = 0$, and

$$\varphi_2(\chi_\ell(\xi, \xi\eta_1)) = \xi(1 - \eta_1),$$

$$\varphi_3(\chi_\ell(\xi, \xi\eta_1)) = \xi\eta_1.$$

Analogously, if $i \in \{4, 5\}$, then $\varphi_i(\chi_\ell(\xi, \xi\eta_1)) = 0$ and so

$$\begin{aligned} \varphi_4(\chi_m(\xi\eta_2, \xi\eta_2\eta_3)) &= -\xi\eta_2(1-\eta_3), \\ \varphi_5(\chi_m(\xi\eta_2, \xi\eta_2\eta_3)) &= -\xi\eta_2\eta_3. \end{aligned}$$

Thus, defining the functions $\psi_k^{(1)} \colon [0,1]^3 \to \mathbb{R}$ ($k \in \{1, \ldots, 5\}$),

$$\begin{aligned} &\psi_1^{(1)}(\eta) = \eta_2 - 1, && \psi_2^{(1)}(\eta) = 1 - \eta_1, && \psi_3^{(1)}(\eta) = \eta_1, \\ &\psi_4^{(1)}(\eta) = -\eta_2(1-\eta_3), && \psi_5^{(1)}(\eta) = -\eta_2\eta_3, \end{aligned}$$

we may write

$$\begin{aligned} \int_{D_1} F_{ij}(\hat{z})\, d\hat{z} &= \int_{[0,1]} \int_{[0,1]^3} \frac{\psi_i^{(1)}(\eta)\psi_j^{(1)}(\eta)}{\left| B_\ell \begin{pmatrix} \xi \\ \xi\eta_1 \end{pmatrix} - B_m \begin{pmatrix} \xi\eta_2 \\ \xi\eta_2\eta_3 \end{pmatrix} \right|^{2+2s}} \xi^5 \eta_2 \, d\eta \, d\xi \\ &= \left(\int_0^1 \xi^{3-2s} d\xi \right) \left(\int_{[0,1]^3} \frac{\psi_i^{(1)}(\eta)\psi_j^{(1)}(\eta)}{|d^{(1)}(\eta)|^{2+2s}} \eta_2 \, d\eta \right) \\ &= \frac{1}{4-2s} \left(\int_{[0,1]^3} \frac{\psi_i^{(1)}(\eta)\psi_j^{(1)}(\eta)}{|d^{(1)}(\eta)|^{2+2s}} \eta_2 \, d\eta \right), \end{aligned}$$

where we have defined the function

$$d^{(1)}(\eta) = B_\ell \begin{pmatrix} 1 \\ \eta_1 \end{pmatrix} - B_m \begin{pmatrix} \eta_2 \\ \eta_2\eta_3 \end{pmatrix}.$$

Observe that in the first line of last equation (or equivalently, in (A.2.2)), the integrand is singular at the origin. The key point in the identity above is that the singularity of the integral is explicitly computed. The function $d^{(1)}$ does not vanish on $[0,1]^3$, and therefore the last integral involves a regular integrand that is easily estimated by means of a Gaussian quadrature rule.

In a similar fashion, the integrals over D_2 take the form

$$\int_{D_2} F_{ij}(\hat{z})\, d\hat{z} = \frac{1}{4-2s} \left(\int_{[0,1]^3} \frac{\psi_i^{(2)}(\eta)\psi_j^{(2)}(\eta)}{|d^{(2)}(\eta)|^{2+2s}} \eta_2 \, d\eta \right),$$

where

$$\psi_1^{(2)}(\eta) = 1 - \eta_2, \qquad \psi_2^{(2)}(\eta) = \eta_2(1-\eta_3), \qquad \psi_3^{(2)}(\eta) = \eta_2\eta_3,$$

$$\psi_4^{(2)}(\eta) = \eta_1 - 1, \qquad \psi_5^{(2)}(\eta) = -\eta_1,$$

and

$$d^{(2)}(\eta) = B_\ell \begin{pmatrix} \eta_2 \\ \eta_2\eta_3 \end{pmatrix} - B_m \begin{pmatrix} 1 \\ \eta_1 \end{pmatrix}.$$

Let $p_1, ..., p_{n_G} \in [0,1]^3$ be a set of quadrature points and $w_1, ..., w_{n_G}$ their respective weights, and set $h \in \{1,2\}$. Then, applying the mentioned quadrature rule in the cube,

$$\int_{[0,1]^3} \frac{\psi_i^{(h)}(\eta)\psi_j^{(h)}(\eta)}{|d^{(h)}(\eta)|^{2+2s}}\, \eta_2 \, d\eta \approx \sum_{k=1}^{n_G} w_k \frac{\psi_i^{(h)}(p_k)\psi_j^{(h)}(p_k)}{|d^{(h)}(p_k)|^{2+2s}}\, p_{k,2},$$

where $p_{k,2}$ denotes the second coordinate of the point p_k. The right hand side only depends on ℓ and m through $d^{(h)}$. So, $I_{\ell,m}^{i,j}$ may also be efficiently computed using vectorized operations.

A.2.3 Edge-touching elements

Proceeding similarly, we compute next the case where $\overline{T_\ell} \cap \overline{T_m}$ is an edge. Now there are only 4 nodal basis functions involved, and the local numbering is such that the first two nodes correspond to the endpoints of the shared edge, the third is the one in T_ℓ but not in T_m and the last one is the node in T_m but not in T_ℓ (Figure A.2).

In this case, the parametrization of the elements we are considering is such that both χ_ℓ and χ_m map $[0,1] \times \{0\}$ to the common edge between T_ℓ and T_m. Therefore, if we consider $\hat{z} = (\hat{y}_1 - \hat{x}_1, \hat{y}_2, \hat{x}_2)$, the singularity of the integrand is localized at $\hat{z} = 0$:

$$I_{\ell,m}^{i,j} = 4|T_\ell||T_m| \int_0^1 \int_{-\hat{x}_1}^{1-\hat{x}_1} \int_0^{\hat{z}_1+\hat{x}_1} \int_0^{\hat{x}_1} F_{ij}(\hat{x}_1, \hat{z}_3, \hat{x}_1 + \hat{z}_1, \hat{z}_2)\, d\hat{z}\, d\hat{x}_1.$$

We decompose the domain of integration as $\cup_{k=1}^5 D_k$, where

$$\begin{aligned}
D_1 = \{(\hat{x}_1, \hat{z})\colon & -1 \le \hat{z}_1 \le 0,\ 0 \le \hat{z}_2 \le 1 + \hat{z}_1, \\
& 0 \le \hat{z}_3 \le \hat{z}_2 - \hat{z}_1,\ \hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1\}, \\
D_2 = \{(\hat{x}_1, \hat{z})\colon & -1 \le \hat{z}_1 \le 0,\ 0 \le \hat{z}_2 \le 1 + \hat{z}_1, \\
& \hat{z}_2 - \hat{z}_1 \le \hat{z}_3 \le 1,\ \hat{z}_3 \le \hat{x}_1 \le 1\}, \\
D_3 = \{(\hat{x}_1, \hat{z})\colon & 0 \le \hat{z}_1 \le 1,\ 0 \le \hat{z}_2 \le \hat{z}_1, \\
& 0 \le \hat{z}_3 \le 1 - \hat{z}_1,\ \hat{z}_3 \le \hat{x}_1 \le 1 - \hat{z}_1\}, \\
D_4 = \{(\hat{x}_1, \hat{z})\colon & 0 \le \hat{z}_1 \le 1,\ \hat{z}_1 \le \hat{z}_2 \le 1, \\
& 0 \le \hat{z}_3 \le \hat{z}_2 - \hat{z}_1,\ \hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1 - \hat{z}_1\}, \\
D_5 = \{(\hat{x}_1, \hat{z})\colon & 0 \le \hat{z}_1 \le 1,\ \hat{z}_1 \le \hat{z}_2 \le 1,
\end{aligned}$$

$$\hat{z}_2 - \hat{z}_1 \leq \hat{z}_3 \leq 1 - \hat{z}_1,\ \hat{z}_3 \leq \hat{x}_1 \leq 1 - \hat{z}_1\}.$$

Consider the mappings $T_k\colon [0,1] \times [0,1]^3 \to D_k$ ($k \in \{1, \ldots, 5\}$),

$$T_1\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi\\ -\xi\eta_1\eta_2\\ \xi\eta_1(1-\eta_2)\\ \xi\eta_1\eta_3\end{pmatrix}, \qquad T_2\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi\\ -\xi\eta_1\eta_2\eta_3\\ \xi\eta_1\eta_2(1-\eta_3)\\ \xi\eta_1\end{pmatrix},$$

$$T_3\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi(1-\eta_1\eta_2)\\ \xi\eta_1\eta_2\\ \xi\eta_1\eta_2\eta_3\\ \xi\eta_1(1-\eta_2)\end{pmatrix}, \qquad T_4\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi(1-\eta_1\eta_2\eta_3)\\ \xi\eta_1\eta_2\eta_3\\ \xi\eta_1\\ \xi\eta_1\eta_2(1-\eta_3)\end{pmatrix},$$

$$T_5\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi(1-\eta_1\eta_2\eta_3)\\ \xi\eta_1\eta_2\eta_3\\ \xi\eta_1\eta_2\\ \xi\eta_1(1-\eta_2\eta_3)\end{pmatrix},$$

with Jacobian determinants given by

$$|JT_1| = \xi^3\eta_1^2, \quad |JT_h| = \xi^3\eta_1^2\eta_2,\ h \in \{2, \ldots, 5\}.$$

Then, over D_h it holds that

$$\int_{D_h} F_{ij} = \frac{1}{4-2s}\int_{[0,1]^3} \frac{\psi_i^{(h)}(\eta)\psi_j^{(h)}(\eta)}{|d^{(h)}(\eta)|^{2+2s}}\, J^{(h)}(\eta)\, d\eta,$$

where

$$\begin{aligned}
\psi_1^{(1)}(\eta) &= -\eta_1\eta_2, & \psi_2^{(1)}(\eta) &= \eta_1(1-\eta_3),\\
\psi_3^{(1)}(\eta) &= \eta_1\eta_3, & \psi_4^{(1)}(\eta) &= -\eta_1(1-\eta_2),\\
\psi_1^{(2)}(\eta) &= -\eta_1\eta_2\eta_3, & \psi_2^{(2)}(\eta) &= -\eta_1(1-\eta_2),\\
\psi_3^{(2)}(\eta) &= \eta_1, & \psi_4^{(2)}(\eta) &= -\eta_1\eta_2(1-\eta_3),\\
\psi_1^{(3)}(\eta) &= \eta_1\eta_2, & \psi_2^{(3)}(\eta) &= -\eta_1(1-\eta_2\eta_3),\\
\psi_3^{(3)}(\eta) &= \eta_1(1-\eta_2), & \psi_4^{(3)}(\eta) &= -\eta_1\eta_2\eta_3,\\
\psi_1^{(4)}(\eta) &= \eta_1\eta_2\eta_3, & \psi_2^{(4)}(\eta) &= \eta_1(1-\eta_2),\\
\psi_3^{(4)}(\eta) &= \eta_1\eta_2(1-\eta_3), & \psi_4^{(4)}(\eta) &= -\eta_1,\\
\psi_1^{(5)}(\eta) &= \eta_1\eta_2\eta_3, & \psi_2^{(5)}(\eta) &= -\eta_1(1-\eta_2),\\
\psi_3^{(5)}(\eta) &= \eta_1(1-\eta_2\eta_3), & \psi_4^{(5)}(\eta) &= -\eta_1\eta_2.
\end{aligned}$$

Moreover, the functions $d^{(h)}$ are given by

$$d^{(1)}(\eta) = B_\ell \begin{pmatrix} 1 \\ \eta_1\eta_3 \end{pmatrix} - B_m \begin{pmatrix} 1-\eta_1\eta_2 \\ \eta_1(1-\eta_2) \end{pmatrix},$$
$$d^{(2)}(\eta) = B_\ell \begin{pmatrix} 1 \\ \eta_1 \end{pmatrix} - B_m \begin{pmatrix} 1-\eta_1\eta_2\eta_3 \\ \eta_1\eta_2(1-\eta_3) \end{pmatrix},$$
$$d^{(3)}(\eta) = B_\ell \begin{pmatrix} 1-\eta_1\eta_2 \\ \eta_1(1-\eta_2) \end{pmatrix} - B_m \begin{pmatrix} 1 \\ \eta_1\eta_2\eta_3 \end{pmatrix},$$
$$d^{(4)}(\eta) = B_\ell \begin{pmatrix} 1-\eta_1\eta_2\eta_3 \\ \eta_1\eta_2(1-\eta_3) \end{pmatrix} - B_m \begin{pmatrix} 1 \\ \eta_1 \end{pmatrix},$$
$$d^{(5)}(\eta) = B_\ell \begin{pmatrix} 1-\eta_1\eta_2\eta_3 \\ \eta_1(1-\eta_2\eta_3) \end{pmatrix} - B_m \begin{pmatrix} 1 \\ \eta_1\eta_2 \end{pmatrix},$$

and the Jacobians are

$$J^{(1)}(\eta) = \eta_1^2, \quad J^{(h)}(\eta) = \eta_1^2\eta_2, \ h \in \{2, \dots, 5\}.$$

As in the case of vertex-touching elements, the problem is reduced to computing integrals in the unit cube. Let $p_1, ..., p_{n_G} \in [0,1]^3$ the quadrature points, and $w_1, ..., w_{n_G}$ their respective weights. For $h \in \{1, \dots, 5\}$ we have

$$\int_{[0,1]^3} \frac{\psi_i^{(h)}(\eta)\psi_j^{(h)}(\eta)}{|d^{(h)}(\eta)|^{2+2s}} J^{(h)}(\eta)\, d\eta \approx \sum_{k=1}^{n_G} w_k \frac{\psi_i^{(h)}(p_k)\psi_j^{(h)}(p_k)}{|d^{(h)}(p_k)|^{2+2s}} J^{(h)}(p_k).$$

Once more, we may take advantage of *MATLAB*®'s vectorized operations because the right hand side only depends on ℓ and m through $d^{(h)}$.

A.2.4 Identical elements

In the same spirit as before, we consider $\hat{z} = \hat{y} - \hat{x}$, so that

$$I_{\ell,\ell}^{i,j} = 4|T_\ell|^2 \int_0^1 \int_0^{\hat{x}_1} \int_{-\hat{x}_1}^{1-\hat{x}_1} \int_{-\hat{x}_2}^{\hat{z}_1+\hat{x}_1-\hat{x}_2} F_{ij}(\hat{x}_1, \hat{x}_2, \hat{x}_1+\hat{z}_1, \hat{x}_2+\hat{z}_2)\, d\hat{z}_2\, d\hat{z}_1\, d\hat{x}_2\, d\hat{x}_1.$$

Let us decompose the integration region into

$$
\begin{aligned}
D_1 &= \{(\hat{x},\hat{z})\colon -1 \le \hat{z}_1 \le 0,\ -1 \le \hat{z}_2 \le \hat{z}_1, \\
&\qquad -\hat{z}_2 \le \hat{x}_1 \le 1,\ -\hat{z}_2 \le \hat{x}_2 \le \hat{x}_1\}, \\
D_2 &= \{(\hat{x},\hat{z})\colon 0 \le \hat{z}_1 \le 1,\ \hat{z}_1 \le \hat{z}_2 \le 1, \\
&\qquad \hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1 - \hat{z}_1,\ 0 \le \hat{x}_2 \le \hat{z}_1 - \hat{z}_2 + \hat{x}_1\}, \\
D_3 &= \{(\hat{x},\hat{z})\colon -1 \le \hat{z}_1 \le 0,\ \hat{z}_1 \le \hat{z}_2 \le 0, \\
&\qquad -\hat{z}_1 \le \hat{x}_1 \le 1,\ -\hat{z}_2 \le \hat{x}_2 \le \hat{x}_1 + \hat{z}_1 - \hat{z}_2\}, \\
D_4 &= \{(\hat{x},\hat{z})\colon 0 \le \hat{z}_1 \le 1,\ 0 \le \hat{z}_2 \le \hat{z}_1, \\
&\qquad 0 \le \hat{x}_1 \le 1 - \hat{z}_1,\ 0 \le \hat{x}_2 \le \hat{x}_1\}, \\
D_5 &= \{(\hat{x},\hat{z})\colon -1 \le \hat{z}_1 \le 0,\ 0 \le \hat{z}_2 \le 1 + \hat{z}_1, \\
&\qquad \hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1,\ 0 \le \hat{x}_2 \le \hat{x}_1 + \hat{z}_1 - \hat{z}_2\}, \\
D_6 &= \{(\hat{x},\hat{z})\colon 0 \le \hat{z}_1 \le 1,\ -1 + \hat{z}_1 \le \hat{z}_2 \le 0, \\
&\qquad -\hat{z}_2 \le \hat{x}_1 \le 1 - \hat{z}_1,\ -\hat{z}_2 \le \hat{x}_2 \le \hat{x}_1\}.
\end{aligned}
\tag{A.2.3}
$$

We begin by considering the first two sets. Making the change of variables $(\hat{x}',\hat{z}') = (\hat{x},-\hat{z})$ on D_1 and $(\hat{x}',\hat{z}') = (\hat{x}+\hat{z},\hat{z})$ on D_2, both regions are transformed into

$$
D_1' = \{(\hat{x}',\hat{z}')\colon 0 \le \hat{z}_1' \le 1,\ \hat{z}_1' \le \hat{z}_2' \le 1,\ \hat{z}_2' \le \hat{x}_1' \le 1,\ \hat{z}_2' \le \hat{x}_2' \le \hat{x}_1'\},
$$

so that

$$
\begin{aligned}
4|T_\ell|^2 \int_{D_1\cup D_2} F_{ij}(\hat{x},\hat{x}+\hat{z}) &= 4|T_\ell|^2 \int_{D_1'} F_{ij}(\hat{x}',\hat{x}'-\hat{z}') + F_{ij}(\hat{x}'-\hat{z}',\hat{x}')\, d\hat{x}'\, d\hat{z}' \\
&= 8|T_\ell|^2 \int_{D_1'} F_{ij}(\hat{x}',\hat{x}'-\hat{z}')\, d\hat{x}'\, d\hat{z}',
\end{aligned}
$$

because

$$
F_{ij}(\hat{x}',\hat{x}'-\hat{z}') = \frac{(\hat{\varphi}_i(\hat{x}') - \hat{\varphi}_i(\hat{x}'-\hat{z}'))(\hat{\varphi}_j(\hat{x}') - \hat{\varphi}_j(\hat{x}'-\hat{z}'))}{|B_\ell(\hat{z}')|^{2+2s}} = F_{ij}(\hat{x}'-\hat{z}',\hat{x}').
$$

Next, consider the four-dimensional simplex

$$
D = \{w\colon 0 \le w_1 \le 1,\ 0 \le w_2 \le w_1,\ 0 \le w_3 \le w_2,\ 0 \le w_4 \le w_3\},
$$

the map $T_1\colon D \to D_1'$,

$$
\begin{pmatrix} \hat{x}' \\ \hat{z}' \end{pmatrix} = T_1 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1, \\ w_1 - w_2 + w_3, \\ w_4, \\ w_3 \end{pmatrix}, \quad |JT_1| = 1,
$$

and the Duffy-type transform $T : [0,1]^4 \to D$,

$$w = T\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi, \\ \xi\eta_1, \\ \xi\eta_1\eta_2, \\ \xi\eta_1\eta_2\eta_3 \end{pmatrix}, \quad |JT| = \xi^3\eta_1^2\eta_2. \tag{A.2.4}$$

The composition of these two changes of variables allows to write the variables in F_{ij} in terms of (ξ,η) in the following way:

$$\hat{x}' = \begin{pmatrix} \xi \\ \xi(1-\eta_1+\eta_1\eta_2) \end{pmatrix}, \ \hat{z}' = \begin{pmatrix} \xi\eta_1\eta_2\eta_3 \\ \xi\eta_1\eta_2 \end{pmatrix}, \ \hat{x} - \hat{z}' = \begin{pmatrix} \xi(1-\eta_1\eta_2\eta_3) \\ \xi(1-\eta_1) \end{pmatrix}.$$

Observe that

$$\Lambda_k^{(1)}(\xi,\eta) := \hat{\varphi}_k(\hat{x}') - \hat{\varphi}_k(\hat{x}'-\hat{z}') = \begin{cases} -\xi\eta_1\eta_2\eta_3 & \text{if } k=1, \\ -\xi\eta_1\eta_2(1-\eta_3) & \text{if } k=2, \\ \xi\eta_1\eta_2 & \text{if } k=3. \end{cases}$$

Thus,

$$\begin{aligned} 4|T_\ell|^2 \int_{D_1\cup D_2} F_{ij}(\hat{x},\hat{x}+\hat{z}) &= 8|T_\ell|^2 \int_D F_{ij}(w_1, w_1-w_2+w_3, w_4, w_3)\, dw = \\ &= 8|T_\ell|^2 \int_{[0,1]^4} \frac{\Lambda_i^{(1)}(\xi,\eta)\,\Lambda_j^{(1)}(\xi,\eta)}{\left|B_\ell\begin{pmatrix} \xi\eta_1\eta_2\eta_3 \\ \xi\eta_1\eta_2 \end{pmatrix}\right|^{2+2s}} \xi^3\eta_1^2\eta_2\, d\xi\, d\eta. \end{aligned}$$

Finally, as the functions $\Lambda_k^{(1)}$ may be rewritten as $\Lambda_k^{(1)}(\xi,\eta) = \xi\eta_1\eta_2\psi_k^{(1)}(\eta_3)$, where

$$\psi_1^{(1)}(\eta_3) = -\eta_3, \quad \psi_2^{(1)}(\eta_3) = -(1-\eta_3), \quad \psi_3^{(1)}(\eta_3) = 1,$$

we obtain

$$\begin{aligned} &4|T_\ell|^2 \int_{D_1\cup D_2} F_{ij}(\hat{x},\hat{x}+\hat{z}) = \\ &= 8|T_\ell|^2 \int_0^1 \xi^{3-2s} d\xi \int_0^1 \eta_1^{2-2s} d\eta_1 \int_0^1 \eta_2^{1-2s} d\eta_2 \int_0^1 \frac{\psi_i^{(1)}(\eta_3)\psi_j^{(1)}(\eta_3)}{\left|B_\ell\begin{pmatrix} \eta_3 \\ 1 \end{pmatrix}\right|^{2+2s}} d\eta_3. \end{aligned}$$

The first three integrals above are straightforwardly calculated by hand, and the last one involves a regular integrand, so that it is easily estimated by means of a Gaussian quadrature rule.

It still remains to perform similar calculations on the rest of the sets in (A.2.3). Consider the new variables $(\hat{x}', \hat{z}') = (\hat{x}, -\hat{z})$ on D_3, $(\hat{x}', \hat{z}') = (\hat{x}+\hat{z}, \hat{z})$ on D_4, $(\hat{x}', \hat{z}') = (\hat{x}+\hat{z}, \hat{z})$ on D_5 and $(\hat{x}', \hat{z}') = (\hat{x}, -\hat{z})$ on D_6, so that

$$4|T_\ell|^2 \int_{D_3 \cup D_4} F_{ij}(\hat{x}, \hat{x}+\hat{z}) = 8|T_\ell|^2 \int_{D_2'} F_{ij}(\hat{x}', \hat{x}'-\hat{z}')\, d\hat{x}'\, d\hat{z}',$$
$$4|T_\ell|^2 \int_{D_5 \cup D_6} F_{ij}(\hat{x}, \hat{x}+\hat{z}) = 8|T_\ell|^2 \int_{D_3'} F_{ij}(\hat{x}', \hat{x}'-\hat{z}')\, d\hat{x}'\, d\hat{z}',$$

where

$$D_2' = \{(\hat{x}', \hat{z}') \colon 0 \le \hat{z}_1' \le 1,\ 0 \le \hat{z}_2' \le \hat{z}_1',\ \hat{z}_1' \le \hat{x}_1' \le 1,\ \hat{z}_2' \le \hat{x}_2' \le \hat{x}_1' - \hat{z}_1' + \hat{z}_2'\},$$
$$D_3' = \{(\hat{x}', \hat{z}') \colon -1 \le \hat{z}_1' \le 0,\ 0 \le \hat{z}_2' \le 1+\hat{z}_1',\ \hat{z}_2' \le \hat{x}_1' \le 1+\hat{z}_1',\ \hat{z}_2' \le \hat{x}_2' \le \hat{x}_1'\}.$$

These domains are transformed into $[0,1]^4$ by the respective composition of the transformations $T_h \colon D \to D_h'$ $(h = 1, 2)$

$$T_2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 - w_3 + w_4 \\ w_3 \\ w_4 \end{pmatrix}, \quad T_3 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1 - w_4 \\ w_2 - w_4 \\ -w_4 \\ w_3 - w_4 \end{pmatrix},$$

and the Duffy transformation (A.2.4). Simple calculations lead finally to

$$4|T_\ell|^2 \int_{D_3 \cup D_4} F_{ij}(\hat{x}, \hat{x}+\hat{z}) = \frac{8|T_\ell|^2}{(4-2s)(3-2s)(2-2s)} \int_0^1 \frac{\psi_i^{(2)}(\eta_3)\psi_j^{(2)}(\eta_3)}{\left|B_\ell \begin{pmatrix} 1 \\ \eta_3 \end{pmatrix}\right|^{2+2s}} d\eta_3,$$
$$4|T_\ell|^2 \int_{D_5 \cup D_6} F_{ij}(\hat{x}, \hat{x}+\hat{z}) = \frac{8|T_\ell|^2}{(4-2s)(3-2s)(2-2s)} \int_0^1 \frac{\psi_i^{(3)}(\eta_3)\psi_j^{(3)}(\eta_3)}{\left|B_\ell \begin{pmatrix} \eta_3 \\ 1-\eta_3 \end{pmatrix}\right|^{2+2s}} d\eta_3,$$

where

$$\psi_1^{(2)}(\eta_3) = -1, \qquad \psi_2^{(2)}(\eta_3) = 1-\eta_3, \qquad \psi_3^{(2)}(\eta_3) = \eta_3,$$
$$\psi_1^{(3)}(\eta_3) = \eta_3, \qquad \psi_2^{(3)}(\eta_3) = -1, \qquad \psi_3^{(3)}(\eta_3) = 1-\eta_3.$$

For the sake of simplicity of notation, we write

$$d^{(1)}(x) := \left|B_\ell \begin{pmatrix} x \\ 1 \end{pmatrix}\right|^{2+2s}, \qquad d^{(2)}(x) := \left|B_\ell \begin{pmatrix} 1 \\ x \end{pmatrix}\right|^{2+2s},$$
$$d^{(3)}(x) := \left|B_\ell \begin{pmatrix} x \\ 1-x \end{pmatrix}\right|^{2+2s}.$$

In order to estimate the integrals in the unit interval, we use a Gaussian quadrature rule. Let $p_1, \dots, p_{n_G} \in [0,1]$ the quadrature points, and $w_1, ..., w_{n_G}$ their respective weights. Considering the integrals over the domains D'_h ($h \in \{1,2,3\}$), we may write

$$\int_0^1 \frac{\psi_i^{(h)}(\eta)\psi_j^{(h)}(\eta)}{d^{(h)}(\eta)} d\eta \approx \sum_{k=1}^{n_G} w_k \frac{\psi_i^{(h)}(p_k)\psi_j^{(h)}(p_k)}{d^{(h)}(p_k)}.$$

As before, we take advantage of the fact that the integrand only depends on ℓ through its denominator.

A.2.5 Complement

As for the integrals $J_\ell^{i,j}$, recalling the notation given by Definition 1.2.19, notice that

$$J_\ell^{i,j} = \int_{T_\ell} \varphi_i(x)\varphi_j(x)\,\omega_B^s(x)\,dx,$$

with $\omega_B^s(x) = \int_{B^c} \frac{1}{|x-y|^{2+2s}}\,dy$ (cf. Definition 1.2.19) and $B = B(0,R)$. Therefore all we need is an accurate computation of $\omega_B^s(x)$ for each quadrature point used in $T_\ell \subset \bar{\Omega}$ (notice that $\omega_B^s(x)$ is a smooth function up to the boundary of Ω since $|x-y| > d(\bar{\Omega}, B^c) > 0$).

Taking this into account, we observe that it is possible to take advantage of the fact that $\omega_B^s(x)$ is a radial function that can be either quickly computed on the fly or even precomputed with an arbitrary degree of precision.

Recalling parametrization (A.2.1), we aim to compute

$$J_\ell = 2|T_\ell| \int_{\hat{T}} \hat{\varphi}_i(\hat{x})\hat{\varphi}_j(\hat{x})\omega_B^s(\chi_\ell(\hat{x}))\,d\hat{x}.$$

The integral above may be calculated by a Gaussian quadrature rule in the reference element $\hat{T}$, provided that the values of ω_B^s at the quadrature points are computed.

Observe that the function ω_B^s is radial (see Figure A.3) and therefore it suffices to estimate it on points of the form $x = (x_1, 0)$, where $x_1 > 0$. For a fixed point x and given $\theta \in [0, 2\pi]$, let $\rho_0(\theta)$ be the distance between x and the intersection of the ray starting from x with angle θ with respect to the horizontal axis. Then, it is simple to verify that

$$\rho_0(\theta, x) = -x_1 \cos\theta + \sqrt{R^2 - x_1^2 \sin^2\theta},$$

and therefore, integrating in polar coordinates,

$$\omega_B^s(x) = \frac{1}{2s} \int_0^{2\pi} \frac{1}{\rho_0(\theta, x)^{2s}}\,d\theta.$$

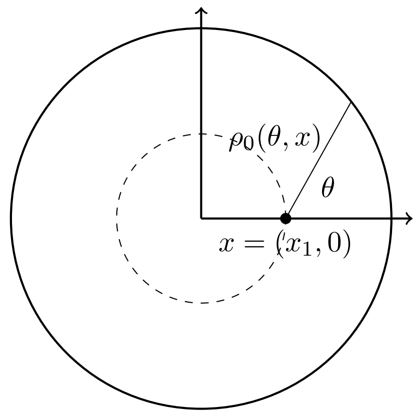

Figure A.3: Computing $\omega_B^s(x)$ in a point of $B = B(0,R)$. Due to the symmetry, the value of ω_B^s is the same along the dashed circle, hence we may assume that $x = (x_1, 0)$ and $0 \leq x_1 < R$. For any $0 \leq \theta \leq \pi$, the function ρ_0 is given by $\rho_0(\theta, x) = -x_1 \cos\theta + \sqrt{R^2 - x_1^2 \sin^2\theta}$.

In order to compute $J_\ell^{i,j}$ we perform two nested quadrature rules: one over $\hat{T}$ and, for each quadrature point p_k in $\hat{T}$, another one to estimate $\omega_B^s(p_k)$ over $[0, 2\pi]$. Let us denote by $n_{\hat{G}}$ the number of quadrature points in $\hat{T}$ and a n_G point one on $[0, 2\pi]$. Let $p_1, \ldots, p_{n_{\hat{G}}} \in \hat{T}$, $\theta_1, \ldots, \theta_{n_G} \in [0, 2\pi]$ be these quadrature nodes, and $w_1, \ldots, w_{n_{\hat{G}}}$, $W_1, \ldots, W_{n_G}$ their respective weights. Applying the rules we obtain

$$J_\ell^{i,j} \approx \frac{|T_\ell|}{s} \sum_{k=1}^{n_{\hat{G}}} w_k \hat{\varphi}_i(p_k) \hat{\varphi}_j(p_k) \sum_{q=1}^{n_G} \frac{W_q}{\rho_0(\theta_q, \chi_\ell(p_k))^{2s}}.$$

In the same fashion as for the other computations, the previous expression may be efficiently calculated by writing it as the product of a pre-computed matrix (that only depends on the choice of the quadrature rules) times a vector that depends on the elements under consideration.

Appendix B

Balayage problem

The purpose of this appendix is to describe the method utilized to compute solutions for problems posed in a ball. In [76, Chapter I], the so-called *balayage problem* on a ball for the fractional Laplacian is studied. This allows to obtain an explicit expression for the Poisson kernel of the fractional Laplacian in such a domain. Indeed, let $\Omega = B(0,r) \subset \mathbb{R}^n$ for some $r > 0$ and let $g : \Omega^c \to \mathbb{R}$. Then, a solution to

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega^c, \end{cases} \tag{B.0.1}$$

is given by

$$u(x) = \int_{\Omega^c} g(y)\, P(x,y)\, dy, \tag{B.0.2}$$

where

$$P(x,y) = \frac{\Gamma(n/2)\sin(\pi s)}{\pi^{n/2+1}} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{1}{|x-y|^2}, \ x \in \Omega, \ y \in \Omega^c.$$

Even though (B.0.2) is an explicit expression of the solution to (B.0.1), computing its values is not a straightforward task. For the sake of simplicity, we set $r = 1$ and $n = 2$, as has been the case in the experiments we carried out in examples 5.5.2 and 5.5.3. More precisely, we aim to compute the map

$$B(0,1) \ni x \mapsto \phi(x) := \int_{B(0,1)^c} \frac{g(y)}{(|y|^2-1)^s |x-y|^2}\, dy,$$

that allows to evaluate the solution in $B(0,1)$,

$$u(x) = \frac{\sin(\pi s)}{\pi^2} (1 - |x|^2)^s \phi(x).$$

Besides the hypotheses from Theorem 5.4.4, we assume that g is radial, namely, that $g(y) = \tilde{g}(|y|)$ for some function $\tilde{g} : [0,\infty) \to \mathbb{R}$. Because of the radiality of g and because Ω is a ball, the solution u is also a radial function. This means that it suffices to compute ϕ over points of the form $x = (x_1, 0)$ for $x_1 \in [0, 1)$.

Integrating in polar coordinates, we obtain

$$\phi(x) = \int_1^\infty \frac{\tilde{g}(\rho)\,\rho}{(\rho^2-1)^s} \int_{\partial B_1} \frac{1}{|x-\rho v|^2} d\sigma(v)\, d\rho.$$

Next, we parametrize the sphere $v = (\cos\theta, \sin\theta)$ with $\theta \in [0, 2\pi)$, and apply the Law of cosines to obtain

$$|x - \rho v|^2 = \rho^2 + x_1^2 - 2x_1\rho\cos\theta.$$

Making the change of variables $\tilde{\rho} = \rho^2 - 1$, we arrive at

$$\phi(x) = \frac{1}{2}\int_0^\infty \frac{\tilde{g}(\sqrt{1+\tilde{\rho}})}{\tilde{\rho}^s} \psi(\tilde{\rho})\, d\tilde{\rho},$$

where

$$\psi(\tilde{\rho}) = \int_0^{2\pi} \frac{1}{\tilde{\rho} + 1 + x_1^2 - 2x_1\sqrt{1+\tilde{\rho}}\,\cos\theta} d\theta.$$

Observe that as $x_1 \to 1^-$ and $\tilde{\rho} \to 0$, the integrand in $\psi(\tilde{\rho})$ becomes singular. Luckily, this integral can be explicitly calculated after basic manipulations. Indeed, since for every $a > 1$ it holds that

$$\int_0^{2\pi} \frac{1}{a - \cos\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2-1}},$$

we obtain

$$\psi(\rho) = \frac{2\pi}{\tilde{\rho} + 1 - x_1^2}.$$

Thus, it holds that

$$\phi(x) = \pi \int_0^\infty \frac{\tilde{g}(\sqrt{1+\tilde{\rho}})}{\tilde{\rho}^s(\tilde{\rho}+1-x_1^2)}\, d\tilde{\rho}.$$

In order to compute the integral with respect to $\tilde{\rho}$, we split the interval as $(0,\infty) = (0,1) \cup [1,\infty)$. The first part is singular at the origin, while the second part involves integration on an unbounded domain. Making respectively the changes of variables $\tilde{\rho} = z^\alpha$ and $\tilde{\rho} = z^{-\beta}$ for some $\alpha, \beta > 0$, we obtain

$$\phi(x) = \pi\alpha \int_0^1 \frac{\tilde{g}\left(\sqrt{1+z^\alpha}\right) z^{\alpha(1-s)-1}}{z^\alpha + 1 - x_1^2} dz + \pi\beta \int_0^1 \frac{\tilde{g}\left(\sqrt{1+z^{-\beta}}\right) z^{\beta(s-1)-1}}{z^{-\beta}+1-x_1^2} dz.$$

The integrals above are computed by using a Gaussian quadrature rule in the interval $[0, 1]$, and therefore, using (B.0.2), we are able to evaluate

$$u(x) = \frac{\sin(\pi s)}{\pi^2}\,(1-|x|^2)^s \phi(x).$$

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