Algunos problemas de análisis sobre cúspides exteriores

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Resumen

En esta tesis estudiamos distintos problemas de análisis sobre dominios con cúspides exteriores. Principalmente: la densidad de funciones suaves en espacios de Sóbolev, el problema de extensión y la desigualdad de Korn. Se sabe que muchos de los resultados clásicos de análisis en espacios de Sóbolev estandar, necesarios para el trabajo con ecuaciones en derivadas parciales y para la aproximación de sus soluciones usando métodos numéricos, no son ciertos en dominios con singularidades, como las cúspides exteriores. Por ello se hace necesario trabajar con espacios de Sóbolev con pesos, donde los pesos son tomados de modo de compensar la singularidad del dominio.

Las distintas nociones de cúspide exterior con las que trabajamos prescinden de una descripción precisa del borde del dominio, aunque le impongan algunas restricciones. En primer lugar, introducimos el concepto de cúspide normal, cuya definición está basada en la descomposición de Whitney del dominio e incluye una propiedad de uniformidad por bandas que establece cierta regularidad local sobre el borde. Esta definición nos permite probar los siguientes resultados:

- La densidad de funciones suaves hasta el borde del dominio, en el espacio de Sóbolev $W^{k,p}(\Omega)$.
- Teoremas de extensión en los que se construyen operadores: $E : W^{k,p}(\Omega) \longrightarrow W^{k,p}_{\sigma}(\mathbb{R}^n)$, donde σ es un peso apropiado.

Luego introducimos las nociones de cadenas de rectángulos y cadenas de cuasi-rectángulos. Las cadenas de cuasi-rectángulos nos permiten definir una clase muy general de dominios que incluyen a las cúspides normales, pero admiten condiciones más laxas sobre el borde. Para estas cadenas probamos:

- Desigualdades de Poincaré sin pesos.
- Desigualdades de tipo Korn con pesos.

En algunos casos exhibimos contraejemplos que muestran que los pesos obtenidos son óptimos. También estudiamos estos problemas considerando que el espacio original es un Sóbolev pesado: $W^{k,p}_{\omega}(\Omega)$.

Palabras clave: Cúspides exteriores, operadores de extensión, dominios de extensión, desigualdad de Korn, desigualdad de Poincaré, espacios de Sóbolev con peso.

Some problems of analysis on external cusps

Abstract

In this thesis we study several problems of analysis on domains with external cusps. Mainly: the density of smooth functions on Sobolev spaces, the extension problem and Korn's inequality. It is known that many classical analysis results on standard Sobolev spaces, which are necessary for the study of partial differential equations and for approximating their solutions using numerical methods, do not hold on singular domains, such as external cusps. Hence, it is necessary to work with weighted Sobolev spaces, where the weights are taken in a way that somehow compensates the singularity of the domain.

The different notions of external cusp that we handle avoid any precise description of the domain's boundary, even when they impose some restrictions. In the first place, we introduce the concept of normal cusp, which definition is based on the Whitney decomposition of the domain, and includes a sectional unifomity property that establishes some local regularity on the boundary. This notion allows us to prove the following results:

- The density of smooth functions up to the boundary of the domain, in the Sobolev space $W^{k,p}(\Omega)$.
- Extension theorems where we build operators of the form $E : W^{k,p}(\Omega) \longrightarrow W^{k,p}_{\sigma}(\mathbb{R}^n)$, being σ a proper weight.

Afterwards, we introduce the notions of chains of rectangles and chains of quasi-rectangles. Chains of quasi-rectangles allow us to define a very general class of domains, that includes normal cusps, but admits more relaxed conditions on the boundary. For these chains, we prove:

- Unweighted Poincaré inequalities.
- Weighted Korn inequalities.

In some cases we exhibit counterexamples that show that the obtained weights are optimal. We also study these problems considering that the original space is a weighted Sobolev space $W^{k,p}_{\omega}(\Omega)$.

Keywords: External cusps, extension operator, extension domain, Korn inequality, Poincaré inequality, weighted Sobolev spaces.

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Contents

1

Introducción

1.1 Dominios con cúspides exteriores

Dado $\Omega \subset \mathbb{R}^n$ un dominio acotado, coloquialmente decimos que tiene una cúspide exterior en x_0 si $x_0 \in \partial\Omega$ y Ω se angosta al acercarse a x_0 de modo que ningún cono con vértice en x_0 está contenido en Ω. A lo largo esta tesis asumiremos que $x_0 = \mathbf{0}$. En la bibliografía se presentan distintas definiciones de cúspide exterior. Las cúspides más simples son las llamadas cúspides de tipo potencia:

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} : \; |x'| < x_n^{\gamma}\}, \tag{1.1.1}$$

siendo γ algún número real mayor que 1.

Esta noción se generaliza naturalmente a dominios cuyo perfil está descripto por una función φ, cuyas características implican un comportamiento cuspidal:

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} : \; |x'| < \varphi(x_n)\}, \tag{1.1.2}$$

donde $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ es una función creciente y derivable tal que $\varphi(0) = 0$ y $\varphi'(0) = 0$, ó, más generalmente, φ es Lipschitz y $\frac{\varphi(t)}{t} \longrightarrow 0$ $(t \longrightarrow 0^+)$.

Si llamamos $B' = B^{n-1}(\mathbf{0}, 1)$ a la bola $n-1$ dimensional con centro en el origen y radio 1, y aB' es la dilatación de B' por a, (i.e.: $aB' = B^{n-1}(\mathbf{0}, a)$) está claro que (1.1.2) puede escribirse:

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} : \; x' \in \varphi(x_n)B'\}.$$

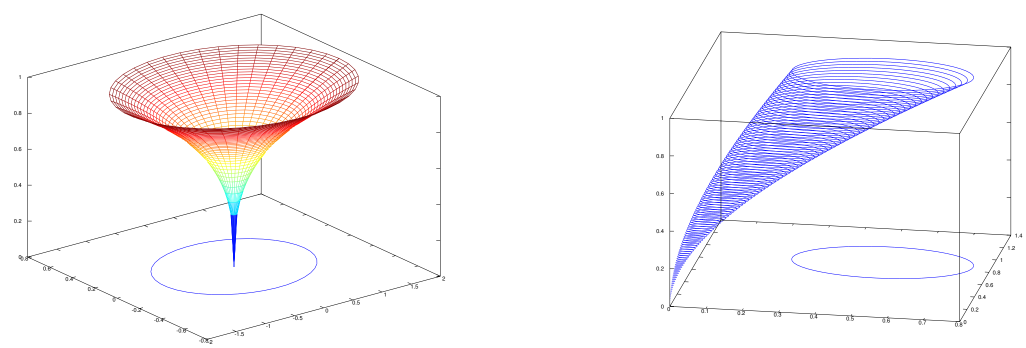
Maz'ya y Poborchiĭ, en [Maz'ya and Poborchiĭ, 1997], generalizan esta idea, e introducen la siguiente definición de cúspide experior:

Definición A. *Sea $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ un dominio con borde compacto $\partial\Omega$. Asumimos que $\mathbf{0} \in \partial\Omega$ y que $\partial\Omega \setminus \{\mathbf{0}\}$ es localmente el gráfico de una función Lipschitz. Decimos que Ω tiene una cúspide exterior en el origen si existe un entorno del origen $U \subset \mathbb{R}^n$, tal que*

$$U \cap \Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} : x' \in \varphi(x_n)\varpi\},$$

donde $\varpi \subset \mathbb{R}^{n-1}$ es un dominio Lipschitz acotado y $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ es una función Lipschitz creciente tal que $\frac{\varphi(t)}{t} \to 0$ $(t \to 0^+)$ y $\varphi(0) = 0$.

La Definición A implica una importante generalización en la medida en que contempla dominios que no incluyen el eje vertical, sino que se comportan de manera tangencial a él. Sin embargo, impone aún una restricción importante: todo corte horizontal de Ω presenta la misma forma (ϖ), escalada de acuerdo a la altura.



(a) Cúspide que contiene el eje z

(b) Cúspide tangencial al eje z

Figura 1.1: Cúspides de Maz'ya

En esta tesis introducimos una noción de cúspide que generaliza la Definición A (ver las Definiciones 3.2.1 y 3.3.1). Nuestra definición está basada en la existencia de una cadena de cubos en la descomposición de Whitney del dominio, que forma lo que llamamos la *espina* del dominio, su columna vertebral. Esta cadena de cubos se ubica aproximadamente en el centro del dominio. Por ejemplo, en el caso de una cúspide de tipo potencia, estaría formada por cubos que tocan el eje vertical. Además, se angosta al aproximarse al origen, y la velocidad de este angostamiento da el comportamiento cuspidal del dominio. Sobre el borde, en lugar de la Lipschitzianidad local de la Definición A, imponemos una condición de uniformidad por secciones. Los dominios uniformes [Martio and Sarvas, 1979, Martio, 1980, Jones, 1981, Smith et al., 1994, Väisälä, 1988] incluyen a los Lipschitz, y admiten la construcción de operadores de extensión [Jones, 1981]. En este sentido, la condición de uniformidad por secciones representa una hipótesis de regularidad bastante laxa sobre el borde del dominio que permite definir un operador de extensión localmente.

Para simplicar las demostraciones presentamos dos definiciones ligeramente distintas. En 3.2.1 introducimos las cúspides *normales*, cuya espina contiene al eje vertical. Estas cúspides mantienen cierta simetría respecto del eje, como en (1.1.2). Las cúspides *curvas* (Definición 3.3.1) cumplen con las mismas propiedades que las normales, pero pueden ser tangenciales al eje, como las que satisfacen la Definición A.

La principal virtud de las cúspides normales es que, en tanto no involucran una descripción del perfil del dominio, nos permiten probar que los pesos necesarios para compensar la singularidad no dependen del detalle del borde, sino sólo de la velocidad a la que el dominio se angosta al aproximarse al origen. La definición de cúspide normal puede interpretarse como un análogo de la Definición A, en donde la función φ no representa un perfil preciso de la cúspide sino que simplemente la interpola en algunos puntos, dando así una descripción de la velocidad del angostamiento. Además, según esta definición, φ puede no ser monótona. Finalmente, el requisito de uniformidad por secciones constituye una condición mucho más general que la Lipschitzianidad local de la Definición A. A modo de ejemplo, probamos que una cúspide cumpliendo con la Definición A, pero donde ϖ es un dominio uniforme, no necesariamente Lipschitz, es una cúspide normal o curva.

1.2 Dominios de Extensión

Sea Ω un dominio en $\mathbb{R}^n$. $W^{k,p}(\Omega)$ es el espacio de Sóbolev de funciones con derivadas débiles de orden α para todo α tal que $|\alpha| \leq k$, con la norma:

$$\|f\|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)}.$$

Decimos que Ω es un dominio de extensión de Sóbolev (E.D.S., por sus siglas en inglés) si existe un operador lineal y acotado:

$$E : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n),$$

tal que $Ef|_\Omega = f$ para toda $f \in W^{k,p}(\Omega)$.

La existencia de un operador de extensión es de suma utilidad, en tanto implica que muchos resultados válidos para $W^{k,p}(\mathbb{R}^n)$ son heredados por $W^{k,p}(\Omega)$. Un ejemplo clásico de esta situación está dado por los teoremas de inmersión, que pueden demostrarse primero en $\mathbb{R}^n$ y luego, a través de un argumento de extensión, para ciertos dominios. Pueden encontrarse esta y otras aplicaciones de los teoremas de extensión en la bibliografía clásica de espacios de Sóbolev. Por ejemplo: [Adams and Fournier, 2003, Burenkov, 1998, Evans, 1998, Maz'ya, 2011, Maz'ya and Poborchiĭ, 1997].

Es sabido que los dominios suaves son E.D.S. De hecho, al ser suave, el borde del dominio puede ser localmente aplanado a través de transformaciones regulares y el operador de extensión puede construirse aplicando argumentos de reflexión (ver [Adams and Fournier, 2003, Maz'ya, 2011]). Por otro lado usando la fórmula de representación de Sóbolev en un cono e integrales singulares, Calderón [Calderón, 1968] probó que los dominios Lipschitz son E.D.S. para $1 < p < \infty$. Este resultado fue extendido por Stein al rango $1 \leq p \leq \infty$, usando un procedimiento de reflexión promediada [Stein, 1970].

Las técnicas de reflexión son un enfoque natural para extender funciones. Dominios más generales requieren técnicas de reflexión más complejas. En este contexto, Jones, en [Jones, 1981], estudió los dominios (ε, δ) también llamados *localmente uniformes*, que habían sido introducidos en [Martio and Sarvas, 1979] y forman una clase más general que los Lipschitz.

Jones probó que todo dominio (ε, δ) es un E.D.S. En términos de las descomposiciones de Whitney $\mathcal{W}$ y $\mathcal{W}^c$, de Ω y $(\Omega^c)^o$ respectivamente, la idea de Jones se basa en que los dominios (ε, δ) satisfacen las siguientes propiedades:

(a) Los cubos de Whitney $Q \in \mathcal{W}^c$ cerca de Ω tienen un cubo "reflejado" $Q^* \in \mathcal{W}$, de tamaño similar y cercano a Q.

(b) Los reflejados $Q_1^*, Q_2^* \in \mathcal{W}$ de cubos vecinos $Q_1, Q_2 \in \mathcal{W}^c$ pueden unirse a través de una cadena de cubos en $\mathcal{W}$.

Gracias a esto, una aproximación polinomial de f en Q^* puede utilizarse para definir la extensión de f en Q. En la Figura 1.2 mostramos dos cubos vecinos y sus reflejados, junto con una possible cadena uniéndolos.

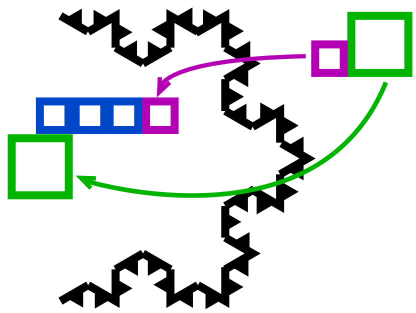


Figura 1.2: Cubos reflejados y cadena

En el caso de las cúspides exteriores, en cambio, se conocen contraejemplos que muestran que no es posible construir operadores de extensión en el sentido clásico, por lo que se hace necesario extender a espacios de Sóbolev con pesos.

En este sentido, Maz'ya y Poborchiĭ [Maz'ya and Poborchiĭ, 1997] probaron el siguiente teorema de extensión para cúspides cumpliendo con la Definición A.

Teorema A. *Sea $\Omega \subset \mathbb{R}^n$ un dominio con una cúspide exterior en el origen, según la Definición A. Entonces, existe un operador de extensión*

$$\Lambda : W^{kp}(\Omega) \to W^{kp}_{\sigma}(\mathbb{R}),$$

donde el peso σ puede tomarse según las siguientes condiciones:

(a) Si $kp < n-1$, ó $k = n-1$ y $p = 1$, y φ satisface:

$$\frac{\varphi(t)}{t} \text{ es no decreciente.} \tag{1.2.1}$$

entonces,

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^{kp} & x \in \Omega^c \end{cases}$$

(b) *Si $kp > n-1$, y φ es tal que:*

$$\exists C_\varphi \text{ constante} : \qquad \varphi(2t) \leq C_\varphi \varphi(t) \tag{1.2.2}$$

entonces,

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^{n-1} & x \in \Omega^c \end{cases}$$

(c) *Si $kp = n-1$, $1 < p < \infty$, φ es tal que vale (1.2.1) y además:*

$$\exists \delta > 0 : \qquad \varphi(t + \varphi(t)) = \varphi(t)[1 + O(\varphi(t)/t)^\delta] \qquad \text{as } t \to 0 \tag{1.2.3}$$

entonces,

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^{kp} \log\left(\frac{\varphi(|x|)}{|x|}\right)^{\frac{1}{p'}} & x \in \Omega^c \end{cases}$$

(d) *Asumiendo (1.2.2), si $\tilde{\sigma}$ es un peso radial no decreciente, tal que existe un operador de extensión: $\tilde{\Lambda} : W^{kp}(\Omega) \to W^{kp}_{\tilde{\sigma}}(\mathbb{R}^n)$, entonces:*

$$\tilde{\sigma}(x) \leq C\sigma(x) \quad \forall x \in U \setminus \Omega,$$

donde U es un entorno del origen y σ se toma según el caso. Para el peso del item (b) *asumimos que $\mathbf{0} \in \varpi$.*

Entre otras aplicaciones, un resultado de estas características es crucial , por ejemplo, para la construcción de mallas triangulares (o tetraedrales) apropiadas para la aplicación del método de elementos finitos para la resolución de ecuaciones elípticas en derivadas parciales. Cualquier triangulación de una cúspide exterior Ω produce un dominio poligonal que excede el borde de Ω. Pero en tanto los resultados clásicos de extensión no son válidos sobre cúspides, la solución de la ecuación sobre el dominio poligonal es *menos* regular que la solución exacta del problema original. Una consecuencia de este hecho es que las mallas cuasi-uniformes no permiten obtener órdenes óptimos de convergencia, como sucede en caso de dominios suaves. En [Acosta et al., 2007] y [Acosta and Armentano, 2011], se muestra que el orden óptimo de convergencia puede recuperarse utilizando mallas graduadas, donde la graduación de la malla se realiza de acuerdo al peso σ del operador de extensión.

En el Apéndice A presentamos otra posible aplicación de los teoremas de extensión deduciendo de ellos desigualdades de Korn con pesos para cúspides normales.

En esta tesis presentamos una serie de teoremas de extensión que generalizan al Teorema A en varios sentidos. Por un lado, valen para cúspides normales y curvas, que son más generales que las contempladas en la Definición A. Por otro, probamos que puede prescindirse de las condiciones sobre los parámetros k, p y n, impuestas en los incisos (a) y (b) del Teorema A. Finalmente, tratamos el caso de espacios de Sóbolev con pesos, obteniendo operadores de extensión de la forma: $E : W^{k,p}_\omega(\Omega) \longrightarrow W^{k,p}_{\omega\sigma}(\mathbb{R}^n)$.

Siguiendo los argumentos de Maz'ya, nuestro operador de extensión se construye en tres etapas. En la primera se extiende a una *doble cúspide*. Esta extensión local tiene por objeto independizar el resto del proceso del detalle del borde, y utiliza la uniformidad por secciones para aplicar una adaptación de las ideas de [Jones, 1981]. Al no resolver la singularidad del dominio, esta etapa no requiere de ningún peso. La segunda etapa extiende a un cono, usando sólo la información de la cadena de cubos central del dominio. El peso que surge naturalmente es el estrictamente necesario para compensar la velocidad del angostamiento de la cadena de cubos. Finalmente, en la tercera etapa se completa la extensión a un entorno del origen, radialmente.

Para poder garantizar que la extensión de la primera etapa se encuentra con la función original en $\partial\Omega$ de manera tal que las derivadas débiles se mantienen en L^p, la demostración se realiza primero para funciones suaves, en $C^\infty(\bar{\Omega} \setminus \{\mathbf{0}\})$ y se generaliza luego a $W^{k,p}(\Omega)$ a través de un argumento de densidad. Para ello es necesario demostrar que las funciones de $W^{k,p}(\Omega)$ pueden aproximarse por funciones en C^∞. Dado que el problema de aproximación por funciones suaves tiene interés en sí mismo, probamos el teorema de densidad separadamente, en el Capítulo 4.

1.3 Desigualdad de Korn

Dado un campo vectorial $u \in W^{k,p}(\Omega)^n$, la desigualdad de Korn establece que

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p(\Omega)^{n\times n}}, \tag{1.3.1}$$

donde $\varepsilon(u)$ es la parte simétrica de la matriz diferencial de u, Du. Es decir:

$$\varepsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

En el contexto de las ecuaciones de elasticidad lineal, $u(x)$ representa el desplazamiento del punto $x \in \Omega$, al ser Ω sometido a deformaciones, mientras que $\varepsilon(u)$ es el tensor de deformaciones. La desigualdad de Korn es fundamental para probar la coercividad de la forma bilineal asociada a las ecuaciones de elasticidad, lo que permite garantizar la existencia de soluciones, pero también la convergencia de los métodos numéricos aplicados para aproximarlas.

Está claro que la desigualdad (1.3.1) podría ser falsa. Basta tomar, por ejemplo, un campo vectorial u cuya matriz diferencial fuese antisimétrica. Por lo tanto, es necesario imponer condiciones adicionales sobre u. Korn, [Korn, 1906, 1909] probó, en el caso particular $p = 2$, la validez de (1.3.1) para funciones de traza nula, siendo Ω un abierto cualquiera. Este resultado es conocido como el primer caso de la desigualdad. El llamado segundo caso se refiere a campos u que satisfacen:

$$\int_\Omega \frac{Du - Du^t}{2} = 0, \tag{1.3.2}$$

y para ellos la validez de (1.3.1) depende de la naturaleza del dominio Ω.

El segundo caso de la desigualdad de Korn está fuertemente relacionado con el *caso general* que establece que:

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)^{n\times n}}\Big\}. \tag{1.3.3}$$

Está claro que, para funciones que satisfacen (1.3.2), (1.3.1) implica (1.3.3). Para una u cualquiera, vale la misma implicación, independientemente de las carecterísticas del dominio, a través de un sencillo argumento que puede verse en [Brenner and Scott, 2008], y que adaptamos en el Capítulo 6. Por otro lado, (1.3.1) puede deducirse de (1.3.3) utilizando argumentos de compacidad algo más complejos, que dependen del dominio considerado (ver, por ejemplo [Kikuchi and Oden, 1988]).

Se conocen diversas demostraciones de esta desigualdad para dominios no singulares. Friederichs [Friederichs, 1937],[Friederichs, 1947] la prueba en algunos casos particulares en espacios de 2 y 3 dimensiones. Nitsche, en [Nitsche, 1981] la demuestra para dominios Lipschitz, utilizando argumentos de extensión. En [Kondratiev and Oleinik, 1989] los autores tratan dominios estrellados respecto de una bola y prueban que la constante de la desigualdad está acotada en términos del cociente entre el diámetro del dominio y el diámetro de la bola. En un artículo reciente, [Durán, 2012], se prueba el segundo caso de la desigualdad para dominios estrellados respecto de una bola para $p = 2$ usando la continuidad de la inversa a derecha del operador divergencia, y se obtiene una expresión explícita para la constante. Para $n = 2$, la constante se puede acotar por $\frac{R}{\rho}$ por un término logarítmico, donde R y ρ son los radios de Ω y de la bola, respectivamente. En [Costabel and Dauge, 2013] se muestra que el término logarítmico puede ser eliminado. Por otra parte, en [Durán, 2012] el autor prueba que la constante para dominios convexos en $\mathbb{R}^n$ es $\frac{R}{\rho}$. Este hecho resultará de utilidad más adelante. Debemos mencionar también [Durán and Muschietti, 2004], donde los autores prueban que (1.3.3) vale para dominios uniformes usando el operador de extensión construido en [Jones, 1981] para estos dominios. En el Apéndice A mostramos, siguiendo a [Durán and Muschietti, 2004], cómo las técnicas de extensión que desarrollamos en el Capítulo 5 pueden adaptarse para probar desigualdades de Korn sobre cúspides normales. Finalmente, en [Acosta et al., 2006b] se prueba la desigualdad de Korn para dominios de John, como un corolario de la existencia de una inversa a derecha para el operador divergencia. Se conocen también demostraciones que utilizan la teoría de integrales singulares, debidas a Govert, Fichera y Ting. Sus argumentos se siguen en [Kikuchi and Oden, 1988]. Otras referencias clásicas son [Fichera, 1974], [Horgan, 1995].

A pesar de estos resultados, se sabe que la desigualdad de Korn no vale en cúpisdes exteriores [Acosta et al., 2012]. Esto puede ser resuelto, como en el caso del problema de extensión, utilizando pesos apropiados para obtener una desigualdad de la forma:

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p_\sigma(\Omega)^{n\times n}}\Big\}. \tag{1.3.4}$$

Los principales antecedentes que seguimos en lo concerniente a la desigualdad de Korn sobre cúspides exteriores son: [Acosta et al., 2006a], [Durán and López García, 2010b] y [Acosta et al., 2012].

En [Durán and López García, 2010b] se tratan cúspides de tipo potencia y se prueba la existencia de una inversa a derecha para el operador divergencia. Como corolario, se deduce la siguiente desigualdad de Korn con pesos (ver [Durán and López García, 2010b, Theorem 6.2]):

Teorema B. *Dado Ω un dominio de la forma (1.1.1), $1 < p < \infty$, $B \subset \Omega$ una bola abierta y $\beta \geq 0$; existe una constante C, dependiendo sólo de Ω, B, p y β, tal que para toda $u \in W^{1,p}_{d^{p\beta}}(\Omega)^n$:*

$$\|Du\|_{L^p_{d^{p\beta}}(\Omega)^{n\times n}} \leq C\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{d^{p(\beta+1-\gamma)}}(\Omega)^{n\times n}}\},$$

donde $d = d(x)$ es la distancia al origen, y γ es la potencia de la cúspide.

Para el caso sin pesos $W^{1,p}(\Omega)^n$, tomamos $\beta = 0$, y el peso a la derecha debido al comportamiento cuspidal de Ω es $d^{p(1-\gamma)}$.

La optimalidad de este resultado se prueba en [Acosta et al., 2012], donde los autores trabajan con cúspides de perfil φ y prueban el siguiente teorema:

Teorema C. *Sea Ω una cúspide de perfil φ, según (1.1.2), $\beta_1, \beta_2 \in \mathbb{R}$, $1 < p < \infty$ y B una bola compactamente contenida en Ω. Si hay una constante C tal que:*

$$\|Dv\|_{L^p_{(\varphi')^{p\beta_1}}(\Omega)^{n\times n}} \leq C\{\|v\|_{L^p(B)^n} + \|\varepsilon(v)\|_{L^p_{(\varphi')^{p\beta_2}}(\Omega)^{n\times n}}\},$$

para toda $v \in W^{1,p}_{(\varphi')^{p\beta_1}}(\Omega)^n$, entonces $\beta_1 \geq \beta_2 + 1$.

Es importante observar que para cúspides de tipo potencia, $\varphi'(t) = \gamma t^{\gamma-1}$, y por lo tanto, la desigualdad de Korn del Teorema B se corresponde con el caso en que $\beta_1 = \beta_2 + 1$ en el Teorema C y el peso del miembro derecho resulta, en este sentido, el mejor posible.

Quisiéramos también mencionar [Nazarov, 2012], donde se prueban desigualdades de Korn con pesos anisotrópicos para cúspides en $\mathbb{R}^3$ que satisfacen la Definición A, tomando $\varphi(z) = z^\gamma$, aunque el autor menciona que pueden tratarse casos más generales utilizando las mismas ideas. Finalmente en [Acosta et al., 2006a] se demuestran desigualdades de Korn con pesos para dominios Hölden-α, que incluyen a las cúspides de tipo potencia. En este caso, como un dominio Hölder-α puede tener muchas singularidades, los pesos que surgen naturalmente dependen de la distancia al borde.

Si bien muchas de las demostraciones de Korn se realizan a través de argumentos de extensión ([Nitsche, 1981],[Durán and Muschietti, 2004]), se sabe que la desigualdad vale incluso para dominios que no admiten extensión. Por ejemplo, en [Acosta et al., 2006b] los autores construyen una solución para el problema de la divergencia sobre dominios de John. El segundo caso de la desigualdad de Korn puede deducirse fácilmente de este resultado, y como ya señalamos, el segundo caso implica el caso general, por lo cual ambos valen en dominios de John. Es interesante observar que el problema de extensión puede no tener sentido para un dominio de John. Esto nos hizo sospechar que puede haber cúspides exteriores más generales que las normales para las cuales una desigualdad de Korn con pesos es cierta.

En efecto, teniendo esto en mente, en el Capítulo 6 presentamos un abordaje del problema de Korn que excede el caso de las cúspides normales. Allí probamos desigualdades de Poincaré y de Korn con pesos para cadenas de rectángulos. Una cadena de rectángulos es una unión de rectángulos que satisfacen ciertas propiedades. Esencialmente: cada rectángulo tiene sólo dos vecinos (el anterior y el siguiente), y cualesquiera dos rectángulos vecinos son comparables entre sí. Nuestra demostración está basada fundamentalmente en una versión discreta de una desigualdad de Hardy.

Una cadena de rectángulos puede formar una cúspide exterior, si los rectángulos se angostan apropiadamente al aproximarse a un punto. Sin embargo, la noción de cadena de réctangulos abarca también muchos otros dominios, no necesariamente singulares. Lo interesante de nuestra técnica es que puede aplicarse con facilidad a cadenas de subdominios cuya forma sea sólo *aproximadamente* rectangular. Para ello introducimos la noción de cadena de cuasi-rectángulos y deducimos desigualdades de Poincaré y de Korn para dominios de este tipo. A modo de ejemplo, mostramos que pueden construirse cúspides exteriores cuyo borde sea, por secciones, el de un dominio de John. Así, definimos la noción de cúspide localmente John, que generaliza la de cúspide normal (o curva), en tanto todo dominio uniforme es un dominio de John. La validez de una desigualdad de Korn con pesos para cúspides localmente John se deduce inmediatamente de los resultados obtenidos para cadenas de cuasi rectángulos.

Nuestros resultados se generalizan facilmente a la situación en que el campo u está en un espacio de Sóbolev con peso $W^{k,p}_\omega(\Omega)$. Para el caso de las cúspide localmente John, analizamos específicaemente pesos ω dependiendo de la distancia a la cúspide y de la distancia al borde del dominio. Es interesante observar que para pesos de la forma $\omega = (\varphi')^{p\beta}$ generalizamos el Teorema B probando la desigualdad de Korn incluso para algunos valores negativos de β.

1.4 Resumen

Comenzamos el Capítulo 2 definiendo los espacios de Sóbolev con peso. Luego introducimos la notación que usaremos para referirnos a cubos y rectángulos y damos una demostración clásica del teorema de descomposición de Whitney. A continuación, presentamos los dominios localmente uniformes y los dominios de John, probando algunas de sus propiedades más importantes. Finalmente, demostramos una serie de resultados referidos a aproximaciones polinomiales sobre cubos y rectángulos, que son de uso extensivo a lo largo de la tesis.

En el Capítulo 3 introducimos la noción de cadena de rectángulos que nos permite luego definir cúspides normales y cúspides curvas. Concluimos el capítulo con un ejemplo que muestra que toda cúspide que satisfaga la Definición A es necesariamente una cúspide normal (o curva).

En el Capítulo 4 probamos que $C^\infty(\bar{\Omega} \setminus \{\mathbf{0}\})$ es denso en $W^{k,p}(\Omega)$, siendo Ω una cúspide normal o curva. Tratamos también el caso pesado $W^{k,p}_\omega(\Omega)$, y probamos que la densidad sigue valiendo cuando el peso ω puede ser aproximado por constantes en bandas horizontales. Este resultado se generaliza en la Sección 5.3, donde se concluye que $C^\infty(\bar{\Omega})$ es denso en $W^{k,p}(\Omega)$.

El Capítulo 5 está dedicado a la construcción del operador de extensión. Desarrollamos primero el caso de las cúspides normales, porque es más simple y permite evitar algunos tecnicismos que oscurecen las demostraciones sin aportar ninguna idea de fondo. Para las cúspides curvas desarrollamos una *etapa cero* que, de manera análoga a la etapa uno de la extensión para cúspides normales, permite extender funciones desde una cúspide curva a una normal.

En el Capítulo 6 probamos desigualdades de Poincaré y desigualdades de Korn con pesos para cadenas de rectángulos y cadenas de cuasi-rectángulos. También definimos cúspides localmente John, como un caso particular de cadenas de cuasi-rectángulos y mostramos que nuestros resultados generalizan el Teorema B.

Finalmente, mostramos cómo puede obtenerse la desigualdad de Korn para cúspides normales a través de argumentos de extensión. Dado que este resultado es menos general, y su demostración más intrincada, que los dados en el Capítulo 6, lo incluimos en el Apéndice A.

Es importante señalar que el contenido de la presente tesis ha dado lugar a dos artículos. El primero de ellos contiene fundamentalmente las definiciones de cúspides normales y curvas, y los teoremas de extensión desarrollados en el Capítulo 5 y ha sido publicado en [Acosta and Ojea, 2012]. El segundo, [Acosta and Ojea, 2014], que introduce las nociones de cadenas de rectángulos y de cuasi-rectángulos y establece las desigualdades de Korn que aquí presentamos en el Capítulo 6, ha sido recientemente remitido para su publicación.

1

Introduction

1.1 Domains with external cusps

Given $\Omega \subset \mathbb{R}^n$ a bounded domain, roughly speaking we say that Ω has an exterior cusp at x_0 if $x_0 \in \partial\Omega$ and Ω narrows as it approaches x_0 in a way that prevents any cone with vertex at x_0 to be contained in Ω. Throughout this thesis we assume $x_0 = \mathbf{0}$. Many different definitions of external cusps are considered in the bibliography. The simplest cusps are *power type cusps*:

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ |x'| < x_n^\gamma\}, \tag{1.1.1}$$

being γ some real number $\gamma > 1$.

This notion is naturally generalized to domains with a profile depicted by a function φ with cuspidal behaviour.

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ |x'| < \varphi(x_n)\}, \tag{1.1.2}$$

where $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a derivable function such that $\varphi(0) = 0$ and $\varphi'(0) = 0$, or, more generally, φ is Lipschitz and $\frac{\varphi(t)}{t} \longrightarrow 0$ $(t \longrightarrow 0^+)$.

If we denote $B' = B^{n-1}(\mathbf{0}, 1)$ the $n-1$ dimensional ball with center at the origin an radius 1, and aB' is the dilatation of B' by a, (i.e.: $aB' = B^{n-1}(\mathbf{0}, a)$) it is clear that (1.1.2) can be written:

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ x' \in \varphi(x_n)B'\}.$$

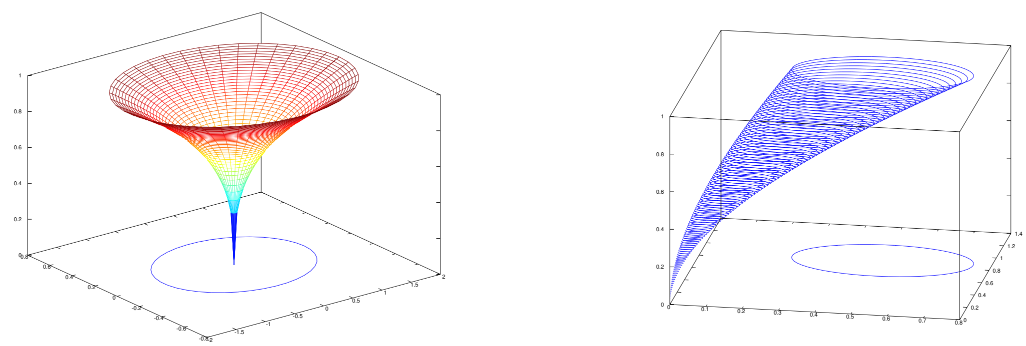
Maz'ya and Poborchiĭ, in [Maz'ya and Poborchiĭ, 1997], generalize this idea, and introduce the following definition of external cusp:

Definition A. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain with compact boundary $\partial\Omega$. Assume that $\mathbf{0} \in \partial\Omega$ and that $\partial\Omega \setminus \{\mathbf{0}\}$ is locally the graph of a Lipschitz function. We say that Ω has an exterior cusp at the origin if there is a neighbourhood of the origin $U \subset \mathbb{R}^n$, such that*

$$U \cap \Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x' \in \varphi(x_n)\varpi\},$$

where $\varpi \subset \mathbb{R}^{n-1}$ is a bounded Lipschitz domain $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a Lipschitz nondecreasing function such that $\frac{\varphi(t)}{t} \to 0$ $(t \to 0^+)$ and $\varphi(0) = 0$.

Definition A implies an important generalization, as long as it admits domains that do not include the vertical axis, but are tangential to it. However, it imposes an important restriction, yet: every horizontal slice of Ω presents the same shape (ϖ), scaled according to the height.



(a) Cusp containing the verical axis

(b) Cusp tangential to the vertical axis

Figure 1.1: Maz'ya's cusps.

In this thesis, we introduce a notion of cusp that generalizes Definition A (see Definitions 3.2.1 and 3.3.1). Our definition is based on the existence of a chain of cubes in the Whitney decomposition of the domain that forms what we call its *spine*. This chain of cubes is placed approximately at the center of the domain. For example, in the case of power type cusp, it would be formed by cubes touching the vertical axis. Furthermore, it narrows towards the origin, and the speed of this narrowing gives the cuspidal behaviour of the domain. On the boundary, instead of the local Lipschitzianity asked in Definition A, we impose a condition of uniformity by stripes. Uniform domains [Martio and Sarvas, 1979, Martio, 1980, Jones, 1981, Smith et al., 1994, Väisälä, 1988] include the Lipschitz ones, and admit the construction of extension operators [Jones, 1981]. In this sense, our sectional uniformity condition constitutes a rather weak regularity hypothesis, that allows us to define extension operators locally.

In order to simplify calculations, we present two slightly different definitions. In 3.2.1 we introduce *normal* cusps, which spine contains the vertical axis. These cusps are somehow symmetric with respect to the axis, as (1.1.2). *Curved* cusps (Definition 3.3.1), on the other hand, satisfy the same properties than normal cusps, but are allowed to be tangential to the axis, like those satisfying Definition A.

The main virtue of normal cusps is that, since their definition does not involve any description of the domain's profile, they allow us to prove that the weights necessary to compensate the singularity do not depend on the detail of the boundary, but only on the speed of the narrowing toward the origin. Our definition of normal cusp can be interpreted as an analogous of Definition A, where the function φ does not represent the precise profile of the cusp, but simply interpolates it in some points, giving a description of the speed of the narrowing.

Furthermore, according to this definition, φ does not need to be monotonous. Finally, the requierement of sectional uniformity constitutes a much more general condition that the local Lipschitzianity of Definition A. As an example, we prove that a cusp satisfying Definition A, but taking ϖ a uniform domain, not necessarily Lipschitz, is a normal or curved cusp.

1.2 Extension Domains

Let Ω be a domain in $\mathbb{R}^n$. $W^{k,p}(\Omega)$ is the Sobolev space of functions having weak derivatives of order α for every α such that $|\alpha| \leq k$, with the norm:

$$\|f\|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)}.$$

We say that Ω is an extension domain of Sobolev (E.D.S.) if there exists a linear bounded operator:

$$E : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n),$$

such that $Ef|_\Omega = f$ for every $f \in W^{k,p}(\Omega)$.

The existence of an extension operator is very useful, since it implies that many results valid for $W^{k,p}(\mathbb{R}^n)$ are inherited by $W^{k,p}(\Omega)$. A classical example of this situation is given by embedding theorems, that can be proved in $\mathbb{R}^n$ in the first place, and then, through an extension argument, for certain domains. This and other applications of extension theorems can be found in the classic literature regarding Sobolev spaces. For example: [Adams and Fournier, 2003, Burenkov, 1998, Evans, 1998, Maz'ya, 2011, Maz'ya and Poborchiĭ, 1997].

It is well known that smooth domains are E.D.S. In fact, since the boundary of a smooth domain can be locally flattened by means of a regular transformation, the extension operator can be constructed applying a simple reflection method (see [Adams and Fournier, 2003, Maz'ya, 2011]). On the other hand, by using the so called Sobolev representation formula in a cone and singular integrals Calderón [Calderón, 1968] showed that Lipschitz domains are also E.D.S. for $1 < p < \infty$. This result was extended to the range $1 \leq p \leq \infty$ by Stein [Adams and Fournier, 2003, Stein, 1970] by using an appropriate averaged reflection procedure.

Reflection type techniques are a natural approach for dealing with extension of functions. More complex ways of reflection are needed in order to handle more general domains. In this context Jones, in [Jones, 1981], studied (ε, δ) domains, also called *locally uniform domains*, that had been introduced in [Martio and Sarvas, 1979], and form a broader class than Lipschitz domains. Jones proved that every (ε, δ) domain is an E.D.S. In terms of the Whitney decompositions $\mathcal{W}$ and $\mathcal{W}^c$, of Ω and $(\Omega^c)^o$ respectively, Jones's idea hinges on the fact that (ε, δ) domains enjoy the following properties:

(a) Whitney cubes $Q \in \mathcal{W}^c$ near Ω have a "reflected" cube $Q^* \in \mathcal{W}$, of similar size and near Q.

(b) reflected cubes $Q_1^*, Q_2^* \in \mathcal{W}$ of neighboring cubes $Q_1, Q_2 \in \mathcal{W}^c$ can be joined by a bounded chain of touching cubes in $\mathcal{W}$.

Thanks to this, an appropriate polynomial approximation of f in Q^* can be used to define the extension of f in Q. In Figure 1.2 we show two touching cubes and their reflected cubes, along with a possible chain of cubes joining them.

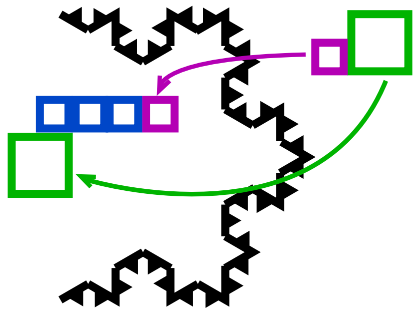


Figure 1.2: Reflected cubes and chain.

In the case of external cusps, on the contrary, there are counterexamples that show that it is not possible to build an extension operator in the classical sense. Hence, it is necessary to perform an extension to a weighted Sobolev space.

In this sense, Maz'ya and Poborchiĭ [Maz'ya and Poborchiĭ, 1997] proved the following extension theorem for external cusps satisfying Definition A.

Theorem A. *Let $\Omega \subset \mathbb{R}^n$ a domain with an external cusp at the origin, according to Definition A. Then, there exists an extension operator:*

$$\Lambda : W^{kp}(\Omega) \to W^{kp}_{\sigma}(\mathbb{R}),$$

where σ can be taken according to the following conditions:

(a) If $kp < n-1$, or $k = n-1$ and $p = 1$, and φ satisfies

$$\frac{\varphi(t)}{t} \text{ is nondecreasing.} \tag{1.2.1}$$

then,

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^{kp} & x \in \Omega^c \end{cases}$$

(b) If $kp > n-1$, and φ is such that:

$$\exists C_{\varphi} \text{ constant} : \qquad \varphi(2t) \leq C_{\varphi}\varphi(t) \tag{1.2.2}$$

then,

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^{n-1} & x \in \Omega^c \end{cases}$$

(c) *If $kp = n-1$, $1 < p < \infty$, φ is such that* (1.2.1) *and:*

$$\exists \delta > 0 : \qquad \varphi(t+\varphi(t)) = \varphi(t)[1 + O(\varphi(t)/t)^{\delta}] \qquad \text{as } t \to 0 \tag{1.2.3}$$

then,

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^{kp} \log\left(\frac{\varphi(|x|)}{|x|}\right)^{\frac{1}{p'}} & x \in \Omega^c \end{cases}$$

(d) *Assuming* (1.2.2)*, if $\tilde{\sigma}$ is a radial nondecreasing weight such that there is an extension operator: $\tilde{\Lambda} : W^{kp}(\Omega) \to W^{kp}_{\tilde{\sigma}}(\mathbb{R}^n)$, then:*

$$\tilde{\sigma}(x) \leq C\sigma(x) \quad \forall x \in U \setminus \Omega,$$

where U is a neighbourhood of the origin and σ is taken according to the case. For the weight in item (b) *we assume $\mathbf{0} \in \varpi$.*

Among several applications, such a result is crucial, for example in the construction of triangular (or tetraedral) meshes for the application of the finite element method for the resolution of elliptic partial differential equations. Any triangulation of an external cusp Ω produces a polygonal domain that exceeds the boundary of Ω. But since the classical extension results do not hold for cuspidal domains, the solution on the polygonal domain is *less* regular than the exact solution of the original problem. A consequence of this fact is that quasi-uniform meshes do not lead to optimal orders of convergence, as it happens in the case of smooth domains. In [Acosta et al., 2007] and [Acosta and Armentano, 2011], it is shown that the optimal order of convergence can be recovered using graded meshes, where the graduation is performed according to the weight σ of the extension operator.

In Appendix A we present another possible application of extension theorems, obtaining a weighted Korn inequality for normal cusps.

In this thesis we present a series of extension theorems that generalizes Theorem A. Our results are valid for normal and curved cusps, that are more general than the ones satisfying Definition A. On the other hand, we prove that the conditions on the parameters k, p and n on items (a) and (b) of Theorem A are not necessary. Finally, we treat the case of weighted Sobolev spaces, obtaining extension operators of the form: $E : W^{k,p}_{\omega}(\Omega) \longrightarrow W^{k,p}_{\omega\sigma}(\mathbb{R}^n)$.

Following [Maz'ya and Poborchiĭ, 1997], our extension operator is built in three stages. The first one extends to a *double cusp*. This local extension is meant to make the rest of the process independent of the detail of the boundary. Here, we use the sectional uniformity property of normal and curved cusps in order to apply an adaptation of the ideas of [Jones, 1981]. Since the singularity of the domain is not solved, this stage does not requiere a weight. Second stage extends to a cone, using only the information of the chain of central cubes of the domain. The weight that appears is the strictly necessary to compensate the speed of the narrowing of the chain of cubes. Finally, the third stage extends to a neighborhood of the origin, radially.

In order to guarraty that the extension of the first stage meets the original function on $\partial\Omega$ in a way such that the weak derivatives remain in L^p, the proof is performed first for smooth functions, in $C^\infty(\bar{\Omega} \setminus \{\mathbf{0}\})$ and is generalized later to $W^{k,p}(\Omega)$ through a density argument. For doing this, it is necessary to prove that functions in $W^{k,p}(\Omega)$ can be approximated by functions in C^∞. Since the problem of the approximation by smooth functions is interesting in itself, we prove a density theorem separately, in Chapter 4.

1.3 Korn's inequality

Given a vector field $u \in W^{k,p}(\Omega)^n$, Korn's inequality establishes that

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p(\Omega)^{n\times n}}, \tag{1.3.1}$$

where $\varepsilon(u)$ is the symmetric part of the differential matrix of u, Du. In other words:

$$\varepsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

In the context of linear elasticity equations, $u(x)$ represents the displacement of the point $x \in \Omega$, when the body Ω is under deformation, whereas $\varepsilon(u)$ is the strain tensor. Korn's inequality is a funtamental tool for proving the coercivity of the bilinear form associated with elasticity equations, which allows to prove the existence of solutions, but also the convergence of the numerical methods applied to approximate them.

It is clear that inequality (1.3.1) could be false. It is enough to take, for example, a vector field u such that Du is skew-symmetric. Hence, it is necessary to impose additional conditions on u. Korn, [Korn, 1906, 1909] proved, in the case $p = 2$, the validity of (1.3.1) for functions with null trace, being Ω any open set. This result is known as the first case of the inequality. The so called second case consider fields u that satisfy:

$$\int_\Omega \frac{Du - Du^t}{2} = 0, \tag{1.3.2}$$

and for them, the validity of (1.3.1) depends on the nature of the domain Ω.

The second case of Korn's inequality is closely related with the *general case*, that establishes:

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)^{n\times n}}\Big\} \qquad \forall u \in W^{1,p}(\Omega). \tag{1.3.3}$$

For functions that satisfy (1.3.2), (1.3.1) implies (1.3.3) trivially. For any $u \in W^{1,p}(\Omega)$, the same implication holds, regardless the shape of the domain Ω. The proof requieres a simple argument that can be seen in [Brenner and Scott, 2008], or in Chapter 6. On the other hand, (1.3.1) can be deduced from (1.3.3) using a more complex compactness argument that depends on the domain (see, for example [Kikuchi and Oden, 1988]).

Many proofs of Korn's inequality are known for non singular domains. Friederichs proves it in [Friederichs, 1937],[Friederichs, 1947] for some particular cases in spaces of 2 and 3 dimensions. Nitsche, in [Nitsche, 1981] proves it for Lipschitz domains, using extension arguments. In [Kondratiev and Oleinik, 1989] the authors treat domains starshaped with respect to a ball and prove that the constant in the inequality is bounded in terms of the quotient between the diameter of the domain and the diameter of the ball. In a recent paper, [Durán, 2012], the second case is proved for domains star shaped with respect to a ball, for $p = 2$, using the continuity of the right inverse of the divergence operator, and an explicit expression for the constant is found. For $n = 2$, the constant behaves as $\frac{R}{\rho}$ times a logarithmic term, where R and ρ are the radii of Ω and the ball respectively. In [Costabel and Dauge, 2013] it is shown that the logarithmic term can be removed. On the other hand, in [Durán, 2012] the author proves that the constant for convex domains in $\mathbb{R}^n$ is $\frac{R}{\rho}$, a fact that will be useful later. We should also mention [Durán and Muschietti, 2004], where the authors prove that (1.3.3) holds on uniform domains, using the extension operator built in [Jones, 1981] for such domains. In the Appendix A we show how the extension procedure developed in Chapter 5 can be applied to prove weighted Korn inequalities for normal cusps. Finally, in [Acosta et al., 2006b], Korn's inequality is proved for John domains, as a corollary of the existence of a right inverse for the divergence operator. The inequality has also been proved using Calderón-Zygmund inequalities for singular integrals, by Govert, Fichera and Ting. Their arguments are followed in [Kikuchi and Oden, 1988]. Other classical references are [Fichera, 1974], [Horgan, 1995].

Despite these results, it is well known that Korn's inequality does not hold on external cusps [Acosta et al., 2012]. This can be solved, as in the extension problem, using appropriate weights in order to obtain an inequality of the form:

$$\|u\|_{L^p(\Omega)^n} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p_\sigma(\Omega)^{n\times n}}\Big\}. \tag{1.3.4}$$

The main precedents that we follow regarding Korn's inequality on external cusps are: [Durán and López García, 2010b], [Acosta et al., 2012] and [Acosta et al., 2006a].

In [Durán and López García, 2010b] power type cusps are treated, and the existence of a right inverse for the divergence operator is proved. As a corollary, the following weighted Korn inequality is obtained (see [Durán and López García, 2010b, Theorem 6.2]):

Theorem B. *Given Ω a domain of the form (1.1.1), $1 < p < \infty$, $B \subset \Omega$ an open ball and $\beta \geq 0$; there exists a constant C, depending only on Ω, B, p and β, such that for every $u \in W^{1,p}_{d^{p\beta}}(\Omega)^n$:*

$$\|Du\|_{L^p_{d^{p\beta}}(\Omega)^{n\times n}} \leq C\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{d^{p(\beta+1-\gamma)}}(\Omega)^{n\times n}}\},$$

where $d = d(x)$ is the distance to the origin and γ is the exponent of the cusp.

For the unweighted case, $W^{1,p}(\Omega)^n$, we take $\beta = 0$, and the weight on the right hand side, due to the cuspidal behaviour of Ω is $d^{p(1-\gamma)}$.

The optimality of this result is proved in [Acosta et al., 2012], where the authors work with cusps with profile φ and prove the following theorem:

Theorem C. *Let Ω be a cusp with profile φ, according to (1.1.2), $\beta_1, \beta_2 \in \mathbb{R}$, $1 < p < \infty$ and B a ball compactly contained in Ω. If there is a constant C such that:*

$$\|Dv\|_{L^p_{(\varphi')^{p\beta_1}}(\Omega)^{n\times n}} \leq C\{\|v\|_{L^p(B)^n} + \|\varepsilon(v)\|_{L^p_{(\varphi')^{p\beta_2}}(\Omega)^{n\times n}}\},$$

for every $v \in W^{1,p}_{(\varphi')^{p\beta_1}}(\Omega)^n$, *then* $\beta_1 \geq \beta_2 + 1$

It is important to observe that for a power type cusp $\varphi'(t) = \gamma t^{\gamma-1}$, and therefore, Korn's inequality in Theorem B corresponds with the case $\beta_1 = \beta_2 + 1$ in Theorem C, and the weight on the right hand side is, in this sense, the best possible.

We would also like to mention [Nazarov, 2012], where weighted anisotropic Korn inequalities are proved for exterior peaks in $\mathbb{R}^3$, satisfying Definition A, taking $\varphi(z) = z^\gamma$, although the author mentions that more general cases could be treated in the same way. Finally, in [Acosta et al., 2006a], the authors prove weighted Korn inequalities for Hölder-α domains, that include power type cusps. In this case, since a Hölder-α domain may have many singularities, the weight that naturally arises for them depends on the distance to the boundary.

In spite of the fact that many proofs of Korn's inequality are carried out using extension arguments, ([Nitsche, 1981],[Durán and Muschietti, 2004]), it is known that the inequality holds even for domains that do not admit an extension operator. For example, in [Acosta et al., 2006b] the authors provide a solution for the divergence problem on John domains. The second case of Korn's inequality can be obtained from this result and, as we commented above, the general case can be easily deduced from the second. It is interesting to observe that the extension problem can be senseless in a John domain. This induced us to suspect that there could be external cusps of a more general kind than normal or curved cusps, for which a weighted Korn inequality holds.

Indeed, bearing this in mind, in Chapter 6 we present an approach for the problem of Korn's inequality that exceeds the case of normal or curved cusps. There, we prove Poincaré and weighted Korn inequalities for chains of rectangles. A chain of rectangles is a union of rectangles satisfying certain properties. Essentially: each rectangle has two neighbours (a preceding and a subsequent one), and any two neighbouring rectangles are comparables. Our proof is based mainly on a discrete version of a Hardy type inequality.

A chain of rectangles may form an external cusp, if the rectangles narrow properly as they approach a point. However, the notion of chain of rectangles also includes many other domains, not necessarily singular. An important aspect of our technique is that it can be easily applied to chains of subdomains which shape is only *approximately* rectangular. For that, we introduce the notion of chain of quasi-rectangles, and we deduce Poincaré and weighted Korn inequalities for these domains. As an example, we show that with chains of quasi-rectangles it is possible to build external cusps which boundary is, by stripes, the one of a John domain. In this way we define the notion of locally John cusps, that generalizes normal and curved cusps, since every uniform domain is a John domain. The validity of a weighted Korn inequality for locally John cusps is an inmediate consequence of the results obtained for chains of quasi-rectangles.

Our results can be easily generalized to the situation where the field u belongs to the weighted Sobolev space $W^{k,p}_\omega(\Omega)$. For the case of locally John cusps, we analyze specifically weights ω depending on the distance to the cusp and on the distance to the boundary of the domain. It is interesting to notice that for weights of the form $\omega = (\varphi')^{p\beta}$ we generalize

Theorem B, proving Korn's inequality even for some negative values of β.

1.4 Summary

We begin Chapter 2 defining weighted Sobolev spaces. Afterwards we introduce the notation that we use for cubes and rectangles, and we provide a classical proof of Whitney's decomposition theorem. Later, we present locally uniform domains and John domains, proving some of their most important properties. Finally, we prove several results regarding polynomial approximations on cubes an rectangles that are extensively used along the thesis.

In Chapter 3 we introduce the notion of chain of rectangles, that allows us to define normal and curved cusps. We finish this chapter with an example that shows that every external cusp satisfying Definition A is necessarily a normal or curved cusp:

In Chapter 4 we prove that $C^\infty(\bar{\Omega} \setminus \{\mathbf{0}\})$ is dense in $W^{k,p}(\Omega)$, being Ω a normal or curved cusp. We also treat the weighted case $W^{k,p}_\omega(\Omega)$, and we prove that the density holds when the weight ω can be approximated by constants over horizontal stripes. This result is generalized in Section 5.3, where we conclude that $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.

Chapter 5 is devoted to the construction of the extension operator. We first develop the case of normal cusps, because it is simpler and because in this way we avoid some technical details that blur the proof not providing any important idea. For curved cusps we develop a *stage zero* that works like the first stage of the process for normal cusps, and allows us to extend from a curved cusp to a normal one.

In Chapter 6 we prove Poincaré and weighted Korn inequalities for chains of rectangles and quasi-rectangles. We also define locally John cusps, as a particular case of chain of quasi-rectangles, and show that our results are a generalization of Theorem B.

Finally, we show how Korn's inequality for normal cusps can be obtained using extension arguments. Since this result is less general, and its proof more cumbersome, than the ones in Chapter 6, we include it in the Appendix A.

It is important to notice that the content of the present thesis has been included in two articles. The first one contains mainly the definitions of normal and curved cusps, and the extension theorems developed in Chapter 5, and has been published in [Acosta and Ojea, 2012]. The second one, [Acosta and Ojea, 2014], which introduces the notions of chains of rectangles and quasi-rectangles and establishes the Korn's inequalities that are presented here in Chapter 6, has been recently summited for its publication.

2

Preliminaries

This chapter is meant to present the basic definitions and the notation that is used throughout this work. We divide it in four sections. The first three are introductory and deal with the definitions of weighted Sobolev spaces, the notation that we use for cubes and rectangles, which leads to the Whitney decomposition theorem, and the notions of uniform and John domains, respectively. In the last one we define two different polynomial approximations for Sobolev functions and we prove their main properties.

2.1 Weighted Sobolev Spaces

Let Ω be an open connected set in $\mathbb{R}^n$, and $f : \Omega \longrightarrow \mathbb{R}$ a locally integrable function. We say that $h_\alpha \in L^1_{loc}(\Omega)$ is a weak derivative of f of order α, with $\alpha = (\alpha_1, \cdots \alpha_n) \in \mathbb{N}_0^n$ a multi-index if :

$$\int_\Omega f D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega h_\alpha \phi \quad \forall \phi \in C_0^\infty,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We denote $h_\alpha = D^\alpha u$.

Sometimes we write

$$\nabla^m f := \sum_{\alpha:|\alpha|=m} D^\alpha f.$$

Let $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a locally integrable nonnegative function. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the weighted Sobolev space $W^{k,p}_\omega(\Omega)$ is the space of functions f defined in Ω having weak derivatives of order α, for $|\alpha| \leq k$, $D^\alpha f \in L^p_{loc}(\Omega)$, and satisfying (for $p < \infty$):

$$\|f\|^p_{W^{k,p}_\omega(\Omega)} := \sum_{|\alpha|\leq k} \|\omega^{\frac{1}{p}} D^\alpha f\|^p_{L^p(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega \omega(x) |D^\alpha f(x)|^p dx < \infty. \tag{2.1.1}$$

The natural extension is taken for $p = \infty$

In the first chapters we deal with scalar fields, proving the density of smooth functions in Sobolev spaces, and the extendability of Sobolev functions, both on domains with external cusps. These results can be easily extended to vector fields, handling each coordinate separately. However, in the last chapter we work specifically with vector fields, in the context of the equations of linear elasticity. Hence let us denote, for a constant matrix $A \in \mathbb{R}^{n\times n}$:

$$|A|^p = \sum_{i=1}^{n}\sum_{j=1}^{n} |A_{i,j}|^p.$$

Furthermore, let us define $W^{k,p}_{\omega}(\Omega)^n$, the space of vector fields $u : \Omega \to \mathbb{R}^n$, with weak derivatives of order α for all $0 \le |\alpha| \le k$, equipped with the norm:

$$\|u\|^p_{W^{k,p}_{\omega}(\Omega)^n} = \sum_{i=1}^{n}\sum_{|\alpha|\le k} \|\omega^{\frac{1}{p}} D^{\alpha} u_i\|^p_{L^p(\Omega)}.$$

We also take the L^p norm of a matrix field $A : \Omega \longrightarrow \mathbb{R}^{n\times n}$ as:

$$\|A\|^p_{L^p_{\omega}(\Omega)^{n\times n}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \|\omega^{\frac{1}{p}} A_{i,j}\|^p_{L^p(\Omega)}.$$

In the unweighted case ($\omega \equiv 1$) we write $\|u\|_{W^{1,p}(\Omega)}$, $\|u\|_{W^{1,p}(\Omega)^n}$ and $\|A\|_{L^p(\Omega)^{n\times n}}$, respectively. p' stands for the conjugate exponent of p: $\frac{1}{p} + \frac{1}{p'} = 1$.
Sometimes we write $⨍_S f$ to denote the mean value of f on S:

$$⨍_S f = \frac{1}{|S|}\int_S f.$$

Troughout this thesis, we denote with C a generic constant that may change from line to line. Most of the times, this constant depends on the general parameters (n and p), and on the measure or the parameters of the domain being considered. Sometimes we specify this dependance for the sake of clarity.

2.2 Cubes and rectangles

We say that two positive numbers are C comparable, and we denote it $a \underset{C}{\sim} b$, if

$$\frac{1}{C} a \le b \le Ca.$$

Sometimes we write simply $a \sim b$, omitting the constant C.
For a collection of sets $\mathcal{C}$, we denote $\cup\mathcal{C}$, the union of all the sets in $\mathcal{C}$, i.e.:

$$\cup\mathcal{C} = \bigcup_{S\in\mathcal{C}} S.$$

Given two sets A and B, we denote $A \equiv B$ if they differ in measure zero.

Given a rectangle $R \subset \mathbb{R}^n$ with edges parallels to the coordinate axis, the size vector of R is denoted with $\vec{\ell}(R) = (\ell_1(R), \ell_2(R), \ldots, \ell_n(R))$, where $\ell_i(R)$ is the length of R's i-th edge. For a cube Q we use $\ell(Q)$ to denote the length of any of its edges, and for a rectangle R, we define $\ell_M(R) := \max_{1\le i\le n}\{\ell_i(R)\}$ and $\ell_m(R) := \min_{1\le i\le n} \ell_i(R)$. Sometimes we deal with rectangles with $n-1$ short edges, that we denote $\ell(R)$, and a long one, that we denote $L(R)$. Our rectangles are sometimes closed, sometimes open, and sometimes semi-open sets. We hope this will be clear from the context, in each case.

A pair of rectangles R_1, R_2 are called *C-comparable*, and we write $R_1 \underset{C}{\sim} R_2$, if there is a constant C, such that $\ell_i(R_1) \underset{C}{\sim} \ell_i(R_2)$ for $1 \le i \le n$.

We say that R_1 and R_2 are *touching rectangles* if $R_1^o \cap R_2^o = \emptyset$ and $\bar{R}_1 \cap \bar{R}_2 = F$ with F a face of R_1 or R_2.

For a rectangle R, we denote its center with c_R. Our external cusps are defined in terms of rectangles that are placed vertically, one below the other, so we introduce a special notation for the upper and lower faces of a cube or rectangle, and for the respective x_n coordinate: If $c_R = (c_1, \cdots, c_n)$ the upper face F_R^u of R is given by

$$F_R^u = \Big\{(x_1, \cdots, x_n) \in \bar{R} : x_n = c_n + \frac{1}{2}\ell_n(R)\Big\},$$

and analogously is defined the lower face F_R^l.

For a rectangle R, centered in $c_R = (c_1, \cdots, c_n)$ we denote $z_R = c_n - \frac{1}{2}\ell_n(R)$ (the last coordinate of points belonging to F_R^l). Anagolously, we denote $\bar{z}_R$ the last coordinate of the points belonging to F_R^u.

We denote by aR ($a > 1$), the expanded rectangle centered at c_R with edges $\ell_i(aR) = a\ell_i(R)$. Sometimes we consider the special case of *horizontal* expansions, so we denote:

$$a \star R = \Big\{(x', x_n) \in \mathbb{R}^n : \ x' \in aR', z_R \le x_n < \bar{z}_R\Big\},$$

being R' the projection of R into the space $\mathbb{R}^{n-1}$ corresponding to the first $n-1$ coordinates.

Finally, throughout this work $\hat{x}_n$ stands for the x_n axis.

Now, we can establish the following well known decomposition Theorem, due to Whitney:

Theorem 2.2.1 (Whitney)**.** *Let $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$, be an open set. Then, there is a collection $\mathcal{W} = \mathcal{W}(\Omega) := \{Q_j\}$ of (countably) infinite dyadic closed cubes such that $\Omega = \cup\mathcal{W}$, and:*

(a) $Q_j^o \cap Q_k^o = \emptyset \quad \forall Q_j, Q_k \in \mathcal{W}(\Omega)$,

(b) $\sqrt{n}\ell(Q_j) \le d(Q_j, \partial\Omega) \le 4\sqrt{n}\ell(Q_j) \quad \forall Q_j$,

(c) if $Q_j \cap Q_k \ne \emptyset$ *then:* $\ell(Q_j) \le 4\ell(Q_k)$.

Proof. Let $\mathcal{W}_0$ be the set of all the cubes in $\mathbb{R}^n$ of unit length whose vertices have integer coordinates. From $\mathcal{W}_0$ we can obtain a two tails chain of sets, given by: $\mathcal{W}_k = 2^{-k}\mathcal{W}_0$,

$k \in \mathbb{Z}$. $\mathcal{W}_{k+1}$ can be constructed dividing every cube $Q \in \mathcal{W}_k$ into 2^n equal cubes. Naturally, the edges of the cubes in $\mathcal{W}_k$ have length 2^{-k}.

We also consider

$$\Omega_k = \{x \in \Omega : \ c2^{-k} < d(x, \partial\Omega) \leq c2^{-k+1}\},$$

where c is a constant that we specify later. Observe that the size of the cubes in $\mathcal{W}_k$ is 2^{-k} and the distance of every point in Ω_k to $\partial\Omega$ is proportional to 2^{-k}. Consequently, in order to obtain (b), it is natural to consider the cubes in $\mathcal{W}_k$ that intersects Ω_k. Hence, we define:

$$\widetilde{\mathcal{W}} = \bigcup_k \Big\{Q \in \mathcal{W}_k : \ Q \cap \Omega_k \neq \emptyset\Big\}.$$

Now we can select the value of c: given $Q \in \widetilde{\mathcal{W}}$, we have that $Q \in \mathcal{W}_k$, for some k and there is a point $x \in Q \cap \Omega_k$. Then:

$$d(Q, \partial\Omega) \leq d(x, \partial\Omega) \leq c2^{-k+1} = 2c\ell(Q),$$

so, we can take $c = 2\sqrt{n}$, and we have: $d(Q, \partial\Omega) \leq 4\sqrt{n}\ell(Q)$. On the other hand:

$$d(Q, \partial\Omega) \geq d(x, \partial\Omega) - \sqrt{n}\ell(Q) \geq c2^{-k} - \sqrt{n}\ell(Q) = \sqrt{n}\ell(Q).$$

In order to complete the proof we need to exclude from $\widetilde{\mathcal{W}}$ the redundant cubes. First, suppose $Q_1 \in \mathcal{W}_{k_1}$, $Q_2 \in \mathcal{W}_{k_2}$, $Q_1 \cap Q_2 \neq \emptyset$ and $Q_1, Q_2 \in \widetilde{\mathcal{W}}$. Then, one of them is contained in the other. In particular, if $k_1 \geq k_2$, $Q_1 \subset Q_2$.

Now, for every $Q \in \widetilde{\mathcal{W}}$, we take the maximal cube in $\widetilde{\mathcal{W}}$ that contains Q. Given Q_1, Q_2 such that $Q \subset Q_1, Q_2$, we have that $Q_1 \cap Q_2 \neq \emptyset$, and then, thanks to the previous observation, $Q_1 \subset Q_2$ or vice versa. Since $d(Q, \partial\Omega) < \infty$, the maximal cube referred above is unique. We define $\mathcal{W}$, the set of all these maximal cubes in $\widetilde{\mathcal{W}}$.

Every cube $Q \in \mathcal{W}$ satisfies (a), since every cube in $\widetilde{\mathcal{W}}$ does. Morever, it is clear that $\mathcal{W}$ also satisfies (b) and (c). □

Remark 2.2.2. *One can easily observe, from the proof, that for any pair of open sets A and B, with $A \subset B$, every cube $Q \in \mathcal{W}(A)$ is contained in some cube $\widetilde{Q} \in \mathcal{W}(B)$.*

2.3 Domains

2.3.1 Smooth domains and the cone condition

It is well known that some important properties of the Sobolev space $W^{k,p}(\Omega)$ depend strongly on the nature of the domain Ω[1].The classical theory of Sobolev spaces treats *smooth* domains, providing several notions of *smoothness*. Here we state only three important classical definitions before passing to more general domains that are the basis of our external cusps.

[1]We use the word *domain* to denote "open connected set in $\mathbb{R}^n$"

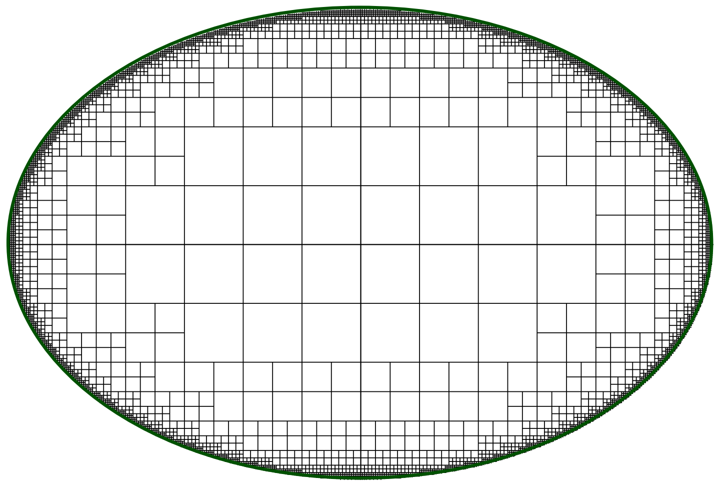


Figure 2.1: Whitney decomposition of an ellipse

Definition 2.3.1. *A domain $D \subset \mathbb{R}^n$ is called a Lipschitz domain if its boundary is locally the graph of a Lipschitz function. More precisely, if there are numbers ε, M and N, a finite collection of open sets $\{U_i\}$ and one of functions $\{f_i\}$, such that:*

(i) If $x \in D$ and $d(x, \partial D) < \varepsilon$ then $x \in U_i$ for some i.

(ii) A point x cannot belong to more than N sets U_i.

(iii) $D \cap U_i$ can be represented by the inequality $x_n < f_i(x_1, \ldots, x_{n-1})$ in some Cartesian coordinate system, for $(x_1, \ldots, x_{n-1})$ in some domain in $\mathbb{R}^{n-1}$.

(iv) All the funcitons f_i satisfy the Lipschitz condition with constant M:

$$|f_i(\xi) - f_i(\eta)| \leq M|\xi - \eta|.$$

Sometimes domains satisfying Definition 2.3.1 are called *strong Lipschitz* (for example, [Adams and Fournier, 2003]) or $C^{0,1}$-domains ([Maz'ya and Poborchiĭ, 1997]). Since we are stating this definition just for the sake of completeness, we prefer to use the simpler name of Lipschitz domains.

Definition 2.3.2. *A domain D belongs to the class C (or has a C boundary) if ∂D is locally the graph of a continuos function. More precisely, if it satisfies Definition 2.3.1, but taking f_i just continuous, and not necessarily Lipschitz.*

A more geometric notion that have proved to be a very useful tool is the so called cone condition:

Definition 2.3.3. *Given a parameter ξ, let us consider the cone*

$$K = \left\{ x \in \mathbb{R}^n : \ (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < \xi x_n \right\}.$$

We say that a domain $D \subset \mathbb{R}^n$ satisfies the cone condition if for every $x \in D$ there is a cone K_x which is the result of a rigid movement of K, such that $K_x \subset D$.

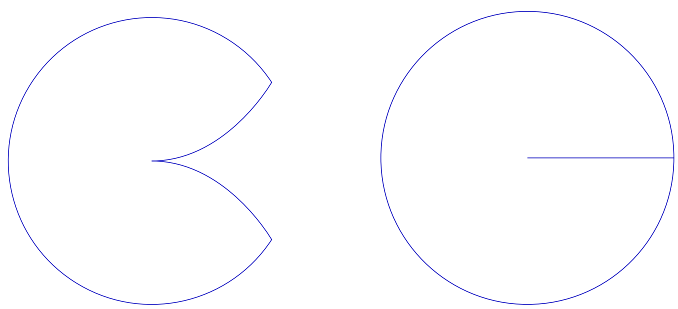
This definition remains the same if we state the existence of the cones K_x for every x in ∂D or in $\bar{D}$, instead of $x \in D$.

It is well known that every Lipschitz domain satisfies the cone condition. Moreover a domain is Lipschitz if and only if it satisfies the *uniform cone condition*[2]. However, there are domains that satisfy the cone condition but are not Lipchitz. Consider, for example, an inner cusp in $\mathbb{R}^2$ (see Figure 2.2):

$$\Omega_\alpha = B(0,1) \setminus \{(x,y) \in \mathbb{R}^2 : \ |y| < x^\alpha\} \qquad \alpha > 1. \tag{2.3.1}$$

And the limit case Ω_∞, which corresponds with a ball without a segment:

$$\Omega_\infty = B(0,1) \setminus \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1, \ y = 0\}. \tag{2.3.2}$$



(a) Ω_α (b) Ω_∞

Figure 2.2: Inner cusp and limit case.

These examples show that the cone property can be satisfied by singular domains. Moreover, it is clear that both Ω_∞ and $(\Omega_\infty^c)^o$ satisfy the cone condition. But it is also clear that $\partial\Omega_\infty$ cannot be represented by a Lipschitz function in any neighborhood of the point $(1,0)$.

On the other hand, external cusps of power type and profile cusps, are in the class C, but do not fulfill the cone condition. Furthermor, take a *bent* cusp, such as:

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \ 0 < y < 1, \ y^2 < x < 2y^2\}.$$

This domain, whose singularity is esentially of the same kind that the one in a power type cusp, is not in the class C, and does not satisfy the cone condition.

[2]We say that D satisfies the *uniform* cone condition if it satisfies the cone condition in the following way: there is a collection of open sets $C = \{U_i\}$ that covers D and a corresponding collection of cones K_i that are rotations of a fixed cone K with vertex at the origin, and such that $x + K_i \subset D$ for every $x \in U_i$. The proof of the equivalence between the uniform cone condition and the Lipschitzianity of the boundary can be seen in [Grisvard, 1985, Theorem 1.2.2.2].

2.3.2 John domains

John domains were first introduced by Fritz John in [John, 1961], and named after him by Martio and Sarvas in [Martio and Sarvas, 1979]. These domains can be understood as a generalization of the notion of domains star shaped with respect to a ball. Essentially, in a John domain there is a distinguished point x_0, and every point x can be joined with x_0 through a *twisted cone*:

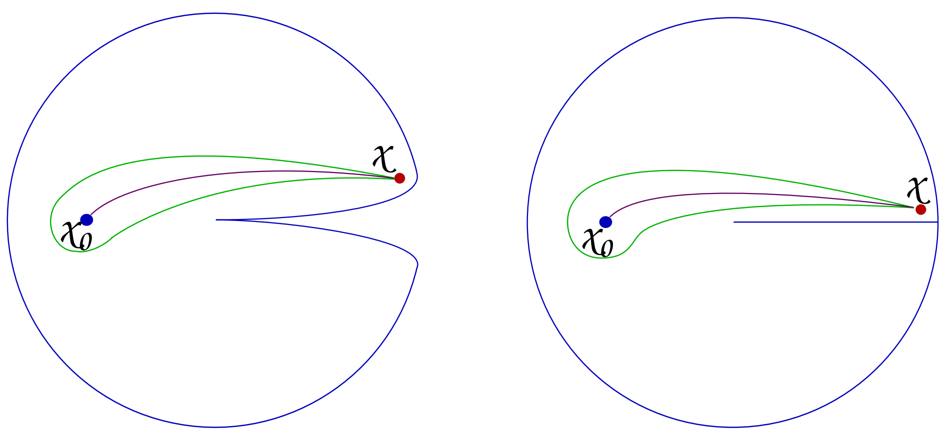
Definition 2.3.4. *Let $0 < \alpha \leq \beta < \infty$. A domain $D \subset \mathbb{R}^n$ is called a John domain with parameters α and β if there is a point $x_0 \in D$ (the John-center of D) such that for every $x \in D$ there is a rectifiable curve with parametrization by arc length $\gamma : [0, \ell] \to D$ such that $\gamma(0) = x$ and $\gamma(\ell) = x_0$, and:*

$$\ell \leq \beta, \tag{2.3.3}$$

$$d(\gamma(t), \partial D) \geq \frac{\alpha}{\ell} t \quad \forall t \in [0, \ell]. \tag{2.3.4}$$

Given $x \in D$, and its correspondant curve γ, the set $\cup_t B(\gamma(t), \frac{\alpha}{\ell} t)$ is a *twisted cone* with its axis depicted by the curve γ.

The inner cusps Ω_α defined in (2.3.1) and the limit case Ω_∞, in (2.3.2), are good examples of John-domains, and are shown along with examples of twisted cones, in Figure 2.3. Another interesting example is the fractal known as the Koch snowflake. In Figure 2.4(a), we show the snowflake and a possible twisted cone. It is important to observe that every domain satisfying the cone condition is a John domain. On the other hand, Koch snowflake does not satisfy the cone condition, which proves that the converse is not true.



(a) The inner cusp Ω_α (b) The limit case Ω_∞

Figure 2.3: Examples of John domains.

Ω_∞ is also a good example for showing that some of the problems that we tackle in this thesis can be unsolvable, if the domain is particularly evil: it is clear that no set of smooth functions up to the boundary of Ω_∞ could be dense in $W^{k,p}(\Omega_\infty)$. Moreover, the extension problem is also meaningless in Ω_∞, and consequently in a general John domain.

2.3.3 Uniform domains

The concept of (ε, δ)-*uniform domain* was introduced in [Martio and Sarvas, 1979]. There are several equivalent definitions for this kind of domains (see, for example, [Martio, 1980], [Smith et al., 1994], [Väisälä, 1988]). Here, we state the definition used in [Jones, 1981] (which is presented in [Martio, 1980]), because it fits better with the extension procedure developed in Chapter 5.

Definition 2.3.5. *(Locally Uniform Domains) D is a (ε, δ) domain if for all $x, y \in D$ with $|x - y| < \delta$ there is a rectifiable curve γ joining x and y such that:*

$$\ell(\gamma) < \frac{|x-y|}{\varepsilon}, \tag{2.3.5}$$

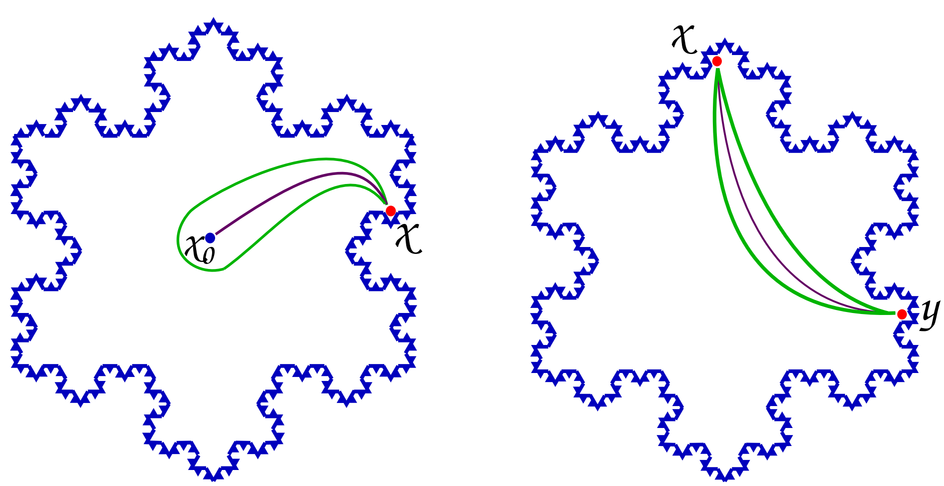
$$d(z, \partial D) > \frac{\varepsilon |x-z||z-y|}{|x-y|} \qquad \forall z \in \gamma. \tag{2.3.6}$$

where $\ell(\gamma)$ denotes the length of γ.

If $\delta > diam(D)$, we say that D is a uniform *domain.*

Roughly speaking, a uniform domain D admits, for every pair of points $x, y \in D$, a *cigar* that joins them and remains inside D. This *cigar* is the neighborhood of the curve γ defined by (2.3.6), which is fat in the center of γ and thin at its endpoints. A unique parameter ε controls the fatness of every cigar. In Figure 2.4(b), we show an example of cigar.

Uniform domains include Lipschitz domains, but they form a much larger class. In fact, if D is uniform, ∂D could be very rough. The Koch fractal snowflake mentioned earlier is uniform (see Figure 2.4(b)).



(a) Is John domain - Twisted cone (b) Is uniform domain - Cigar

Figure 2.4: Koch snowflake.

On the other hand some simpler domains with an isolated singular point are not uniform. In inner cusps like Ω_α, in (2.3.1), property (2.3.5) fails: for every election of ε we can pick a pair of points x, y as near of each other as needed, in order to force the curve γ to be larger than requested in (2.3.5). This shows that a John domain need not to be uniform, even when every uniform domain is a John domain.

Complementarily, the kind of domain that we are interested in, external cusps, are not uniform either: for every fixed value of ε we can take a point x and another point y as near of the tip of the cusp as needed in order to force the cigar to touch the boundary of the domain, so property (2.3.6) fails. In Figure 2.3.3 we show an external cusp of power type, with a cigar that exceeds the boundary of the domain.

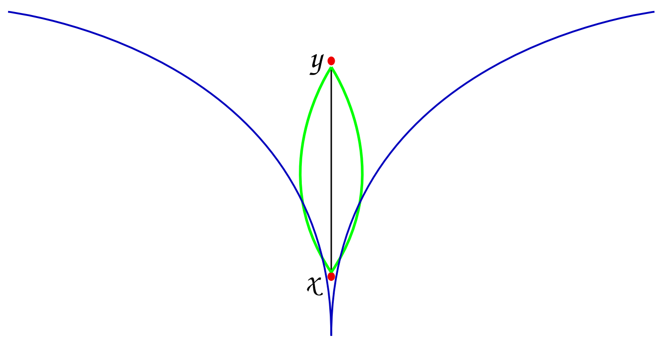


Figure 2.5: External cusps are not uniform.

The Koch snowflake example shows that the boundary of a uniform domain can be very intricated. However, the *measure* of this boundary have to be null. In order to prove this, let us recall the following well known result:

Lemma 2.3.6. *Let $E \subset \mathbb{R}^n$ be a measurable set, then almost every point of E is a* density *point of E. In other words:*

$$\lim_{Q \searrow x} \frac{|Q \cap E|}{|Q|} = 1 \quad \textit{for almost every } x \in E.$$

Proof. It is a simple corollary of Lebesgue's differentiation theorem for the indefinite integral (see, for example, [Wheeden and Zygmund, 1977]). This theorem states that if f is locally integrable in $\mathbb{R}^n$, then, for almost every x,

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f(y)dy = f(x).$$

The results follows by taking $f = \chi_E$, and $x \in E$. □

Lemma 2.3.7. *If D is a (ε, δ) domain, then $|\partial D| = 0$.*

Proof. Let $x_0 \in \partial D$, $y \in D$, and consider a cube Q centered at x_0 and such that $\ell(Q) \leq \frac{1}{2}|y - x_0|$. Take $x \in D$ such that $|x - x_0| \leq \frac{1}{8}\ell(Q)$, and γ the curve joining x and y. There is a $z \in \gamma$ such that $|z - x| = \frac{1}{8}\ell(Q)$. Then:

$$d(z, \partial D) \geq \varepsilon \frac{|x - z||z - y|}{|x - y|} \geq \varepsilon \frac{1}{100}\ell(Q).$$

Then $|D \cap Q| \geq C\varepsilon^n |Q|$. Consequently, Properties (2.3.5) and (2.3.6) allow us to estimate the measure of $D \cap Q$. Now, let us observe that:

$$1 = \frac{|Q|}{|Q|} = \frac{|Q \cap D| + |Q \cap D^c|}{|Q|} \geq C\varepsilon^n + \frac{|Q \cap D^c|}{|Q|}.$$

And then:

$$\frac{|Q \cap D^c|}{|Q|} \leq 1 - C\varepsilon^n < 1.$$

And this happens for every $Q \ni x_0$, and for every $x_0 \in \partial D$. Hence, ∂D is a subset of points of D^c that are not density points of D^c. Consequently, $|\partial D| = 0$. $\square$

2.4 Polynomial approximations

Throughout this work we deal extensively with polynomial approximations of Sobolev functions on cubes and on rectangles. We begin this section stating some general results on polynomials, and afterwards we present two different kinds of polynomial approximations, and prove some of their most important properties.

For a polynomial P, deg$P$ stands for the degree of P.

Lemma 2.4.1. *Let R be a rectangle, P a polynomial with deg$(P) \leq k$, then:*

$$\|P\|_{L^\infty(R)} \leq \frac{C}{|R|^{\frac{1}{p}}}\|P\|_{L^p(R)}, \qquad 1 \leq p \leq \infty,$$

with C depending only on k.

Proof. Let $\hat{Q} = [-\frac{1}{2}, \frac{1}{2}]^n$. Let $F : \hat{Q} \to R$ be the linear application: $F : \hat{x} \to x$, that maps $\hat{Q}$ onto R: $F(\hat{x}) = \vec{\ell}(R) \cdot \hat{x}^t + c_R$. Observe that $|DF| = |R|$. We consider the polynomial $\hat{P}$ defined on $\hat{Q}$ as $\hat{P}(\hat{x}) = P(F(\hat{x}))$. Notice that $deg(\hat{P}) = deg(P)$. Changing variables, we obtain:

$$\begin{aligned}\|P\|_{L^\infty(R)} &= \|\hat{P}\|_{L^\infty(\hat{Q})} \leq \hat{C}\|\hat{P}\|_{L^p(\hat{Q})} \\ &= \hat{C}\Big(\int_{\hat{Q}} |\hat{P}(\hat{x})|^p d\hat{x}\Big)^{\frac{1}{p}} \leq \hat{C}\Big(\int_R |P(x)|^p \frac{1}{|R|} dx\Big)^{\frac{1}{p}},\end{aligned}$$

where the first inequality follows from the equivalence of norms in the finite dimensional space of polynomials of degree $\leq k$ defined on $\hat{Q}$. $\square$

Lemma 2.4.2. *Let R and Q be rectangles such that $R \subset Q$, and P a polynomial with $degP \le k$. Then, there exists a constant C, depending only on k, such that:*

$$\|P\|_{L^p(Q)} \le C\Big(\frac{|Q|}{|R|}\Big)^{\frac{1}{p}} \sum_{|\alpha|\le k} \|D^\alpha P\|_{L^p(R)} \vec{\ell}(Q)^\alpha.$$

Proof. We may assume $0 \in R$. Let $q \in Q$ such that $\|P\|_{L^\infty(Q)} = |P(q)|$, then:

$$\begin{aligned}\|P\|_{L^p(Q)} &\le \|P\|_{L^\infty(Q)}|Q|^{\frac{1}{p}} = |P(q)||Q|^{\frac{1}{p}} \le |Q|^{\frac{1}{p}} \sum_{|\alpha|\le k} |D^\alpha P(0)| \frac{|q^\alpha|}{\alpha!} \\ &\le C|Q|^{\frac{1}{p}} \sum_{|\alpha|\le k} \|D^\alpha P\|_{L^\infty(R)} \vec{\ell}(Q)^\alpha \le C\Big(\frac{|Q|}{|R|}\Big)^{\frac{1}{p}} \sum_{|\alpha|\le k} \|D^\alpha P\|_{L^p(R)} \vec{\ell}(Q)^\alpha.\end{aligned}$$

□

The following corollary is derived from Lemma 2.4.2 using a simple inverse inequality:

Corollary 2.4.3. *Let $R \subset Q$ rectangles such that $Q \sim R$, and P a polynomial with $deg(P) \le k$. Then, there exists a constant C, depending only on k such that:*

$$\|P\|_{L^p(Q)} \le C\|P\|_{L^p(R)}.$$

Remark 2.4.4. *A version of Corollary 2.4.3 is proved in [Jones, 1981, Lemma 2.1]. In our case we need to compare polynomials in rectangles that are not of similar size (a fact that eventually leads to the weights involved in the extension) and we need the less comfortable variant given in Lemma 2.4.2.*

Definition 2.4.5. *Let $f \in W^{k,p}(\Omega)$, and $S \subset \Omega$ a set of positive measure, we denote with $P_{k-1}(S)$ (or just $P(S)$ if the degree is clear from the context) the unique polynomial of degree $k-1$ such that:*

$$\int_S D^\alpha(f - P_{k-1}(S)) = 0 \quad \textit{for all } \alpha, \textit{ with } |\alpha| \le k-1.$$

Naturally, $P(S)$ depends on the function f, so we should write $P(S)(f)$, but we prefer the simpler notation $P(S)$, since the function f will be clear from the context.

Let us recall the well known Poincaré inequality:

Theorem 2.4.6 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $f \in W^{1,p}(\Omega)$ such that $\frac{1}{|\Omega|}\int_\Omega f = 0$, then, there is a constant C_P depending on n, p and Ω, such that:*

$$\|f\|_{L^p(\Omega)} \le C_P\|Df\|_{L^p(\Omega)}$$

The validity of Poincaré inequality depends on the domain Ω. Here we are mainly interested on the following version, regarding convex domains, where the constant C_P can be expressed in terms of the diameter of Ω:

Lemma 2.4.7. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and $f \in W^{1,p}(\Omega)$ such that $\frac{1}{|\Omega|}\int_\Omega f = 0$, then there is a constant C depending on n and p such that:*

$$||f||_{L^p(\Omega)} \leq Cdiam(\Omega)||\nabla f||_{L^p(\Omega)}.$$

Proof. We proceed by a density argument, asuming $f \in C^1(\Omega)$. Being Ω convex we have:

$$f(x) - f(y) = \int_0^1 \nabla f(tx + (1-t)y)(x-y)dt.$$

Now integrating with respect to y and multiplying by $\frac{1}{|\Omega|}$:

$$f(x) - \frac{1}{|\Omega|}\int_\Omega f(y)dy = \frac{1}{|\Omega|}\int_\Omega\int_0^1 \nabla f(tx + (1-t)y)(x-y)dtdy.$$

Hence, since $\int u = 0$,

$$\begin{aligned}
|f(x)|^p &\leq \frac{diam(\Omega)^p}{|\Omega|^p}\Big(\int_\Omega\int_0^1 \big|\nabla f(tx+(1-t)y)\big|dtdy\Big)^p \\
&\leq \frac{diam(\Omega)^p}{|\Omega|^p}\int_\Omega\int_0^1 \big|\nabla f(tx+(1-t)y)\big|^p dtdy|\Omega|^{\frac{p}{p'}} \\
&= \frac{diam(\Omega)^p}{|\Omega|}\int_\Omega\int_0^1 \big|\nabla f(tx+(1-t)y)\big|^p dtdy.
\end{aligned}$$

Now let us extend ∇f by 0 outside Ω, and call F such extension. Then:

$$\begin{aligned}
&||f||^p_{L^p} \leq \frac{diam(\Omega)^p}{|\Omega|}\int_\Omega\int_\Omega\int_0^1 \big|F(tx+(1-t)y)\big|^p dtdydx = \\
&= \frac{diam(\Omega)^p}{|\Omega|}\Big\{\underbrace{\int_\Omega\int_\Omega\int_0^{\frac{1}{2}} \big|F(tx+(1-t)y)\big|^p dtdydx}_{I} + \underbrace{\int_\Omega\int_\Omega\int_{\frac{1}{2}}^1 \big|F(tx+(1-t)y)\big|^p dtdydx}_{II}\Big\}.
\end{aligned}$$

Now:

$$\begin{aligned}
I &= \int_\Omega\int_0^{\frac{1}{2}}\int_\Omega \big|F(tx+(1-t)y)\big|^p dydtdx \leq \int_\Omega\int_0^{\frac{1}{2}}\int_{\mathbb{R}^n} \big|F(tx+(1-t)y)\big|^p dydtdx \\
&= \int_\Omega\int_0^{\frac{1}{2}}\int_{\mathbb{R}^n} \big|F((1-t)y)\big|^p dydtdx = \int_\Omega\int_0^{\frac{1}{2}}\int_{\mathbb{R}^n} |F(z)|^p \frac{1}{(1-t)^n}dzdtdx \\
&\leq C|\Omega|||F||^p_{L^p(\mathbb{R}^n)} = C|\Omega|||\nabla f||^p_{L^p(\Omega)}.
\end{aligned}$$

II can be estimated in the same way, and the result follows. $\square$

The constant obtained in Lemma 2.4.7 is not optimal, but it is enough for our needs. Optimal estimates for convex domains can be seen, for example, in [Payne and Weinberger, 1960], for $p = 2$, and in [Acosta and Durán, 2003], for $p = 1$. For other values of p the optimal constant is unknown but some sharp bounds are provided in [Chua and Wheeden, 2006].

In particular, for a rectangle R, we can apply Poincaré inequality to $D^\alpha(P(R) - f)$, obtaining:

$$\|D^\alpha(P(R) - f)\|_{L^p(R)} \leq C\ell_M(R)^{k-|\alpha|}\|\nabla^k f\|_{L^p(R)}. \tag{2.4.1}$$

So, $P(R)$ has good approximation properties if R is a rectangle (regardless the eccentricity of R). In the spirit of [Jones, 1981, Lemma 2.2] we also need such a result for the union of two touching rectangles of similar size.

Lemma 2.4.8. *Let R_1, R_2 rectangles such that $R_1 \sim R_2$. Assume that either R_1 and R_2 are touching or $R_1 \subseteq R_2$ (renumbering if necessary). Then, for any $f \in W^{k,p}(R_1 \cup R_2)$:*

$$\|f - P(R_1 \cup R_2)\|_{L^p(R_1\cup R_2)} \leq C\ell_M(R_1)^k \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(R_1\cup R_2)}.$$

Proof. Clearly it is enough to prove the result in the case $k = 1$. If $R_1 \subseteq R_2$ (or vice versa) then the result follows by Lemma 2.4.7. Let us then treat the case of touching rectangles. Define $f_{R_1\cup R_2} = \frac{1}{|R_1\cup R_2|}\int_{R_1\cup R_2} f$, then $P(R_1 \cup R_2) = f_{R_1\cup R_2}$. Write

$$\|f - P(R_1 \cup R_2)\|^p_{L^p(R_1\cup R_2)} = \|f - P(R_1 \cup R_2)\|^p_{L^p(R_1)} + \|f - P(R_1 \cup R_2)\|^p_{L^p(R_2)}.$$

We now show how to deal with the first term (the other follows analogously). We have

$$\|f - P(R_1 \cup R_2)\|_{L^p(R_1)} \leq \frac{|R_1|}{|R_1| + |R_2|}\|f - P(R_1)\|_{L^p(R_1)} + \frac{|R_2|}{|R_1| + |R_2|}\|f - P(R_2)\|_{L^p(R_1)}.$$

The first term is fine. For the other term we write

$$\|f - P(R_2)\|_{L^p(R_1)} \leq \|f - P(R_1)\|_{L^p(R_1)} + \|P(R_1) - P(R_2)\|_{L^p(R_1)},$$

and again the first term is all right. In order to treat $\|P(R_1) - P(R_2)\|_{L^p(R_1)}$ observe that, since R_1 and R_2 are touching, there exist rectangles R_3 and R_4 such that $R_3 \subset R_1 \cup R_2 \subset R_4$, $R_1 \sim R_1 \cap R_3 \sim R_2 \cap R_3 \sim R_2 \sim R_3 \sim R_4$, then (using, for instance, Corollary 2.4.3)

$$\|P(R_1) - P(R_2)\|_{L^p(R_1)} \leq C\|P(R_1) - P(R_3)\|_{L^p(R_1\cap R_3)} + \|P(R_3) - P(R_2)\|_{L^p(R_4)},$$

and

$$\|P(R_1) - P(R_3)\|_{L^p(R_1\cap R_3)} \leq \|P(R_1) - f\|_{L^p(R_1)} + \|f - P(R_3)\|_{L^p(R_3)},$$

while (using again Corollary 2.4.3)

$$\|P(R_3) - P(R_2)\|_{L^p(R_4)} \leq C\|P(R_3) - P(R_2)\|_{L^p(R_2\cap R_3)} \leq C\|f - P(R_2)\|_{L^p(R_2)} + \|P(R_3) - f\|_{L^p(R_3)}.$$

The lemma follows. □

The polynomial $P(R)$ is a very simple and useful approximation of f over R, and it is the one used for the construction of the extension operators in Chapter 5. However, in order to prove the density of smooth functions, we need to introduce a more sofisticated approximation, that enjoys some specific properties.

Let us state the well known Sobolev Representation Formula for star shaped domains, that can be seen, for example in [Maz'ya and Poborchiĭ, 1997, Section 1.5, Theorem 1], or in [Maz'ya, 2011, Section 1.1, Theorem 1]:

Theorem 2.4.9. *Let Ω be a star shaped domain with respect to the ball $B(z, \delta)$, k a positive integer and take $f \in W^{k,p}(\Omega)$. Then:*

$$f(x) = \delta^{-1} \sum_{|\beta|<k} \Big(\frac{x-z}{\delta}\Big)^{\beta} \int_{B(0,\delta)} \varphi_\beta\Big(\frac{y}{\delta}\Big) f(y+z) + \sum_{|\alpha|=k} \frac{g_\alpha(x; r, \theta)}{r^{n-1}} D^\alpha f(y) dy,$$

where $r = |y-x|$, $\theta = \frac{y-x}{r^n}$, $g_\alpha \in C^\infty$, $|g_\alpha| \le c\Big(\frac{diam(\Omega)}{\delta}\Big)^{n-1}$, and:

$$\varphi_\beta(y) = \sum_{|\gamma| \le k-1-|\beta|} \frac{(n+k-1)!}{(n+|\gamma+\beta|)!(k-1-|\gamma+\beta|)!} \frac{(-1)^{|\beta|}}{\beta!\gamma!} y^\gamma D^{\beta+\gamma} h(y),$$

$h \in C_0^\infty(B(0,1))$ and $\int h = 1$.

This Theorem induce the definition of another projection onto $\mathcal{P}_{k-1}$:

Definition 2.4.10. *Given a cube Q, consider the ball $B = B(c_Q, \ell(Q)) \subset Q$, and a function $f \in W^{k,p}(Q)$, we define:*

$$\pi(Q)(f)(x) = \ell(Q)^{-1} \sum_{|\beta|<k} \Big(\frac{x-c_Q}{\ell(Q)}\Big)^{\beta} \int_B \varphi_\beta\bigg(\frac{y}{\ell(Q)}\bigg) f(y+c_Q) dy,$$

where φ_β is the one in Theorem 2.4.9.

When it is clear from the context, we denote $\pi(Q)$ instead of $\pi(Q)(f)$. $\pi(Q)$ is a polynomial approximation of f over Q, that enjoys a very important propery that $P(Q)$ does not satisfy:

Theorem 2.4.11. *Let Q be a cube contained in a domain Ω, $f \in W^{k,p}(\Omega)$, $1 \le p \le \infty$ and γ: $|\gamma| < k$, then:*

$$\|D^\gamma \pi(Q)\|_{L^p(Q)} \le C \|D^\gamma f\|_{L^p(Q)}.$$

Proof. It is clear that, since

$$\varphi_\beta = \sum_{|\gamma| \le k-1-|\beta|} C_{\beta,\gamma} y^\gamma D^{\beta+\gamma} h,$$

and $h \in C_0^\infty(B(0,1))$, then $\varphi_\beta = D^\beta \psi_\beta$, with $\psi_\beta \in C_0^\infty(B(0,1))$. Consequently:

$$\left| \int_B \varphi_\beta\Big(\frac{y}{\ell(Q)}\Big) f(y+c_Q) \right| = \left| \int_B D^\beta \psi_\beta\Big(\frac{y}{\ell(Q)}\Big) f(y+c_Q) \right|$$
$$\leq \left| \int_B \ell(Q)^{|\gamma|} D^{\beta-\gamma}\psi_\beta\Big(\frac{y}{\ell(Q)}\Big) D^\gamma f(y+c_Q) \right| \leq C\ell(Q)^{|\gamma|} \|D^\gamma f\|_{L^1(B)}.$$

On the other hand:

$$D^\gamma \pi(Q)(x) = \ell(Q)^{-1} \sum_{\gamma\leq\beta, |\beta|<k} C\Big(\frac{x-c_Q}{\ell(Q)}\Big)^{\beta-\gamma} \int_B \varphi_\beta\Big(\frac{y}{\ell(Q)}\Big) f(y),$$

and then, taking into account that $|B| \sim |Q|$:

$$|D^\gamma \pi(Q)(x)| \leq C\ell(Q)^{-n} \|D^\gamma f\|_{L^1(B)} \leq C\ell(Q)^{-n} \|D^\gamma f\|_{L^p(B)} |B|^{\frac{1}{p'}} \leq \frac{C}{|B|^{\frac{1}{p}}} \|D^\gamma f\|_{L^p(B)}.$$

Which lead us to:

$$\|D^\gamma \pi(Q)\|_{L^p(Q)} \leq \frac{C}{|B|^{\frac{1}{p}}} |Q|^{\frac{1}{p}} \|D^\gamma f\|_{L^p(B)} \leq C\|D^\gamma f\|_{L^p(Q)}.$$

□

The projection $\pi(Q)$ also satisfies an approximation property analogous to (2.4.1):

Theorem 2.4.12. *Let Q be a cube in Ω, $1 \leq p \leq \infty$, $\gamma : |\gamma| < k$, then:*

$$\|D^\gamma(\pi(Q) - f)\|_{L^p(Q)} \leq C\ell(Q)^{k-|\gamma|} \|\nabla^k f\|_{L^p(Q)}.$$

Proof. We alternate $P(Q)$, in order to deduce the Theorem from (2.4.1).

$$\|D^\gamma(\pi(Q)(f) - f)\|_{L^p(Q)} \leq \underbrace{\|D^\gamma(\pi(Q)(f) - P(Q))\|_{L^p(Q)}}_{I} + \underbrace{\|D^\gamma(P(Q) - f)\|_{L^p(Q)}}_{II}.$$

II can be bounded immediatly by applying (2.4.1). For I, observe that $\pi(Q)(P(Q)) = P(Q)$, so, applying Theorem 2.4.11 and (2.4.1):

$$I = \|D^\gamma(\pi(Q)(f - P(Q)))\|_{L^p(Q)} \leq C\|D^\gamma(f - P(Q))\|_{L^p(Q)} \leq C\ell(Q)^{k-|\gamma|} \|\nabla^k f\|_{L^p(Q)}.$$

□

Remark 2.4.13. *In [Chua, 1992], the author proves that these Theorems hold in the weighted case $W^{k,p}_\omega$, when ω is in the class of Muckenhoupt (see Section 4.2), and he uses these results for proving the density of smooth functions on $W^{k,p}_\omega(D)$, for D a uniform domain. Our arguments are essentially Chua's, but we state only the unweighted case for the sake of simplicity.*

3

Normal and curved cusps

In this work, we provide several definitions for external cusps. The central ideal of all of them is to describe the cuspidal behaviour of the domain through a chain of rectangles that narrows toward the origin faster than any cone. This chain is somehow the *core* of the cusp, and contains all the essential information about its cuspidal singularity. *Stepped cusps*, that are presented in the Appendix A, are nothing more but such a chain. The other definitions correspond to domains that grow around this chain, satisfying certain properties. In this Chapter we introduce the notions of *normal* and *curved* cusps. Both of them satisfy a sectional uniformity property. Roughly speaking: a horizontal stripe of a normal or curved cusp is a uniform domain. The only difference between a normal and a curved cusp is that the first is straight, and contains an axis, whereas the latter can be tangential to an axis.

In the first section we present the definitions and main properties concerning chains of rectangles. The second and third ones are devoted to the definitions of normal and curved cusps, respectively. Finally, in the forth section of this Chapter we present a few examples. Particularly, we show that every external cusp satisfying Definition A is a normal (or curved) cusp.

3.1 Chains of rectangles

In this section we introduce some basic definitions on chains of rectangles that are extensively used in the sequel.

Definition 3.1.1. *A (finite or countable) collection of rectangles $\mathcal{R} = \{R_i\}$ for which $\sum |R_i| < \infty$, is called a* chain of rectangles *if:*

a) $\bar{R}_i \cap \bar{R}_j = \emptyset$ *for* $|i - j| > 1$.

b) For any i, R_i and R_{i+1} are touching.

c) There is a constant C such that $R_i \underset{C}{\sim} R_{i+1}$ for every i.

Remark 3.1.2. *Given a chain of rectangles $\mathcal{R} = \{R_i\}$, we have the following important facts:*

- *since the rectangles R_i and R_{i+1} are touching and C-comparable, there exists a rectangle $R_{i,i+1} \subset \bar{R}_i \cup \bar{R}_{i+1}$ and a constant $\widetilde{C}$ depending only on C such that:*

$$R_{i,i+1} \underset{\widetilde{C}}{\sim} (R_{i,i+1} \cap R_i) \underset{\widetilde{C}}{\sim} R_i \underset{\widetilde{C}}{\sim} (R_{i,i+1} \cap R_{i+1}) \underset{\widetilde{C}}{\sim} R_{i+1}.$$

- *Naturally, this implies that the same relation stands for the measure of the rectangles:*

$$|R_{i,i+1}| \underset{\widetilde{C}}{\sim} |R_{i,i+1} \cap R_i| \underset{\widetilde{C}}{\sim} |R_i| \underset{\widetilde{C}}{\sim} |R_{i,i+1} \cap R_{i+1}| \underset{\widetilde{C}}{\sim} |R_{i+1}|.$$

Definition 3.1.3. *Any collection of intermediate rectangles $R_{i,i+1}$ enjoying properties like those in Remark 3.1.2 is denoted $\mathcal{R}_I = \{R_{i,i+1}\}$.*

The existence of a chain $\mathcal{R}_I$ is crucial for proving most of the results included in this thesis. There are many properties that are well known to hold for cubes or rectangles, and we are able to prove them for chains passing from the rectangle R_i to R_{i+1} through the intermediate one $R_{i,i+1}$.

However, for describing cusps that are tangencial to an axis it is useful to define:

Definition 3.1.4. *A collection of rectangles $\mathcal{R} = \{R_i\}$ is called a* quasi-chain of rectangles *if it satisfies:*

a) $\bar{R}_i \cap \bar{R}_j = \emptyset$ *for* $|i - j| > 1$.

b) $\bar{R}_i \cap \bar{R}_{i+1} \neq \emptyset$, $R_i^o \cap R_{i+1}^o = \emptyset$.

c) There is a constant C such that $R_i \underset{C}{\sim} R_{i+1}$ for every i.

The only difference between a chain and a quasi-chain is that in a chain two consecutive rectangles touch each other in a face, whereas in a quasi-chain this contact can be performed in an edge or a corner. Naturally, in a quasi-chain cannot be guaranteed the existence of an intermediate chain.

3.2 Normal cusps

Normal cusps are defined in terms of a particular chain of cubes belonging to the Whitney decomposition of the domain Ω. This chain $\mathcal{S} = \{S_i\}_i$ has cubes placed one under the other along the x_n axis, i.e: $S_i \cap S_{i+1} = F^u_{i+1}$. In this context, we write z_i instead of z_{S_i} to denote the x_n coordinate of the floor of S_i. Furthermore, for $z > 0$, we write $S(z)$ the cube in $\mathcal{S}$ at height z. In other words:

$$S(z) = S_i \in \mathcal{S} : \; z_i \leq z < z_i + \ell(S_i).$$

Observe that there is only one cube $S(z)$ for each z. On the other hand, i_z stands for the index of the cube $S(z)$ in $\mathcal{S}$, i.e.: $S(z) = S_{i_z}$. Finally, let us denote $\mathcal{W}$ and $\mathcal{W}^c$ the Whitney decompositions of Ω and $(\Omega^c)^o$ respectively.

Definition 3.2.1 (Normal Cusp)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mathbf{0} \in \partial\Omega$. Let $\varepsilon > 0$ and $K > 1$ be given parameters. We say that Ω has a $(\varepsilon, K)-$*normal *external cusp (or outer peak) at the origin if it satisfies:*

(i) There exists a chain of cubes $\mathcal{S} = \{S_i\}_{i=1}^{\infty} \subset \mathcal{W}$ increasingly numbered towards the origin (i.e.: $d(S_{i+1}, 0) \leq d(S_i, 0)$), such that

$$S_i \cap S_{i+1} = F^u_{S_{i+1}}, \tag{3.2.1}$$

and

$$d(S_i, \mathbf{0}) \to 0 \quad (i \to \infty). \tag{3.2.2}$$

(ii) We have that

$$\{x \in \Omega : \ x_n < z\} \subset \bigcup_{i=i_z}^{\infty} \Omega_i \ \textit{for any} \ z_1 > z > 0, \tag{3.2.3}$$

with $\Omega_i = KS_i \cap \Omega$.

(iii) For every pair of points $x, y \in \Omega_i \cup \Omega_{i+1}$, there is a rectifiable curve, $\gamma \subset \Omega$, joining x and y, and satisfying:

$$\ell(\gamma) \leq \frac{1}{\varepsilon}|x - y|, \tag{3.2.4}$$

$$d(z, \partial\Omega) \geq \varepsilon \frac{|x - z||z - y|}{|x - y|} \quad \forall z \in \gamma. \tag{3.2.5}$$

(iv) We have

$$\frac{\ell(S_i)}{z_i} \longrightarrow 0 \qquad (i \to \infty). \tag{3.2.6}$$

The set $\mathcal{S}$ is named the **spine** *of Ω.*

Condition (3.2.1) implies that the chain is decreasing, i.e.: $\ell(S_{i+1}) \leq \ell(S_i)$. This last fact is not really necessary, but is assumed for the sake of simplicity: the sizes of the cubes in $\mathcal{S}$ could oscillate, as long as its oscillation is controlled by some universal parameter, depending only on Ω.

On the other hand, conditions (3.2.1) and (3.2.2) imply that every cube S_i of $\mathcal{S}$ touches $\hat{x}_n$, while (3.2.3) guarantees that Ω's behavior (its narrowing toward the origin) is faithfully represented by the behavior of the chain $\mathcal{S}$: a fixed expansion of the tails of $\mathcal{S}$ reaches the whole boundary of Ω below certain height z, and consequently $\partial\Omega$ narrows toward the origin as fast as $\ell(S(z))$. In other words, the function $\ell(S(z))$ plays the role of $\varphi(z)$ in Definition A.

Finally, conditions (3.2.4) and (3.2.5) constitute what we call the sectional uniformity property of Ω. They provide some regularity to the boundary of Ω and exclude the existence of non connected components.

Condition (3.2.6) is stated in order to exclude cones and other non-singular domains from our definition of cusp. However, it is important to notice that our extension theorems (see

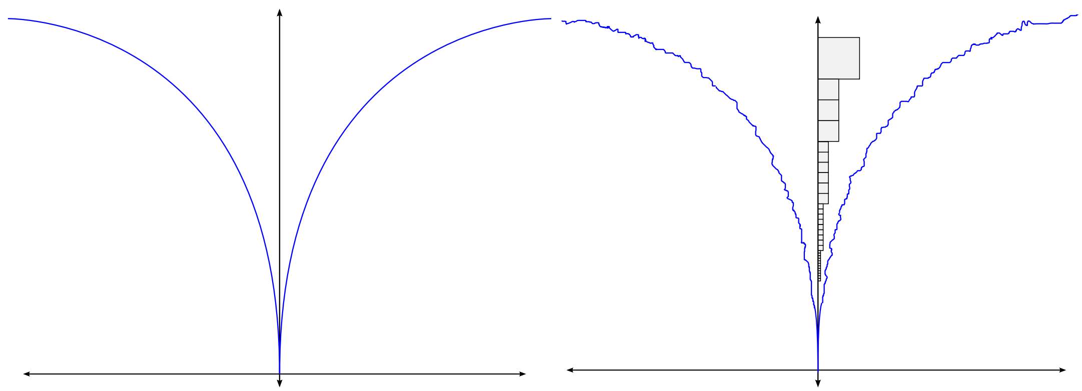


Figure 3.1: Cusp of power type vs. *Normal* cusp with its *spine*

Theorems 5.1.1 and 5.2.1) stand even for domains where (3.2.6) is not fulfilled. In that cases the weight turns to be a constant, and a classical (unweighted) extension is obtained.

Now, we state some important results on normal cusps.

Since external cusps are not uniform domains, it is clear that the sectional uniformity properties (3.2.4) and (3.2.5) do not hold for every x, y in Ω. However, they can be extended to larger bands:

Proposition 3.2.2. *Let us define $\Omega'_i = K'S_i \cap \Omega$, for some $K' > K$, such that Ω'_i contains a finite number of Ω_i's. Then, Properties (3.2.4) and (3.2.5) stand for every $x, y \in \Omega'_i \cap \Omega'_{i+1}$, with some $\varepsilon' > 0$.*

Proof. It is sufficient to prove that (3.2.4) and (3.2.5) stand for every $x, y \in \Omega_i \cup \Omega_{i+1} \cup \Omega_{i+2}$. Naturally, the only interesting case is $x \in \Omega_i \setminus \Omega_{i+1}$, and $y \in \Omega_{i+2}$.

Observe that $d(x, S_{i+1}) \leq \ell(S_i) + \sqrt{n}K\ell(S_i) \leq C\ell(S_{i+1})$ with C depending only on n and K. Consequently, $|x - v| \leq C\ell(S_{i+1})$ for every $v \in S_{i+1}$. The same results hold for y. Furthermore, since $x \in \Omega_i \setminus \Omega_{i+1}$, $|x - y| \geq C\ell(S_{i+1})$.

Now, let w be the center of S_{i+1}. Then, since $S_{i+1} \subset \Omega_{i+1}$, we have curves γ_1 and γ_2 given by Definition 3.2.1, joining x and w, and y and w respectively. We take $\gamma = \gamma_1 \cup \gamma_2$. Then

$$\ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2) \leq \varepsilon(|x - w| + |y - w|) \leq C|x - y|.$$

It only remains to prove (3.2.5). First, let $z \in \gamma \cap S_{i+1}$, then, we have that:

$$\frac{|x - z||z - y|}{|x - y|} \leq C\frac{\ell(S_{i+1})^2}{\ell(S_{i+1})} \leq C\ell(S_{i+1}) \leq Cd(z, \partial\Omega),$$

where in the last step we used that S_{i+1} is a Whitney cube of Ω.

Finally, let us take $z \in \gamma \setminus S_{i+1}$. We can assume $z \in \gamma_1$. We have:

$$d(z, \partial\Omega) \geq \varepsilon\frac{|x - z||z - w|}{|x - w|} \geq C\frac{|x - z|\ell(S_{i+1})}{\ell(S_{i+1})} \geq C\frac{|x - z||z - y|}{|x - y|}.$$

The result follows, takin ε' the worst of the constants C involved in the previous inequalities. □

The following Lemma is a fundamental property of uniform domains. Since we apply it to normal cusps, we state it in terms of the sets Ω_i:

Lemma 3.2.3. *Let Ω be a normal cusp, and $\mathcal{W}$ its Whitney decomposition. Let $Q_1, Q_2 \in \mathcal{W}$, be such that, for some i: $Q_j \cap (\Omega_i \cup \Omega_{i+1}) \neq \emptyset$, $j = 1, 2$, and $d(Q_1, Q_2) \leq C\ell(Q_1)$. Then there is a constant $\widetilde{C} = \widetilde{C}(\varepsilon, n, K)$ and a chain of cubes $\mathcal{F}_{1,2} = \{V_1 := Q_1, V_2, \dots, V_r := Q_2\} \subset \mathcal{W}$ such that $r \leq \widetilde{C}$ and $\ell(V_j) \underset{\widetilde{C}}{\sim} \ell(Q_1)$, for every j.*

Proof. There is a curve γ joining Q_1 and Q_2 with $\ell(\gamma) \leq Cd(Q_1, Q_2) \leq C\ell(Q_1)$. Observe that here C denotes different constants, but all of them independent of the cubes considered. Let us consider, then, the chain

$$\mathcal{F}_{1,2} = \{V_1 = Q_1, V_2, \dots, V_r = Q_2\} \subset \mathcal{W},$$

of cubes touching γ. We need a lower bound for the size of V_j: $\ell(V_2) \geq \frac{1}{4}\ell(Q_1)$. Analogously, $\ell(V_{r-1}) \geq C\ell(Q_1)$. If $1 < j < r$, let us take $z \in \gamma \cap V_j$. Then:

$$d(z, \partial\Omega) \geq \varepsilon \frac{|x - z||z - y|}{|x - y|} \geq C \frac{\ell(Q_1)^2}{\ell(Q_1)} \geq C\ell(Q_1).$$

It follows that no more than C cubes can be placed along γ, and then $r \leq C$. Once again $\widetilde{C}$ is the worst of the constants C. □

Remark 3.2.4. *It is important to observe that Lemma 3.2.3 is a consequence of the sectional uniformity properties (3.2.4) and (3.2.5). Hence, regarding Proposition 3.2.2, it holds for every pair of cubes Q_1 and Q_2 such that $d(Q_1, Q_2) \leq C\ell(Q_1)$ as long as they are contained in some band $K'S_i \cap \Omega$.*

3.3 Curved cusps

Normal cusps are, somehow, "symmetric" with respect to $\hat{x}_n$. More precisely, *normal* cusps are those that grow around an axis, which is placed approximately at its center; see Figure 3.1. The following definition includes cusps that are *tangential* to a certain axis, which is not necessarily interior to the domain:

Definition 3.3.1 (Curved Cusp)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mathbf{0} \in \partial\Omega$. Let $\varepsilon > 0$ and $K > 1$ be given parameters. We say that Ω has a $(\varepsilon, K)-$*curved *external cusp (or outer peak) at the origin if there exists a quasi-chain of cubes $\mathcal{S} = \{S_i\}_i$, $S_i \in \mathcal{W}$, increasingly numbered towards the origin, such that $z_{i+1} < z_i$, and satisfying:*

$$d(S_i, \hat{x}_n) \leq C_\Omega \ell(S_i) \qquad \textit{for some } C_\Omega, \tag{3.3.1}$$

$$\ell(S_{i+1}) \le \ell(S_i), \tag{3.3.2}$$

and if Ω *satisfies conditions (ii), (iii) and (iv) of Definition 3.2.1.*

Condition (3.3.1), along with the fact that $\mathcal{S}$ is a quasi-chain (and not a chain), constitutes a relaxation of condition (3.2.1). Since the spine is now a quasi-chain, it is not forced to be straight and parallel to $\hat{x}_n$, but is allowed to approximate it asymptotically. As we remarked earlier, condition (3.3.2) is not necessary, but comfortable. Since we abandoned property (3.2.1), (3.3.2) is not implicit any more, and so we include it in the definition of *curved* cusps. Finally, we ask $\{S_i\}$ to satisfy: $z_{i+1} < z_i$ because this implies that for every height z there is a unique cube $S(z)$, which is necessary for the correct statement of condition (ii).

Remark 3.3.2. *Since* $\mathcal{S}$ *is a quasi-chain, we cannot conclude the existence of intermediate cubes joining* S_i *and* S_{i+1}*. However,* $\mathcal{S}$ *is formed by Whitney cubes, and hence it is easy to see that there is a chain* $\widetilde{\mathcal{S}} = \{\widetilde{S}_i\}$ *with* $\widetilde{S}_i \in \mathcal{W}$*, such that* $\mathcal{S} \subset \widetilde{\mathcal{S}}$*.*

3.4 Examples

Below we show that the class of normal and curved cusps is broader than the class of cusps satisfying Definition A.

Generally speaking our results can be understood in the following way: the role of the "profile" function φ in Definition A can be relaxed in the sense that it can just describe the speed of the narrowing of Ω towards the origin (i.e. if the *spine* of Ω decreases as φ: $\ell(S(z)) \sim \varphi(z)$) provided that $\partial\Omega \setminus \{\mathbf{0}\}$ remains smooth enough.

For example, consider the domain:

$$\Omega = \{(x, z) \in \mathbb{R}^2 : \quad z^3 < x < z^2\}.$$

This domain does not satisfy Definition A: its narrowing cannot be described by a profile function φ, since the two curves that form the boundary of Ω approach the origin at different speeds. However, it is easy to see that it is an *external curved cusp*. In fact, the chain $\widetilde{\mathcal{S}}$ is formed by all the cubes in $\mathcal{W}$ that intersect the central curve: $\gamma(z) = \frac{z^3+z^2}{2}$, and the spine $\mathcal{S}$ can be obtained by substracting from $\widetilde{\mathcal{S}}$ redundant cubes, if any.

The second example is general, and it constitutes the proof of the fact that every domain satisfying Definition A is a normal or curved cusp, so we devote a few lines to it:

Definition 3.4.1. *Let* $\Omega \subset \mathbb{R}^n$ *(*$n \ge 2$*) be a domain with compact boundary* $\partial\Omega$*. Assume that* $\mathbf{0} \in \partial\Omega$*. We say that* Ω *has a* restricted *external cusp at the origin if there exists a neighborhood of* $\mathbf{0}$*,* $U \subset \mathbb{R}^n$ *such that*

$$U \cap \Omega = \{(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} : x \in \varphi(z)\varpi\},$$

where $\varpi \subset \mathbb{R}^{n-1}$ *is a bounded* uniform *domain and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lipschitz increasing function such that* $\varphi(0) = 0$ *and* $\frac{\varphi(t)}{t} \to 0$ *(*$t \to 0^+$*).*

Definition 3.4.1 is exactly like Definition A, but the domain $\varpi \subset \mathbb{R}^{n-1}$ is asked to be uniform instead of Lipschitz. Therefore, it is clear that every outer peak satisfying Definition A is a *restricted* external cusp.

Claim 3.4.2. *Every* restricted *cusp satisfies Definition 3.3.1 (or 3.2.1).*

We sketch the proof of this claim through a series of observations:

Given Ω a restricted cusp, let us define Ω_z the set of points of Ω at height z and the boundary of this set $\partial\Omega_z := \{(x,z) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ x \in \varphi(z)\partial\varpi\}$.

Observe that the distance from a point $(x,z) \in \Omega$ to $\partial\Omega$ is equivalent to its distance to $\partial\Omega_z$. Indeed, it is clear that $d((x,z),\partial\Omega) \leq d((x,z),\partial\Omega_z)$. On the other hand, let us denote $x = \varphi(z)\zeta$, for some $\zeta \in \varpi$. Let $(x_0,z_0) = (\phi(z_0)\zeta_0, z_0) \in \partial\Omega$ be such that $d((x,z),\partial\Omega) = d((x,z),(x_0,z_0))$. Naturally, $\widetilde{x_0} = (\varphi(z)\zeta_0, z)$ is in $\partial\Omega_z$. Then

$$
\begin{aligned}
d((x,z),\partial\Omega_z) \leq |x-\widetilde{x_0}| = |\varphi(z)\zeta - \varphi(z)\zeta_0| &\leq |\varphi(z)\zeta - \varphi(z_0)\zeta_0| + |\varphi(z_0)-\varphi(z)||\zeta_0| \\
&\leq |\varphi(z)\zeta - \varphi(z_0)\zeta_0| + C_\varphi C_\varpi|z_0-z| \leq C(|\varphi(z)\zeta - \varphi(z_0)\zeta_0| + |z_0-z|) \\
&\leq Cd((x,z),(x_0,z_0)) = Cd((x,z),\partial\Omega),
\end{aligned}
$$

where C_φ is the Lipschitz constant of φ and $C_\varpi = \sup\{\|\xi\| : \ \xi \in \varpi\}$.

Let r_ϖ be the inner radius of ϖ:

$$r_\varpi = \sup_{x\in\varpi} \inf_{y\in\partial\varpi} d(x,y),$$

and $c_\varpi \in \varpi$ a point such that $B(c_\varpi, r_\varpi) \subset \varpi$.

Let us consider the curve $\Gamma : \mathbb{R}_+ \to \mathbb{R}^n$, $\Gamma(t) = (\varphi(t)c_\varpi, t)$ that describes the "center" of Ω. Let $\widetilde{\mathcal{S}}$ be the set of all cubes $S \in \mathcal{W} = \mathcal{W}(\Omega)$ such that $S \cap \Gamma(t) \neq \emptyset$. Let $\mathcal{S} = \{S_i\}_{i=1}^\infty$ be a subset of $\widetilde{\mathcal{S}}$ such that $S_i \cap S_{i+1} \neq \emptyset$ and $z_{S_{i+1}} < z_{S_i}$ (this is possible because $\frac{\varphi(t)}{t} \to 0$). $\mathcal{S}$ is the spine of Ω.

Since φ is Lipschitz, we have:

$$\varphi(z + C\varphi(z)) - \varphi(z) \leq C_\varphi(z + C\varphi(z) - z) = C\varphi(z).$$

Then:

$$\varphi(z + C\varphi(z)) \leq C\varphi(z). \tag{3.4.1}$$

On the other hand $d(\Gamma(t),\partial\Omega_t) = r_\varpi\varphi(t)$, and consequently $d(\Gamma(t),\partial\Omega) \sim \varphi(t)$. Taking this into account, (3.4.1) implies that $\ell(S_i) \sim \varphi(z_i)$.

Properties (3.3.1) and (3.3.2) (as well as (3.2.1) and (3.2.2) when c_ϖ can be taken equal to 0) follow easily from the definition of $\mathcal{S}$. The covering property (3.2.3) is a consequence of (3.4.1). Since ϖ is a fixed bounded domain, there is a radius R_ϖ such that $\varpi \subset B(c_\varpi, R_\varpi)$. This radius scaled to the section Ω_z is $\varphi(z)R_\varpi$, but $\varphi(z)$ is essentially the length $\ell(S(z))$. Taking (3.4.1) into consideration, this implies that there is a constant K (depending on r_ϖ, R_ϖ and n), such that KS_i covers the slice of Ω between heights z_i and $z_i + \ell(S_i)$, $\forall i$. Thence, (3.2.3) follows.

The last thing to prove, then, is that uniformity properties (3.2.4) and (3.2.5) stand for every restricted cusp. We use the following result stated by Smith, Stanoyevitch and Stegenga in [Smith et al., 1994]:

Lemma 3.4.3. *Let Ω_1 and Ω_2 be uniform domains with finite diameters. Then $\Omega_1 \times \Omega_2$ is a uniform domain.*

Remark 3.4.4. *The definition of* uniform domain *used in [Smith et al., 1994] (for the proof of this lemma) is slightly different than that stated here. For the equivalence between both see [Väisälä, 1988] and [Martio, 1980].*

In Definition 3.2.1, Properties (3.2.4) and (3.2.5) are requiered for points in $\Omega_i \cup \Omega_{i+1}$. We prove that they stand in every slice between heights $z - C\varphi(z)$ and $z + C\varphi(z)$, for every fixed constant C. Our proof is based on the following idea: since φ is Lipschitz, the set:

$$\Omega \cap \{(x,z) \in \mathbb{R}^n : \ z \in (z_0 - C\varphi(z_0), z_0 + C\varphi(z_0))\},$$

is almost the cylinder:

$$\widehat{\Omega}_0 := \varphi(z_0)\varpi \times (z_0 - C\varphi(z_0), z_0 + C\varphi(z_0)), \tag{3.4.2}$$

which is uniform thanks to Lemma 3.4.3. In that lemma, the ε parameter of $\Omega_1 \times \Omega_2$ depends on the respective values of the parameters of Ω_1 and Ω_2 and on the quotient $\frac{diam(\Omega_1)}{diam(\Omega_2)}$. Since in (3.4.2) $diam(\Omega_1) \sim diam(\Omega_2)$, we may assume that the same ε stands for the cylinder for every z_0.

Let $z_0 > 0$ be a fixed number and C_0 a constant such that $C_0 < \frac{z_0}{\varphi(z_0)}$. Observe that since $\frac{t}{\varphi(t)} \to \infty$ as $t \to 0$, the constant C_0 chosen for a certain z_0 remains useful for every $z < z_0$. Let us denote:

$$\Omega_0 = \Omega \cap \{(x,z) \in \Omega : \ z_0 - C_0\varphi(z_0) < z < z_0 + C_0\varphi(z_0)\}.$$

We want to prove that Ω_0 is uniform. We associate points in $\widehat{\Omega}_0$ with points in Ω_0 at the same heights, so we denote $(\widehat{x}, z)$ the points in $\widehat{\Omega}_0$ and (x,z) those in Ω_0. Let $F : \widehat{\Omega}_0 \to \Omega_0$ be the function:

$$F(\widehat{x}, z) = \Big(\frac{\varphi(z)}{\varphi(z_0)}\widehat{x}, z\Big).$$

Suppose $\zeta \in \varpi$ is such that $\varphi(z_0)\zeta = \widehat{x}$. Then $x = \frac{\varphi(z)}{\varphi(z_0)}\widehat{x} = \varphi(z)\zeta$, and $F(\widehat{x}, z) = (x,z) \in \Omega_0$. F is obviously bijective, with

$$F^{-1}(x,z) = \Big(\frac{\varphi(z_0)}{\varphi(z)}x, z\Big).$$

Now we prove that both F and F^{-1} are Lipschitz with constants independent of z_0 (this, in turn, shows that Ω_0 is uniform). We show only the case F^{-1} since the proof for F is similar. Let us consider (x,z), $(y,w) \in \Omega_0$, $x = \varphi(z)\zeta$, $y = \varphi(w)\xi$ for some $\zeta, \xi \in \varpi$.

$$|F^{-1}(x,z) - F^{-1}(y,w)| = |(\varphi(z_0)\zeta - \varphi(z_0)\xi, z-w)| \leq \underbrace{|\varphi(z_0)\zeta - \varphi(z_0)\xi|}_{I} + \underbrace{|z-w|}_{II}.$$

And

$$I \leq \varphi(z_0)\Big|\frac{\varphi(z)}{\varphi(z)}\zeta - \frac{\varphi(w)}{\varphi(w)}\xi\Big| = \varphi(z_0)\Big|\frac{\varphi(w)x - \varphi(z)y}{\varphi(z)\varphi(w)}\Big|.$$

Since $z, w \in (z_0 - C\varphi(z_0), z_0 + C\varphi(z_0))$, Equation (3.4.1) implies that $\varphi(z_0) \sim \varphi(z)$ so:

$$I \leq C\Big|\frac{\varphi(w)x - \varphi(z)y}{\varphi(w)}\Big|.$$

On the other hand:

$$\begin{aligned}|\varphi(w)x - \varphi(z)y| &\leq |\varphi(w)x - \varphi(w)y| + |\varphi(w)y - \varphi(z)y| \leq \varphi(w)|x-y| + |\varphi(w) - \varphi(z)||y| \\ &\leq \varphi(w)|x-y| + C_\varphi|w-z||\varphi(w)\xi| \leq C_\varphi C_\varpi \varphi(w)\{|x-y| + |w-z|\}.\end{aligned}$$

Hence: $I \leq C\{|x-y| + |w-z|\}$, and consequently

$$|F^{-1}(x,z) - F^{-1}(y,w)| \leq C\{|x-y| + |w-z|\} \leq C|(x,z) - (y,w)|.$$

So, F^{-1} is Lipschitz with a Lipschitz constant depending only on the constants C_0, C_φ and C_ϖ.

Remark 3.4.5. *We do not really need Ω_0 to be* uniform *as a separate domain (with its* floor *and its* roof *as parts of the boundary): we just need to prove that the curve joining two points in Ω_0 satisfy property* (3.2.5)*, which is given in terms of the distance to the boundary of Ω. But $d((x,z), \partial\Omega) \geq d((x,z), \partial\Omega_0)$, $\forall (x,z) \in \Omega_0$, so* (3.2.5) *stands.*

This complete the proof of Claim 3.4.2. Since the class of domains given by Definition 3.4.1 is broader than that of Definition A, we can state the following:

Corollary 3.4.6. *Every domain satisfying Definition A is an external cusp in terms of Definition 3.3.1 (or Definition 3.2.1).*

4

Approximation by smooth functions

The density of smooth functions on a certain Sobolev space is a very useful tool for many pourposes, since it allows to prove different properties arguing first for smooth functions, that are easier to treat, and then, by a simple density argument, for functions in the correspondant Sobolev space.

Given a bounded open set D, it is well known that every function $f \in W^{k,p}(D)$ can be approximated by a sequence of functions $f_m \in C^\infty(D)$ (see, for example, [Evans, 1998, Section 5.3.2.]). Observe that, in this case, no conditions on the boundary of D are imposed. We are interested in a stronger version of this result, which is the approximation by smooth functions up to the boundary of D, i.e. we want f_m to be in $C^\infty(\bar{D})$ or, in other words, we want f_m to be the restriction to $\bar{D}$ of functions $\widetilde{f_m}$ in $C^\infty(\mathbb{R}^n)$, or in $C_0^\infty(\mathbb{R}^n)$. The possibility of approximating Sobolev functions by smooth ones up to ∂D depends on the nature of the domain considered. The classical literature concerning Sobolev spaces includes results of this kind for domains with Lipschitz or C^1 boundary (e.g.: [Evans, 1998, Section 5.3.3], [Maz'ya, 2011, Section 1.1.6], [Burenkov, 1998, Chapter 2]), and, more generally for domains satisfying the segment property, or equivanlently, with C boundary (e.g. [Adams and Fournier, 2003, Theorem 3.22],[Kufner, 1985, Chapter 7]).

In this Chapter, we prove that given a normal (or curved) cusp Ω, and $f \in W^{k,p}(\Omega)$, there is a function $g \in C^\infty(\mathbb{R}^n_+)$, with $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, as close to f as needed. Observe that such a result does not involve the whole boundary of Ω, but excludes precisely the singular point at the tip of the cusp. The density of $C^\infty(\bar{\Omega})$ is proved in Section 5.3 as a corollary of the first stage of the extension process given in Theorems 5.1.1 and 5.2.1.

Our approximation theorem is a simple corollary of the density of smooth functions on uniform domains, that is proved in [Jones, 1981]. We reproduce Jones's proof just for the sake of completeness. For normal and curved cusps, we prove first an unweighted density theorem and afterwards, an easy weighted generalization, for weights that can be considered constants by bands.

4.1 The unweighted case

4.1.1 Uniform domains

In [Jones, 1981, Proposition 4.4], the author proves that every function in $W^{k,p}(D)$, with D a uniform domain, can be approximated by functions in $C^\infty(\bar{D})$. In other words, he proves that given $f \in W^{k,p}(D)$, and $\eta > 0$, there is a funcion $g_\eta \in C^\infty(\mathbb{R}^n)$ such that $\|f - g_\eta\|_{W^{k,p}(D)} < \eta$. The main tool for obtaining this result is the existence of a chain of cubes similar to the one in Lemma 3.2.3. An inmediate consequence of this fact is that if Ω is a normal (or curved) cusp we can approximate $f \in W^{k,p}(\Omega)$ by C^∞ functions on the bands $\Omega_i = KS_i \cap \Omega$. Pasting this local approximations, we construct a smooth approximation of f over all Ω.

Some technical aspects of the proof of Jones's density theorem, only sketched in [Jones, 1981], are developed in detail in [Chua, 1992], although the author ommits a few details (particularly, those fully explained in Jones [1981]).

For the sake of completeness, we include here a complete proof of this result. We follow mostly [Chua, 1992], although we introduce some little modifications. We prove:

Proposition 4.1.1. *Let D be a ε-uniform domain, and let $f \in W^{k,p}(\Omega)$. Then, for every $\eta > 0$ there is a function $g \in C^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{W^{k,p}(D)} < \eta$.*

We avoid the weighted version of this result (which is proved, for weights in the A_p class of Muckenhoupt, in [Chua, 1992]), since the weights that we are interested in can be treated in a very simple way (see Section 4.2).

Let $\rho = 2^{-m}$ a number that will be fixed later. Let $\mathcal{Q} = \{Q_j\}$ be the collection of all diadic cubes with $\ell(Q_j) = \rho$ and $Q_j \subset D$. We define:

$$\mathcal{Q}' = \left\{ Q_j \in \mathcal{Q} : \ Q_j \subset V_k \text{ for some } V_k \in \mathcal{W},\ \ell(V_k) \geq \frac{15\sqrt{n}}{\varepsilon}\rho \right\}.$$

For $Q_j \in \mathcal{Q}'$, we denote $\widetilde{Q_j} = \frac{601n}{\varepsilon^2} Q_j$, and $\widetilde{\widetilde{Q_j}} = \frac{1202n}{\varepsilon^2} Q_j$. For simplicity, we assume $\varepsilon < 1$.

Lemma 4.1.2. *ρ can be taken small enough so that $D \subset \cup_{Q_j \in \mathcal{Q}'} \widetilde{Q_j}$.*

Proof. Given $z \in D$, let

$$m_z = \inf \left\{ d(z, V) : \ V \in \mathcal{W}(D),\ \ell(V) \geq \frac{15\sqrt{n}\rho}{\varepsilon} \right\}.$$

If $m_z \leq \frac{600n\rho}{\varepsilon^2}$, then there is some $Q_j \in \mathcal{Q}'$ and $w \in Q_j$ such that $|z - w| \leq \frac{600n\rho}{\varepsilon^2}$. And then:

$$|z - c(Q_j)| \leq |z - w| + \sqrt{n}\rho \leq \frac{600n}{\varepsilon^2}\rho + \sqrt{n}\rho \leq \frac{601n\rho}{\varepsilon^2}.$$

And then, $z \in \frac{601n}{\varepsilon^2} Q_j = \widetilde{Q_j}$.

Now, we want to prove that $m_z \leq \frac{600n\rho}{\varepsilon^2}$ for every $z \in D$. Suppose that for some z, $m_z > \frac{600n\rho}{\varepsilon^2}$. Then, let $x \in D$ be such that $|x - z| = \frac{1}{2}\min\{r, m_z\}$, where r is the radius of D. Take γ the curve joining x and z, and $x_0 \in \gamma$ such that $|x_0 - z| = \frac{1}{2}|x - z|$. Observe that if V_{x_0} is the cube in $\mathcal{W}(D)$ that contain x_0, then:

$$4\sqrt{n}\ell(V_{x_0}) \geq d(V_{x_0}, \partial(D)) \geq d_\partial(x_0) - \sqrt{n}\ell(V_{x_0}).$$

Hence:

$$5\sqrt{n}\ell(V_{x_0}) \geq d_\partial x_0 \geq \varepsilon\frac{|x - x_0||x_0 - z|}{|x - z|} \geq \varepsilon\frac{|x - z|}{4} = \frac{\varepsilon}{8}\min\{r, m_z\}.$$

Now, if we take ρ such that $\frac{\varepsilon}{8}r > \frac{75n}{\varepsilon}\rho$, then: $\ell(V_{x_0}) \geq \frac{15\sqrt{n}}{\varepsilon}\rho$. Which is a contradiction, since $d(z, V_{x_0}) \leq \frac{m_z}{4}$. □

Lemma 4.1.3. *If $Q_1, Q_2 \in \mathcal{Q}'$ are such that $\widetilde{\widetilde{Q_1}} \cap \widetilde{\widetilde{Q_2}} \neq \emptyset$, then there is a chain of cubes in $\mathcal{Q}$ connecting Q_1 and Q_2, $\mathcal{F} = \{U_1 = Q_1, U_2, \ldots, U_r = Q_2\}$, with $r \leq C$, for some C independent of Q_1 and Q_2.*

Proof. The arguments are similar to those used to prove Lemma 3.2.3. Let us consider γ the curve joining Q_1 and Q_2, and let $z \in \gamma$. We can assume that $d(z, Q_1) \leq d(z, Q_2)$. If $d(z, Q_1) \leq \frac{10\sqrt{n}}{\varepsilon}\rho$, then

$$5\sqrt{n}\ell(V_z) \geq d_\partial(z) \geq d_\partial(Q_1) - d(z, Q_1) \geq \frac{15\sqrt{n}}{\varepsilon}\rho - \frac{10\sqrt{n}}{\varepsilon}\rho \geq \frac{5\sqrt{n}}{\varepsilon}\rho.$$

Hence,

$$\ell(V_z) \geq \frac{\rho}{\varepsilon}.$$

Anf if $d(z, Q_1) > \frac{10\sqrt{n}}{\varepsilon}\rho$, then:

$$5\sqrt{n}\ell(V_z) \geq d_\partial(z) \geq \varepsilon\frac{d(z, Q_1)d(z, Q_2)}{d(Q_1, Q_2)} \geq \varepsilon\frac{d(z, Q_1)}{2} > 5\sqrt{n}\rho.$$

Consequently, in any case, $\ell(V_z) \geq \rho$, for all $z \in \gamma$. Therefore, the collection of cubes $U_j \in \mathcal{Q}'$ such that $U_j \cap \gamma \neq \emptyset$ contain a chain as the one desired. □

For each $Q_j \in \mathcal{Q}'$, let $\pi_j = \pi(Q_j)(f)$, the polynomial approximation of f given by Definition 2.4.10. Let us also construct functions $\varphi_j \in C_0^\infty(\widetilde{\widetilde{Q_j}})$, satisfying:

$$0 \leq \varphi_j \leq 1, \quad 0 \leq \sum_{Q_j \in \mathcal{Q}'} \varphi_j \leq 1, \quad \text{and} \sum_{Q_j \in \mathcal{Q}'} \varphi_j(x) = 1, \ \forall x \in \bigcup_{Q_j \in \mathcal{Q}'} \widetilde{Q_j}.$$

With:

$$|D^\alpha \varphi_j| \leq \frac{C}{r^{|\alpha|}}.$$

We define

$$g_0 = \sum_{Q_j \in \mathcal{Q}'} \pi_j \varphi_j.$$

g_0 will approximate f near $\partial\Omega$, so we will define the approximation g as a smooth combination of g_0 and another function that approximates f in the interior of D. However, before proving Proposition 4.1.1, let us state two lemmas concerning the local approximation properties of the polynomials π_j:

Lemma 4.1.4. *If $Q_j \in \mathcal{Q}'$, $0 \le |\alpha| \le k$, then:*

$$\|D^\alpha \pi_j\|_{L^p(\widetilde{\widetilde{Q}}_j)} \le C\|D^\alpha f\|_{L^p(Q_j)} + C\rho^{k-|\alpha|}\|\nabla^k f\|_{L^p(Q_j)}.$$

Proof. Applying Corollary 2.4.3 and Theorem 2.4.12:

$$\begin{aligned}\|D^\alpha \pi_j\|_{L^p(\widetilde{\widetilde{Q}}_j)} &\le C\|D^\alpha \pi_j\|_{L^p(Q_j)} \le C\|D^\alpha f\|_{L^p(Q_j)} + C\|D^\alpha(\pi_j - f)\|_{L^p(Q_j)} \\ &\le C\|D^\alpha f\|_{L^p(Q_j)} + C\rho^{k-|\alpha|}\|\nabla^k f\|_{L^p(Q_j)}.\end{aligned}$$

□

Lemma 4.1.5. *If $Q_0 \in \mathcal{Q}'$, $0 \le |\alpha| \le k$. We denote $\mathcal{F}_0$ the collection of cubes formed by all the chains $\mathcal{F}_{0,j}$ between Q_0 and Q_j for some $Q_j \in \mathcal{Q}'$ such that $\widetilde{\widetilde{Q}}_j \cap \widetilde{\widetilde{Q}}_0 \neq \emptyset$. Then:*

$$\sum_{Q_j \in \mathcal{Q}' :\, \widetilde{\widetilde{Q}}_j \cap \widetilde{\widetilde{Q}}_0 \neq \emptyset} \|D^\alpha((\pi_0 - \pi_j)\varphi_j)\|_{L^p(Q_0)} \le C\rho^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup \mathcal{F}_0)}.$$

Proof. Fix Q_j and take $\mathcal{F}_{0,j} = \{U_1 = Q_0, \ldots, U_r = Q_j\}$ the chain joining Q_0 and Q_j. We have:

$$\begin{aligned}
&\|D^\gamma(\pi_0 - \pi_j)\|_{L^p(Q_0)} \le \sum_{i=1}^{r-1} \|D^\gamma(\pi(U_i) - \pi(U_{i+1}))\|_{L^p(Q_0)} \le C\sum_{i=1}^{r-1} \|D^\gamma(\pi(U_i) - \pi(U_{i+1}))\|_{L^p(U_i \cup U_{i+1})} \\
&\le C\sum_{i=1}^{r-1} \Big\{\|D^\gamma(\pi(U_i) - \pi(U_i \cup U_{i+1}))\|_{L^p(U_i)} + \|D^\gamma(\pi(U_i \cup U_{i+1}) - \pi(U_{i+1}))\|_{L^p(U_{i+1})}\Big\} \\
&\le C\sum_{i=1}^{r-1} \Big\{\|D^\gamma(\pi(U_i) - f)\|_{L^p(U_i)} + \|D^\gamma(f - \pi(U_{i+1}))\|_{L^p(U_{i+1})} + \|D^\gamma(f - \pi(U_i \cup U_{i+1}))\|_{L^p(U_i \cup U_{i+1})}\Big\} \\
&\le C\sum_{i=1}^{r-1} \Big\{\ell(U_i)^{k-|\gamma|}\|\nabla^k f\|_{L^p(U_i)} + \ell(U_{i+1})^{k-|\gamma|}\|\nabla^k f\|_{L^p(U_{i+1}} + L(U_i \cup V_{i+1})^{k-|\gamma|}\|\nabla^k f\|_{L^p(U_i \cup U_{i+1})}\Big\} \\
&\le C\sum_{i=1}^{r-1} \rho^{k-|\gamma|}\|\nabla^k f\|_{L^p(U_i \cup U_{i+1})} \le \rho^{k-|\gamma|}\sum_{i=1}^{r-1}\left(\|\nabla^k f\|^p_{L^p(U_i \cup U_{i+1})}\right)^{\frac{1}{p}} r^{\frac{1}{p'}} \le C\rho^{k-|\gamma|}\|\nabla^k f\|_{L^p(\cup \mathcal{F}_{0,j})}.
\end{aligned}$$

Observe that in the last step it is crucial the fact that the number r of cubes in the chain $\mathcal{F}_{0,j}$ is bounded by a constant independent of Q_0 and Q_j.

Now:

$$\sum_{Q_j\in\mathcal{Q}':\,\widetilde{\widetilde{\widetilde{Q}}}_j\cap\widetilde{\widetilde{\widetilde{Q}}}_0\neq\emptyset}\|D^\alpha((\pi_0-\pi_j)\varphi_j)\|_{L^p(Q_0)}\leq\sum_{Q_j\in\mathcal{Q}':\,\widetilde{\widetilde{\widetilde{Q}}}_j\cap\widetilde{\widetilde{\widetilde{Q}}}_0\neq\emptyset}\sum_{\gamma\leq\alpha}\|D^\gamma(\pi_0-\pi_j)D^{\alpha-\gamma}\varphi_j\|_{L^p(Q_0)}$$
$$\leq C\sum_{Q_j\in\mathcal{Q}':\,\widetilde{\widetilde{\widetilde{Q}}}_j\cap\widetilde{\widetilde{\widetilde{Q}}}_0\neq\emptyset}\sum_{\gamma\leq\alpha}\rho^{|\gamma|-|\alpha|}\rho^{k-|\gamma|}\|\nabla^k f\|_{L^p(\cup\mathcal{F}_{0,j})}\leq C\rho^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup\mathcal{F}_0)}.$$

In the last step we use that a cube U participates at most in a finite number of chains $\mathcal{F}_{0,j}$ bounded by a constant that depends only on D, but not on Q_0. This fact is a consequence of the finiteness of the chains (Lemma 4.1.3). □

Now we can prove the Proposition:

Proof of Proposition 4.1.1. Let $D_s=\{x\in D:\ d(x,\partial D)\geq s\}$, for some $s\in(0,1)$. We take s such that $\|f\|_{W^{k,p}(D\setminus D_{2s})}<\eta$. Let $\psi\in C_0^\infty(D_{s/2})$, such that $\psi(x)=1,\ \forall x\in D_s$, and $|D^\alpha\psi|\leq C(|\alpha|)s^{-|\alpha|}$. Now, let $\xi\in C_0^\infty(B(0,1))$ such that $\int_{\mathbb{R}^n}\xi=1$, and take $\xi_t=t^{-n}\xi(\frac{x}{t})$. We can choose t in order to obtain:

$$\|f-f*\xi_t\|_{W^{1,p}(D_{s/2})}\leq\eta s^k.$$

Finally, put $g_1=g_0(1-\psi)$ and $g_2=(f*\xi_t)\psi$. Then, $g=g_1+g_2$ is the desired approximation of f. In fact, it is clear that $g\in C^\infty(\mathbb{R}^n)$, and $|g|\leq M$ for some constant M. Furthermore:

$$\|D^\alpha(f-g)\|_{L^p(D)}\leq\underbrace{\|D^\alpha(f-g)\|_{L^p(D_s)}}_{I}+\underbrace{\|D^\alpha(f-g)\|_{L^p(D\setminus D_s)}}_{II}.$$

$$I=\|D^\alpha(f-(g_1+g_2))\|_{L^p(D_s)}=\|D^\alpha(f-g_2)\|_{L^p(D_s)}=\|D^\alpha(f-f*\xi_t)\|_{L^p(D_s)}<\eta s^k.$$

For II, observe that:

$$D^\alpha(f-(g_1+g_2))=\underbrace{\sum_{\beta\leq\alpha}C_{\alpha,\beta}(D^{\alpha-\beta}\psi)(D^\beta(f-f*\xi_t))}_{II'}+\underbrace{\sum_{\beta\leq\alpha}C_{\alpha,\beta}(D^{\alpha-\beta}(1-\psi))(D^\beta(f-g_0))}_{II''}.$$

And:

$$\|II'\|_{L^p(D\setminus D_s)}\leq C\frac{1}{s^{|\alpha|-|\beta|}}s^k\eta=C\eta.$$

The only thing left is to estimate II''. We solve separately the cases $\beta=\alpha$ and $\beta<\alpha$:
Case $\beta<\alpha$: The advantage of this case is that $D^{\alpha-\beta}(1-\psi)=0$ on $D\setminus D_{s/2}$. We can take ρ small enough so $D_{s/2}\setminus D_s\subset\cup_{Q\in\mathcal{Q}'}Q$ (for this we need at least $\rho<s/2$):

$$\|(D^{\alpha-\beta}(1-\psi))(D^\beta(f-g_0))\|^p_{L^p(D\setminus D_s)}\leq Cs^{-|\alpha-\beta|p}\|D^\beta(f-g_0)\|^p_{L^p(D_{s/2}\setminus D_s)}$$
$$\leq Cs^{-|\alpha-\beta|p}\sum_{Q_0\cap(D_{s/2}\setminus D_s)\neq\emptyset}\|D^\beta(f-g_0)\|^p_{L^p(Q_0)}.$$

And:

$$\begin{aligned}\|D^\beta(f-g_0)\|^p_{L^p(Q_0)} &\le C\|D^\beta(f-\pi_0)\|^p_{L^p(Q_0)} + C\Big\|D^\beta \sum_{j:\widetilde{\widetilde{Q_j}}\cap\widetilde{\widetilde{Q_0}}\neq\emptyset}(\pi_j-\pi_0)\varphi_j\Big\|^p_{L^p(Q_0)}\\ &\le C\rho^{(k-|\beta|)p}\|\nabla^k f\|^p_{L^p(Q_0)} + C\rho^{(k-|\beta|)p}\|\nabla^k f\|^p_{L^p(\cup\mathcal{F}_0)} \le C\rho^{(k-|\beta|)p}\|\nabla^k f\|^p_{L^p(\cup\mathcal{F}_0)}.\end{aligned}$$

Now we need to estimate the summation of $\|\nabla^k f\|^p_{L^p(\cup\mathcal{F}_0)}$ over all the sets $\mathcal{F}_0$, for all $Q_0\in\mathcal{Q}'$ such that $Q_0\cap(D_{s/2}\setminus D_s)\neq\emptyset$. For this matter, we can take ρ small enough so that if $Q_0\cap(D_{s/2}\setminus D_s)\neq\emptyset$ and Q_j such that $\widetilde{\widetilde{Q_0}}\cap\widetilde{\widetilde{Q_j}}\neq\emptyset$, then $\cup\mathcal{F}_{0,j}\subset D\setminus D_{2s}$. In this way:

$$\|(D^{\alpha-\beta}(1-\psi))(D^\beta(f-g_0))\|_{L^p(D\setminus D_s)} \le Cs^{-|\alpha-\beta|}\rho^{k-|\beta|}\|\nabla^k f\|_{L^p(D\setminus D_{2s})} \le C\eta.$$

Which conclude the analysis of this first case.
Case $\beta=\alpha$: In this case we cannot restrict the domain, because $1-\psi$ does not necessarily vanish in any subset of $D\setminus D_s$. On the other hand, no derivative is applied on $1-\psi$, and hence the factor $s^{-|\alpha-\beta|}$ in the previous case does not appear here.

$$\|D^\beta(f-g_0)\|_{L^p(D\setminus D_s)} \le \|D^\beta f\|_{L^p(D\setminus D_s)} + \|D^\beta g_0\|_{L^p(D\setminus D_s)} \le \eta + \|D^\beta g_0\|_{L^p(D\setminus D_s)}.$$

$$\|D^\beta g_0\|_{L^p(D\setminus D_s)} \le \sum_{Q_0\subset D\setminus D_s}\Big\|D^\beta\sum_j\big(\pi_j\varphi_j\big)\Big\|_{L^p(\widetilde{Q_0})}.$$

But, since $\sum\varphi_j=1$ on Q_0,

$$D^\beta\sum_j\big(\pi_j\varphi_j\big) \le D^\beta\pi_0 + D^\beta\sum_j(\pi_j-\pi_0)\varphi_j,$$

so:

$$\begin{aligned}\|D^\beta g_0\|_{L^p(D\setminus D_s)} &\le C\sum_{Q_0\subset D\setminus D_s}\left(\|D^\beta\pi_0\|_{L^p(\widetilde{Q_0})} + \Big\|\sum_j D^\beta\big((\pi_j-\pi_0)\varphi_j\big)\Big\|^p_{L^p(\widetilde{Q_0})}\right)\\ &\le C\sum_{Q_0\subset D\setminus D_s}\left(\|D^\beta f\|_{Q_0} + \rho^{k-|\beta|}\|\nabla^k f\|_{L^p(\cup\mathcal{F}_0)}\right) \le C\|f\|_{W^{k,p}(D\setminus D_{2s})} < C\eta.\end{aligned}$$

□

4.1.2 External cusps

As we commented earlier, the density of smooth functions on Sobolev spaces for uniform domains can be used as a local approximation result for normal cusps.

Given Ω a normal (or curved) cusp, with spine $\mathcal{S}=\{S_i\}$, we define:

$$\check{\Omega}_i = \Omega\cap\Big\{x=(x',x_n):\quad z_i-\frac{\ell(S_{i+1})}{2}\le x_n < z_{i-1}+\frac{\ell(S_{i-1})}{2}\Big\},$$

$$\check{\Omega}_i' = \Omega \cap \Big\{x = (x', x_n) : \quad z_i - \frac{\ell(S_{i+1})}{2} \le x_n < z_i + \frac{\ell(S_i)}{2}\Big\}.$$

The proof of Proposition 4.1.1 relies on the existence of the chain given by Lemma 4.1.3. Furthermore, thanks to Proposition 3.2.2 we can affirm that this kind of chain also exists in $\check{\Omega}_i$. Consequently, we obtain the following:

Corollary 4.1.6. *Given Ω a normal (or curved) cusp, with $\check{\Omega}_i$ the sets defined above, and $f \in W^{k,p}(\Omega)$. For every $\eta > 0$, there is a function $g_i \in C^\infty(\mathbb{R}^n)$ such that*

$$\|f - g_i\|_{W^{k,p}(\check{\Omega}_i)} < \eta.$$

Now we can prove the main result of this Chapter:

Theorem 4.1.7. *Let Ω be a normal or curved cusp, and $f \in W^{k,p}(\Omega)$. Given $\eta > 0$, there is a function $g \in C^\infty(\mathbb{R}^n_+)$ such that $\|f - g\|_{W^{k,p}(\Omega)} < C\eta$*

Proof. Let us consider $g_i \in C^\infty$ such that $\|f - g_i\|_{W^{k,p}(\check{\Omega}_i)} < \frac{\eta}{2^i}\ell(S_i)^k$.

Let $\{\psi_i\}_i$ be a partition of the unity in the segment $(0, z_1]$, such that

$$\psi_i \in C_0^\infty\left(\left[z_i - \frac{\ell(S_{i+1})}{2}, z_{i-1} + \frac{\ell(S_{i-1})}{2}\right]\right),$$

$\sum \psi_i(t) \equiv 1$ $\forall t \in (0, z_1]$, and $|D^r\psi_i| \le \frac{C}{\ell(S_i)^r}$.

Let us define

$$g(x) = \sum_{i=2}^{\infty} g_i(x)\psi_i(x_n).$$

Observe that, in $\check{\Omega}_i'$, $\psi_i + \psi_{i+1} \equiv 1$. And then:

$$\begin{aligned}\|D^\alpha(f - g)\|_{L^p(\check{\Omega}_i')} &\le \|D^\alpha(f - (\psi_i g_i + \psi_{i+1}g_{i+1}))\|_{L^p(\check{\Omega}_i')}\\ &\le \|D^\alpha(\psi_i(f - g_i))\|_{L^p(\check{\Omega}_i')} + \|D^\alpha(\psi_{i+1}(f - g_{i+1}))\|_{L^p(\check{\Omega}_i')}.\end{aligned}$$

But,

$$\begin{aligned}\|D^\alpha(\psi_i(f - g_i))\|_{L^p(\check{\Omega}_i')} &\le \Big\|\sum_{\beta\le\alpha} D^{\alpha-\beta}\psi_i D^\beta(f - g_i)\Big\|_{L^p(\check{\Omega}_i')}\\ &\le \sum_{\beta\le\alpha} \frac{C}{\ell(S_i)^{|\alpha|-|\beta|}}\|D^\beta(f - g_i)\|_{L^p(\check{\Omega}_i')}\\ &\le \sum_{\beta\le\alpha} \frac{C}{\ell(S_i)^{|\alpha|-|\beta|}}\frac{\eta}{2^i}\ell(S_i)^k \le C\frac{\eta}{2^i}.\end{aligned}$$

Consequently:

$$\|f - g\|^p_{W^{k,p}(\Omega)} = \sum_{i=1}^{\infty} \|f - g\|^p_{W^{k,p}(\check{\Omega}_i')} \le \sum_{i=1}^{\infty} C\frac{\eta^p}{2^{pi}} \le C\eta^p.$$

$\square$

4.2 The weighted case

In [Muckenhoupt, 1972], the author introduces the class of A_p weights, also known as the Muckenhoupt class. We say that a nonnegative function ω is in the class A_p if:

$$\sup_{Q\subset\mathbb{R}^n \text{cube}} \frac{1}{|Q|}\Big(\int_Q \omega(x)dx\Big)\Big(\int_Q \omega(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \leq C < \infty.$$

Chua ([Chua, 1992]) proves Proposition 4.1.1 for the weighted Sobolev space $W^{k,p}_\omega(\Omega)$, where ω is an A_p weight. In fact, the proof is exactly the one that we provide in Section 4.1.1, since the approximation properties of the polinomials $\pi(Q)$ (Theorems 2.4.11 and 2.4.12), hold for the weighted case, with A_p weights. We avoid this approach for the sake of simplicity, and because our interest on the weighted case is mostly devoted to weights depending on the distance to the tip of the cusp, which can be treated in a very easy way. However, let us state that Theorem 4.1.7 holds in $W^{k,p}_\omega(\Omega)$, for ω an A_p weight. This fact is recalled in Chapter 5 for stating a weighted extension theorem. This said, let us now prove a very simple weighted density theorem for weights depending on the distance to the tip of the cusp.

Definition 4.2.1. *Let Ω be a normal or curved cusp, and $\omega : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ a nonnegative integrable function. We say that ω is an admissible weight for Ω if there is a constant C such that:*

$$\omega(x) \underset{C}{\sim} \omega_i \underset{C}{\sim} \omega_{i+1} \quad \forall x \in \check{\Omega}_i,\ \forall i$$

where ω_i and ω_{i+1} are constants that approximates ω on $\check{\Omega}_i$ and $\check{\Omega}_{i+1}$ respectively.

Example 4.2.1. *Let us consider $\widehat{\omega} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$, a nonnegative, monotonous, integrable function, such that $\widehat{\omega}(2t) \underset{C}{\sim} \widehat{\omega}(t)$ for some constant C independent of t. Take $\omega : \Omega \longrightarrow \mathbb{R}_{\geq 0}$, $\omega(x) = \omega(|x|)$. Then, ω is an admissible weight for Ω.*

The following is the weighted version of Theorem 4.1.7, and it is proved merely pulling out of the integrals the constants approximating the weight on each $\check{\Omega}_i$:

Theorem 4.2.2. *Let Ω be a normal or curved cusp, $f \in W^{k,p}_\omega(\Omega)$, with ω an admissible weight for Ω. Given $\eta > 0$, there is a function $g \in C^\infty(\mathbb{R}^n_+)$ such that $\|f - g\|_{W^{k,p}_\omega(\Omega)} < \eta$.*

Proof. Let ψ_i be the partition of the unity constructed in Theorem 4.1.7, and g_i functions in $C^\infty(\mathbb{R}^n)$ such that,

$$\|f - g_i\|_{W^{k,p}(\check{\Omega}_i)} < \frac{\eta}{\omega_i 2^i}\ell(S_i)^k.$$

We define

$$g(x) = \sum_{i=2}^{\infty} \psi_i(x) g_i(x).$$

Then:

$$\begin{aligned}\|D^\alpha(f-g)\|^p_{L^p_\omega(\Omega)} &= \|D^\alpha(f-g)\omega^{\frac{1}{p}}\|^p_{L^p(\Omega)} = \sum_i \|D^\alpha(f-g)\omega^{\frac{1}{p}}\|^p_{L^p(\check{\Omega}'_i)}\\ &\le C\sum_i \omega_i \|D^\alpha(f-g)\|^p_{L^p(\check{\Omega}'_i)} \le C\sum_i \omega_i\|D^\alpha(f-g)\|^p_{L^p(\check{\Omega}'_i)}.\end{aligned}$$

And now, following like in Theorem 4.1.7:

$$\le \sum_i \sum_{\beta\le\alpha} \frac{C}{\ell(S_i)^{-|\alpha-\beta|}}\omega_i \frac{\eta}{\omega_i 2^i}\ell(S_i)^k \le C\eta.$$

□

5

Extension Theorems

Given a domain Ω, a linear bounded operator E is called an extension operator on $W^{k,p}(\Omega)$, if

$$E : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n),$$

and $Ef|_{\Omega} = f$ for every $F \in W^{k,p}(\Omega)$. The existence of an extension operator implies that many properties that hold in $W^{k,p}(\mathbb{R}^n)$ are inherited by $W^{k,p}(\Omega)$. As we commented earlier, a classical example of this situation is given by embedding results (see, for example [Burenkov, 1998, Sections 4.2 and 4.7], [Adams and Fournier, 2003, Section 5.7]). In the sequel we provide another results that can be obtained from extension theorems. For example, we prove the density of smooth functions up to the tip of a normal cusp. Also in Appendix A we prove weighted Korn inequalities for external cusps, using the extension operator constructed in this Chapter.

If Ω is such that $W^{k,p}(\Omega)$ admits an extension operator, we say that Ω is an extension domain for Sobolev spaces (E.D.S.). Simple examples ([Stein, 1970]) show that external cusps are not E.D.S. Therefore, we can only expect to obtain *weighted* extension operators of the form:

$$\Lambda : W^{k,p}(\Omega) \longrightarrow W^{k,p}_{\sigma}(\mathbb{R}^n),$$

where σ is a weight that somehow compensates the singularity of the domain.

Remark 5.0.3. *The narrowing of a cuspidal domain Ω allows functions in $W^{k,p}(\Omega)$ to go to infinity very rapidly at the tip of the cusp. This forces the weight that naturally arises in the extension process to vanish at the origin. Moreover, for this reason we are not able to guarranty that the extension of a function f, Λf has weak derivatives at the origin. Particularly, we cannot prove that Λf is in L^1_{loc} of a neighborhood of the origin. We can prove, however, that our extension and its derivatives are in $L^1_{loc}(G \setminus \{\mathbf{0}\})$, being G a neighborhood of the origin. This induce us to define a special weighted Sobolev space where the extension belongs. Following Maz'ya and Poborchiĭ, we state the following definition: given G a domain containing the origin $\mathbf{0}$, and ω a weight that vanishes at $\mathbf{0}$ and is bounded away from that point, we say that a function g is in $W^{k,p}_{\omega}(G)$ if g has weak derivatives of order α for $|\alpha| \leq k$, defined in G setminus$\{\mathbf{0}\}$, and $\sum_{\alpha} \|\omega^{\frac{1}{p}} D^{\alpha} g\|_{L^p(G)} < \infty$.*

This Chapter is organized as follows: in the first Section we construct two weighted extension operators for normal cusps assuming two different extra conditions on the domain. In this way, we prove Theorem 5.1.1, which is a generalization of Theorem A, stated in the Introduction and due to Maz'ya and Poborchiĭ, (see [Maz'ya and Poborchiĭ, 1997, Chapter 5]). Our extension operators are constructed in three stages, the first of which is a simple adaptation of the techniques used by Jones [Jones, 1981] for proving the extendability of functions on uniform domains. In Section 5.2, we introduce a *zero* stage, that allows us to extend functions from a curved cusp to a normal one. In this way we prove Theorem 5.2.1, that extends Theorem 5.1.1 to the case of curved cusps. In Section 5.3 we use the first stage of the extension process to prove the density of smooth functions up to the tip of the cusp, extending the result obtained in Chapter 4. Finally, in Section 5.4, we prove a weighted version of our extension results, obtaining an operator of the form:

$$\Lambda : W^{k,p}_{\omega}(\Omega) \longrightarrow W^{k,p}_{\omega\sigma}(\Omega).$$

We focus our analysis on weights depending on the distance to the tip of the cusp, but we also consider weights depending on the distance to the boundary.

5.1 Extension for normal cusps in the unweighted case

The aim of this section is to build an extension operator for normal cusps in the unweighted case $W^{k,p}$. For doing this, we need to introduce two extra conditions, which are generalizations of the ones requiered by Maz'ya and Poborchiĭ in Theorem A.

The first condition is:

$$\frac{\ell(S_i)}{z_i} \leq C\frac{\ell(S_j)}{z_j} \quad \forall i > j \quad C \text{ constant}, \tag{5.1.1}$$

which is a generalization of (1.2.1). We use it to prove item *(a)* in Theorems 5.1.1 and 5.2.1. The second condition is:

$$\ell(S_j) \leq K\ell(S_i) \quad \forall i > j \text{ such that } d(S_i, 0) > \frac{1}{2}d(S_j, 0), \tag{5.1.2}$$

which is a generalization of (1.2.2), and it is necessary for the proof of item *(b)* in both theorems.

Now, we can state the main result of this Section:

Theorem 5.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external normal cusp at the origin.*

(a) If $kp \neq 1$ or $kp = 1$ and the spine $\mathcal{S}$ satisfies (5.1.1), there is an extension operator

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_{\sigma}(\mathbb{R}^n),$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^{kp} & x \in \Omega^c \end{cases}$$

(b) If the spine $\mathcal{S}$ satisfies (5.1.2), there is an extension operator

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_{\sigma}(\mathbb{R}^n),$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^{n-1} & x \in \Omega^c \end{cases}$$

(c) Assuming (5.1.2) stands, if $\tilde{\sigma}$ is such that there is $\tilde{\Lambda} : W^{k,p}(\Omega) \to W^{k,p}_{\tilde{\sigma}}(\mathbb{R}^n)$, an extension operator, then

$$\tilde{\sigma}(x) \leq C\sigma(x) \quad \forall x \in U \setminus \Omega,$$

where U a neighborhood of the origin, and σ is taken as in item (a) *when $kp < n - 1$ and as in item* (b) *when $kp > n - 1$.*

Remark 5.1.2. *Observe that Theorem A imposes conditions on the relationship between the parameters k, n and p. For example, if an external cusp satisfies property* (1.2.1) *(but not* (1.2.2)*) then, it admits the application of Theorem A* only *if $kp < n - 1$. This is not the case of Theorem 5.1.1, where the extension operators are built regardless of the values of k, n and p (except for the particular case $kp = 1$). Moreover, observe that item (b) in Theorem 5.1.1, when $kp \neq 1$, provides an extension operator that does not requiere any additional hyphothesis on the domain.*

Below we provide a detailed proof for items *(a)* and *(b)* of Theorem 5.1.1. Item *(c)* is proved later by a simple adaptation of the counterexample given in [Maz'ya and Poborchiĭ, 1997] for proving the optimality of the weight in Theorem A. Observe that we do not prove the optimality of the weight in the critical case $kp = n-1$. In fact, we believe that the weight is not sharp for $kp = n - 1$, and that an equivalence for the weight obtained in Theorem A in this case should stand for normal cusps, although we were not able to prove it, so far. The case of curved cusps requires a little modification of our arguments (similar to that needed in [Maz'ya and Poborchiĭ, 1997]) and it is presented later in Theorem 5.2.1, in the next section.

Let us notice that thanks to item *(iii)*, in Definition 3.2.1, and the results on extension for locally uniform domains proved in [Jones, 1981], it is clear that it is enough to construct an extension operator Λ for functions f such that $supp(f) \subset D_r = \{x = (x_1, \cdots, x_n) : |x_n| < r/2\}$, where $r << \sum_{i=1}^{\infty} \ell(S_i)$. Our operator Λ is defined in a set of cubes belonging to $\mathcal{W}^c$. Let us call $\mathcal{W}_2 \subset \mathcal{W}^c$ to the set of cubes belonging to $\mathcal{W}^c$ and contained in D_r. We divide $\mathcal{W}_2$ in three parts related to three different stages of the extension process.

$$\mathcal{W}_3 = \left\{Q \in \mathcal{W}_2 : \ \ z_Q > 0 \ \text{ and } \ \ell(Q) \leq \left(\frac{\varepsilon}{5\sqrt{n}}\frac{K-1}{K}\right)\ell(S(z_Q))\right\}, \tag{5.1.3}$$

$$\mathcal{W}_4 = \{Q \in \mathcal{W}_2 \setminus \mathcal{W}_3 : \quad z_Q > 0 \ \text{ and } \ \ell(Q) \leq z_Q/(8\sqrt{n})\}, \tag{5.1.4}$$

$$\mathcal{W}_5 = \{Q \in \mathcal{W}_2 \setminus (\mathcal{W}_3 \cup \mathcal{W}_4)\}. \tag{5.1.5}$$

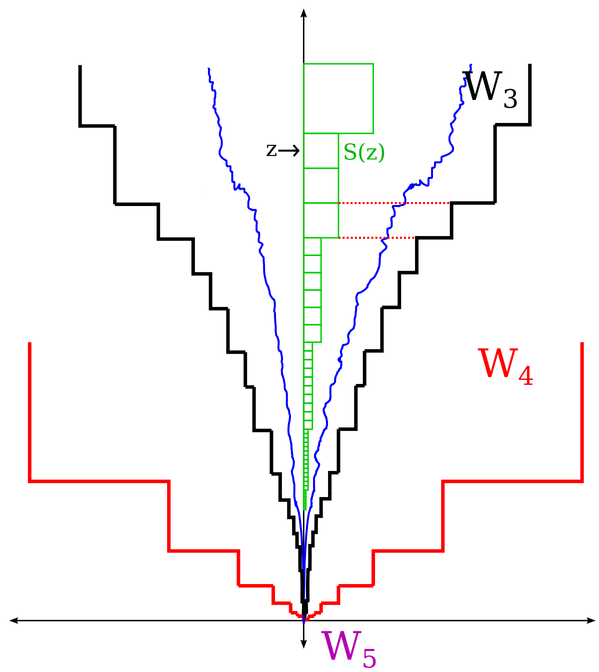


Figure 5.1: $\mathcal{W}_3$, $\mathcal{W}_4$ and $\mathcal{W}_5$ for a normal cusps.

In Figure 5.1 we show schematically the area covered by each set $\mathcal{W}_i$.

Furthermore, let us denote Q_j the cubes in $\mathcal{W}_3$, so: $\mathcal{W}_3 = \{Q_j\}_j$, and similarly, $\{T_j\}_j$ the cubes in $\mathcal{W}_4$, and $\{U_j\}_j$ those in $\mathcal{W}_5$. Finally, let $\{\xi_j\}_j$, $\{\phi_j\}_j$ and $\{\psi_j\}_j$ be a partition of the unity on $\cup\mathcal{W}_2$, such that $\xi_j, \phi_j, \psi_j \in C_0^\infty$; $\operatorname{sop}(\xi_j) \subset \frac{17}{16}Q_j$, $\operatorname{sop}(\phi_j) \subset \frac{17}{16}T_j$, $\operatorname{sop}(\psi_j) \subset \frac{17}{16}U_j$, and:

$$\sum_j \xi_j(x) + \sum_j \phi_j(x) + \sum_j \psi_j(x) = 1 \qquad \forall x \in \cup\mathcal{W}_2.$$

As usual, we may also assume that:

$$|D^\alpha \xi_j(x)| \le \frac{C}{\ell(Q_j)^{|\alpha|}}, \qquad |D^\alpha \phi_j(x)| \le \frac{C}{\ell(T_j)^{|\alpha|}}, \qquad |D^\alpha \psi_j(x)| \le \frac{C}{\ell(U_j)^{|\alpha|}}.$$

Observe that $\Omega \cup (\cup\mathcal{W}_3)$ is an expanded cusp, broader than Ω, but with the same kind of singularity. On the other hand $\Omega \cup (\cup\mathcal{W}_3) \cup (\cup\mathcal{W}_4)$ contains a cone with vertex at the origin, and therefore it is not a singular domain. Finally, the addition of the cubes in $\mathcal{W}_5$ completes a neighbourhood of the origin.

In each stage of the extension process we define the extension operator in one of these sets of cubes (in the first stage, in $\mathcal{W}_3$, etc.). Since the first stage does not solve the singularity of Ω, the weight only appears in the second stage.

Our construction is based on the ideas used in [Jones, 1981] for proving an extension theorem for uniform domains. Jones shows that for a uniform domain D, every cube Q in

$\mathcal{W}(\mathbb{R}^n \setminus \bar{D})$ near D has a reflected cube $Q^* \in \mathcal{W}(D)$ such that (a) $\ell(Q^*) \underset{C}{\sim} \ell(Q)$ and (b) $d(Q, Q^*) \leq C\ell(Q)$, for some universal constant C. These facts allows him to define the extension Λf on Q through a polynomial that approximates f on Q^*.

The difficulty for applying this technique to a normal cusp Ω is that it is only possible to define reflected cubes Q^* with the previous properties for cubes Q in a close neighbourhood of the domain. Specifically, we are able to do this just for the cubes in $\mathcal{W}_3$. On the contrary the cubes in $\mathcal{W}_4$ are bigger than any cube in $\mathcal{W}$ *near* them. Consequently, for each cube $Q \in \mathcal{W}_4$, we are forced to choose between two options: defining a reflected *set* $S(Q)$, that is not necessarily a cube, or finding a cube Q^* comparable to Q, but far from it. These two options lead to the two different versions of the extension operator, and to the two different weights, in items *(a)* and *(b)* of Theorem 5.1.1. Finally, the extension to the cubes in $\mathcal{W}_5$ is permorfed radially, and preserves the weight obtained for $\mathcal{W}_4$.

5.1.1 First stage

This stage follows closely the reflection method given in [Jones, 1981]. It is based on Lemma 5.1.5, where the existence of a reflected cube for every $Q \in \mathcal{W}_3$ is proved. We need to state a previous lemma:

Lemma 5.1.3. *Given Ω an external normal cusp, with parameters ε, K, there is a constant $\widetilde{K}$ (that could be taken $\widetilde{K} = \frac{K(K+1)}{2}$) such that, if $x \in \Omega$, and*

$$z_i - \frac{K-1}{2}\ell(S_i) \leq x_n \leq z_i + \frac{K+1}{2}\ell(S_i),$$

then, $x \in \widetilde{K}S_i$.

Proof. Let us take $j = i_{x_n}$. We suppose $j < i$ (the complementary case is analogous). Property (3.2.3) implies that $KS_j \ni x$. On the other hand $z_j \leq z_i + \frac{K+1}{2}\ell(S_i)$. But, since $\ell(S_j) \geq \ell(S_i)$, we have $\ell(S_j) = 2^N \ell(S_i)$ for some $N \in \mathbb{N}_0$. The largest size for S_j is obtained when the cubes in $\mathcal{S}$ grow exponentially between S_i and S_j. In that case:

$$z_j - z_i = \sum_{m=0}^{N-1} 2^m \ell(S_i) \leq \frac{K+1}{2}\ell(S_i),$$

and $2^N \leq \frac{K+1}{2}$, which leads us to conclude that

$$\ell(S_j) \leq \frac{K+1}{2}\ell(S_i).$$

But then $x \in K\frac{K+1}{2}S_i$, since $x \in KS_j$. □

Remark 5.1.4. *Proposition 3.2.2 implies that properties (3.2.4) and (3.2.5) hold for finite unions of sets Ω_i (and not only for $\Omega_i \cup \Omega_{i+1}$). Therefore we may apply both properties for $\widetilde{\Omega}_i \cup \widetilde{\Omega}_{i+1}$, where $\widetilde{\Omega}_i = \widetilde{K}S_i \cap \Omega$. On the other hand (3.2.4) implies that the curve given by item* (iii) *in Definition 3.2.1 is contained in a finite union of sets Ω_i (or in a universal dilation of S_i).*

Lemma 5.1.5. *For each $Q \in \mathcal{W}_3$ there is a cube $Q^* \in \mathcal{W}$ such that:*

$$\frac{1}{4}\ell(Q) \leq \ell(Q^*) \leq \ell(Q), \tag{5.1.6}$$

$$d(Q^*, Q) \leq C\ell(Q). \tag{5.1.7}$$

Proof. Let i be such that $z_Q \in [z_i, z_{i-1})$, and $x \in \Omega$ such that $d(Q, x) \leq 5\sqrt{n}\ell(Q)$. We may assume that $\frac{\varepsilon}{\sqrt{n}K} < \frac{1}{2}$. In this case, observe that

$$x_n \geq z_Q - 5\ell(Q) \geq z_i - 5\frac{\varepsilon(K-1)}{5\sqrt{n}K}\ell(S_i) \geq z_i - \frac{K-1}{2}\ell(S_i).$$

The right hand term of the equation is exactly the floor of the expanded cube KS_i. On the other hand:

$$\begin{aligned} x_n \leq z_{i-1} + 5\ell(Q) &\leq z_{i-1} + 5\frac{\varepsilon(K-1)}{5\sqrt{n}K}\ell(S_i) \\ &\leq z_i + \ell(S_i) + \frac{K-1}{2}\ell(S_i) = z_i + \frac{K+1}{2}\ell(S_i), \end{aligned}$$

and the right term is the roof of the expanded cube KS_i. Consequently, $x \in \widetilde{\Omega}_i$. Let $y \in \widetilde{\Omega}_i$ be such that $|x - y| = \frac{5\sqrt{n}}{\varepsilon}\ell(Q)$. Note that this is possible because:

$$|x - y| = \frac{5\sqrt{n}}{\varepsilon}\ell(Q) \leq \frac{K-1}{K}\ell(S_i) < \operatorname{diam}(\widetilde{\Omega}_i).$$

Let, then, γ be the curve given by properties (3.2.4) and (3.2.5). If $\xi \in \gamma$ is such that $|x - \xi|, |\xi - y| \geq \frac{|x-y|}{2}$, we have: $d_{\partial\Omega}(\xi) \geq \frac{\varepsilon}{4}|x - y| = \frac{5\sqrt{n}}{4}\ell(Q)$. If $S \in \mathcal{W}$, $S \ni \xi$, then:

$$4\sqrt{n}\ell(S) \geq d(S, \partial\Omega) \geq d(\xi, \partial\Omega) - \sqrt{n}\ell(S) \geq \frac{5\sqrt{n}}{4}\ell(Q) - \sqrt{n}\ell(S).$$

Therefore

$$\ell(S) \geq \frac{1}{4}\ell(Q).$$

Let us consider all the cubes $T \in \mathcal{W}$ satisfying $\ell(T) \geq \frac{1}{4}\ell(Q)$ and take Q^* to be the one that minimizes the distance to Q. Then $\ell(Q^*) \leq \ell(Q)$. On the other hand

$$d(Q^*, Q) \leq d(S, Q) \leq \frac{1}{\varepsilon}|x - y| + d(x, Q) \leq \Big(\frac{5\sqrt{n}}{\varepsilon^2} + 5\sqrt{n}\Big)\ell(Q).$$

This completes the proof of the lemma. □

Corollary 5.1.6. *If $Q_1, Q_2 \in \mathcal{W}_3$, $Q_1 \cap Q_2 \neq \emptyset$, then $d(Q_1^*, Q_2^*) \leq C\ell(Q_1)$.*

The following lemma is a simple corollary of Lemma 3.2.3:

Lemma 5.1.7. *Given $Q_1, Q_2 \in \mathcal{W}_3$, $Q_1 \cap Q_2 \neq \emptyset$, there is a constant $C = C(\varepsilon, n, K)$ and a chain of cubes $\mathcal{F}_{1,2} = \{V_1 := Q_1^*, V_2, \ldots, V_r := Q_2^*\} \subset \mathcal{W}$ such that $r \leq C$ and $\ell(V_i) \sim \ell(Q_1)$, $\forall i$.*

Lemmas 5.1.5 and 5.1.7 are the essential tools for proving Jones's extension theorem for uniform domains. Therefore, they are the key of the first stage of our extension process.

For each $Q_j \in \mathcal{W}_3$ let us define $P_{Q_j} = P(Q_j^*)$. The first term of the extension operator is:

$$\Lambda_1 f(x) = \chi_\Omega(x) f(x) + \sum_{Q_j \in \mathcal{W}_3} P_{Q_j}(x)\xi_j(x).$$

Thanks to Lemmas 5.1.5 and 5.1.7, and Corollary 5.1.6, this operator can be bounded using the same arguments that are employeed in [Jones, 1981, Chua, 1992] for the case of uniform domains.

Remark 5.1.8. *In this first stage, and in particular during the proof of the next lemma, we could invoke Corollary 2.4.3. However, in order to be consistent with the rest of the stages we show how to use Lemma 2.4.2 instead.*

Lemma 5.1.9. *If $Q \in \mathcal{W}_3$ far from $\mathcal{W}_4$ (i.e. $Q \in \mathcal{W}_3$ is surrounded by cubes in $\mathcal{W}_3$) , then:*

$$\|D^\alpha \Lambda_1 f\|_{L^p(Q)} \leq C\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup\mathcal{F}(Q))} + \|f\|_{W^{k,p}(Q^*)}\Big\},$$

where $\mathcal{F}(Q)$ is the set of all the cubes that participate in a chain $\mathcal{F}_j(Q)$, connecting Q^* with Q_j^*, for $Q_j \cap Q \neq \emptyset$.

Proof. We have

$$\|D^\alpha \Lambda_1 f\|_{L^p(Q)} = \Big\|D^\alpha \sum_{Q_j \cap Q \neq \emptyset} P_{Q_j}\xi_j\Big\|_{L^p(Q)} \leq \underbrace{\Big\|D^\alpha \sum_{Q_j \cap Q \neq \emptyset} (P_{Q_j} - P_Q)\xi_j\Big\|_{L^p(Q)}}_{I} + \underbrace{\Big\|D^\alpha P_Q\Big\|_{L^p(Q)}}_{II}.$$

The second term is easily bounded by means of Lemma 2.4.2, taking into account that Q and Q^* can be included inside an auxiliary cube $\tilde{Q}$, $Q \sim \tilde{Q} \sim Q^*$. Alternating the derivatives of f we get

$$\begin{aligned} II &\leq C \sum_{|\gamma+\alpha|<k} \ell(Q)^{|\gamma|}\|D^{\gamma+\alpha}P_Q\|_{L^p(Q^*)} \\ &\leq C \sum_{|\gamma+\alpha|<k} \ell(Q)^{|\gamma|}\Big\{\|D^{\gamma+\alpha}(P_Q - f)\|_{L^p(Q^*)} + \|D^{\gamma+\alpha} f\|_{L^p(Q^*)}\Big\} \leq \\ &\leq C\|\nabla^k f\|_{L^p(Q^*)}\ell(Q)^{k-|\alpha|} + \|f\|_{W^{k,p}(Q^*)} \leq C\|f\|_{W^{k,p}(Q^*)}. \end{aligned}$$

On the other hand:

$$\begin{aligned} I &\leq C \sum_{Q_j \cap Q \neq \emptyset} \sum_{\beta \leq \alpha} \|D^{\alpha-\beta}\xi_j D^\beta (P_{Q_j} - P_Q)\|_{L^p(Q)} \\ &\leq C \sum_{Q_j \cap Q \neq \emptyset} \sum_{\beta \leq \alpha} \frac{1}{\ell(Q)^{|\alpha-\beta|}} \|D^\beta (P_{Q_j} - P_Q)\|_{L^p(Q)}. \end{aligned}$$

For each j, let us alternate the polynomials associated to the cubes of the chain between Q_j^* and Q^*, given by Lemma 5.1.7. We set $\mathcal{F}_j = \{T_1 = Q^*, T_2, \dots, T_r = Q_j^*\}$ and obtain:

$$\begin{aligned} &\|D^\beta (P_{Q_j} - P_Q)\|_{L^p(Q)} \leq \sum_{i=1}^{r-1} \|D^\beta (P(T_{i+1}) - P(T_i))\|_{L^p(Q)} \\ &\leq \sum_{i=1}^{r-1} \left\{ \|D^\beta (P(T_{i+1}) - P(T_i \cup T_{i+1}))\|_{L^p(Q)} + \|D^\beta ((P(T_i \cup T_{i+1}) - P(T_i))\|_{L^p(Q)} \right\} \\ &\leq C \sum_{i=1}^{r-1} \left\{ \|D^\beta (P(T_{i+1}) - P(T_i \cup T_{i+1}))\|_{L^p(T_{i+1})} + \|D^\beta ((P(T_i \cup T_{i+1}) - P(T_i)\|_{L^p(T_i)} \right\} \\ &\leq C \sum_{i=1}^{r-1} \left\{ \|D^\beta (P(T_{i+1}) - f)\|_{L^p(T_{i+1})} + \|D^\beta (f - P(T_i \cup T_{i+1}))\|_{L^p(T_i \cup T_{i+1})} + \|D^\beta (f - P(T_i))\|_{L^p(T_i)} \right\} \\ &\leq C \sum_{i=1}^{r-1} \ell(Q)^{k-|\beta|} \|\nabla^k f\|_{L^p(T_i \cup T_{i+1})} \leq \ell(Q)^{k-|\beta|} \|\nabla^k f\|_{L^p(\cup \mathcal{F}_j)}. \end{aligned}$$

And then:

$$I \leq C\ell(Q)^{k-|\alpha|} \|\nabla^k f\|_{L^p(\cup \mathcal{F}(Q))}.$$

□

Finally, let us observe that from Lemmas 5.1.5 and 5.1.7 it follows that:

$$\Big\| \sum_{\substack{Q_l \in \mathcal{W}_3 \\ Q_l \cap Q_j \neq \emptyset}} \chi_{\cup \mathcal{F}_{jl}} \Big\|_\infty \leq C < \infty \qquad \text{for all } Q_j \in \mathcal{W}_3, \tag{5.1.8}$$

$$\Big\| \sum_{Q_j \in \mathcal{W}_3} \chi_{\cup \mathcal{F}(Q_j)} \Big\|_\infty \leq C < \infty \tag{5.1.9}$$

This means that each cube Q_j^* is used at most a fixed number of times, then

$$\|D^\alpha \Lambda_1 f\|^p_{L^p(\cup \mathcal{W}_3)} = \sum_{Q \in \mathcal{W}_3} \|D^\alpha \Lambda_1 f\|^p_{L^p(Q)} \leq C \|f\|^p_{W^{k,p}(\Omega)}.$$

Therefore

$$\|D^\alpha \Lambda_1 f\|_{L^p(\cup \mathcal{W}_3)} \leq C \|f\|_{W^{k,p}(\Omega)}. \tag{5.1.10}$$

hence, the operator (Λ_1) is bounded *far from* $\mathcal{W}_4$.

5.1.2 Second stage

This stage, where the extension operator is defined over $\mathcal{W}_4$, is the heart of the extension process. The first stage was essentially a translation of Jones' theorem, which extends functions to an *expanded cusp*, where no weight is needed. Second stage, on the other hand, extends functions to a *cone*: the cuspidal behaviour of Ω is compensated here by a weight. Stage three, in turn, completes the extension to a neighborhood of the origin, but does not contain any interesting idea.

Let us begin stating some properties of $\mathcal{W}_4$ itself. Let T be a cube in $\mathcal{W}_4$, and $S_i = S(z_T)$. Observe that from the definition of $\mathcal{W}_3$ we know that $\ell(T) > C\ell(S_i)$, with the constant $C = \frac{\varepsilon}{5\sqrt{n}}\frac{K-1}{K}$. In order to simplify notation in subsequent calculations we set $C = 1$ and assume that $\ell(T) \geq \ell(S_i)$.

Let $\widetilde{\mathcal{W}_4}$ denote the Whitney decomposition of $\mathbb{R}^n \setminus \hat{x}_n$. Observe that the structure of $\widetilde{\mathcal{W}_4}$ is very simple: cubes grow exponentially as we move away from the axis. Since the positive semiaxis of $\hat{x}_n$ is contained in Ω, Remark 2.2.2 implies that for every cube $T \in \mathcal{W}_4$, there is a cube $\widetilde{T} \in \widetilde{\mathcal{W}_4}$, such that $T \subset \widetilde{T}$. The following lemma proves that in fact $\ell(T) \sim \ell(\widetilde{T})$, $\forall T \in \mathcal{W}_4$.

Lemma 5.1.10. *There is a constant C such that $d(T, \hat{x}_n) \leq C\ell(T)$, for all $T \in \mathcal{W}_4$*

Proof. Let $x^* \in \partial\Omega$ be such that $d(T, \partial\Omega) = d(T, x^*)$. Let γ be the segment joining T and x^*, and $Q \in \mathcal{W}_3$, the nearest cube to T such that $Q \cap \gamma \neq \emptyset$. It is clear that

$$\sqrt{n}\ell(Q) \leq d(Q, \partial\Omega) \leq d(T, \partial\Omega) \leq 4\sqrt{n}\ell(T).$$

Let us denote $x^q \in \partial\Omega$ the point such that $d(Q, \partial\Omega) = d(Q, x^q)$. Then:

$$\begin{aligned} d(T, \hat{x}_n) &\leq d(T, Q) + \sqrt{n}\ell(Q) + d(Q, \hat{x}_n) \leq 4\sqrt{n}\ell(T) + 4n\ell(T) + d(Q, x^q) + d(x^q, \hat{x}_n) \\ &\leq C\ell(T) + d(x^q, \hat{x}_n) \leq C\ell(T) + \widetilde{K}\ell(S(z_Q)). \end{aligned}$$

Consequently, if $\ell(S(z_Q)) \leq C\ell(T)$, the result is proved.

Let us denote $I = i_{z_Q}$. Furthermore, let $T_1 \in \mathcal{W}_4$ be such that $T_1 \cap Q \neq \emptyset$ and $T_1 \cap \gamma \neq \emptyset$. Then, $\frac{1}{4}\ell(Q) \leq \ell(T_1) \leq 4\ell(Q)$. Suppose that $\ell(Q) < \frac{1}{16}\ell(S_I)$. Then

$$z_{T_1} \geq z_Q - \ell(T_1) \geq z_I - 4\ell(Q) > z_I - \frac{1}{4}\ell(S_I) \geq z_I - \ell(S_{I+1}) \geq z_{I+1}.$$

But, since $T_1 \in \mathcal{W}_4$,

$$\ell(Q) \geq \frac{1}{4}\ell(T_1) \geq \frac{1}{4}\ell(S(z_{T_1})) \geq \frac{1}{4}\ell(S_{I+1}) \geq \frac{1}{16}\ell(S_I).$$

which is a contradiction. Consequently, $\ell(T) \geq C\ell(Q) \geq C\ell(S_I)$, and the result follows. □

Remark 5.1.11. *A much simpler proof for this lemma can be provided assuming property* (5.1.2)*. However, item* (a) *in Theorem 5.1.1 can be proved without* (5.1.2)*, and so we prefer to detail the general proof.*

As we stated above, Lemma 5.1.10 shows that $\ell(T) \sim \ell(\widetilde{T})$, $\forall T \in \mathcal{W}_4$. This fact implies that the number of cubes of a certain size in $\mathcal{W}_4$ is comparable with the number of cubes of the same size in $\widetilde{\mathcal{W}}_4$. In some passages of this stage, we estimate the number of cubes in $\mathcal{W}_4$ by the number of cubes in $\widetilde{\mathcal{W}}_4$, which is easier to count.

In this second stage a weight is needed in order to bound the norm of the extension operator: we provide two different versions for the extension to the cubes in $\mathcal{W}_4$, the first one is *horizontal* (each cube will be associated with a set at the same height), leading to the weight $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{n-1}$ corresponding to item (b) in Theorem 5.1.1. Property (5.1.2) is needed in this case. The second version is *vertical* giving another possible weight: $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{kp}$ as in item (a) in Theorem 5.1.1. Property (5.1.2) is not needed for this version.

First version: dimensional-horizontal weight

For each cube $T_j \in \mathcal{W}_4$ let us define

$$S(T_j) = \bigcup \left\{ S_i : z_{T_j} \leq z_i < z_{T_j} + \ell(T_j) \right\}.$$

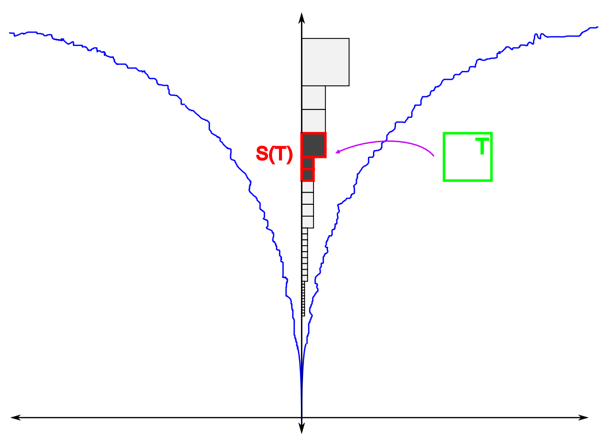


Figure 5.2: *Reflected* tower: second stage's first version.

Remark 5.1.12. *$S(T_j)$ is the* reflected *set of T_j as well as Q_j^* is the reflected cube for Q_j in the first stage. Observe that $S(T_j)$ is not a cube, nor a rectangle. However, normality property (3.2.1) implies that it is a* tower *of cubes, eventually of different sizes. Since cubes in $\mathcal{W}_4$ are far from Ω, T_j will be larger than the S_i's in $S(T_j)$. Nevertheless, the dyadic nature of cubes in Whitney decompositions implies that its height is exactly $\ell(T_j)$. Finally, if $S(T_j) = \{S_{I_j}, S_{I_j+1} \ldots, S_{I_j+N_j}\}$, property (5.1.2) guarantees that $\frac{\ell(S_{I_j})}{\ell(S_{I_j+N_j})} \leq C < \infty$. Therefore, for each T_j there is a pair of rectangles R_j^1 and R_j^2 such that:*

$$R_j^1 \subset S(T_j) \subset R_j^2,$$

$$\vec{\ell}(R_j^1) = (\ell(S_{I_j+N_j}), \dots, \ell(S_{I_j+N_j}), \ell(T_j)),$$
$$\vec{\ell}(R_j^2) = (\ell(S_{I_j}), \dots, \ell(S_{I_j}), \ell(T_j)),$$

satisfying: $\frac{\ell_i(R_j^2)}{\ell_i(R_j^1)} \le C$ *for all* T_j *and* $i = 1, \dots, n$*; i.e.:* $R_j^1 \sim R_j^2$.

Let us define, for each $T_j \in \mathcal{W}_4$, $P_{T_j} = P(R_j^1)$. Our extension operator is

$$\Lambda_2 f(x) = \sum_{T_j \in \mathcal{W}_4} P_{T_j}(x)\phi_j(x). \tag{5.1.11}$$

The following lemma is equivalent to Lemma 5.1.9. However, since $\mathcal{W}_4$ is far from Ω, a weight is needed:

Lemma 5.1.13. *If* $T \in \mathcal{W}_4$ *(far from* $\mathcal{W}_3$ *and* $\mathcal{W}_5$*), then:*

$$\|D^\alpha \Lambda_2 f\|_{L^p(T)} \le C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \|f\|_{W^{k,p}(\cup \mathcal{F}(T))},$$

where $\mathcal{F}(T)$ *is the family of all the towers* $S(T_j)$ *with* $T_j \cap T \neq \emptyset$*, and* R^1 *is the rectangle in* $S(T)$ *provided by Remark 5.1.12.*

Proof. As we procceded in Lemma 5.1.9, we alternate the polynomial corresponding to T, P_T:

$$\|D^\alpha \Lambda_2 f\|_{L^p(T)} = \Big\|D^\alpha \sum_{T_j \cap T \neq \emptyset} P_{T_j}\phi_j\Big\|_{L^p(T)} \le \underbrace{\Big\|D^\alpha \sum_{T_j \cap T \neq \emptyset} (P_{T_j} - P_T)\phi_j\Big\|_{L^p(T)}}_{I} + \underbrace{\Big\|D^\alpha P_T\Big\|_{L^p(T)}}_{II}.$$

Since $d(T, S(T)) \le C\ell(T)$, the second term can be bounded by means of Lemma 2.4.2, by considering an auxiliary cube $\tilde{T} \sim T$, such that $T, S(T) \subset \tilde{T}$:

$$II \le C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \sum_{\gamma : |\gamma + \alpha| < k} \ell(T)^{|\gamma|} \|D^{\alpha+\gamma} P_T\|_{L^p(S(T))}.$$

If we go on like in Lemma 5.1.9:

$$II \le C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \|f\|_{W^{k,p}(S(T))}.$$

On the other hand:

$$I \le C \sum_{T_j \cap T \neq \emptyset} \sum_{\beta \le \alpha} \frac{1}{\ell(T)^{|\alpha-\beta|}} \|D^\beta (P_{T_j} - P_T)\|_{L^p(T)},$$

and $T \cap T_j \neq \emptyset$, implies that $S(T) \cap S(T_j) \neq \emptyset$ and $R^1 \sim R_j^1$. In fact, $S(T) \subset S(T_j)$ or $S(T_j) \subset S(T)$ (which imply $R^1 \subset R_j^1$ or $R_j^1 \subset R^1$ resp.), or $S(T)$ and $S(T_j)$ form a new, longer tower where $S(T)$ is over $S(T_j)$, or vice versa (which implies that R^1 and R^1_j are touching). We show only the case that leads to touching rectangles (the other cases follow similarly):

$$\|D^\beta(P_{T_j} - P_T)\|_{L^p(T)} \leq \|D^\beta(P_{T_j} - P(R^1 \cup R^1_j)\|_{L^p(T)} + \|D^\beta(P_T - P(R^1 \cup R^1_j))\|_{L^p(T)}$$

$$\leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \sum_{\gamma:|\gamma+\beta|<k} \ell(T)^{|\gamma|}\Big\{\|D^{\beta+\gamma}(P_T - P(R^1 \cup R^1_j))\|_{L^p(R^1)} + \|D^{\beta+\gamma}(P_{T_j} - P(R^1 \cup R^1_j))\|_{L^p(R^1_j)}\Big\}$$

$$\leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \sum_{\gamma:|\gamma+\beta|<k} \ell(T)^{|\gamma|}\Big\{\|D^{\beta+\gamma}(P_T - f)\|_{L^p(R^1)} + \|D^{\gamma+\beta}(f - P(R^1 \cup R^1_j))\|_{L^p(R^1)} + \|D^{\beta+\gamma}(P_{T_j} - f)\|_{L^p(R^1_j)} + \|D^{\gamma+\beta}(f - P(R^1 \cup R^1_j))\|_{L^p(R^1_j)}\Big\}.$$

Applying Lemma 2.4.8 we obtain:

$$\|D^\beta(P_{T_j} - P_T)\|_{L^p(T)} \leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \sum_{\gamma:|\gamma+\beta|<k} \ell(T)^{|\gamma|}\ell(R^1)^{k-|\gamma|-|\beta|} \sum_{\tau:|\tau|=k} \|D^\tau f\|_{L^p(S(T)\cup S(T_j))}.$$

And, consequently:

$$\begin{aligned}
I &\leq C \sum_{T_j\cap T\neq\emptyset} \sum_{|\beta|\leq|\alpha|} \frac{1}{\ell(T)^{|\alpha-\beta|}} C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \sum_{\gamma:|\gamma+\beta|\leq k} \ell(T)^{|\gamma|}\ell(R^1)^{k-|\gamma|-|\beta|} \sum_{\tau:|\tau|=k} \|D^\tau f\|_{L^p(S(T)\cup S(T_j))} \\
&\leq C \sum_{T_j\cap T\neq\emptyset} \sum_{|\beta|\leq|\alpha|} \ell(T)^{|k|-|\alpha|} C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \sum_{\gamma:|\gamma+\beta|<k} \sum_{\tau:|\tau|=k} \|D^\tau f\|_{L^p(S(T)\cup S(T_j))} \\
&\leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \|f\|_{W^{k,p}(\mathcal{F}(T))}.
\end{aligned}$$

□

Lemma 5.1.9 bounds the norm of the extension operator in all the cubes in $\mathcal{W}_3$ *far from* $\mathcal{W}_4$. That is, in all cubes $Q \in \mathcal{W}_3$ such that all the neighbours of Q are in $\mathcal{W}_3$. Lemma 5.1.13 does the same thing for cubes in $\mathcal{W}_4$, *far from* $\mathcal{W}_3$. Let us consider now cubes in the frontier of these sets: let $Q \in \mathcal{W}_3$ and $T \in \mathcal{W}_4$ be such that $Q \cap T \neq \emptyset$. Notice that $\frac{1}{4} \leq \frac{\ell(Q)}{\ell(T)} \leq 4$. Furthermore:

$$4\sqrt{n}\ell(Q) \geq d(Q, \partial\Omega) \geq d(T, \partial\Omega) - \sqrt{n}\ell(Q) \geq \sqrt{n}\ell(T) - \sqrt{n}\ell(Q),$$

and then

$$\ell(T) \leq 5\ell(Q) \leq C\ell(S_I),$$

where S_I is the cube in the *spine* of Ω such that $z_Q \in [z_I, z_{I-1})$. This implies that $\ell(Q^*) \sim \ell(T)$, and since $S_I \cap S(T) \neq \emptyset$, by means of lemma 5.1.7, there is a chain of cubes joining Q^* and $S(T)$. Hence, the proof for the following lemma is the same that the one for Lemmas 5.1.9 and 5.1.13.

Lemma 5.1.14. *Let $Q \in \mathcal{W}_3$ and $T \in \mathcal{W}_4$ be such that $Q \cap T \neq \emptyset$, then:*

$$\|D^\alpha(\Lambda_1 + \Lambda_2)f\|_{L^p(Q)} \leq C\|f\|_{W^{k,p}(\mathcal{F}(Q))},$$

$$\|D^\alpha(\Lambda_1 + \Lambda_2)f\|_{L^p(T)} \leq C\|f\|_{W^{k,p}(\mathcal{F}(T))}.$$

We need to prove that the norm of the extension is bounded as in Lemma 5.1.13 all over $\mathcal{W}_4$ and not only in a particular cube. Let us pick, then, a cube $S_i \in \mathcal{S}$. A simple comparison with $\widetilde{\mathcal{W}_4}$ implies that the number of cubes T_j, with $\ell(T_j) = 2^m\ell(S_i)$, such that $S_i \subset S(T_j)$ is bounded by a constant independent of S_i. Furthermore, such a comparison allows us to bound the possible values of m, for each i: $0 \leq m \leq \log\left(\frac{z_i}{\ell(S_i)}\right)$, where $\log = \log_2$.

Proposition 5.1.15. *If we denote $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{n-1}$, then:*

$$\|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|_{L^p(\cup\mathcal{W}_4)} \leq C\|f\|_{W^{k,p}(\cup\mathcal{S})}$$

for every α, $|\alpha| \leq k$.

Proof. We can take σ as constant in each cube $T \in \mathcal{W}_4$: $\sigma_T \sim \left(\frac{\ell(S(z_T))}{z_T}\right)^{n-1}$. Then

$$\begin{aligned}\|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|^p_{L^p(\cup\mathcal{W}_4)} &= \sum_{T:T\in\mathcal{W}_4} \|\sigma^{\frac{1}{p}} D^\alpha f\|^p_{L^p(T)} \leq C \sum_{T:T\in\mathcal{W}_4} \left(\frac{\ell(S(z_T))}{z_T}\right)^{n-1} \|D^\alpha f\|^p_{L^p(T)} \\ &\leq C \sum_{T:T\in\mathcal{W}_4} \left(\frac{\ell(S(z_T))}{z_T}\right)^{n-1} \frac{|T|}{|R^1|} \|f\|^p_{W^{k,p}(\cup\mathcal{F}(T))}.\end{aligned}$$

Now, since $\ell_i(R^1) \sim \ell(S(z_T))$ for $i = 1, \ldots, n-1$, and $\ell_n(R^1) = \ell(T)$, we have:

$$\begin{aligned}\|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|^p_{L^p(\cup\mathcal{W}_4)} &\leq C \sum_{T:T\in\mathcal{W}_4} \left(\frac{\ell(T)}{z_T}\right)^{n-1} \|f\|^p_{W^{k,p}(\cup\mathcal{F}(T))} \\ &= C \sum_{T:T\in\mathcal{W}_4} \sum_{S:S\in\mathcal{F}(T)} \left(\frac{\ell(T)}{z_T}\right)^{n-1} \|f\|^p_{W^{k,p}(S)} \\ &= C \sum_{S:S\in\mathcal{S}} \sum_{T:\mathcal{F}(T)\ni S} \left(\frac{\ell(T)}{z_T}\right)^{n-1} \|f\|^p_{W^{k,p}(S)}.\end{aligned}$$

Given a fixed cube $S \in \mathcal{S}$, the cubes $T \in \mathcal{W}_4$ such that $S \in \mathcal{F}(T)$, can be classified by their sizes: $\ell(T) = 2^m\ell(S)$, where $0 \leq m \leq M = \log(\frac{z_S}{\ell(S)})$. Furthermore, $z_T \sim z_S$ for every $T \in \mathcal{W}_4$ such that $S \in \mathcal{F}(T)$. Finally, the comparison between cubes in $\mathcal{W}_4$ and cubes in $\widetilde{\mathcal{W}_4}$, guarantees that, given a cube $S \in \mathcal{S}$, there is a bound C, depending only on the dimension n, such that

$$\#\{T \in \mathcal{W}_4 : \ \ S \in \mathcal{F}(T) \ \ \ell(T) = 2^m\ell(S)\} \leq C.$$

Then:

$$\|\sigma^{\frac{1}{p}} D^\alpha f\|^p_{L^p(\cup \mathcal{W}_4)} \leq C \sum_{S:S\in\mathcal{S}} \sum_{m=1}^{M-1} \sum_{\substack{T:\mathcal{F}(T)\ni S\\ \ell(T)=2^m\ell(S)}} \left(\frac{\ell(T)}{z_T}\right)^{n-1} \|f\|^p_{W^{k,p}(S)}$$
$$\leq C \sum_{S:S\in\mathcal{S}} z_S^{1-n}\ell(S)^{n-1}\Big(\sum_{m=1}^{M-1}(2^{n-1})^m\Big)\|f\|^p_{W^{k,p}(S)},$$

and the result follows by recalling that $M \approx \log(\frac{z_S}{\ell(S)})$. □

This result concludes the first version of the second stage of the extension.

Second version: derivative-vertical weight

This version of the extension is based on a different construction of the *reflected* set of a cube in $\mathcal{W}_4$. For each T, we find some $T^* \in \mathcal{S}$ such that $\ell(T^*) \sim \ell(T)$, but T^* is *far* above T. The weight in this case is due to the distance between T and T^*.

Let us consider $\tilde{T}$ a cube belonging to $\widetilde{\mathcal{W}_4}$ (the Whitney decomposition of $\mathbb{R}^n \setminus \hat{x}_n$) such that $\tilde{T} \cap T \neq \emptyset$, for some $T \in \mathcal{W}_4$. Thanks to Lemma 5.1.10 only a finite number (the number does not depend on $\tilde{T}$) of cubes belonging to $\mathcal{W}_4$ are contained in $\tilde{T}$. We can now pack the elements of $\mathcal{W}_4$ in cylinders of the form $\eta(\tilde{T}) = Q' \times \mathbb{R}$, where $Q' \subset \mathbb{R}^{n-1}$ is the projected face $F^u_{\tilde{T}}$ of $\tilde{T}$ into $\mathbb{R}^{n-1}$. We identify cylinders given by cubes $\tilde{T}$ sharing the projection Q'. In this way each cube $T \in \mathcal{W}_4$ belongs to only one cylinder. Moreover, cubes inside the cylinder $\eta(\tilde{T})$ are equivalent, i.e $T_1, T_2 \in \eta(\tilde{T})$ implies that $T_1 \sim \tilde{T} \sim T_2$. For each $T_j \in \mathcal{W}_4$, we denote with $\tau(T_j)$ the set of cubes in $\mathcal{W}_4$ that share the cylinder with T_j. The set τ is called a tower.

Let us consider T^1 one of the upper cubes in $\tau(T^1)$. We define $T^* = S(z_{T^1})$ for every $T \in \tau(T^1)$. This situation is represented in Figure 5.3.

It is important that, with this definition, for every $T \in \mathcal{W}_4$ we have $T^* \in \mathcal{S}$, and $T^* \sim T$. However, the distance between T and T^* could be *large*, particularly in the x_n direction. In fact, since $d(T^1, T^N) \sim z_{T^1}$, we have $d(T^1, T^{1*}) \sim \ell(T^1)$, but $d(T^N, T^{N*}) \sim z_{T^1}$ (where T^1 and T^N are upper and lower cubes in a certain tower τ).

As in the first version of the second stage extension, we define $P_{T_j} = P(T_j^*)$ and

$$\Lambda_2 f(x) = \sum_{T_j\in\mathcal{W}_4} P_{T_j}(x)\phi_j(x).$$

Observe that, if $T_j, T \in \mathcal{W}_4$, and $T_j \cap T \neq \emptyset$, the tops of the towers $\tau(T)$ and $\tau(T_j)$ could be setted at very different heights (and so would be the heights of the reflected cubes T_j^* and T^*). This is so because of the following fact:

Remark 5.1.16. *Suppose $S \in \mathcal{S}$ is the higher cube of a certain size. Let us denote $\#(S)$ the number of cubes with edges of lengh* exactly *$\ell(S)$. Then, since $\mathbf{0} \in \partial\Omega$, $z_S - \#(S)\ell(S) > 0$, and consequently, $\#(S) \leq \frac{z_S}{\ell(S)}$. However, no better estimate can be provided (in fact, it is easy*

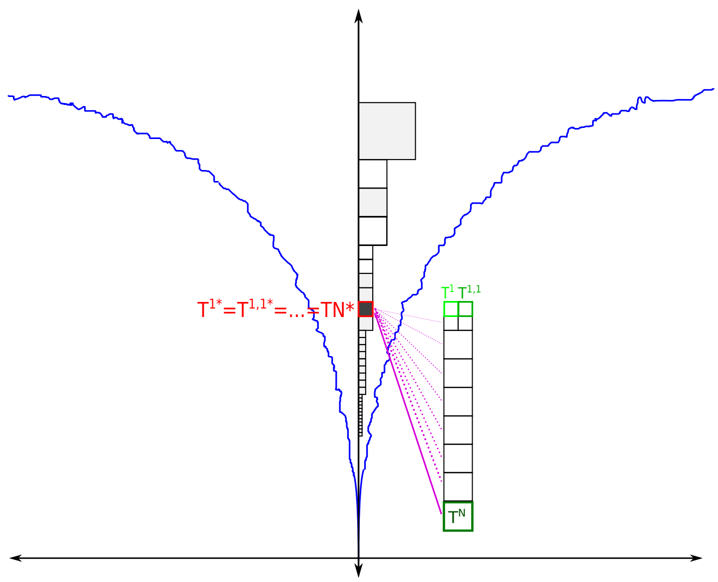


Figure 5.3: *Reflected* cubes: second stage's second version.

to see that for cusps with profile $\varphi(z) = z^{\nu}$, there are $\sim z^{1-\nu}$ cubes with edges z^{ν}), so the worst case, that there could be $\sim \frac{z_S}{\ell(S)}$ cubes in $\mathcal{S}$ with side $\ell(S)$, should be assumed to hold.

Consequently, the shape of $\mathcal{W}_4$ could show *long steps*. When two towers touching each other are in the edge of a long step, their heights are very different. This situation is represented in Figure 5.4. In this figure, two touching towers are shown, where reflected cubes are far from each other. Therefore, the chain (in $\mathcal{S}$) joining the reflected cubes for each tower is large.

Lemma 5.1.17. *For every cube $T \in \mathcal{W}_4$:*

$$\|D^{\alpha}\Lambda_2 f\|_{L^p(T)} \leq C\ell(T)^{k-|\alpha|}\left(\frac{\ell_n(\tau(T))}{\ell(T)}\right)^{k-\frac{1}{p}}\|f\|_{W^{k,p}(\cup\mathcal{F}(T))},$$

where $\mathcal{F}(T)$ is the family of all the cubes in all the chains connecting T^* and T_j^* for $T_j \cap T \neq \emptyset$.

Proof.

$$\|D^{\alpha}\Lambda_2 f\|_{L^p(T)} \leq \underbrace{\Big\|D^{\alpha}\sum_{T_j\cap T\neq\emptyset}(P_{T_j} - P_T)\phi_j\Big\|_{L^p(T)}}_{I} + \underbrace{\Big\|D^{\alpha}P_T\Big\|_{L^p(T)}}_{II}.$$

As usual:

$$I \leq C\sum_{\beta\leq\alpha}\frac{1}{\ell(T)^{|\alpha|-|\beta|}}\sum_{T_j\cap T\neq\emptyset}\|D^{\beta}(P_{T_j} - P_T)\|_{L^p(T)}.$$

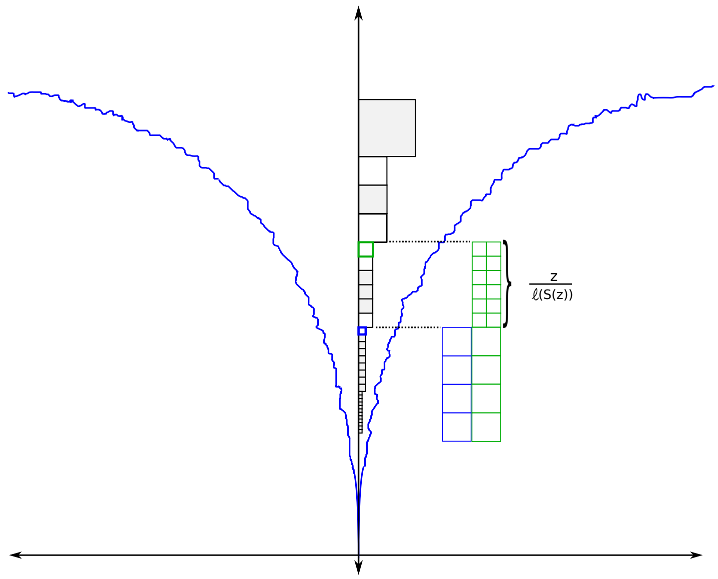


Figure 5.4: *Long steps* imply long chains.

Let us denote $\mathcal{F}_j(T) = \{T^* = S^1, S^2 \dots, T_j^* = S^M\}$ the chain of cubes joining T^* and T_j^*, then:

$$\|D^\beta(P_{T_j} - P_T)\|_{L^p(T)} \leq \sum_{l=1}^{M-1} \|D^\beta(P(S^{l+1}) - P(S^l))\|_{L^p(T)}.$$

Now, if we denote R_l the minimal rectangle containing T and S^l, we have $\ell_i(R_l) \sim \ell(T)$, $i = 1, \dots, n-1$, and $\ell_n(R) \leq \ell_n(\tau(T))$.

$$\|D^\beta(P(S^{l+1}) - P(S^l))\|_{L^p(T)} \leq C\Big\{\|D^\beta(P(S^{l+1}) - P(S^{l+1} \cup S^l))\|_{L^p(T)} + \\ \|P(S^{l+1} \cup S^l) - P(S^l))\|_{L^p(T)}\Big\}$$

$$\leq C|T|^{\frac{1}{p}}\Big\{\|D^\beta(P(S^{l+1}) - P(S^{l+1} \cup S^l))\|_{L^\infty(R_{l+1})} \\ + \|D^\beta(P(S^{l+1} \cup S^l) - P(S^l))\|_{L^\infty(R_l)}\Big\}$$

$$\leq C|T|^{\frac{1}{p}} \sum_{|\gamma+\beta|<k} \Big\{\frac{\ell(R^l)^{|\gamma|}}{|S^{l+1}|^{\frac{1}{p}}}\|D^{\beta+\gamma}(P(S^{l+1}) - P(S^{l+1} \cup S^l)\|_{L^p(S^{l+1})} \\ + \frac{\ell(R^l)^{|\gamma|}}{|S^l|^{\frac{1}{p}}}\|D^{\beta+\gamma}(P(S^l) - P(S^{l+1} \cup S^l)\|_{L^p(S^l)}\Big\}$$

$$\leq C \sum_{|\gamma+\beta|<k} \ell_n(\tau(T))^{|\gamma|} \ell(S^l)^{k-|\beta|-|\gamma|} \|\nabla^k f\|_{L^p(S^l \cup S^{l+1})}.$$

Consequently:

$$\begin{aligned} I &\leq C \sum_{T_j \cap T \neq \emptyset} \sum_{\beta \leq \alpha} \frac{1}{\ell(T)^{|\alpha|-|\beta|}} \sum_{l=1}^{M-1} \sum_{|\gamma+\beta|<k} \ell_n(\tau(T))^{|\gamma|} \ell(S^l)^{k-|\beta|-|\gamma|} \|\nabla^k f\|_{L^p(S^l \cup S^{l+1})} \\ &\leq C \ell(T)^{k-|\alpha|} \Big(\frac{\ell_n(\tau(T))}{\ell(T)} \Big)^{k-1} \sum_{l=1}^{M-1} \|\nabla^k f\|_{L^p(S^l \cup S^{l+1})}. \end{aligned}$$

Applying the Hölder inequality gives

$$I \leq C \ell(T)^{k-|\alpha|} \Big(\frac{\ell_n(\tau(T))}{\ell(T)} \Big)^{k-1} M^{\frac{1}{p'}} \|\nabla^k f\|_{L^p(\cup \mathcal{F}(T))},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. But M is the number of cubes in the chain joining T^* and T_j^*, which we saw that could be as large as $\frac{\ell_n(\tau(T))}{\ell(T)}$, and then:

$$I \leq C \ell(T)^{k-|\alpha|} \Big(\frac{\ell_n(\tau(T))}{\ell(T)} \Big)^{k-\frac{1}{p}} \|\nabla^k f\|_{L^p(\cup \mathcal{F}(T))}.$$

II could be bounded by means of the same ideas.

□

A proposition equivalent to 5.1.15 can now be easily proved:

Proposition 5.1.18. *If we denote $\sigma(x) = \Big(\frac{\ell(S(|x|))}{|x|}\Big)^{kp}$, then:*

$$\|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|_{L^p(\cup \mathcal{W}_4)} \leq C \|f\|_{W^{k,p}(\cup \mathcal{S})}.$$

Proof. As we did in Proposition 5.1.15, let us observe that the weight σ could be considered constant in each cube $T \in \mathcal{W}_4$, $\sigma_T \sim \Big(\frac{\ell(S(z_T))}{z_T}\Big)^{kp}$. Then:

$$\begin{aligned} \|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|^p_{L^p(\cup \mathcal{W}_4)} &= \sum_{T \in \mathcal{W}_4} \|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|^p_{L^p(T)} \leq C \sum_{T \in \mathcal{W}_4} \Big(\frac{\ell(S(z_T))}{z_T} \Big)^{kp} \|D^\alpha \Lambda_2 f\|^p_{L^p(T)} \\ &\leq C \sum_{T \in \mathcal{W}_4} \Big(\frac{\ell(S(z_T))}{z_T} \Big)^{kp} \ell(T)^{(k-|\alpha|)p} \Big(\frac{\ell_n(\tau(T))}{\ell(T)} \Big)^{kp-1} \|f\|^p_{W^{k,p}(\cup \mathcal{F}(T))} \\ &= C \sum_{S : S \in \mathcal{S}} \sum_{T : \mathcal{F}(T) \ni S} \Big(\frac{\ell(S(z_T))}{z_T} \Big)^{kp} \ell(T)^{(k-|\alpha|)p} \Big(\frac{\ell_n(\tau(T))}{\ell(T)} \Big)^{kp-1} \|f\|^p_{W^{k,p}(S)}. \end{aligned}$$

Now, observe that if we fix a cube $S \in \mathcal{S}$, every cube $T \in \mathcal{W}_4$ such that $S \in \mathcal{F}(T)$ satisfies: $\ell(T) \sim \ell(S)$ and $\ell_n(\tau(T)) \leq z_S$. By considering this and $|\alpha| \leq k$, we obtain:

$$\|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|^p_{L^p(\cup W_4)} \le C \sum_{S:S\in\mathcal{S}} \sum_{T:\mathcal{F}(T)\ni S} \left(\frac{\ell(S(z_T))}{z_T}\right)^{kp} \left(\frac{z_S}{\ell(T)}\right)^{kp-1} \|f\|^p_{W^{k,p}(S)} = \circledast.$$

The rest of the proof is done in two separate cases. If $kp = 1$, using property (5.1.1) we obtain:

$$\circledast = C \sum_{S:S\in\mathcal{S}} \sum_{T:\mathcal{F}(T)\ni S} \frac{\ell(S(z_T))}{z_S} \|f\|^p_{W^{k,p}(S)} \le C \sum_{S:S\in\mathcal{S}} \sum_{T:\mathcal{F}(T)\ni S} \frac{\ell(S)}{z_S} \|f\|^p_{W^{k,p}(S)}.$$

But, for a fixed $S \in \mathcal{S}$, the number of cubes T such that $S \in \mathcal{F}(T)$ is at most $C\frac{z_S}{\ell(S)}$ and then:

$$\le C \sum_{S:S\in\mathcal{S}} \frac{z_S}{\ell(S)} \frac{\ell(S)}{z_S} \|f\|^p_{W^{k,p}(S)} \le C \sum_{S:S\in\mathcal{S}} \|f\|^p_{W^{k,p}(S)} = C\|f\|^p_{W^{k,p}(\cup\mathcal{S})}.$$

On the other hand, if $kp \ne 1$, property (5.1.1) can be avoided. Proceeding in a similar way than we did in Proposition 5.1.15, we clasify the cubes $T \in \mathcal{W}_4$ such that $S \in \mathcal{F}(T)$ according to their heights z_T. Observe that the minimum possible z_T is $\ell(S)$, whereas the maximum possible z_T is $z_S/\ell(S)$. Then:

$$\circledast \le C \sum_{S:S\in\mathcal{S}} \sum_{T:\mathcal{F}(T)\ni S} z_T^{-kp} z_S^{kp-1} \ell(S) \|f\|^p_{W^{k,p}(S)} = C \sum_{S:S\in\mathcal{S}} \left(\sum_{m=1}^{z_S/\ell(S)} \sum_{\substack{T:\mathcal{F}(T)\ni S \\ z_T = m\ell(S)}} z_T^{-kp} \right) z_S^{kp-1} \ell(S) \|f\|^p_{W^{k,p}(S)}.$$

But the number of cubes T at the same height is bounded by a constant depending only on the dimension n (because of the comparison with cubes in $\widetilde{\mathcal{W}_4}$), then:

$$\begin{aligned}
&= C \sum_{S:S\in\mathcal{S}} \left(\sum_{m=1}^{z_S/\ell(S)} m^{-kp} \right) z_S^{kp-1} \ell(S)^{-kp+1} \|f\|^p_{W^{k,p}(S)} \\
&\sim C \sum_{S:S\in\mathcal{S}} \left(\frac{z_S}{\ell(S)}\right)^{-kp+1} z_S^{kp-1} \ell(S)^{-kp+1} \|f\|^p_{W^{k,p}(S)} \\
&= C \sum_{S:S\in\mathcal{S}} \|\nabla^k f\|^p_{W^{k,p}(S)} \le C\|f\|^p_{W^{k,p}(\cup\mathcal{S})}.
\end{aligned}$$

The key point of this approach is the estimation of $\sum m^{-kp}$, where we use that $kp \ne 1$. □

5.1.3 Third Stage

This stage is devoted to define our extension operator in the cubes of $\mathcal{W}_5$. We explain the construction of the reflected sets for each version of the extension, but we do not enter into details since the ideas are exactly the same given in Lemmas 5.1.13 and 5.1.17 and Propositions 5.1.15 and 5.1.18, according to the case.

For the first (dimensional) version of the extension, let us define:

$$S(U) = \bigcup \{S_i : \ell(U) \le z_i < 2\ell(U)\}.$$

It is clear that $d(U, S(U)) \le C\ell(U)$. On the other hand, $S(U)$ is a tower of cubes that admits an interior rectangle R^1, with $\ell_i(R^1) \sim \ell(S(\ell(U)))$ for $i = 1, \dots, n-1$ and $\ell_n(R^1) \sim \ell(U)$. Because of property (5.1.2), there is an exterior rectangle $R^2 \supset S(U)$ such that $R^2 \sim R^1$. Hence, Remark 5.1.12 holds for cubes in $\mathcal{W}_5$, and so do Lemma 5.1.13 and Proposition 5.1.15. As we did earlier, we define $P_{U_j} = P(R^1_j)$. The last thing to notice is that if $T \in \mathcal{W}_4$ and $U \in \mathcal{W}_5$ are such that $T \cap U \neq \emptyset$, then $d(S(U), S(T)) \le C\ell(T)$, and then there is a finite chain of towers that join $S(U)$ and $S(T)$. This guarantees that the transition between $\mathcal{W}_4$ and $\mathcal{W}_5$ is smooth.

For the second (derivative) version, let us define $U^* = S_i$ the cube in $\mathcal{S}$ with i the maximum index such that $\ell(S_i) \ge \ell(U)$. This implies $\ell(U^*) \sim \ell(U)$, which is the essential property of the reflected cube in this case. On the other hand, $d(U, U^*) \le C z_{U^*}$. Once again, we define $P_{U_j} = P(U^*_j)$. It is clear that if $T \in \mathcal{W}_4$ and $U \in \mathcal{W}_5$ are such that $T \cap U \neq \emptyset$ and $U^* \sim T^*$, then $d(U^*, T^*) \le C z_{T^*}$, and so the norm of the extension can be bounded in the frontier between $\mathcal{W}_4$ and $\mathcal{W}_5$ as we did in $\mathcal{W}_4$.

As we did in the previous sections, let us define, for both versions:

$$\Lambda_3 f(x) = \sum_{U_j \in \mathcal{W}_5} P_{U_j}(x)\psi_j(x).$$

The last matter that we need to deal with is the superposition induced by this definitions of reflected sets. In the previous stage we introduced $\widetilde{\mathcal{W}_4}$ in order to help us *counting* some sets of cubes in $\mathcal{W}_4$. Similarly, let us introduce now $\widetilde{\mathcal{W}_5} = \mathcal{W}(\mathbb{R}^n \setminus \{\mathbf{0}\})$. Thanks to Remark 2.2.2 we may define, for every $U \in \mathcal{W}_5$, $\widetilde{U}$ the cube in $\widetilde{\mathcal{W}_5}$ such that $U \subset \widetilde{U}$. On the other hand the ideas exposed earlier (see Lemma 5.1.10) lead us to conclude that $U \sim \widetilde{U}$. The number of cubes in $\widetilde{\mathcal{W}_5}$ with edges of a certain length 2^{-l} are bounded by a constant depending only on the dimension n. The same holds for cubes in $\mathcal{W}_5$. Consequently, after this third stage, every cube in $\mathcal{S}$ is loaded with at most a bounded quantity of cubes of the exterior of Ω.

Our complete extension operator is, then,

$$\Lambda f(x) = \Lambda_1 f(x) + \Lambda_2 f(x) + \Lambda_3 f(x).$$

For every $x = (x', x_n) \in \mathcal{W}_4$, $x_n \sim |x|$. We use this fact to write the weight in terms of $|x|$ instead of x_n. Since the third term of the extension is also radial, the weight can be taken

$$\left(\frac{\ell(S(|x|))}{|x|}\right)^{\gamma}$$

for every $x \in \cup\mathcal{W}_2$, where γ is the exponent corresponding to the case.

5.1.4 $D^\alpha \Lambda f$ is in $W^{1,\infty}_{loc}$

Observe that the proof of Lemma 2.3.7 can be easily applied to prove that $|\partial\Omega| = 0$ for every normal (or curved) cusp Ω. Consequently Λf is defined almost everywhere in a neighborhood of the origin. In order to complete the proof of items (a) and (b) in Theorem 5.1.1, we need to prove that Λf has weak derivatives of all orders α, $0 \le |\alpha| \le k$ everywhere except, perhaps, at the origin. Hence, we only need to consider the first term of the extension operator: $\Lambda_1 f$. Let us denote, then, $\widehat{\Omega} = \Omega \cup (\cup \mathcal{W}_3)$, the expanded cusp covered by the first stage of the extension process. It is enough to prove that $D^\alpha \Lambda_1 f \in W^{1,\infty}_{loc}(\widehat{\Omega})$. We proceed following a density argument: taking into account Theorem 4.1.7, we may assume that f is the restriction to Ω of a function in $C^\infty(\mathbb{R}^n_+)$. Moreover, if we take $\eta > 0$ we have: $\|D^\alpha f\|_{L^\infty(\widehat{\Omega}\setminus B(\mathbf{0},\eta))} \le M$, $0 \le |\alpha| \le k$. Recall that a function is in $W^{1,\infty}_{loc}$ if and only if it is locally Lipschitz[1]. Therefore, in order to guarranty the existence of weak derivatives of $\Lambda_1 f$ we only need to prove that $D^\alpha \Lambda_1 f$ is Lipschitz, for every α, $0 \le |\alpha| \le k-1$. Our proof is esentially the same that the one provided by [Jones, 1981] and [Chua, 1992] for proving the same thing for uniform domains:

Proposition 5.1.19. *$D^\alpha \Lambda_1 f$ is locally Lipschitz, for $|\alpha| < k$.*

Proof. As we stated earlier, we assume $f \in C^\infty(\mathbb{R}^n_+)$, and $\|D^\beta f\|_{L^\infty(\widehat{\Omega}\setminus B(\mathbf{0},\eta))} \le M, \forall |\beta| \le k$. We begin proving that $D^\alpha \Lambda_1 f$ is continuous. It is clear that we only need to prove the continuity in $\partial\Omega$. Let us define, for every $Q_j \in \mathcal{W}_3$:

$$f_j = \frac{1}{|Q_j^*|} \int_{Q_j^*} D^\alpha f$$

Given $x \in \partial\Omega \setminus B(\mathbf{0}, 2\eta)$, we show that:

$$\|D^\alpha \Lambda_1 f - f_j\|_{L^\infty(Q_j)} \longrightarrow 0, \quad \text{as } \; Q_j \longrightarrow x.$$

Alternating $P_j = P(Q_j^*)$, we obtain:

$$\begin{aligned}
\|D^\alpha \Lambda_1 f - f_j\|_{L^\infty(Q_j)} &\le \|D^\alpha P_j - f_j\|_{L^\infty(Q_j)} + \|D^\alpha(\Lambda_1 f - P_j)\|_{L^\infty(Q_j)} \\
&= \underbrace{\|D^\alpha P_j - f_j\|_{L^\infty(Q_j)}}_{I} + \underbrace{\|D^\alpha \sum_k (P_k - P_j)\xi_k\|_{L^\infty(Q_j)}}_{II}.
\end{aligned}$$

where the last summation involves all the cubes $Q_k \in \mathcal{W}_3$ such that $\bar{Q}_k \cap \bar{Q}_j \neq \emptyset$. Now, for I:

$$\begin{aligned}
I \le C\|D^\alpha P_j - f_j\|_{L^\infty(Q_j^*)} &\le C\{\|D^\alpha(P_j - f)\|_{L^\infty(Q_j^*)} + \|D^\alpha f - f_j\|_{L^\infty(Q_j^*)}\} \\
&\le C\{\ell(Q_j)^{k-|\alpha|}\|\nabla^k f\|_{L^\infty(Q_j^*)} + \ell(Q_j)\|\nabla D^\alpha f\|_{L^\infty(Q_j^*)}\} \\
&\le CM\ell(Q_j) \longrightarrow 0.
\end{aligned}$$

[1] See [Evans and Gariepy, 1992, Theorem 4.2.3].

On the other hand, II was estimated in Lemma 5.1.9, so:

$$II \leq C\ell(Q_j)^{k-|\alpha|}\|\nabla^k f\|_{L^\infty(\cup\mathcal{F}(Q_j))} \longrightarrow 0.$$

Observe that we are assuming that Q_j is small enough so $\mathcal{F}(Q_j) \subset \Omega \setminus B(\mathbf{0}, \eta)$.

Thence: $D^\alpha\Lambda_1 f$ is continuous. Now, we want to deduce from this fact the local Lipschitzianity of $D^\alpha\Lambda_1 f$, far from the origin. In other words, if $\mathcal{K}$ is a compact set such that $\mathbf{0} \notin \mathcal{K}$, and $x, y \in \mathcal{K}$, we need to show:

$$|D^\alpha\Lambda_1 f(x) - D^\alpha\Lambda_1 f(y)| \leq C_{\mathcal{K}}|x - y|. \tag{5.1.12}$$

But this fact follows inmediatly, since $D^\beta\Lambda_1 f$ is continuous and bounded on $\widehat{\Omega} \setminus B(\mathbf{0}, \eta)$ for every $|\beta| \leq k$, where η is chosen such that $\mathcal{K} \subset \widehat{\Omega} \setminus B(\mathbf{0}, \eta)$. □

This implies that $D^\alpha\Lambda_1 f$ are in fact the weak derivatives of $\Lambda_1 f$, when $f \in C^\infty(\mathbb{R}^n_+)$. The result follows for $f \in W^{k,p}(\Omega)$ by means of a simple density argument, taking into account that Λ_1 is an unweighted extension.

This completes the proof of Theorem 5.1.1, except item *(c)*, that is proved below.

5.1.5 Optimality of the weights

We prove item *(c)* in Theorem 5.1.1 by showing two examples of functions f that need a weight at least as small as σ. These examples are exactly the ones proposed by Maz'ya and Poborchiĭ for proving item *(c)* in Theorem A (see [Maz'ya and Poborchiĭ, 1997, Theorem 5.2 and Theorem 5.4]), we state them for the sake of completeness.

Derivative weight

For this case we assume $kp < n - 1$.

Take $g \in C_0^\infty(0, 3)$ such that $g(t) = 1$, $\forall t \in (1, 2)$. Let $\rho > 0$ be a small number, and consider:

$$f_\rho(x) = g\left(\frac{x_n}{\rho}\right).$$

Clearly, $f_\rho \in W^{k,p}(\Omega)$. Moreover:

$$\|D^\alpha f_\rho\|_{L^p(\Omega)} \leq C\frac{1}{\rho^{|\alpha|}}\rho^{\frac{1}{p}}\ell(S(3\rho))^{\frac{n-1}{p}}. \tag{5.1.13}$$

Now, let us suppose that there is an extension operator $\widetilde{\Lambda} : W^{k,p} \longrightarrow W^{k,p}_{\widetilde{\sigma}}(\mathbb{R}^n)$, with $\widetilde{\sigma}(x) = \widetilde{\sigma}(|x|)$ nondecreasing. Then:

$$\|f_\rho\|^p_{W^{k,p}(\Omega)} \geq C\|\widetilde{\sigma}^{\frac{1}{p}}D^\alpha\widetilde{\Lambda}f_\rho\|^p_{L^p(\mathbb{R}^n)} \geq C\widetilde{\sigma}(\rho)\int_\rho^{2\rho}\|D^\alpha\widetilde{\Lambda}f_\rho(\cdot, x_n)\|^p_{L^p(\mathbb{R}^{n-1})}dx_n = \circledast.$$

Now we apply the imbedding $W^{k,p}(R^{n-1}) \subset L^q(\mathbb{R}^{n-1})$, for $q = \frac{(n-1)p}{n-1-kp}$[2]. In this way, taking $\Omega_z = \{x \in \Omega : \ x_n = z\}$:

$$\circledast \geq C\widetilde{\sigma}(\rho) \int_\rho^{2\rho} \|f_\rho(\cdot, x_n)\|^p_{L^q(\Omega_{x_n})} dx_n \sim C\widetilde{\sigma}(\rho)\rho\ell(S(\rho))^{\frac{p}{q}} = C\widetilde{\sigma}(\rho)\rho\ell(S(\rho))^{n-1-kp}.$$

And then, considering the worst case in (5.1.13), in which $|\alpha| = k$, we have:

$$C_1\widetilde{\sigma}(\rho)^{\frac{1}{p}}\rho^{\frac{1}{p}}\ell(S(\rho))^{\frac{n-1-kp}{p}} \leq \|f_\rho\|_{W^{k,p}(\Omega)} \leq C_2\rho^{\frac{1}{p}-k}\ell(S(3\rho))^{\frac{n-1}{p}}.$$

Finally, thanks to Property (5.1.2), $\ell(S(3\rho)) \sim \ell(S(\rho))$. Then:

$$\widetilde{\sigma}(\rho) \leq C\left(\frac{\ell(S(\rho))}{\rho}\right)^{kp} = C\sigma(\rho).$$

Dimensional weight

We asume $kp > n - 1$

Take $g \in C_0^\infty(2,5)$ such that $g(t) = 1$, $\forall 3 < t < 4$. Let $\rho > 0$ be a small number, and consider:

$$f_\rho(x) = g\left(\frac{x_n}{\rho}\right).$$

Once again, $f_\rho \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$. Now, let us suppose that there is an extension operator $\widetilde{\Lambda} : W^{k,p} \longrightarrow W^{k,p}_{\widetilde{\sigma}}(\mathbb{R}^n)$, with $\widetilde{\sigma}(x) = \widetilde{\sigma}(|x|)$ nondecreasing. Let us take

$$\Pi_\rho = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ \rho < x_n < 2\rho\}.$$

Then, arguing as in the previous case, we have:

$$\|\widetilde{\sigma}^{\frac{1}{p}} D^\alpha \widetilde{\Lambda} f_\rho\|_{L^p(\mathbb{R}^n)} \leq \|f_\rho\|_{W^{k,p}(\Omega)} \leq C\rho^{\frac{1}{p}-k}\ell(S(5\rho))^{\frac{n-1}{p}}.$$

Now, in order to obtain the optimality of the dimensional weight we need to prove that

$$\|\widetilde{\sigma}^{\frac{1}{p}} \nabla^k \widetilde{\Lambda} f_\rho\|_{L^p(\mathbb{R}^n)} \geq C\rho^{\frac{n}{p}-k}\widetilde{\sigma}(\rho)^{\frac{1}{p}}. \tag{5.1.14}$$

In fact, observe that in that case,

$$C_1\rho^{\frac{n}{p}-k}\widetilde{\sigma}(\rho)^{\frac{1}{p}} \leq C_2\rho^{\frac{1}{p}-k}\ell(S(4\rho))^{\frac{n-1}{p}},$$

which leads to:

$$\widetilde{\sigma}(\rho) \leq C\left(\frac{\ell(S(\rho))}{\rho}\right)^{n-1} = C\sigma(\rho).$$

[2]For a proof of this classical result, which is a consequence of the Gagliardo-Nirenberg-Sobolev inequality, see for example: [Maz'ya and Poborchiĭ, 1997, Theorem 1.8.1], [Adams and Fournier, 2003, Theorem 4.12], [Brezis, 2010, Corollary 9.13].

The proof of (5.1.14) is much more complicated than the corresponding inequality in the derivative case. The reason is the lack of sharpness of the imbedding used earlier. Now, we use the imbedding $W^{k,p} \subset L^\infty$.

Note that the function $\widetilde{\Lambda} f_\rho(x', \cdot)$ is in $W^{k,p}(\mathbb{R})$ for almost every $x' \in \mathbb{R}^{n-1}$. Now, fixing the value of x', we can take the polynomial in the variable x_n: $\pi(x', x_n) = \pi(\widetilde{\Lambda} f_\rho(x', \cdot))([\rho, 2\rho])(x_n)$, that approximates the function $\widetilde{\Lambda} f_\rho(x', \cdot)$ on the interval $[\rho, 2\rho]$, according to definition 2.4.10. Thanks to the approximation property given by Theorem 2.4.12:

$$\|\widetilde{\Lambda} f_\rho(x', \cdot) - \pi(x', \cdot)\|^p_{L^p(\rho,2\rho)} \le C\rho^{kp} \int_\rho^{2\rho} \left| \frac{\partial^k \widetilde{\Lambda} f_\rho}{\partial {x_n}^k}(x', z)\right|^p dz.$$

$\pi(x', \cdot)$ can be written:

$$\pi(x', x_n) = \sum_{m=1}^{k-1} a_m(x')\Big(x_n - \frac{3}{2}\rho\Big)^m,$$

with:

$$a_m(x') = \frac{1}{\rho^{1+k}} \int_\rho^{2\rho} \varphi_m\Big(\frac{t}{\rho}\Big)\widetilde{\Lambda} f_\rho(x', t)dt.$$

Let us denote:

$$h_\rho(x') = \frac{1}{\rho}\int_{2\rho}^{5\rho} \Big(\widetilde{\Lambda} f_\rho(x', z) - \pi(x', z)\Big)dz, \qquad x' \in \mathbb{R}^{n-1}.$$

Since $\hat{x}_n$ is contained in every normal cusp, we have that $\widetilde{\Lambda} f_\rho(0, x_n) = f_\rho(0, x_n)$. But taking into account that $supp(f_\rho) \subset (2\rho, 3\rho)$, we obtain that if $x' = 0$ the coefficients a_m equal zero. Consequently:

$$h_\rho(0) = \frac{1}{\rho}\int_{2\rho}^{5\rho} f_\rho(0, z)dz = \int_2^5 g(z)dz > 0.$$

Now, take B'_ρ the $n-1$ dimensional ball of radius ρ. Sobolev's imbedding $W^{k,p}(B'_\rho) \subset C(B'_\rho) \cap L^\infty(B'_\rho)$, along with the estimation $\|u\|_{W^{k,p}(B(0,1))} \le C\{\|u\|_{L^p(B(0,1))} + \|\nabla^k u\|_{L^p(B(0,1))}\}$ and a scaling argument, leads to:

$$C|B'_\rho|^{\frac{1}{p}}\|h_\rho\|_{L^\infty(B'_\rho)} \le \|h_\rho\|_{L^p(B'_\rho)} + \rho^k\|\nabla^k h_\rho\|_{L^p(B'_\rho)}.$$

Hence:

$$C \le \underbrace{\rho^{-\frac{n-1}{p}}\|h_\rho\|_{L^p(\mathbb{R}^{n-1})}}_{I} + \underbrace{\rho^{k-\frac{n-1}{p}}\|\nabla^k h_\rho\|_{L^p(\mathbb{R}^{n-1})}}_{II}. \tag{5.1.15}$$

We need to bound I and II. For I, we apply Hölder inequality to h_ρ:

$$|h_\rho(x')| \le \frac{1}{\rho}\rho^{\frac{1}{p'}}\|\widetilde{\Lambda} f_\rho(x', \cdot) - \pi(x', \cdot)\|_{L^p(\rho,2\rho)} = C\frac{1}{\rho^{\frac{1}{p}}}\rho^k\Big(\int_\rho^{2\rho}\left|\frac{\partial^k \widetilde{\Lambda} f_\rho}{\partial {x_n}^k}(x', z)\right|^p dz\Big)^{\frac{1}{p}}.$$

And then:

$$I \leq \rho^{-\frac{n-1}{p}-\frac{1}{p}+k}\|\nabla^k\widetilde{\Lambda}f_\rho\|_{L^p(\Pi_\rho)} = C\rho^{k-\frac{n}{p}}\|\nabla^k\widetilde{\Lambda}f_\rho\|_{L^p(\Pi_\rho)}.$$

On the other hand, for II, take α, $|\alpha| = k$, then:

$$D^\alpha h_\rho(x') = \frac{1}{\rho}\int_\rho^{2\rho} D^\alpha_{x'}\Big(\widetilde{\Lambda}f_\rho(x',z) - \pi(x',z)\Big)dz.$$

But applying Hölder inequality to $D^\alpha a_m(x')$, we obtain:

$$|D^\alpha a_m(x')| \leq \frac{1}{\rho^{k+1}}\rho^{\frac{1}{p'}}\|D^\alpha_{x'}\widetilde{\Lambda}f_\rho(x',\cdot)\|_{L^p(\rho,2\rho)} = \rho^{-k-\frac{1}{p}}\|D^\alpha_{x'}\widetilde{\Lambda}f_\rho(x',\cdot)\|_{L^p(\rho,2\rho)}.$$

Which leads to:

$$|D^\alpha_{x'}P(x',z)| \leq C\rho^-\frac{1}{p}\|D^\alpha_{x'}\widetilde{\Lambda}f_\rho(x',\cdot)\|_{L^p(\rho,2\rho)}.$$

And finally:

$$II \leq C\rho^{k-\frac{n-1}{p}}\rho^{-\frac{1}{p}}\|\nabla^k\widetilde{\Lambda}f_\rho\|_{L^p(\Pi_\rho)} = C\rho^{k-\frac{n}{p}}\|\nabla^k\widetilde{\Lambda}f_\rho\|_{L^p(\Pi_\rho)}.$$

This fulfill our needs, since, by the monotonicity of $\widetilde{\sigma}$:

$$\|\widetilde{\sigma}^{\frac{1}{p}}\nabla^k\widetilde{\Lambda}f_\rho\|_{L^p(\mathbb{R}^n)} \geq \|\widetilde{\sigma}^{\frac{1}{p}}\nabla^k\widetilde{\Lambda}f_\rho\|_{L^p(\Pi_\rho)} \geq C\widetilde{\sigma}(\rho)^{\frac{1}{p}}\rho^{\frac{n}{p}-k},$$

and (5.1.15) is proved.

5.2 Extension for curved cusps in the unweighted case

Theorem 5.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external curved cusp at the origin.*

(a) If $kp \neq 1$ or $kp = 1$ and the spine $\mathcal{S}$ satisfies (5.1.1), there is an extension operator

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_\sigma(\mathbb{R}^n),$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^{kp} & x \in \Omega^c \end{cases}$$

(b) If the spine $\mathcal{S}$ satisfies (5.1.2), there is an extension operator

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_\sigma(\mathbb{R}^n),$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^{n-1} & x \in \Omega^c \end{cases}$$

(c) *In case (a), assuming* (5.1.2) *stands, if* $\tilde{\sigma}$ *is such that there is an extension operator* $\tilde{\Lambda} : W^{k,p}(\Omega) \to W^{k,p}_{\tilde{\sigma}}(\mathbb{R}^n)$*, then*

$$\tilde{\sigma}(x) \leq C\sigma(x) \quad \forall x \in U \setminus \Omega,$$

being U *a neighborhood of the origin.*

In order to prove Theorem 5.2.1 We introduce a *stage zero*, consisting of an extension from a curved cusp Ω to a larger domain that includes a *normal* cusp $\widehat{\Omega}$. Functions defined on $\widehat{\Omega}$ will be extended as in Theorem 5.1.1. The most important fact to mention is that after the stage one, the distance of cubes in $\mathcal{W}_4$ and $\mathcal{W}_5$ to Ω are comparable with the distance of them to $\widehat{\Omega}$, and so will be the weights.

5.2.1 Stage *zero*

Let Ω be a curved cusp, and $\mathcal{S} = \{S_i\}_{i=1}^{\infty}$ its spine. Then $d(S_i, \hat{x}_n) \leq C_\Omega \ell(S_i)$, and we may take $C_\Omega \geq K$. Assuming $\ell(S_i) \leq 1$, let us consider:

$$\widetilde{\Omega} = \bigcup_i 4(C_\Omega + 1) \star S_i.$$

Recall that $C \star S_i$ is the *horizontal* dilatation of S_i:

$$C \star S_i = CS_i \cap \{x = (x', x_n) : \quad z_i \leq x_n \leq z_i + \ell_i\}.$$

Clearly, $\Omega \subset \bigcup_i C_\Omega S_i$. Even more: let us take S'_i the horizontal traslation of S_i to $\hat{x}_n$, so $S'_i \cap S'_{i+1} = F^u_{S'_{i+1}}$, and $z_{S'_i} = z_{S_i}$. Then, if we denote $\widehat{\Omega} = \bigcup_i 2C_\Omega S'_i$, we have:

$$\Omega \subset \widehat{\Omega} \subset \widetilde{\Omega}.$$

Lemma 5.1.5 can be reproduced in order to find a *reflected* cube for every $Q \in \mathcal{W}_2$ such that $Q \subset \widetilde{\Omega}$, eventually relaxing the definition of $\mathcal{W}_3$ with a larger constant. Consequently, a first (unweighted) extension can be performed as in stage one. Let us denote $\Lambda_0 f$ the extension of f to $\widetilde{\Omega}$, and let us take $\widehat{f} : \widehat{\Omega} \to \mathbb{R}$, $\widehat{f} = \Lambda_0 f|_{\widehat{\Omega}}$. Observe that $\widehat{\Omega}$ is a *normal* cusp. Then, we can extend $\widehat{f}$ as in Theorem 5.1.1. Let us denote $\widehat{\mathcal{W}_3}$, $\widehat{\mathcal{W}_4}$, $\widehat{\mathcal{W}_5}$, the subsets of the Whitney decomposition of the exterior of $\widehat{\Omega}$ corresponding to stage one, two and three respectively. If we denote $\widehat{\mathcal{S}} = \{\widehat{S}_i\}_i$ the spine of $\widehat{\Omega}$ (observe that $\widehat{S}_i$ is not necessarily S'_i, but they are equivalent), we have $\widehat{S}_i \sim S_i$. Now, if we take $T \in \widehat{\mathcal{W}_4}$, such that $z_i \leq z_T < z_{i-1}$, then

$$\ell(T) \geq C\ell(\widehat{S}_i) \geq C\ell(S_i). \tag{5.2.1}$$

Furthermore,

$$d(T, S_i) \leq Cd(T, \widehat{S}_i) \leq C\ell(T), \tag{5.2.2}$$

and

$$d(T,\partial\widehat{\Omega}) \le d(T,\partial\Omega) \le Cd(T,\widehat{S}_i) \le Cd(T,\partial\widehat{\Omega}). \tag{5.2.3}$$

The weight of the extension operator based on $\widehat{\Omega}$ is expressed in terms of $\widehat{S}_i$, but these inequalities allow us to change it for S_i, and then items *(a)* and *(b)* in Theorem 5.2.1 are proved.

For item *(c)*, the same function f_ρ taken in the previous section for normal cusps provides the optimality of the weight in item *(a)*.

5.3 Approximation by smooth functions up to the tip of the cusp

In Chapter 4, we prove that every function $f \in W^{k,p}(\Omega)$, with Ω a normal or curved cusp, can be approximated in Ω by functions in $C^\infty(\mathbb{R}^n_+)$. This fact was used to prove the extension theorems that constitute the main results of this chapter. In particular, it was used in Proposition 5.1.19 to prove that the extension meets the function smoothly in $\partial\Omega \setminus \{\mathbf{0}\}$. Now, we can use the first stage of the extension process for normal cusps (or stage zero for curved cusps) to prove that, in fact, the smooth approximation can be performed up to the tip of the cusp.

It is a well known fact that $C^\infty(\bar{D})$ is dense in $W^{k,p}(D)$ for every domain D of class C . The proof can be seen, for example, in [Maz'ya, 2011, Theorem 1.2] and [Maz'ya and Poborchiĭ, 1997, Theorem 1.4.2.1]. Also in [Adams and Fournier, 2003, Theorem 3.22]. There, the authors work with domains satisfying the segment condition. We say that D satisfies the segment condition if for every $x \in \partial D$ there is a neighborhood of x, U_x and a nonzero vector y_x such that for every $z \in \bar{D} \cap U_x$, the points $z + ty_x$ belongs to D for $0 < t < 1$. It is important to notice that domains satisfying the segment condition are exactly the domains of class C. It is clear that every domain with continuos boundary satisfies the segment condition. The converse is proved, for example, in [Maz'ya and Poborchiĭ, 1997, Theomem 1.3].

Observe that both the segment condition and the belonging to the class C, hold for every power type cusp satisfying (1.1.1), and for every profile cusp sastisfying (1.1.2). Moreover, they hold for every external cusp according to Definition A, as long as $\mathbf{0} \in \varpi$. Consequently, the density of smooth functions up to the boundary stands for all these domains.

Now, consider a normal cusp Ω. Take $\widehat{\Omega} = \cup_i KS_i$. Thanks to item *(iii)* in Definition 3.2.1, we have that $\Omega \subset \widehat{\Omega}$. On the other hand we may assume $\widehat{\Omega} \subset \Omega \cup (\cup \mathcal{W}_3)$ (eventually it could be necessary to take another constant in the definition of $\mathcal{W}_3$ in order to allow bigger cubes). Hence, we have a profile cusp $\widetilde{\Omega}$ contained in $\widehat{\Omega}$ with profile given, for example, by the interior polygonal φ that interpolates $K\ell(S(z))$. This profile cusp contains Ω. Morever, the first stage of the extension process provides an extension unweighted operator $\widetilde{\Lambda} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\widetilde{\Omega})$. So, given $f \in W^{k,p}(\Omega)$, we have $\widetilde{\Lambda} f \in W^{k,p}(\widetilde{\Omega})$. Now, taking into account the previous discusion, we obtain the density of $C^\infty(\bar{\widetilde{\Omega}})$ in $W^{k,p}(\widetilde{\Omega})$. Hence, for every $\eta > 0$ there is some $g_\eta \in C^\infty(\mathbb{R}^n)$ such that $\|\widetilde{\Lambda} f - g_\eta\|_{W^{k,p}(\widetilde{\Omega})} < \eta$. But then:

$$\|f - g_\eta\|_\Omega \le \|\widetilde{\Lambda} f - g_\eta\|_{W^{k,p}(\widetilde{\Omega})} < \eta,$$

and g_η approximates f up to the boundary of Ω, including the tip of the cusp.

It is clear that the same argument stands for curved cusps, with the previous application of stage zero extension. In this way, we have proved:

Theorem 5.3.1. *Let Ω be a normal or curved cusp. Then, for every $f \in W^{k,p}(\Omega)$, and every $\eta > 0$, there is a function $g_\eta \in C^\infty(\mathbb{R}^n)$ such that $\|f - g_\eta\|_{W^{k,p}(\Omega)} < \eta$.*

5.4 Extensions in the weighted case

In this section we prove that the extension from normal or curved cusps can be performed in the weighted case, obtaining an extension operator of the form $\Lambda : W^{k,p}_\omega(\Omega) \longrightarrow W^{k,p}_{\omega\sigma}(\mathbb{R}^n)$, for some particular weights. We begin discusing a few technical details. This discusion leads us to restrict our analysis to two kind of weights: weights depending on the distance to the cusp, and weights depending on the distance to the boundary. Our arguments are very simple, since we follow the line of reasoning used in Theorem 4.2.2: We work with weights that can be approximated by constants in each cube, and that therefore, can be pulled out or in the norms. Hence, in this Section we do not state the complete proof of almost any result, but we limit our exposition to the arguments that allow the application of this trivial technique.

5.4.1 Discussion

For all measurable set $S \subset \mathbb{R}^n$, let $\omega(S)$ be the measure induced by the weight ω:

$$\omega(S) = \int_S \omega.$$

We say ω is doubling if for every cube $Q \in \mathbb{R}^n$, $\omega(2Q) \leq C\omega(Q)$ with C independent of Q.

In [Chua, 1992, 1994] the author adapts Jones's techniques for proving an extension theorem for locally uniform domains in the weighted case. Essentially, he proves:

Theorem 5.4.1. *Let D be an (ε, δ) connected domain, $1 \leq p < \infty$. Suppose that ω is* doubling, *$\omega^{-\frac{1}{p-1}}$ is locally integrable and $Lip^{k-1}_{loc}(\mathbb{R}^n)$ is dense on $W^{k,p}_\omega(D)$. Finally, suppose that for every cube Q and every $f \in Lip_{loc}(\mathbb{R}^n)$*

$$\|f - f_{Q,\omega}\|_{L^p_\omega(Q)} \leq C\ell(Q)\|\nabla f\|_{L^p_\omega(Q)}, \tag{5.4.1}$$

where $f_{Q,\omega} = \frac{1}{\omega(Q)} \int f d\omega$. Then an extension operator $\Lambda : W^{k,p}_\omega(D) \to W^{k,p}_\omega(\mathbb{R}^n)$ exists.

The density of smooth functions (in this case, $Lip^{k-1}_{loc}(\mathbb{R}^n)$, the set of locally Lipschitz functions with $k - 1$ weak derivatives in $\mathbb{R}^n$) is used for proving that the extension meets properly the function in ∂D, just as we do in Section 5.1.4. Property (5.4.1) is just a weighted Poincaré inequality. A simpler but stronger hyphothesis, that implies all the requirements on the weight, is that $\omega \in A_p$.

Chua's extension operator is constructed as Jones's one: For each cube $Q_j \in \mathcal{W}^c$ *near* the domain, a *reflected* cube $Q_j^* \in \mathcal{W}$ is found (as in Lemma 5.1.5). Given $f \in W^{k,p}_\omega(\Omega)$, a suitable polynomial $P_j = P(Q_j^*, \omega)$, that can be constructed thanks to (5.4.1), is associated to Q_j. Thence, the operator is the smooth summation of all the $\{P_j\}_j$.

The doubling condition is crucial for Chua's arguments to hold: since $d(Q, Q^*) \leq C\ell(Q)$, a bounded expansion of Q, $\widetilde{Q} = cQ$, contains both cubes Q and Q^*. But ω being doubling, $\omega(\widetilde{Q}) \leq C\omega(Q)$. This allows the comparison between the values of the weight ω over Q and over Q^*. Therefore, the *weighted* norm of the extension in Q can be bounded by the *weighted* norm of the function in Q^* just as in Lemma 5.1.9.

Since the first stage of our extension process agrees with the ideas used by Jones for uniform domains, Chua's techniques could be applied. However, second stage presents a very different situation. Reflected sets for cubes in $\mathcal{W}_4$ do not fulfill properties (5.1.6) and (5.1.7), that are the ones used by both Jones and Chua. In the *dimensional-horizontal* version, the reflected set of Q is a *tower* $S(Q)$, not a cube, and whereas $d(Q, S(Q)) \sim \ell(Q)$, the edges $\ell_i(S(Q))$ are not equivalent to $\ell(Q)$, so (5.1.6) fails. Consequently, the values of the weight ω over Q cannot be estimated by its values over $S(Q)$. On the other hand: in the *derivative-vertical* version, the reflected set of Q is a cube Q^*, with $\ell(Q^*) \sim \ell(Q)$, but it may happen that $d(Q, Q^*) >> \ell(Q)$, so (5.1.7) fails. In this case, no bounded fixed expansion of Q could reach Q^*, and the doubling property of ω is useless.

Furthermore, another important problem should be pointed out: the weight σ, which compensates the singularity of the outer peak, appears as a consequence of the asymmetries between a cube and its reflected set, expressed in the failure of one of the reflection properties, (5.1.6) or (5.1.7). Then, the value of σ in a certain cube Q can be estimated as long as the measures of Q, the reflected set of Q and the distance between them are known. In other words: the values of σ over Q, in the weighted case, depend on the measures $\omega(Q)$ and $\omega(S(Q))$ (or $\omega(Q^*)$). But if these magnitudes remain unknown, no general expression can be found for σ. Particularly, the general estimates over $\mathcal{W}_4$, given by Propositions 5.1.15 and 5.1.18, are not possible to obtain.

All these facts lead us to conclude that no results can be given, following our techniques, for the weighted problem, when the weight is completely unknown, and just a few very general properties are assumed about it.

However, it is noteworthy that some particular weights can be easily integrated into our extension process. We present here two examples involved in several applications: weights depending on the distance to the boundary of Ω, that fit easily with the *Derivative Version* of the extension, and weights depending on the distance to $\mathbf{0}$ (the tip of the cusp), that are naturally adapted for the *Dimensional Version*.

Finally, let us comment another aspect of the problem. In Theorem 5.4.1, Chua assumes that ω is defined over the whole space $\mathbb{R}^n$. In other words, ω is supposed to be defined over all cubes in $\mathcal{W}^c$, and the extension process has to fit with it. This approach is possible and comfortable for uniform domains, that don't requiere an *extension weight* σ. The most general version of the problem, however, would be to consider a weight defined just over the domain Ω. The extension operator, in this case, should extend both the weight and the function. From this point of view, Theorem 5.4.1 proves that the extension for uniform domains can be performed for every definition of the weight ω outside Ω, as long as it remains doubling and satisfies property (5.4.1) all over $\mathbb{R}^n$.

For our weighted extension, we proceed accordingly to this last general idea: we assume ω is defined only over Ω, and we set its values outside Ω in order to preserve the weight σ, obtaining an extension operator

$$\Lambda : W^{k,p}_{\omega}(\Omega) \longrightarrow W^{k,p}_{\omega\sigma}(\mathbb{R}^n),$$

where the ω on the right side is a particular definition of ω on $\mathbb{R}^n$ taken from the large set of all possible weights $\widetilde{\omega}$ that satisfy $\widetilde{\omega}(x) = \omega(x)$, $\forall x \in \Omega$.

We analyze each type of weight separately.

5.4.2 Weights depending on $d(x, 0) = |x|$

We have already observed, that near the origin $|x| \sim x_n$, $\forall x \in \Omega$. Moreover, the same thing holds close enough to Ω, and in particular in the sets $\cup\mathcal{W}_3$ and $\cup\mathcal{W}_4$. Let $\omega : \mathbb{R}^n \to \mathbb{R}$, $\omega \geq 0$. As in Example 4.2.1, we denote $\omega(x) = \hat{\omega}(|x|)$, and we assume that $\hat{\omega} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a monotonic function that satisfies $\hat{\omega}(2t) \sim \hat{\omega}(t)$. Notice that the only interesting case is that either $\hat{\omega}(t) \to 0$ or $\hat{\omega}(t) \to \infty$ when $t \to 0$, since otherwise the weighted space agrees with the already treated case of $W^{k,p}$. Let us mention that $W^{k,p}_{\omega}(D)$ is a Banach space [Kufner, 1985, Theorem 3.6] for any open set D.

Since we are considering weights that are admissible for normal or curved cusps (see Definition 4.2.1), we have, in particular, that given a cube $Q \in \mathcal{W}(\Omega)$, $\omega(x) \underset{C}{\sim} \omega_Q$, $\forall x \in Q$ for some constant ω_Q.

The first stage of the extension is trivial: for every $x \in \cup\mathcal{W}_3$, let us set $\omega(x) = \hat{\omega}(x_n)$. Lemma 5.1.5 guarantees that $d(Q, Q^*) \leq C\ell(Q)$, $\forall Q \in \mathcal{W}_3$. This implies $z_{Q^*} \sim z_Q$, and consequently, the constant approximations of ω in Q and Q^* are comparable: $\omega_Q \sim \omega_{Q^*}$. Furthermore, it is easy to see that $\omega_Q \sim \omega_S$ for every $S \in \mathcal{F}(Q)$. This facts are the key tool for our weighted extension process:

Lemma 5.4.2. *If $Q \in \mathcal{W}_3$ is far from $\mathcal{W}_4$, then:*

$$\|D^\alpha \Lambda f\|_{L^p_\omega(Q)} \leq C\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p_\omega(\mathcal{F}(Q))} + \|f\|_{W^{k,p}_\omega(Q^*)}\Big\}.$$

Proof. Just applying the constant approximation of the weight and Lemma 5.1.9:

$$\begin{aligned}\|D^\alpha \Lambda f\|_{L^p_\omega(Q)} &= \|\omega D^\alpha \Lambda f\|_{L_p(Q)} \leq C\omega_Q \|D^\alpha \Lambda f\|_{L^p(Q)} \\ &\leq C\omega_Q\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\mathcal{F}(Q))} + \|f\|_{W^{kp}(Q^*)}\Big\} \\ &\leq C\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p_\omega(\mathcal{F}(Q))} + \|f\|_{W^{kp}_\omega(Q^*)}\Big\}.\end{aligned}$$

□

For the second stage we use essentially the same idea: the weight, being approximately constant over every cube, can be pulled in or out integrals, so the weighted norm can be estimated using the non-weighted lemmas proved in Section 5.1.2.

However, ω should be defined differently for each version.

Version one: dimensional-horizontal weight:

Let us set $\omega(x) = \hat{\omega}(x_n)$, $\forall x \in \cup\mathcal{W}_4$. In this case, it is clear that $\omega_T \sim \omega_{S(T)}$, $\forall Q \in \mathcal{W}_4$. The weighted form of Lemma 5.1.13 can be proved exactly as Lemma 5.4.2, so the next proposition follows, completing the second stage for this version:

Proposition 5.4.3. *If we denote* $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{n-1}$, *then:*

$$\|\sigma^{\frac{1}{p}} D^\alpha f\|_{L^p_\omega(\mathcal{W}_4)} \le C\|f\|_{W^{k,p}_\omega(\mathcal{S})}.$$

Exactly the same ideas can be used for the third stage.

Version two: derivative-vertical weight:

In this case, we need to define ω over $\mathcal{W}_4$ in a different way. In order to preserve the simple technique used earlier, we want to set $\omega_T \sim \omega_{T^*}$.

For every cube $T \in \mathcal{W}_4$, we define $\omega(x) = \omega_{T^*}$, $\forall x \in T$. In other words, ω is constant over each cylinder $\eta(\widetilde{T})$. It is important to note that if $T_1 \cap T_2 \neq \emptyset$, $z_{T_1^*} \le z_{T_2^*}$, then $d(T_1^*, T_2^*) \le z_{T_1^*}$, and then $z_{T_2^*} \le 2z_{T_1^*}$, which lead us to conclude $\omega_{T_1^*} \sim \omega_{T_2^*}$. This allows to prove the weighted version of Lemma 5.1.17 as we proved Lemma 5.4.2 and, consequently, to state the following:

Proposition 5.4.4. *If we denote* $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{kp}$, *then:*

$$\|\sigma^{\frac{1}{p}} D^\alpha \Lambda_2 f\|_{L^p_\omega(\cup\mathcal{W}_4)} \le C\|f\|_{W^{k,p}_\omega(\mathcal{S})}.$$

Finally, for the third stage, we may define $\omega(x) = \omega_{S(U)}$ (or $\omega(x) = \omega_{U^*}$), for every $x \in S(U)$ (or U^*), for every $U \in \mathcal{W}_5$. With this definitions, we can state the following weighted extension theorem:

Theorem 5.4.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external normal cusp at the origin. Let $\hat{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonic function satisfying $\hat{\omega}(2t) \sim \hat{\omega}(t)$, and consider the weighted Sobolev space $W^{k,p}_\omega(\Omega)$, where $\omega(x) = \hat{\omega}(|x|)$, $\forall x \in \Omega$.*

a) If $kp \neq 1$ or $kp = 1$ and the spine $\mathcal{S}$ satisfies (5.1.1), there is an extension of ω over Ω^c such that there exists an extension operator

$$\Lambda : W^{kp}_\omega(\Omega) \to W^{kp}_{\omega\sigma}(\mathbb{R}^n),$$

where

$$\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{kp}.$$

b) If the spine $\mathcal{S}$ satisfies (5.1.2), there is an extension of ω over Ω^c such that there exist an extension operator

$$\Lambda : W^{kp}_{\omega}(\Omega) \to W^{kp}_{\omega\sigma}(\mathbb{R}^n),$$

where

$$\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{n-1}.$$

Observe that the density of $C^\infty(\mathbb{R}^n_+)$ in $W^{k,p}_{\omega}(\Omega)$ was proved in Section 4.2 for admissible weights, such as the ones considered here.

5.4.3 Weights depending on $d(\cdot, \partial\Omega)$ - the derivative case

We obviously have:

$$d(x, \partial\Omega) \sim \ell(Q) \quad \forall x \in Q, \quad \forall Q \in \mathcal{W} \cup \mathcal{W}^c. \tag{5.4.2}$$

Let us set $\hat{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ a monotonic function such that $\hat{\omega}(2t) \sim \hat{\omega}(t)$. And let $\omega : \mathbb{R}^n \to \mathbb{R}^n$, be the weight $\omega(x) = \hat{\omega}(d(x, \partial\Omega))$. This implies that ω can be taken as a constant ω_Q over every cube $Q \in \mathcal{W} \cup \mathcal{W}^c$.

This leads us to the following corollary of Lemma 5.1.9 (which proof is exactly as the one of Lemma 5.4.2):

Lemma 5.4.6. *Let Ω be a domain satisfying Definition 3.2.1, then:*

$$\|D\Lambda_1 f\|_{L^p_\omega(Q)} \leq C\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p_\omega(\mathcal{F}(Q))} + \|D^\alpha f\|_{L^p_\omega(Q^*)}\Big\}.$$

This lemma says, esentially, that the first stage of the extension can be performed, with weights depending only on $d(\cdot, \partial\Omega)$, just copying the procedure for the unweighted case. The correspondant results for the second stage depend on the version used. Since the technique is always the same (the weight goes out the norm, and the unweighted result is applied), we limit our exposition to the proper extension of the weight for each case:

Version one: dimensional-horizontal weight:

The problem for this version is that $d(T, \partial\Omega) \nsim d(S(T), \partial\Omega)$. So, we need to define ω over $\mathcal{W}_4$ in order to obtain $\omega_T \sim \omega_{S(T)}$. This can be done setting $\omega_T = \omega(d(S(T), \partial\Omega))\ \forall T \in \mathcal{W}_4$, and $\omega(x) = \omega_T\ \forall x \in T$. This guarantees the desired property: $\omega_T \sim \omega_{S(T)}$. Morever, if $T_1 \cap T_2 \neq \emptyset$, $\omega_{T_1} \sim \omega_{T_2}$. With this definition the second stage of the extension process can be performed.

Derivative-vertical weight:

For the second stage, let us recall that in the *derivative version*, $\ell(Q) \sim \ell(Q^*)$, and then $d(Q, \partial\Omega) \sim d(Q^*, \partial\Omega)$. Consequently, the definition of the weight over $\mathcal{W}_4$ is the natural: $\omega(x) = \hat{\omega}(d(x, \partial\Omega))$. This fact is enough to complete the second stage.

For the third stage, the weight ω is defined over $\mathcal{W}_5$ just as for weights depending on x_n: $\omega_U = \omega_{S(U)}$ or $\omega_U = \omega_{U^*}$ $\forall U \in \mathcal{W}_5$, and $\omega(x) = \omega_U$, $\forall x \in U$.

In this way, we can state the following Theorem:

Theorem 5.4.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external normal cusp at the origin. Let $\hat{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonic function satisfying $\hat{\omega}(2t) \sim \hat{\omega}(t)$, and consider the weighted Sobolev space $W^{k,p}_\omega(\Omega)$, where $\omega(x) = \hat{\omega}(d(x, \partial\Omega))$, $\forall x \in \Omega$. Finally, suppose that $C^\infty(\mathbb{R}^n_+)$ is dense in $W^{k,p}_\omega(\Omega)$. Then:*

a) If $kp \neq 1$ or $kp = 1$ and the spine $\mathcal{S}$ satisfies (5.1.1), there is an extension of ω over Ω^c such that there exists an extension operator

$$\Lambda : W^{kp}_\omega(\Omega) \to W^{kp}_{\omega\sigma}(\mathbb{R}^n),$$

where

$$\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{kp}.$$

b) If the spine $\mathcal{S}$ satisfies (5.1.2), there is an extension of ω over Ω^c such that there exists an extension operator

$$\Lambda : W^{kp}_\omega(\Omega) \to W^{kp}_{\omega\sigma}(\mathbb{R}^n),$$

where

$$\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{n-1}.$$

Observe that the density of smooth functions is included as a hyphotesis of this last Theorem. The reason is that we cannot guarranty that such a result holds for this kind of weight. We may, however, make a few comments on the issue. Let us first state the following:

Definition 5.4.8. *For $0 \leq m \leq n$, a set F is called $m-$regular, if there exists a positive constant C such that*

$$C^{-1}r^m < \mathcal{H}^m(B(x,r) \cap F) < Cr^m,$$

for all $x \in F$ and $0 < r \leq diam(F)$. Where $\mathcal{H}^m$ stands for the m dimensional Hausdorff measure and the restriction $0 < r \leq diam(F)$ is eliminated if F is a set with only one point.

Let us mention that some self similiar fractals such as the Koch curve are $m - regular$ with $m \notin \mathbb{N}$ (in fact $m = \log(4)/\log(3)$ in the Koch example).

As we commented in Section 5.3, for a uniform domain D, a general and simple condition that guarantees the density of $C^\infty(\bar{D})$ in $W^{k,p}_\omega(D)$, is that $\omega \in A_p$ (see Chua [1992]). Under extra assumptions on the boundary of D it is possible to find conditions for which weights of the type $d(\cdot, \partial D)^{\mu}$ belong to A_p. Indeed, in [Durán and López García, 2010a] the authors prove that such a weight is in A_p when $-(n-m) < \mu < (n-m)(p-1)$ provided that ∂D is a compact set contained in an $m - regular$ set.

In such a case we can replicate Theorem 4.1.7 by using a weighted version of Proposition 4.1.1. Therefore the density assumption in Theorem 5.4.5 can be removed.

Let us observe that for a "good" domain D, one expects $m = n-1$, therefore the range $-1 < \mu < p-1$ is precisely the one for which the extension problem makes sense and it is non-trivial. Indeed, on the one hand if $\mu \geq p-1$, then $\omega^{-\frac{1}{p-1}} \notin L^1_{loc}(\mathbb{R}^n)$ and the weighted global space can not be defined in the standard way. On the other, taking for instance D Lipschitz and $\mu \leq -1$ it can be shown that $C_0^{\infty}(D)$ is dense in $W^{k,p}_{\omega}(D)$ [Kufner, 1985], and therefore functions in that space can be extended by 0.

6

Korn and Poincaré inequalities

In this Chapter we work mainly with chains of rectangles, and with chains of quasi-rectangles, which are a generalization of the formers. These classes of domains include some cuspidal domains, as long as we allow the rectangles to narrow faster than any cone. However, many non cuspidal, and even non singular, domains can be described through a chain of rectangles (or quasi-rectangles). Our technique is based on a discrete Hardy type inequality that allows us to pass from one rectangle to another.

We begin studying chains of rectangles for the sake of simplicity, but the treatment for chains of quasi-rectangles is exactly the same. We finish this Chapter presenting some examples of chains of quasi-rectangles with cuspidal behaviour.

6.1 Preliminaries

The following lemma is a fundamental tool in the sequel. It is a discrete version of a well known weighted one-dimensional Hardy type inequality:

Lemma 6.1.1. *Let $\{u_i\}_i$, $\{v_i\}_i$ be sequences of non-negative weights; and let $1 < p \leq q < \infty$. Then the inequality:*

$$\Big[\sum_{j=1}^{\infty} u_j\Big(\sum_{i=1}^{j} b_i\Big)^q\Big]^{\frac{1}{q}} \leq c\Big[\sum_{j=1}^{\infty} v_j b_j^p\Big]^{\frac{1}{p}}$$

holds for every non-negative sequences $\{b_i\}_i$ if and only if:

$$\mathbf{A} = \sup_{k>0}\Big(\sum_{j=k}^{\infty} u_j\Big)^{\frac{1}{q}}\Big(\sum_{j=0}^{k} v_j^{1-p'}\Big)^{\frac{1}{p'}} < \infty.$$

The constant c is $c = M\mathbf{A}$, where M depends only on p and q.

This result (see, for example [Kufner and Persson, 2003]), can be easily obtained from its continuous (integral) version, that can be seen in [Kufner and Persson, 2003, page 3], [Maz'ya, 2011, Theorem 1.3/2]

The following Lemma, is a particular case of the previous one:

Lemma 6.1.2. *Let $\{r_i\}_i$ and $\mathbf{a} = \{a_i\}_i$ be sequences such that $\{r_i\}_i \geq 0$, $\sum_i r_i = r < \infty$ and $\{a_i r_i\}_i$ is summable. Let us denote*

$$\bar{a} = \frac{1}{r} \sum_j a_j r_j.$$

Then the inequality:

$$\Big(\sum_{j=1}^{\infty} |a_j - \bar{a}|^p r_j \Big)^{\frac{1}{p}} \leq c \Big(\sum_{j=1}^{\infty} |a_{j+1} - a_j|^p r_{j+1} \Big)^{\frac{1}{p}}, \tag{6.1.1}$$

holds if

$$\mathbf{A} = \sup_{k>0} \Big(\sum_{j=k}^{\infty} r_j \Big)^{\frac{1}{p}} \Big(\sum_{j=0}^{k} r_j^{1-p'} \Big)^{\frac{1}{p'}} < \infty. \tag{6.1.2}$$

The constant c is $c = M\mathbf{A}$ where M depends only on p.

Proof. Let us define the norm:

$$\|\mathbf{a}\|_p = \Big(\sum_i |a_i|^p r_i \Big)^{\frac{1}{p}}.$$

From Hölder's inequality, it holds $|\bar{a}| r \leq \|\mathbf{a}\|_p r^{\frac{1}{p'}}$ and then $\|\mathbf{a} - \bar{a}\|_p \leq 2\|\mathbf{a}\|_p$. Applying this last inequality with $\mathbf{a}$ replaced by $\mathbf{a} - a_0$, we obtain

$$\|\mathbf{a} - \bar{a}\|_p \leq 2\|\mathbf{a} - a_0\|_p.$$

Therefore:

$$\sum_i |a_i - \bar{a}|^p r_i \leq 2^p \sum_i |a_i - a_0|^p r_i \leq 2^p \sum_i \Big(\sum_{j=1}^{i} |a_j - a_{j-1}| \Big)^p r_i$$

And we conclude applying Lemma 6.1.1 with $u_i = v_i = r_i$, $q = p$ and $b_i = |a_i - a_{i-1}|$. $\square$

6.2 Poincaré and Korn inequalities for chains of rectangles

In this section we give a necessary condition for Korn's inequality to hold on chains of rectangles (recall Definition 3.1.1). This abstract result is a consequence of Lemma 6.1.2.

Definition 6.2.1. *Given a chain of open rectangles $\mathcal{R} = \{R_i\}$, and calling $\mathcal{R}_I = \{R_{i,i+1}\}$ the chain of intermediate rectangles given by Remark 3.1.2, an $\mathcal{R}-$linked domain Ω is any open set such that $\cup(\mathcal{R}_I \cup \mathcal{R}) \subset \Omega$ and $\Omega \equiv (\cup\mathcal{R})$.*

Now, we can state the main result of this section:

Theorem 6.2.2 (Second Case of Korn's Inequality for Chains of Rectangles)**.** *Let $\mathcal{R} = \{R_i\}$ be a chain of rectangles, and let C_i be the constants for the second case of Korn's inequality on R_i. Then for any $\mathcal{R}$ − linked domain Ω, and any $u \in W^{1,p}(\Omega)^n$ such that $\int_\Omega \frac{Du-Du^t}{2} = 0$ we have*

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C(1+\mathbf{A})\|\varepsilon(u)\|_{L^p_\sigma(R)^{n\times n}},$$

where $\mathbf{A}$ is defined in (6.1.2) with $r_j = |R_j|$, and the weight σ is constant on each R_i being $\sigma|_{R_i} = C_i^p$.

Proof. Let

$$A^i = \frac{1}{2|R_i|}\int_{R_i}(Du - Du^t).$$

Then:

$$\|Du\|^p_{L^p(\Omega)^{n\times n}} = \sum_i \|Du\|^p_{L^p(R_i)^{n\times n}} \leq \underbrace{C\sum_i \|Du - A^i\|^p_{L^p(R_i)^{n\times n}}}_{I} + \underbrace{C\sum_i \|A^i\|^p_{L^p(R_i)^{n\times n}}}_{II}.$$

I leads to

$$I \leq C\sum_i C_i^p\|\varepsilon(u)\|^p_{L^p(R_i)^{n\times n}} \leq C\sum_i \|\varepsilon(u)\|^p_{L^p_\sigma(R_i)^{n\times n}} = C\|\varepsilon(u)\|^p_{L^p_\sigma(\Omega)^{n\times n}}.$$

For II, apply inequality (6.1.1) with $r_j = |R_j|$. Let us observe that $\sum |R_i|A^i = 0$, therefore taking

$$\mathbf{A} = \sup_{k>0}\Big(\sum_{j\geq k}|R_j|\Big)^{\frac{1}{p}}\Big(\sum_{j\leq k}|R_j|^{1-p'}\Big)^{\frac{1}{p'}},$$

we have

$$II = C\sum_i |A^i|^p|R_i| \leq C\mathbf{A}^p\sum_i |A^{i+1} - A^i|^p|R_{i+1}|,$$

where C is a constant depending on n and p. For each i, let us consider the intermediate rectangle $R_{i,i+1}$. Calling

$$A^{i,i+1} = \frac{1}{2|R_{i,i+1}|}\int_{R_{i,i+1}} Du - Du^t,$$

we get, using extensively Remark 3.1.2,

$$\begin{aligned}
II &\leq C\mathbf{A}^p \sum_i \left\{|A^{i+1} - A^{i,i+1}|^p + |A^{i,i+1} - A^i|^p\right\}|R_{i+1}| \\
&\leq C\mathbf{A}^p \sum_i \left\{|A^{i+1} - A^{i,i+1}|^p|R_{i+1} \cap R_{i,i+1}| + |A^{i,i+1} - A^i|^p|R_i \cap R_{i,i+1}|\right\} \\
&= C\mathbf{A}^p \sum_i \left\{\|A^{i+1} - A^{i,i+1}\|^p_{L^p(R_{i+1}\cap R_{i,i+1})} + \|A^i - A^{i,i+1}\|^p_{L^p(R_i\cap R_{i,i+1})}\right\} \\
&\leq C\mathbf{A}^p \sum_i \left\{\|A^{i+1} - Du\|^p_{L^p(R_{i+1})^{n\times n}} + \|Du - A^{i,i+1}\|^p_{L^p(R_{i,i+1})^{n\times n}} + \|Du - A^i\|^p_{L^p(R_i)^{n\times n}}\right\} \\
&\leq C\mathbf{A}^p \sum_i C_i^p \|\varepsilon(u)\|^p_{L^p(R_{i+1}\cup R_i)^{n\times n}} \\
&\leq C\mathbf{A}^p \sum_i C_i^p \|\varepsilon(u)\|^p_{L^p(R_i^{n\times n})},
\end{aligned}$$

where, in the last inequality we use that for each R_i, $\overline{R}_i \cap \overline{R}_j = \emptyset$ if $|i - j| > 1$. Therefore

$$II \leq C\mathbf{A}^p \|\varepsilon(u)\|^p_{L^p_\sigma(\Omega)^{n\times n}},$$

and the Theorem follows. □

By using scaling arguments, it is straightforward to check that the constant in the second case of Korn's inequality for cubes is the same, regardless of the size of the cube. Taking this into account, the following Lemma is a consequence of Theorem 6.2.2. It provides a sharp estimate for the constant on rectangles.

Lemma 6.2.3 (Korn inequality for rectangles)**.** *Let $R \subset \mathbb{R}^n$ be the rectangle with $n-1$ short edges of length ℓ and a long edge of length L. Then, for every $u \in W^{1,p}(R)^n$ such that $\int_R Du$ is symmetric:*

$$\|Du\|_{L^p(R)^{n\times n}} \leq C\frac{L}{\ell}\|\varepsilon(u)\|_{L^p(R)^{n\times n}},$$

with C depending only on n and p.

Proof. For the sake of simplicity, let us assume $\frac{L}{\ell} \in \mathbb{N}$.

We can decompose R in $N = \frac{L}{\ell}$ touching cubes:

$$R = \bigcup_{i=1}^N Q_i,$$

with $\ell(Q_i) = \ell$ for all i. Now, we can apply Theorem 6.2.2, taking $R_i = Q_i$ for $i = 1, \ldots, N$, and $R_i = \emptyset$ for $i > N$. We only need to estimate the value of $\mathbf{A}$.

$$\mathbf{A} = \sup_{0<k\leq\frac{L}{\ell}} \left(\sum_{j=1}^{k}|Q_j|\right)^{\frac{1}{p}}\left(\sum_{j=k}^{\frac{L}{\ell}}|Q_j|^{1-p'}\right)^{\frac{1}{p'}} = \sup_{0<k\leq\frac{L}{\ell}} \left(k|Q_1|\right)^{\frac{1}{p}}\left(\left(\frac{L}{\ell}-k\right)|Q_1|^{1-p'}\right)^{\frac{1}{p'}}$$
$$= \sup_{0<k\leq\frac{L}{\ell}} k^{\frac{1}{p}}\left(\frac{L}{\ell}-k\right)^{\frac{1}{p'}} \leq \frac{L}{\ell}.$$

The result follows since the weight σ is constant.

□

Example 6.2.1. *Take, for $n=2$, $u(x,y) = (-xy, \frac{x^2}{2})$, defined over $R = [0,L]\times[-\frac{\ell}{2},\frac{\ell}{2}]$. Then, we have that*

$$\|Du\|^p_{L^p(R)^{n\times n}} = C\ell L^{p+1} \quad \text{and} \quad \|\varepsilon(u)\|^p_{L^p(R)^{n\times n}} = C\ell^{p+1}L,$$

where the constants denoted by C are not the same, but depend only on p. Then:

$$\frac{\|Du\|^p_{L^p(R)^{n\times n}}}{\|\varepsilon(u)\|^p_{L^p(R)^{n\times n}}} \leq C\left(\frac{L}{\ell}\right)^p.$$

Hence, the estimation of Lemma 6.2.3 is sharp.

Remark 6.2.4. *In a more general context, the constant for any convex domain Ω can be bounded taking the quotient between the diameter of Ω and the diameter of a maximal ball contained in Ω [Durán, 2012, Theorem 4.2]. Even when in [Durán, 2012] that result is stated only for $p=2$, the same proof works for $1<p<\infty$. It is important to notice that this implies that given a rectangle R with edges $\ell_i(R)$, eventually all different, Korn's constant in the second case can be taken $\frac{\ell_M(R_i)}{\ell_m(R_i)}$*[1]*. Our technique, however, only produces such a constant when the rectangle has $n-1$ short equal edges and one long edge.*

Remark 6.2.5. *The previous remark implies that the constants C_i in Theorem 6.2.2 can be taken as follows:*

$$C_i = \frac{\ell_M(R_i)}{\ell_m(R_i)}. \tag{6.2.1}$$

and therefore $\sigma|_{R_i} = \left(\frac{\ell_M(R_i)}{\ell_m(R_i)}\right)^p$.

Theorem 6.2.2 can be straightforwardly extended to some weighted spaces. We work with weights that are *admissible* for chains of rectangles in a similar sense than the one used in Definition 4.2.1 for normal cusps:

Definition 6.2.6. *Let $\mathcal{R} = \{R_i\}$, be a chain of rectangles, and Ω an $\mathcal{R}-$linked domain. We say that ω is an admissible weight in Ω if there is a constant C such that for any $x\in R_i$*

$$\omega(x) \underset{C}{\sim} \omega_{R_i} \underset{C}{\sim} \omega_{R_{i+1}} \; \forall i. \tag{6.2.2}$$

being ω_{R_i} appropriate constants.

[1]Recall that we denote $\ell_M(R)$ and $\ell_m(R)$ the largest and shortest edge of R, respectively.

Now, we can prove the following elementary generalization of Theorem 6.2.2:

Theorem 6.2.7 (Second Case of weighted Korn's Inequality for Chains of Rectangles)**.** *Let $\mathcal{R} = \{R_i\}$ be a chain of rectangles and Ω an $\mathcal{R}-$ linked domain.*

Let $u \in W^{1,p}_\omega(\Omega)^n$, with ω an admissible weight, be such that

$$\frac{1}{\omega(\Omega)} \int_\Omega \frac{Du - Du^t}{2} \omega = 0.$$

If $\sum_i \omega(R_i) = r < \infty$,

$$\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \le C(1 + \mathbf{A}_\omega)\|\varepsilon\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}},$$

where $\sigma|_{R_i}$ can be taken as in Theorem 6.2.2, and

$$\mathbf{A}_\omega := \sup_{k>0} \Big(\sum_{j\ge k} \omega(R_j)\Big)^{\frac{1}{p}} \Big(\omega(R_j)^{1-p'}\Big)^{\frac{1}{p'}}. \tag{6.2.3}$$

Proof. Let:

$$A^i = \frac{1}{|R_i|} \int_{R_i} \frac{Du - Du^t}{2} \quad \text{and} \quad A^i_\omega = \frac{1}{\omega(R_i)} \int_{R_i} \frac{Du - Du^t}{2} \omega.$$

We take:

$$\|Du\|^p_{L^p_\omega(\Omega)^{n\times n}} = \sum_i \|Du\|^p_{L^p_\omega(R_i)^{n\times n}} \le C\Big\{\underbrace{\sum_i \|Du - A^i_\omega\|^p_{L^p_\omega(R_i)^{n\times n}}}_{(a)} + \underbrace{\sum_i \|A^i_\omega\|^p_{L^p_\omega(R_i)^{n\times n}}}_{(b)}\Big\}.$$

For (a) we write

$$\|Du - A^i_\omega\|_{L^p_\omega(R_i)^{n\times n}} \le \underbrace{\|Du - A^i\|_{L^p_\omega(R_i)^{n\times n}}}_{I} + \underbrace{\|A^i - A^i_\omega\|_{L^p_\omega(R_i)^{n\times n}}}_{II},$$

and for I, we can take the weight off the norms

$$\begin{aligned} I^p &= \|Du - A^i\|^p_{L^p_\omega(R_i)^{n\times n}} \le \omega^p_{R_i}\|Du - A^i\|^p_{L^p(R_i)^{n\times n}} \le C\omega^p_{R_i}\|\varepsilon(u)\|^p_{L^p_\sigma(R_i)} \\ &\le C\|\varepsilon(u)\|^p_{L^p_{\omega\sigma}(R_i)}. \end{aligned}$$

On the other hand

$$\begin{aligned} II^p &= \|A^i - A^i_\omega\|^p_{L^p_\omega(R_i)^{n\times n}} = \omega(R_i)\Big|A^i - \frac{1}{\omega(R_i)} \int_{R_i} \frac{Du - Du^t}{2}\omega(x)dx\Big|^p \\ &= \omega(R_i)\Big|\frac{1}{\omega(R_i)} \int_{R_i} \Big(A^i - \frac{Du - Du^t}{2}\Big)\omega(x)dx\Big|^p \\ &\le C\omega(R_i)^{1-p}\Big\{\Big|\int_{R_i} (A^i - Du)\omega(x)dx\Big|^p + \Big|\int_{R_i} (Du - \frac{Du - Du^t}{2})\omega(x)dx\Big|^p\Big\} \\ &= C\omega(R_i)^{1-p}\Big\{\Big|\int_{R_i} (A^i - Du)\omega(x)^{\frac{1}{p}}\omega(x)^{\frac{1}{p'}}dx\Big|^p + \Big|\int_{R_i} \varepsilon(u)\omega(x)^{\frac{1}{p}}\omega(x)^{\frac{1}{p'}}dx\Big|^p\Big\}. \end{aligned}$$

Applying Hölder inequality in both terms,

$$\begin{aligned}
II^p &\leq C\omega(R_i)^{1-p}\Big\{\|A^i - Du\|^p_{L^p_\omega(R_i)^{n\times n}}\omega(R_i)^{\frac{p}{p'}} + \|\varepsilon(u)\|^p_{L^p_\omega(R_i)^{n\times n}}\omega(R_i)^{\frac{p}{p'}}\Big\}\\
&= C\Big\{I^p + \|\varepsilon(u)\|^p_{L^p_\omega(R_i)^{n\times n}}\Big\} \leq C\|\varepsilon(u)\|^p_{L^p_{\omega\sigma}(R_i)}.
\end{aligned}$$

On the other hand, for (b), let us observe that

$$\sum_i \omega(R_i)A^i_\omega = 0,$$

and that:

$$\sum_i \|A^i_\omega\|^p_{L^p_\omega(R_i)^{n\times n}} = \sum_i \omega(R_i)|A^i_\omega|^p.$$

Consequently, Lemma 6.1.2 with $a_i = A^i_\omega$ and $r_i = \omega(R_i)$, yields

$$\sum_i \|A^i_\omega\|^p_{L^p_\omega(R_i)^{n\times n}} \leq C\mathbf{A}_\omega \sum_{i=1}^\infty |A^{i+1}_\omega - A^i_\omega|^p\omega(R_{i+1}) \leq C\mathbf{A}_\omega \sum_{i=1}^\infty \|A^{i+1}_\omega - A^i_\omega\|^p_{L^p_\omega(R_{i+1})},$$

Now we may proceed like in Theorem 6.2.2, alternating $A^{i,i+1}_\omega$, the weighted average of $\frac{Du-Du^t}{2}$ on an overlaping rectangle $R_{i,i+1}$, afterwards alternating Du, and finally applying the estimates for (a). We leave the final details to the reader. □

Observe that Theorem 6.2.2 is a Corollary of the previous theorem taking $\omega \equiv 1$. However, Theorem 6.2.7 does not provide information unless $\mathbf{A}_\omega < \infty$. A simple way to bound $\mathbf{A}_\omega$ involves a reasonable decay for $\omega(R_i)$.

Corollary 6.2.8. *Under the same hypotheses of Theorem 6.2.7. Assume that for any k,*

$$\omega(R_{k+1}) \leq \alpha\omega(R_k) \quad \text{with } 0 \leq \alpha < 1. \tag{6.2.4}$$

Then for any $u \in W^{1,p}_\omega(\Omega)^n$ such that $\frac{1}{\omega(\Omega)}\int_\Omega \frac{Du-Du^t}{2}\omega = 0$, we have

$$\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p_{\omega\sigma}(R)^{n\times n}},$$

where the weight σ is constant on each element of $\mathcal{R}$, and can be taken as $\sigma|_{R_i} = \Big(\frac{\ell_M(R_i)}{\ell_m(R_i)}\Big)^p$.

Proof. From (6.2.1), we know that $C_i = \frac{\ell_M(R_i)}{\ell_m(R_i)}$. Thence only remains to show that $\sum_i \omega(R_i) < \infty$ and $\mathbf{A}_\omega < C$. These follow from the bounds $\omega(R_k) \leq \alpha^{k-i}\omega(R_i)$ for $0 \leq i \leq k$ and $\omega(R_i) \leq \alpha^{i-k}\omega(R_k)$ for $i \geq k$. Indeed

$$\mathbf{A}_\omega = \sup_{k>0}\Big(\sum_{j=k}^\infty \omega(R_j)\Big)^{\frac{1}{p}}\Big(\sum_{j=0}^k \omega(R_j)^{1-p'}\Big)^{\frac{1}{p'}} \leq \omega(R_k)^{1/p}\Big(\sum_{j=0}^\infty \alpha^j\Big)^{\frac{1}{p}}\omega(R_k)^{1/p'-1}\Big(\sum_{j=0}^k \alpha^{(p'-1)j}\Big)^{\frac{1}{p'}}$$

then

$$\mathbf{A}_\omega \leq \Big(\frac{1}{1-\alpha}\Big)^{1/p}\Big(\frac{1}{1-\alpha^{p'-1}}\Big)^{1/p'},$$

and the Corollary follows. □

Everything done so far for the second case of Korn's inequality for chains of rectangles can be done for Poincaré inequality following step by step the arguments given above. Since the constant in Poincaré inequality for rectangles (and in general for convex domains) depends only on the diameter of the rectangle, the weight involved in the inequality can be weakened as it is stated below.

Theorem 6.2.9 (Poincaré inequality for Chains of Rectangles)**.** *Let $\mathcal{R} = \{R_i\}$ be a chain of rectangles and Ω a $\mathcal{R}-$linked domain. Let ω be an admissible weight (see (6.2.2)), such that for any k, $\omega(R_{k+1}) \leq \alpha\omega(R_k)$ with $0 \leq \alpha < 1$. Then if $u \in W^{1,p}_{\omega}(\Omega)^n$, and $\frac{1}{\omega(\Omega)}\int_\Omega u\omega = 0$, we have*

$$\|u\|_{L^p_\omega(\Omega)^n} \leq C\|Du\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}},$$

where the weight σ is constant on each R_i and can be taken as $\sigma|_{R_i} = \ell_M(R_i)^p$.

The following version will be useful in the sequel.

Corollary 6.2.10. *With the same hypotheses of Theorem 6.2.9, assume that B is a ball in Ω such that $B \cap R_j \neq \emptyset$ only for a finite number of rectangles. Then, for every $u \in W^{1,p}_\omega(\Omega)^n$, we have:*

$$\|u\|_{L^p_\omega(\Omega)^n} \leq C\Big\{\|u\|_{L^p(B)^n} + \|Du\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}}\Big\}$$

where the weight σ is constant on each R_i and can be taken as $\sigma|_{R_i} = \ell_M(R_i)^p$.

Proof. For the sake of clarity we write the case $\omega \equiv 1$.

$$\|u\|_{L^p(\Omega)^n} \leq \|u - u_B\|_{L^p(\Omega)^n} + \|u_B\|_{L^p(\Omega)^n} \leq \underbrace{\|u - u_\Omega\|_{L^p(\Omega)^n}}_{I} + \underbrace{\|u_\Omega - u_B\|_{L^p(\Omega)^n}}_{II} + \underbrace{\|u_B\|_{L^p(\Omega)^n}}_{III}.$$

Applying Theorem 6.2.9:

$$I \leq C\|Du\|_{L^p_\sigma(\Omega)^{n\times n}}.$$

On the other hand,

$$III^p = \int_\Omega \Big(\fint_B u\Big)^p = \frac{|\Omega|}{|B|^p}\Big(\int_B u\Big)^p \leq \frac{|\Omega|}{|B|^p}|B|^{\frac{p}{p'}}\int_B u^p = \frac{|\Omega|}{|B|}\|u\|^p_{L^p(B)^n}.$$

For II, applying Hölder inequality:

$$|u_\Omega - u_B| \leq \frac{1}{|B|}\int_B |u_\Omega - u| \leq \frac{|B|^{\frac{1}{p'}}}{|B|}\|u - u_\Omega\|_{L^p(B)} \leq \frac{1}{|B|^{\frac{1}{p}}}\|u - u_\Omega\|_{L^p(\Omega)^n} \leq \frac{C}{|B|^{\frac{1}{p}}}\|Du\|_{L^p(\Omega)^{n\times n}_\sigma},$$

then

$$II \leq C\frac{|\Omega|^{\frac{1}{p}}}{|B|^{\frac{1}{p}}}\|Du\|_{L^p(\Omega)^{n\times n}_\sigma},$$

and the lemma follows for $\omega \equiv 1$.

The general case follows similarly using (6.2.2), and taking into account that B only meets a finite number of rectangles and then $\|u\|_{L^p_\omega(B)^n} \underset{C}{\sim} \|u\|_{L^p(B)^n}$.

∎

We now prove the general case of Korn's inequality for chains of rectangles. Our proof is a straigthforward adaptation of the classic argument given in [Brenner and Scott, 2008]. Let us notice that we require that $\ell_M(R_i) \leq C$ for any i. That is in order to remove the weigth σ from the Poincaré inequality given above.

Theorem 6.2.11 (General Case of Korn's inequality for Chains of Rectangles). *Let $\mathcal{R} = \{R_i\}$ be a chain of rectangles, and Ω an $\mathcal{R}-$linked domain. Consider a weight ω such that (6.2.2) holds, and assume that $\omega(R_{k+1}) \leq \alpha\omega(R_k)$ with $0 \leq \alpha < 1$ and that $\ell_M(R_i) < C$, for any i. If B is a ball such that $B \subset \Omega$, and B meets only a finite number of rectangles R_i then for any $u \in W^{1,p}_\omega(\Omega)^n$, we have*

$$\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}}\Big\}, \tag{6.2.5}$$

where the weight σ is constant on each element of $\mathcal{R}$, and can be taken as $\sigma|_{R_i} = \left(\frac{\ell_M(R_i)}{\ell_m(R_i)}\right)^p$.

Proof. Again, let us focus first on the case $\omega \equiv 1$. Consider the space of rigid movements:

$$RM(\Omega)^n = \{v \in W^{1,p}(\Omega)^n : \quad \varepsilon(v) = 0\},$$

every function in RM can be written as

$$v(x) = a + Mx,$$

where $M \in \mathbb{R}^{n\times n}$ is skew symmetric. On the other hand, a complement of RM in $W^{1,p}$ can be defined as follows

$$\widehat{W}^{1,p}(\Omega)^n = \left\{w \in W^{1,p}(\Omega)^n : \quad \fint_B w = 0, \quad \fint_\Omega \frac{Dw - Dw'}{2} = 0\right\}.$$

In fact, given $u \in W^{1,p}(\Omega)^n$, we can take $v \in RM(\Omega)^n$:

$$v = a + M(x - \bar{x}),$$

with

$$a = \fint_B u \quad \text{and} \quad m_{ij} = \frac{1}{2}\fint_\Omega \Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big),$$

being $\bar{x}$ the center of B. Obviously $w = u - v \in \widehat{W}^{1,p}(\Omega)^n$, and in particular

$$W^{1,p}(\Omega)^n = RM(\Omega)^n \oplus \widehat{W}^{1,p}(\Omega)^n.$$

Moreover, it is clear by definition that

$$\|v\|_{W^{1,p}(\Omega)^n} \leq C\|u\|_{W^{1,p}(\Omega)^n} \qquad \|w\|_{W^{1,p}(\Omega)^n} \leq C\|u\|_{W^{1,p}(\Omega)^n}.$$

Now we prove the theorem by contradiction. If (6.2.5) does not hold, there is a sequence $\{u_n\} \subset W^{1,p}(\Omega)^n$ such that

$$\|Du_n\|_{L^p(\Omega)^{n\times n}} = 1 \tag{6.2.6}$$

but,

$$\|u_n\|_{L^p(B)^n} + \|\varepsilon(u_n)\|_{L^p_\sigma(\Omega)^{n\times n}} < \frac{1}{n}. \tag{6.2.7}$$

If we write

$$u_n = v_n + w_n,$$

with $v_n \in RM(\Omega)^n$ and $w_n \in \widehat{W}^{1,p}(\Omega)^n$, w_n admits both Poincaré inequality in B, and second case of Korn inequality in Ω:

$$\begin{aligned}\|w_n\|_{W^{1,p}(\Omega)^n} = \|w_n\|_{L^p(\Omega)^n} + \|Dw_n\|_{L^p(\Omega)^{n\times n}} &\le C\Big(\|w_n\|_{L^p(B)^n} + \|Dw_n\|_{L^p(\Omega)^{n\times n}}\Big)\\ &\le C\|Dw_n\|_{L^p(\Omega)^{n\times n}} \le C\|\varepsilon(w_n)\|_{L^p_\sigma(\Omega^{n\times n})} < C\frac{1}{n}.\end{aligned}$$

And then, $w_n \longrightarrow 0$ in $W^{1,p}$. On the other hand, v_n belongs to the finite dimensional space $RM(\Omega)^n$ and is bounded on Ω. Consequently, there is a sub sequence, called again v_n, such that $v_n \longrightarrow v \in RM(\Omega)^n$ strongly in $W^{1,p}(B)^n$. As $w_n \longrightarrow 0$, we have that

$$u_n \longrightarrow v \in RM(B)^n \quad \text{in } W^{1,p}(\Omega)^n.$$

But because of (6.2.7), $\|v\|_{L^p(B)^n} = 0$, and v is a linear function, so $v \equiv 0$ on Ω, which contradicts (6.2.6), and the result follows in the case $\omega \equiv 1$. The general case can be treated by the same means defining the appropriate weighted versions

$$RM_\omega(\Omega)^n = \{v \in W^{1,p}_\omega(\Omega)^n : \quad \varepsilon(v) = 0\},$$

and

$$\widehat{W}^{1,p}_\omega(\Omega)^n = \Big\{\widetilde{v} \in W^{1,p}_\omega(\Omega)^n : \quad \int_B \widetilde{v}\omega = 0, \quad \int_\Omega \frac{D\widetilde{v} - D\widetilde{v}'}{2}\omega = 0\Big\}.$$

□

6.3 Korn and Poincaré Inequalities for Chains of Quasi-Rectangles

The job done for chains of rectangles can be easily generalized for chains of more general sets, all we have to do is to set appropriate hypotheses.

Definition 6.3.1. *Let $\mathcal{V} = \{\Omega_i\}$ be a (finite or countable) collection of disjoint open sets. Assume that there exists a chain of rectangles $\mathcal{R} = \{R_i\}$, and such that $R_i \subset \Omega_i \subset CR_i$, for a fixed constant C. Finally assume that $C_{K_i} \le C\frac{\ell_M(R_i)}{\ell_m(R_i)}$ and $C_{P_i} \le C\ell_M(R_i)$ being C_{K_i} and C_{P_i} the constants for the Korn's second inequality and Poincaré inequality for Ω_i respectively. Then $\mathcal{V} = \{\Omega_i\}$ is called a chain of quasi-rectangles*

Definition 6.3.2. *Given a chain of quasi-rectangles $\mathcal{V}$, a $\mathcal{V}-$linked domain Ω is any open set such that $\cup(\mathcal{R}_I \cup \mathcal{V}) \subset \Omega$ and $\Omega \equiv (\cup\mathcal{V})$. Here $\mathcal{R}_I$ is a collection of intermediate rectangles associated to $\mathcal{R}$.*

Definition 6.3.3. *Let $\mathcal{V} = \{\Omega_i\}$, be a chain of quasi-rectangles, and Ω a $\mathcal{V}-$ linked domain. We say that ω is an admissible weight in Ω if for any $x \in \Omega_i$*

$$\omega(x) \underset{C}{\sim} \omega_{\Omega_i} \underset{C}{\sim} \omega_{\Omega_{i+1}} \ \forall i. \tag{6.3.1}$$

being ω_{Ω_i} appropriate constants.

Remark 6.3.4. *From Definitions 6.3.1, 6.3.2 and 6.3.3 one can readily find that any proof given in the previous section for $\mathcal{R}-$linked domains can be carried out for $\mathcal{V}-$linked domains. For this reason we state all the results of this section without further analisys.*

Theorem 6.3.5 (Second Case of weighted Korn's Inequality for Chains of Quasi-Rectangles)**.** *Let $\mathcal{V} = \{\Omega_i\}$ be a chain of quasi-rectangles and Ω a $\mathcal{V}-$ linked domain. Let $u \in W^{1,p}_\omega(\Omega)^n$, with ω an admissible weight (see (6.3.1)), be such that*

$$\frac{1}{\omega(\Omega)}\int_\Omega \frac{Du - Du^t}{2}\omega = 0.$$

Assume that for any k, $\omega(R_{k+1}) \le \alpha\omega(R_k)$ with $0 \le \alpha < 1$. Then

$$\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \le C\|\varepsilon\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}}$$

where $\sigma|_{R_i}$ can be taken as in Theorem 6.2.2.

Theorem 6.3.6 (Poincaré inequality for Chains of Quasi-Rectangles)**.** *Let $\mathcal{V} = \{\Omega_i\}$ be a chain of quasi-rectangles and Ω a $\mathcal{V}-$ linked domain. Let ω be an admissible weight such that for any k, $\omega(R_{k+1}) \le \alpha\omega(R_k)$ with $0 \le \alpha < 1$. Then if $u \in W^{1,p}_\omega(\Omega)^n$, and $\frac{1}{\omega(\Omega)}\int_\Omega u\omega = 0$, we have*

$$\|u\|_{L^p_\omega(\Omega)} \le C\|Du\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}},$$

where the weight σ is constant on each R_i and can be taken as $\sigma|_{\Omega_i} = C^p_{P_i}$.

Corollary 6.3.7. *With the same hypotheses of Theorem 6.3.6, assume that B is a ball such that $B \subset \Omega$, and $B \cap \Omega_j \neq \emptyset$ only for a finite number of rectangles. Then, for every $u \in W^{1,p}_\omega(\Omega)^n$, we have:*

$$\|u\|_{L^p_\omega(\Omega)^n} \le C\Big\{\|u\|_{L^p(B)^n} + \|Du\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}}\Big\}$$

where the weight σ is constant on each Ω_i and can be taken as $\sigma|_{\Omega_i} = C^p_{P_i}$.

Theorem 6.3.8 (General Case of Korn's inequality for Chains of Quasi-Rectangles)**.** *Let $\mathcal{V} = \{\Omega_i\}$ be a chain of rectangles, and Ω an $\mathcal{V}-$ linked domain. Consider an admissible weight ω and assume that $\omega(R_{k+1}) \le \alpha\omega(R_k)$ with $0 \le \alpha < 1$ and that $\max_j \ell_j(R_i) < C$, for any i. If B is a ball such that $B \subset \Omega$, and B meets only a finite number of rectangles Ω_i then for any $u \in W^{1,p}_\omega(\Omega)^n$, we have*

$$\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \le C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}}\Big\}, \tag{6.3.2}$$

where the weight σ is constant on each element of $\mathcal{V}$, and can be taken as $\sigma|_{\Omega_i} = C^p_{K_i}$.

Remark 6.3.9. *All these results can be proved exactly like the ones for chains of rectangles, except for a subtle detail: we impose the decreasing measure condition (6.2.4) on the rectangles R_i and not on the subdomains Ω_i, as it would be natural. This is possible because of the relationship between the measures of Ω_i and R_i. Indeed, since ω is admissible and $|R_i| \leq |\Omega_i| \leq C|R_i|$, we have*

$$\omega(\Omega_i) \leq C\omega_i|\Omega_i| \leq C\omega_i|R_i| \leq C\omega(R_i),$$

and

$$\omega(R_i) \leq C\omega_i|R_i| \leq C\omega_i|\Omega_i| \leq C\omega(\Omega_i).$$

And consequently:

$$\mathbf{A}_\omega = \sup_{k>0}\Big(\sum_{j=k}^{\infty}\omega(\Omega_j)\Big)^{\frac{1}{p}}\Big(\sum_{j=0}^{k}\omega(\Omega_j)^{1-p'}\Big)^{\frac{1}{p'}} \leq C\sup_{k>0}\Big(\sum_{j=k}^{\infty}\omega(R_j)\Big)^{\frac{1}{p}}\Big(\sum_{j=0}^{k}\omega(R_j)^{1-p'}\Big)^{\frac{1}{p'}}.$$

So, if the decreasing property (6.2.4) is imposed on the rectangles R_i we have that $\mathbf{A}_\omega$ is finite.

6.4 Chains of John quasi-rectangles

The previous Section seems to provide a generalization of the results obtained in Section 6.2. However, there is an important question that remains unanswered: *what is a quasi-rectangle?* In other words: let R be a rectangle, and Ω a domain such that $R \subset \Omega \subset CR$, and suppose that both Poincaré and the second case of Korn's inequalities hold in Ω. The question is: how general can Ω be if we ask the constants of these inequalities for Ω to be proportional to the respective constants for R? Here, we provide examples that give a partial answer to this question, showing that chains of quasi-rectangles form, indeed, a very general class of domains.

Definition 6.4.1. *We say that $\mathcal{U} = \{\Omega_i\}$ is a chain of quasi-cubes if it is a chain of quasi-rectangles, where the rectangle contained in Ω_i is a cube Q_i, for every i.*

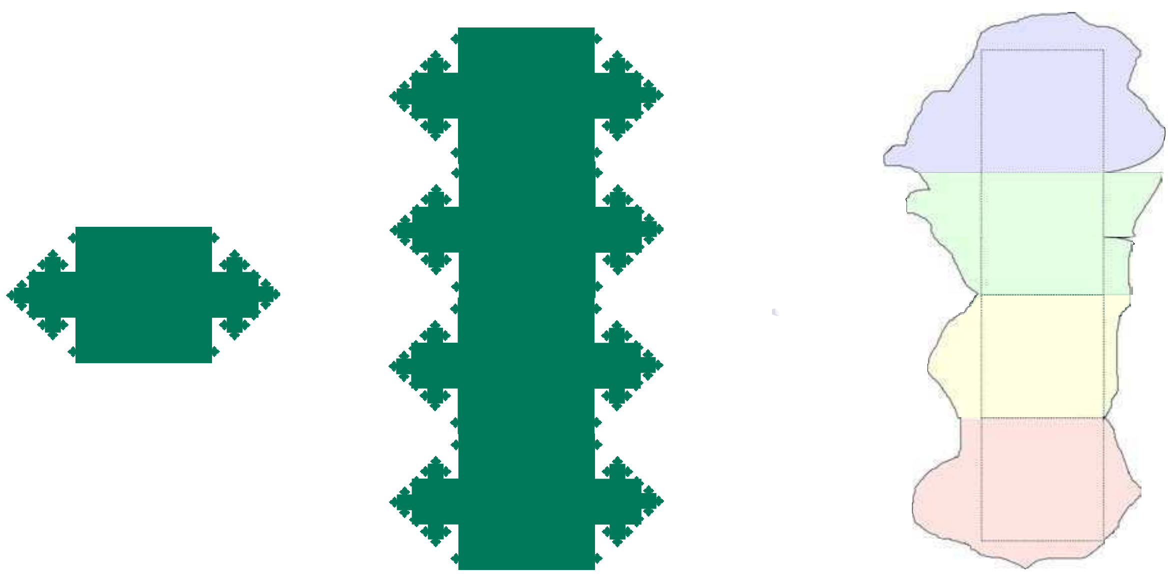
Now, consider Ω an $\mathcal{U}$ – *linked* domain, being $\mathcal{U} = \{\Omega_i\}$, $i = 1, \dots, N$ a finite chain of quasi-cubes where all the cubes Q_i are placed along a straight line and have the same size $\ell(Q_i) = \ell$. If we apply Theorem 6.3.8 to Ω, recalling that the constant $\mathbf{A}$ satisfies $\mathbf{A} \leq CN$ (see Lemma 6.2.3), we have that the constant in the second case of Korn's inequality for Ω is $C_K \leq CN$. But the number of cubes N is precisely $\frac{\ell_M(R)}{\ell_m(R)}$ where R is the rectangle formed by the union of the cubes Q_i. In the same way, the Poincaré constant for $\mathcal{U}$ is $C_P \leq CN\ell(Q) = C\ell_M(R)$. Hence: Ω is a quasi-rectangle.

Consequently, in order to build a quasi-rectangle we only need to put together a finite number of almost cubic domains, where Korn and Poincaré inequalities hold.

Remark 6.4.2. *Recall that the second case of Korn's inequality holds on John domains [Acosta et al., 2006b]. Morever, improved Poincaré inequalities also stand on such domains: [Hurri-Syrjänen, 1994] and [Drelichman and Durán, 2008]. This motivates the following definition:*

Definition 6.4.3. *Let Ω be a $\mathcal{U}$ − linked domain with $\mathcal{U} = \{\Omega_i\}$, $i = 1, \ldots, N$ a chain of quasi-cubes placed along a straight line, satisfying $\ell(Q_i) = \ell$. We say that Ω is a* John quasi-rectangle *if every subdomain Ω_i is a John domain with respect to the center of Q_i.*

The previous discusion implies that in every John quasi rectangle, $C_K = C\frac{\ell_M(R)}{\ell_m(R)}$, and $C_P = C\ell_M(R)$, where R is the rectangle formed by the union of the cubes Q_i.



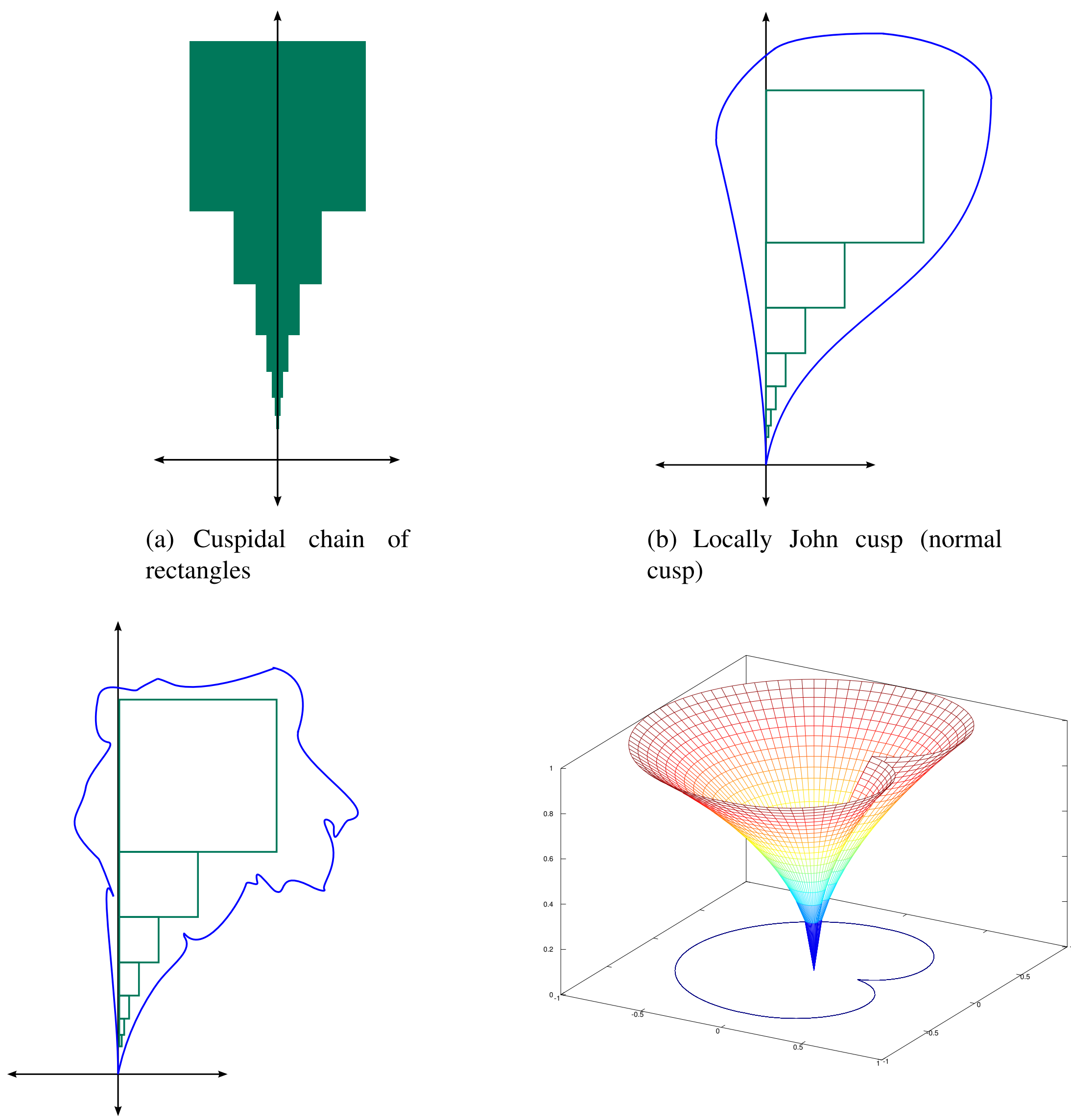
(a) A quasi-cube (b) A Quasi-rectangle (c) Another Quasi-rectangle

Figure 6.1: Quasi-rectangles

In Figure 6.1, we show a quasi-cube and two quasi-rectangles. The quasi-cube is a self-similar fractal like the Koch snowflake, without the upper and lower ramifications. The first quasi-rectangle is a tower formed by four quasicubes like the one in Figure 6.1(a). Finally, in Figure 6.1(c), we show another quasi-rectangle, formed by quasi-cubes of different shapes, but similar aspect-ratio. The quasicube in 6.1(a) is a uniform domain, and then, Korn's inequality stands there. On the other hand, the quasicubes in 6.1(c) are not all of them uniform (observe that some of them have inner cusps), but are John domains. Consequently, both these quasi-rectangles are in fact John quasi-rectangles.

Naturally, we can consider chains of John quasi-rectangles, where the results of the previous section can be applied, as long as Property (6.2.4) is satisfied. Since we are particularly interested in domains having external cusps, we show a class of domains, linked by chains of John quasi-rectangles, where the chain narrows toward the origin forming an external cusp. Korn's inequalities can be derived for these domains, that generalize Theorem B.

Definition 6.4.4. *Let $\mathcal{V} = \{\Omega_i\}$ be a chain of John quasi-rectangles with rectangles $\mathcal{R} = \{R_i\}$, and suppose that R_i has $n-1$ short edges of length ℓ_i and one of length L_i. Furthermore, suppose that the rectangles R_i are placed one above the other, along the x_n axis, so that $\bar{R}_{i+1} \cap \bar{R}_i = F^u_{R_{i+1}}$, and let z_i be the x_n coordinate of the points in the floor of R_i. Finally, let us assume that $|R_{i+1}| \leq \alpha |R_i|$ for some $\alpha < 1$. Now, let $\varphi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ be a nondecreasing C^1 function such that φ' is nondecreasing, and $\varphi(0) = \varphi'(0) = 0$ and such that $\varphi(z_i) = \ell_i$. Then, a $\mathcal{V}-$ linked domain Ω is called a* locally John cusp.



(a) Cuspidal chain of rectangles

(b) Locally John cusp (normal cusp)

(c) Locally John cusp

(d) Locally John cusp by dilatation

Figure 6.2: Examples of locally John cusps

It is clear that a locally John cusp Ω is, in fact, an external cusp, and that the function φ gives the cuspidal behaviour of Ω. In Figure 6.2 we show examples of locally John cusps. Figure 6.2(a) is just a chain of rectangles satisfying: $L_{i+1} \sim \frac{1}{\sqrt{2}} L_i$ and $\ell_i = z_i^2$. Figure 6.2(b) shows an external cusp with locally smooth boundary away from the origin. The interior chain of rectangles is like the one in 6.2(a), but leant. In is important to notice that these are a normal and a curved cusp, respectively. On the other hand, Figure 6.2(c) is a perturbation of 6.2(b), formed by a chain of John quasi-rectangles. Finally, observe that arguing like in Section 3.4, we can prove that a domain satisfying Definition A, but taking $\varpi \subset \mathbb{R}^{n-1}$ a John domain with respect to the center of a cube included in ϖ, is a locally John cusp. This is the case of Figure 6.2(d), where we take $\varphi(t) = t^2$ and ϖ an inner cusp.

The unweighted results of the previous section can be immediatly applied to a locally John cusp Ω, obtaining an unweighted Poincaré inequality, and a weighted Korn one, both in the second and the general case.

Moreover, if we take a weight ω which is a nondecreasing function of x_n (or $|x|$), we have:

$$\omega(R_{i+1}) \le \Big\{\max_{[z_{i+1},z_i]} \varphi\Big\}|R_{i+1}| \le \alpha \Big\{\min_{[z_i,z_{i-1}]} \varphi\Big\}|R_i| \le \alpha\omega(R_i),$$

and the decreasing property (6.2.4) is fulfilled. In this way we can consider some particularly interesting weights. For example, being φ' non-decreasing, we can take weights of the form: $\omega(x) = (\varphi')^{p\beta}$ with $\beta \ge 0$. On the other hand, we can also take weights of the form $\omega(x) = x_n^{p\beta}$, being $\beta \ge 0$. In this way, we obtain the following Theorem. Let us denote $L(a)$ and $\ell(a)$ the lengths of the edges $L(R)$ and $\ell(R)$, being R the rectangle at height a.

Theorem 6.4.5. *Let Ω be a locally John cusp, and $\sigma(x) = \left(\frac{\ell(|x|)}{L(|x|)}\right)^{-p}$. Then the inequality:*

$$\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \le C\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}}\}$$

holds for weights of the form:

(a)
$$\omega(x) = \gamma x_n^{p\beta}, \qquad \beta \ge 0$$

(b)
$$\omega(x) = (\varphi')^{p\beta}, \qquad \beta \ge 0$$

It is important to observe that, if φ is such that $\varphi(z_{i-1}) - \varphi(z_i) \sim \varphi(z_i)$, then:

$$\frac{\ell_i}{L_i} = \frac{\varphi(z_i)}{z_{i-1} - z_i} \sim \frac{\varphi(z_{i-1}) - \varphi(z_i)}{z_{i-1} - z_i} \sim \varphi'(z_i).$$

Hence, $\sigma \sim (\varphi')^{-p}$ and item (b) in Theorem 6.4.5 is a generalization of Theorem B. In fact, as in Theorem B, the weight on the left hand side is $(\varphi')^{p\beta}$, whereas the one on the right hand side is $(\varphi')^{p(\beta-1)}$. Here, φ is not forced to be a power function and it does not depict the precise profile of Ω but only provides a qualitative description of its cuspidal behaviour, allowing the boundary of Ω to be locally John. It is also noteworthy that the critical case of Theorem C is reached.

On the other hand, let us consider a profile cusp satisfying a definition similar to Definition A, but taking ϖ a John domain. An example can be seen in Figure 6.2(d). Moreover, let us suppose $\varphi(z) = z^\gamma$ for some $\gamma > 1$. We show how the rectangles can be chosen in order to prove that such a cusp is a locally John one.

Let us take

$$z_i = \frac{1}{2^i}.$$

The rectangle R_i is placed at height z_i, and the length of its edges is

$$\ell_i = \varphi(z_i) = \frac{1}{2^{i\gamma}}, \quad \text{and} \quad L_i = z_{i-1} - z_i = \frac{1}{2^i}.$$

Let us consider a weight of the form:

$$\omega(x) = \left(\frac{\ell_i}{L_i}\right)^{p\beta} = \frac{1}{2^{i(\gamma-1)p\beta}} \quad \forall x \in R_i.$$

Then

$$\begin{aligned}\omega(R_{i+1}) &= \frac{1}{2^{(i+1)(\gamma-1)p\beta}}|R_{i+1}| = \frac{1}{2^{(i+1)(\gamma-1)p\beta}}\frac{1}{2^{(i+1)\gamma(n-1)}}\frac{1}{2^{i+1}} = \frac{1}{2^{(i+1)\left((\gamma-1)p\beta+\gamma(n-1)+1\right)}}\\ &= \frac{1}{2^{(\gamma-1)p\beta+\gamma(n-1)+1}}\omega(R_i).\end{aligned}$$

Hence, the decresing property (6.2.4) is satisfied when

$$\frac{1}{2^{(\gamma-1)p\beta+\gamma(n-1)+1}} < 1,$$

or, in other words:

$$(\gamma-1)p\beta + \gamma(n-1) + 1 > 0,$$

which leads us to:

$$\beta > -\frac{1+\gamma(n-1)}{(\gamma-1)p}$$

Since $\omega \sim (\varphi')^{p\beta}$, we can express the weight in terms of φ', obtaining the following result:

Theorem 6.4.6. *Let Ω be an external cusp satisfying a definition like Definition A, but taking $\varpi \subset \mathbb{R}^{n-1}$ a John domain, and $\varphi(z) = z^\gamma$, with $\gamma > 1$. Then:*

$$\|Du\|_{L^p_\omega(\Omega)} \leq C\Big\{\|u\|_{L^p(B)} + \|\varepsilon(u)\|_{L^p_{\omega\sigma}(\Omega)}\Big\},$$

with:

$$\sigma(x) = (\varphi'(x))^{-p} \quad \textit{and} \quad \omega(x) = (\varphi'(x))^{p\beta},$$

being $\beta > -\frac{1+\gamma(n-1)}{(\gamma-1)p}$.

This result is also a generalization of Theorem B. It imposes more restrictions than Theorem 6.4.5 on the boundary of Ω, but it admits a negative range for the exponent β. On the other hand, the critical case $\beta_1 = \beta_2 + 1$ in Theorem C is once again reached. It is important to notice that the counterexamples proposed in [Acosta et al., 2012] for proving Theorem C (like the ones provided by Maz'ya for the extension problem that we reproduce in Section 5.1.5), are given in terms of functions that depend only on the last coordinate and on the profile function φ. Consequently, they are independent of the boundary of the cusp, and can be easily adapted for locally John cusps.

Appendix A

Korn inequality for normal cusps using extension arguments

The results obtained in the last Chapter, regarding Korn and Poincaré inequalities for chains of rectangles and quasi-rectangles, can be applied to many general domains. In fact, we prove in Theorems 6.4.5 and 6.4.6 that Theorem B can be generalized to cusps with John boundary by stripes.

This was not, however, our first approach. In [Durán and Muschietti, 2004], the authors prove Korn's inequality for uniform domains using an adaptation of Jones's extension operator. Having proved the extension theorems of Chapter 5, and bearing [Durán and Muschietti, 2004] in mind, we begin our work on this subject hoping that the arguments developed in [Durán and Muschietti, 2004] could be adapted to prove Korn's inequality for normal and curved cusps. In the course of our research, we found that the notion of quasi-rectangle enables us to prove far more general results, such as Theorems 6.3.8, 6.4.5 and 6.4.6.

Since we find the extension approach interesting, we include in this Appendix an sketched proof of Korn's inequalities for normal cusps, following an adaptation of the extension arguments used in [Durán and Muschietti, 2004]. It is important to take into account that the concept of quasi-rectangle is not needed here. We only use the results of Section 6.2, regarding chains of rectangles.

Our arguments follow this line of reasoning: the spine $\mathcal{S}$ of a normal cusp Ω can be seen as a chain of rectangles, as long as we pack together all the cubes S_i of the same size. Consequently, we could apply Korn inequalities for chains of rectangles to $\mathcal{S}$. Moreover, we could apply them to a fixed dilatation of $\mathcal{S}$ that covers Ω. Then, an extension argument can be used in order to prove Korn inequalities for normal cusps. For doing this, we begin stating the definition of stepped cusp. Stepped cusps are ment to reproduce the behaviour of the spine of a normal cusp, in order to simplify the extension process.

Let $\mathcal{R} = \{R_i\}$ be a chain of rectangles, such that R_i has $n-1$ short edges of length ℓ_i and one of length L_i. Observe that the Korn constant on R_i satisfies: $C_{K_i} \leq C\frac{L_i}{\ell_i}$, $\forall i$. In order to produce a cuspidal behaviour in $\mathcal{R}$, let us add a few hyphotheses on the rectangles R_i:

$$\bar{R}_i \cap \bar{R}_{i+1} = F^u_{R_{i+1}}, \tag{A.1}$$

$$z_i \searrow 0. \tag{A.2}$$

This implies that $\mathcal{R}$ is a tower that approaches the origin. Furthermore, we can suppose that the shape of $\mathcal{R}$ is like the one of the spine of a normal cusp. This can be expressed through the following conditions:

$$\frac{1}{4}\ell_i \leq \ell_{i+1} \leq \frac{1}{2}\ell_i \quad \forall i, \tag{A.3}$$

$$\frac{1}{2}L_i \leq L_{i+1} \leq L_i \quad \forall i. \tag{A.4}$$

These properties establish a rule for the narrowing of the rectangles. The constants involved in both of them can be chosen in a different way, or even be expressed as abstract constants c_1, c_2 and c_3, c_4. The actual values $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{2}$, $c_4 = 1$, have been selected arbitrarily, but not mindlessly. Property (A.3) corresponds with the narrowing given in a spine of a normal cusp, formed by Whitney cubes. On the other hand, the constants in Property (A.4) simplify some calculations. Finally let us impose a last requirement in order to exclude non-singular domains:

$$\frac{\ell_i}{z_i} \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty. \tag{A.5}$$

Definition A.1. *Let $\mathcal{R} = \{R_i\}$ be a chain of rectangles such that the edges of R_i corresponding to the $n-1$ first coordinate axis have length ℓ_i, whereas the length of the n-th edge is L_i (we assume $L_i > \ell_i$). An $\mathcal{R}-$linked domain Ω is a* stepped cusp *if the rectangles R_i are placed at positive heights $\{z_i\}_i$ and satisfy properties* (A.1) *to* (A.5).

Figure 6.2(a) is a stepped cusp.

Note that:

$$|R_{i+1}| = \ell_{i+1}^{n-1} L_{i+1} \leq \frac{1}{2^{n-1}} \ell_i^{n-1} L_i = \frac{1}{2^{n-1}} |R_i|.$$

Then, the results obtained in Section 6.4 in the unweighted case can be applied to stepped cusps, and hence Korn and Poincaré inequalities hold on them. A similar analysis can be performed for the weighted case, considering weights like those in Theorem 6.4.5.

Now we want to extend these results to normal cusps.

Let us take Ω a normal cusp, and :

$$\widehat{\Omega} = \bigcup_i \widehat{K} \star S_i. \tag{A.6}$$

Lemma 5.1.3 shows that $\widehat{K}$ can be chosen depending only on the parameter K of Ω and such that $\Omega \subset \widehat{\Omega}$. It is clear that $\widehat{\Omega}$ is a union of rectangles. In order to study these rectangles, let us define $\{i_k\}_k$ the sequence of indices such that the subsequence of $\mathcal{S}$, $\{S_{i_k}\}_k$ is formed with the first cubes of each size, it is: $\ell(S_{i_k}) < \ell(S_{i_k-1})$ and $\ell(S_{i_{k-1}}) = \ell(S_{i_k-1})$. Now, we can define:

$$R_k = \bigcup\{\widehat{K} \star S \, : \; \ell(S) = \ell(S_{i_k})\},$$

and we have that $\widehat{\Omega} = \cup_k R_k$. Morever, it is clear that R_k is a rectangle with $n-1$ edges of length

$$\ell(R_k) = \widehat{K}\ell(S_{i_k}),$$

and one edge (the x_n edge) of length

$$L(R_k) = \ell(S_{i_k})\#\{S \in \mathcal{S} : \; \ell(S) = \ell(S_{i_k})\}.$$

Since S_i and S_{i+1} are placed one above the other, the same thing happens to R_k and R_{k+1}.

It is also clear that, by definition $\{R_k\}_k$ satisfies Properties (A.1) and (A.2). On the other hand, since we are dealing with Whitney cubes, we have:

$$\frac{1}{4}\ell(R_k) = \frac{1}{4}\ell(S_{i_k}) \leq \ell(S_{i_{k+1}}) = \ell(R_{k+1}) \leq \frac{1}{2}\ell(S_{i_k}) = \frac{1}{2}\ell(R_k),$$

so $\{R_k\}_k$ satisfies Property (A.3). Property (A.5) is obviously satisfied thanks to Property (3.2.6).

Finally, in order to guarranty that $\widehat{\Omega}$ satisfies Property (A.4), we have to impose one extra hypothesis on Ω. Let us denote:

$$\#_k = \#\{S \in \mathcal{S} : \quad \ell(S) = \ell(S_{i_k})\}.$$

The natural translation of Property (A.4) into the language of normal cusps is:

$$\frac{1}{2}\ell(S_{i_k})\#_k \leq \ell(S_{i_{k+1}})\#_{k+1} \leq \ell(S_{i_k})\#_k. \tag{A.7}$$

Now, we have proved the following:

Lemma A.2. *Let Ω be a normal cusp satisfying Property* (A.7)*, and let $\widehat{\Omega}$ be the domain defined in* (A.6)*. Then, $\widehat{\Omega}$ is a stepped cusp.*

In order to prove Korn's inequality for normal cusps, we need to provide an extension operator from the normal cusp Ω to the stepped cusp $\widehat{\Omega}$, *that preserves the norm of $\varepsilon(u)$*. This last requirement forces us to introduce a little modification in the extension operator presented in Chapter 5. Particularly, we take another polynomial approximation, that fulfills all our needs. We follow [Durán and Muschietti, 2004], where the authors prove Korn's inequality for uniform domains using Jones's extension operator modified with the proper polynomial approximation on cubes.

For every cube $T \in \mathcal{W}(\Omega)$, let us define:

$$P_T(x) = a + M(x - x_T), \tag{A.8}$$

where $a \in \mathbb{R}^n$ and $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ are defined by:

$$a = \fint_T u \qquad M_{i,j} = \frac{1}{2}\fint_T \Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big),$$

and x_T is the center of T.

It is easy to check that $\varepsilon(P_T) = 0$, so $P_T \in RM(T)^n$. Consequently, $u - P_T$ satisfies:

$$\fint_T u - P_T = 0, \tag{A.9}$$

and

$$\fint_T D(u - P_T) \text{ is symmetric.} \tag{A.10}$$

This allows the application of Poincaré's and second case Korn's inequalities on T:

$$\|u - P_T\|_{L^p(T)^n} \le C\ell(T)\|D(u - P_T)\|_{L^p(T)^{n\times n}} \le C\ell(T)\|\varepsilon(u)\|_{L^p(T)^{n\times n}}. \tag{A.11}$$

Furthermore,

$$\|DP_T\|_{L^\infty(T)^{n\times n}} \le C\|Du\|_{L^\infty(T)^{n\times n}},$$

and then,

$$\|D(u - P_T)\|_{L^\infty(T)^{n\times n}} \le C\|Du\|_{L^\infty(T)^{n\times n}}.$$

The last inequalities are needed for proving the existence of weak derivatives of the extended function (as in Section 5.1.4). Property (A.11), on the other hand, allows us to prove a suitable equivalent of Lemma 5.1.9. Observe that we only need the first stage of the extension process, since:

$$\Omega \subset \widehat{\Omega} \subset \Omega \cup (\cup \mathcal{W}_3),$$

(eventually adjusting the constants in the definition of $\widehat{\Omega}$ and $\mathcal{W}_3$).

The extension operator is defined as in Chapter 5, but using these new polynomials.:

$$Eu(x) = \chi_\Omega(x) f(x) + \sum_{Q_j \in \mathcal{W}_3} P_{Q_j^*}(x)\varphi_j(x),$$

where φ_j is a partition of the unity associated with the cubes $Q_j \in \mathcal{W}_3$.

The following Lemma is equivalent to Lemma 5.1.9, and it is analogous to Lemma 2.4 in [Durán and Muschietti, 2004]:

Lemma A.3. *Let $u \in W^{1,p}(\Omega)^n$, and Q a cube in $\mathcal{W}_3$, then:*

$$\|Eu\|_{L^p(Q)^n} \leq C\Big\{\|u\|_{L^p(Q^*)^n} + \ell(Q)\|\varepsilon(u)\|_{L^p(\mathcal{F}(Q))^{n\times n}}\Big\}, \tag{A.12}$$

$$\|\varepsilon(Eu)\|_{L^p(Q)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p(\mathcal{F}(Q))^{n\times n}}, \tag{A.13}$$

$$\|Eu\|_{L^\infty(Q)^n} \leq C\Big\{\|u\|_{L^\infty(Q^*)^n} + \ell(Q)\|\varepsilon(u)\|_{L^\infty(\mathcal{F}(Q))^{n\times n}}\Big\}, \tag{A.14}$$

$$\|D(Eu)\|_{L^\infty(Q)^{n\times n}} \leq C\|Du\|_{L^\infty(\mathcal{F}(Q))^{n\times n}}. \tag{A.15}$$

Proof. For (A.12):

$$\|Eu\|_{L^p(Q)^n} = \Big\| \sum_{j:Q_j\cap Q_i\neq\emptyset} P_{Q_j^*}\phi_j\Big\|_{L^p(Q)^n} \leq \underbrace{\sum_j \Big\|(P_{Q_j^*} - P_{Q^*})\phi_j\Big\|_{L^p(Q)^n}}_{I} + \underbrace{\|P_{Q^*}\|_{L^p(Q)^n}}_{II}.$$

II can be easily bounded by means of Corollary 2.4.3 and (A.11):

$$\begin{aligned} II &\leq C\|P_{Q^*}\|_{L^p(Q^*)^n} \leq C\{\|P_{Q^*} - u\|_{L^p(Q^*)^n} + \|u\|_{L^p(Q^*)^n}\} \\ &\leq C\{\|u\|_{L^p(Q^*)^n} + \ell(Q)\|\varepsilon(u)\|_{L^p(Q^*)^{n\times n}}\}. \end{aligned}$$

On the other hand, for I we use Lemma 5.1.7: for a fixed j, let $\mathcal{F}_j = \{S_1 = Q^*, \dots, S_r = Q_j^*\}$ be the chain of cubes that join Q^* with Q_j^*. Then:

$$\begin{aligned} \|(P_{Q_j^*} - P_{Q^*})\phi_j\|_{L^p(Q)^n} &\leq \sum_{i=1}^r \|P_{S_i} - P_{S_{i+1}}\|_{L^p(Q)^n} \\ &\leq \sum_{i=1}^r \{\|P_{S_i} - P_{S_i\cup S_{i+1}}\|_{L^p(Q)^n} + \|P_{S_i\cup S_{i+1}} - P_{S_{i+1}}\|_{L^p(Q)^n}\} \\ &\leq C\sum_{i=1}^r \{\|P_{S_i} - P_{S_i\cup S_{i+1}}\|_{L^p(S_i)^n} + \|P_{S_i\cup S_{i+1}} - P_{S_{i+1}}\|_{L^p(S_{i+1})^n}\} \\ &\leq C\sum_{i=1}^r \{\|P_{S_i} - u\|_{L^p(S_i)^n} + \|u - P_{S_i\cup S_{i+1}}\|_{L^p(S_i\cup S_{i+1})^n} + \|u - P_{S_{i+1}}\|_{L^p(S_{i+1})^n}\} \\ &\leq C\sum_{i=1}^r \ell(S_i)\|\varepsilon(u)\|_{L^p(S_i\cup S_{i+1})^{n\times n}} \leq C\ell(Q)\|\varepsilon(u)\|_{L^p(\cup\mathcal{F}_j)^{n\times n}}, \end{aligned}$$

where the last inequality is obtained applying Hölder inequality and taking into account that $r \leq C$ for a universal constant C.

For (A.13), let us denote $P^k_{Q^*}$ the k-th component of P_{Q^*}. Since $\varepsilon(P_{Q^*}) = 0$, we have that:

$$\varepsilon_{km}(P_{Q_i^*}\phi_i) = \frac{1}{2}\Big(P^k_{Q_i^*}\frac{\partial\phi_i}{\partial x_m} + P^m_{Q_i^*}\frac{\partial\phi_i}{\partial x_k}\Big),$$

since on Q^*, $Eu = P_{Q^*} + \sum_j P_{Q_j^*}\phi_j$:

$$\varepsilon(Eu) = \sum_{j:Q_j\cap Q\neq\emptyset} \varepsilon((P_{Q_j^*} - P_{Q^*})\phi_j),$$

and then:

$$\begin{aligned}\|\varepsilon(Eu)\|_{L^p(Q^*)^{n\times n}} &\leq \sum_j \frac{1}{\ell(Q)}\|P_{Q_j^*} - P_{Q^*}\|_{L^p(Q)^{n\times n}} \leq C\sum_j \frac{1}{\ell(Q)}\|P_{Q_j^*} - P_{Q^*}\|_{L^p(Q^*)^{n\times n}} \\ &\leq C\sum_j \|\varepsilon(u)\|_{L^p(\mathcal{F}_j)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p(\mathcal{F}(Q))^{n\times n}}.\end{aligned}$$

The arguments for (A.14) and (A.15) are anologous. □

Now, we can sum over all the cubes in $\mathcal{W}_3$, as it is done in (5.1.10). We write the following Corollary adding and admissible weight. The proof is the same than the one of (5.1.10):

Corollary A.4. *Let ω be an admissible weight for Ω. If $u \in W^{1,\infty}(\Omega)^n$, then:*

$$\|Eu\|_{L^p_\omega(\cup\mathcal{W}_3)^n} \leq C\|u\|_{L^p_\omega(\Omega)^n},$$

$$\|\varepsilon(Eu)\|_{L^p_\omega(\cup\mathcal{W}_3)^{n\times n}} \leq C\|\varepsilon(Eu)\|_{L^p_\omega(\Omega)^{n\times n}}.$$

And:

$$\|Eu\|_{W^{1,\infty}(\cup\mathcal{W}_3)^n} \leq C\|u\|_{W^{1,\infty}(\Omega)^n}.$$

Finally, since:

$$Eu \in W^{1,p}((\cup\mathcal{W}_3)\cup\Omega)^n \qquad \text{for every } u \in W^{1,p}(\Omega)^n,$$

and $\widehat{\Omega} \subset (\cup\mathcal{W}_3)\cup\Omega$, we have:

$$Eu \in W^{1,p}(\widehat{\Omega})^n.$$

Being $\widehat{\Omega}$ a stepped cusp this allows us to state the following:

Theorem A.5. *Let Ω be a normal cusp satisfying Property (A.7), $u \in W^{1,p}(\Omega)^n$ and B a ball contained in Ω. Then:*

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_\sigma(\Omega)^{n\times n}}\},$$

where

$$\sigma(x) = (\#\{S \in \mathcal{S} : \ \ell(S) = \ell(S(|x|))\})^p.$$

Proof. Let us denote $\widetilde{\sigma}$ the weight corresponding to $\widehat{\Omega}$ according to Theorem 6.2.11. If we denote $\{R_i\}_i$ the set of rectangles that defines $\widehat{\Omega}$, we have that:

$$\ell(R_i) = \ell(S_j),$$

for some $S_j \in \mathcal{S}$, and:

$$L(R_i) = \ell(R_i) \cdot \#\{S \in \mathcal{S} : \ \ell(S) = \ell(S_j)\}.$$

Thence,

$$\widetilde{\sigma}(x) = \sigma(x) \quad \forall x \in \Omega,$$

so $\widetilde{\sigma}$ is an extension of σ on $\widehat{\Omega}$.

Now, applying Theorem 6.2.11 for Eu and Corollary A.4:

$$\begin{aligned}
\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \leq \|D(Eu)\|_{L^p_\omega(\widehat{\Omega})^{n\times n}} &\leq C\Big\{\|Eu\|_{L^p(B)^n} + \|\varepsilon(Eu)\|_{L^p_{\omega\widetilde{\sigma}}(\widehat{\Omega})^{n\times n}}\Big\} \\
&\leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{\omega\widetilde{\sigma}}(\Omega)^{n\times n}}\Big\} \leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{\omega\sigma}(\Omega)^{n\times n}}\Big\}.
\end{aligned}$$

□

Note that the case of curved cusps can be treated in the same way, applying stage zero, instead of the first stage.

The weighted form of this result, with weights like those in Theorem 6.4.5, can be proved in the same way.

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Index