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Propiedades de regularidad homológica de variedades de banderas cuánticas y álgebras asociadas

Zadunaisky Bustillos, Pablo

2014

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UNIVERSIDAD DE BUENOS AIRES

Facultad de Ciencias Exactas y Naturales

Departamento de Matemática

Propiedades de regularidad homológica de variedades de banderas cuánticas y álgebras asociadas

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Propiedades de regularidad homológica de variedades de banderas cuánticas y álgebras asociadas

Los objetos de estudio de esta tesis pertenecen a dos familias de "variedades no conmutativas", es decir álgebras $\mathbb{N}$-graduadas conexas noetherianas a las que consideramos, siguiendo la perspectiva de la geometría no conmutativa, como análogos de anillos de coordenadas homogéneas sobre ciertas variedades proyectivas.

La primera familia es la de las variedades tóricas cuánticas, subálgebras graduadas de toros cuánticos. Clasificamos estas álgebras y estudiamos en detalle sus propiedades de regularidad homológica, definidas por Artin-Schelter, Zhang, Van den Bergh, etc. La segunda familia es la de las álgebras conocidas como variedades de banderas cuánticas y otras álgebras asociadas, análogos no conmutativos de las álgebras de coordenadas homogeneas de las variedades de banderas y de sus subvariedades de Schubert. Demostramos que los miembros de esta segunda familia pueden filtrarse de forma que sus álgebras graduadas asociadas son variedades tóricas cuánticas. Luego probamos que las propiedades de regularidad homológica de las álgebras de las variedades de bandera y de Schubert cuánticas se deducen de las propiedades de las variedades tóricas cuánticas.

Palabras clave: Variedades de banderas cuánticas, variedades tóricas cuánticas, Cohen-Macaulay, Gorenstein, complejos dualizantes

Homological regularity properties of quantum flag varieties and related algebras

The objects of study of this thesis are two families of "noncommutative varieties", that is noetherian connected $\mathbb{N}$-graded algebras which, following the general notions of noncommutative geometry, we regard as analogues of homogeneous coordinate rings of certain projective varieties.

The first family is that of quantum toric varieties, which are graded subalgebras of quantum tori. We classify these algebras and study their homological regularity properties as defined by Artin-Schelter, Zhang, Van den Bergh, etc. The second family is that of quantum flag varieties and associated algebras, noncommutative analogues of the homogeneous coordinate rings of flag varieties and their Schubert subvarieties. We show that the members of this second family can be endowed with a filtration such that their associated graded algebras are quantum toric varieties. We then show that the homological regularity properties of quantum flag and Schubert varieties can be deduced from those of quantum toric varieties.

Keywords: Quantum flag varieties, quantum toric varieties, Cohen-Macaulay algebras, Gorenstein algebras, dualizing complexes.

Régularité homologique des variétés de drapeaux quantiques et de quelques algèbres liées.

Deux familles d'algèbres noethériennes connexes constituent les objets d'étude de cette thèse; on les regarde, suivant les idées générales de la géométrie non commutative, comme des anneaux de coordonnées homogènes de certaines variétés projectives.

La première famille est celle des variétés toriques quantiques, autrement dit les sousalgèbres graduées de tores quantiques. Nous classifions ces algèbres et nous étudions ses propriétés de régularité homologique suivant notamment Artin-Schelter, Zhang et Van den Bergh. La deuxième famille est celle des variétés de drapeaux quantiques et leurs sousvariétés de Schubert. Nous démontrons que les algèbres appartenant a cette deuxième famille possèdent une filtration tel que leur graduée associé est une variété torique quantique. En suite nous démontrons que les propriétés de régularité homologique des variétés de drapeaux quantiques et des variétés de Schubert se déduisent de celles des variétés toriques quantiques.

Mots clés: Variétés des dreapeaux quantiques, variétés toriques quantiques, algèbres du type Cohen-Macaulay, algèbres du type Gorenstein, complexe dualisant.

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Buenos Aires - París - Buenos Aires - París - Buenos Aires...

2009-2014.

Para Mauricio y Jaime, mis abuelos

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Introducción

Este trabajo está escrito bajo la perspectiva conocida informalmente como "geometría proyectiva no conmutativa", es decir, la idea de que las álgebras $\mathbb{N}$-graduadas asociativas pero no necesariamente conmutativas pueden ser consideradas álgebras de coordenadas homogéneas sobre "variedades proyectivas no conmutativas". A partir de este principio general se intenta adaptar las técnicas del caso conmutativo, conservando en la medida de lo posible la intuición geométrica, al no conmutativo. Muchos ejemplos de álgebras a las que se aplica esta idea se obtienen a través del proceso conocido como cuantización. Otra vez de manera informal, un álgebra conmutativa se "cuantiza" deformando su multiplicación a través de un parámetro continuo, lo cual resulta muchas veces en álgebras no conmutativas. El álgebra original puede recuperarse tomando el límite cuando el parámetro de deformación tiende a 0.

Un ejemplo clásico de este proceso son las variedades de banderas sobre un cuerpo k, o más precisamente sus álgebras de coordenadas homogéneas. Una variedad de bandera (generalizada) es el cociente de un grupo algebraico semisimple G por un subgrupo parabólico P; el ejemplo clásico es el de $G = \mathsf{SL}_n(k)$ y P el subgrupo de las matrices triangulares inferiores en G. El nombre proviene del hecho que en este caso el cociente G/P es una variedad proyectiva y sus puntos están en correspondencia biyectiva con las banderas completas del espacio k^n; otras elecciones de G y P parametrizan ciertos tipos de banderas específicas dentro del mismo espacio. Las variedades de bandera más sencillas son, además del ejemplo ya mencionado, las grassmanianas, que parametrizan los subespacios de k^n de una dimensión fija. Todas estas variedades tienen inmersiones naturales en espacios proyectivos, llamadas inmersiones de Plücker. El anillo de coordenadas homogéneas $\mathcal{O}[G/P]$ correspondiente a la inmersión de Plücker puede realizarse de manera natural como una subálgebra del álgebra de funciones regulares sobre G, que por resultados clásicos puede identificarse con el dual de Hopf del álgebra $U(\mathfrak{g})$, donde $\mathfrak{g}$ es el álgebra de Lie del grupo G. Las variedades de bandera, o mejor dicho sus álgebras de coordenadas homogéneas, tienen versiones cuánticas definidas de forma independiente por Ya. Soĭbel′man en [Soĭ92] y por V. Lakshmibai y N. Reshetikin en [LR92], como ciertas subálgebras del dual de Hopf del álgebra envolvente cuántica $U_q(\mathfrak{g})$. Recordamos los detalles de esta definición en el capítulo 6.

Las variedades de banderas son consideradas uno de los principales ejemplos de

variedades proyectivas. Su estudio es un punto de encuentro entre la geometría algebraica, la teoría de representaciones (de grupos finitos, grupos algebraicos, álgebras de Lie...) y la combinatoria[1]. La topología y la geometría de estas variedades han sido ampliamente estudiadas a través de sus subvariedades de Schubert y Richardson (una variedad de Richardson es la intersección de una variedad de Schubert con una variedad de Schubert opuesta), las cuales también poseen versiones cuánticas.

Siguiendo el principio general que afirma que las propiedades homológicas son "invariantes por deformación", conjeturamos que las variedades de banderas cuánticas y sus respectivas subvariedades de Schubert y Richardson deberían tener propiedades de regularidad homológica similares a las de sus contrapartes clásicas. Dado que las definiciones y las técnicas clásicas no se aplican directamente a los objetos cuánticos, para convertir esta idea general en un enunciado formal es necesario adaptar ambas al contexto no conmutativo.

Recordemos algunas de las técnicas clásicas desarrolladas para el estudio de las variedades de banderas. El estudio sistemáticos de los anillos de coordenadas homogéneas correspondientes a las inmersiones de Plücker de las grassmanianas en espacios proyectivos fue iniciado por Hodge y desarrollado por de Concini, Eisenbud y Procesi en [DCEP] (la introducción de esta referencia tiene una excelente reseña histórica), dando origen tiempo después a la actual Teoría de Monomios Estándar, cuyo objetivo es extender el trabajo de Hodge a inmersiones arbitrarias de variedades de banderas y subvariedades de Schubert, e inclusive variedades proyectivas más generales, en espacios proyectivos. La idea original de Hodge fue aprovechar la estructura combinatoria subyacente a las relaciones de Plücker; Eisenbud llamó *straightening laws* (literalmente "leyes de enderezamiento", aunque en esta introducción nos referiremos a ellas como "reglas de reescritura") a las relaciones con restricciones combinatorias similares, y definió la clase de Álgebras con Reglas de Reescritura (a partir de ahora ASLs por su sigla en inglés), axiomatizando el análisis de Hodge.

En base a estos y otros resultados de la Teoría de Monomios Estándar, Gonciulea y Lakshmibai [GL96] encontraron una deformación de las grassmanianas clásicas en variedades tóricas afines, con el objetivo explícito de estudiar las propiedades estables por deformación de las variedades originales a través de sus degeneraciones tóricas[2]. En términos puramente algebraicos esto se corresponde con definir una filtración sobre el álgebra de coordenadas homogéneas de las inmersiones de Plücker de las grassmanianas de forma que el álgebra graduada asociada sea un álgebra de semigrupo. Esta construcción fue generalizada después a ciertas variedades de Schubert en artículos de R. Chirivi [Chi00], R. Dehy y R. Yu [DY01], y finalmente por P. Caldero [Cal02] a todas las subvariedades de Schubert de la variedad de banderas completas.

[1]Una muestra de esto es que el libro "Flag Varieties" [BL09], cuyo objetivo principal es presentar los resultados más básicos del estudio de las variedades de bandera, define qué es una grassmaniana después de ciento cincuenta páginas de preliminares.

[2]Una variedad tórica es una variedad algebraica con un toro denso, cuya acción sobre sí mismo se extiende a toda la variedad.

En el artículo [LR06], T. Lenagan y L. Rigal definieron la clase de *quantum graded ASLs* (Álgebras graduadas cuánticas con leyes de reescritura, para nosotros simplemente ASL cuánticas) con el objetivo de estudiar las deformaciones cuánticas de las grassmanianas de tipo A y sus subvariedades de Schubert. Las ASL cuánticas son una versión no conmutativa de las ASL: a las leyes de reescritura clásicas se agregan ciertas *leyes de conmutación*, con restricciones combinatorias muy similares a las de aquellas. El programa original de esta tesis era demostrar que esta estructura combinatoria se puede utilizar para adaptar el método de degeneración de Gonciulea y Lakshmibai y así continuar con el estudio de las subvariedades de Richardson de las grassmanianas cuánticas, principalmente sus propiedades de regularidad homológicas, y eventualmente generalizar este método a otras variedades de banderas cuánticas. Las propiedades de regularidad en las que estamos interesado son las de AS-Cohen-Macaulay, AS-Gorenstein y AS-regular, generalizaciones de las nociones clásicas de Cohen-Macaulay, Gorenstein y regular para anillos conmutativos locales, (ver el capítulo 3 para la definición de estas propiedades) y la de poseer un complejo dualizante balanceado (ver capítulo 4).

La ejecución de este programa puede dividirse en tres etapas. La primera fue mostrar que el método de degeneración era adecuado para el estudio de las propiedades homológicas que nos interesan. La solución de este problema resultó ser técnica pero directa. El contexto es el siguiente: partimos de un álgebra graduada (nuestra "variedad no conmutativa") con una filtración por espacios vectoriales graduados, y queremos estudiarla a partir de su álgebra graduada asociada (nuestra "variedad degenerada"). Notar que el álgebra graduada asociada es bi-graduada: la primera componente de la bi-graduación proviene del hecho de que es un álgebra graduada asociada, y la segunda de la graduación del álgebra original, que permanece precisamente porque las capas de nuestra filtración son subespacios graduados. Demostramos la existencia de una sucesión espectral que relaciona los funtores $\underline{\mathrm{Ext}}$ del álgebra graduada asociada con los de la original, que además es compatible con *ambas* graduaciones. Los principales resultados son el Teoremas 2.4.8, que describe la sucesión espectral, y los Teoremas 3.2.13 y 4.2.12 donde se demuestra que las propiedades de regularidad homológica que nos interesan se transfieren de la variedad degenerada a la original.

El paso siguiente fue encontrar las filtraciones adecuadas para las variedades de banderas cuánticas. Esta fue la etapa más sencilla. Ya en su trabajo [Calo2], P. Caldero demuestra la existencia de una degeneración de las variedades de Schubert de la variedad de banderas completas filtrando la variedad de banderas completas cuántica y demostrando que esta filtración es compatible con la especialización al caso clásico. Podríamos decir que las variedades de banderas cuánticas estaban simplemente esperando ser degeneradas. El método de Gonciulea y Lakshmibai requiere de un poco más de trabajo para ser adaptado al caso cuántico, pero casi todos los resultados necesarios se encuentran en los artículos [LR06] y [LR08]. Nos interesaba adaptar ambos métodos dado que el enfoque de Caldero funciona para las variedades de Schubert cuánticas arbitrarias, pero solamente sobre cuerpos de característica cero y parámetro

trascendentes sobre $\mathbb{Q}$, mientras que el de Gonciulea y Lakshmibai funciona solo para variedades de Richardson de grassmanianas de tipo A, pero en cuerpos de cualquier característica y con parámetro de deformación arbitrario. Los resultados principales son el Teorema 6.1.5, sobre la degeneración de las variedades de Richardson cuánticas de las grassmanianas de Tipo A, y el Corolario 6.2.4 sobre la degeneración de las variedades de Schubert de las variedades de banderas arbitrarias.

La última etapa fue el estudio de las variedades degeneradas. Este resultó ser el paso más complicado, y la mayor parte de la tesis está dedicada a desarrollar las herramientas necesarias para ello. Como dijimos antes, las variedades de banderas cuánticas y sus variedades asociadas efectivamente degeneran, pero ¿qué se obtiene al final de este proceso? La respuesta es que uno obtiene un análogo cuántico de las variedades tóricas que se obtenían al degenerar las variedades de banderas clásicas (qué más si no...). Una variedad tórica afín es el espectro de un álgebra de semigrupo $k[S]$, donde S es un subsemigrupo finitamente generado de $\mathbb{Z}^{r+1}$, con $r \geq 0$; estos semigrupos son llamados *afines*. Las variedades tóricas afines cuánticas son deformaciones no conmutativas de las álgebras de estos semigrupos; merecen este nombre porque tienen un "toro cuántico denso", es decir que si invertimos todos los elementos homogéneos no nulos de este álgebra, que resultan ser normales y regulares, obtenemos un toro cuántico[3].

Fijemos un semigrupo afín S y consideremos su álgebra de semigrupo $k[S]$, donde k es un cuerpo cualquiera (si el cuerpo no es algebricamente cerrado entonces puede haber más variedades tóricas además de los espectros de áglebras de semigrupos). Dado un 2-cociclo sobre S con coeficientes en $k^\times$, que notamos α, podemos construir una variedad tórica afín cuántica que notamos $k^\alpha[S]$; de esta manera se obtienen todas las variedades tóricas afines cuánticas, la demostración de este hecho se encuentra en el capítulo 5. Dado que estamos considerando al álgebra $k[S]$ como el anillo de coordenadas homogéneas de una variedad proyectiva, este posee una graduación natural sobre $\mathbb{N}$, y las propiedades en las que nos interesamos (Cohen-Macaulay, Gorenstein, regularidad, etc.) son propiedades de la categoría de $k[S]$-módulos $\mathbb{Z}$-graduados, $\mathsf{Mod}^{\mathbb{Z}}\, k[S]$. Las propiedades análogas para álgebras no conmutativas, presentadas en los capítulos 3 y 4, dependen de la categoría $\mathsf{Mod}^{\mathbb{Z}}\, k^\alpha[S]$ (el álgebra $k^\alpha[S]$ tiene el mismo espacio vectorial graduado subyacente que $k[S]$). En este momento nos encontramos frente a un problema porque no conocemos ninguna forma de transferir información entre las categorías $\mathsf{Mod}^{\mathbb{Z}}\, k[S]$ y $\mathsf{Mod}^{\mathbb{Z}}\, k\alpha[S]$ directamente.

Sin embargo las álgebras $k[S]$ y $k^\alpha[S]$ tienen una $\mathbb{Z}^{r+1}$-graduación, más fina que la original, y podemos considerar las categorías de módulos $\mathbb{Z}^{r+1}$-graduados sobre ellas. Por un teorema de J. Zhang, las categorías $\mathsf{Mod}^{\mathbb{Z}^{r+1}}\, k[S]$ y $\mathsf{Mod}^{\mathbb{Z}^{r+1}}\, k^\alpha[S]$ son isomorfas. A continuación, tomando una construcción debida a A. Polishchuk y L. Positselski [PP11], en la que a cada morfismo $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ le asignan tres funtores

[3]Señalamos que no somos los primeros en considerar estos objetos. En el preprint [Ing], C. Ingalls demuestra resultados similares trabajando sobre cuerpos algebraicamente cerrados usando técnicas muy distantas a las nuestras.

$\varphi_!, \varphi_* : \mathsf{Mod}^{\mathbb{Z}^{r+1}} k \longrightarrow \mathsf{Mod}^{\mathbb{Z}} k$ y $\varphi^* : \mathsf{Mod}^{\mathbb{Z}} k \longrightarrow \mathsf{Mod}^{\mathbb{Z}^{r+1}} k$, llamados *funtores de cambio de graduación*, que inducen funtores correspondientes entre las categorías de $k[S]$ y $k^\alpha[S]$-módulos graduados. Así obtenemos el siguiente diagrama:

$$\begin{array}{ccc} \mathsf{Mod}^{\mathbb{Z}^r} k[S] & \rightleftarrows & \mathsf{Mod}^{\mathbb{Z}^r} k^\alpha[S] \\ {\scriptstyle\varphi_!}\downarrow \uparrow {\scriptstyle\varphi^*} \downarrow {\scriptstyle\varphi_*} & & {\scriptstyle\varphi_*}\downarrow \uparrow {\scriptstyle\varphi^*} \downarrow {\scriptstyle\varphi_!} \\ \mathsf{Mod}^{\mathbb{Z}} k[S] & & \mathsf{Mod}^{\mathbb{Z}} k^\alpha[S] \end{array}$$

Los funtores de cambio de graduación son exactos, y φ^* es adjunto a derecha de $\varphi_!$ y a izquierda de φ_*, lo que permite transferir mucha información entre las categorías de objetos $\mathbb{Z}^{r+1}$ y $\mathbb{Z}$-graduados. Por ejemplo, demostramos que la dimensión global y la dimensión inyectiva de $k[S]$ (o de $k^\alpha[S]$) pueden leerse en ambas categorías. Otros invariantes más sutiles (dimensión local finita, propiedad χ) también pueden leerse al nivel de la categoría de módulos $\mathbb{Z}^{r+1}$-graduados.

Vemos entonces que si bien no hay un camino a través del cual transferir la información directamente entre $\mathsf{Mod}^{\mathbb{Z}} k[S]$ y $\mathsf{Mod}^{\mathbb{Z}} k^\alpha[S]$, podemos construir uno pasando por las categorías $\mathsf{Mod}^{\mathbb{Z}^{r+1}} k[S]$ y $\mathsf{Mod}^{\mathbb{Z}^{r+1}} k^\alpha[S]$. En este caso el principio general de la estabilidad de las propiedades homológicas por deformación se puede expresar de manera muy concreta: toda propiedad que pueda leerse en la categoría de módulos $\mathbb{Z}^{r+1}$-graduados es invariante por deformación por 2-cociclos. Los funtores de cambio de graduación son una herramienta muy útil para entender cómo se refleja una propiedad $\mathbb{Z}$-graduada al nivel $\mathbb{Z}^{r+1}$-graduado. El principal resultado en este sentido es la Proposición 5.2.12, donde se resumen las propiedades de regularidad de las variedades tóricas cuánticas.

Después de este resumen global de la estrategia, señalamos que el principal resultado de la tesis es el Corolario 5.3.8, que afirma que las álgebras con degeneraciones tóricas, en particular las variedades de banderas cuánticas, heredan las propiedades de regularidad homológicas de las variedades tóricas clásicas asociadas, las cuales a su vez dependen únicamente del semigrupo subyacente.

A continuación detallamos los contenidos de cada capítulo.

El capítulo 1 incluye algunos resultados clásicos de álgebra homológica y teoría de semigrupos. El objetivo de este capítulo es servir de referencia para los resultados más utilizados en capítulos poseriores y fijar notación a ser usada en el resto de la tesis.

El capítulo 2 trata sobre k-álgebras graduadas sobre un grupo cualquiera G. Fijada una k-álgebra graduada A, primero recordamos los resultados generales sobre la categoría de A-módulos G-graduados, que notamos $\mathsf{Mod}^G A$. A continuación recordamos varias construcciones sobre estas categorías. La primera es la de los twists de Zhang, una generalización de las torciones por 2-cociclos que a partir de A y un cierto sistema de automorfismos de A como G-espacio vectorial graduado produce una nueva álgebra ${}^\tau A$, con el mismo espacio vectorial graduado subyacente que A y categoría de módulos G-graduados isomorfa a la de A, pero una multiplicación distinta.

Luego definimos los funtores de cambio de graduación en este contexto general, donde cualquier morfismo de grupos $\varphi : G \longrightarrow H$ induce una H-graduación en A y funtores como los mencionados anteriormente. Recordamos también la definición de funtores de torsión asociados a un ideal graduado $\mathfrak{a}$, y estudiamos invariantes homológicos del par $(A, \mathfrak{a})$; utilizando los funtores de cambio de graduación demostramos que estos invariantes puede leerse tanto en la categoría $\mathsf{Mod}^G A$ como en $\mathsf{Mod}^H A$. Terminamos el capítulo con la definición de las álgebras GF, es decir álgebras $\mathbb{N}$-graduadas con una filtración cuyas capas son subespacios vectoriales graduados. Redemostramos algunos resultados clásicos de la teoría de álgebras filtradas en este contexto, llegando finalmente a la sucesión espectral que relaciona los espacios $\underline{\mathsf{Ext}}$ graduados del álgebra GF con los de su álgebra graduada asociada.

En el capítulo 3 nos restringimos a trabajar sobre álgebras $\mathbb{N}^{r+1}$-graduadas conexas, es decir, de componente $(0, \ldots, 0)$ igual a k. Después de repasar las propiedades principales de las álgebras $\mathbb{N}^{r+1}$-graduadas conexas (completamente análogas a las de las $\mathbb{N}$-graduadas conexas) recordamos la definición de las propiedades de AS-Cohen-Macaulay, AS-Gorenstein y AS-regular para álgebras $\mathbb{N}$-graduadas conexas, y proponemos análogos $\mathbb{N}^{r+1}$-graduados. Demostramos que estas propiedades son invariantes por cambio de graduación (es decir que un álgebra A tiene alguna de estas propiedades si y solo si el álgebra $\varphi_!(A)$ la tiene) y que son estables por deformación (es decir que si el álgebra graduada asociada tiene alguna de estas propiedades, el álgebra original también la tiene).

El capítulo 4 se ocupa de una noción de regularidad mucho más técnica, la de poseer un complejo dualizante balanceado. Un complejo dualizante es un objeto de la categoría derivada $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes A^\circ)$, así que comenzamos repasando generalidades sobre categorías derivadas (estos resultados solo se utilizan en este capítulo). A continuación adaptamos la definición de complejos dualizantes al contexto de álgebras $\mathbb{N}^{r+1}$-graduadas y señalamos muchas de sus propiedades básicas. Demostramos que la propiedad de poseer un complejo dualizante balanceado es invariante por cambio de graduación. Demostramos además en el Corolario 4.2.8 un criterio de existencia de complejos dualizantes balanceados idéntico al de M. Van den Bergh (ver [VdB97, Proposition 6.3]) para el caso $\mathbb{N}^{r+1}$-graduado conexo, que deducimos del criterio para álgebras $\mathbb{N}$-graduadas y del hecho de que poseer un complejo dualizante balanceado es invariante por cambio de graduación. Con este criterio demostramos que poseer un complejo dualizante balanceado también es una propiedad invariante por twists de Zhang y estable por deformación.

En el capítulo 5 presentamos la noción de una degeneración tórica cuántica. Comenzamos con un repaso general de las propiedades de las variedades tóricas afines, y definimos las variedades tóricas afines cuánticas como las álgebras $\mathbb{Z}^{r+1}$-graduadas cuyo anillo de fracciones homogéneas es isomorfo a un toro cuántico. Clasificamos las variedades tóricas afines cuánticas y probamos que esta clase coincide con la de los twists de Zhang $\mathbb{Z}^{r+1}$-graduados de las álgebras de semigrupo afines. Por los resultados demostrados en los capítulos anteriores, las propiedades de una variedad

tórica afín cuántica son las mismas que las de su correspondiente variedad tórica afín clásica. Luego presentamos la noción de una S-álgebra, donde S es un semigrupo afín positivo (es decir, contenido en $\mathbb{N}^{r+1}$). Los semigrupos afines positivos tienen una única presentación minimal, y una S-álgebra es un álgebra con relaciones cuya forma está dictada por esta presentación minimal, siguiendo el modelo de las ASLs. Toda S-álgebra graduada es una álgebra GF, y su álgebra graduada asociada es isomorfa a una variedad tórica afín cuántica regraduada. Una vez más usamos los resultados de los capítulos anteriores para deducir las propiedades homológicas de las S-álgebras a partir de las propiedades de las álgebras de semigrupo conmutativas. Concluimos definiendo dos familias particulares de S-álgebras, las ASL cuánticas simétricas y las álgebras con una S-base homogénea.

Finalmente en el capítulo 6 recordamos la definición de las variedades de banderas cuánticas y de sus subvariedades de Schubert y Richardson. Probamos que las grassmanianas de tipo A y sus subvariedades de Schubert y Richardson son ASLs cuánticas simétricas sobre un cuerpo cualquiera y un parámetro de deformación arbitrario, y que las variedades de Schubert de variedades de banderas arbitrarias, son álgebras con S-bases homogéneas cuando el cuerpo de base es de característica cero y el parámetro de deformación es trascendente sobre $\mathbb{Q}$.

Introduction

The results in this thesis belong to the theory informally known as noncommutative projective geometry, that is, the study of not necessarily commutative $\mathbb{N}$-graded algebras by applying techniques borrowed from the commutative world. The name arises from the fact that commutative noetherian $\mathbb{N}$-graded algebras correspond to projective varieties over the ground field, which gives the subject a distinctively geometric flavor.

A typical example of a classical object that guides the study of a quantum analogue are flag varieties and the associated Schubert and Richardson varieties. Their noncommutative analogues[4] are called *quantum flag varieties*. Following the general principle that "homological properties are stable under quantization", one is led to consider these properties for quantum flag varieties. We expect them to be analogous in some sense to those of (coordinate rings of) flag varieties, but in order to prove that this is the case we have to adapt our techniques to the quantum setting.

Flag varieties are widely considered some of the most important examples of projective varieties. Their study lies at the intersection of algebraic geometry, representation theory (of finite groups, algebraic groups, Lie algebras) and combinatorics.[5] At the same time, their topology and geometry is well understood in terms of its Schubert and Richardson subvarieties (a Richardson variety is the intersection of a Schubert variety with an opposite Schubert variety). Quantum flag varieties were introduced independently by Ya. Soĭbel′man in [Soĭ92] and by V. Lakshmibai and N. Reshetikin in [LR92]. See chapter 6 for the definition of quantum flag, Schubert and Richardson varieties.

The systematic study of the homogeneous coordinate rings corresponding to the Plücker embedding of grassmannians in projective space was started by Hodge and continued by de Concini, Eisenbud and Procesi in [DCEP] (see the introduction of this book for the history of the subject), and eventually gave birth to the still active field of Standard Monomial Theory, which extends Hodge's work to the study of arbitrary flag varieties and their Schubert subvarieties. The idea behind this approach is to take advantage of strongly combinatorial conditions in the Plücker relations; Eisenbud calls

[4] That is, quantizations of their homogeneous coordinate rings.

[5] It seems quite telling that a recent book dedicated to the study of flag varieties only introduces grassmannians after a hundred and fifty pages of preliminaries.

relations obeying such combinatorial constraints *straightening relations*, and introduced the notion of an Algebra with Straightening Laws (henceforth ASL) as an abstraction of Hodge's analysis.

Applying these and other results from Standard Monomial Theory, Gonciulea and Lakshmibai found a deformation of classical grassmannians into toric varieties in [GL96], with the explicit objective of studying deformation invariant properties of the former by analyzing the latter; in purely algebraic terms this amounted to finding a filtration on the corresponding homogeneous coordinate ring whose associated graded ring is a semigroup ring. This construction was later generalized by several people such as R. Chirivi [Chi00], R. Dehy and R. Yu [DY01] and P. Caldero [Cal02].

In the article [LR06] T. Lenagan and L. Rigal introduced the notion of a quantum graded algebra with a straightening law in order to study quantum grassmannians and their quantum Schubert varieties. This is a noncommutative version of classical ASLs in the sense that, along with straightening relations, the algebras are also endowed with *commutation relations* which obey similar combinatorial constraints. The original program for this thesis was to prove that this combinatorial structure could be used to adapt the degeneration method of Gonciulea and Lakshmibai to the study of quantum grassmannians, mainly of their homological regularity properties (see chapters 3 and 4 for details), and eventually to more general quantum flag varieties.

The task was divided in three stages. The first was to show that the degeneration method was well suited for the study of the homological properties we were interested in. This turned out to be a straightforward, though slightly technical, question: there is a spectral sequence relating the degenerated noncommutative projective varieties to the original ones. Since we are filtering a graded algebra the associated graded algebra has two gradings, one arising from the filtration and the other coming from the grading of the original algebra. The fact that the mentioned spectral sequence is compatible with *both* gradings is essential for the proofs.

The next step was to prove that quantum flag and Schubert varieties have an adequate filtration, but this turned out to be the simplest part of the problem. Caldero proved in [Cal02] that the complete flag varieties and their Schubert varieties have toric degenerations by filtering the quantum flag variety and then showing that this filtration behaves well with respect to specialization, so in a sense quantum flag varieties were waiting to be degenerated. Gonciulea and Lakshmibai's method requires more work in order to be adapted to the quantum setting, but most of this is already done in [LR06]. We extended both methods since Caldero's proof works for all quantum Schubert varieties of arbitrary quantum flag varieties, but under the hypothesis that the underlying field is a transcendental extension of $\mathbb{Q}$, while Lakshmibai and Gonciulea's idea works in arbitrary characteristic, but only for quantum Schubert and Richardson subvarieties of quantum grassmannians in type A.

The last step, the study of the degenerated varieties, turned out to be more complicated and the technical bulk of the thesis is dedicated to developing the necessary

tools. As we said before, quantum flag varieties and their associates do degenerate, but into what? Since classical flag varieties degenerate into affine toric varieties, the answer is: into *quantum affine toric varieties* (of course). An affine toric variety is the spectrum of a semigroup algebra $k[S]$ where S is a finitely generated subsemigroup of $\mathbb{Z}^{r+1}$ for some $r \geq 0$; such semigroups are called *affine*. Quantum affine toric varieties are noncommutative deformations of semigroup algebras; they are worthy of the name since they have a "dense quantum torus", in the sense that localizing at the set of its homogeneous elements, which happen to be normal and regular, one obtains a quantum torus.[6].

Consider an affine semigroup S and its semigroup algebra $k[S]$, where k is a field (if k is not algebraically closed then there may be other toric varieties aside from the spectra of semigroup rings). Given a 2-cocycle over S with coefficients in $k^\times$ which we denote α, there is a corresponding quantum toric variety denoted $k^\alpha[S]$ and all quantum affine toric varieties are of this form, see chapter 5 for details. Since we are thinking of $k[S]$ as the homogeneous coordinate ring of a projective variety, it is naturally $\mathbb{N}$-graded, and the properties we are interested in (Cohen Macaulayness, Gorensteinness, smoothness...) are read from the category of $\mathbb{Z}$-graded $k[S]$-modules $\mathsf{Mod}^{\mathbb{Z}}\, k[S]$. The analogous properties for noncommutative algebras are introduced in chapters 3 and 4, and are read from the category $\mathsf{Mod}^{\mathbb{Z}}\, k^\alpha[S]$, with the $\mathbb{N}$-grading of $k^\alpha[S]$ induced by that of $k[S]$. We run into a problem here since there is a priori no link between the categories $\mathsf{Mod}^{\mathbb{Z}}\, k[S]$ and $\mathsf{Mod}^{\mathbb{Z}}\, k^\alpha[S]$.

However the algebras $k[S]$ and $k^\alpha[S]$ are also $\mathbb{Z}^{r+1}$-graded so we may consider $\mathbb{Z}^{r+1}$-graded modules over them. By a theorem due to J. Zhang, the categories $\mathsf{Mod}^{\mathbb{Z}^{r+1}}\, k[S]$ and $\mathsf{Mod}^{\mathbb{Z}^{r+1}}\, k^\alpha[S]$ are isomorphic. In order to relate $\mathbb{Z}$-graded and $\mathbb{Z}^{r+1}$-graded objects we borrow a construction due to A. Polishchuk and L. Positselski from [PP11], where to every group morphism $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ we assign three functors, $\varphi_!, \varphi_* : \mathsf{Mod}^{\mathbb{Z}^{r+1}}\, k \longrightarrow \mathsf{Mod}^{\mathbb{Z}}\, k$ and $\varphi^* : \mathsf{Mod}^{\mathbb{Z}}\, k \longrightarrow \mathsf{Mod}^{\mathbb{Z}^{r+1}}$, known as the *change of grading functors* which induce corresponding functors between the categories of graded $k[S]$ and $k^\alpha[S]$ modules. We obtain a diagram as follows:

$$\begin{array}{ccc} \mathsf{Mod}^{\mathbb{Z}^i r+1}\, k[S] & \rightleftarrows & \mathsf{Mod}^{\mathbb{Z}^{r+1}}\, k^\alpha[S] \\ \varphi_! \downarrow \quad \uparrow \varphi^* \quad \downarrow \varphi_* & & \varphi_* \downarrow \quad \uparrow \varphi^* \quad \downarrow \varphi_! \\ \mathsf{Mod}^{\mathbb{Z}}\, k[S] & & \mathsf{Mod}^{\mathbb{Z}}\, k^\alpha[S] \end{array}$$

The change of grading functors are exact, and furthermore φ^* is right adjoint to $\varphi_!$ and left adjoint to φ_*, which allows to transfer a lot of information between the categories of $\mathbb{Z}$-graded and $\mathbb{Z}^{r+1}$-graded modules. For example, they allow us to prove that the global and injective dimension of $k[S]$ (or $k^\alpha[S]$) can be checked in either cate-

[6]Incidentally, we were not the first to consider such objects. In the widely circulated preprint [Ing], C. Ingalls finds similar results over algebraically closed fields, although the techniques used there are quite different.

gory. More subtle invariants (property χ, finite local dimension) are also shown to be readable at the $\mathbb{Z}^{r+1}$-graded level.

So even though there is no direct road from $\mathsf{Mod}^{\mathbb{Z}}\, k[S]$ to $\mathsf{Mod}^{\mathbb{Z}}\, k^{\alpha}[S]$, there is a way to transfer information from one category to the other. In this instance the general principle that "homological properties are stable under quantization" can be given a very concrete meaning: it is enough for the homological property, usually read from the category $\mathsf{Mod}^{\mathbb{Z}}\, k[S]$, to be visible at the level of $\mathbb{Z}^{r+1}$-graded modules since at these level both algebras behave identically. The change of grading functors provide a good tool to see how a $\mathbb{Z}$-graded property reflects in the $\mathbb{Z}^{r+1}$-graded context.

We now give an outline of the structure of the thesis:

Chapter 1 sets notation that will be used for the rest of the thesis and recalls some general results for further reference.

In chapter 2 we work with algebras graded over an arbitrary group G. After reviewing the general properties of the category of G-graded modules, we introduce several constructions. We first recall some results on Zhang twists, which are certain systems of graded automorphisms which allow to "twist" the multiplication of a G-graded algebra A to obtain a new algebra ${}^{\tau}A$, whose category of G-graded modules is isomorphic to that of A. Then we introduce the change of grading functors, and finally to each G-graded ideal $\mathfrak{a}$ of A we associate the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}} : \mathsf{Mod}^{G} A \longrightarrow \mathsf{Mod}^{G} A$, and prove that it commutes with twisting of modules and with $\varphi_!$ and φ^*. From this we deduce that several cohomological invariants of the pair $(A, \mathfrak{a})$ are invariant by twisting and by change of grading. We finish the chapter by re-proving several known results for filtered algebras in the case where the original algebra is graded and the filtration is by graded subspaces.

In chapter 3 we recall the notions of AS-Cohen-Macaulay, AS-Gorenstein and AS-regular algebras, which were originally defined for connected $\mathbb{N}$-graded algebras. We propose analogous conditions for $\mathbb{N}^{r+1}$-graded algebras and show that these conditions are stable by change of grading and by twisting.

Chapter 4 is dedicated to a much more technical notion, that of a dualizing complex. Since this is an object of the derived category $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}}\, A \otimes A^{\circ})$, we begin with some general results on derived categories and extend the change of grading functors to this setting. After introducing dualizing complexes, we prove that the property of having a (balanced) dualizing complex is invariant by twists and also by change of grading.

In chapter 5 we introduce quantum toric degenerations. We begin with a general discussion of quantum affine toric varieties and characterize them as twists of affine semigroup algebras, so by the results of previous chapters they inherit good homological properties from the corresponding commutative objects. We then introduce a class of connected $\mathbb{N}$-graded algebras, which we call S-algebras, with a presentation modelled on that of an affine semigroup S in the spirit of quantum graded ASLs. An

S-algebra has a natural filtration whose associated graded ring is a quantum toric variety, so it inherits the nice homological properties of the latter. We finish by presenting two subfamilies of the class of S-algebras, symmetric quantum ASLs and algebras with an S-basis.

Finally in chapter 6 we review the definitions of quantum flag varieties and their Schubert and Richardson varieties. We prove that grassmannians and their Schubert and Richardson varieties in type A are symmetric quantum ASLs for an arbitrary field and quantum parameter. Assuming the underlying field is a transcendental extension of $\mathbb{Q}$, we also show that arbitrary quantum flag and Schubert varieties have S-bases, and hence they degenerate to quantum affine toric varieties.

Chapter 1

Generalities

In this chapter we fix the notation and some general results of homological algebra we will use in the sequel.

We denote by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the ring of integers, and the fields of rational, real and complex numbers respectively. We denote by $\mathbb{N}$ the set of natural numbers, which includes 0, and by $\mathbb{N}^*$ the set of positive natural numbers. Given a finite set I, we denote its cardinality by $|I|$.

Throughout this document k denotes a field. All vector spaces are k-vector spaces and unadorned tensor products are always over k. Unless explicitly stated, all algebras are unital associative k-algebras. Modules will always be *left* modules unless otherwise specified. Ideals on the other hand will be *two-sided* ideals.

Given $n \in \mathbb{N}$ and $O \in \{k, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots\}$, we denote by $e_i \in O^n$ the n-uple with a 1 in the i-th coordinate and 0's in all others. The set $\{e_1, \ldots, e_n\}$ is called simply *the canonical basis* of O^n.

1.1 Semigroups

A *semigroup* is a set S with an associative binary operation and a neutral element. A *semigroup morphism* is a function between two semigroups that is compatible with the operations and neutral elements in the obvious sense. Even though many of the statements in this section hold for arbitrary semigroups, we will use additive notation because it is better suited for the subject at hand. Also, all the semigroups that appear in the sequel are commutative, and this will save us from translating multiplicative notation to additive notation later on.

Definition 1.1.1. Given a semigroup S, a *congruence* on S is an equivalence relation $R \subset S \times S$ such that for every $s \in S$ and every $(t, t') \in R$, it is $(s + t, s + t') \in R$ and $(t + s, t' + s) \in R$.

Given a semigroup S and a congruence R on it, the *quotient semigroup* S/R is defined as the set of equivalence classes of this relation with the operation $[s] + [s'] = [s + s']$, where $[s]$ denotes the class of s in S/R. There is a semigroup morphism from S to S/R, with the obvious universal property. Given a set $L \subset S \times S$, the set $R(L)$ of congruence relations containing L is nonempty, so we define the *congruence relation generated by* L as $\langle L \rangle = \bigcap_{R \in R(L)} R$. For more details, see [CP61, section 1.5].

Given a set X, the *free semigroup* on X, which we denote by $F(X)$, is the set of words on X with concatenation as the operation. A semigroup is said to be finitely generated if it is isomorphic to a semigroup of the form $F(X)/R$ with X finite and R a congruence on $F(X)$. If R is generated by a finite subset L, then the semigroup is said to be *finitely presented*.

A group G is called the *enveloping group* of S if there is a semigroup morphism $i : S \longrightarrow G$ such that for every group H and every semigroup morphism $f : S \longrightarrow H$, there exists a unique morphism $\tilde{f} : G \longrightarrow H$ such that $f = \tilde{f} \circ i$.

Any semigroup has an enveloping group; it can be presented as the free group generated by the underlying set of S divided by the normal semigroup generated by all the elements of the form $s + s' - s''$, with $s'' = s + s'$ in S. Since the enveloping group solves a universal problem, it is unique up to isomorphism.

A semigroup is said to be *left cancellative* if $s + s' = s + s''$ implies $s' = s''$. Right cancellative semigroups are defined analogously. A *cancellative* semigroup is both left and right cancellative. A commutative semigroup S is cancellative if and only if the canonical morphism from S into its enveloping group is injective, see [CP61, section 1.10].

1.2 Homological matters

We assume the general theory of derived functors and homological algebra as developed in [Wei94]. We quote some well known results for future reference.

Throughout this section $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ denote abelian categories. We use cohomological notation, i.e. increasing indices, for complexes over abelian categories. Given a left, resp. right, exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, for every $i \geq 0$ we denote by $\mathcal{R}^i F$, resp. $\mathcal{L}^i F$, the i-th right, resp. left, derived functor of F if it exists. We say that F *reflects exactness* if given a complex $A^\bullet$ of objects of $\mathcal{A}$, $F(A^\bullet)$ is exact if and only if $A^\bullet$ is exact.

Given a diagram of functors and abelian categories as follows

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D}, \end{array}$$

we will say that the diagram *commutes* if there exists a natural isomorphism $T \circ F \cong G \circ S$.

The following proposition recalls two classical results due to Grothendieck.

Proposition 1.2.1. *If $\mathcal{A}$ has arbitrary coproducts, then it has arbitrary direct limits. If $\mathcal{A}$ has arbitrary coproducts and a projective generator, and direct limits over $\mathcal{A}$ are exact, then $\mathcal{A}$ has enough projective and injective objects.*

Proof. See [Gro57, sections 1.8 - 1.10]. □

If $\mathcal{A}$ has enough projective, resp. injective, objects, then every object A of $\mathcal{A}$ has a projective, resp. injective, resolution. The *projective dimension* of A, denoted by $\mathrm{pdim}_{\mathcal{A}} A$, is the minimal length of a projective resolution of A. The *injective dimension* of A, denoted by $\mathrm{injdim}_{\mathcal{A}} A$, is defined analogously.

Given left exact functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$, the derived functors of the composition $G \circ F$ are related to the composition of the derived functors by a spectral sequence. We will not use this result in all its generality, and instead prove a simple special case for future reference.

Lemma 1.2.2. *Suppose the categories $\mathcal{A}$ and $\mathcal{B}$ have enough injective objects. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be two covariant left exact functors. If F is exact and sends injective objects to G-acyclic objects, then $\mathcal{R}^i(G \circ F) \cong \mathcal{R}^i G \circ F$ for all $i \geq 0$.*

Proof. Given an object A of $\mathcal{A}$, choose an injective resolution $A \longrightarrow I^\bullet$. Since F is exact and sends injective objects to G-acyclic objects, $F(A) \longrightarrow F(I^\bullet)$ is a G-acyclic resolution of $F(A)$, so by definition $\mathcal{R}^i(G \circ F)(A) \cong H^i(G(F(I^\bullet))) \cong \mathcal{R}^i G(F(A))$. The naturality of this isomorphism is a consequence of the general theory of δ-functors. □

We say that a functor $\tilde{F} : \mathcal{A} \longrightarrow \mathcal{B}$ *extends* $F : \mathcal{C} \longrightarrow \mathcal{D}$, or that F *induces* $\tilde{F}$, if there is a commutative diagram of functors as follows

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\tilde{F}} & \mathcal{B} \\ {\scriptstyle O}\downarrow & & \downarrow{\scriptstyle O'} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D}, \end{array}$$

where O and O' reflect exactness.

Proposition 1.2.3. *Suppose $\tilde{F}$ extends F. If F is left exact then $\tilde{F}$ is also left exact, and if furthermore O sends injective objects to F-acyclic objects, then for every $i \geq 0$ the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{R}^i\tilde{F}} & \mathcal{B} \\ {\scriptstyle O}\downarrow & & \downarrow{\scriptstyle O'} \\ \mathcal{C} & \xrightarrow{\mathcal{R}^i F} & \mathcal{D}. \end{array}$$

Proof. Since F is left exact and O reflects exactness, $F \circ O = O' \circ \tilde{F}$ is left exact. Now since O' reflects exactness, $\tilde{F}$ is left exact. Finally, since O is exact, $\mathcal{R}^i(O' \circ \tilde{F}) = O' \circ \mathcal{R}^i\tilde{F}$, and since O sends injective objects to F-acyclic objects, Lemma 1.2.2 implies that $\mathcal{R}^i(O' \circ \tilde{F}) = \mathcal{R}^i(F \circ O) \cong \mathcal{R}^iF \circ O$. □

Given two functors $S : \mathcal{A} \longrightarrow \mathcal{B}$ and $T : \mathcal{B} \longrightarrow \mathcal{A}$, we say that (S, T) is an *adjoint pair* of functors, or that S is a left adjoint for T, or that T is a right adjoint for S, if for every pair of objects A of $\mathcal{A}$ and B of $\mathcal{B}$ there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(S(A), B) \cong \mathrm{Hom}_{\mathcal{A}}(A, T(B)).$$

The following is a well known characterization of adjoint pairs of functors.

Proposition 1.2.4. *Let $S : \mathcal{A} \longrightarrow \mathcal{B}$ and $T : \mathcal{B} \longrightarrow \mathcal{A}$ be two functors. The following conditions are equivalent:*

1. *The pair (S, T) is a pair of adjoint functors.*
2. *There exist natural transformations $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow TS$ and $\epsilon : ST \Rightarrow \mathrm{Id}_B$, such that for every object A of $\mathcal{A}$ and every object B in $\mathcal{B}$ the transformations*

$$S(A) \xrightarrow{S(\eta)} STS(A) \xrightarrow{\epsilon S} S(A) \qquad\qquad T(B) \xrightarrow{\eta T} TST(B) \xrightarrow{T(\epsilon)} T(B)$$

are identities.

Proof. See [Wei94, Theorem A.6.2]. □

The natural transformations η and ϵ of Proposition 1.2.4 are called the unit and counit of the adjoint pair (S, T), respectively.

Next we summarize some general properties of adjoint pairs of functors.

Proposition 1.2.5. *Let $S : \mathcal{A} \longrightarrow \mathcal{B}$ and $T : \mathcal{B} \longrightarrow \mathcal{A}$ be two functors such that (S, T) is an adjoint pair. Then*

1. *The functor S is right exact and preserves direct limits. The functor T is left exact and preserves inverse limits.*
2. *If T is exact then S sends projective objects to projective objects. If S is exact then T sends injective objects to injective objects.*
3. *If $\mathcal{A}$ and $\mathcal{B}$ have enough projective objects, resp. injectives, and S, resp. T, is exact, then for every object A of $\mathcal{A}$, resp B of $\mathcal{B}$, we have $\mathrm{pdim}_{\mathcal{B}} S(A) \leq \mathrm{pdim}_{\mathcal{A}} A$, resp. $\mathrm{injdim}_{\mathcal{A}} T(B) \leq \mathrm{injdim}_{\mathcal{B}} B$.*

Proof. 1. See [Wei94, Theorems 2.6.1 and 2.6.10].

2. Let P be a projective object of $\mathcal{A}$. By hypothesis, there is an isomorphism $\mathsf{Hom}_{\mathcal{B}}(S(P), -) \cong \mathsf{Hom}_{A}(P, T(-))$. Since T is exact by hypothesis, this is an exact functor, which implies that $S(P)$ is a projective object of $\mathcal{B}$. A similar reasoning works for the other case.

3. Let A be any object of $\mathcal{A}$, and let $P^{\bullet} \longrightarrow A$ be a projective resolution of A of length $\mathrm{pdim}_{\mathcal{A}} A$. Since S is exact, the previous item implies that the complex $S(P^{\bullet}) \longrightarrow S(A)$ is a projective resolution of $S(A)$ in $\mathcal{B}$ of length $\mathrm{pdim}_{\mathcal{A}} A$, which proves the desired inequality. A similar argument works for T and injective dimension.

□

Chapter 2

Graded and filtered algebras

In this chapter we study G-graded algebras, where G is an arbitrary group. In the next chapters we will only work with $\mathbb{Z}^{r+1}$-graded algebras for some $r \geq 0$, but we prove here several results which are valid in this general context.

The chapter is organized as follows: section 2.1 reviews the main definitions and basic properties of G-graded algebras and the category of G-graded modules. In section 2.2 we associate to any group morphism $\varphi : G \longrightarrow H$ three functors between the categories of G and H-graded modules over a G-graded algebra and study their homological properties, and in section 2.3 we associate to each graded ideal a graded torsion functor. All these functors feature prominently in the following chapters. Finally Section 2.4 extends some classical results from the theory of filtered algebras to $\mathbb{N}$-graded algebras filtered by graded vector spaces.

Throughout this chapter, G denotes a group and G° denotes its opposite group. Also for every algebra A we denote its opposite algebra by A°.

2.1 Graded rings and modules

In this section we review some basic facts on graded algebras and graded modules over them. We follow the presentation given in [NVO04].

Definition 2.1.1. A G-*graded algebra* is an algebra A together with a set of vector subspaces $\{A_g \mid g \in G\}$, such that $A = \bigoplus_{g\in G} A_g$ and $A_g A_{g'} \subset A_{gg'}$ for all $g, g' \in G$.

If A is a G-graded algebra, a G-*graded* A-*module*, or simply a graded module if A and G are clear from the context, is an A-module M together with a set of vector subspaces $\{M_g \mid g \in G\}$, such that $M = \bigoplus_{g\in G} M_g$ and $A_g M_{g'} \subseteq M_{gg'}$ for all $g, g' \in G$.

We consider k to be a G-graded algebra with $k_{1_G} = k$.

For the rest of this section A denotes a G-graded algebra. Setting $A^\circ_g = A_{g^{-1}}$ for every $g \in G$ gives A° the structure of a G°-graded algebra. Thus a *right* G-graded A-module is the same as a left G°-graded A°-module, and all definitions and results in the sequel apply to right graded modules.

2.1.1 The category of graded modules

In this subsection we review some facts on the category of G-graded modules over A. For proofs and more details, see [NVO04, Chapter 2].

Let M be a G-graded A-module. For every $g \in G$ the subspace $M_g \subset M$ is called the *homogeneous component* of degree g of M. We say that M is *locally finite* if M_g is finite dimensional for all $g \in G$. The *support* of the module M is the set $\operatorname{supp} M = \{g \in G \mid M_g \neq 0\}$. If the support of A is contained in a subsemigroup $S \subset G$, we will make a slight abuse of notation and say that A is an S-graded algebra.

An element $m \in M$ is said to be *homogeneous* if $m \in M_g$ for some $g \in G$; in this case we say that g is the *degree* of m, which we denote by $\deg m$. Any element $m \in M$ can be written in a unique way as a finite sum $\sum_{g\in G} m_g$, where m_g is a homogeneous element of degree g. The *support* of m is the set $\operatorname{supp} m = \{g \in G \mid m_g \neq 0\}$, and these elements m_g with $g \in \operatorname{supp} m$ are called the *homogeneous components* of m. Since the support of any element is finite, M is finitely generated as an A-module if and only if it has a finite set of homogeneous generators.

Let M be a G-graded A-module and let $M' \subset M$ be a submodule. We say that M' is a *graded submodule* if $M' = \bigoplus_{g\in G} M' \cap M_g$, or equivalently if M' has a set of homogeneous generators. Setting $M'_g = M' \cap M_g$ for every $g \in G$ gives M' the structure of a G-graded module. A *graded ideal* of A is an ideal that is also a graded submodule of A.

The algebra A is said to be *graded left noetherian* if every graded left ideal of A is finitely generated. If A is left noetherian and graded, then clearly it is graded left noetherian. By [CQ88, Theorem 2.2], if G is a polycyclic-by-finite group then the converse is also true, that is, if A is graded left noetherian then it is left noetherian. Recall that a group G is said to be *polycylic-by-finite* if there exists a finite chain of groups $\{e\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$ such that $G_i \triangleleft G_{i+1}$ for all i and G_{i+1}/G_i is either a finite group or isomorphic to $\mathbb{Z}$.

Given two G-graded A-modules N and M, an A-linear morphism $f : N \longrightarrow M$ is said to be *homogeneous of degree* g if $f(N_{g'}) \subseteq M_{g'g}$ for all $g' \in G$. We denote by $\mathsf{Mod}^G A$ the category of G-graded A-modules with homogeneous A-linear morphisms of degree 1_G, and by $\mathsf{mod}^G A$ the full subcategory of finitely generated objects of $\mathsf{Mod}^G A$. The direct sum of two G-graded A-modules has a natural G-graded module structure, and so do kernels and cokernels of homogeneous morphisms, so $\mathsf{Mod}^G A$ is an abelian

category. Given two objects M and N of $\mathsf{Mod}^G A$ we write $\mathsf{Hom}^G_A(N, M)$ for the vector space $\mathsf{Hom}_{\mathsf{Mod}^G A}(N, M)$.

Any $f \in \mathsf{Hom}^G_A(N, M)$ induces a linear map $f_g : N_g \longrightarrow M_g$ for every $g \in G$, which we call its *homogeneous component of degree* g. Let K be another object of $\mathsf{Mod}^G A$. A sequence of morphisms $K \xrightarrow{f} N \xrightarrow{f'} M$ in $\mathsf{Mod}^G A$ is exact if and only if it is exact as a sequence of A-modules, if and only if its homogeneous components $K_g \xrightarrow{f_g} N_g \xrightarrow{f'_g} M_g$ form an exact sequence of vector spaces for all $g \in G$. In particular, f is a monomorphism if and only if each of its homogeneous components is a monomorphism, and similar statements hold for epimorphisms and isomorphism.

Denote by $\mathcal{O} : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}\, A$ the forgetful functor that sends each object M of $\mathsf{Mod}^G A$ to its underlying A-module. By the discussion in the previous paragraph, the functor $\mathcal{O}$ reflects exactness.

Lemma 2.1.2. *Let A be a G-graded algebra.*

1. *The category $\mathsf{Mod}^G A$ has arbitrary direct and inverse limits.*
2. *The functor $\mathcal{O}$ is exact and has an exact right adjoint. In particular, it commutes with direct limits and sends projective objects to projective objects.*
3. *Direct limits are exact in $\mathsf{Mod}^G A$.*

Proof.

1. By Proposition 1.2.1, it is enough to show that $\mathsf{Mod}^G A$ has arbitrary direct sums and products. Given a family of objects $\mathcal{M} = \{M^i\}_{i \in I}$, for every $g \in G$ we define $S_g = \bigoplus_{i \in I} M^i_g$ and $P_g = \prod_{i \in I} M^i_g$. Let $S = \bigoplus_{g \in G} S_g$ and let $P = \bigoplus_{g \in G} P_g$. Since $S = \bigoplus_{i \in I} \mathcal{O}(M^i)$, S has a natural A-module structure, and P is an A-submodule of $\prod_{i \in I} \mathcal{O}(M^i)$. It is immediate that the previous decompositions turn S and P into G-graded A-modules. The fact that S is a direct sum and P a direct product for the family $\mathcal{M}$ in $\mathsf{Mod}^G A$ can be checked directly.
2. See [NVO04, Theorem 2.5.1]. We prove a generalization of this result in Proposition 2.2.6.
3. Since $\mathcal{O}$ reflects exactness and commutes with direct limits, the result follows from the fact that direct limits are exact in $\mathsf{Mod}\, A$.

□

For every $g \in G$ we denote by $M[g]$ the object of $\mathsf{Mod}^G A$ whose underlying A-module is equal to M and whose homogeneous components are given by $M[g]_{g'} = M_{g'g}$ for every $g' \in G$. We refer to this new object as the g-*shift* of M. For any morphism $f \in \mathsf{Hom}^G_A(N, M)$, the morphism $f[g] \in \mathsf{Hom}^G_A(N[g], M[g])$ is the A-linear map with homogeneous components $f[g]_{g'} = f_{g'g}$. The functor $-[g] : \mathsf{Mod}^G_A \longrightarrow \mathsf{Mod}^G_A$ is an autoequivalence.

An A-linear morphism $f : N \longrightarrow M$ is homogeneous of degree g if and only if $f \in \mathsf{Hom}_A^G(N, M[g])$. This allows us to consider homogeneous morphisms of arbitrary degree as morphisms of $\mathsf{Mod}^G A$, and inspires the following definition.

Definition 2.1.3. For any two objects N and M of $\mathsf{Mod}^G A$, set

$$\underline{\mathsf{Hom}}_A^G(N, M) = \bigoplus_{g \in G} \mathsf{Hom}_A^G(N, M[g]) \subseteq \mathsf{Hom}_A(\mathcal{O}(N), \mathcal{O}(M)).$$

The vector space $\underline{\mathsf{Hom}}_A^G(N, M)$ is thus a G-graded vector space, called the *enriched homomorphism space* of $\mathsf{Mod}^G A$.

A G-graded A-module M is said to be *graded-free* if it is isomorphic to a direct sum of shifts of A. Graded-free modules are projective objects of $\mathsf{Mod}^G A$ and in fact $\bigoplus_{g \in G} A[g]$ is a projective generator of $\mathsf{Mod}^G A$. By Proposition 1.2.1 and item 3 of Lemma 2.1.2, the category $\mathsf{Mod}^G A$ has enough projective and injective objects. It is clear from Definition 2.1.3 that M is projective, resp. injective, in $\mathsf{Mod}^G A$ if and only if the functor $\underline{\mathsf{Hom}}_A^G(M, -)$, resp. $\underline{\mathsf{Hom}}_A^G(-, M)$, is exact. We write $\mathrm{pdim}_A^G M$ and $\mathrm{injdim}_A^G M$ for the projective and injective dimensions of M in Mod_A^G, respectively. The *graded global dimension* of A is the supremum of the projective dimensions of objects of $\mathsf{Mod}^G A$.

By abuse of notation, we also denote by $\mathcal{O}$ the forgetful functor from $\mathsf{Mod}^G k$ to $\mathsf{Mod}\, k$. The following lemma is a well known result, see for example [NVO04, Corollary 2.4.7].

Lemma 2.1.4. *Let N and M be G-graded A-modules and suppose N is finitely generated. Then*

$$\mathcal{O}(\underline{\mathsf{Hom}}_A^G(N, M)) = \mathsf{Hom}_A(\mathcal{O}(N), \mathcal{O}(M))$$

If A is left noetherian, there exist natural isomorphisms of vector spaces

$$\mathcal{O}(\mathcal{R}^i \underline{\mathsf{Hom}}_A^G(N, M)) \cong \mathsf{Ext}_A^i(\mathcal{O}(N), \mathcal{O}(M)).$$

We will generalize this result in Proposition 2.2.11.

2.1.2 Graded bimodules

It is a well known fact that a G-graded algebra A is a comodule algebra over the Hopf algebra $k[G]$, and that G-graded A-modules are relative $(A, k[G])$-Hopf modules, see [Mon93, Example 8.5.3]. The tensor product over A of two relative $(A, k[G])$-Hopf modules has a natural $k[G]$-comodule structure, that is, it is again G-graded. Given a left G-graded A-module M and a right G-graded A-module N, the homogeneous components of $N \otimes_A M$ are given by

$$(N \otimes_A M)_g = \left\langle n \otimes_A m \mid n \in N_{g'}, m \in M_{g''} \text{ such that } g'g'' = g \right\rangle \quad \text{for all } g \in G.$$

For any $i \geq 0$, the i-th derived functor of the tensor product in $\mathsf{Mod}^G A$ is denoted by $\underline{\mathrm{Tor}}_i^A$. The usual adjunction is valid in $\mathsf{Mod}^G A$, that is, if O is any G-graded vector space then

$$\underline{\mathrm{Hom}}_k^G(N \otimes_A M, O) \cong \underline{\mathrm{Hom}}_A^G(M, \underline{\mathrm{Hom}}_k^G(N, O)).$$

For a proof of this fact see [NVO04, Proposition 2.4.9].

In particular, the enveloping algebra $A^e = A \otimes A^\circ$ has a natural G-grading, so we may consider G-graded A^e-modules, or equivalently G-graded A-bimodules. There are obvious functors $\Lambda : \mathsf{Mod}^G A^e \longrightarrow \mathsf{Mod}^G A$ and $P : \mathsf{Mod}^G A^e \longrightarrow \mathsf{Mod}^G A^\circ$ which assign to every G-graded A-bimodule M its underlying left or right A-module, respectively.

Lemma 2.1.5. *The functors Λ and P reflect exactness, and send projective objects to projective objects, and injective objects to injective objects.*

Proof. It is clear from the definition that Λ reflects exactness. Consider A^e as a graded $A^e - A$-bimodule. Given an A^e-bimodule M, the G-graded vector-space $\underline{\mathrm{Hom}}_{A^e}^G(A^e, M)$ has a left A-module structure induced by the right A-module structure of A^e, and the natural map $\underline{\mathrm{Hom}}_{A^e}^G(A^e, M) \longrightarrow \Lambda(M)$ is an isomorphism of G-graded left A-modules. In particular Λ has a left adjoint, given by $A^e \otimes_A -$. Since A^e is free over A, this functor is exact, so by item 2 of Proposition 1.2.5, Λ sends injective objects to injective objects. Again, since A^e is free over A, every projective A^e-module is projective as a left A-module, so Λ maps projective objects to projective objects. An analogous argument works for P. □

Let B and C stand for either A or k. Let N be an $A \otimes B^\circ$-module and M a $A \otimes C^\circ$-module. Then the $B \otimes C^\circ$-module structure of $\underline{\mathrm{Hom}}_A^G(N, M)$ is compatible with its G-grading. This functor is different from the usual $\underline{\mathrm{Hom}}_A^G$, since its domain is different. However, the following proposition justifies in some measure the abuse of notation.

Proposition 2.1.6. *Let B and C denote either A or k, and let N be an $A \otimes B^\circ$-module and M be an $A \otimes C^\circ$-module. We denote by O, resp O', the functor that sends $A \otimes B^\circ$-modules, resp. $A \otimes C^\circ$-modules, to their underlying A-modules.*

The G-graded $B \otimes C^\circ$-module $\mathcal{R}^i \underline{\mathrm{Hom}}_A^G(N, M)$ is naturally isomorphic as a G-graded B-module to $\mathcal{R}^i \underline{\mathrm{Hom}}_A^G(N, O'(M))$, as a G-graded C°-module to $\mathcal{R}^i \underline{\mathrm{Hom}}_A^G(O(N), M)$, and as a G-graded vector space to $\underline{\mathrm{Hom}}_A^G(O(N), O'(M))$.

Proof. Notice that when $B = A$, the functor O is equal to Λ, while for $A = k$ it is simply

Id_A^G. The following diagram commutes

$$\begin{array}{ccc} \mathsf{Mod}^G A \otimes C^\circ & \xrightarrow{\underline{\mathrm{Hom}}_A^G(N,-)} & \mathsf{Mod}^G B \otimes C^\circ \\ {\scriptstyle O}\downarrow & & \downarrow{\scriptstyle O'} \\ \mathsf{Mod}^G A & \xrightarrow[\underline{\mathrm{Hom}}_A^G(O(N),O'(-))]{} & \mathsf{Mod}^G k, \end{array}$$

and we may apply Proposition 1.2.3 because in either case O sends injective $A \otimes B^\circ$-modules to injective A-modules. The isomorphism along with the naturality on the second variable follow from said Proposition. To obtain the naturality on the first variable, consider the corresponding diagram leaving M fixed and follow the same reasoning. □

2.1.3 Zhang twists

In this subsection we review a construction by J. Zhang from [Zha96], where the reader can find the missing proofs and further information. This construction can be seen as a generalization of twisting graded algebras by 2-cocycles as defined in Section 5.2.1. The main definition is that of a *twisting system on* A, which allows one to define a new graded algebra structure on the underlying graded vector space of A, with the property that the category of G-graded modules of this new algebra is *isomorphic* to that of A. We will use these results in the following chapters to study the homological regularity properties of a family of algebras related to 2-cocycle twists of semigroup algebras.

Definition 2.1.7. [Zha96, Definitions 2.1, 4.1] A *left twisting system* on A is a set $\tau = \{\tau_g \mid g \in G\}$ of G-graded k-linear automorphisms of A, such that for all $g, g', g'' \in G$

$$\tau_{g''}(\tau_{g'}(a)a') = \tau_{g'g''}(a)\tau_{g''}(a') \qquad \text{for all } a \in A_g, a' \in A_{g'}. \tag{†}$$

A *right twisting system* is defined analogously with the previous condition replaced by

$$\tau_{g''}(a\tau_g(a')) = \tau_{g''}(a)\tau_{g''g}(a').$$

If τ is a left twisting system on A, then it is a right twisting system on A° with its G°-graded structure. Thus every result on left twisting systems has an analogue for right twisting systems.

Given a left twisting system τ on A, one can define a new G-graded algebra ${}^\tau A$ with the same underlying G-graded vector space as A and multiplication given by

$$a \cdot_\tau a' = \tau_{g'}(a)a' \qquad \text{for all } g' \in G, a \in A,\ a' \in A_{g'}.$$

The unit of the algebra ${}^\tau A$ is $\tau_{1_G}^{-1}(1)$. Of course, condition (†) is tailor-made so that this new product is associative. The algebra ${}^\tau A$ is called the *left twist of* A *by* τ.

For each object M of $\mathsf{Mod}^G A$ there is a corresponding object of $\mathsf{Mod}^G {}^\tau A$, denoted by ${}^\tau M$, with the same underlying G-graded vector space as M and ${}^\tau A$-module structure given by

$$a \cdot_\tau m = \tau_{g'}(a)m \qquad \text{for all } g' \in G, a \in A, m \in M_{g'}.$$

Once again, condition (†) guarantees that this action is associative. If $f : M \longrightarrow M'$ is a homogeneous morphism of G-graded A-modules, then the same function defines a homogeneous ${}^\tau A$-linear morphism from ${}^\tau M$ to ${}^\tau M'$. Thus this construction defines a functor $\mathcal{F}^\tau : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}^G {}^\tau A$.

If $B = {}^\tau A$, then $\tau^{-1} = \{\tau_g^{-1} \mid g \in G\}$ is a left twisting system on B, and in fact ${}^{\tau^{-1}}B = A$. Furthermore, $\mathcal{F}^\tau$ and $\mathcal{F}^{\tau^{-1}}$ are inverses of each other. This is the main result of this section, so we state it as a theorem.

Theorem 2.1.8. *The functor $\mathcal{F}^\tau : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}^G {}^\tau A$ is an isomorphism of categories.*

Proof. See [Zha96, Theorem 3.1]. □

Finally we quote a result that allows to study *right* G-graded ${}^\tau A$-modules as right twists of G-graded A-modules.

Theorem 2.1.9. *Suppose τ is a left twist on A. For every $g, g' \in G$, and every $a \in A_g$ set*

$$\nu_{g'}(a) = \tau_{(g'g)^{-1}}\tau_{g^{-1}}^{-1}(a).$$

The set $\nu = \{\nu_g \mid g \in G\}$ is a right twisting system on A, and

$$\theta : {}^\tau A \longrightarrow A^\nu$$
$$a \in {}^\tau A_g \longmapsto \tau_{g^{-1}}(a) \in A^\nu_g$$

is a G-graded algebra isomorphism.

Proof. See [Zha96, Theorem 4.3]. □

The isomorphism θ induces an isomorphism between the categories of right G-graded ${}^\tau A$-modules and the category of right G-graded A^ν-modules, which is itself isomorphic to $\mathsf{Mod}^{G^\circ} A^\circ$. Given a right G-graded A-module M, we write M^ν_θ for the right ${}^\tau A$-module with action defined by

$$m \cdot_{{}^\tau A} a = m \cdot_{A^\nu} \theta(a) = m\nu_{g'}(\theta(a)) \qquad \text{for all } a \in {}^\tau A, m \in M_{g'}.$$

In particular, the induced right ${}^\tau A$-module A^ν_θ is isomorphic to ${}^\tau A$. This fact is part of the proof of [Zha96, Theorem 4.3]. This isomorphism will play an important role in Chapter 3, where we will prove that certain homological properties of A that depend on the categories of right and left A-modules transfer to left twists of A.

2.2 Change of grading functors

Throughout this section, G and H denote groups, $\varphi : G \longrightarrow H$ is a group morphism, and $L = \ker \varphi$. We will now introduce three functors between the categories of G and H-graded vector spaces.

2.2.1 Change of grading for vector spaces

Let M be a G-graded vector space. The morphism φ induces an H-grading on the underlying vector space of M, and we denote this new H-graded vector space by $\varphi_!(M)$. Its homogeneous component of degree $h \in H$ is given by

$$\varphi_!(M)_h = \bigoplus_{g \in \varphi^{-1}(h)} M_g.$$

If M' is another G-graded vector space and $f : M \longrightarrow M'$ is a homogeneous morphism of degree 1_G, we denote by $\varphi_!(f)$ the morphism from $\varphi_!(M)$ to $\varphi_!(M')$ with the same underlying linear transformation as f. It is immediate to check that this is a morphism of H-graded vector spaces; its homogeneous components are $\varphi_!(f)_h = \bigoplus_{g \in \varphi^{-1}(h)} f_g$ for every $h \in H$.

Next, we define $\varphi_*(M)$ to be the H-graded vector space with homogeneous component of degree h given by

$$\varphi_*(M)_h = \prod_{g \in \varphi^{-1}(h)} M_g,$$

and $\varphi_*(f) : \varphi_*(M) \longrightarrow \varphi_*(M')$ to be the morphism whose homogeneous components are $\varphi_*(f)_h = \prod_{g \in \varphi^{-1}(h)} f_g$ for every $h \in H$.

Finally, given an H-graded vector space N, let $\varphi^*(N)$ be the G-graded vector space with homogeneous components $\varphi^*(N)_g = N_{\varphi(g)} u_g$ for every $g \in G$, where u_g is simply a placeholder to keep track of the degree of an element in $\varphi^*(N)$. If N' is another H-graded vector space and $f : N \longrightarrow N'$ is a homogeneous morphism of degree 1_H, we define $\varphi^*(f) : \varphi^*(N) \longrightarrow \varphi^*(N')$ to be the morphism with homogeneous components given by the assignation

$$n u_g \in \varphi^*(N)_g \longmapsto f(n) u_g \in \varphi^*(N')_g \qquad \text{for every } g \in G.$$

It is clear that these three assignations are functorial. We refer to the functors $\varphi_!, \varphi_* : \mathsf{Mod}^G k \longrightarrow \mathsf{Mod}^H k$ and $\varphi^* : \mathsf{Mod}^H k \longrightarrow \mathsf{Mod}^G k$ as the *change of grading functors*. In the next section we will show that similar functors exist for graded A-modules, where A is a G-graded algebra.

Now we provide a few simple examples of the behavior of the change of grading functors.

Example 2.2.1. 1. If $\varphi : G \longrightarrow \{e\}$ is the trivial morphism, then $\varphi_! : \mathsf{Mod}^G k \longrightarrow \mathsf{Mod}\, k$ is the forgetful functor that assigns to each G-graded vector space its underlying vector space. Evidently φ_* assigns to every G-graded vector space the product of its homogeneous components. Finally φ^* assigns to each vector space a G-graded vector space such that each homogeneous component is a copy of the original space.

2. If $r \geq 1$, $\varphi : \mathbb{Z}^r \longrightarrow \mathbb{Z}$ is the morphism that sends each r-uple $(\xi^1, \ldots, \xi^r)$ to $\xi^1 + \ldots + \xi^r$ and $A = k[x_1, \ldots x_r]$ with the natural $\mathbb{Z}^r$-grading, then $\varphi_!(A)$ is the polynomial algebra in r variables graded by total degree. On the other hand, $\varphi^*(k[x])$ is isomorphic to the $\mathbb{Z}^r$-graded subspace of $k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ spanned by elements of total degree greater than or equal to 0.

We begin our study of the change of grading functors with a basic proposition.

Proposition 2.2.2. *The change of grading functors reflect exactness.*

Proof. Since a complex of G-graded vector spaces is exact if and only if it is exact as a complex of vector spaces, and $\varphi_!$ does not change the underlying linear structures of objects and morphisms, it reflects exactness. Since complexes of graded vector spaces are exact if and only if they are exact at each homogeneous component, φ^* also reflects exactness. Finally, using the fact that direct products reflect exactness over $\mathsf{Mod}\, k$, we see that φ_* also reflects exactness. □

Definition 2.2.3. A G-graded A-module M is said to be φ-*finite* if for every $h \in H$ the set $\operatorname{supp} M \cap \varphi^{-1}(h)$ is finite.

There is a natural transformation $\eta : \varphi_! \Rightarrow \varphi_*$. For every G-graded vector space M and each $h \in H$, the map $\eta(M)_h : \varphi_!(M)_h \longrightarrow \varphi_*(M)$ is given by the natural inclusion of the direct sum of a family into its direct product. Evidently η is an isomorphism if and only if M is φ-finite.

Remark 2.2.4. As we mentioned before, the category $\mathsf{Mod}^G k$ is equivalent to the category of comodules over the group coalgebra $k[G]$, and any morphism $\varphi : G \longrightarrow H$ induces a coalgebra morphism $\varphi : k[G] \longrightarrow k[H]$ in an obvious way. In [Doi81], Y. Doi assigns to each morphism of coalgebras $\varphi : A \longrightarrow B$ two functors, $-_\varphi : \mathsf{CoMod}\, A \longrightarrow \mathsf{CoMod}\, B$ and $-^\varphi : \mathsf{CoMod}\, B \longrightarrow \mathsf{CoMod}\, A$; if $A = k[G]$ and $B = k[H]$ then $-_\varphi = \varphi_!$ and $-^\varphi = \varphi^*$. The functor φ_* was introduced by A. Polishchuk and L. Positselski in [PP11], along with the notation we use for the change of grading functors.

2.2.2 Change of grading for A-modules

Throughout this subsection A denotes a G-graded algebra. Applying $\varphi_!$, we obtain the H-graded algebra $\varphi_!(A)$. We will usually write A for $\varphi_!(A)$, since the context will always make it clear which grading we are considering.

We denote by $F^G : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}^G k$ the functor that sends a G-graded A-module to its underlying G-graded vector space. Evidently this functor reflects exactness. Given an object M and a morphism f of $\mathsf{Mod}^G A$, we write $\varphi_!(M)$ for $\varphi_!(F^G(M))$, and $\varphi_!(f)$ for $\varphi_!(F^G(f))$. A similar convention applies to φ_* and φ^*.

We denote by $\varphi_!^A(M)$ the object of $\mathsf{Mod}^H A$ with underlying H-graded vector space $\varphi_!(M)$ and the same A-module structure as M. If f is a morphism of $\mathsf{Mod}^G A$, then $\varphi_!(f)$ is a homogeneous A-linear morphism, so setting $\varphi_!^A(f) = \varphi_!(f)$, we have defined a functor $\varphi_!^A : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}^H A$.

The vector space $\varphi_*(M)$ can also be endowed with an A-module structure. Recall that for every $h, h' \in H$, an element $a \in A_h$ can be written as a finite sum $\sum_{g \in \varphi^{-1}(h)} a_g$, with $a_g \in A_g$, and an element $m \in \varphi_*(M)_{h'}$ is given by a family $m = (m_{g'})_{g' \in \varphi^{-1}(h')}$ with $m_{g'} \in M_{g'}$. Thus it is enough to define the action of the homogeneous element a_g on m, which is defined to be

$$a_g \cdot m = (a_g m_{g'})_{g' \in \varphi^{-1}(h')}.$$

Since φ is a morphism, this is an element of $\varphi_*(M)_{hh'}$ so the action is homogeneous. We denote this H-graded A-module by $\varphi_*^A(M)$. We also set $\varphi_*^A(f) = \varphi_*(f)$. Since f is A-linear so is $\varphi_*^A(f)$, and once again we have defined a functor $\varphi_*^A : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}^H A$.

Finally, given an object N of $\mathsf{Mod}^H A$, the H-graded vector space $\varphi^*(N)$ can be endowed with an A-module structure, defined as follows: Given $g, g' \in G$, for every $a \in A_g$ and $nu_{g'} \in \varphi^*(N)_{g'}$ we set $a(nu_{g'}) = (an)u_{gg'}$. Notice that $a \in A_{\varphi(g)}$ and by definition $n \in N_{\varphi(g')}$, so $an \in N_{\varphi(gg')}$ and the action is well defined. If f is a morphism of $\mathsf{Mod}^H A$, we set $\varphi_A^*(f) = \varphi^*(f)$, and once again the A-linearity of f implies that of $\varphi_A^*(f)$. We have thus defined a functor $\varphi_A^* : \mathsf{Mod}^H A \longrightarrow \mathsf{Mod}^G A$.

We also refer to $\varphi_!^A$, φ_*^A and φ_A^* as the change of grading functors. They fit into a nice triad of commutative diagrams

$$\begin{array}{ccc} \mathsf{Mod}^G A & \xrightarrow{\varphi_!^A} & \mathsf{Mod}^H A \\ {\scriptstyle F^G}\downarrow & & \downarrow{\scriptstyle F^H} \\ \mathsf{Mod}^G k & \xrightarrow{\varphi_!} & \mathsf{Mod}^H k, \end{array} \qquad \begin{array}{ccc} \mathsf{Mod}^G A & \xrightarrow{\varphi_*^A} & \mathsf{Mod}^H A \\ {\scriptstyle F^G}\downarrow & & \downarrow{\scriptstyle F^H} \\ \mathsf{Mod}^G k & \xrightarrow{\varphi_*} & \mathsf{Mod}^H k, \end{array} \qquad \begin{array}{ccc} \mathsf{Mod}^H A & \xrightarrow{\varphi_A^*} & \mathsf{Mod}^G A \\ {\scriptstyle F^H}\downarrow & & \downarrow{\scriptstyle F^G} \\ \mathsf{Mod}^H k & \xrightarrow{\varphi^*} & \mathsf{Mod}^G k. \end{array}$$

Since the forgetful functors F^G and F^H reflect exactness, and so do the change of grading functors $\varphi_!$, φ_* and φ^*, the same is true for $\varphi_!^A$, φ_*^A and φ_*^A. There is also a natural transformation $\eta^A : \varphi_!^A \longrightarrow \varphi_*^A$ such that $F^H(\eta^A(M)) = \eta(F^H(M))$, and $\eta(M)$ is an isomorphism if and only if M is a φ-finite A-module.

Recall that we have set $L = \ker \varphi$. We begin our study of the change of grading functors over A with the following lemma.

Lemma 2.2.5. *Let M be an object of* $\mathsf{Mod}^G A$.

1. *There exist natural isomorphisms*

$$\varphi_A^* \varphi_!^A(M) \cong \bigoplus_{l \in L} M[l] \qquad \text{and} \qquad \varphi_A^* \varphi_*^A(M) \cong \prod_{l \in L} M[l].$$

2. *For every* $g \in G$, *we have*

$$\varphi_!^A(M[g]) = \varphi_!^A(M)[\varphi(g)], \qquad \varphi_*^A(M[g]) = \varphi_*^A(M)[\varphi(g)] \qquad \text{and}$$
$$\varphi_A^*(N)[g] = \varphi_A^*(N[\varphi(g)]).$$

Proof. We write $\equiv_L$ for equivalence mod L, that is, given $g, g' \in G$ we write $g \equiv_L g'$ if and only if $\varphi(g) = \varphi(g')$.

1. By definition, for every $g \in G$ there are vector space isomorphisms

$$\varphi_A^* \varphi_!^A(M)_g = \left(\bigoplus_{g \equiv_L g'} M_{g'} \right) u_g = \left(\bigoplus_{l \in L} M_{gl} \right) u_g \cong \bigoplus_{l \in L} M[l]_g = \left(\bigoplus_{l \in L} M[l] \right)_g,$$
$$\varphi_A^* \varphi_*^A(M)_g = \left(\prod_{g \equiv_L g'} M_{g'} \right) u_g = \left(\prod_{l \in L} M_{gl} \right) u_g \cong \prod_{l \in L} M[l]_g = \left(\prod_{l \in L} M[l] \right)_g.$$

   Their naturality is clear from the definitions, and it is routine to prove that they are A-linear.

2. Fix $h \in H$. By definition,

$$\varphi_!^A(M)[\varphi(g)]_h = \varphi_!^A(M)_{h\varphi(g)} = \bigoplus_{g' \in \varphi^{-1}(h)} M_{g'g} = \varphi_!^A(M[g])_h.$$

   The proofs of the other claims are analogous.

□

In the next proposition we prove that $\varphi_!^A$ and φ_*^A are left and right adjoints to φ_A^*, respectively. This result is stated in [PP11, section 2.5]. The fact that $\varphi_!$ is left adjoint to φ^* was proved in [Doi81, Proposition 6].

Proposition 2.2.6. *Let* A *be a* G*-graded algebra and* $\varphi : G \longrightarrow H$ *a group morphism.*

1. *The functor* $\varphi_!^A$ *is left adjoint to* φ_A^*.
2. *The functor* φ_*^A *is right adjoint to* φ_A^*.

Proof. We will use Proposition 1.2.4, and prove the existence of a unit and a counit for both adjunctions. We denote the identity functor of $\mathrm{Mod}^G A$ by Id_A^G.

1. We define a unit $\iota : \mathrm{Id}_A^G \Rightarrow \varphi_A^* \varphi_!^A$ and a counit $\pi : \varphi_!^A \varphi_A^* \Rightarrow \mathrm{Id}_A^H$. Set $\iota(M)$ and $\pi(N)$ to be the morphisms whose homogeneous components are

$$\begin{aligned} \iota(M)_g : M_g &\longrightarrow \varphi_A^* \varphi_!^A(M)_g & \pi(N)_h : \varphi_!^A \varphi_A^*(N)_h &\longrightarrow N_h \\ m &\longmapsto m u_g & \sum_{g' \in \varphi^{-1}(h)} n_{g'} u_{g'} &\longmapsto \sum_{g' \in \varphi^{-1}(h)} n_{g'} \end{aligned}$$

for every $g \in G$ and $h \in H$. Notice that using the isomorphism of item 1 of Lemma 2.2.5, $\iota(M)$ is the natural inclusion of M in the direct sum $\bigoplus_{l \in L} M[l]$.

With these definitions naturality is obvious, and the homogeneous components of the corresponding unit and counit equations are

$$\varphi_!^A(M)_h \xrightarrow{\varphi_!^A(\iota(M))} \varphi_!^A \varphi_A^* \varphi_!^A(M)_h \xrightarrow{\pi(\varphi_!^A(M))} \varphi_!^A(M)_h$$

$$m = \sum_{g \in \varphi^{-1}(h)} m_g \longmapsto \sum_{g \in \varphi^{-1}(h)} m_g u_g \longmapsto \sum_{g \in \varphi^{-1}(h)} m_g = m,$$

$$\varphi_A^*(N)_g \xrightarrow{\iota(\varphi_A^*(N))} \varphi_A^* \varphi_!^A \varphi_A^*(N)_g \xrightarrow{\varphi_A^*(\pi(N))} \varphi_A^*(N)_g$$

$$n u_g \longmapsto (n u_g) u_g \longmapsto \pi(n u_g) u_g = n u_g.$$

The result follows.

2. Again, we define a unit $\nu : \mathrm{Id}_A^H \Rightarrow \varphi_*^A \varphi_A^*$ and a counit $\rho : \varphi_A^* \varphi_*^A \Rightarrow \mathrm{Id}_A^G$. Using item 1 of Lemma 2.2.5 we see that every homogeneous element $m \in \varphi_A^* \varphi_*^A(M)_g$ of degree $g \in G$ is of the form $(m_{gl})_{l \in L} u_g$, with $m_{gl} \in M_{gl}$. We define the homogeneous components of $\nu(N)$ and $\pi(M)$ as

$$\begin{aligned} \nu(N)_h : N_h &\longrightarrow \varphi_*^A \varphi_A^*(N)_h & \rho(M)_g : \varphi_A^* \varphi_*^A(M)_g &\longrightarrow M_g \\ n &\longmapsto (n u_{g'})_{g' \in \varphi^{-1}(h)} & (m_{gl})_{l \in L} u_g &\longmapsto m_g \end{aligned}$$

for every $g \in G$ and $h \in H$.

Once again naturality is immediate, and the homogeneous components of the

corresponding unit and counit equations are given by

$$\varphi_A^*(N)_g \xrightarrow{\varphi_A^*(\nu(N))} \varphi_A^*\varphi_*^A\varphi_A^*(N)_g \xrightarrow{\rho(\varphi_A^*(N))} \varphi_A^*(N)_g$$

$$nu_g \longmapsto (nu_{gl})_{l\in L} u_g \longmapsto nu_g$$

$$\varphi_*^A(M)_h \xrightarrow{\nu(\varphi_*^A(M))} \varphi_*^A\varphi_A^*\varphi_*^A(M)_h \xrightarrow{\varphi_*^A(\rho(M))} \varphi_*^A(M)_h$$

$$(m_{g'})_{g'\in\varphi^{-1}(h)} \longmapsto ((m_{g'})_{g'\in\varphi^{-1}(h)} u_{g''})_{g''\in\varphi^{-1}(h)} \longmapsto (m_{g''})_{g''\in\varphi^{-1}(h)},$$

so we are done.

□

The following corollary restates the adjunctions between the change of grading functors in terms of the spaces of enriched homomorphisms of $\mathsf{Mod}^G A$ and $\mathsf{Mod}^H A$.

Corollary 2.2.7. *Let* M *be an object of* $\mathsf{Mod}^G A$ *and let* N *be an object of* $\mathsf{Mod}^H A$*. There exist natural isomorphisms*

$$\varphi^*(\underline{\mathsf{Hom}}_A^H(\varphi_!^A(M), N)) \cong \underline{\mathsf{Hom}}_A^G(M, \varphi_A^*(N)),$$
$$\varphi^*(\underline{\mathsf{Hom}}_A^H(N, \varphi_*^A(M))) \cong \underline{\mathsf{Hom}}_A^G(\varphi_A^*(N), M).$$

Proof. By Proposition 2.2.6 and item 2 of Lemma 2.2.5, for every $g \in G$ there exist natural isomorphisms

$$\begin{aligned}\varphi^*(\underline{\mathsf{Hom}}_A^H(\varphi_!^A(M), N))_g &= \mathsf{Hom}_A^H(\varphi_!^A(M), N[\varphi(g)]) \cong \mathsf{Hom}_A^G(M, \varphi_A^*(N[\varphi(g)])) \\ &\cong \mathsf{Hom}_A^G(M, \varphi_A^*(N)[g]) = \underline{\mathsf{Hom}}_A^G(M, \varphi_A^*(N))_g.\\ \varphi^*(\underline{\mathsf{Hom}}_A^H(N, \varphi_*^A(M)))_g &= \mathsf{Hom}_A^H(N, \varphi_*^A(M)[\varphi(g)]) \cong \mathsf{Hom}_A^H(N, \varphi_*^A(M[g])) \\ &\cong \mathsf{Hom}_A^G(\varphi_*^A(N), M[g]) = \underline{\mathsf{Hom}}_A^G(\varphi_*^A(N), M)_g.\end{aligned}$$

□

Remark 2.2.8. The functor $\varphi_!^A$ is evidently a generalization of the forgetful functor $\mathcal{O} : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}\, A$, so it might seem strange that it is *left* adjoint to φ_A^*, which is in a sense a "free object functor". This can be explained by the fact that the G-graded structure comes from the k[G]-comodule structure of M. It is quite common for comodule categories to exhibit a behavior that is dual to that of module categories. For example, free comodules are injective objects, and Hom functors have right adjoints.

By item 3 of Proposition 1.2.5, the fact that $\varphi_!^A$ is left adjoint to φ_A^* implies that it sends projective objects to projective objects, and since φ_*^A is right adjoint to the same functor, it sends injective objects to injective objects. The following proposition refines these results.

Proposition 2.2.9. *Let M be an object of $\mathsf{Mod}^G A$. The following hold:*

$$\operatorname{pdim}_A^H \varphi_!^A(M) = \operatorname{pdim}_A^G M \qquad \operatorname{injdim}_A^H \varphi_!^A(M) \geq \operatorname{injdim}_A^G M$$
$$\operatorname{pdim}_A^H \varphi_*^A(M) \geq \operatorname{pdim}_A^G M \qquad \operatorname{injdim}_A^H \varphi_*^A(M) = \operatorname{injdim}_A^G M.$$

If M is φ-finite, equality holds in all cases.

Proof. Since $\varphi_!^A$ and φ_A^* have exact right adjoints, item 3 of Proposition 1.2.5 implies that

$$\operatorname{pdim}_A^G \varphi_A^* \varphi_!^A(M) \leq \operatorname{pdim}_A^H \varphi_!^A(M) \leq \operatorname{pdim}_A^G M.$$

By item 1 of Lemma 2.2.5, there is an isomorphism $\varphi_A^* \varphi_!^A(M) \cong \bigoplus_{l \in L} M[l]$. Since $\operatorname{pdim}_A^G M = \operatorname{pdim}_A^G M[l]$ for all $l \in L$, and the projective dimension of a direct sum is the supremum of the projective dimensions of the summands, the projective dimension of $\varphi_A^* \varphi_!^A(M)$ equals that of M, so all inequalities in the previous sequence are in fact equalities. Using again Proposition 1.2.5 and the fact that M is a direct factor of $\varphi_A^* \varphi_!^A(M)$, we see that $\operatorname{injdim}_A^H \varphi_!^A(M) \geq \operatorname{injdim}_A^G \varphi_A^* \varphi_!^A(M) \geq \operatorname{injdim}_A^G M$.

The last two formulas can be proved by dual arguments, using the isomorphism $\varphi_A^* \varphi_*^A(M) \cong \prod_{l \in L} M[l]$ from item 1 of Lemma 2.2.5. If M is φ-finite then $\varphi_!^A(M) = \varphi_*^A(M)$, so equality holds in all cases. □

The following example shows that the inequalities proved in Proposition 2.2.9 are sharp.

Example 2.2.10. Let $A = k[x, x^{-1}]$ with the obvious $\mathbb{Z}$-grading. This algebra is graded-simple, so all objects of $\mathsf{Mod}^{\mathbb{Z}} A$ are projective and injective. On the other hand, since A is a PID, an A-module is injective if and only if it is divisible, which is clearly not the case for $k[x, x^{-1}]$. Hence $\varphi_!^A(A)$ is not an injective object of $\mathsf{Mod}\, A$. Also $\varphi_*^A(A)$ is not a projective object of $\mathsf{Mod}\, A$, since the element $(x^n)_{n \in \mathbb{Z}} \in \varphi_*^A(A)$ is annihilated by $x - 1 \in A$, and projective objects of $\mathsf{Mod}\, A$ are torsion free. Finally, since $\varphi_A^* \varphi_!^A(A)$ is projective and $\varphi_A^* \varphi_*^A(A)$ is injective in $\mathsf{Mod}^{\mathbb{Z}} A$, the inequalities obtained for φ_A^* from item 3 of Proposition 1.2.5 are also sharp.

We summarize the results of the previous proposition in Table 2.1.

We finish this section relating the derived functors of Hom_A^G with those of Hom_A^H through the change of grading functors.

	$\varphi_!$	φ_*	φ^*
pdim	is preserved	may increase (preserved for φ-finite)	may decrease
injdim	may increase (preserved for φ-finite)	is preserved	may decrease

Table 2.1: Homological behaviour of the change of grading functors

Proposition 2.2.11. *Let* M *and* N *be objects of* $\mathsf{Mod}^G A$. *For every* $i \geq 0$ *there exists a morphism*

$$D^i(N, M) : \varphi_!(\mathcal{R}^i \underline{\mathsf{Hom}}_A^G(N, M)) \longrightarrow \mathcal{R}^i \underline{\mathsf{Hom}}_A^H(\varphi_!^A(N), \varphi_!^A(M)),$$

which is natural in both variables. If N *is finitely generated,* $D^0(N, M)$ *is an isomorphism, and if furthermore* A *is noetherian,* $D^i(N, M)$ *is an isomorphism for all* $i \geq 0$.

Proof. Let $D(N, M) : \varphi_!(\underline{\mathsf{Hom}}_A^G(N, M)) \longrightarrow \underline{\mathsf{Hom}}_A^H(\varphi_!^A(N), \varphi_!^A(M))$ be the morphism given by the inclusion of H-graded vector spaces

$$\varphi_!(\underline{\mathsf{Hom}}_A^G(N, M)) \subset \underline{\mathsf{Hom}}_A^H(\varphi_!^A(N), \varphi_!^A(M)),$$

i.e. a G-homogeneous morphism $f : N \longrightarrow M$ is sent to $\varphi_!^A(f)$. Since this is an inclusion, it is obviously natural in both variables.

Fix N for a moment. By the dual version of [Wei94, Theorem 2.4.7], the family of functors $\{\mathcal{R}^i\varphi_!(\underline{\mathsf{Hom}}_A^G(N, -))\}$ is a universal cohomological δ-functor, and hence there exists a unique natural transformation

$$D^i(N, -) : \varphi_!(\mathcal{R}^i \underline{\mathsf{Hom}}_A^G(N, -)) \longrightarrow \mathcal{R}^i \underline{\mathsf{Hom}}_A^H(\varphi_!^A(N), \varphi_!^A(-))$$

such that $D^0(N, -) = D(N, -)$ (We have used the fact that $\mathcal{R}^i\varphi_!(\underline{\mathsf{Hom}}_A^G(N, M)) = \varphi_!(\mathcal{R}^i \underline{\mathsf{Hom}}_A^G(N, M))$ since $\varphi_!$ is exact).

We now show how to find the morphisms D^i explicitly. Let $P^\bullet$ be a projective resolution of N in $\mathsf{Mod}^G A$. The natural transformation D induces a morphism of complexes

$$D(P^\bullet, M) : \varphi_!(\underline{\mathsf{Hom}}_A^G(P^\bullet, M)) \longrightarrow \underline{\mathsf{Hom}}_A^H(\varphi_!^A(P^\bullet), \varphi_!^A(M)).$$

Since $\varphi_!^A$ is exact and sends projective objects to projective objects, $\varphi_!^A(P^\bullet)$ is a projective resolution of $\varphi_!^A(N)$ in $\mathsf{Mod}^H A$, so taking the cohomology of bothe complexes we obtain a morphism

$$H^i(D(P^\bullet, M)) : \varphi_!(\mathcal{R}^i \underline{\mathsf{Hom}}_A^G(N, M)) \longrightarrow \mathcal{R}^i \underline{\mathsf{Hom}}_A^H(\varphi_!^A(N), \varphi_!^A(M))$$

(do not confuse the cohomology functors H^i with the group H!) which is natural in M because both $D(P^\bullet, -)$ and H^i are natural. Thus the family of natural transformations

$\{H^i(D(P^\bullet,-))\}$ is a morphism of cohomological δ-functors extending $H^0(D(P^\bullet,-)) = D(N,-)$, and hence can be identified with $D^i(N,M)$ as presented above. In particular, it is independent of the chosen resolution. If $f' : N \longrightarrow N'$ is a morphism of G-graded A-modules, choosing a projective resolution $Q^\bullet \longrightarrow N'$ and lifting f to a morphism $\tilde{f} : P^\bullet \longrightarrow Q^\bullet$, it is easy prove that the morphism is also natural in the first variable.

Finally, if N is finitely generated then $D(N,M)$ is an isomorphism by Lemma 2.1.4. Furthermore, if A is noetherian we can choose the resolution $P^\bullet$ to consist of finitely generated projective modules, in which case the morphism $D(P^\bullet,M)$ is an isomorphism of complexes since the inclusion $\varphi_!(\underline{\mathrm{Hom}}^G_A(P^i,M)) \subset \underline{\mathrm{Hom}}^H_A(\varphi_!^A(P^i),\varphi_!^A(M))$ is an equality for all $i \leq 0$. Thus under these conditions, $D^i(N,M)$ is an isomorphism for all $i \geq 0$. □

The following corollary states that the map of the last proposition is also compatible with possible extra structures on the $\underline{\mathrm{Hom}}^G_A$ modules.

Corollary 2.2.12. *Let* B *and* C *be either* A *or* k*, and let* N *be an* $A \otimes B^\circ$*-module and* M *an* $A \otimes C^\circ$*-module. Then for every* $i \geq 0$ *there exists a natural morphism of* $B \otimes C^\circ$*-modules.*

$$E^i(N,M) : \varphi_!^{B\otimes C^\circ}(\mathcal{R}^i \underline{\mathrm{Hom}}^G_A(N,M)) \longrightarrow \mathcal{R}^i \underline{\mathrm{Hom}}^H_A(\varphi_!^{A\otimes B^\circ}(N), \varphi_!^{A\otimes C^\circ}(M)).$$

If N *is finitely generated as a left* A*-module then* $E^0(N,M)$ *is an isomorphism, and if furthermore* A *is noetherian* $E^i(N,M)$ *is an isomorphism for all* $i \geq 0$*.*

Proof. Notice that there is an inclusion of H-graded $B \otimes C^\circ$-modules

$$\varphi_!^{B\otimes C^\circ}(\underline{\mathrm{Hom}}^G_A(N,M)) \subset \underline{\mathrm{Hom}}^H_A(\varphi_!^{A\otimes B^\circ}(N), \varphi_!^{A\otimes C^\circ}(M)).$$

Denote by $F^G : \mathrm{Mod}^G\, B \otimes C^\circ \longrightarrow \mathrm{Mod}^G\, k$ the obvious forgetful functor, and by O, resp. O' the functor assigning to each $A \otimes B^\circ$-module, resp. $A \otimes C^\circ$-module, its underlying A-module. Recall from Lemma 2.1.5 that given a projective resolution $P^\bullet$ of N as G-graded $A \otimes B^\circ$-module, $O(P^\bullet)$ is a projective resolution of $O(N)$. The natural transformation $E^i(N,M)$ can be defined adapting the arguments of the proof of Proposition 2.2.11, and it follows from the definitions that $F^G(E^i(N,M)) = D^i(O(N),O'(M))$ for all $i \geq 0$. The rest of the statement follows from Proposition 2.2.11. □

2.3 Torsion and local cohomology

We keep the notation from the previous sections, so G denotes a group and A denotes a G-graded algebra. We fix a G-graded two-sided ideal $\mathfrak{a} \subset A$, and for every G-graded A-module M we define its $\mathfrak{a}$*-torsion submodule* as

$$\Gamma^G_\mathfrak{a}(M) = \{x \in M \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}.$$

An element $x \in \Gamma_{\mathfrak{a}}^G(M)$ is said to be an $\mathfrak{a}$-*torsion element* of M.

Since $\mathfrak{a}^n$ is a graded ideal for all $n \in \mathbb{N}$, the torsion module $\Gamma_{\mathfrak{a}}^G(M)$ is a graded submodule of M. If $f : N \longrightarrow M$ is a morphism of G-graded A-modules then $f(\Gamma_{\mathfrak{a}}^G(N)) \subset \Gamma_{\mathfrak{a}}^G(M)$, so we set $\Gamma_{\mathfrak{a}}^G(f) : \Gamma_{\mathfrak{a}}^G(N) \longrightarrow \Gamma_{\mathfrak{a}}^G(M)$ to be the restriction and corestriction of f to the torsion submodules of N and M, respectively. This assignation defines the $\mathfrak{a}$-*torsion functor* $\Gamma_{\mathfrak{a}}^G : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}^G A$.

Definition 2.3.1. For every $i \geq 0$, the i-th derived functor of $\Gamma_{\mathfrak{a}}^G$ is called the i-*th local cohomology functor* with respect to $\mathfrak{a}$. For every G-graded A-module M, we refer to $\mathcal{R}^i\Gamma_{\mathfrak{a}}^G(M)$ as its i-*th local cohomology module* with respect to $\mathfrak{a}$.

We denote by $\Gamma_{\mathfrak{a}^\circ}^G$ the analogous functor defined for G-graded right A-modules. As usual, we only work with left modules.

If M is a G-graded A-bimodule then the set of $\mathfrak{a}$-torsion elements of M is a sub bimodule of M, so taking left $\mathfrak{a}$-torsion submodules defines a functor $\Gamma : \mathsf{Mod}^G A^e \longrightarrow \mathsf{Mod}^G A^e$. Using the notation from subsection 2.1.2 it is clear that $\Lambda \circ \Gamma = \Gamma_{\mathfrak{a}}^G \circ \Lambda$, so by Proposition 1.2.3 the underlying G-graded left A-module of $\mathcal{R}^i\Gamma(M)$ is naturally isomorphic to $\mathcal{R}^i\Gamma_{\mathfrak{a}}^G(\Lambda(M))$ for all $i \geq 0$. Thus we can make a slight abuse of notation and write $\Gamma_{\mathfrak{a}}^G$ for Γ.

2.3.1 Local cohomology

From this point on to the end of this section B stands for either A or k, and in both cases we will denote by $\Gamma_{\mathfrak{a}}^G$ the torsion functor over $\mathsf{Mod}^G A \otimes B^\circ$. The following is a standard result.

Lemma 2.3.2. *The torsion functor* $\Gamma_{\mathfrak{a}}^G : \mathsf{Mod}^G A \otimes B^\circ \longrightarrow \mathsf{Mod}^G A \otimes B^\circ$ *is left exact, and for every* $i \geq 0$ *there exists an isomorphism*

$$\mathcal{R}^i\Gamma_{\mathfrak{a}}^G \cong \varinjlim_n \mathcal{R}^i \underline{\mathsf{Hom}}_A^G(A/\mathfrak{a}^n, -).$$

Proof. Let M be a G-graded $A \otimes B^\circ$ module and fix $n \in \mathbb{N}$. For every $a \in A$ we denote by $[a]$ its class modulo $\mathfrak{a}^n$. Let $\varepsilon_n(M) : \underline{\mathsf{Hom}}_A^G(A/\mathfrak{a}^n, M) \longrightarrow \Gamma_{\mathfrak{a}}^G(M)$ be the $A \otimes B^\circ$-linear morphism that to each $f \in \underline{\mathsf{Hom}}_A^G(A/\mathfrak{a}^n, M)$ assigns $f([1])$. This is well defined since $\mathfrak{a}^n f([1]) = f(\mathfrak{a}^n[1]) = f([0]) = 0$. The morphism f is determined by its value at $[1]$, so the assignation $\varepsilon_n(M)$ is injective. Thus ε_n is a natural transformation from $\underline{\mathsf{Hom}}_A^G(A/\mathfrak{a}^n, -)$ to $\Gamma_{\mathfrak{a}}^G$.

On the other hand, for every $x \in \Gamma_{\mathfrak{a}}^G(M)$ there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n x = 0$, so there is an A-linear morphism $f_x : A/\mathfrak{a}^n \longrightarrow M$ such that $f_x([1]) = x$, i.e. x is in the image of ε_n for $n \gg 0$. This shows that the natural map

$$\varepsilon(M) = \varinjlim_n \varepsilon_n(M) : \varinjlim_n \underline{\mathsf{Hom}}_A^G(A/\mathfrak{a}^n, M) \longrightarrow \Gamma_{\mathfrak{a}}^G(M)$$

is both injective and surjective, which proves the case $i = 0$. Since $\underline{\mathrm{Hom}}_A^G(A/\mathfrak{a}^n, -)$ is left exact for every n and direct limits are exact over $\mathrm{Mod}^G A \otimes B^\circ$, the torsion functor is left exact.

Finally, ε is a natural isomorphism between left exact functors, so for every $i \geq 0$ there exists an isomorphism between the derived functors

$$\mathcal{R}^i\Gamma_\mathfrak{a}^G \cong \mathcal{R}^i \varinjlim_n \underline{\mathrm{Hom}}_A^G(A/\mathfrak{a}^n, -) \cong \varinjlim_n \mathcal{R}^i \underline{\mathrm{Hom}}_A^G(A/\mathfrak{a}^n, -),$$

where the second isomorphism holds because direct limits are exact over $\mathrm{Mod}^G A \otimes B^\circ$. □

Let H be a group and let $\varphi : G \longrightarrow H$ be a group morphism. Notice that $\mathfrak{a}$ is also an H-graded ideal for the H-grading induced by φ on A. The following lemma examines the relation between the change of grading functors from section 2.2 and the derived functors of $\Gamma_\mathfrak{a}^G$.

Proposition 2.3.3. *Write $\Gamma_\mathfrak{a}^H$ for the $\mathfrak{a}$-torsion functor on H-graded $A \otimes B^\circ$-modules.*

1. *There exists a natural transformation $\varphi_!^{A\otimes B^\circ} \circ \mathcal{R}^i\Gamma_\mathfrak{a}^G \Rightarrow \mathcal{R}^i\Gamma_\mathfrak{a}^H \circ \varphi_!^{A\otimes B^\circ}$, which is an isomorphism for $i = 0$. If A is noetherian then it is an isomorphism for all $i \geq 0$, that is, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Mod}^G A \otimes B^\circ & \xrightarrow{\mathcal{R}^i\Gamma_\mathfrak{a}^G} & \mathrm{Mod}^G A \otimes B^\circ \\ \downarrow{\scriptstyle \varphi_!^{A\otimes B^\circ}} & & \downarrow{\scriptstyle \varphi_!^{A\otimes B^\circ}} \\ \mathrm{Mod}^H A \otimes B^\circ & \xrightarrow{\mathcal{R}^i\Gamma_\mathfrak{a}^H} & \mathrm{Mod}^H A \otimes B^\circ \end{array}$$

2. *For every $i \geq 0$, the following diagram of functors commutes*

$$\begin{array}{ccc} \mathrm{Mod}^G A \otimes B^\circ & \xrightarrow{\mathcal{R}^i\Gamma_\mathfrak{a}^G} & \mathrm{Mod}^G A \otimes B^\circ \\ \uparrow{\scriptstyle \varphi^*_{A\otimes B^\circ}} & & \uparrow{\scriptstyle \varphi^*_{A\otimes B^\circ}} \\ \mathrm{Mod}^H A \otimes B^\circ & \xrightarrow{\mathcal{R}^i\Gamma_\mathfrak{a}^H} & \mathrm{Mod}^H A \otimes B^\circ \end{array}$$

Proof. 1. Let M be a G-graded $A \otimes B^\circ$-module. Since $\varphi_!^{A\otimes B^\circ}(M)$ has the same underlying $A \otimes B^\circ$-module structure as M, it is clear that $\Gamma_\mathfrak{a}^H(\varphi_!^{A\otimes B^\circ}(M)) = \varphi_!^{A\otimes B^\circ}(\Gamma_\mathfrak{a}^G(M))$; the functor $\varphi_!^{A\otimes B^\circ}$ does not change the underlying linear functions of morphisms, so $\Gamma_\mathfrak{a}^H \circ \varphi_!^{A\otimes B^\circ} = \varphi_!^{A\otimes B^\circ} \circ \Gamma_\mathfrak{a}^G$. From this we immediately deduce that $\mathcal{R}^i(\Gamma_\mathfrak{a}^H \circ \varphi_!^{A\otimes B^\circ}) \cong \mathcal{R}^i(\varphi_!^{A\otimes B^\circ} \circ \Gamma_\mathfrak{a}^G) \cong \varphi_!^{A\otimes B^\circ} \circ \mathcal{R}^i\Gamma_\mathfrak{a}^G$ for all $i \geq 0$, where the last equallity follows from the fact that $\varphi_!^{A\otimes B^\circ}$ is exact.

Now suppose A is noetherian. Let J be an injective G-graded $A \otimes B^\circ$-module. By Corollary 2.2.12, for all $i, n \geq 0$ there exists an isomorphism

$$\varphi_!^{A\otimes B^\circ}(\mathcal{R}^i \underline{\mathrm{Hom}}_A^G(A/\mathfrak{a}^n, J)) \xrightarrow{\cong} \mathcal{R}^i \underline{\mathrm{Hom}}_A^H(\varphi_!^{A\otimes B^\circ}(A/\mathfrak{a}^n), \varphi_!^{A\otimes B^\circ}(J)).$$

By Lemma 2.1.5 J is injective when considered as a left A-module, and by Proposition 2.1.6 the modules displayed above are equal to zero for all n and all $i \geq 1$. Now using Lemma 2.3.2 we get

$$\mathcal{R}^i\Gamma_\mathfrak{a}^H(\varphi_!^{A\otimes B^\circ}(J)) \cong \varinjlim_n \mathcal{R}^i \underline{\mathrm{Hom}}_A^H(\varphi_!^{A\otimes B^\circ}(A/\mathfrak{a}^n), \varphi_!^{A\otimes B^\circ}(J)) = 0,$$

for all $i \geq 1$, so $\varphi_!^{A\otimes B^\circ}(J)$ is a $\Gamma_\mathfrak{a}^H$-acyclic object. Thus we may apply Lemma 1.2.2 to obtain the last step in the following chain of isomorphisms

$$\varphi_!^{A\otimes B^\circ} \circ \mathcal{R}^i\Gamma_\mathfrak{a}^G \cong \mathcal{R}^i(\varphi_!^{A\otimes B^\circ} \circ \Gamma_\mathfrak{a}^G) \cong \mathcal{R}^i(\Gamma_\mathfrak{a}^H \circ \varphi_!^{A\otimes B^\circ}) \cong \mathcal{R}^i\Gamma_\mathfrak{a}^H \circ \varphi_!^{A\otimes B^\circ}.$$

2. We first claim that $\varphi^*_{A\otimes B^\circ} \circ \Gamma_\mathfrak{a}^H = \Gamma_\mathfrak{a}^G \circ \varphi^*_{A\otimes B^\circ}$. For any H-graded $A \otimes B^\circ$-module N choose a homogeneous element $xu_g \in \varphi^*_{A\otimes B^\circ}(N)_g$. Then xu_g is $\mathfrak{a}$-torsion if and only if there exists $n \geq 0$ such that for every homogeneous element $a \in \mathfrak{a}^n$ of degree g' it is $a(xu_g) = (ax)u_{g'g} = 0$, which happens if and only if $ax = 0$ for any homogeneous element of $\mathfrak{a}^n$. Since $\mathfrak{a}^n$ is generated by homogeneous elements this happens if and only if $x \in N$ is $\mathfrak{a}$-torsion. It is routine to verify the equality of both compositions on morphisms.

   Since $\varphi^*_{A\otimes B^\circ}$ is an exact functor that sends injective objects to injective objects (see Propositions 1.2.5 and 2.2.6), we may apply Lemma 1.2.2 as in the previous item.

□

We now introduce some invariants associated to local cohomology. Let M be a G-graded A-module. The *depth* and *local dimension* of M with respect to $\mathfrak{a}$ are defined as

$$\operatorname{depth}_\mathfrak{a}^G M = \inf\{i \in \mathbb{N} \mid \mathcal{R}^i\Gamma_\mathfrak{a}^G(M) \neq 0\}$$
$$\operatorname{ldim}_\mathfrak{a}^G M = \sup\{i \in \mathbb{N} \mid \mathcal{R}^i\Gamma_\mathfrak{a}^G(M) \neq 0\}.$$

The *local $\mathfrak{a}$-dimension of A as a graded algebra* is

$$\operatorname{lcd}_\mathfrak{a}^G A = \sup\{\operatorname{ldim}_\mathfrak{a}^G M \mid M \text{ is an object of } \mathsf{Mod}^G A\}.$$

From item 1 of Proposition 2.3.3 it follows that if A is noetherian then depth and ldim are independent of the gradings, and $\operatorname{lcd}_\mathfrak{a}^G A \leq \operatorname{lcd}_\mathfrak{a}^H A$. The following lemma refines this result.

Lemma 2.3.4. *Suppose A is noetherian.*

1. *We have an equality* $\operatorname{lcd}_a^H A = \operatorname{lcd}_a^G A$.

2. *Local cohomology functors commute with direct sums.*

3. *If* $\operatorname{lcd}_a^G A$ *is finite, then* $\operatorname{lcd}_a^G A = \operatorname{ldim}_{\mathcal{A}}^G A$.

Proof. 1. Suppose there exists an H-graded A-module N such that $\mathcal{R}^i\Gamma_a^H(N)$ is not zero. Then there exists $h \in H$ with $\mathcal{R}^i\Gamma_a^H(N)_h \neq 0$. Since $\underline{\mathsf{Hom}}_A$ and injective limits commute with shifts, Lemma 2.3.2 implies that so do local cohomology functors. Using Proposition 2.3.3 we obtain

$$\mathcal{R}^i\Gamma_a^G(\varphi_A^*(N[h]))_{1_G} \cong \varphi_A^*(\mathcal{R}^i\Gamma_a^H(N[h]))_{1_G} \cong \mathcal{R}^i\Gamma_a^H(N[h])_{1_H} \cong \mathcal{R}^i\Gamma_a^H(N)_h \neq 0,$$

which proves that $\operatorname{lcd}_a^H A \leq \operatorname{lcd}_a^G A$; notice that the hypothesis of noetherianity is not used.

On the other hand if A is noetherian and there exists a G-graded A-module M such that $\mathcal{R}^i\Gamma_a^G(M) \neq 0$, then $\mathcal{R}^i\Gamma_a^H(\varphi_!^A(M)) = \varphi_!^A(\mathcal{R}^i\Gamma_a^G(M)) \neq 0$, which proves the other inequality.

2. Let $\{M_t\}_{t\in T}$ be a family of G-graded A-modules. It is routine to check that $\Gamma_a^G(\bigoplus_{t\in T} M_t) = \bigoplus_{t\in T}\Gamma_a^G(M_t)$. Now since A is noetherian, the direct sum of graded injective modules is again graded injective (the proof found in [Lam99, Theorem 3.46] adapts readily to the graded case), so choosing for each M_t an injective resolution $I_t^\bullet$, we see that $\bigoplus_t I_t^\bullet$ is an injective resolution of $\bigoplus_{t\in L} M_t$. Thus for each $i \geq 0$.

$$\mathcal{R}^i\Gamma_a^G\left(\bigoplus_{t\in T} M_t\right) \cong H^i\left(\Gamma_a^G\left(\bigoplus_{t\in T} I_t^\bullet\right)\right) = \bigoplus_{t\in T} H^i(\Gamma_a^G(I_t^\bullet)) \cong \bigoplus_{t\in T}(\mathcal{R}^i\Gamma_a^G(M_t)).$$

3. Let N be a graded A-module such that $n = \operatorname{ldim}_a^G N = \operatorname{lcd}_a^G A$. There is a short exact sequence of graded A-modules $0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$ with F graded free. Looking at the associated long exact sequence for local cohomology, we obtain

$$\mathcal{R}^n\Gamma_a^G(F) \longrightarrow \mathcal{R}^n\Gamma_a^G(N) \longrightarrow \mathcal{R}^{n+1}\Gamma_a^G(K) = 0.$$

In particular $\mathcal{R}^n\Gamma_a^G(F) \neq 0$, and since local cohomology functors commute with direct sums, $\operatorname{ldim}_a^G F = \operatorname{ldim}_a^G A = n$.

□

We finish this section considering the relation between local cohomology and Zhang twists. Recall that given a twisting system τ over A, there exists an isomorphism of abelian categories $\mathcal{F}^\tau : \mathsf{Mod}^G A \longrightarrow \mathsf{Mod}^G {}^\tau A$.

Proposition 2.3.5. *Let τ be a twisting system on A, and let $\mathfrak{b} = {}^{\tau}\mathfrak{a}$. For every $i \geq 0$ the following diagram commutes*

$$\begin{array}{ccc} \mathsf{Mod}^G A & \xrightarrow{\mathcal{R}^i\Gamma_{\mathfrak{a}}^G} & \mathsf{Mod}^G A \\ {\scriptstyle \mathcal{F}^\tau}\downarrow & & \downarrow{\scriptstyle \mathcal{F}^\tau} \\ \mathsf{Mod}^G {}^{\tau}A & \xrightarrow{\mathcal{R}^i\Gamma_{\mathfrak{b}}^G} & \mathsf{Mod}^G {}^{\tau}A. \end{array}$$

Proof. It is immediate from the definitions that $\Gamma_{\mathfrak{b}}^G \circ \mathcal{F}^\tau = \mathcal{F}^\tau \circ \Gamma_{\mathfrak{a}}^G$. Since $\mathcal{F}^\tau$ is exact, $\mathcal{R}^i(\Gamma_{\mathfrak{b}} \circ F^\tau) = \mathcal{R}^i(\mathcal{F}^\tau \circ \Gamma_{\mathfrak{a}}^G) = \mathcal{F}^\tau \circ \mathcal{R}^i\Gamma_{\mathfrak{a}}^G$. Also $\mathcal{F}^\tau$ sends injective objects to injective objects, so we may apply Lemma 1.2.2 and conclude that $\mathcal{R}^i(\Gamma_{\mathfrak{b}}^G \circ \mathcal{F}^\tau) \cong \mathcal{R}^i\Gamma_{\mathfrak{b}}^G \circ \mathcal{F}^\tau$. □

2.4 Filtered algebras

In this section we review a few general results on filtered algebras. We are particularly interested in graded algebras endowed with a filtration by graded subspaces, which we call GF-algebras. This situation arises often in the study of Hodge algebras, see for example [BH93, chapter 6] or [DCEP]. The associated graded algebra of a GF-algebra is an $\mathbb{N}^2$-graded algebra, and this grading plays an important role in the sequel.

2.4.1 Filtrations indexed by semigroups

Throughout this subsection we assume that S is a finitely generated subsemigroup of $\mathbb{Z}^{r+1}$ for some $r \geq 0$. We denote by $<$ the restriction of the lexicographic order of $\mathbb{Z}^{r+1}$ to S.

Definition 2.4.1. An *S-filtered algebra* is an algebra A along with a family of subspaces $\mathcal{F} = \{F_s A \mid s \in S\}$, such that $F_s A \cdot F_{s'} A \subset F_{s+s''} A$ for every $s, s' \in S$ and $F_{s'} A \subset F_s A$ whenever $s' < s$.

If A is an S-filtered algebra, an *filtered A-module*, or simply a filtered module if S and A are clear from the context, is an A-module along with a family of subspaces $\{F_s M \mid s \in \mathbb{Z}^{r+1}\}$, such that $F_s A \cdot F_{s'} M \subset F_{s+s'} M$ for every $s \in S$, $s' \in \mathbb{Z}^{r+1}$, and $F_{s'} M \subset F_s M$ if $s' < s$.

Given a filtered A-module M, the subspaces $F_s M$ are called the *layers* of the filtration. We denote by $F_{<s} M$ the space $\sum_{s'<s} F_{s'} M$. Just as for graded algebras, we say that M is S-filtered if $F_s M = 0$ whenever $s \notin S$.

The *associated S-graded algebra* of A, denoted by $\operatorname{gr} A$, is the S-graded algebra with homogeneous components

$$(\operatorname{gr} A)_s = \frac{F_s A}{F_{<s} A} \qquad \text{for every } s \in S.$$

By definition of a filtered algebra, for every $s, s' \in S$ the multiplication of A restricts to $F_sA \times F_{s'}A \longrightarrow F_{s+s'}A$, which in turn induces a map $\frac{F_sA}{F_{<s}A} \times \frac{F_{s'}A}{F_{<s'}A} \longrightarrow \frac{F_{s+s'}A}{F_{<s+s'}A}$. This defines the product of $\operatorname{gr} A$ over homogeneous elements, which is then extended by bilinearity to the whole space.

Given a filtered A-module, its *associated graded module* is the $\mathbb{Z}^{r+1}$-graded $\operatorname{gr} A$-module with homogeneous components

$$(\operatorname{gr} M)_s = \frac{F_sM}{F_{<s}M} \qquad \text{for every } s \in \mathbb{Z}^{r+1}.$$

By definition of a filtered module, for every $s \in S$, $s' \in \mathbb{Z}^{r+1}$, the action of A on M induces a linear morphism $F_sA \times F_{s'}M \longrightarrow F_{s+s'}M$, which in turn induces an action of $\operatorname{gr} A$ on $\operatorname{gr} M$.

Definition 2.4.2. A filtration on a vector space M is said to be:

(E) *exhaustive* if $\bigcup_{s\in\mathbb{Z}^{r+1}} F_sM = M$,

(B) *bounded below* if there exists $s' \in \mathbb{Z}^{r+1}$ such that $F_sM = \{0\}$ for all $s < s'$,

(D) *discrete* if $\bigcap_{s\in\mathbb{Z}^{r+1}} F_sM = \{0\}$.

If M is a filtered module and its filtration is exhaustive and discrete, then for every non-zero element $m \in M$ there exists $p(m) \in \mathbb{Z}^{r+1}$ such that $m \in F_{p(m)}M$ and $\notin F_{<p(m)}M$. The class of m in $(\operatorname{gr} M)_{p(m)}$ is a nonzero element denoted by $\operatorname{gr} m$.

We consider k to be a filtered algebra setting $F_0k = k$. The following is a standard result on filtered vector spaces.

Lemma 2.4.3. *Suppose $S \subset \mathbb{N}^{r+1}$ is a positive affine semigroup. Let V be an S-filtered vector space with an exhaustive filtration $\mathcal{F}$, and let $\{v_i \mid i \in I\}$ be a subset of V. Then $\{\operatorname{gr} v_i\}_{i\in I}$ is a homogeneous basis of $\operatorname{gr}_{\mathcal{F}} V$ if and only if for every $s \in S$ the set $\{v_i \mid i \in I\} \cap F_sV$ is a basis of F_sV.*

Proof. First we set some notation. For each $s \in S$ set

$$I_s = \{i \in I \mid v_i \in F_sV\}, \qquad I_{<s} = \cup_{t<s} I_t \qquad \text{and} \qquad I_s^\circ = I_s \setminus I_{<s}.$$

We point out that $\{\operatorname{gr} v_i\}$ is a basis of $\operatorname{gr} V$ if and only if the set $\{\operatorname{gr} v_i \mid i \in I_s^\circ\}$ is a basis of the homogeneous component $(\operatorname{gr}_{\mathcal{F}} V)_s$. The if part of the statment follows.

We now prove that $\{v_i \mid i \in I_s\}$ is a basis of F_sV by induction on the totally ordered set S. The case $s = 0$ is clear since $\operatorname{gr} V_0 = F_0V$. Let $s \in S$. Suppose the result holds for all $t < s$, and that there exist scalars λ_i with $i \in I_s$ such that

$$0 = \sum_{i\in I_s} \lambda_i v_i = \sum_{i\in I_{<s}} \lambda_i v_i + \sum_{i\in I_s^\circ} \lambda_i v_i.$$

Reducing this equality modulo $F_{<s}V$ we get

$$0 = \sum_{i \in I_s^\circ} \lambda_i \operatorname{gr} v_i$$

which implies that $\lambda_i = 0$ for $i \in I_s^\circ$. On the other hand, there are at most finitely many $i \in I_{<s}$ such that $\lambda_i \neq 0$, so there exists $t < s$ such that $\lambda_i \neq 0$ implies $i \in I_t$. By inductive hypothesis the set $\{v_i \mid i \in I_t\}$ is linearly independent, so $\lambda_i = 0$ for all $i \in I_s$ and $\{v_i \mid i \in I_s\}$ is linearly independent.

Let $v \in F_sV$. If $v \in F_tV$ for some $t < s$ then the inductive hypothesis guarantees that v is in the vector space generated by $\{v_i \mid i \in I_t\}$. If not, then $\operatorname{gr} v \in (\operatorname{gr}_{\mathcal{F}} V)_s$, and there are scalars λ_i for $i \in I_s^\circ$ such that

$$\operatorname{gr} v = \sum_{i \in I_s^\circ} \lambda_i \operatorname{gr} v_i.$$

This implies that $v - \sum_{i \in I_s^\circ} \lambda_i v_i \in F_tV$ for some $t < s$, and by inductive hypothesis this element lies in the vector space generated by $\{v_i \mid i \in I_t\}$. Hence F_sV is generated by $\{v_i \mid i \in I_s\}$. □

2.4.2 GF-algebras and modules

We now focus on the case where A is $\mathbb{N}$-graded *and* $\mathbb{N}$-filtered. Most of the material in this section is adapted from [NVO79, chapter I].

Definition 2.4.4. A *GF-algebra* (as in "graded *and* filtered") is an $\mathbb{N}$-graded algebra A with a filtration $\mathcal{F} = \{F_pA\}_{p \in \mathbb{N}}$ such that each layer is a graded subspace of A.

We say that an A-module M is a *GF-module* over A if it is both a $\mathbb{Z}$-graded and a $\mathbb{Z}$-filtered module, and each layer of its filtration is a graded subspace of M.

As a trivial example, we consider k to be a GF-algebra with $k_0 = F_0k = k$. A GF-vector space is a $\mathbb{Z}$-graded vector space with a filtration by graded subspaces.

For the rest of this section A denotes a GF-algebra. We write $\underline{\operatorname{Hom}}_A$ instead of $\underline{\operatorname{Hom}}_A^{\mathbb{Z}}$ and for every $i \geq 0$ we denote by $\underline{\operatorname{Ext}}_A^i$ its i-th derived functor.

Let M be a GF-module over A. Given a vector subspace $V \subset M$, the filtration on M induces a filtration on V by setting $F_pV = V \cap F_pM$ for every $p \in \mathbb{Z}$. In particular every homogeneous component of M is a filtered subspace. Recall that if the filtration on M is exhaustive and discrete, then for each $m \in M \setminus \{0\}$ we denote by $p(m)$ the smallest integer such that $m \in F_{p(m)}M$ and $m \notin F_{<p(m)}M$. Given a homogeneous element $m \in M$ we denote by $\delta(m)$ the ordered pair $(p(m), \deg m)$.

We say that M is a *GF-free* module if it is a GF-module with a homogeneous basis

$\{e_i\}_{i\in I}$ and an exhaustive and discrete filtration such that, if $\delta(e_i) = (p_i, d_i)$, then

$$F_pM_d = \sum_{i\in I} F_{p-p_i}A_{d-d_i}e_i.$$

The GF-module M is *locally finite* if F_pM_d is a finite dimensional vector space for all $p, d \in \mathbb{Z}$. Finally M is *GF-finite* if it is generated by a finite set of homogeneous elements $m_1, \dots, m_r$ with $\delta(m_i) = (p_i, d_i)$, such that

$$F_pM_d = \sum_{i=1}^{r} F_{p-p_i}A_{d-d_i}m_i.$$

Let M be a finitely generated graded A-module. Choosing a finite set of homogeneous generators $m_1, \dots, m_r$, we can put a filtration on M setting

$$F_pM = \sum_{i=1}^{r} F_pAm_i \qquad \text{for every } p \in \mathbb{Z}.$$

It is clear by definition that this gives M the structure of a GF-finite module. This simple construction will be used repeatedly in the next chapter, so we put it down as a lemma.

Lemma 2.4.5. *Assume A is a GF-algebra and let M be a finitely generated graded A-module. Then M can be given the structure of a GF-finite module and its associated graded module is finitely generated over* $\operatorname{gr} A$.

Given two GF-modules M and N, the vector space $\operatorname{Hom}_A(N, M)$ is filtered in the following way: for every $p \in \mathbb{Z}$, set

$$F_p \operatorname{Hom}_A(N, M) = \{f \in \operatorname{Hom}_A(N, M) \mid f(F_qN) \subset F_{q+p}M \text{ for all } q \in \mathbb{Z}\}.$$

This induces a filtration on the vector subspace $\underline{\operatorname{Hom}}_A(N, M) \subset \operatorname{Hom}_A(N, M)$. With this filtration, the $\mathbb{Z}$-graded vector space $\underline{\operatorname{Hom}}_A(N, M)$ becomes a GF-vector space.

A morphism $f \in \operatorname{Hom}_A^{\mathbb{Z}}(N, M)$ such that $f(F_pN) \subset F_pM$ for every $p \in \mathbb{Z}$ is called a *GF-morphism*. Equivalently, f is a GF-morphism if and only if $f \in F_0 \underline{\operatorname{Hom}}_A(N, M)_0$. A GF-morphism f is *strict* if $f(F_pN) = \operatorname{Im} f \cap F_pM$ for all $p \in \mathbb{Z}$. Notice that this condition is stronger than that of being a GF-morphism. A GF-module M is finite if and only if there is a GF-finite and free module F and a strict epimorphism $F \longrightarrow M$.

Since the GF-algebra A is filtered, we may consider its associated graded algebra

$$\operatorname{gr} A = \bigoplus_{p=0}^{\infty} \frac{F_pA}{F_{p-1}}A = \bigoplus_{(p,d)\in\mathbb{N}^2} \frac{F_pA_d}{F_{p-1}A_d}.$$

This decomposition gives $\operatorname{gr} A$ the structure of an $\mathbb{N}^2$-graded algebra. By [MR01, Theorem 1.6.9], if $\operatorname{gr} A$ is noetherian then so is A.

Given a GF-module M, its associated graded module

$$\operatorname{gr} M = \bigoplus_{p\in\mathbb{Z}} \frac{F_p M}{F_{p-1} M} = \bigoplus_{(p,d)\in\mathbb{Z}^2} \frac{F_p M_d}{F_{p-1} M_d}$$

is a $\mathbb{Z}^2$-graded $\operatorname{gr} A$-module. If the filtration on M is exhaustive and discrete, then for every homogeneous element $m \in M$ we have that $\operatorname{gr} m \in (\operatorname{gr} M)_{\delta(m)}$. If $f \in \operatorname{Hom}_A(N, M)$ with $\delta(f) = (p, d)$, then $\operatorname{gr} f : \operatorname{gr} N \longrightarrow \operatorname{gr} M$ is a homogeneous $\operatorname{gr} A$-linear morphism of degree (p, d).

Now we present some technical results on GF-modules to be used in the sequel.

Lemma 2.4.6. *Let A be a GF-algebra and let K, M and N be GF-modules. Suppose the filtrations on A and M are exhaustive and discrete.*

1. *Let $(*) : K \xrightarrow{f} M \xrightarrow{g} N$ be a complex, where f and g are GF-morphisms. Its associated graded complex $\operatorname{gr} K \xrightarrow{\operatorname{gr} f} \operatorname{gr} M \xrightarrow{\operatorname{gr} g} \operatorname{gr} N$ is exact if and only if $(*)$ is exact and f, g are strict.*

2. *If M is GF-free with basis $\{m_i\}_{i\in I}$, then $\operatorname{gr} M$ is a $\mathbb{Z}^2$-graded $\operatorname{gr} A$-free module with basis $\{\operatorname{gr} m_i\}_{i\in I}$.*

3. *If $\operatorname{gr} M$ is generated over $\operatorname{gr} A$ by the set $\{\operatorname{gr} m_i \mid i \in I\}$, then M is generated over A by the set $\{m_i \mid i \in I\}$. Moreover, M is GF-finite if and only if $\operatorname{gr} M$ is finitely generated.*

4. *If $\operatorname{gr} A$ is noetherian then A is noetherian.*

5. *There exists a resolution of M by GF-free modules with strict differentials. Furthermore, if $\operatorname{gr} A$ is noetherian and $\operatorname{gr} M$ is finitely generated over $\operatorname{gr} A$, the GF-free modules in the resolution can be chosen to be GF-finite.*

Proof. 1. Notice that the gradings play no role in this statement, so we may refer to the filtered case, which proved in [NVO79, 4.4, item 5].

2. Let $p_i = p(m_i)$ and $d_i = \deg m_i$. By definition, for every $(p, d) \in \mathbb{Z}^2$

$$F_p M_d = \bigoplus_{i\in I} F_{p-p_i} A_{d-d_i} m_i$$

so

$$\operatorname{gr} M_{(p,d)} = \bigoplus_{i\in I} \frac{F_{p-p_i} A_{d-d_i} m_i}{F_{p-p_i-1} A_{d-d_i} m_i} \cong \bigoplus_{i\in I} (\operatorname{gr} A)_{(p-p_i,d-d_i)} \operatorname{gr} m_i.$$

Thus $\operatorname{gr} M$ is a graded-free $\operatorname{gr} A$-module with basis $\{\operatorname{gr} m_i \mid i \in I\}$.

3. Let F be the GF-free module with basis $\{e_i \mid i \in I\}$ and $\delta(e_i) = \delta(m_i)$. Since the filtration on A is exhaustive and discrete, the same holds for the filtration on F. Let $f : F \longrightarrow M$ be the map defined by setting $f(e_i) = m_i$, and consider its associated graded morphism $\operatorname{gr} f : \operatorname{gr} F \longrightarrow \operatorname{gr} M$. Since $\operatorname{gr} f(\operatorname{gr} e_i) = \operatorname{gr} m_i$, the map $\operatorname{gr} f$ is surjective by hypothesis, so item 1 implies that f is a strict epimorphism. In particular M is generated by the set $\{m_i \mid i \in I\}$.

   If $\operatorname{gr} M$ is finitely generated, the module F can be taken to be GF-finite and free in the previous argument, and so there exists a strict epimorphism from a GF-finite and free module onto M. This is equivalent to the fact that M is a GF-finite module. Conversely, suppose we have a GF-finite and free module F and a strict epimorphism $F \longrightarrow M$. By passing to the associated graded modules we get an epimorphism from $\operatorname{gr} F$ to $\operatorname{gr} M$, and since $\operatorname{gr} F$ is finitely generated and free, $\operatorname{gr} M$ is finitely generated.

4. It is enough to show that A is graded noetherian. If $I \subset A$ is a graded ideal then it is a GF-module with the filtration induced by that of A. Now the ideal $\operatorname{gr} I$ is finitely generated since $\operatorname{gr} A$ is noetherian, and by item 3 this implies that I is finitely generated.

5. The first part of item 3 shows that for every GF-module M there exists a GF-free module F and a strict epimorphism $F \longrightarrow M$, so applying the usual procedure to construct a free resolution we obtain a resolution by GF-free modules with strict differentials.

   If $\operatorname{gr} A$ is noetherian and $\operatorname{gr} M$ is finitely generated, then A is a noetherian algebra and M a finitely generated A-module. The second part of item 3 shows that every finitely generated module has a GF-finite and free cover, so we may apply the usual procedure to construct finite and free resolutions over noetherian algebras to obtain the desired resolution.

□

The next lemma relates the $\mathbb{Z}$-graded $\underline{\operatorname{Hom}}$ spaces over A with the $\mathbb{Z}^2$-graded $\underline{\operatorname{Hom}}$ spaces over $\operatorname{gr} A$. We write $\underline{\operatorname{Hom}}_{\operatorname{gr} A}$ for $\underline{\operatorname{Hom}}^{\mathbb{Z}^2}_{\operatorname{gr} A}$, and we denote by $\underline{\operatorname{Ext}}^i_{\operatorname{gr} A}$ its i-th derived functor for every $i \geq 0$.

Lemma 2.4.7. *Let A be a GF-algebra and let M and N be two GF-modules, with N GF-finite. Assume all filtrations are exhaustive and discrete.*

1. *The filtration $\{F_p \operatorname{Hom}_A(N, M)\}_{p \in \mathbb{Z}}$ on $\operatorname{Hom}_A(N, M)$ is exhaustive. If the filtration on M is bounded below, then the filtration on $\operatorname{Hom}_A(N, M)$ is also bounded below.*

2. *If N is GF-free, then there exists a natural $\mathbb{Z}^2$-graded vector space isomorphism*

$$\varphi : \operatorname{gr} \operatorname{Hom}_A(N, M) \longrightarrow \underline{\operatorname{Hom}}_{\operatorname{gr} A}(\operatorname{gr} N, \operatorname{gr} M)$$

given by the following assignation: for every $f \in F_{p(f)} \operatorname{Hom}_A(N, M)$ *and* $x \in F_{p(x)}N$

$$\varphi(\operatorname{gr} f)(\operatorname{gr} x) = \overline{f(x)} \in F_{p(f)+p(x)}M/F_{p(f)+p(x)-1}M.$$

Proof. 1. Let $n_1, \ldots, n_r$ be a finite set of generators of N and set $p_i = p(n_i)$ for $1 \leq i \leq r$. Let $f \in \operatorname{Hom}_A(N, M)$ and let $q_i = p(f(n_i))$. If $t = \max_i\{q_i - p_i\}$ then

$$f(F_pN) = \sum_i f(F_{p-p_i}An_i) = \sum_i F_{p-p_i}Af(n_i) \subset \sum_i F_{p-p_i+q_i}M \subset F_{p+t}M.$$

Hence $f \in F_t \operatorname{Hom}_A(N, M)$. If we assume the filtration on M is bounded below, then there exists $q \in \mathbb{Z}$ such that $F_qM = 0$. Let $p_0 = \max\{p_i, 1 \leq i \leq r\}$. If $f \in F_{q-p_0} \operatorname{Hom}_A(N, M)$, then $f(n_i) \in F_{q-p_0+p_1}M = 0$, so $F_{q-p_0} \operatorname{Hom}_A(N, M) = 0$ and the filtration on $\operatorname{Hom}_A(N, M)$ is bounded below.

2. It is clear from the definition that φ is a graded morphism. Since GF-free modules are filtered projective in the sense of [NVO79, Section I.5], φ is an isomorphism by [NVO79, Lemma 6.4].

□

The following theorem relates the $\mathbb{Z}$-graded $\underline{\operatorname{Ext}}$-modules over A with the $\mathbb{Z}^2$-graded $\underline{\operatorname{Ext}}$-modules over $\operatorname{gr} A$. We will use this result in chapter 3 to transfer some homological properties from $\operatorname{gr} A$ to A.

Theorem 2.4.8. *Let* A *be a GF-algebra, and let* M *and* N *be two GF-modules with* N *GF-finite. Suppose that all filtrations are exhaustive and discrete, that the filtration on* M *is bounded below, and that* $\operatorname{gr} A$ *is noetherian. Then for each* $d \in \mathbb{Z}$ *there is a convergent spectral sequence:*

$$E(N, M)_d : E^1_{p,q} = \underline{\operatorname{Ext}}^{-p-q}_{\operatorname{gr} A}(\operatorname{gr} N, \operatorname{gr} M)_{(p,d)} \Rightarrow \underline{\operatorname{Ext}}^{-p-q}_A(N, M)_d \qquad p, q \in \mathbb{Z},$$

and the filtration of the $\underline{\operatorname{Ext}}$*-group on the right hand side is bounded below and exhaustive.*

Proof. By item 4 of Lemma 2.4.6 there is a GF-free and finite resolution of N with strict differentials

$$\ldots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow N \longrightarrow 0$$

Each P^i has a finite and bounded below filtration, so item 1 of Lemma 2.4.7 implies that for every $i \leq 0$ the vector space $\underline{\operatorname{Hom}}_A(P^i, M)$ is a GF-vector space and its filtration is exhaustive and bounded below.

Fix $d \in \mathbb{Z}$. Since the differentials in the resolution of N are GF-morphisms, $\underline{\operatorname{Hom}}_A(P^\bullet, M)_d$ is a complex of filtered vector spaces. By [Wei94, 5.5.1.2] there exists a spectral sequence with page one equal to

$$E^1_{p,q} = H_{p+q}(F_p \underline{\operatorname{Hom}}_A(P^\bullet, M)_d / F_{p-1} \underline{\operatorname{Hom}}_A(P^\bullet, M)_d) \qquad p, q \in \mathbb{Z},$$

that converges to

$$H_{p+q}(\underline{\mathrm{Hom}}_A(P^\bullet, M)_d) = \underline{\mathrm{Ext}}_A^{-p-q}(N, M)_d.$$

By item 1 of Lemma 2.4.6, the complex

$$\ldots \longrightarrow \mathrm{gr}\, P^{-2} \longrightarrow \mathrm{gr}\, P^{-1} \longrightarrow \mathrm{gr}\, P^{0} \longrightarrow \mathrm{gr}\, N \longrightarrow 0$$

is exact, and by item 2 of the same lemma it is a free resolution of $\mathrm{gr}\, N$ as a $\mathbb{Z}^2$-graded $\mathrm{gr}\, A$-module. Finally, by item 2 of Lemma 2.4.7 there is a natural isomorphism

$$F_p \underline{\mathrm{Hom}}_A(P^\bullet, M)_d / F_{p-1} \underline{\mathrm{Hom}}_A(P^\bullet, M)_d \cong \underline{\mathrm{Hom}}_{\mathrm{gr}\, A}(\mathrm{gr}\, P^\bullet, \mathrm{gr}\, M)_{(p,d)}$$

and so $E^1_{p,q} \cong \underline{\mathrm{Ext}}^{-p-q}_{\mathrm{gr}\, A}(\mathrm{gr}\, N, \mathrm{gr}\, M)_{(p,d)}$. This completes the proof. □

Theorem 2.4.8 is a generalization of the spectral sequence found in [Bjö89, section 3], which deals with filtered algebras. MathOverflow user Ralph kindly pointed out and provided a proof of the existence of this spectral sequence. The proof above is taken from [Ral] with minor modifications to take into account the graded structure.

The following is an immediate corollary of the previous result

Corollary 2.4.9. *Suppose* M *and* N *are GF-modules over* A *satisfying the hypotheses of Theorem 2.4.8. Fix* $d \in \mathbb{Z}$ *and* $i \in \mathbb{N}$. *If* $\underline{\mathrm{Ext}}^i_{\mathrm{gr}\, A}(\mathrm{gr}\, N, \mathrm{gr}\, M)_{(p,d)} = 0$ *for all* $p \in \mathbb{Z}$, *then* $\underline{\mathrm{Ext}}^i_A(N, M)_d = 0$.

Proof. By hypothesis, all the entries in the i-th diagonal of the first page of $E(N, M)_d$ are equal to zero, so the same is true for the infinity page. By Theorem 2.4.8, the component $\underline{\mathrm{Ext}}^i_A(N, M)_d$ has a bounded below and exhaustive filtration such that its associated graded module is zero. By item 3 of Lemma 2.4.6, $\underline{\mathrm{Ext}}^i_A(N, M)_d = 0$. □

Chapter 3

Homological regularity of connected algebras

In their classic paper [AS87], M. Artin and W. Schelter introduced the notion of AS-regular algebras. Several other properties from the commutative world, such as being Gorenstein, Cohen-Macaulay, having dualizing complexes, etc., were borrowed afterwards by noncommutative geometry to help in the study and classification of connected $\mathbb{N}$-graded algebras. In order to proceed with this program, the theory of local cohomology for connected $\mathbb{N}$-graded algebras was developed by analogy with commutative local rings. Examples of the application of local cohomology for noncommutative algebras can be found in many articles, such as [Yek92], [AZ94], [Jør97], etc. In this chapter, we show that some of these ideas adapt painlessly to the connected $\mathbb{N}^{r+1}$-graded case.

The chapter is organized as follows: In section 3.1 we review the general properties of $\mathbb{N}^{r+1}$-graded algebras. In section 3.2 we review the noncommutative version of some regularity conditions over graded rings, show their relation to change of gradings, and use these results to prove that said properties are stable by Zhang twists and by passing to associated graded rings.

3.1 Connected graded algebras

Let $r \geq 0$. An $\mathbb{N}^{r+1}$-graded algebra A is said to be *connected* if its component of degree $(0, \ldots, 0)$ is equal to k. Just as in the $\mathbb{N}$-graded case, connected graded algebras have several properties in common with local rings.

We assume for the rest of this section that A is a connected $\mathbb{N}^{r+1}$-graded algebra.

We write "graded" instead of "$\mathbb{Z}^{r+1}$-graded", $\underline{\mathrm{Hom}}_A$ instead of $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}$, and for every $i \geq 0$ we denote by $\underline{\mathrm{Ext}}_A^i$ its i-th derived functor. We denote by $\mathfrak{m}$ the ideal generated by the homogeneous elements of nonzero degree, which is the only maximal graded ideal of A. Since $A/\mathfrak{m} \cong k$, we consider k as a left and right graded A-module through this isomorphism. In particular, A has a notion of rank. As discussed in subsection 2.1.1, A is graded noetherian if and only if it is noetherian.

3.1.1 General properties

We begin with an $\mathbb{N}^{r+1}$-graded version of Nakayama's Lemma.

Lemma 3.1.1 (Nakayama's Lemma)**.** *Let M be a finitely generated graded A-module. The following are equivalent:*

1. $M = 0$.
2. $M = \mathfrak{m}M$.
3. $k \otimes_A M = 0$.

If N is another graded A-module and $f : N \longrightarrow M$ is a morphism of graded modules, f is epic if and only if $1 \otimes_A f : k \otimes_A N \longrightarrow k \otimes_A M$ is epic.

Proof. The equivalence between 2 and 3 is clear since $k \otimes_A M \cong M/\mathfrak{m}M$, and obviously 1 implies the other two. We prove that 2 implies 1 by contradiction. Suppose M is finitely generated and $M \neq 0$. Setting the lexicographic order on $\mathbb{Z}^{r+1}$, we may consider $\xi = \min \operatorname{supp} M$. The minimality of ξ implies that $(\mathfrak{m}M)_\xi = 0$, so $(M/\mathfrak{m}M)_\xi \neq 0$, in particular $M \neq \mathfrak{m}M$.

Since the functor $k \otimes_A -$ is right exact, $k \otimes_A (M/f(N)) = \operatorname{coker} 1 \otimes_A f$, so by the first part of the lemma, $1 \otimes_A f$ is epic if and only if $M/f(N) = 0$. □

The following fact is a classical consequence of Nakayama's Lemma.

Lemma 3.1.2. *Suppose A is noetherian. For any finitely generated graded A-module M there exists a resolution $P^\bullet \longrightarrow M$ by finitely generated graded-free modules of length $\mathrm{pdim}_A^{\mathbb{Z}^{r+1}} M$.*

Proof. For every graded A-module N we set $P(N) = A \otimes (N/\mathfrak{m}N)$. This tensor product has a grading as described in subsection 2.1.2, which is compatible with the A-module structure, so it is a graded free module. If $\overline{n}_1, \ldots, \overline{n}_r$ is a basis of $N/\mathfrak{m}N$ we define a morphism $P(N) \longrightarrow N$ by setting $1 \otimes \overline{n}_i \mapsto n_i$ for all i, which by Nakayama's lemma is an epimorphism.

We set $\Omega^0 M = M$ and $P^0 = P(M)$. For every $i \geq 0$ we define recursively $\Omega^{-i-1}M = \ker(P(\Omega^{-i}M) \longrightarrow \Omega^{-i}M)$ and $P^{-i-1} = P(\Omega^{-i-1}M)$. Since A is noetherian, $\Omega^\bullet M$ and

$P^\bullet$ are finitely generated for all $\bullet \leq 0$. We set $d^{-i} : P^{-i-1} \longrightarrow P^{-i}$ to be the composition of the morphisms $P^{-i-1} \longrightarrow \Omega^{-i-1}M \longrightarrow P^{-i}$. This defines a complex $P^\bullet$ which is a graded free resolution of M, with i-th syzygy $\Omega^{-i}M$. Since $\dim_k k \otimes P^{-i} = \dim_k k \otimes \Omega^{-i}M = \dim \underline{\mathrm{Tor}}_i^A(k, M)$, if $i > \mathrm{pdim}_A^{\mathbb{Z}^{r+1}} M$ then $\underline{\mathrm{Tor}}_i^A(k, M) = 0$ and $P^{-i} = 0$, so the length of this resolution is equal to $\mathrm{pdim}_A^{\mathbb{Z}^{r+1}} M$. □

Let $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ be a group morphism. Recall from section 2.2 that we denote by $\varphi_!(A)$ the $\mathbb{Z}$-graded algebra with the same underlying algebra structure as A, and with homogeneous components given by

$$\varphi_!(A)_d = \bigoplus_{\varphi(\xi)=d} A_\xi \qquad \text{for all } d \in \mathbb{Z}.$$

We say that A is φ-*connected* if $\varphi_!(A)$ is a connected $\mathbb{N}$-graded algebra. We denote by $\sigma : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ the group morphism that sends each r-uple $\xi = (\xi^0, \ldots, \xi^r) \in \mathbb{Z}^{r+1}$ to $\xi^0 + \ldots + \xi^r \in \mathbb{Z}$. Notice that any connected $\mathbb{N}^{r+1}$-graded algebra is σ-connected.

The following proposition relates the graded global and injective dimensions of A to ungraded invariants of A.

Proposition 3.1.3. *Let $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ be a group morphism, and suppose A is noetherian and φ-connected.*

1. *A finitely generated graded A-module is locally finite and φ-finite.*
2. *The following equalities hold:*

$$\mathrm{injdim}_A^{\mathbb{Z}^{r+1}} A = \mathrm{injdim}_{\varphi_!(A)}^{\mathbb{Z}} \varphi_!^A(A) = \mathrm{injdim}_A A.$$

3. *The following equalities hold:*

$$\mathrm{pdim}_A^{\mathbb{Z}^{r+1}} k = \mathrm{pdim}_{\varphi_!(A)}^{\mathbb{Z}} \varphi_!^A(k) = \mathrm{pdim}_A k,$$

and these numbers are equal to the graded global dimension of A.

Proof. 1. First we prove the statement on the local finitude of finitely generated modules. Let M be a finitely generated graded module and let $\xi \in \mathbb{Z}^{r+1}$. Since $M_\xi \subset \varphi_!^A(M)_{\varphi(\xi)}$, and $\varphi_!(A)$ is a connected $\mathbb{N}$-graded algebra, it is enough to prove the result for connected $\mathbb{N}$-graded algebras.

Let B be a noetherian connected $\mathbb{N}$-graded algebra, and for every $n \in \mathbb{N}$ let $B_{\geq n}$ be the ideal generated by elements of degree greater than or equal to n. Since B is noetherian, $B_{\geq n}$ is finitely generated for every n, so $B_{\geq n}/B_{\geq n+1} \cong B_n$ is finitely generated over $B_0 = k$. This shows that B is locally finite, and hence so is any finitely generated graded-free module. Since every finitely generated

graded module is the homomorphic image of a finitely generated graded-free module, it must be locally finite.

Now since $\varphi_!(A)$ is noetherian connected $\mathbb{N}$-graded, $\varphi_!^A(M)$ is locally finite, so for every $n \in \mathbb{Z}$ the vector space $\varphi_!^A(M)_n = \bigoplus_{\xi \in \varphi^{-1}(n)} M_\xi$ is finite dimensional, so there are only finitely many ξ in the fiber of n with M_ξ nonzero.

2. By item 1 we may apply Proposition 2.2.9 to deduce that the graded injective dimension of A is equal to the graded injective dimension of $\varphi_!^A(A)$. By [Lev92, Lemma 3.3], this in turn is equal to the injective dimension of A as A-module.

3. Once again we apply Proposition 2.2.9 to prove the equality of the projective dimensions; notice that this is true even if A is not noetherian since k is trivially φ-finite. Clearly the graded global dimension of A is at least $\operatorname{pdim}_A^{\mathbb{Z}^{r+1}} k$, so without loss of generality we may assume that this number is finite. Lemma 3.1.2 implies that the graded projective dimension of any finitely generated graded right A-module is bounded by the projective dimension of k, in particular $\operatorname{pdim}_{A^\circ}^{\mathbb{Z}^{r+1}} k \leq \operatorname{pdim}_{A^\circ}^{\mathbb{Z}^{r+1}} k$ and so k has finite projective dimension as a $\mathbb{Z}^{r+1}$-graded right A-module. By symmetry these two numbers are equal and bound the graded projective dimension of all finitely generated graded left modules. The result then follows from the graded version of [Wei94, Theorem 4.1.2].

□

Remark 3.1.4. Item 3 of Proposition 3.1.3 can be improved. By a classical result, the projective dimension of k is equal to the global dimension of A, no gradings involved. For a proof of this fact the reader is referred to [Ber05].

3.2 Local cohomology and regularity conditions

We keep the conventions from the previous section. In particular, A denotes a connected $\mathbb{N}^{r+1}$-graded algebra and $\mathfrak{m}$ denotes its unique maximal graded ideal. Also we write $\underline{\mathrm{Hom}}_A$ for $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}$ and $\underline{\mathrm{Ext}}_A^i$ for its i-th derived functor. We also assume from now on that A is noetherian.

Let B denote either A or k. In section 2.3 we introduced the torsion functor $\Gamma_{\mathfrak{m}}^{\mathbb{Z}^{r+1}} : \mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ \longrightarrow \mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ$ and its derived functors $\mathcal{R}^i\Gamma_{\mathfrak{m}}^{\mathbb{Z}^{r+1}}$, which are called the *local cohomology functors*. As explained in that section, we are justified in using the same notation for the two different functors thanks to Proposition 1.2.3.

In order to lighten up notation we will write $\Gamma_{\mathfrak{m}}$ for $\Gamma_{\mathfrak{m}}^{\mathbb{Z}^{r+1}}$ and $H_{\mathfrak{m}}^i$ for its i-th derived functor. We write $\Gamma_{\mathfrak{m}^\circ}$ for the torsion functor associated to the connected graded algebra A°.

Recall also that we have defined some invariants related to local cohomology. If M is a graded A-module, its *depth* and *local dimension* are

$$\operatorname{depth} M = \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(M) \neq 0\}$$
$$\operatorname{ldim} M = \sup\{i \in \mathbb{N} \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

The *local dimension of* A *as a graded algebra* is

$$\operatorname{lcd} A = \sup\{\operatorname{ldim} M \mid M \text{ is an object of } \mathsf{Mod}^{\mathbb{Z}^{r+1}} A\}.$$

By item 1 of Proposition 2.3.3 and item 1 of Lemma 2.3.4, these numbers are invariant by change of grading. If M is a graded A-bimodule we sometimes write $\operatorname{depth}_{\mathfrak{m}} M$ or $\operatorname{depth}_{\mathfrak{m}^\circ} M$, and $\operatorname{ldim}_{\mathfrak{m}} M$ or $\operatorname{ldim}_{\mathfrak{m}^\circ} M$ to clarify which structure we are considering.

As mentioned in the introduction to this chapter, M. Artin and W. Schelter adapted the notions of regular and Gorenstein rings to connected $\mathbb{N}$-graded algebras. They are now called AS-regular and AS-Gorenstein in their honour. There is also a notion of AS-Cohen-Macaulay algebras, introduced by M. Van den Bergh in [VdB97, section 8].

Definition 3.2.1. Let B be a noetherian connected $\mathbb{N}$-graded algebra with maximal ideal $\mathfrak{n}$.

1. We say that B is *left, resp right, AS-Cohen-Macaulay* if $\operatorname{depth}_{\mathfrak{n}} B = \operatorname{ldim}_{\mathfrak{n}} B$, resp. $\operatorname{depth}_{\mathfrak{n}^\circ} B = \operatorname{ldim}_{\mathfrak{n}^\circ} B$. We say that B is *AS-Cohen-Macaulay* if

   $$\operatorname{depth}_{\mathfrak{n}} B = \operatorname{ldim}_{\mathfrak{n}} B = \operatorname{ldim}_{\mathfrak{n}^\circ} B = \operatorname{depth}_{\mathfrak{n}^\circ} B.$$

2. We say that B is *left AS-Gorenstein* if $\operatorname{injdim}^{\mathbb{Z}}_B B = d < \infty$ and there exists $\ell \in \mathbb{Z}$, called the *left Gorenstein parameter of* B, such that

   $$\underline{\operatorname{Ext}}^i_B(k, B) \cong \begin{cases} k[\ell] & \text{if } i = d, \\ 0 & \text{otherwise,} \end{cases}$$

   where the isomorphism is of graded vector spaces. We say that B is *right AS-Gorenstein* if an analogous condition holds for B as a right graded module over itself. We say that B is *AS-Gorenstein* if it is both left an right AS-Gorenstein, with the same graded injective dimension and Gorenstein parameters in both cases.

3. We say that B is *left AS-regular* if it is left AS-Gorenstein and it has finite graded global dimension, and analogously for right AS-regular. We say that B is *AS-regular* if it is AS-Gorenstein, has finite graded global dimension.

The following remark clarifies the relation between these properties and their classical counterparts.

Remark 3.2.2. 1. If A is a commutative noetherian connected $\mathbb{N}$-graded algebra of finite Krull dimension, then A is Cohen-Macaulay if and only if the local algebra $A_{\mathfrak{m}}$ is Cohen-Macaulay, see [BH93, Exercise 2.1.27]. By Grothendieck's vanishing theorem [BH93, Theorem 3.5.7] and [BH93, Remark 3.6.18], the local algebra $A_{\mathfrak{m}}$ is Cohen-Macaulay if and only if the local cohomology modules $H^i_{\mathfrak{m}}(A)$ are zero except when i is equal to the Krull dimension of A. This shows that the notions of Cohen-Macaulay and AS-Cohen-Macaulay coincide in the commutative, finite-dimensional case. A similar statement holds for AS-Gorenstein and AS-regular algebras.

2. By definition, if A is AS-regular then it is AS-Gorenstein. Suppose now that A is AS-Gorenstein. Then $\operatorname{depth} A \leq \operatorname{ldim} A \leq \operatorname{injdim}_A^{\mathbb{Z}^{r+1}} A = \operatorname{depth} A$, from which we immediately deduce that A is left AS-Cohen-Macaulay. A similar argument shows that A is right AS-Cohen-Macaulay, and since the injective dimensions of A as a left or right A-module coincide, A is AS-Cohen-Macaulay.

3.2.1 The Artin-Schelter conditions for $\mathbb{N}^{r+1}$-graded algebras

We now focus on studying the relation between the properties from Definition 3.2.1 and $\mathbb{N}^{r+1}$-graded algebras. We will show that they are stable by changing the grading of the algebra, and also by twisting the algebras by Zhang twists.

Recall that we denote by $\sigma : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ the morphism that assigns to each $r+1$-uple the sum of its components. Notice that A is necessarily σ-connected. We will say that A has the *left AS-CM property* if $\operatorname{depth}_{\mathfrak{m}} A = \operatorname{ldim}_{\mathfrak{m}} A$; the right AS-CM property is defined analogously. We will say that A has the *left AS-G property* with parameter $\ell \in \mathbb{Z}^{r+1}$ if it has finite graded injective dimension d over itself and

$$\underline{\operatorname{Ext}}^i(k, A) \cong \begin{cases} k[\ell] & \text{if } i = d; \\ 0 & \text{otherwise;} \end{cases}$$

the right AS-G property is analogous. Finally, we say that A has the *left AS-r* property if it has the left AS-G property and finite graded global dimension, with its corresponding right analogue. The following three propositions show that these properties can be considered as suitable analogues of the AS-Cohen-Macaulay, AS-Gorenstein and AS-regular properties of $\mathbb{N}$-graded algebras.

Proposition 3.2.3. *The following are equivalent:*

1. A *has the left, resp. the right, resp. both the left and the right AS-CM property;*

2. *For all group morphisms* $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}^{t+1}$, *where* $t \geq 0$, *such that* A *is* φ*-connected, the algebra* $\varphi_!(A)$ *has the left, resp. the right, resp. both the left and the right AS-CM property;*

3. *The algebra* $\sigma_!(A)$ *is left, resp. right, resp. both left and right AS-Cohen-Macaulay.*

Proof. As we have stated before, the depth and local dimension of a module are invariant under change of grading for noetherian algebras. The result follows immediately from this. □

Proposition 3.2.4. *The following are equivalent:*

1. A *has the left AS-G property with parameter* ℓ*;*
2. *For all group morphisms* $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}^{t+1}$*, where* $t \geq 0$*, such that* A *is* φ*-connected,* $\varphi_!(A)$ *has the left AS-G property with parameter* $\varphi(\ell)$*;*
3. *The algebra* $\sigma_!(A)$ *is left AS-Gorenstein with parameter* $\sigma(\ell)$*.*

Furthermore, any of the previous conditions is equivalent to its corresponding right condition, with the same injective dimension and Gorenstein parameter.

Proof. The equivalence of the three conditions follows from Propositions 2.2.9 and 2.2.11. Of course an analogous argument shows that the corresponding right properties are equivalent. By a result of J. Zhang, a connected $\mathbb{N}$-graded algebra is left AS-Gorenstein if and only if it is right AS-Gorenstein, see [Zha97, Corollary 1.2]. □

Proposition 3.2.5. *The following are equivalent:*

1. *The algebra* A *has the left AS-r property.*
2. *For all group morphisms* $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}^{t+1}$*, where* $t \geq 0$*, such that* A *is* φ*-connected,* $\varphi_!(A)$ *has the left AS-r property* $\varphi(\ell)$*.*
3. *The algebra* $\sigma_!(A)$ *is left AS-regular.*

Furthermore, any of the previous conditions is equivalent to its corresponding right condition.

Proof. The equivalence of these properties, including the fact that they are equivalent to their right counterparts, follows from item 3 of Proposition 3.1.3 and Proposition 3.2.4. □

In view of the last three propositions, we will say that an $\mathbb{N}^{r+1}$-graded algebra is AS-Cohen-Macaulay if it has both the left and right AS-CM properties; we will say that it is AS-Gorenstein if it has both the left and right AS-G properties with the same graded injective dimension and parameter in both cases; and that it is AS-regular if it has both the left and right AS-r properties, with the same graded global dimension in both cases.

Remark 3.2.6. One can prove even stronger versions of the previous propositions, namely that if A can be endowed with two different connected gradings with maximal ideal $\mathfrak{m}$, then A has the left or right AS-CM property with respect to one of the gradings if and only if it has said property with respect to the other grading. Similar results holds for the AS-G and AS-r properties. These results can be proved by using the forgetful functors from the corresponding categories of graded modules to the category of A-modules, just as we used the change of grading functors in the previous proofs.

Recall from section 2.1.3 that a left twisting system on A is a set of graded k-linear automorphisms $\tau = \{\tau_\xi \mid \xi \in \mathbb{Z}^{r+1}\}$, satisfying

$$\tau_\xi(\tau_{\xi'}(a)a') = \tau_{\xi'+\xi}(a)\tau_\xi(a')$$

for all $\xi, \xi' \in \mathbb{Z}^{r+1}$ and all $a \in A$ and $a' \in A_{\xi'}$. Given a left twist τ on A, the connected $\mathbb{N}^{r+1}$-graded algebra ${}^\tau A$ is the algebra with the same underlying graded vector space as A, and product defined by

$$a \cdot_\tau a' = \tau_{\xi'}(a)a' \qquad \text{for all } a \in A,\ a' \in A_{\xi'}.$$

The category of $\mathbb{Z}^{r+1}$-graded ${}^\tau A$ modules is isomorphic to $\mathsf{Mod}^{\mathbb{Z}^{r+1}} A$ by Theorem 2.1.8. Since A is noetherian, this implies that ${}^\tau A$ is $\mathbb{Z}^{r+1}$-graded noetherian, and by [CQ88, Theorem 2.2] it is noetherian.

A property is said to be *twisting invariant* if it is true for A if and only if it is true for ${}^\tau A$. In [Zha96, Theorem 5.11], J. Zhang proved that being AS-Gorenstein and AS-regular are twisting invariant properties for connected $\mathbb{N}$-graded algebras. This is not surprising since the homological regularity properties of A are defined in terms of the category of graded A-modules. We will now prove a similar result for $\mathbb{N}^{r+1}$-graded connected algebras, but first we need a technical result.

Lemma 3.2.7. *Let τ be a left twisting system on A and let M be an object in $\mathsf{Mod}^{\mathbb{Z}^{r+1}} A$. For every $n \in \mathbb{N}$ there exists a natural isomorphism of $\mathbb{Z}^{r+1}$-graded k-vector spaces*

$$T(M) : \underline{\mathsf{Hom}}_A(A/\mathfrak{m}^n, M) \longrightarrow \underline{\mathsf{Hom}}_{{}^\tau A}({}^\tau A/({}^\tau\mathfrak{m})^n, {}^\tau M).$$

Proof. Given an element $m \in M$ we will denote by ${}^\tau m$ the corresponding element in ${}^\tau M$.

Fix $n \in \mathbb{N}$ and $d \in \mathbb{Z}$. We denote by $[1]$ the class of 1 in the quotient $A/\mathfrak{m}^n$. For every $f \in \underline{\mathsf{Hom}}_A(A/\mathfrak{m}^n, M)_d$ the element $f([1]) \in M_d$ is annihilated by $\mathfrak{m}^n$, and every element of M_d that is annihilated by $\mathfrak{m}^n$ is the image of one such morphism. On the other hand by definition of the ${}^\tau A$-module structure on ${}^\tau M$, an element $m \in M_d$ is annihilated by $\mathfrak{m}^n$ if and only if the corresponding element ${}^\tau m \in {}^\tau M_d$ is annihilated by $({}^\tau\mathfrak{m})^n$. Hence we can define $T(M)(f)$ to be the only ${}^\tau A$-linear function such that $T(M)(f)([{}^\tau 1]) = {}^\tau f([1])$. This is clearly an isomorphism, whose inverse can be found by repeating the above construction but considering A as a twist of ${}^\tau A$. □

We are now ready to prove the result announced above.

Proposition 3.2.8. *Let τ be a left twisting system on A. Then A is AS-Cohen-Macaulay, resp. AS-Gorenstein, resp. AS-regular, if and only if ${}^{\tau}A$ is AS-Cohen-Macaulay, resp. AS-Gorenstein, resp. AS-regular.*

Furthermore, if A is AS-Gorenstein then the injective dimension and the Gorenstein parameters of A and ${}^{\tau}A$ coincide, and if A is AS-regular then the graded global dimensions of A and ${}^{\tau}A$ also coincide.

Proof. Recall from Theorem 2.1.9 that there is a right twist ν on A and a graded algebra isomorphism $\theta : {}^{\tau}A \longrightarrow A^{\nu}$, and that the right ${}^{\tau}A$-module ${}^{\tau}A$ is isomorphic to the induced right ${}^{\tau}A$-module A^{ν}_{θ}.

By Proposition 2.3.5, for every $i \geq 0$ there exists an isomorphism of $\mathbb{Z}^{r+1}$-graded ${}^{\tau}A$-modules $H^i_{\tau_m}({}^{\tau}A) \cong {}^{\tau}H^i_m(A)$, so $\operatorname{depth}_m A = \operatorname{depth}_{\tau_m} {}^{\tau}A$ and $\operatorname{ldim}_m A = \operatorname{ldim}_{{}^{\tau}A} {}^{\tau}A$. Since local cohomology also commutes with right twists, and it obviously commutes with the isomorphism of categories induced by θ, we see that $H^i_{\tau_{m^\circ}}({}^{\tau}A) \cong H^i_{(m^{\nu})^\circ}(A^{\nu}_{\theta}) \cong H^i_m(A)^{\nu}_{\theta}$ for all i. Thus A is AS-Cohen-Macaulay if and only if ${}^{\tau}A$ is.

By Proposition 3.2.4, it is enough to prove that if A is left AS-Gorenstein then ${}^{\tau}A$ is AS-Gorenstein. Since $\mathcal{F}^{\tau}$ is an isomorphism of categories $\operatorname{injdim}^{\mathbb{Z}^{r+1}}_{{}^{\tau}A} {}^{\tau}A = \operatorname{injdim}^{\mathbb{Z}^{r+1}}_{A} A$. Let $I^{\bullet}$ be an injective resolution of A as A-module. Then ${}^{\tau}I$ is an injective resolution of ${}^{\tau}A$ as module over itself, so by Lemma 3.2.7 $\underline{\operatorname{Hom}}_A(k, I^{\bullet})$ and $\underline{\operatorname{Hom}}_{{}^{\tau}A}(k, {}^{\tau}I^{\bullet})$ are isomorphic as complexes of graded vector spaces. This implies that their cohomologies are also isomorphic as graded vector spaces, that is

$$\underline{\operatorname{Ext}}^i_A(k, A) \cong \underline{\operatorname{Ext}}^i_{{}^{\tau}A}(k, {}^{\tau}A)$$

for all $i \geq 0$. Thus A is AS-Gorenstein if and only if ${}^{\tau}A$ is, and in that case their graded injective dimensions and their parameters coincide.

Finally since $\mathcal{F}^{\tau}$ is an isomorphism of categories, A and ${}^{\tau}A$ have the same graded global dimension, so one is AS-regular if and only if the other is AS-regular. □

3.2.2 The Artin-Schelter conditions for GF-algebras

It is a classical result in the commutative setting that if A is a filtered algebra and its associated graded algebra is Cohen-Macaulay, Gorenstein or regular, then so is A, see for example [BH93, Theorem 4.5.7]. In this subsection we prove that, given a technical condition, similar results hold for GF-algebras which were introduced in subsection 2.4.2.

The technical condition mentioned in the previous paragraph is the following.

Definition 3.2.9. Suppose A is noetherian and let M be a graded A-module. We say M has *property* χ if for every $i \geq 0$ the vector space $\underline{\operatorname{Ext}}^i_A(k, M)$ has finite dimension. We

say that A has *property χ as a graded algebra* if every finitely generated graded A-module has property χ.

Remark 3.2.10. Properties χ° and χ were introduced in [AZ94, Definitions 3.2 and 3.7]. By [AZ94, Proposition 3.1(1) and Proposition 3.11(2)], both properties agree with the one defined above when A is a noetherian connected $\mathbb{N}$-graded algebra. In [AZ94, Proposition 3.11 (3)] it is shown that if A is $\mathbb{N}$-graded and commutative then it has property χ. The proof adapts word for word to the $\mathbb{N}^{r+1}$-graded case.

Property χ appears naturally when trying to transfer notions from algebraic geometry to the study of noncommutative connected graded algebras. Informally, a noetherian algebra with property χ is expected to have an homological behavior close to that of commutative noetherian algebras. Evidence for this claim can be found in [AZ94], [VdB97], [Jør97], [JZ00], etc.

From this point on, A denotes a noetherian connected $\mathbb{N}$-graded algebra. Recall that for every $n \in \mathbb{N}$, we denote by $A_{\geq n}$ the ideal generated by all homogeneous elements of degree greater than or equal to n. Now we summarize various results relating property χ and local cohomology for noetherian connected $\mathbb{N}$-graded algebras.

Lemma 3.2.11. *Let M be a graded A-module.*

1. *For every $l \in \mathbb{N}$ there exist $n, n' \in \mathbb{N}$ such that $\mathfrak{m}^n \subset A_{\geq l}$ and $A_{\geq n'} \subset \mathfrak{m}^l$.*
2. *For every $i \geq 0$ there exist natural isomorphisms*
$$H^i_{\mathfrak{m}} \cong \varinjlim_n \operatorname{Ext}^i_A(A/A_{\geq n}, -).$$
3. *The following equality holds:* $\operatorname{depth}_{\mathfrak{m}} M = \inf\{i \in \mathbb{N} \mid \underline{\operatorname{Ext}}^i_A(k, M) \neq 0\}$.
4. *If M has property χ, then for every $d_0 \in \mathbb{Z}$ and every $i \geq 0$ there exists $n_0 \in \mathbb{N}$ such that*
$$H^i_{\mathfrak{m}}(M)_d \cong \underline{\operatorname{Ext}}^i_A(A/A_{\geq n}, M)_d \qquad \text{for all } n \geq n_0,\ d \geq d_0.$$
5. *The A-module M has property χ if and only if for each $i \geq 0$ there exists $d_i \in \mathbb{N}$ such that $H^i_{\mathfrak{m}}(M)_d = 0$ for $d \geq d_i$.*

Proof. 1. Since $\mathfrak{m} = A_{\geq 1}$, it is clear that $\mathfrak{m}^l \subset A_{\geq l}$ for every $l \in \mathbb{N}$. For the other inclusion, let $\mathcal{M}$ be a finite set of homogeneous generators of $\mathfrak{m}$. An element of $A_{\geq n'} \subset \mathfrak{m}$ can be written as a linear combination of monomials in the elements of $\mathcal{M}$, each monomial of degree greater than n'. Let $D = \max\{\deg a \mid a \in \mathcal{M}\}$. If $a_1 a_2 \dots a_t$ is a monomial of degree at least n', then $n' \leq \deg a_1 a_2 \dots a_t \leq Dt$. Taking $n' = Dl$ we see that $t \geq l$, so $a_1 a_2 \dots a_t \in \mathfrak{m}^l$ and $A_{\geq n'} \subset \mathfrak{m}^l$.

2. Item 1. implies that

$$\Gamma_{\mathfrak{m}}(M) = \{m \in M \mid A_{\geq n} m = 0 \text{ for } n \gg 0\}.$$

Using this equality, the proof of Lemma 2.3.2 can be adapted to prove the existence of the desired isomorphisms.

3. Let d be the infimum defined in the statement. For every $n \in \mathbb{N}$ and every $i \geq 0$ consider the long exact sequence

$$\begin{aligned}\operatorname{Ext}_A^{i-1}(A_{\geq n}/A_{\geq n+1}, M) \longrightarrow \operatorname{Ext}_A^i(A/A_{\geq n}, M) \xrightarrow{\pi_n^i} \\ \longrightarrow \operatorname{Ext}_A^i(A/A_{\geq n+1}, M) \longrightarrow \operatorname{Ext}_A^i(A_{\geq n}/A_{\geq n+1}, M). \quad (*)\end{aligned}$$

Suppose $i < d$. Since $\operatorname{Ext}_A^i(k, M) = 0$ and $A_{\geq n}/A_{\geq n+1}$ is isomorphic as A-module to a direct sum of copies of k, the exactness of sequence (*) implies that the map π_n^i is an isomorphism for all n. By induction, we see that $\operatorname{Ext}_A^i(A/A_{\geq n}, M) = 0$ for all n, so

$$H_{\mathfrak{m}}^i(M) \cong \varinjlim_n \operatorname{Ext}_A^i(A/A_{\geq n}, M) = 0.$$

The exactness of (*) also implies that π_n^d is injective for all n, and this in turn implies that the natural morphism $\operatorname{Ext}_A^d(A/A_{\geq 1}, M) \hookrightarrow H_{\mathfrak{m}}^d(M)$ is injective, so $H_{\mathfrak{m}}^d(M) \neq 0$.

4. By [AZ94, Proposition 3.11], properties χ and χ° are equivalent for locally finite algebras. The result then follows from [AZ94, Proposition 3.5 (1)].

5. See [AZ94, Corollary 3.6(3)].

□

From this point on we assume that A is a GF-graded algebra as defined in section 2.4.2; we assume furthermore that $F_0 A = k$, so its associated graded algebra $\operatorname{gr} A$ is a connected $\mathbb{N}^2$-graded algebra, with maximal ideal $\operatorname{gr}\mathfrak{m}$.

Before we go on, we fix some more notation. We denote by $\pi : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ the projection to the second coordinate, and write $B = \pi_!(\operatorname{gr} A)$. Since A is a GF-algebra, for every $d \in \mathbb{N}$ the vector space A_d is filtered by $F_p A_d = F_p A \cap A_d$, and the vector spaces B_d and $\operatorname{gr}(A_d)$ are equal; in particular, B is connected with maximal ideal $\mathfrak{n} = \pi_!(\operatorname{gr}\mathfrak{m})$. Notice also that the inclusion $A_{\geq n} \hookrightarrow A$ is a strict morphism, so item 1 of Lemma 2.4.6 implies that $B/B_{\geq n} \cong \pi_!^{\operatorname{gr} A}(\operatorname{gr}(A/A_{\geq n}))$ as graded B-modules.

We recall that by [MR01, Theorem 1.6.9], if $\operatorname{gr} A$, or equivalently B, is noetherian then so is A and by item 1 of Proposition 3.1.3 A, $\operatorname{gr} A$ and B are locally finite.

Lemma 3.2.12. *Suppose* $\operatorname{gr} A$ *is noetherian. Let* M *be a finitely generated graded* A*-module. Set a GF-module structure on* M *as in Lemma 2.4.5. Then following hold:*

1. $\operatorname{depth}_{\operatorname{gr}\mathfrak{m}} \operatorname{gr} M \leq \operatorname{depth}_{\mathfrak{m}} M$.

2. $\operatorname{pdim}_A^{\mathbb{Z}} M \leq \operatorname{pdim}_{\operatorname{gr} A}^{\mathbb{Z}^2} \operatorname{gr} M$ *and* $\operatorname{injdim}_A^{\mathbb{Z}} M \leq \operatorname{injdim}_{\operatorname{gr} A}^{\mathbb{Z}^2} \operatorname{gr} M$.

3. *If* $\operatorname{gr} M$ *has property* χ*, so does* M.

4. *If* $\operatorname{gr} M$ *has property* χ*, then* $\operatorname{ldim} M \leq \operatorname{ldim} \operatorname{gr} M$.

Proof. Recall from Theorem 2.4.8 that under the hypotheses of the lemma, for any finite GF-module N and any $d \in \mathbb{Z}$ there exists a spectral sequence

$$E(N, M)_d : E^1_{p,q} = \underline{\operatorname{Ext}}^{-p-q}_{\operatorname{gr} A}(\operatorname{gr} N, \operatorname{gr} M)_{(p,d)} \Rightarrow \underline{\operatorname{Ext}}^{-p-q}_A(N, M)_d \qquad p, q \in \mathbb{Z}.$$

1. Let $i < \operatorname{depth}_{\operatorname{gr}\mathfrak{m}} \operatorname{gr} M$. By item 3 of Lemma 3.2.11, $\underline{\operatorname{Ext}}^i_{\operatorname{gr} A}(\operatorname{gr} k, \operatorname{gr} M) = 0$ since $\operatorname{gr} k = k$. Corollary 2.4.9 then implies $\underline{\operatorname{Ext}}^i_A(k, M) = 0$, so $i < \operatorname{depth} M$.

2. The module $P(M) = A \otimes M/\mathfrak{m}M$ is a graded A-module, and as in Lemma 3.1.2 there is a map $P(M) \longrightarrow M$, call it ϵ. We may give $P(M)$ the structure of a GF-module by setting $F_p P(M) = \epsilon^{-1}(F_p M)$ for every $p \in \mathbb{Z}$. Given a basis $\overline{x}_1, \ldots, \overline{x}_t$ of $M/\mathfrak{m}M$, it is immediate that

$$F_p P(M) = \sum_{i=1}^{t} F_{p-p_i} A(1 \otimes \overline{x}_i),$$

   where p_i is such that $x \in F_{p_i} M \setminus F_{p_i - 1} M$, so $P(M)$ is a GF-free module.

   By definition the morphism $\epsilon : P(M) \longrightarrow M$ is strict, so the resolution $P^\bullet$ from Lemma 3.1.2 is a resolution of M by finitely generated GF-free modules with strict differentials, and its length is equal to the projective dimension of M. According to items 1 and 2 of Lemma 2.4.6 $\operatorname{gr} P^\bullet$ is a projective resolution of $\operatorname{gr} M$, from which it follows that $\operatorname{pdim}_{\operatorname{gr} A}^{\mathbb{Z}^2} \operatorname{gr} M \leq \operatorname{pdim}_A^{\mathbb{Z}} M$.

   Now let N be a finitely generated graded A-module, and give it a GF-module structure as in Lemma 2.4.5. If $i > \operatorname{injdim}_{\operatorname{gr} A}^{\mathbb{Z}^2} \operatorname{gr} M$, then $\underline{\operatorname{Ext}}^i_{\operatorname{gr} A}(\operatorname{gr} N, \operatorname{gr} M) = 0$ and by Corollary 2.4.9 $\underline{\operatorname{Ext}}^i_A(N, M) = 0$. Using the graded version of [Wei94, Theorem 4.1.2] we obtain $\operatorname{injdim}_A^{\mathbb{Z}} M \leq \operatorname{injdim}_{\operatorname{gr} A}^{\mathbb{Z}^2} \operatorname{gr} M$.

3. Suppose $\operatorname{gr} M$ has property χ, that is $\underline{\operatorname{Ext}}^i_{\operatorname{gr} A}(k, \operatorname{gr} M)$ is finite dimensional for all $i \geq 0$, and fix $i \in \mathbb{N}$. By hypothesis, for all $d \in \mathbb{Z}$ the i-th diagonal of the first page of $E(k, M)_d$ has at most a finite number of nonzero entries, each of finite dimension. Furthermore, there are no nonzero entries except for finitely many d's. The same is true of the infinity page, so the associated graded vector space of $\underline{\operatorname{Ext}}^i_A(k, M)_d$ is finite dimensional, and equal to zero for all but finitely many d's and by item 3 of Lemma 2.4.6, the same is true for $\underline{\operatorname{Ext}}^i_A(k, M)_d$. It follows that $\underline{\operatorname{Ext}}^i_A(k, M)$ is a finite dimensional k-vector space.

4. Let $i > \operatorname{ldim} \operatorname{gr} M$ and let $d \in \mathbb{Z}$. We will prove that $H^i_{\mathfrak{m}}(M)_d = 0$. By item 4 of Lemma 3.2.11, there exists $n_0 \in \mathbb{N}$ such that

$$H^i_{\mathfrak{m}}(M)_d \cong \underline{\operatorname{Ext}}^i_A(A/A_{\geq n}, M)_d \qquad \text{for all } n \geq n_0,$$

so it is enough to prove that $\underline{\operatorname{Ext}}^i_A(A/A_{\geq n}, M)_d = 0$ for n large enough.

Let $\tilde{M} = \pi^{\operatorname{gr} A}_!(\operatorname{gr} M)$. By item 1 of Proposition 2.3.3, $H^i_n(\tilde{M}) = 0$. Once again by item 4 of Lemma 3.2.11, there exists $n_1 \in \mathbb{N}$ such that

$$0 = H^i_n(\tilde{M})_d \cong \underline{\operatorname{Ext}}^i_B(B/B_{\geq n}, \tilde{M})_d \qquad \text{for all } n \geq n_1.$$

Applying Proposition 2.2.11 we obtain

$$\bigoplus_{p \in \mathbb{Z}} \underline{\operatorname{Ext}}^i_{\operatorname{gr} A}(\operatorname{gr}(A/A_{\geq n}), \operatorname{gr} M)_{(p,d)} \cong \underline{\operatorname{Ext}}^i_B(B/B_{\geq n}, \tilde{M})_d = 0$$

for all $n \geq n_1$. Taking n greater than both n_0 and n_1, the result follows from Corollary 2.4.9.

□

We now prove that the regularity properties of $\operatorname{gr} A$ transfer to A. Notice that A° is naturally a GF-algebra, and that $\operatorname{gr} A^\circ = (\operatorname{gr} A)^\circ$.

Theorem 3.2.13. *Suppose that* $\operatorname{gr} A$ *is noetherian and that* $\operatorname{gr} A$ *and* $\operatorname{gr} A^\circ$ *have property* χ *as graded algebras. Then* A *and* A° *have property* χ *as graded algebras. Furthermore if* $\operatorname{gr} A$ *is AS Cohen-Macaulay, resp. AS-Gorenstein, resp. AS-regular, then* A *is AS-Cohen-Macaulay, resp. AS-Gorenstein, resp. AS-regular.*

Proof. Let M be a finitely generated graded A-module and endow it with the structure of a GF-module as in Lemma 2.4.5. Since $\operatorname{gr} A$ has property χ as a graded algebra, $\operatorname{gr} M$ has property χ, and by item 3 of Lemma 3.2.12 so does M. This proves that A has property χ as a graded algebra. An analogous argument works for A°.

Suppose $\operatorname{gr} A$ is AS-Cohen-Macaulay. Since $\operatorname{gr} A$ has property χ over $\operatorname{gr} A$ and $\operatorname{gr} A^\circ$, we can apply items 1. and 4. of Lemma 3.2.12 to get the series of inequalities

$$\operatorname{depth}_{\operatorname{gr}\mathfrak{m}} \operatorname{gr} A \leq \operatorname{depth}_{\mathfrak{m}} A \leq \operatorname{ldim}_{\mathfrak{m}} A \leq \operatorname{ldim}_{\operatorname{gr}\mathfrak{m}} \operatorname{gr} A,$$
$$\operatorname{depth}_{\operatorname{gr}\mathfrak{m}^\circ} \operatorname{gr} A \leq \operatorname{depth}_{\mathfrak{m}^\circ} A \leq \operatorname{ldim}_{\mathfrak{m}^\circ} A \leq \operatorname{ldim}_{\operatorname{gr}\mathfrak{m}^\circ} \operatorname{gr} A,$$

The hypothesis implies that all these numbers are equal, so A is AS-Cohen-Macaulay.

Suppose now that $\operatorname{gr} A$ is AS-Gorenstein with Gorenstein parameter $\ell = (\ell^1, \ell^2) \in \mathbb{Z}^2$ and injective dimension d. By Proposition 3.2.4 it is enough to prove that A is left AS-Gorenstein. By item 2 of Lemma 3.2.12 we see that $\operatorname{injdim}^{\mathbb{Z}}_A A \leq d$. By the condition on the modules $\underline{\operatorname{Ext}}^i_{\operatorname{gr} A}(k, \operatorname{gr} A)$, the spectral sequence $E(k, A)_t$ collapses at

page 1 for all $t \in \mathbb{Z}$, since $E^1_{p,q} = 0$ for all $p, q \in \mathbb{Z}$ save for $p = \ell^1$ and $-p - q = d$. We thus obtain graded vector space isomorphisms

$$\underline{\mathrm{Ext}}^i_A(k, A) \cong \begin{cases} k[\ell^2] & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

These implies $\mathrm{injdim}^{\mathbb{Z}}_A A = d$, so A is AS-Gorenstein with the same injective dimension and Gorenstein parameter $\pi(\ell)$. Notice that B is also AS-Gorenstein with the same injective dimension and Gorenstein parameter.

Finally, suppose $\mathsf{gr}\, A$ is AS-regular. By the previous argument A is AS-Gorenstein, and by item 2 of Lemma 3.2.12 $\mathrm{pdim}^{\mathbb{Z}}_A k \leq \mathrm{pdim}^{\mathbb{Z}^2}_B k < \infty$, so k has finite graded projective dimension over A. The result then follows from Proposition 3.2.5. □

Chapter 4

Dualizing complexes

Dualizing complexes for connected graded algebras were introduced by A. Yekutieli to answer a question posed by M. Artin on the local cohomology modules of AS-regular connected $\mathbb{N}$-graded algebras. They have proven to be a very useful tool for the study of connected graded algebras and their homological properties. In this chapter we review the definitions and basic results, and show that they hold in the $\mathbb{N}^{r+1}$-graded setting. We then use these results to prove that having a (balanced) dualizing complex is a twisting invariant property, and that it transfers from associated graded algebras to GF-algebras.

Dualizing complexes are objects of the derived category $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$, so in section 4.1 we give a brief review of general results of derived categories, and then extend some results on local cohomology and the change of grading functors to the derived setting. Then in section 4.2 we prove the main results of this chapter, mostly $\mathbb{N}^{r+1}$-graded analogues of known results.

The enveloping algebra $A^e = A \otimes A^\circ$ is an $\mathbb{N}^{r+1}$-graded algebra with the natural grading for the tensor product of graded vector spaces, as defined in subsection 2.1.2. Notice that the functor $\varphi_!$ is compatible with tensor products, and that $\varphi_!(A^e) = \varphi_!(A)^e$.

4.1 Derived categories

All undefined objects and notations in this section are taken from [Har66, chapter I], which we use as our main reference on derived categories.

4.1.1 Generalities

Throughout this subsection, $\mathcal{A}$ and $\mathcal{B}$ denote two abelian categories with enough projective and injective objects, and all functors between them will be additive functors. We denote the homotopy category of $\mathcal{A}$ by $\mathcal{K}(\mathcal{A})$, and its derived category by $\mathcal{D}(\mathcal{A})$. We identify objects of $\mathcal{A}$ with complexes in $\mathcal{D}(\mathcal{A})$ concentrated in degree 0. For every object $M^\bullet$ of $\mathcal{D}(\mathcal{A})$ and any $n \in \mathbb{Z}$, we denote by $H^n(M^\bullet)$ the n-th cohomology module of $M^\bullet$. We also denote by $M^\bullet(n)$ the complex $M^{\bullet+n}$, the *n-th translate* of M. The complex $M^\bullet$ is said to be left bounded, resp. right bounded, resp. bounded, if there exists $n_0 \in \mathbb{Z}$ such that $M^n = 0$ for $n \leq n_0$, resp. $n \geq n_0$, resp. $n \geq |n_0|$. We denote the full subcategories of left bounded, right bounded and bounded objects of $\mathcal{D}(\mathcal{A})$ by $\mathcal{D}^+(\mathcal{A}), \mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$, respectively.

Given an object $M^\bullet$ of $\mathcal{D}(\mathcal{A})$, a *projective resolution* of $M^\bullet$ is a complex $P^\bullet$ of projective objects of $\mathcal{A}$ and a quasi-isomorphism $p : P^\bullet \longrightarrow M^\bullet$. An *injective resolution* of $M^\bullet$ is a complex $I^\bullet$ of injective objects of $\mathcal{A}$ along with a quasi-isomorphism $i : M^\bullet \longrightarrow I^\bullet$.

Since $\mathcal{A}$ has enough injectives, every object $M^\bullet$ of $\mathcal{D}^+(\mathcal{A})$ has an injective resolution by [Har66, Lemma 4.6]. The *injective dimension* of a complex is defined as the minimal length of injective resolutions of $M^\bullet$. By [Har66, Proposition 7.6] and its proof, the injective dimension of $M^\bullet$ is equal to the minimum $i \in \mathbb{N}$ such that $H^i(\mathrm{Hom}_{\mathcal{A}}(A, M^\bullet)) = 0$ for all objects A of $\mathcal{A}$.

Recall that an additive functor $\Delta : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$ is called a *∂-functor* if it commutes with the translation functor of the derived category and takes distinguished triangles to distinguished triangles. For example, if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is exact then F induces a ∂-functor in the corresponding derived categories; we will also denote by F the induced functor $F : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$.

Every left exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ has a right derived functor $\mathcal{R}F : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{B})$ that is calculated as follows: for every object $M^\bullet$ of $\mathcal{D}^+(\mathcal{A})$ choose an injective resolution $M^\bullet \longrightarrow I^\bullet$; then $\mathcal{R}F(M^\bullet) = F(I^\bullet)$. We write $\mathcal{R}^iF(M^\bullet) = H^i(\mathcal{R}F(M^\bullet))$. As usual, injective objects can be replaced with F-acyclic objects, see [Har66, Theorem 5.1 and Corollary 5.3]. This implies the following lemma, which is a derived version of Lemma 1.2.2 and can be proved using the same argument.

Lemma 4.1.1. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be two covariant left exact functors, where $\mathcal{C}$ is an abelian category. If F is exact and sends injective objects to G-acyclic objects, then $\mathcal{R}(G \circ F) \cong \mathcal{R}G \circ F$.*

Given a subcategory $\mathcal{C}$ of $\mathcal{A}$, we say that $\mathcal{C}$ is *closed under extensions* if whenever there exist objects C', C'' of $\mathcal{C}$ and an exact sequence of objects of $\mathcal{A}$

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0,$$

then C is also an object of $\mathcal{C}$. Notice that in [Har66] these categories are called *thick* subcategories. If $\mathcal{C}$ is closed under extensions, the full subcategory $\mathcal{D}_{\mathcal{C}}(\mathcal{A})$ formed by

the objects of $\mathcal{D}(\mathcal{A})$ whose cohomology modules lie in $\mathcal{C}$ is a full triangulated subcategory of $\mathcal{D}(\mathcal{A})$. We write $\mathcal{D}^+_{\mathcal{C}}(\mathcal{A})$, resp. $\mathcal{D}^-_{\mathcal{C}}(\mathcal{A})$, resp. $\mathcal{D}^b_{\mathcal{C}}(\mathcal{A})$, for the full subcategories of $\mathcal{D}_{\mathcal{C}}(\mathcal{A})$ formed by left bounded, resp. right bounded, resp. bounded, objects of $\mathcal{D}_{\mathcal{C}}(\mathcal{A})$.

Proposition 4.1.2. *Let $D : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$ be a ∂-functor and let $\mathcal{A}'$ be a subcategory of $\mathcal{A}$, closed under extensions.*

1. *Suppose there exist a ∂-functor $E : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$ and a natural transformation $\eta : D \longrightarrow E$ such that for every object A of $\mathcal{A}'$ the map $\eta(A)$ is an isomorphism. Then $\eta(X^\bullet)$ is an isomorphism for every object $X^\bullet$ of $\mathcal{D}^b_{\mathcal{A}'}(\mathcal{A})$.*
2. *Let $\mathcal{B}'$ be a subcategory of $\mathcal{B}$, and suppose that it is closed under extensions. If D sends objects of $\mathcal{A}'$ to $\mathcal{D}_{\mathcal{B}'}(\mathcal{B})$, then for every object $X^\bullet$ of $\mathcal{D}^b_{\mathcal{A}'}(\mathcal{A})$ the complex $F(X^\bullet)$ is in $\mathcal{D}_{\mathcal{B}'}(\mathcal{B})$.*

Proof. See [Har66, Chapter 1, Propositions 7.1 (i) and 7.3 (i)]. □

4.1.2 The category $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$

We now return to the study of connected $\mathbb{N}^{r+1}$-graded algebras. For the rest of this section A denotes a connected $\mathbb{N}^{r+1}$-graded algebra, and we focus on the derived category of $\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e$.

We start by reviewing the definitions of the $\underline{\mathsf{Hom}}_A$ functors defined at the level of complexes, following closely [Yek92, section 2]. Notice that the reference works with $\mathbb{Z}$-graded modules instead of $\mathbb{Z}^{r+1}$-graded ones, but the proofs can be adapted to this new context almost word by word.

Let B and C be two connected $\mathbb{N}$-graded algebras. Recall from subsection 2.1.2 that given a graded $A \otimes B^\circ$-module N and a graded $A \otimes C^\circ$-module M, the enriched homomorphism space $\underline{\mathsf{Hom}}^{\mathbb{Z}^{r+1}}_A(N, M)$ is a graded $B \otimes C^\circ$-module. Given $N^\bullet$ and $M^\bullet$ objects of $\mathcal{K}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)$ and $\mathcal{K}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ)$ respectively, the complex $\underline{\mathsf{Hom}}^{\mathbb{Z}^{r+1}}_A(N^\bullet, M^\bullet)$ is the object of $\mathcal{K}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} B \otimes C^\circ)$ whose n-th component is

$$\underline{\mathsf{Hom}}^{\mathbb{Z}^{r+1}}_A(N^\bullet, M^\bullet)^n = \prod_{p \in \mathbb{Z}} \underline{\mathsf{Hom}}^{\mathbb{Z}^{r+1}}_A(N^p, M^{p+n}),$$

where the product is taken in the category of graded modules (see item 1 of Lemma 2.1.2), and whose differential is given by

$$d^n = \prod_{p \in \mathbb{Z}} ((-1)^{n+1} \underline{\mathsf{Hom}}^{\mathbb{Z}^{r+1}}_A(d^p_N, M^{p+n}) + \underline{\mathsf{Hom}}^{\mathbb{Z}^{r+1}}_A(N^p, d^{p+n}_M)).$$

Thus we obtain a bifunctor

$$\underline{\mathsf{Hom}}^{\mathbb{Z}^{r+1}}_A : \mathcal{K}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)^\circ \times \mathcal{K}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ) \longrightarrow \mathcal{K}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} B \otimes C^\circ).$$

Remark 4.1.3. Notice that our sign conventions differ from those of [Yek92]. A. Yekutieli follows [Har66] in his sign conventions, but as shown in [Conoo] this leads to an inconsistency in the definition of the natural transformation τ (see section 4.2). This last reference corrects the error, so we follow the conventions found there.

By a reasoning similar to that of [Wei94, 2.7.5], for every $\xi \in \mathbb{Z}^{r+1}$ and every $n \in \mathbb{N}$ the cohomology group $H^n(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet))$ is the space of homotopy classes of homogeneous A-linear morphisms of complexes from $N^\bullet$ to $M^\bullet[\xi](n)$.

Theorem 4.1.4. *1. The functor $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}$ has a right derived functor*

$$\mathcal{R}\,\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}} : \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)^\circ \times \mathcal{D}^+(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ) \longrightarrow \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} B \otimes C^\circ).$$

When $M^\bullet$ is an object of $\mathcal{D}^+(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ)$ such that M^i is injective as left A-module for each $i \in \mathbb{Z}$, then $\mathcal{R}\,\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet) \cong \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet)$ for every object $N^\bullet$ of $\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)^\circ$.

2. The functor $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}$ also has a right derived functor

$$\mathcal{R}\,\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}} : \mathcal{D}^-(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)^\circ \times \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ) \longrightarrow \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} B \otimes C^\circ).$$

When $N^\bullet$ is an object of $\mathcal{D}^-(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)^\circ$ such that N^i is projective as left A-module for each $i \in \mathbb{Z}$, then $\mathcal{R}\,\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet) \cong \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet)$ for every object $M^\bullet$ of $\mathcal{D}^+(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ)$.

3. These derived functors coincide over $\mathcal{D}^-(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)^\circ \times \mathcal{D}^+(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ)$.

Proof. See [Yek92, Theorem 2.2]. □

Of course one may define in the same way a bifunctor

$$\underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}} : \mathcal{K}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} B \otimes A^\circ)^\circ \times \mathcal{K}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} C \otimes A^\circ) \longrightarrow \mathcal{K}(\mathrm{Mod}^{\mathbb{Z}^{r+1}} C \otimes B^\circ),$$

with the same properties as $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}$. We will only be interested in the cases where B, C are either A or k.

As in subsection 2.1.2 we denote by Λ and P the functors that assign to each $\mathbb{Z}^{r+1}$-graded A^e-module its underlying left and right $\mathbb{Z}^{r+1}$-graded A-module, respectively. Recall from Lemma 2.1.5 that Λ and P are exact functors and send injective objects to injective objects.

Lemma 4.1.5. *Let $N^\bullet$ be an object of $\mathcal{D}^+(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A^e)$. The objects $\Lambda(N^\bullet)$ and $P(N^\bullet)$ have finite injective dimension if and only if there is an object $I^\bullet$ of $\mathcal{D}^b(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A^e)$ isomorphic to $N^\bullet$ such that $\Lambda(I^p)$ and $P(I^p)$ are injective for all $p \in \mathbb{Z}$.*

Proof. The proof is analogous to [Yek92, Proposition 2.4]. □

In section 2.3 we introduced the torsion functor $\Gamma_{\mathfrak{m}} : \mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e \longrightarrow \mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e$ that extends the torsion functor for A-modules, in the sense that $\Gamma_{\mathfrak{m}}$ commutes with Λ. Of course, $\Gamma_{\mathfrak{m}^\circ}$ commutes with P.

Proposition 4.1.6. *The following diagrams of functors commute*

$$\begin{array}{ccc} \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e) & \xrightarrow{\mathcal{R}\Gamma_{\mathfrak{m}}} & \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e) \\ {\scriptstyle\Lambda}\downarrow & & \downarrow{\scriptstyle\Lambda} \\ \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A) & \xrightarrow{\mathcal{R}\Gamma_{\mathfrak{m}}} & \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A), \end{array} \qquad \begin{array}{ccc} \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e) & \xrightarrow{\mathcal{R}\Gamma_{\mathfrak{m}^\circ}} & \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^\circ) & \xrightarrow{\mathcal{R}\Gamma_{\mathfrak{m}^\circ}} & \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^\circ). \end{array}$$

Proof. We have already seen in section 2.3 that $\Lambda \circ \Gamma_{\mathfrak{m}} = \Gamma_{\mathfrak{m}} \circ \Lambda$, from which we can deduce that

$$\Lambda \circ \mathcal{R}\Gamma_{\mathfrak{m}} \cong \mathcal{R}(\Lambda \circ \Gamma_{\mathfrak{m}}) \cong \mathcal{R}(\Gamma_{\mathfrak{m}} \circ \Lambda) \cong \mathcal{R}\Gamma_{\mathfrak{m}} \circ \Lambda,$$

where the last isomorphism follows from Lemma 4.1.1 and the fact that Λ sends injective objects to injective objects. The proof is the same for P and $\Gamma_{\mathfrak{m}^\circ}$. □

For the sake of legibility we will omit the functors Λ and P when the context makes it clear in which category we are working.

For every graded A-module M the graded vector space $M^* = \underline{\mathsf{Hom}}_k^{\mathbb{Z}^{r+1}}(M, k)$ has a natural $\mathbb{Z}^{r+1}$-graded *right* A-module structure. This space is called the *Matlis dual* of M. More generally, the functor $\underline{\mathsf{Hom}}_k^{\mathbb{Z}^{r+1}}(-, k)$ induces a functor

$$\underline{\mathsf{Hom}}_k^{\mathbb{Z}^{r+1}}(-, k) : \mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)^\circ \longrightarrow \mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} B \otimes A^\circ).$$

Suppose M is a graded A-module. The usual adjunction between $\underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}$ and graded tensor products gives

$$\underline{\mathsf{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(-, M^*) \cong \underline{\mathsf{Hom}}_k^{\mathbb{Z}^{r+1}}(- \otimes_A M, k).$$

Hence the functor $\underline{\mathsf{Hom}}_{A^\circ}(-, M^*)$ is exact, that is M^* is injective, if and only if M is a flat A-module. If A is noetherian and M is finitely generated then Lemma 3.1.2 implies that M^* is injective if and only if M is free. Notice that the Matlis dual of the algebra A^* is injective both as left and right graded module.

4.1.3 Change of grading between derived categories

Before we begin our discussion of dualizing complexes we prove derived versions of Propositions 2.3.3 and 2.2.12. In this subsection $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ denotes a group morphism such that A is φ-connected. By a slight abuse of notation we denote by Λ the functor that assigns to each graded $A \otimes B^\circ$-module its underlying graded A-module.

It is clear from the definitions that Λ and P commute with the change of grading functor $\varphi_!^{A\otimes B^\circ}$. We will use this fact throughout this section without further mention. Notice that since $\varphi_!^{A\otimes B^\circ}$ is an exact functor, it induces a ∂-functor between the corresponding derived categories which we also denote by $\varphi_!^{A\otimes B^\circ}$.

Proposition 4.1.7. *Suppose A is noetherian*

1. *The following diagram of functors commutes*

$$\begin{array}{ccc} \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A\otimes B^\circ) & \xrightarrow{\mathcal{R}\Gamma_{\mathfrak{m}}} & \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A\otimes B^\circ) \\ \downarrow{\scriptstyle \varphi_!^{A\otimes B^\circ}} & & \downarrow{\scriptstyle \varphi_!^{A\otimes B^\circ}} \\ \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}} \varphi_!(A\otimes B^\circ)) & \xrightarrow{\mathcal{R}\Gamma_{\mathfrak{m}}} & \mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}} \varphi_!(A\otimes B^\circ)). \end{array}$$

2. *The following diagram of functors commutes*

$$\begin{array}{ccc} \mathcal{D}^b_{lf}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A\otimes B^\circ) & \xrightarrow{-^*} & \mathcal{D}^b_{lf}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A\otimes B^\circ) \\ \downarrow{\scriptstyle \varphi_!^{A\otimes B^\circ}} & & \downarrow{\scriptstyle \varphi_!^{A\otimes B^\circ}} \\ \mathcal{D}^b_{lf}(\mathsf{Mod}^{\mathbb{Z}} \varphi_!(A\otimes B^\circ)) & \xrightarrow{-^*} & \mathcal{D}^b_{lf}(\mathsf{Mod}^{\mathbb{Z}} \varphi_!(A\otimes B^\circ)), \end{array}$$

   where $-^*$ is Matlis duality and lf *denotes the subcategory of locally finite $A\otimes B^\circ$-modules.*

Proof. 1. This is proved just like item 1 of Proposition 2.3.3, replacing Lemma 1.2.2 by its derived version Lemma 4.1.1.

2. By definition, for every $n\in\mathbb{Z}$

$$\varphi_!^{A\otimes B^\circ}(M^*)_n = \bigoplus_{\varphi(\xi)=n} \mathsf{Hom}_k(M_{-\xi},k) \subset \mathsf{Hom}_k\left(\bigoplus_{\varphi(\xi)=n} M_{-\xi},k\right) = \varphi_!^{A\otimes B^\circ}(M)^*_n.$$

   Denote by $\eta(M) : \varphi_!^{A\otimes B^\circ}(M^*) \longrightarrow \varphi_!^{A\otimes B^\circ}(M)^*$ the natural transformation induced by this inclusion. Notice that $\eta(M)$ is an isomorphism if and only if M is φ-finite, in particular if M is locally finite.

   Since $\varphi_!^{A\otimes B^\circ}$ and Matlis duality are exact functors, they induce ∂-functors in the corresponding derived categories, which are the ones that appear in the diagram. For every complex $X^\bullet$ in $\mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A\otimes B^\circ)$ there is a morphism $\eta(X^\bullet) : \varphi_!^{A\otimes B^\circ}((X^\bullet)^*) \longrightarrow \varphi_!^{A\otimes B^\circ}(X^\bullet)^*$, and by Proposition 4.1.2 it is an isomorphism whenever $X^\bullet$ is a bounded complex with locally finite cohomology modules.

□

Proposition 4.1.8. *Let $R^\bullet$ and $S^\bullet$ be objects of $\mathcal{D}^b(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)$ and $\mathcal{D}^b(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes C^\circ)$ respectively. There exists a morphism of complexes,*

$$F(R^\bullet, S^\bullet) : \varphi_!^{B\otimes C^\circ}(\mathcal{R}\,\underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}(R^\bullet, S^\bullet)) \longrightarrow \mathcal{R}\,\underline{\mathsf{Hom}}^{\mathbb{Z}}_{\varphi_!(A)}(\varphi_!^{A\otimes B^\circ}(R^\bullet), \varphi_!^{A\otimes C^\circ}(S^\bullet)),$$

which is natural in $R^\bullet$ and $S^\bullet$. If A is noetherian and $\Lambda(R^\bullet)$ has finitely generated cohomology modules, then $F(R^\bullet, S^\bullet)$ is an isomorphism.

Proof. Let $P^\bullet$ be a projective resolution of $R^\bullet$. As we have seen several times before, $\varphi_!^{A\otimes B^\circ}(P^\bullet)$ is a projective resolution of $\varphi_!^{A\otimes B^\circ}(R^\bullet)$, so by Theorem 4.1.4

$$\mathcal{R}\,\underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}(R^\bullet, S^\bullet)) \cong \underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}(P^\bullet, S^\bullet)),$$
$$\mathcal{R}\,\underline{\mathsf{Hom}}^{\mathbb{Z}}_{\varphi_!(A)}(\varphi_!^{A\otimes B^\circ}(R^\bullet), \varphi_!^{A\otimes C^\circ}(S^\bullet))) \cong \underline{\mathsf{Hom}}^{\mathbb{Z}}_{\varphi_!(A)}(\varphi_!^{A\otimes B^\circ}(P^\bullet), \varphi_!^{A\otimes C^\circ}(S^\bullet)))$$

We define $F(R^\bullet, S^\bullet)$ as the morphism induced by the inclusion

$$\varphi_!^{B\otimes C^\circ}(\underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}(P^i, S^j)) \subset \underline{\mathsf{Hom}}^{\mathbb{Z}}_{\varphi_!(A)}(\varphi_!^{A\otimes B^\circ}(P^i), \varphi_!^{A\otimes C^\circ}(S^j))) \qquad i, j \in \mathbb{Z}.$$

We write F instead of $F(R^\bullet, S^\bullet)$ to alleviate notation. The naturality of this morphism and the fact that it is independent from the chosen projective resolution can be proved as in Proposition 2.2.11.

Suppose now that A is noetherian. Notice that if $R^\bullet$ and $S^\bullet$ are concentrated in homological degree 0 then $F(R^\bullet, S^\bullet) = E(R^0, S^0)$, where E is the natural transformation defined in Corollary 2.2.12. If furthermore R^0 is finitely generated then this map is an isomorphism by the same corollary. Applying Proposition 4.1.2 to the natural transformation $F(R^0, -)$, we see that $F(R^0, S^\bullet)$ is also an isomorphism whenever $S^\bullet$ is a bounded complex. Now consider the natural transformation $F(-, S^\bullet)$; we have just proved that this is an isomorphism whenever the first variable is evaluated in a complex $R^\bullet$ concentrated in homological degree zero such that R^0 is finitely generated as left A-module. Since the class of $A \otimes B^\circ$-modules which are finitely generated as left A-modules is closed by extensions, we may again apply Proposition 4.1.2 and deduce that $F(R^\bullet, S^\bullet)$ is an isomorphism whenever $R^\bullet$ is a bounded complex whose cohomology modules are finitely generated as left A-modules. □

4.2 Dualizing complexes

In this section we study dualizing complexes over $\mathbb{N}^{r+1}$-graded algebras. Since dualizing complexes were originally defined for $\mathbb{N}$-graded algebras, see [Yek92] most results in the literature are proved in this context. Whenever the proofs can be adapted to the $\mathbb{N}^{r+1}$-graded case without effort we simply give a reference. However, in order to prove a general existence result as that given by M. Van den Bergh in [VdB97, Proposition 6.3] one would have to re-write most of said paper in the $\mathbb{N}^{r+1}$-graded context;

in this case we offer an alternative, namely using the change of grading functors to deduce the desired result from the $\mathbb{N}$-graded case.

Throughout this $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ is a group morphism such that A is φ-connected.

4.2.1 Dualizing complexes for $\mathbb{N}^{r+1}$-graded algebras

Suppose we are given a bounded below complex $R^\bullet$ of A^e-bimodules such that $\Lambda(R^\bullet)$ and $P(R^\bullet)$ have finite injective dimension. We can associate to $R^\bullet$ the functors

$$D = \mathcal{R}\,\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(-, R^\bullet) : \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}}\, A \otimes B^\circ) \longrightarrow \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}}\, B \otimes A^\circ),$$
$$D^\circ = \mathcal{R}\,\underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(-, R^\bullet) : \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}}\, B \otimes A^\circ) \longrightarrow \mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}}\, A \otimes B^\circ).$$

There exists a natural transformation from the identity to the composition $D^\circ D$, which we denote by τ and is defined as follows: using Lemma 4.1.5 we replace $R^\bullet$ by a resolution $I^\bullet$, where each I^i is injective, so using Theorem 4.1.4 we see that it is enough to define a morphism of complexes

$$\tau(M^\bullet) : M^\bullet \longrightarrow \underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(M^\bullet, I^\bullet), I^\bullet).$$

Given a homogeneous element $m \in M^p$ and $q \in \mathbb{Z}$, we define $\tau(M^\bullet)(m)$ to be the morphism of complexes induced by the assignation

$$(f^t)_{t\in\mathbb{Z}} \in \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(M^\bullet, I^\bullet)^q = \prod_{t\in\mathbb{Z}} \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(M^t, I^{t+q}) \mapsto (-1)^{pq} f^p(m) \in M^{q+p}.$$

The sign $(-1)^{pq}$ appears to make this assignation into a morphism of complexes. If $M^\bullet$ is a complex of bimodules then we may repeat the construction and $\tau(M^\bullet)$ becomes a morphism of complexes of bimodules, etc. We may also consider complexes of graded right A-modules, interchanging D and D°. See Appendix A for further details.

If we consider A as a complex of bimodules concentrated in homological degree 0 and identify $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(A, I^\bullet)$ with $I^\bullet$ we obtain a morphism $\tau(A) \longrightarrow D^\circ D(A) \cong \mathcal{R}\,\underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(R^\bullet, R^\bullet)$. The image of an homogeneous element $a \in A_\xi$ by $\tau(A)$ is the A-linear morphism from $I^\bullet$ to itself given by right multiplication by a, which is in turn a lifting to the injective resolution of the endomorphism of $R^\bullet$ given by right multiplication by a. Applying [Har66, Theorem 6.4] we see that

$$\begin{aligned}
\mathrm{H}^i(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(I^\bullet, I^\bullet)) &= \bigoplus_{\xi\in\mathbb{Z}^{r+1}} \mathrm{H}^i\left(\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_A(I^\bullet, I[\xi]^{\bullet+p})\right) \\
&= \bigoplus_{\xi\in\mathbb{Z}^{r+1}} \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}}\, A)}(R^\bullet, R[\xi]^\bullet(i)) \\
&= \bigoplus_{\xi\in\mathbb{Z}^{r+1}} \mathrm{Ext}^i_{\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}^{r+1}}\, A)}(R^\bullet, R[\xi]^\bullet).
\end{aligned}$$

Hence, the image of the morphism $H^0(\tau(A))$ is the subspace of endomorphisms of $R^\bullet$ as an object of the derived category $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A)$ induced by right multiplication by a for all $a \in A$.

We are now ready to define dualizing complexes. The following definition is adapted from [Yek92, Definition 3.3].

Definition 4.2.1. A $\mathbb{Z}^{r+1}$*-graded dualizing complex over* A is an object $R^\bullet$ of the derived category $\mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$ such that:

1. Both $\Lambda(R^\bullet)$ and $P(R^\bullet)$ have finite injective dimension.
2. Both $\Lambda(R^\bullet)$ and $P(R^\bullet)$ have finitely generated cohomology modules.
3. The natural morphisms

$$A \longrightarrow \mathcal{R}\,\underline{\mathsf{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(R^\bullet, R^\bullet) \qquad \text{and} \qquad A^\circ \longrightarrow \mathcal{R}\,\underline{\mathsf{Hom}}_{A}^{\mathbb{Z}^{r+1}}(R^\bullet, R^\bullet)$$

   are isomorphisms in $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$.

A dualizing complex $R^\bullet$ is said to be *balanced* if $\mathcal{R}\Gamma_{\mathfrak{m}}(R^\bullet) \cong A^*$ and $\mathcal{R}\Gamma_{\mathfrak{m}^\circ}(R^\bullet) \cong A^*$ in $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$.

From the introduction to this subsection, we see that a dualizing complex is a complex of bimodules without self extensions in $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$ and such that all its endomorphism as a complex of left or right A-modules are given by right or left multiplication by elements of A.

For every object $R^\bullet$ of $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$ and every $i \in \{1, 2, 3\}$ we will say that condition $\mathsf{DC}(i)$ holds for $R^\bullet$ if it complies with items 1 to i of Definition 4.2.1.

Before moving on to study the effect of change of grading functors on dualizing complexes, we point out some of their nice properties.

Proposition 4.2.2. *Suppose* $R^\bullet$ *is an object of* $\mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$ *such that condition* $\mathsf{DC}(2)$ *holds for it.*

1. *The functors* D *and* D° *restrict to functors between the categories* $\mathcal{D}^b_{lf}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A \otimes B^\circ)$ *and* $\mathcal{D}^b_{lf}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} B \otimes A^\circ)$.
2. *Condition* $\mathsf{DC}(3)$ *holds for* $R^\bullet$ *if and only if for every object* $M^\bullet$ *of* $\mathcal{D}^b_{fg}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A)$ *the map* $\tau(M^\bullet)$ *is an isomorphism.*

Proof. The proofs found in [Yek92, Propositions 3.4 and 3.5] adapt to the $\mathbb{Z}^{r+1}$-graded case. An alternative proof for the second item is to notice that, through the isomorphism described in Proposition 4.1.8, we can identify the maps $\tau(\varphi_!^{A\otimes B^\circ}(M^\bullet))$ and $\varphi_!^{A\otimes B^\circ}(\tau(M^\bullet))$, so one is an isomorphism if and only if the other is. □

If an object $R^\bullet$ in $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$ is a dualizing complex then the complex $\varphi_!^{A^e}(R^\bullet)$ is a good candidate for a dualizing complex over $\varphi_!(A)$. The following proposition clarifies the relation between both complexes.

Proposition 4.2.3. *Suppose A is noetherian and let $R^\bullet$ be an object of $\mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$. Then for every $i \in \{1,2,3\}$, condition $\mathsf{DC}(i)$ holds for $R^\bullet$ if and only if it holds for $\varphi_!^{A^e}(R^\bullet)$.*

Proof. We recall that Λ and P commute with the change of grading functors.

$\mathsf{DC}(1)$ It is clear that the cohomology modules of $\Lambda(R^\bullet)$ are finitely generated if and only if the cohomology modules of $\varphi_!^A(\Lambda(R^\bullet)) = \Lambda(\varphi_!^{A^e}(R^\bullet))$ are also finitely generated, since this fact is independent of the grading. The same holds for $P(R^\bullet)$.

$\mathsf{DC}(2)$ We will prove that the injective dimensions of $\Lambda(\varphi_!^{A^e}(R^\bullet)) = \varphi_!^A(\Lambda(R^\bullet))$ and $\Lambda(R^\bullet)$ coincide; a similar result holds by symmetry for $P(R^\bullet)$, which clearly implies the desired result. Since the cohomology modules of $\Lambda(R^\bullet)$ are finitely generated, they are φ-finite by item 1 of Proposition 3.1.3. By item 1 of Proposition 4.1.2, the natural transformation $\varphi_!^A(\Lambda(R^\bullet)) \longrightarrow \varphi_*^A(\Lambda(R^\bullet))$ is a quasi-isomorphism, so we reduce the problem to prove that the injective dimension of $\Lambda(R^\bullet)$ is equal to that of $\varphi_*^A(\Lambda(R^\bullet))$.

Recall from Proposition 2.2.9 that a graded A-module is injective if and only if its image by φ_*^A is injective. Let d and d' be the injective dimensions of $\Lambda(R^\bullet)$ and $\varphi_*^A(\Lambda(R^\bullet))$ respectively, and let $I^\bullet$ be an injective resolution of $R^\bullet$ of length d. Then $\varphi_*^A(I^\bullet)$ is an injective resolution of $\varphi_*^A(\Lambda(R^\bullet))$, so $d' \leq d$. Now let $\sigma_{\leq d'}$ denote the truncation of complexes at position d' as defined in [Har66, section 7.1]. Since φ_*^A reflects exactness, we see that $\sigma_{\leq d'}(\varphi_*^A(I^\bullet)) = \varphi_*^A(\sigma_{\leq d'}(I^\bullet))$. Since d' is the injective dimension of $\varphi_*^A(\Lambda(R^\bullet))$, the complex $\sigma_{\leq d'}(\varphi_*^A(I^\bullet)) = \varphi_*^A(\sigma_{\leq d'}(I^\bullet))$ is an injective resolution of it, which in turn implies that $\sigma_{\leq d'}(I^\bullet)$ is an injective resolution of $R^\bullet$, so $d' = d$.

$\mathsf{DC}(3)$ Using the morphism defined in Proposition 4.1.8 we can identify $\tau(\varphi_!^{A^e}(A))$ with $\varphi_!^{A^e}(\tau(A))$ and analogously for $\tau(A^\circ)$; see Proposition A.0.6 for details. Thus one is an isomorphism if and only if the other is.

□

By [Yek92, Theorem 3.9], a dualizing complex is unique up to tensor product with an invertible bimodule. The notion of a balanced dualizing complex is introduced to distinguish a particular class of dualizing complexes. We extend the previous result to cover this case.

Corollary 4.2.4. *Suppose $R^\bullet$ is an object of $\mathcal{D}^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$. Then $\varphi_!^{A^e}(R^\bullet)$ is a balanced dualizing complex over $\varphi_!(A)$ if and only if there exists $\xi \in \ker \varphi$ such that $R[\xi]^\bullet$ is a balanced dualizing complex over A.*

Proof. By Proposition 4.2.3 $R^\bullet[\xi]$ is a dualizing complex if and only if $\varphi_!^{A^e}(R[\xi]^\bullet) = \varphi_!^{A^e}(R^\bullet)$ is one. Furthermore, if either $\varphi_!^{A^e}(R^\bullet)$ or $R[\xi]^\bullet$ is balanced then $H^i(\mathcal{R}\Gamma_m(R^\bullet)) = 0$ for all $i \neq 0$. Replacing $R^\bullet$ by a finite resolution $I^\bullet$, where each I^p is injective as left and right graded A-module, we see that the vertical maps in the following diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \Gamma_m(I^{-1}) & \xrightarrow{d^{-1}} & \Gamma_m(I^0) & \xrightarrow{d^0} & \Gamma_m(I^1) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & \frac{\Gamma_m(I^0)}{d^{-1}(\Gamma_m(I^{-1}))} & \longrightarrow & \Gamma_m(I^1) & \longrightarrow & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & H^0(\Gamma_m(I^\bullet)) & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}$$

are quasi-isomorphisms. In particular $\mathcal{R}\Gamma_m(R^\bullet) \cong H^0(\Gamma_m(I^\bullet))$ and, denoting this last module by H, the problem reduces to proving that $\varphi_!^{A^e}(H) \cong \varphi_!^{A^e}(A)^*$ if and only if $H[\xi] \cong A^*$.

The "only if" part follows from item 1 of Proposition 4.1.7. For the "if" part, the hypothesis implies that H is locally finite, and so applying Matlis duality (see Proposition 4.1.7) $\varphi_!^{A^e}(A) \cong \varphi_!^{A^e}(H^*)$. Thus by Proposition 2.2.9 H^* is projective both as left and right graded A-module, and by Lemma 3.1.2 it is free of rank 1 as left and right graded A-module. Since $\varphi_!^{A^e}(H^*) \cong \varphi_!^{A^e}(A)$, the vector space $\bigoplus_{\xi \in \ker \varphi} H^*_\xi$ is one dimensional and contains a central generator of H^* (that is, an element h that generates H^* and such that $ah = ha$ for all $a \in A$), which must be homogeneous of degree ξ for some $\xi \in \ker \varphi$. Thus $H^* \cong A[\xi]$ and applying Matlis duality again we conclude that $H[\xi] \cong A^*$, which implies that the dualizing complex $R[\xi]^\bullet$ is balanced. □

This last Proposition implies that if A has a $\mathbb{Z}^{r+1}$-graded dualizing complex then so does $\varphi_!(A)$. However there there is no obvious way to construct a $\mathbb{Z}^{r+1}$-graded dualizing starting from a $\mathbb{Z}$-graded one, as the image of a finitely generated $\mathbb{Z}$-graded $\varphi_!(A)$-module by φ_A^* is not necessarily finitely generated, so we can not prove a converse result.

If $R^\bullet$ is a balanced dualizing complex over A then, we may fix an isomorphism $\mathcal{R}\Gamma_m(R^\bullet) \cong A^*$, and so for every $M^\bullet$ in $\mathcal{D}^+(\mathrm{Mod}^{\mathbb{Z}^{r+1}} A)$ there exists a natural morphism just as in [Yek92, (4.17)]

$$\theta(M^\bullet) : \mathcal{R}\Gamma_m(M^\bullet) \longrightarrow \mathcal{R}\,\underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(\mathcal{R}\,\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(M^\bullet, R^\bullet), A^*),$$

such that $\varphi_!^A(\theta(M^\bullet)) = \theta(\varphi_!^A(M^\bullet))$ (notice that $\varphi_!^{A^e}(R^\bullet)$ is also a balanced dualizing complex, see Corollary 4.2.4). By [Yek92, Theorem 4.8] this map is an isomorphism for all $M^\bullet$ whose cohomology modules are finitely generated. Setting $M^\bullet = A$ we see that

$$\mathcal{R}\Gamma_m(A) \cong \underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(R^\bullet, A^*) \cong (R^\bullet)^*,$$

and since $R^\bullet$ has locally finite cohomology modules, we can apply Matlis duality to obtain $R^\bullet \cong \mathcal{R}\Gamma_{\mathfrak{m}}(A)^*$. With this observation in mind, there is always a candidate for a *balanced* $\mathbb{Z}^{r+1}$-graded dualizing complex, namely $\mathcal{R}\Gamma_{\mathfrak{m}}(A)^*$. In fact Corollary 4.2.4 can be restated as saying that $\mathcal{R}\Gamma_{\mathfrak{m}}(A)^*$ is a balanced $\mathbb{Z}^{r+1}$-graded dualizing complex over A if and only if $\mathcal{R}\Gamma_{\mathfrak{m}}(\varphi_!^{A^e}(A))^*$ is a balanced $\mathbb{Z}$-graded dualizing complex over $\varphi_!(A)$.

Before moving on, we clarify the relation of the regularity properties studied in section 3.2 with dualizing complexes.

Remark 4.2.5. By [Jør99, Theorem 1.6], an AS-Cohen-Macaulay algebra has a balanced dualizing complex if and only if it is a graded quotient of an AS-Gorenstein algebra. In particular any AS-Gorenstein algebra has a dualizing complex. This shows that the hypothesis that $\operatorname{gr} A$ has property χ in Theorem 3.2.13 is redundant in case $\operatorname{gr} A$ is AS-Gorenstein, or a graded quotient of an AS-Gorenstein algebra.

4.2.2 Existence results for balanced dualizing complexes

Recall that the left local dimension of an algebra A is the cohomological dimension of the functor $\Gamma_{\mathfrak{m}}$ over the category $\mathsf{Mod}^{\mathbb{Z}^{r+1}} A$. By a celebrated result of M. Van den Bergh [VdB97, Proposition 6.3], a connected $\mathbb{N}$-graded noetherian algebra B has a balanced dualizing complex if and only if both B and B° have finite local dimension and property χ. In this section we will deduce from that result an analogous one for noetherian connected $\mathbb{N}^{r+1}$-graded algebras.

First we need some facts on $\mathbb{N}$-graded algebras, so we fix a connected $\mathbb{N}$-graded noetherian algebra B. The following result, known as the noncommutative local duality theorem, shows that the complex $\mathcal{R}\Gamma_{\mathfrak{m}}(B)^*$ has nice dualizing properties under the hypothesis that B has finite local dimension, even if it is not a dualizing complex in the sense of Definition 4.2.1.

Theorem 4.2.6. [VdB97, Theorem 5.1] *Assume that* B *has finite local dimension. Then*

1. *The complex* $\mathcal{R}\Gamma_{\mathfrak{m}}(B)^*$ *has finite injective dimension in* $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}} B)$.
2. *For any object* $M^\bullet$ *of* $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}} B)$ *there is an isomorphism*

$$\mathcal{R}\Gamma_{\mathfrak{m}}(M^\bullet)^* \cong \mathcal{R}\,\underline{\mathsf{Hom}}^{\mathbb{Z}}_B(M^\bullet, \mathcal{R}\Gamma_{\mathfrak{m}}(B)^*).$$

in $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}} B^\circ)$.

The condition that $\operatorname{lcd}^{\mathbb{Z}} B < \infty$ is crucial for this result. Notice that the object $M^\bullet$ is not assumed to be bounded, so $\mathcal{R}\Gamma_{\mathfrak{m}}(M^\bullet)$ may not be well defined unless $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension.

Recall that a B-module M is said to have property χ if $\dim_k \operatorname{Ext}^i_B(k, M) < \infty$ for every $i \geq 0$; by item 5 of Lemma 3.2.11, M has property χ if and only if for every

$i \geq 0$ the homogeneous component $H^{-i}(\mathcal{R}\Gamma_{\mathfrak{m}}(M)^*)_d \cong H^i_{\mathfrak{m}}(M)^*_{-d}$ is 0 for $d \gg 0$, i.e. the cohomology modules of $\mathcal{R}\Gamma_{\mathfrak{m}}(M)^*$ have left bounded grading. The local duality theorem 4.2.6 has the following consequence.

Corollary 4.2.7. *Suppose* B *is a noetherian connected* $\mathbb{N}$*-graded algebra. If* $\operatorname{lcd}^{\mathbb{Z}} B < \infty$ *then* B *has property* χ *as a graded algebra if and only if the left* B*-module* B *has property* χ*.*

Proof. Clearly if B has property χ as a graded algebra then the left B-module B has property χ.

Now suppose the left B-module B has property χ and let $\mathcal{A}$, resp. $\mathcal{B}$, be the subcategory of $\operatorname{Mod}^{\mathbb{Z}} B^e$, resp. $\operatorname{Mod}^{\mathbb{Z}} B^\circ$, formed by objects with left-bounded grading. Let M be a finitely generated $\mathbb{Z}$-graded left B-module; in order to prove that it has property χ we will prove that $\mathcal{R}\Gamma_{\mathfrak{m}}(M)^*$ is an object of $\mathcal{D}_{\mathcal{B}}(\operatorname{Mod}^{\mathbb{Z}} B^\circ)$. By Theorem 4.2.6 there exists an isomorphism in $\mathcal{D}^+(\operatorname{Mod} B^\circ)$

$$\mathcal{R}\Gamma_{\mathfrak{m}}(M)^* \cong \mathcal{R}\,\underline{\operatorname{Hom}}^{\mathbb{Z}}_B(M, \mathcal{R}\Gamma_{\mathfrak{m}}(B)^*)$$

so it is enough to prove that the complex on the right hand side is an object of $\mathcal{D}_{\mathcal{B}}(\operatorname{Mod}^{\mathbb{Z}} B^\circ)$.

The module M is finitely generated so its grading is left-bounded, and for every object N of $\mathcal{A}$ the grading of the B°-module $\underline{\operatorname{Ext}}^i_B(M, N)$ is also left-bounded (this follows from the noetherianity of B). By item 2 of Proposition 4.1.2, the functor $\mathcal{R}\,\underline{\operatorname{Hom}}^{\mathbb{Z}}_B(M, -)$ sends objects in $\mathcal{D}^b_{\mathcal{A}}(\operatorname{Mod} B^e)$ to objects in $\mathcal{D}_{\mathcal{B}}(\operatorname{Mod}^{\mathbb{Z}} B^\circ)$, so we have reduced the problem to proving that $\mathcal{R}\Gamma_{\mathfrak{m}}(B)^*$ is (isomorphic to) an object of $\mathcal{D}^b_{\mathcal{A}}(\operatorname{Mod} B^e)$.

By the preamble to this corollary, the fact that B has property χ implies $\mathcal{R}\Gamma_{\mathfrak{m}}(B)^*$ is an object of $\mathcal{D}^+_{\mathcal{A}}(\operatorname{Mod}^{\mathbb{Z}} B^e)$. Since $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension over $\operatorname{Mod}^{\mathbb{Z}} B^e$ we can take a resolution of B by injective B^e-bimodules and truncate it at position $\operatorname{lcd}^{\mathbb{Z}} B$, thus obtaining a $\Gamma_{\mathfrak{m}}$-acyclic resolution of B of finite length in $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} B^e)$ which we denote by $I^\bullet$. Since $I^\bullet$ is a bounded complex, so is $\mathcal{R}\Gamma_{\mathfrak{m}}(B)^* \cong \Gamma_{\mathfrak{m}}(I^\bullet)^*$, and $\mathcal{R}\Gamma_{\mathfrak{m}}(B)^*$ is isomorphic to an object of $\mathcal{D}^b_{\mathcal{A}}(\operatorname{Mod} B^e)$. □

We return to the study of $\mathbb{N}^{r+1}$-graded algebras. The last theorem allows us to give the following $\mathbb{Z}^{r+1}$-graded version of Van den Bergh's criterion.

Corollary 4.2.8. *Assume* A *is a noetherian connected* $\mathbb{N}^{r+1}$*-graded algebra. If* A *has a* $\mathbb{Z}^{r+1}$*-graded balanced dualizing complex, then it is given by*

$$R^\bullet \cong \mathcal{R}\Gamma_{\mathfrak{m}}(A)^*,$$

and furthermore $\operatorname{lcd}^{\mathbb{Z}^{r+1}} A$ *and* $\operatorname{lcd}^{\mathbb{Z}^{r+1}} A^\circ$ *are finite, and both* A *and* A° *have property* χ *as* $\mathbb{Z}^{r+1}$*-graded algebras.*

Conversely, if A *and* A° *have finite local dimension and property* χ *as* $\mathbb{Z}^{r+1}$*-graded algebras, then* A *has a balanced dualizing complex, given by* $\mathcal{R}\Gamma_{\mathfrak{m}}(A)^*$*.*

Proof. As stated above, the criterion was originally proved for $\mathbb{N}$-graded algebras in [VdB97, Theorem 6.3]. One possibility is to reprove all the results in said paper for $\mathbb{N}^{r+1}$-graded algebras. Here we show how to use the change of grading functors to extend the result.

Remember that $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ is a morphism such that A is φ-connected. By Lemma 2.3.4 and Corollary 4.2.7, A and A° have finite local dimension and property χ as graded algebras if and only if the same holds for $\varphi_!(A)$ and $\varphi_!(A)^\circ$. By Van den Bergh's criterion this happens if and only if $\mathcal{R}\Gamma_{\mathfrak{m}}(\varphi_!^{A^e}(A))^* \cong \varphi_!^{A^e}(\mathcal{R}\Gamma_{\mathfrak{m}}(A)^*)$ is a balanced dualizing complex for $\varphi_!(A)$. By Proposition 4.2.3 this happens if and only if $\mathcal{R}\Gamma_{\mathfrak{m}}(A)^*[\xi]$ is a balanced dualizing complex, where $\xi \in \ker \varphi$. Since every balanced dualizing complex is isomorphic to $\mathcal{R}\Gamma_{\mathfrak{m}}(A)^*$, we see that $\xi = 0$. □

An immediate consequence of the previous corollary is the following.

Corollary 4.2.9. *Every commutative connected $\mathbb{N}^{r+1}$-graded noetherian algebra of finite Krull dimension has a $\mathbb{Z}^{r+1}$-graded balanced dualizing complex.*

Proof. As discussed in Remark 3.2.10, every commutative noetherian algebra has property χ. Also, by Grothendieck's vanishing theorem [BS98, Theorem 6.1.2], the local dimension of the algebra is bounded by its Krull dimension, so in particular it is finite. The result now follows from Corollary 4.2.8. □

We now use Corollary 4.2.8 to prove that having a balanced dualizing complex is a twisting invariant property.

Proposition 4.2.10. *Suppose A is noetherian, and that τ is a left twisting system on A. Then the algebra ${}^\tau A$ has a balanced dualizing complex if and only if A does.*

Proof. We only prove one implication. Let M be a finitely generated graded A-module. By Lemma 3.2.7, the underlying graded vector spaces of $\underline{\mathrm{Ext}}^i_A(k, M)$ and $\underline{\mathrm{Ext}}^i_{{}^\tau A}(k, {}^\tau M)$ coincide, M has property χ if and only if ${}^\tau M$ does. By Proposition 2.3.5, there exist isomorphisms ${}^\tau H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}({}^\tau M)$ for every $i \geq 0$, so the local dimensions of M and ${}^\tau M$ are the same. Hence A has property χ and finite local dimension as a graded algebra if and only if ${}^\tau A$ does. By a mirror argument, $({}^\tau A)^\circ = (A^\circ)^\tau$ has property χ and finite local dimension if and only if A° does. The claim now follows from Corollary 4.2.8. □

Remark 4.2.11. It seems natural to ask whether the fact that A has a dualizing complex implies that ${}^\tau A$ has one. However, we know of no way to twist A-bimodules into ${}^\tau A$-bimodules, so there is no natural candidate for a dualizing complex over ${}^\tau A$.

Suppose now that A is a GF-algebra. We finish this chapter proving that if $\mathrm{gr}\, A$ has a dualizing complex, so does A.

Theorem 4.2.12. *Suppose $\mathrm{gr}\, A$ is noetherian. Then the following hold.*

1. *If* $\operatorname{gr} A$ *has property* χ *as a graded algebra, then* A *has property* χ *and* $\operatorname{lcd}^{\mathbb{Z}} A \leq \operatorname{lcd}^{\mathbb{Z}^2} \operatorname{gr} A$.

2. *If* $\operatorname{gr} A$ *has a balanced dualizing complex, so does* A.

Proof. 1. This follows from item 4 of Lemma 3.2.12.

2. By Corollary 4.2.8 A has a balanced dualizing complex if and only if A and A° have property χ as graded algebras and finite local dimension as graded algebras. Since $(\operatorname{gr} A)^\circ = \operatorname{gr} A^\circ$, the result follows from the previous item.

□

Chapter 5

Quantum affine toric varieties

In the last twenty years there has been a wide interest in "toric degenerations", that is deformations of algebraic varieties into toric varieties. One of the main reasons for this interest is that such a degeneration allows to study deformation-invariant properties of the original varieties by studying the resulting toric varieties. In a purely algebraic context this amounts to finding a filtration on the coordinate ring of the variety such that its associated graded ring is a semigroup ring. The objective of this chapter is to extend this idea to the noncommutative world.

In section 5.1 we review the main properties of affine semigroups and toric varieties. Then in section 5.2 we introduce a family of noncommutative algebras which, by analogy with the commutative case, we call quantum affine toric varieties. They turn out to be Zhang twists of semigroup rings, which allows us to use the results from the previous chapter in the study of their homological properties. Later in section 5.3 we introduce a class of algebras with a filtration such that its associated graded ring is a quantum affine toric variety. We study the homological regularity properties of the original algebras by looking at the corresponding quantum toric varieties.

Let $r \in \mathbb{N}^*$. For every $0 \leq i < j \leq r$ fix $q_{i,j} \in k^\times$, and write $\mathbf{q} = (q_{i,j})_{0 \leq i < j \leq r}$. We denote by $k_{\mathbf{q}}[X_0, X_1, \ldots, X_r]$, or $k_{\mathbf{q}}[X]$ for short, the *quantum affine space* with parameter system $\mathbf{q}$, the algebra with generators X_i for $0 \leq i \leq r$ and relations $X_j X_i = q_{i,j} X_i X_j$ for every $0 \leq i < j \leq r$. It is a classical result that quantum spaces are noetherian domains, see for example [MR01, Chapter 1, Theorem 2.9]. The *quantum torus* $k_{\mathbf{q}}[X_0^{\pm 1}, \ldots, X_r^{\pm 1}]$, or $k_{\mathbf{q}}[X^{\pm 1}]$ for short, is the localization of $k_{\mathbf{q}}[X]$ at the multiplicative set generated by $X_0, \ldots, X_n$. By standard localization theory it is also a noetherian domain, with an obvious $\mathbb{Z}^{r+1}$-grading.

Given $p = (p^0, p^1, \ldots, p^r) \in \mathbb{Z}^{r+1}$ we write X^p for $X_0^{p^0} X_1^{p^1} \ldots X_r^{p^r} \in k_q[X^{\pm 1}]$. If all the $q_{i,j}$'s are equal to 1 then $k_{\mathbf{q}}[X^{\pm 1}]$ is the commutative algebra of Laurent polynomials in

$r+1$-variables, which we denote by $k[X^{\pm 1}]$.

5.1 Affine toric varieties

5.1.1 Affine semigroups

In this subsection we review the basic notions on affine semigroups. Our main references on the subject are [Ful93, Chapter 1] and [BH93, Chapter 6]; we follow the notation of this last book.

Definition 5.1.1. [BH93, Section 6.1] A semigroup S is said to be *affine* if it is finitely generated and isomorphic to a subsemigroup of $\mathbb{Z}^{r+1}$ for some $r \geq 0$.

For the rest of this section S denotes an affine semigroup, and its enveloping group is denoted by $\mathbb{Z}S$. By definition there exists an injective semigroup morphism $S \hookrightarrow \mathbb{Z}^{r+1}$, and $\mathbb{Z}S$ is isomorphic to the group generated by the image of S inside $\mathbb{Z}^{r+1}$. Hence $\mathbb{Z}S$ is a finitely generated torsionless abelian group. The *rank* of S, denoted by $\operatorname{rk} S$, is the rank of $\mathbb{Z}S$.

We denote by $\mathbb{Q}S$ the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}S$ and by $\mathbb{R}S$ the $\mathbb{R}$-vector space $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}S$. We can naturally identify $S, \mathbb{Z}S$ and $\mathbb{Q}S$ with subsets of $\mathbb{R}S$. A linear form φ of $\mathbb{R}S$ is called *rational* if it is the extension to $\mathbb{R}S$ of a linear form over $\mathbb{Q}S$; a hyperplane H is called rational if it is the kernel of a rational form. A *rational convex cone* is a set $D \subset \mathbb{R}S$ of the form $D = \bigcap_{i=1}^{n} \{x \in \mathbb{Z}S \mid \varphi_i(x) \geq 0\}$, where φ_i is a rational linear form over $\mathbb{R}S$ for every i.

We also set

$$\mathbb{R}_+S = \left\{ \sum_{i=1}^{n} a_i s_i \mid s_i \in S,\ a_i \in \mathbb{R}_{\geq 0} \right\} \subset \mathbb{R}S.$$

A hyperplane $H \subset \mathbb{R}S$ is said to be a *supporting hyperplane* of S if $\mathbb{R}_+S$ is contained in the closure of one of the half-spaces defined by H. Given a supporting hyperplane H, we denote by H^+ the corresponding closed half-space containing S. The *face* corresponding to H is the intersection $\tau = H \cap \mathbb{R}_+S$. A face τ is called a *facet* if the $\mathbb{R}$-vector space generated by τ is a supporting hyperplane, and in that case we write $\tau < \mathbb{R}_+S$. We denote by H_τ the supporting hyperplane generated by the facet τ; clearly $\tau = H_\tau \cap \mathbb{R}_+S$. To every facet τ we associate the semigroup $S_\tau = \mathbb{Z}S \cap H_\tau^+$.

The semigroup S is said to be *normal* if the following holds: whenever there exist $n \in \mathbb{N}$ and $z \in \mathbb{Z}S$ such that $nz \in S$, then $z \in S$. We focus now on the class of normal affine semigroups.

Lemma 5.1.2. *Let S be a normal affine semigroup of rank $r+1$.*

1. *For every $\tau < \mathbb{R}_+S$ the hyperplane H_τ is generated by a set of r linearly independent elements of S, and $\mathbb{R}_+S = \bigcap_{\tau<\mathbb{R}_+S} H_\tau^+$, that is $\mathbb{R}_+S$ is a rational convex cone.*

2. *(Gordan's Lemma) $S = \mathbb{Z}S \cap \mathbb{R}_+S$. Conversely, if $G \subset \mathbb{Q}^{r+1}$ is a finitely generated subgroup and D is a rational convex cone, then $G \cap D$ is a normal affine semigroup.*

3. *Let $S^\times \subset S$ be the subset of invertible elements of S. Then S decomposes as $S^\times \oplus S'$, with S' a normal affine semigroup with no invertible elements other than zero.*

4. $S = \bigcap_{\tau<\mathbb{R}_+S} S_\tau$.

5. *For every $\tau < \mathbb{R}_+S$ the semigroup S_τ is normal and isomorphic as a semigroup to $\mathbb{Z}^r \oplus \mathbb{N}$.*

Proof. 1. See [Ful93, Section 1.2, Point (8)].

2. See [BH93, Proposition 6.1.2].

3. See [BH93, Proposition 6.1.3 (a)]

4. Using items 1 and 2, and the fact that $\mathbb{Z}S = \mathbb{Z}S_\tau$, we obtain

$$S = \mathbb{Z}S \cap \mathbb{R}_+S = \bigcap_{\tau<\mathbb{R}_+S} \mathbb{Z}S \cap H_\tau^+ = \bigcap_{\tau<\mathbb{R}_+S} S_\tau.$$

5. Let S_τ° denote the semigroup generated by S plus the inverses of the generators of H_τ given in item 1. It is easy to see that this semigroup is also normal. By definition, the set of invertible elements of S_τ° has rank at least r, and since it is contained in the half-space H_τ^+ this rank must be exactly r. Thus by item 3 S_τ° is isomorphic to $\mathbb{Z}^r \oplus N$, with N a normal semigroup of rank 1. Any normal semigroup of rank 1 is isomorphic to $\mathbb{N}$, so $S_\tau^\circ \cong \mathbb{Z}^r \oplus \mathbb{N}$, and we only have to prove that $S_\tau^\circ = S_\tau$. By Gordan's lemma, $S_\tau^\circ = \mathbb{Z}S_\tau^\circ \cap \mathbb{R}_+S_\tau^\circ \supset \mathbb{Z}S \cap H_\tau^+ = S_\tau$, and since the other inclusion is obvious we are done.

□

The affine semigroup S is said to be *positive* if the only invertible element in it is 0_S. Notice that item 3 of the previous lemma states that every normal affine semigroup is the direct sum of a free abelian group and a positive affine semigroup. Any positive affine semigroup has a natural order, which we denote by $\preceq$, where $s' \preceq s$ holds for any $s, s' \in S$ if and only if there exists $s'' \in S$ such that $s' + s'' = s$.

Lemma 5.1.3. *Let S be a positive affine semigroup of rank $r + 1$.*

1. *The semigroup S has a unique minimal set of generators, formed by the minimal nonzero elements with respect to the order $\preceq$.*

2. *There exists an injective semigroup morphism* $S \longrightarrow \mathbb{N}^{r+1}$.

Proof. See [MS05, Proposition 7.15] for item 1 and [MS05, Corollary 7.23] for item 2. Notice that in the reference given, positive semigroups are called *pointed*, and the term positive is reserved for subsemigroups of $\mathbb{N}^{r+1}$. □

5.1.2 Affine toric varieties

We begin this subsection giving a presentation of the semigroup algebra $k[S]$ by generators and relations. We will show that this is a finitely generated algebra, and recall the main results on the associated varieties when $k = \mathbb{C}$; these are called *toric varieties*. We then discuss some general facts on toric varieties. Our main references are [Ful93] and [CLS11].

Let S be an affine semigroup and choose a set of generators $s_1, \dots, s_n$ of S such that n is minimal. There is an obvious semigroup morphism $\pi : \mathbb{N}^n \longrightarrow S$ defined by the assignation $e_i \mapsto s_i$ for every $1 \leq i \leq n$. Set

$$L(S) = \{(p, p') \in \mathbb{N}^n \times \mathbb{N}^n \mid \pi(p) = \pi(p') \text{ and } p <_{\mathsf{lex}} p'\} \cup \{(0,0)\},$$

where $<_{\mathsf{lex}}$ denotes the total lexicographic order of $\mathbb{N}^n$. Clearly $L(S)$ is a subsemigroup of $\mathbb{N}^n \times \mathbb{N}^n$.

The morphism π induces a surjective morphism $\varphi : k[X_1, \dots, X_n] \longrightarrow k[S]$. Denote by I_L the ideal generated by the elements of the form $X^p - X^{p'}$ for $(p, p') \in L(S)$.

Theorem 5.1.4. *The semigroup algebra* $k[S]$ *is isomorphic to* $k[X]/I_L$.

Proof. See [MS05, Theorem 7.3]. The ideal I_L is defined in a slightly different way in the reference, but it is routine to check that it coincides with the ideal defined above. □

Since $k[X]$ is a noetherian algebra, the ideal I_L is finitely generated, so there exists a finite subset $\{(p_1, p_1'), \dots, (p_m, p_m')\} \subset L(S)$ such that $I_L = \langle X^{p_i} - X^{p_i'} \mid 1 \leq i \leq m \rangle$. The semigroup $L(S)$ is generated as a semigroup by $\{(p_1, p_1'), \dots, (p_m, p_m')\}$, which proves in particular that every affine semigroup is finitely presented.

We can naturally see S as a subsemigroup of its enveloping group, and fix an isomorphism $\mathbb{Z}S \cong \mathbb{Z}^{r+1}$. This datum induces morphisms between the corresponding semigroup algebras $k[S] \subset k[\mathbb{Z}S] \cong k[\mathbb{Z}^{r+1}]$, and this last algebra can be identified with the Laurent polynomial algebra in $r+1$ variables $k[X^{\pm 1}]$. Thus the semigroup algebra $k[S]$ is isomorphic to a $\mathbb{Z}^{r+1}$-graded subalgebra of $k[X^{\pm 1}]$, and furthermore, the Laurent polynomial algebra is isomorphic to the localization of $k[S]$ at the multiplicative set of homogeneous elements. Geometrically this corresponds to the fact that $\operatorname{Spec} k[S]$ has an open dense set isomorphic to a torus; such varieties are known as *toric varieties*, and we will now review some of their main properties. For the sake of simplicity we only consider the case where $k = \mathbb{C}$.

Definition 5.1.5. [CLS11, Definition 1.1.3] An *affine toric variety* is an irreducible affine variety V over $\mathbb{C}$ containing an algebraic torus $T_{r+1} \cong (\mathbb{C}^\times)^{r+1}$ as an open subset, such that the action of T_{r+1} over itself extends to an algebraic action of T_{r+1} on V.

The following result is a characterization of affine toric varieties in purely algebraic terms.

Proposition 5.1.6. *Let S be an affine semigroup. Then* $\operatorname{Spec}\mathbb{C}[S]$ *is an affine toric variety, and every affine toric variety over $\mathbb{C}$ arises this way.*

We will now sketch a proof of this fact. The interested reader can find a complete proof in [CLS11, Theorem 1.1.17].

As discussed before, $\mathbb{C}[X^{\pm 1}]$ is isomorphic to the localization of $\mathbb{C}[S]$ at the multiplicative set $\mathcal{M} = \{X^s \mid s \in S\}$. Since $\mathcal{M}$ is isomorphic to S as a semigroup, choosing a finite set of generators of S and localizing at the corresponding monomials suffices to obtain inverses of all the elements of $\mathcal{M}$, so $\mathbb{C}[X^{\pm 1}]$ is a localization of S at a finite set of elements. This implies that $\operatorname{Spec}\mathbb{C}[S]$ has a principal open subset, isomorphic to $\operatorname{Spec}\mathbb{C}[X^{\pm 1}] \cong (\mathbb{C}^\times)^{r+1}$.

To prove that the action of this torus over itself extends to the whole variety, consider $\mathbb{C}$ as a semigroup with the operation given by multiplication. For every maximal ideal $\mathfrak{m} \subset \mathbb{C}[S]$, the quotient morphism $\pi : \mathbb{C}[S] \longrightarrow \mathbb{C}[S]/\mathfrak{m} \cong \mathbb{C}$ induces a semigroup morphism $S \longrightarrow \mathbb{C}$ that assigns to each $s \in S$ the element $\pi(X^s) \in \mathbb{C}$. Notice that this cannot be the zero morphism, since it sends $X^0 = 1$ to $1 \in \mathbb{C}$. Conversely, any nonzero semigroup morphism $S \longrightarrow \mathbb{C}$ induces an algebra morphism $\mathbb{C}[S] \longrightarrow \mathbb{C}$, and its kernel must be a maximal ideal. Thus the maximal spectrum of $\mathbb{C}[S]$ is in one-to-one correspondence with the set of non-zero semigroup morphisms from S to $\mathbb{C}$, which we denote by $\mathsf{Hom}_{\mathsf{Sgrp}}(S, \mathbb{C})^\times$. This set has a semigroup structure with pointwise multiplication as the operation. Also notice that $\mathsf{Hom}_{\mathsf{Sgrp}}(\mathbb{Z}^{r+1}, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{r+1}$.

These identifications fit in the following diagram

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C}[S] & \longleftrightarrow & \mathsf{Hom}_{\mathsf{Sgrp}}(S, \mathbb{C})^\times \\ \uparrow & & \uparrow{\scriptstyle i_*} \\ \operatorname{Spec}\mathbb{C}[X^{\pm 1}] & \longleftrightarrow & \mathsf{Hom}_{\mathsf{Grp}}(\mathbb{Z}^{r+1}, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{r+1}, \end{array}$$

where i_* sends a morphism $\mathbb{Z}^{r+1} \longrightarrow \mathbb{C}^\times$ to its restriction to S. This is a semigroup morphism and induces an action of $(\mathbb{C}^\times)^{r+1}$ on $\operatorname{Spec}\mathbb{C}[S]$, which restricted to the subvariety $\operatorname{Spec}\mathbb{C}[X^{\pm 1}]$ is given by the natural action of the torus on itself. Thus $\operatorname{Spec}\mathbb{C}[S]$ is a toric variety.

To see that every affine toric variety V is the spectrum of an affine semigroup algebra, one uses the fact that the coordinate algebra $\mathbb{C}[V]$ can be identified with a

subalgebra of $\mathbb{C}[X^{\pm 1}]$. The action of the torus induces a grading on $\mathbb{C}[V]$ by its character lattice $\mathsf{Hom}_{\mathsf{Grp}}(T_{r+1}, \mathbb{C}^\times) \cong \mathbb{Z}^{r+1}$, and since this action is an extension of the action of the torus on itself, the grading is compatible with the $\mathbb{Z}^{r+1}$-grading of $\mathbb{C}[X^{\pm 1}]$ (see [CLS11, Lemma 1.1.16]). Thus $\mathbb{C}[V]$ is isomorphic to a finitely generated $\mathbb{Z}^{r+1}$-graded subalgebra of $\mathbb{C}[X^{\pm 1}]$, i.e. $\mathbb{C}[V] = \mathbb{C}[S]$ where $S = \operatorname{supp} \mathbb{C}[V]$. Since $\mathbb{C}[V]$ is finitely generated, say by elements $X^{s_1}, \dots, X^{s_n}$, the semigroup S is finitely generated by $\{s_1, \dots, s_n\}$.

5.2 Quantum affine toric varieties

In the last section we saw that affine toric varieties are the spectra of affine semigroup algebras. An affine semigroup algebra is a $\mathbb{Z}^{r+1}$-graded and finitely generated integral domain whose ring of homogeneous fractions is isomorphic to $k[X^{\pm 1}]$, and any algebra of this form is isomorphic to an affine semigroup algebra. This inspires the following definition.

Definition 5.2.1. Let A be a noetherian $\mathbb{Z}^{r+1}$-graded algebra. We say that A is a *quantum affine toric variety*, or QA toric variety for short, if it is integral and its homogeneous ring of fractions is isomorphic to a quantum torus. The *support* of A is the set $S(A) = \{\xi \in \mathbb{Z}^{r+1} \mid A_\xi \neq 0\}$.

Notice that the fact that A is integral implies that $S(A)$ is a subsemigroup of $\mathbb{Z}^{r+1}$, while noetherianity implies that it is finitely generated, so the support of a quantum toric variety is always an affine semigroup.

Recall that a twisting system on $k[S]$ is a family of $\mathbb{Z}S$-graded linear automorphisms $\tau = \{\tau_\xi \mid \xi \in \mathbb{Z}S\}$ of the graded vector space $k[S]$ such that

$$\tau_{s''}(\tau_{s'}(X^s)X^{s'}) = \tau_{s'+s''}(X^s)\tau_{s''}(X^{s'}) \qquad \text{for all } s, s' \in S,\ s'' \in \mathbb{Z}^{r+1}. \tag{†}$$

We will now prove that QA toric varieties are precisely Zhang twists of affine semigroup algebras. In order to give a classification of them up to isomorphism we will need to reinterpret Zhang twists as elements of a cohomology group.

5.2.1 Twists and 2-cocycles

Let S be an affine semigroup and let τ be a left twisting system on the semigroup algebra $k[S]$. For every $s, s' \in S$, the element $\tau_{s'}(X^s)$ is a nonzero homogeneous element of degree s, so it must be a nonzero multiple of X^s. Denote by $\alpha_\tau(s, s')$ the only nonzero scalar such that $\tau_{s'}(X^s) = \alpha_\tau(s, s')X^s$. Condition (†), or equivalently associativity of the product of ${}^\tau k[S]$, implies that

$$\alpha_\tau(s, s')\alpha_\tau(s + s', s'') = \alpha_\tau(s, s' + s'')\alpha_\tau(s', s'') \qquad \text{for all } s, s', s'' \in S,$$

that is, $\alpha_\tau : S \times S \longrightarrow k^\times$ is a 2-cocycle over S with coefficients in $k^\times$.

We denote by $C^2 = C^2(S, k^\times)$ the set of all 2-cocycles over S with coefficients in $k^\times$. This set is a commutative group with pointwise multiplication as the operation. Given $\alpha \in C^2$, the *α-twisted semigroup algebra* of S, denoted by $k^\alpha[S]$, is the vector space with basis $\{X^s \mid s \in S\}$ and product defined over the generators by $X^s X^{s'} = \alpha(s, s')X^{s+s'}$, extended by bilinearity. If $\alpha = \alpha_\tau$ for some twisting system τ over $k[S]$ then $k^\alpha[S] = {}^\tau k[S]$ by definition. Notice that the function $\mathbf{1} : S \times S \longrightarrow k^\times$ with constant value 1 is a 2-cocycle over S, in fact it is the neutral element of C^2, so the usual semigroup algebra $k[S]$ is the "trivial" twisted semigroup algebra.

We will eventually show that 2-cocycle twists and Zhang twists of $k[S]$ define the same family of objects up to isomorphism. We begin with the following observation.

Lemma 5.2.2. *Let $\alpha \in C^2$. Then the algebra $k^\alpha[S]$ is a noetherian domain.*

Proof. Fix an embedding $S \hookrightarrow \mathbb{Z}^{r+1}$, and pull back on S the lexicographic order of $\mathbb{Z}^{r+1}$, so S becomes a completely ordered semigroup.

Let $s_1, \ldots, s_n$ be a minimal system of generators of S; clearly the elements X^{s_i} with $1 \leq i \leq n$ generate the algebra $k^\alpha[S]$. Setting $q_{i,j} = \alpha(s_j, s_i)/\alpha(s_i, s_j)$ for all $1 \leq i < j \leq n$ there is a surjective map $k_{\mathbf{q}}[X] \longrightarrow k^\alpha[S]$ defined by the assignation $X_i \mapsto X^{s_i}$. Thus $k^\alpha[S]$ is a quotient of a quantum space, which is noetherian by [MR01, Chapter 1, Theorem 2.9]. Now suppose $a, b \in k[S]$, with $a = X^s + (\text{terms of lower degree})$ and $b = X^{s'} + (\text{terms of lower degree})$. Then $ab = \alpha(s, s')X^{s+s'} + (\text{terms of lower degree}) \neq 0$ which shows that $k^\alpha[S]$ is a domain. □

For the moment we can prove the following partial case of the equivalence between Zhang twists and 2-cocycle twists.

Lemma 5.2.3. *Let S be an affine semigroup.*

1. *For every left twisting system τ on $k[S]$ there exists a 2-cocycle α_τ over S such that ${}^\tau k[S] = k^{\alpha_\tau}[S]$.*
2. *If $S = \mathbb{Z}^{r+1}$ then the assignation of the previous item is bijective.*
3. *Every 2-cocycle twist and every Zhang twist of $k[X^{\pm 1}]$ is isomorphic to a quantum torus, and all quantum tori arise this way.*

Proof. 1. This was already observed above.

2. Fix $\alpha \in C^2(\mathbb{Z}^{r+1}, k^\times)$. For every $\xi \in \mathbb{Z}^{r+1}$ set $\tau(\alpha)_\xi : k[X^{\pm 1}] \longrightarrow k[X^{\pm 1}]$ to be the graded automorphism that sends $X^{\xi'}$ to $\alpha(\xi', \xi)X^{\xi'}$ for every $\xi' \in \mathbb{Z}^{r+1}$. The fact that α is a 2-cocycle implies that $\tau(\alpha) = \{\tau(\alpha)_\xi \mid \xi \in \mathbb{Z}^{r+1}\}$ complies with condition (†). It follows from the definitions that $\alpha_{\tau(\alpha)} = \alpha$ and $\tau(\alpha_\tau) = \tau$.

3. By item 2, Zhang twists and 2-cocycle twists of $k[X^{\pm 1}]$ coincide, so we only need to prove the result for 2-cocycle twists.

   Given $\alpha \in C^2(\mathbb{Z}^{r+1}, k^\times)$, set $q_{ij}(\alpha) = \alpha(e_j, e_i)/\alpha(e_i, e_j)$ for every $0 \leq i < j \leq r$ and $\mathbf{q}(\alpha) = (q_{ij}(\alpha))_{0\leq i<j\leq r}$. There is a well defined morphism of $\mathbb{Z}^{r+1}$-graded algebras $\psi : k_{\mathbf{q}(\alpha)}[X^{\pm 1}] \longrightarrow k^\alpha[X^{\pm 1}]$ that sends X_i to X_i for every $0 \leq i \leq r$. Since $\mathbb{Z}^{r+1}$ is a completely ordered group, $k^\alpha[X^{\pm 1}]$ is a domain by Lemma 5.2.2. Thus for every $\xi \in \mathbb{Z}^{r+1}$ the element $\psi(X^\xi) \in k^\alpha[X^{\pm 1}]$ is a nonzero homogeneous element of degree ξ, which implies that ψ sends a basis to a basis. Hence every 2-cocycle twist of $k[X^{\pm 1}]$ is isomorphic to a quantum torus.

   On the other hand, given a system of parameters $\mathbf{q} = (q_{i,j})_{0\leq i<j\leq r}$, for every $\xi, \xi' \in \mathbb{Z}^{r+1}$ there is a unique nonzero scalar $\alpha_{\mathbf{q}}(\xi, \xi')$ such that $X^\xi \cdot X^{\xi'} = \alpha_{\mathbf{q}}(\xi, \xi')X^{\xi+\xi'}$ in $k_{\mathbf{q}}[X^{\pm 1}]$. The associativity of the product of $k_{\mathbf{q}}[X^{\pm 1}]$ implies that $\alpha_{\mathbf{q}}$ is a 2-cocycle and by definition $\mathbf{q}(\alpha_{\mathbf{q}}) = \mathbf{q}$, so $k_{\mathbf{q}}[X^{\pm 1}]$ is isomorphic to $k^{\alpha_{\mathbf{q}}}[X^{\pm 1}]$.

□

We now focus on proving a similar result for a general affine semigroup S, i.e. that 2-cocycle twists of $k[S]$ are always isomorphic to a Zhang twist of this algebra. Since we have identified $k[X^{\pm 1}]$ and $k[\mathbb{Z}^{r+1}]$, we may speak of 2-cocycle twists of $k[X^{\pm 1}]$, and given $\alpha \in C^2(\mathbb{Z}^{r+1}, k^\times)$ we write $k^\alpha[X^{\pm 1}]$ for $k^\alpha[\mathbb{Z}^{r+1}]$.

Lemma 5.2.4. *Every 2-cocycle twist of an affine semigroup algebra is a QA toric variety.*

Proof. Let S be an affine semigroup, fix $\alpha \in C^2(S, k^\times)$ and let $A = k^\alpha[S]$. Fixing an isomorphism $\mathbb{Z}S \cong \mathbb{Z}^{r+1}$, we can pull back the lexicographic order from $\mathbb{Z}^{r+1}$ to S, so S is a finitely generated and completely ordered semigroup. By Lemma 5.2.2, A is a noetherian domain. Also $k^\alpha[S]$ has an obvious $\mathbb{Z}S$-grading, which we can see as a $\mathbb{Z}^{r+1}$-grading through this isomorphism. For the rest of this proof we identify S with its image in $\mathbb{Z}^{r+1}$.

Let T denote the localization of A at the monomial basis $\mathcal{M} = \{X^s \mid s \in S\}$. Since $\mathcal{M}$ consists of normal regular elements, the natural morphism $A \longrightarrow T$ is injective, and since the elements of $\mathcal{M}$ are homogeneous, the $\mathbb{Z}^{r+1}$-grading of A induces a $\mathbb{Z}^{r+1}$-grading on T (see [NVO04, Proposition 8.1.2]). By standard localization theory, the elements of the form $X^s(X^t)^{-1}$ with $s, t \in S$ are normal and form a set of vector space generators of T.

Notice that if $s, s', t, t' \in S$ are such that $s - t = s' - t'$ then there exists $c \in k^\times$ such that $X^s(X^t)^{-1} = cX^{s'}(X^{t'})^{-1}$. Indeed, since $s + t' = s' + t$, we know that $\alpha(s', t)X^sX^{t'} = \alpha(s, t')X^{s'}X^t$. Multiplying both terms on the left side by $\alpha(s', t)^{-1}(X^t)^{-1}(X^{t'})^{-1}$ and using the fact that monomials commute up to a constant, we get the desired result. Since the homogeneous component T_ξ of degree $\xi \in \mathbb{Z}^{r+1}$ is generated over k by fractions of the form $X^s(X^t)^{-1}$ with $s - t = \xi$, we see that $\dim_k T_\xi$ is at most 1.

Since S generates $\mathbb{Z}^{r+1}$, for every $0 \leq i \leq r$ there exist $s_i, t_i \in S$ such that $e_i = s_i - t_i$. Let $X_i = X^{s_i}(X^{t_i})^{-1}$. The product $X_0^{\xi^0} X_1^{\xi^1} \dots X_r^{\xi^r}$ is a homogeneous element of degree $\xi = (\xi^0, \xi^1, \dots, \xi^r)$, and is nonzero since T is a domain, so it generates the component T_ξ. Thus T is generated as an algebra by $X_0^{\pm 1}, \dots, X_r^{\pm 1}$. As we have already seen, for every $0 \leq i < j \leq r$ there exist $q_{i,j} \in k^\times$ such that $X_j X_i = q_{i,j} X_i X_j$. Writing $\mathbf{q} = (q_{i,j})_{0 \leq i < j \leq r}$, we define $\psi : k_{\mathbf{q}}[X^{\pm 1}] \longrightarrow T$ to be the $\mathbb{Z}^{r+1}$-graded algebra morphism defined by the assignation $X_i \mapsto X_i$. By definition the image of the monomial $X^\xi \in k_{\mathbf{q}}[X^{\pm 1}]$ is $X^\xi \in T$, so ψ sends a basis to a basis, and hence is an isomorphism. □

Let S be a subsemigroup of $\mathbb{Z}^{r+1}$. We denote by $k_{\mathbf{q}}[S]$ the subalgebra of $k_{\mathbf{q}}[X^{\pm 1}]$ generated by monomials of the form X^s with $s \in S$. The following proposition characterizes QA toric varieties.

Proposition 5.2.5. *Let A be a $\mathbb{Z}^{r+1}$-graded algebra. The following are equivalent.*

1. *A is a QA toric variety with support S.*
2. *A is isomorphic to a Zhang twist of $k[S]$.*
3. *A is isomorphic to a 2-cocycle twist of $k[S]$.*

Proof. For $1 \Rightarrow 2$, fix an isomorphism between the homogeneous ring of fractions of A and a quantum torus $k_{\mathbf{q}}[X^{\pm 1}]$; thus we get an injective morphism of $\mathbb{Z}^{r+1}$-graded algebras $\psi : A \longrightarrow k_{\mathbf{q}}[X^{\pm 1}]$. Evidently the image of A is $B = k_{\mathbf{q}}[S] \subset k_{\mathbf{q}}[X^{\pm 1}]$. By item 3 of Lemma 5.2.3 there exists a twisting system τ over $k[X^{\pm 1}]$ such that $k_{\mathbf{q}}[X^{\pm 1}] = {}^{\tau}k[X^{\pm 1}]$. For every $\xi \in \mathbb{Z}^{r+1}$ set $\hat{\tau}_\xi = \tau_\xi|_B$, and denote by $\hat{\tau}$ the twisting system $\{\hat{\tau}_\xi \mid \xi \in \mathbb{Z}^{r+1}\}$. Evidently $B = {}^{\hat{\tau}}k[S]$, and we are finished with this implication.

Item 1 of Lemma 5.2.3 gives us $2 \Rightarrow 3$, while $3 \Rightarrow 1$ is Lemma 5.2.4. □

By the previous proposition, in order to classify QA toric varieties up to isomorphism it is enough to classify the 2-cocycle twists of a semigroup algebra, which we do now. As usual, any function $f : S \longrightarrow k^\times$ induces an element ∂f of C^2 by setting $\partial f(s, s') = \frac{f(s)f(s')}{f(s+s')}$. A cocycle that is equal to ∂f for some $f : S \longrightarrow k^\times$ is called a *2-coboundary*. We denote by $B^2 = B^2(S, k^\times) \subset C^2$ the subgroup of all 2-boundaries. We denote by $H^2 = H^2(S, k^\times)$ the quotient group C^2/B^2. Two elements of C^2 are said to be *cohomologous* if they define the same class in H^2. The following lemma shows that the group $H^2(S, k^\times)$ classifies twistings of $k[S]$ by 2-cocycles up to S-graded (or $\mathbb{Z}^{r+1}$-graded) isomorphism.

Lemma 5.2.6. *Let $\alpha, \beta \in C^2(S, k^\times)$. The algebras $k^\alpha[S]$ and $k^\beta[S]$ are isomorphic if and only if α and β define the same class in $H^2(S, k^\times)$.*

Proof. The existence of an S-graded algebra isomorphism $\varphi : k^\alpha[S] \longrightarrow k^\beta[S]$ is equivalent to the existence of a function $f : S \longrightarrow k^\times$ such that $\varphi(X^s) = f(s)X^s$ and

$$f(s+s')\alpha(s,s')X^{s+s'} = \varphi(X^sX^{s'}) = \varphi(X^s)\varphi(X^{s'}) = f(s)f(s')\beta(s,s')X^{s+s'}.$$

Such a function exists if and only if $\alpha = (\partial f)\beta$, i.e. if and only if α and β define the same class in $H^2(S, k^\times)$. □

It is easy to check that there is a scalar $a \in k^\times$ such that $a = \alpha(s, 0_S) = \alpha(0_S, s') = \alpha(0_S, 0_S)$ for all $s, s' \in S$. A 2-cocycle is said to be normalized if $a = 1$. We denote by C^2_{norm} the subgroup of C^2 formed by normalized cocycles. If α is a normalized cocycle, the unit of $k^\alpha[S]$ is X^{0_S}. Constant functions from $S \times S$ to $k^\times$ are in B^2, so α is cohomologous to $\tilde{\alpha} = \alpha(0_S, 0_S)^{-1}\alpha$. By the previous lemma $k^{\tilde{\alpha}}[S] \cong k^\alpha[S]$, so without loss of generality we can always assume that α is in C^2_{norm}.

Remark 5.2.7. Semigroup cohomology is defined in [CE99, chapter X], in a similar way to group cohomology. Let S be any semigroup and let G be the enveloping semigroup of S. By Proposition 4.1 of the given reference, if the natural morphism $i : S \longrightarrow G$ is injective then it induces an isomorphism $i^* : H^2(G, k^\times) \longrightarrow H^2(S, k^\times)$. Setting $\iota = i \times i : S \times S \longrightarrow G \times G$, this result implies that given $\alpha \in C^2(S, k^\times)$ one can always find a 2-cocycle $\beta \in H^2(G, k^\times)$ such that α and $\beta \circ \iota$ are cohomologous. Hence there is a morphism

$$k^\alpha[S] \xrightarrow{\cong} k^{\beta\circ\iota}[S] \hookrightarrow k^\beta[G]$$

where the last morphism sends X^s to $X^{i(s)}$. From this we deduce that if S is commutative and cancellative, every twist of $k[S]$ by a 2-cocycle is isomorphic to a subalgebra of a twist of $k[G]$ by a 2-cocycle, which is a generalization of Lemma 5.2.4.

5.2.2 Properties of QA toric varieties

We now prove some ring theoretic properties of QA toric varieties. Throughout this section A denotes a QA toric variety and its support is denoted by S. By Proposition 5.2.5, it is isomorphic to $k^\alpha[S]$ for some $\alpha \in C^2(S, k^\times)$, so we will assume $A = k^\alpha[S]$.

We begin this subsection by giving a presentation of QA toric varieties analogous to the one given for semigroup algebras in Theorem 5.1.4. As before we fix a minimal set of generators $s_1, \ldots, s_n$ of S, set $\pi : \mathbb{N}^n \longrightarrow S$ to be the semigroup morphism defined by the assignation $e_i \mapsto s_i$ for all $1 \leq i \leq n$, and fix a minimal set of generators $(p_1, p_1'), \ldots, (p_m, p_m')$ of $L(S) = \{(p, p') \in \mathbb{N}^n \times \mathbb{N}^n \mid \pi(p) = \pi(p') \text{ and } p <_{\mathsf{lex}} p'\} \cup \{(0,0)\}$.

Set $F = k\langle Y_i \mid i = 1, \ldots, n\rangle$ and let $\varphi : F \longrightarrow k^\alpha[S]$ be the morphism that maps Y_i to X^{s_i}. We fix an S-grading on F by setting the degree of Y_i equal to s_i, so φ is an S-graded algebra morphism. Given $p = (p^1, \ldots, p^n) \in \mathbb{N}^n$ we write Y^p for the monomial $Y_1^{p^1}Y_2^{p^2}\ldots Y_n^{p^n} \in F$. Notice that $\deg Y^p = \pi(p)$.

For $1 \leq i < j \leq n$ the element

$$C_{i,j} = \alpha(s_i, s_j)Y_jY_i - \alpha(s_j, s_i)Y_iY_j$$

is in the kernel of φ. Also, for every $s \in S$ and any $p \in \pi^{-1}(s)$ there exists $c_p \in k^\times$ such that $\varphi(Y^p) = \prod_i (X^{s_i})^{p_i} = c_p X^s$. We write $c_l = c_{p_l}$ and $c'_l = c_{p'_l}$ for all $1 \leq l \leq m$, and set

$$S_l = c_l Y^{p'_l} - c'_l Y^{p_l}.$$

These elements also belong to the kernel of φ.

Lemma 5.2.8. *Keep the notation from the previous paragraph. Let* $I \subset F$ *be the ideal generated by the elements* $C_{i,j}$ *and* S_l*, with* $1 \leq i < j \leq n$ *and* $1 \leq l \leq m$*. The twisted semigroup algebra* $k^\alpha[S]$ *is isomorphic to* F/I*.*

Proof. Set $B = F/I$. By abuse of notation we write Y_i and Y^p for the class of Y_i and Y^p in B for all $1 \leq i \leq n$ and all $p \in \mathbb{N}^n$. Denote by $\tilde{\varphi} : B \longrightarrow k^\alpha[S]$ the S-graded morphism induced by φ.

Clearly B is a quotient of a quantum affine space, so the set of monomials $\{Y^p \mid p \in \mathbb{N}^n\}$ is a set of homogeneous generators of B as a vector space. Furthermore $\deg Y^p = \pi(p)$, so for every $s \in S$ the homogeneous component B_s is generated by the set $\{Y^p \mid p \in \pi^{-1}(s)\}$. Suppose $p, p' \in \mathbb{N}^n$ are such that $\pi(p) = \pi(p')$, and assume without loss of generality that $p <_{\text{lex}} p'$, that is $(p, p') \in L(S)$. Then there exist $n_l \in \mathbb{N}$ such that $(p, p') = \sum_l n_l(p_l, p'_l) \in L$, and since $Y^{p_l} = \frac{c_l}{c'_l} Y^{p'_l}$ in B, there exist nonzero scalars d, d', d'' such that

$$Y^p = d\prod_{l=1}^{m}(Y^{p_l})^{n_l} = d\prod_{l=1}^{m}\left(\frac{c_l}{c'_l}Y^{p'_l}\right)^{n_l} = d'\prod_{l=1}^{m}(Y^{p'_l})^{n_l} = d''Y^{p'}.$$

This implies that for every $s \in S$, any two monomials of degree s in B are nonzero multiples of each other. Hence, $\dim_k B_s = 1$ and $\tilde{\varphi}$ is a surjective morphism between S-graded vector spaces whose homogeneous components have the same dimension, which implies that it is an isomorphism. □

We now recall the definition of a maximal order for noncommutative algebras. Our objective is to prove that, just as in the commutative case, a QA toric variety A is a maximal order if and only if S is normal. Recall that a semigroup is normal if for every $n \in \mathbb{N}$ and every $s \in \mathbb{Z}S$ such that $ns \in S$ then $s \in S$.

Definition 5.2.9. [MR01, section 3.1] A k algebra Q is called a *quotient ring* if every regular element in the ring is a unit. A subring $B \subset Q$ is called a *right order* in Q if every element of Q is of the form rs^{-1} for some $r, s \in B$.

Two orders $B_1, B_2 \subset Q$ are said to be *equivalent* if there exist $a, a', b, b' \in Q$ such that $aB_1b \subset B_2$ and $a'B_2b' \subset B_1$. This defines an equivalence relation on orders of a fixed quotient ring Q. An order is said to be a *maximal order* if it is maximal in its class of equivalence, ordered by inclusion.

The following lemma gathers several results on maximal orders. In particular item 1 shows that one may consider the property of being a maximal order as a noncommutative analogue of being integrally closed, at least in the noetherian case.

Lemma 5.2.10. *1. A commutative noetherian integral domain is a maximal order in its quotient field if and only if it is integrally closed.*

2. *Let A be a filtered ring with associated graded ring $\operatorname{gr} A$. If $\operatorname{gr} A$ is a noetherian integral domain and a maximal order, then so is A.*

3. *Let A be an noetherian integral domain and a maximal order in its quotient ring. Then for every automorphism σ of A the Ore extension $A[X, \sigma]$ and its localization $A[X, \sigma][X^{-1}]$ are maximal orders.*

4. *Suppose that A is a noetherian domain, with a family of Ore subsets $\{U_i \subset A \mid i = 1, \dots, n\}$ such that $A = \bigcap_{i=1}^{n} A[U_i^{-1}]$ and $A[U_i^{-1}]$ is a maximal order for all i. Then A is a maximal order.*

Proof. 1. See [MR01, Proposition 5.1.3].

2. See [MR01, 5.1.6].

3. This is a consequence of [MR80, Chapitre V, Corollaire 2.6] and [MR80, Chapitre IV, Proposition 2.1].

4. Set $T_i = A[U_i^{-1}]$ for $1 \leq i \leq n$, and suppose there is an order R equivalent to A such that $A \subset R$. Since the U_i are Ore sets, T_i is in the same equivalence class as $R[U_i^{-1}]$, and since the former is a maximal order, $R[U_i^{-1}] \subset T_i$. Thus

$$R \subset \bigcap_{i=1}^{n} R[U_i^{-1}] \subset \bigcap_{i=1}^{n} T_i = A,$$

which implies that $R = A$.

□

The following lemma shows that QA toric varieties are maximal orders if and only if the corresponding classical varieties are integrally closed.

Lemma 5.2.11. *Let S be an affine semigroup. The following conditions are equivalent:*

1. *The semigroup S is normal.*

2. *For every* $\alpha \in C^2(S, k^\times)$ *the algebra* $k^\alpha[S]$ *is a maximal order.*

3. *The algebra* $k[S]$ *is a maximal order.*

Proof. The implication $2 \Rightarrow 3$ is obvious, and $3 \Rightarrow 1$ is proved in [BH93, chapter 6, page 258]. To prove $1 \Rightarrow 2$ we choose an isomorphism $\mathbb{Z}S \cong \mathbb{Z}^{r+1}$ and restrict to the case where $S \subset \mathbb{Z}^{r+1}$ is a normal subsemigroup.

Recall from Proposition 5.2.5 that for every $\alpha \in C^2(S, k^\times)$ there exists $\mathbf{q} \in (k^\times)^{\binom{r+1}{2}}$ such that $k^\alpha[S] \cong k_{\mathbf{q}}[S]$, so it is enough to prove that every algebra of the form $k_{\mathbf{q}}[S]$ is a maximal order.

By item 4 of Lemma 5.1.2

$$S = \bigcap_{\tau < \mathbb{R}_+ S} S_\tau,$$

from which we deduce that $k_{\mathbf{q}}[S] = \bigcap_{\tau<\mathbb{R}_+S} k_{\mathbf{q}}[S_\tau]$. Recall that for every facet τ, the semigroup S_τ is obtained by adjoining to S the inverses of every element of $S \cap \tau$, so $k_{\mathbf{q}}[S_\tau]$ is the localization of $k_{\mathbf{q}}[S]$ at the multiplicative set generated by $\{X^s \mid s \in \tau\}$; this set consists of regular normal elements, and hence is an Ore set. By item 5 of Lemma 5.1.2, there exists an isomorphism $S_\tau \cong \mathbb{N} \oplus \mathbb{Z}^r$, and through this isomorphism we can identify $k_{\mathbf{q}}[S_\tau]$ with $k_{\mathbf{q}'}[Y_0, Y_1^{\pm 1}, \ldots, Y^{\pm n}]$ for some system of parameters $\mathbf{q}'$. By item 3 of Lemma 5.2.10 $k_{\mathbf{q}}[S_\tau]$ is a maximal order, and item 4 of the same lemma implies that so is $k_{\mathbf{q}}[S]$. □

We now focus on the homological regularity of twisted semigroup algebras. Recall that we denote by $\mathbb{R}_+S \subset \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}S$ the convex cone of S. We denote by $\operatorname{relint} S$ the set of points of S that lie in the topological interior of $\mathbb{R}_+S$. Recall also that we say that S is positive if has no invertible element other than 0, and that in this case S can be embedded in $\mathbb{N}^{r+1}$, see Lemma 5.1.3.

If A is a QA toric variety whose support is a positive affine semigroup, we can use this last result to give A an $\mathbb{N}^{r+1}$-grading, and with this new grading A becomes a connected $\mathbb{N}^{r+1}$-graded algebra. The following result, which is an immediate consequence of Proposition 5.2.5 and the theory from chapters 2 and 3, summarizes the homological regularity properties of QA toric varieties with positive support.

Proposition 5.2.12. *Let A be a QA toric variety. Suppose that its support is a positive affine semigroup and put on A the $\mathbb{N}^{r+1}$-grading described in the preamble to this statement. Then*

1. *A is a noetherian integral domain.*

2. *A has property χ and finite local dimension. It also has a balanced dualizing complex.*

3. *If S is normal then A is AS-Cohen-Macaulay and a maximal order.*

4. *If S is normal and there exists $s \in S$ such that* $\operatorname{relint} S = s + S$, *then A is AS Gorenstein.*

Proof. By Proposition 5.2.5 there exist a twisting system τ over $k[S]$ and a 2-cocycle α over S such that $A \cong {}^{\tau}k[S] = k^{\alpha}[S]$.

1. This is 5.2.2.

2. The algebra $k[S]$ is a finitely generated noetherian commutative algebra of finite Krull dimension; by Corollaries 4.2.8 and 4.2.9 $k[S]$ has property χ, finite local dimension, and a balanced dualizing complex. Since all this properties are twisting invariant (see Proposition 4.2.10), item 2 follows.

3, 4. Recall from Proposition 3.2.8 that being AS-Cohen-Macaulay and AS-Gorenstein are also twisting invariant properties, so we may reduce again to proving the desired results for $A = k[S]$. In that case 3 is due to Hochster and 4 to Danilov and Stanley, see [BH93, Theorem 6.3.5 and Corollary 6.3.8] for proofs.

□

5.3 Quantum affine toric degenerations

In this section we introduce a class of algebras with "quantum toric degenerations", that is, filtrations such that their associated graded rings are quantum toric varieties.

Throughout this section S denotes a positive affine semigroup. From Lemma 5.1.3 we know that S has a unique minimal generating system, and if $\operatorname{rk} S = r + 1$ then S can be embedded inside $\mathbb{N}^{r+1}$. Let $\{s_1, \dots, s_n\}$ be the minimal set of generators of S. As in the previous sections, we denote by $\pi : \mathbb{N}^n \longrightarrow S$ the semigroup morphism defined by the assignation $e_i \mapsto s_i$ and by $L(S)$ the affine semigroup

$$L(S) = \{(p, p') \in \mathbb{N}^n \times \mathbb{N}^n \mid \pi(p) = \pi(p') \text{ and } p <_{\mathsf{lex}} p'\} \cup \{(0,0)\}.$$

Again by Lemma 5.1.3 there exists a unique minimal set of generators $\{(p_1, p_1'), \dots, (p_m, p_m')\}$ of $L(S)$.

5.3.1 Algebras dominated by a semigroup

By a *section* of π we mean a function $t : S \longrightarrow \mathbb{N}^n$ such that $\pi \circ t = \mathrm{Id}_S$. Notice that the minimality of the generating set implies that $t(s_i) = e_i$.

Definition 5.3.1. Let A be a connected $\mathbb{N}$-graded algebra, with homogeneous generators $b_1, \dots, b_n$. Given $p = (p^1, \dots, p^n) \in \mathbb{N}^n$, we write b^p for $\prod_{i=1}^{n} b_i^{p^i} \in A$. Let

$t : S \longrightarrow \mathbb{N}^n$ be a section of π and let $\varphi : S \longrightarrow \mathbb{N}$ be a semigroup morphism such that $\varphi(s_i) > 0$ for every $1 \leq i \leq n$. We say that A is an (S, t, φ)-*dominated algebra* if the following conditions hold:

1. For every $1 \leq i \leq n$ the element b_i is homogeneous of positive degree.
2. The set $\{b^{t(s)} \mid s \in S\}$ is linearly independent.
3. For every $1 \leq i < j \leq n$ there exist $c_{i,j} \in k^\times$ and $c_{i,j}^s \in k$ such that

$$b_j b_i = c_{i,j} b_i b_j + \sum_{\varphi(s) < \varphi(s_i + s_j)} c_{i,j}^s b^{t(s)}.$$

4. For every $1 \leq l \leq m$ there exist $c_l \in k^\times$ and c_l^s such that

$$b^{p'_l} = c_l b^{p_l} + \sum_{\varphi(s) < \varphi(\pi(p_l))} c_l^s b^{t(s)}.$$

We will say that A is an S-dominated algebra if there exist t and φ as in the statement such that A is an (S, t, φ)-dominated algebra.

Remark 5.3.2. From this point on, we will refer to the equalities arising from point 3 of the previous definition as *commutation relations* of A, and to those arising from point 4 as *straightening relations*.

For the rest of this section A denotes an (S, t, φ)-dominated algebra, where t and φ are as in Definition 5.3.1. Our objective is to prove that A can be endowed with the structure of a GF-algebra such that its associated graded algebra is a connected re-grading of a 2-cocycle twist of the semigroup algebra $k[S]$. By the theory developed in subsection 3.2.2 this will imply that A inherits the regularity properties of $k[S]$.

We write $\tilde{\varphi}$ for the composition $\varphi \circ \pi$. Notice that since $\tilde{\varphi}(e_i) > 0$, for any $u \in \mathbb{N}$ there are only finitely many $p \in \mathbb{N}^n$ such that $\tilde{\varphi}(p) \leq u$, and so the subspace

$$F_u A = \langle b^p \mid \tilde{\varphi}(p) \leq u \rangle \subset A$$

is finite dimensional. The next two lemmas show that these are indeed the layers of a filtration as described above.

Lemma 5.3.3. *The family* $\mathcal{F} = \{F_v A\}_{v \in \mathbb{N}}$ *is a GF-filtration of* A. *More precisely, for all* $p, q \in \mathbb{N}^n$ *there exists* $c \in k^\times$ *such that* $b^p b^q \equiv c b^{p+q} \mod F_{\tilde{\varphi}(p+q)-1} A$.

Proof. We wish to prove that $F_u A \cdot F_v A \subset F_{u+v} A$ for all $u, v \in \mathbb{N}$. We proceed by induction on $u + v$, with the base case $u + v = 0$ being obvious.

Now let $u, v \in \mathbb{N}$ and suppose that for all $p, q \in \mathbb{N}^n$ such that $\tilde{\varphi}(p + q) < u + v$ the congruence of the statement holds. Fix $p, q \in \mathbb{N}^n$ such that $\tilde{\varphi}(p + q) = u + v$, and let i

be the largest integer such that $q^i \neq 0$. Thus $b^p b^q = (b^p b^{q-e_i}) b_i$, and since $\tilde{\varphi}(e_i) > 0$ we may use the inductive hypothesis to deduce that

$$b^p b^{q-e_i} = c_1 b^{p+q-e_i} + \sum_{\tilde{\varphi}(r)<\tilde{\varphi}(p+q-e_i)} c_1^r b^r.$$

where c_1 and all c_1^r are nonzero constants. Multiplying each side by b_i we obtain

$$b^p b^{q-e_i} b_i = c_1 b^{p+q-e_i} b_i + \sum_{\tilde{\varphi}(r)<\tilde{\varphi}(p+q-e_i)} c_1^r b^r b_i$$

and since $\tilde{\varphi}(r) + \tilde{\varphi}(e_i) < \tilde{\varphi}(p) + \tilde{\varphi}(q) = u + v$, we may apply the inductive hypothesis to conclude that $b^p b^q \equiv c_1 b^{p+q-e_i} b_j \mod F_{u+v-1}A$. Hence we have reduced the problem to show that there exists a nonzero constant c_2 such that $b^{p+q-e_i} b_i \equiv c_2 b^{p+q} \mod F_{u+v-1}A$, or in other words it is enough to prove the case $q = e_i$.

Let j be the largest integer such that $p^j \neq 0$. If $j \leq i$ then $b^p b_i = b^{p+e_i}$ and there is nothing left to do. On the other hand, if $i < j$ then we may apply the corresponding commutation relation to obtain

$$b^p b_i = b^{p-e_j} b_j b_i = c_{i,j}(b^{p-e_j} b_i) b_j + \sum_{\varphi(s)<\varphi(s_i+s_j)} c_{i,j}^s b^{p-e_j} b^{t(s)}$$

which, again by repeated application of the inductive hypothesis, is congruent to $c_2 b^{p-e_j+e_i} b_j$ modulo $F_{u+v-1}A$. Repeating this procedure we reduce the problem to the case where $i \leq j$, which we have already considered. □

Lemma 5.3.4. *For every $p \in \mathbb{N}^n$ there exists a non-zero constant d such that $b^p \equiv d b^{t(\pi(p))} \mod F_{\tilde{\varphi}(p)-1}A$.*

Proof. Let $s = \pi(p)$. Since $(t(s), p) \in L(S)$, for each $1 \leq l \leq m$ there exist $n_l \in \mathbb{N}$ such that $(t(s), p) = \sum_l n_l(p_l, p_l')$. By repeated application of Lemma 5.3.3, there exist $c_1, c_2 \in k^\times$ such that

$$\prod_{l=1}^m (b^{p_l})^{n_l} \equiv c_1 b^{t(s)} \mod F_{\varphi(s)-1}A \quad \text{and} \quad \prod_{l=1}^m (b^{p_l'})^{n_l} \equiv c_2 b^p \mod F_{\varphi(s)-1}A.$$

On the other hand the straightening relations of the algebra imply that $b^{p_l'} \equiv c_l b^{p_l} \mod F_{\tilde{\varphi}(p_l)-1}A$, which in turn implies that there exists $c_3 \in k^\times$ such that

$$\prod_{l=1}^m (b^{p_l'})^{n_l} \equiv c_3 \prod_{l=1}^m (b^{p_l})^{n_l} \mod F_{\varphi(s)-1}A.$$

This proves that the congruence of the statement holds with $d = c_2^{-1} c_3 c_1$. □

Before we prove that $\mathrm{gr}_{\mathcal{F}} A$ is isomorphic as an algebra to a QA toric variety, we gather several consequences of the previous lemmas.

Proposition 5.3.5. *Let $\mathcal{F} = \{F_u A\}_{u\in\mathbb{N}}$.*

1. *The algebra A is a connected GF-algebra with filtration $\mathcal{F}$.*
2. *For every $u \in \mathbb{N}$ we have $F_u A = \langle b^{t(s)} \mid \varphi(s) \leq u\rangle$, and the set $\{b^{t(s)} \mid s \in S\}$ is a basis of A.*
3. *For every $p \in \mathbb{N}^n$ the class of b^p in $F_{\tilde{\varphi}(p)}A/F_{\tilde{\varphi}(p)-1}A$ is non zero.*
4. *For every $1 \leq l \leq m$ we have $\deg b^{p_l} = \deg b^{p'_l}$.*
5. *The commutation and straightening relations give a presentation of A as a graded algebra.*

Proof. 1. As we have already seen, $\mathcal{F}$ is a filtration and by definition its layers are generated by homogeneous elements, so they are graded sub-vector spaces of A. The fact that $A_0 = F_0 A = k$ follows from the definitions. To see that $\mathcal{F}$ is exhaustive it is enough to show that an arbitrary product of the generators $b_1, \ldots, b_n$ lies in $F_u A$ for some $u \in \mathbb{N}$, which follows from Lemma 5.3.3.

2. By Lemma 5.3.4, every element $b^p \in F_u A$ is congruent modulo $F_{u-1}A$ to a nonzero multiple of $b^{t(\pi(p))}$, so the set $\{b^{t(s)} \mid \varphi(s) \leq u\}$ generates $F_u A$, and $A = \bigcup_{u\in\mathbb{N}} F_u A = \langle b^{t(s)} \mid s \in S\rangle$. Since by hypothesis the set is linearly independent, it is a basis of A.

3. Since $\{b^{t(s)} \mid s \in S\}$ is a linearly independent set, the previous item implies that $b^{t(s)} \notin F_{\varphi(s)-1}A$. The statement now follows from Lemma 5.3.4.

4. Let l be as in the statement. Since the vector space generated by the $b^{t(s)}$ with $c_l^s \neq 0$ is contained in $F_{\tilde{\varphi}(p)-1}A$, the previous item implies that b^{p_l} is linearly independent of the other terms in the right hand side of the corresponding straightening relation. Since the left hand side is homogeneous of degree $\deg b^{p'_l}$, then b^{p_l} and every $b^{t(s)}$ with $c_l^s \neq 0$ must be of degree $\deg b^{p'_l}$.

5. Let F be the free algebra on generators $X_1, \ldots, X_n$, and set $\deg X_i = \deg b_i$ for all $1 \leq i \leq n$. For every $p \in \mathbb{N}^n$ denote by X^p the element $X_1^{p^1} X_2^{p^2} \ldots X_n^{p^n}$, and let $B = F/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by

$$X_j X_i - c_{i,j} X_i X_j - \sum_{\varphi(s)<\varphi(s_i+s_j)} c_{i,j}^s X^{t(s)} \qquad \text{for all } 1 \leq i < j \leq n$$

$$X^{p'_l} - c_l X^{p_l} - \sum_{\varphi(s)<\tilde{\varphi}(p_l)} c_l^s X^{t(s)} \qquad \text{for all } 1 \leq l \leq m.$$

By abuse of notation we denote the class of X^p in B again by X^p.

We claim that B is an (S, t, φ)-dominated algebra. Let $\rho : B \longrightarrow A$ be the morphism induced by the assignation $X_i \mapsto b_i$ for all $1 \leq i \leq n$. Since $\rho(X^{t(s)}) = b^{t(s)}$,

the set $\{X^{t(s)} \mid s \in S\}$ is linearly independent, and it is clear by definition that B has the necessary commutation and straightening relations. Thus $\{X^{t(s)} \mid s \in S\}$ is a basis of B by the previous item, and since ρ maps a basis of B to a basis of A, it is an isomorphism.

□

We are now ready to prove the main result of this section.

Theorem 5.3.6. *There exist a group morphism $\psi : \mathbb{Z}S \longrightarrow \mathbb{Z}^2$ with $\psi(S) \subset \mathbb{N}^2$ and a 2-cocycle α over S such that $\operatorname{gr}_{\mathcal{F}} A \cong \psi_!(k^\alpha[S])$.*

Proof. Let $B = \operatorname{gr}_{\mathcal{F}} A$. For every $p = (p^1, \ldots, p^n) \in \mathbb{N}^n$ we denote by $(\operatorname{gr} b)^p$ the element $(\operatorname{gr} b_1)^{p^1}(\operatorname{gr} b_2)^{p^2} \ldots (\operatorname{gr} b_n)^{p^n}$. Item 3 of Proposition 5.3.5 implies that $(\operatorname{gr} b)^p = \operatorname{gr} b^p$, and so by item 2 of the same proposition and item 2 of Lemma 2.4.6, $\{(\operatorname{gr} b)^{t(s)} \mid s \in S\}$ is a basis of B. Given $s, s' \in S$, we know by Lemma 5.3.4 that $\operatorname{gr} b^{t(s)} \operatorname{gr} b^{t(s')} = \alpha(s, s') \operatorname{gr} b^{t(s+s')}$ for some nonzero constant $\alpha(s, s')$. We have thus defined a function $\alpha : S \times S \longrightarrow k^\times$, and the associativity of the product of B implies that α is a 2-cocycle.

Let $\rho : k^\alpha[S] \longrightarrow B$ be the vector space isomorphism defined by sending X^s to $\operatorname{gr} b^{t(s)}$ for all $s \in S$; it is clear by definition that ρ is multiplicative, so it is an algebra isomorphism. The assignation $S \longrightarrow \mathbb{N}^2$ given by $s \mapsto \deg \operatorname{gr} b^{t(s)} = (\varphi(s), \deg b^{t(s)})$ is additive by item 4 of Proposition 5.3.5, and we denote its extension to the enveloping groups by $\psi : \mathbb{Z}S \longrightarrow \mathbb{Z}^2$. By construction ρ induces an isomorphism of $\mathbb{N}^2$-graded algebras between $\psi_!(k^\alpha[S])$ and B. □

Remark 5.3.7. Notice that the $\mathbb{N}$-filtration $\mathcal{F}$ has an associated graded algebra which has a natural S-filtration, which is much finer (the original filtration of the (S, φ, t)-dominated algebra does not play any role in this fact). This fact can be explained as follows. We endow S with the order $\leq^*$, where $s' \leq^* s$ if and only if $s - s' \in S$ and $\varphi(s') \leq \varphi(s)$. Setting $F_s A = \langle b^p \mid \pi(p) \leq^* s \rangle$ we get an S-filtration of A with respect to the partial order $\leq^*$, with $F_s A / F_{<s} A = \langle \operatorname{gr} b^{t(s)} \rangle$ for all $s \in S$; the associated S-graded algebra is isomorphic to $k^\alpha[S]$ by a similar reasoning as above. The filtration $\mathcal{F}$ can be recovered by setting $F_u A = \bigcup_{\varphi(s) \leq u} F_s A$ for each $u \in \mathbb{N}$. The properties defining the class of (S, t, φ)-dominated algebras guarantee that the graded algebras associated to both filtrations have the same defining relations. We have preferred to work with the $\mathbb{N}$-filtration since the theory developed in section 2.4.2 gets considerably more technical for algebras filtered by partially ordered semigroups.

We now use the transfer results from chapter 3 to show that S-dominated algebras inherit the regularity properties of their associated QA toric varieties.

Corollary 5.3.8. *Let A be an S-dominated algebra. Then the following hold*

1. *A is a noetherian integral domain.*

2. A *has property* χ *and finite local dimension. It also has a balanced dualizing complex.*

3. *If* S *is normal, then* A *is AS-Cohen-Macaulay and a maximal order.*

4. *If* S *is normal and there exists* $s \in S$ *such that* $\operatorname{relint} S = s + S$*, then* A *is AS-Gorenstein.*

Proof. By Proposition 5.2.12, the statement holds when A is a QA toric variety. For a general S-dominated algebra, we use the fact that all the properties we mention transfer from $\operatorname{gr}_{\mathcal{F}} A$ to A.

For the first item, see [MR01, Proposition 1.6.6 and Theorem 1.6.9]. The transfer of the properties from the second item follows from Theorem 4.2.12. Since $k^{\alpha}[S]$ is noetherian and has property χ, the AS Cohen-Macaulay and AS Gorenstein properties transfer by Theorem 3.2.13. □

Remark 5.3.9. Notice that if $k^{\alpha}[S]$ is regular then A is also regular. However it is rare for affine toric varieties to be regular affine varieties. By [CLS11, Theorem 1.3.12], this happens if and only if the minimal system of generators of the associated semigroup is free over $\mathbb{Z}$.

In the next two subsections we introduce two subclasses of S-dominated algebras.

5.3.2 Algebras with S-bases

The following definition is inspired in the results found in [Cal02, section 2]. Throughout this subsection we assume that we have fixed an embedding $S \hookrightarrow \mathbb{N}^{r+1}$. Just for this section, we identify S with its image inside $\mathbb{N}^{r+1}$, and write $<$ for the lexicographic order of $\mathbb{N}^{r+1}$.

Definition 5.3.10. Let B be a connected $\mathbb{N}$-graded algebra. We say that B has a *homogeneous* S*-basis* if B has a basis $\{b_s \mid s \in S\}$ consisting of homogeneous elements of B with the following property: for all $s, s', s'' \in S$ with $s'' < s + s'$ there exist $d_{s,s'}^{s''} \in k$ and $d_{s,s'}^{s+s'} \in k^{\times}$ such that

$$b_s b_{s'} = d_{s,s'}^{s+s'} b_{s+s'} + \sum_{s''<s+s'} d_{s,s'}^{s''} b_{s''}.$$

For the rest of this section B denotes a connected $\mathbb{N}$-graded algebra with an S-basis. Define a function $\alpha : S \times S \longrightarrow k^{\times}$ by $\alpha(s, s') = d_{s,s'}^{s+s'}$ for all $s, s' \in S$. By comparing the two associators $(b_s b_{s'}) b_{s''}$ and $b_s (b_{s'} b_{s''})$, we see that for every $\sigma < s + s' + s''$ there exist $d_\sigma, d'_\sigma \in k$ such that

$$\begin{aligned} &\alpha(s, s')\alpha(s + s', s'') b_{s+s'+s''} + \sum_{\sigma<s+s'+s''} d_\sigma b_\sigma \\ &\qquad = \alpha(s, s' + s'')\alpha(s', s'') b_{s+s'+s''} + \sum_{\sigma<s+s'+s''} d'_\sigma b_\sigma. \end{aligned}$$

Since the standard elements are linearly independent, α must be a 2-cocycle.

For every $s \in S$ set $F_sB = \langle b_{s'} \mid s' \leq s \rangle$. The definition of an S-basis implies that the family $\mathcal{F} = \{F_sB\}_{s\in S}$ is an S-filtration of B. Clearly $F_sB = \langle b_s \rangle \oplus F_{<s}B$, so $\operatorname{gr} b_s$ generates the component of degree s of $\operatorname{gr}_{\mathcal{F}} B$, and it is also clear that $\operatorname{gr} b_s \operatorname{gr} b_{s'} = \alpha(s, s') \operatorname{gr} b_{s+s'}$. Thus the vector space isomorphism $k^\alpha[S] \longrightarrow \operatorname{gr}_{\mathcal{F}} B$ which sends X^s to $\operatorname{gr} b_s$ is an isomorphism of S-graded algebras.

Before proving that B is an S-dominated algebra, we prove a simple lemma.

Lemma 5.3.11. *Let $\mathcal{C} \subset \mathbb{N}^{r+1}$ be a finite set. Then there exists a semigroup morphism $\varphi : \mathbb{N}^{r+1} \longrightarrow \mathbb{N}$ such that for every $c, c' \in \mathcal{C}$, the inequality $c < c'$ holds if and only if $\varphi(c) < \varphi(c')$.*

Proof. The proof is taken from [Calo2, Lemma 3.2]. Since $\mathcal{C}$ is a finite set, there exists a natural number K such that $\mathcal{C}$ is contained inside the $r+1$-cube $[0, K-1]^{r+1}$. The morphism $\varphi : \mathbb{N}^{r+1} \longrightarrow \mathbb{N}$ defined by the assignation $e_i \mapsto K^{r-i}$ for all $0 \leq i \leq r$ sends every element c inside the cube to the integer whose K-adic expansion is c, so it respects the lexicographic order when restricted to the cube. □

We now proceed with the proof of the main result.

Proposition 5.3.12. *Suppose B has an S-basis. Then for every section t of π there exists a semigroup morphism φ such that B is an (S, t, φ)-dominated algebra.*

Proof. Let $b_i = b_{s_i}$ for every $1 \leq i \leq n$ and let $t : S \longrightarrow \mathbb{N}^n$ be any section of π. For every $p \in \mathbb{N}^n$ we write $(\operatorname{gr} b)^p$ for $(\operatorname{gr} b_1)^{p^1} (\operatorname{gr} b_2)^{p^2} \dots (\operatorname{gr} b_n)^{p^n}$. Since $\operatorname{gr}_{\mathcal{F}} B \cong k^\alpha[S]$ as an S-graded algebra, it is a domain and so $\operatorname{gr} b^p = (\operatorname{gr} b)^p$ for all $p \in \mathbb{N}^n$. Furthermore this is a nonzero homogeneous element of degree $\pi(p)$, so $\{\operatorname{gr} b^{t(s)} \mid s \in S\}$ is a homogeneous basis of $\operatorname{gr}_{\mathcal{F}} B$. By Lemma 2.4.3 the set $\{b^{t(s)} \mid s \in S\}$ is a basis of B and $F_sA = \langle b^{t(s')} \mid s' \leq s \rangle$, which proves points 1 and 2 of the definition of an S-algebra dominated by t.

Since $\operatorname{gr}(b_i b_j) = \frac{\alpha(s_i, s_j)}{\alpha(s_j, s_i)} \operatorname{gr}(b_j b_i)$ for all $1 \leq i < j \leq n$ and for every $1 \leq l \leq m$ there exists $c_l \in k^\times$ such that $\operatorname{gr} b^{p'_l} = c_l \operatorname{gr} b^{p_l}$, we deduce that for every $s < s_i + s_j$ or $s < \pi(p_l)$ exist constants $c^s_{i,j}$ and c^s_l such that

$$b_j b_i = \frac{\alpha(s_i, s_j)}{\alpha(s_j, s_i)} b_i b_j + \sum_{s<s_i+s_j} c^s_{i,j} b^{t(s)} \qquad \text{for all } 1 \leq i < j \leq n,$$

$$b^{p'_l} = c_l b^{p_l} + \sum_{s<\pi(p_l)} c^s_l b^{t(s)} \qquad \text{for all } 1 \leq l \leq m.$$

Let $\mathcal{C} \subset S$ be the finite set formed by

- the elements $s_i + s_j$, for $1 \leq i < j \leq n$;

- the elements of the form $\pi(p_l)$ for $1 \leq l \leq m$;

- the elements $s \in S$ such that $c^s_{i,j} \neq 0$ for some $1 \leq i < j \leq n$;

- the elements $s \in S$ such that $c^s_l \neq 0$ for some $1 \leq l \leq m$.

Applying Lemma 5.3.11 we know there exists a morphism $\varphi : \mathbb{N}^{r+1} \longrightarrow \mathbb{N}$ such that for every $s, s' \in \mathcal{C}$, the inequality $s < s'$ holds if and only if $\varphi(s) < \varphi(s')$. Restricting φ to S, it is clear that B is an (S, t, φ)-dominated algebra. □

5.3.3 Lattice semigroups and their dominated algebras

Let $(\Pi, \leq)$ be a partially ordered set (poset). Given $x, y \in \Pi$ the interval $[x, y]$ is the set $[x, y] = \{z \in \Pi \mid x \leq z \leq y\}$. Clearly, it is non-empty if and only if $x \leq y$.

A subset $\Omega \subset \Pi$ is called a Π-*ideal* provided it satisfies the following condition: for all $w \in \Omega$ and all $p \in \Pi$, if $p \leq w$ then $p \in \Omega$. A Π°-*ideal* is an ideal for the reverse order, that is, a subset $\Omega \subset \Pi$ such that whenever there exist $w \in \Omega$ and $p \in \Pi$ such that $w \leq p$, then $p \in \Omega$.

Let $(\Pi, \leq)$ be a finite ordered set. For any $x \in \Pi$, the *rank* of x, denoted $\operatorname{rk} x$, is defined to be the greatest integer t such that there exists a strictly increasing sequence $x_0 < \cdots < x_t = x$ in Π. The rank of Π, denoted $\operatorname{rk} \Pi$, is the maximum of the ranks of the elements of Π.

For every pair of elements $x, y \in \Pi$ we define

$$R(x, y) = \{(z, z') \in \Pi \times \Pi \mid z \leq x, y \leq z'\}.$$

A *lattice* is a poset $(\mathcal{L}, \leq)$ satisfying the following condition: for any $x, y \in \mathcal{L}$, there exist elements $x \wedge y, x \vee y \in \mathcal{L}$ such that $x \wedge y \leq x, y \leq x \vee y$, and for all $z, z' \in \mathcal{L}$ such that $z \leq x, y \leq z'$, then $z \leq x \wedge y \leq x \vee y \leq z'$; in other words, $(x \wedge y, x \vee y) \in R(x, y)$ and $R(x, y) = R(x \wedge y, x \vee y)$. We write $R(x, y)^*$ for $R(x, y) \setminus (x \wedge y, x \vee y)$.

If the ordered set $(\mathcal{L}, \leq)$ is a lattice there exist two binary operations

$$\begin{aligned} \wedge : \mathcal{L} \times \mathcal{L} &\longrightarrow \mathcal{L} & \qquad \vee : \mathcal{L} \times \mathcal{L} &\longrightarrow \mathcal{L} \\ (x, y) &\longmapsto x \wedge y & (x, y) &\longmapsto x \vee y \end{aligned}$$

called *meet* and *joint*, respectively. The lattice $(\mathcal{L}, \leq)$ is said to be *distributive* if the operation $\vee$ is distributive with respect to the operation $\wedge$, that is, if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in \mathcal{L}$; clearly the meet and join operations are commutative. A finite lattice is a lattice whose underlying set is finite. Clearly, a finite lattice has a unique minimal and a unique maximal element, which are denoted by $\hat{0}$ and $\hat{1}$, respectively. If P is a finite poset then the set $J(P)$ of all poset ideals of P is a finite distributive lattice with union as join and intersection as meet.

A sub-lattice of $\mathcal{L}$ is a subposet $\mathcal{L}' \subset \mathcal{L}$ which is stable under the maps $\wedge$ and $\vee$. A morphism of lattices is a morphism of ordered sets which commutes in the obvious way with the join and meet operations.

Let $\mathcal{L}$ be a lattice. An element $z \in \mathcal{L}$ is called *join-irreducible* provided it is not minimal and satisfies the following condition: given $x, y \in \mathcal{L}$ such that $z = x \vee y$, then either $z = x$, or $z = y$. We denote by $\mathcal{L}^{\mathrm{irr}}$ the poset of join-irreducible elements of $\mathcal{L}$ and set $\mathcal{L}_0^{\mathrm{irr}} = \mathcal{L}^{\mathrm{irr}} \cup \{\widehat{0}\}$.

The following result is due to G. Birkhoff. The reader is referred to [Sta97, Section 3.4] for a proof.

Theorem 5.3.13. *(Birkhoff's representation theorem) Let $\mathcal{L}$ be a finite distributive lattice. For every $x \in \mathcal{L}$ the set $J_x = \{y \in \mathcal{L}^{\mathrm{irr}} \mid y \leq x\}$ is a poset ideal of $\mathcal{L}^{\mathrm{irr}}$, and the map $\mathcal{L} \longrightarrow J(\mathcal{L}^{\mathrm{irr}})$ given by $x \mapsto J_x$ is a lattice isomorphism.*

An immediate consequence of this result is that $\operatorname{rk} x + \operatorname{rk} y = \operatorname{rk} x \wedge y + \operatorname{rk} x \vee y$. Indeed, by Birkhoff's representation theorem we can reduce to the case where $\mathcal{L} = J(P)$ for some poset P, in which case the rank of an element is its cardinality, see [Sta97, Proposition 3.4.4]. Now the formula follows from the fact that given $I, J \in J(P)$, and in fact any two subsets of a set, $|I| + |J| = |I \cap J| + |I \cup J|$.

Let $\mathcal{L}$ be a finite distributive lattice, and let A be an algebra equipped with an injective function $[-] : \mathcal{L} \longrightarrow A$. A *standard monomial* in A is a product of the form $[x_0][x_1]\dots[x_n]$ with $x_{i-1} \leq x_i$ for all $1 \leq i \leq n$.

Definition 5.3.14. Let $\mathcal{L}$ be a finite distributive lattice, and let A be a connected $\mathbb{N}$-graded algebra equipped with an injective function $[-] : \mathcal{L} \longrightarrow A$. We say that A is a *symmetric quantum graded algebra with a straightening law over $\mathcal{L}$*, or symmetric quantum ASL for short, if the following hold:

1. The set $\{[x] \mid x \in \mathcal{L}\}$ consists of homogeneous elements of positive degree that generate A as an algebra.
2. The set of standard monomials is linearly independent.
3. For every $x, y \in \mathcal{L}$ there exist $\{c_{x,y}^{z,z'}\}_{(z,z') \in R(x,y)^*} \subset k$ and $c_{x,y} \in k^\times$, such that
$$[y][x] = c_{x,y}[x][y] + \sum_{(z,z') \in R(x,y)^*} c_{x,y}^{z,z'}[z][z'].$$
4. For every pair of incomparable elements $x, y \in \mathcal{L}$ there exist $\{d_{x,y}^{z,z'}\}_{(z,z') \in R(x,y)^*} \subset k$ and $d_{x,y} \in k^\times$, such that
$$[x][y] = d_{x,y}[x \wedge y][x \vee y] + \sum_{(z,z') \in R(x,y)^*} d_{x,y}^{z,z'}[z][z'].$$

Quantum graded algebras with a straightening law over an arbitrary poset were introduced in [LR06, Definition 1.1.1]. Of course, symmetric quantum ASL's over finite distributive lattices are a special case of these.

We now associate to each finite distributive lattice a semigroup $S(\mathcal{L})$. We will soon see that this is a normal affine semigroup, and that any symmetric quantum ASL over $\mathcal{L}$ is dominated by it.

Definition 5.3.15. Let $\mathcal{L}$ be a finite distributive lattice. The *straightening semigroup* of $\mathcal{L}$, denoted by $S(\mathcal{L})$ is the commutative semigroup generated by $\mathcal{L}$ with relations

$$(x + y, x \wedge y + x \vee y) \qquad \text{for all pairs of incomparable elements } x, y \in \mathcal{L}.$$

We identify the elements of $\mathcal{L}$ with their class in $S(\mathcal{L})$.

The proof that the straightening semigroup of a finite distributive lattice can be embedded in $\mathbb{N}^{r+1}$ for $r = \operatorname{rk}\mathcal{L}$ is due to Hibi, and is taken from [Hib87, section 2]. We fix a $\mathcal{L}$ and consider the poset $\mathcal{L}^{\text{irr}} \subset \mathcal{L}$ of irreducible elements of $\mathcal{L}$. By Birkhoff's theorem the assignation $x \in \mathcal{L} \mapsto J_x = \{y \in \mathcal{L}^{\text{irr}} \mid y \leq x\} \in J(\mathcal{L}^{\text{irr}})$ is an isomorphism of distributive lattices, and $|\mathcal{L}^{\text{irr}}| = \operatorname{rk}\mathcal{L}$. We set $r = \operatorname{rk}\mathcal{L}$ and fix a monotone function $\sigma : \mathcal{L}^{\text{irr}} \longrightarrow \{1, \ldots, r\}$. We define $i = i(\sigma) : S(\mathcal{L}) \longrightarrow \mathbb{N}^{r+1}$ to be the morphism induced by the assignation

$$x \in \mathcal{L} \longmapsto e_0 + \sum_{i \in \sigma(J_x)} e_i \in \mathbb{N}^{r+1}.$$

That $i(\sigma)$ is well defined follows from the fact that

$$\begin{aligned} i(x) + i(y) &= e_0 + \sum_{l \in \sigma(J_x)} e_l + e_0 + \sum_{l \in \sigma(J_y)} e_l \\ &= e_0 + \sum_{l \in \sigma(J_x \cap J_y)} e_l + e_0 + \sum_{l \in \sigma(J_x \cup J_y)} e_l \\ &= e_0 + \sum_{l \in \sigma(J_{x \wedge y})} e_l + e_0 + \sum_{l \in \sigma(J_{x \vee y})} e_l = i(x \wedge y) + i(x \vee y). \end{aligned}$$

Notice that there is a unique monotone extension of σ to a monotone function $\mathcal{L}_0^{\text{irr}} \longrightarrow \{0, \ldots, r\}$, which we also denote by σ.

Lemma 5.3.16. *Let $i : S(\mathcal{L}) \longrightarrow \mathbb{N}^{r+1}$ be as above.*

1. *An $r+1$-uple $\xi = (\xi^0, \ldots, \xi^r)$ lies in the image of i if and only if the following holds: whenever $x, y \in \mathcal{L}_0^{\text{irr}}$ and $x \leq y$ then $\xi^{\sigma(x)} \geq \xi^{\sigma(y)}$.*
2. *The function i is injective, and $S(\mathcal{L})$ is a normal semigroup.*
3. *For every $s \in S(\mathcal{L})$ and every $1 \leq l \leq t$ there exists $n_l \in \mathbb{N}$ such that $s = \sum_{l=1}^{t} n_l x_l$, with $x_1 < x_2 < \ldots < x_t$.*

Proof. For every $0 \leq i < j \leq r$ let $S_{i,j} = \{\xi \in \mathbb{N}^{r+1} \mid \xi^i \geq \xi^j\}$, and let

$$S = \bigcap_{\{x,y \in \mathcal{L}_0^{\mathrm{irr}} | x<y\}} S_{\sigma(x),\sigma(y)}.$$

We may rewrite the statement of the first item as $i(S(\mathcal{L})) = S$. Since S is the intersection of normal semigroups, it is itself normal and proving that i is injective with image S implies that $S(\mathcal{L})$ is normal.

By definition $i(x) \in S$ for all $x \in \mathcal{L}$, so the image of i is contained in S. We now define a function $j : S \longrightarrow S(\mathcal{L})$ which will be an inverse to i. We define $j(\xi)$ by recursion on $|\xi| = \xi^0 + \xi^1 + \ldots + \xi^r$. If $|\xi| = 0$ then $\xi = (0, \ldots, 0)$ and we set $j(\xi) = 0$; evidently $i(j(0)) = 0$. Suppose that we have defined $j(\xi')$ and that $i(j(\xi')) = \xi'$ whenever $|\xi'| < |\xi|$. Suppose furthermore that $j(\xi')$ can be written in a unique way as in item 3. The definition of S guarantees that for every $\xi \in S$ the set $\operatorname{supp} \xi = \sigma^{-1}(\{i > 0 \mid \xi^i \neq 0\}) \subset \mathcal{L}^{\mathrm{irr}}$ is a poset ideal of $\mathcal{L}^{\mathrm{irr}}$ and hence equal to J_x for some $x \in L$. Also by definition, $\xi' = \xi - i(x)$ lies in S, so we set $j(\xi) = j(\xi') + x$. Clearly $i(j(\xi)) = \xi$.

By hypothesis $j(\xi')$ can be written in a unique way as in item 3, and $\operatorname{supp} \xi' = J_{x_t}$. Since by definition $J_{x_t} \subseteq \operatorname{supp} \xi = J_x$, we see that $x_t \leq x$, so $j(\xi)$ can be written as $\sum_l n_l x_l + x$ which is of the form described in item 3. Also, since i is injective over the elements of $\mathcal{L}$ by definition, putting any element $x' \in \mathcal{L}$ different from x in this sum gives an element which is not in the preimage of ξ by i, so this writing is unique and we have item 3. Thus i is injective with image S, which proves items 1 and 2. □

For the rest of this section we fix a finite distributive lattice $\mathcal{L}$ and write S for $S(\mathcal{L})$.

From Lemma 5.3.16 we deduce that $\mathcal{L}$ is a minimal set of generators of S. Indeed, recall from Lemma 5.1.3 that the minimal set of generators of S consists of the minimal elements for the order $\preceq$, i.e. the elements of S that cannot be written as a sum of two nonzero elements. Now every element of S has nonzero 0-th coordinate, and $i(x)^0 = 1$ for all $x \in \mathcal{L}$, so $i(x)$ cannot be written as the sum of two nonzero elements of S and hence it is minimal for $\preceq$. Since $\mathcal{L}$ is also a generating set, it must be the minimal set of generators of S.

Let $n = |\mathcal{L}|$ and choose a monotone function $\rho : \mathcal{L} \longrightarrow \{1, \ldots, n\}$. Write $x_i = \rho^{-1}(i)$ for every $1 \leq i \leq n$ and let $\pi : \mathbb{N}^n \longrightarrow S(\mathcal{L})$ be the group morphism defined by $e_i \mapsto x_i$ for every $1 \leq i \leq n$. We denote $e_{i \wedge j} = \pi^{-1}(x_i \wedge x_j)$ and $e_{i \vee j} = \pi^{-1}(x_i \vee x_j)$ for all $1 \leq i < j \leq n$. Notice that by definition $\pi(e_{i \wedge j} + e_{i \vee j}) = \pi(e_i + e_j)$ and since ρ is an order preserving function, $e_i + e_j <_{\mathrm{lex}} e_{i \wedge j} + e_{i \vee j}$ whenever x_i and x_j are incomparable. Let $\mathcal{I} = \{(i, j) \mid 1 \leq i < j \leq n$ such that x_i, x_j are incomparable$\}$. Then

$$L(S) = \sum_{(i,j) \in \mathcal{I}} \mathbb{N}(e_i + e_j, e_{i \wedge j} + e_{i \vee j}),$$

and it is once again clear that elements in these set of generators are minimal for the divisibility order of $L(S)$. Thus the presentation of S given in Definition 5.3.15 is minimal.

Now from item 3 of Lemma 5.3.16 we see that every element $s \in S$ can be written in a unique way as $s = \sum_{i=1}^{n} m_i x_i$ with the condition that the set $\{x_i \mid m_i \neq 0\}$ is a chain of $\mathcal{L}$. This defines a section $t : S \longrightarrow \mathbb{N}^n$ of π, that to each $s \in S$ assigns $t(s) = (m_1, \ldots, m_n)$. We refer to t as the *standard section* of π.

Before proving the main result of this section, namely that all symmetric quantum graded ASL's over $\mathcal{L}$ are $S(\mathcal{L})$-dominated algebras, we prove a technical lemma.

Lemma 5.3.17. *Let i be as above. Then*

$$i(z) + i(z') <_{\mathsf{lex}} i(x) + i(y)$$

for every $x, y \in \mathcal{L}$ and all $(z, z') \in R(x, y)^*$.

Proof. Since $i(x) + i(y) = i(x \wedge y) + i(x \vee y)$ and $R(x, y)^* = R(x \wedge y, x \vee y)^*$, we only consider the case $z < x \leq y < z'$, or equivalently $J_z \subsetneq J_x \subseteq J_y \subsetneq J_{z'}$. The inequality $i(z) + i(z') <_{\mathsf{lex}} i(x) + i(y)$ holds if and only if $i(z') - i(y) <_{\mathsf{lex}} i(x) - i(z)$, that is if and only if

$$\sum_{i \in \sigma(J_{z'} \setminus J_y)} e_i <_{\mathsf{lex}} \sum_{i \in \sigma(J_x \setminus J_z)} e_i.$$

It follows from the inclusions above that the least element of $J_x \setminus J_z$ is strictly smaller than the least element of $J_{z'} \setminus J_y$, which implies the last inequality. □

Theorem 5.3.18. *Let $\mathcal{L}$ be a finite distributive lattice and let S be the image of $S(\mathcal{L})$ by the embedding i defined above. Let $t : S \longrightarrow \mathbb{N}^n$ be the standard section of $\pi : \mathbb{N}^n \longrightarrow S$. Then there exists a semigroup morphism $\varphi : S \longrightarrow \mathbb{N}$ such that every symmetric quantum graded algebra with a straightening law over $\mathcal{L}$ is an (S, t, φ)-dominated algebra.*

Proof. Since the given presentation of S is minimal, points 1 of Definitions 5.3.14 and 5.3.1 are equivalent. Since the set $\{b^{t(s)} \mid s \in S\}$ from Definition 5.3.1 corresponds precisely to the set of standard monomials, we see that points 2 of both definitions are also equivalent.

Let $\mathcal{C}$ be the union of the sets $i(R(x, y))$ for all $x, y \in \mathcal{L}$. By Lemma 5.3.11 there exists a morphism $\varphi : S \longrightarrow \mathbb{N}$ such that for any two elements $c, c' \in \mathcal{C}$, the inequality $c <_{\mathsf{lex}} c'$ holds if and only if $\varphi(c) < \varphi(c')$. Thus points 3 and 4 of Definition 5.3.14 are equivalent to the corresponding points of Definition 5.3.1 and we are done. □

The following result will be useful in the sequel. It shows that certain quotients of symmetric quantum graded ASL's also belong to this class.

Proposition 5.3.19. *Let A be a symmetric quantum graded ASL over a finite distributive lattice $\mathcal{L}$, let $x, y \in \mathcal{L}$ and let I be the ideal generated by the elements $[z]$ with $z \notin [x, y]$. Then A/I is a symmetric quantum graded ASL over $[x, y]$.*

Proof. Consider first the case where $y = \hat{1}$. Then $\{z \mid \notin [x, \hat{1}]\} = \{z \mid x \not\leq z\}$ is a poset ideal; writing I' for the ideal of A generated by $[z]$ with $z \in [c, \hat{1}]$, [LR06, Corollary 1.2.6] states that the quotient algebra A/I' is a quantum graded ASL over $[x, \hat{1}]$, and since A is symmetric, so is A/I'.

On the other hand, the fact that A/I' is symmetric implies that it is also a quantum graded ASL over $[x, \hat{1}]^\circ$, the lattice with the same underlying group as $[x, \hat{1}]$ but with the opposite order. Since $[x, y]$ is a poset ideal of $[x, \hat{1}]^\circ$, the same result implies that A/I is a quantum graded ASL over $[x, y]$, and again it is symmetric since A/I' is symmetric. □

Chapter 6

Toric degeneration of quantum flag varieties and associated varieties

This chapter contains the results that motivated this thesis, namely the study of properties of the quantized homogeneous coordinate rings of some varieties associated to flag varieties. These have been widely studied as good examples of noncommutative projective varieties, and are both a motivating example for the theory and a fertile ground for testing different ideas.

In section 6.1 we review the definitions of quantum grassmannians and their Schubert and Richardson varieties. We prove that grassmannians and their Schubert and Richardson varieties in type A are symmetric quantum ASLs for an arbitrary field and quantum parameter. Then in section 6.2 we recall the definition of arbitrary quantum flag varieties and their Schubert varieties. Assuming the underlying field is a transcendental extension of $\mathbb{Q}$, we show that arbitrary quantum flag and Schubert varieties have S-bases, and hence they degenerate to quantum affine toric varieties.

6.1 Quantum analogues of grassmannians and related varieties.

In this section, we investigate quantum analogues of coordinate rings of Richardson varieties in grassmannians of type A. The final aim is to show that these are symmetric quantum graded ASL's and to derive from this some of their important properties.

6.1.1 Quantum grassmannians, Schubert and Richardson subvarieties

Consider positive integers n, m and a scalar $q \in k^*$. Following [LR06, section 3.1], we let $\mathcal{O}_q(M_{n,m}(k))$ denote the quantum analogue of the affine coordinate ring of the space of $n \times m$ matrices with entries in k. This is the k-algebra with generators X_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq m$, and relations given by

$$X_{it}X_{ij} = q^{-1}X_{ij}X_{it}; \qquad X_{sj}X_{ij} = q^{-1}X_{ij}X_{sj};$$
$$X_{it}X_{sj} = X_{sj}X_{it}; \qquad X_{st}X_{ij} = X_{ij}X_{st} - (q - q^{-1})X_{sj}X_{it};$$

where $1 \leq i < s \leq n$ and $1 \leq j < t \leq m$. If $n = m$, we put $\mathcal{O}_q(M_n(k)) = \mathcal{O}_q(M_{n,m}(k))$.

Recall that there is a *transpose* automorphism of algebras $\mathrm{tr}_v : \mathcal{O}_q(M_{n,m}(k)) \longrightarrow \mathcal{O}_q(M_{m,n}(k))$, defined by the assignation $X_{ij} \mapsto X_{ji}$. With this in mind, from this point on we assume that $n \leq m$. Recall in addition that, if n', m' are positive integers such that $n' \leq m$ and $m' \leq m$, then the assignment $X_{ij} \mapsto X_{ij}$ defines an injective algebra morphism from $\mathcal{O}_q(M_{n',m'}(k))$ to $\mathcal{O}_q(M_{n,m}(k))$.

Let $I = \{i_1 < i_2 < \ldots < i_t\}$ be a subset of $\{1, \ldots, n\}$ and $J = \{j_1 < j_2 < \ldots < j_t\}$ a subset of $\{1, \ldots, m\}$, where $t \leq n$. We associate to such a pair the *quantum minor*

$$[I \mid J] = \sum_{\sigma \in \mathfrak{S}_t} (-q)^{\ell(\sigma)} X_{i_1, j_{\sigma(1)}} X_{i_2, j_{\sigma(2)}} \ldots X_{i_t, j_{\sigma(t)}} \in \mathcal{O}_q(M_{n,m}(k))$$

where $\ell(\sigma)$ denotes the length of the bijection $\sigma \in \mathfrak{S}_t$. An easy induction argument shows that the transpose morphism sends $[I \mid J]$ to $[J \mid I]$. When $I = \{1, \ldots, n\}$ we simply write $[J]$ for $[I \mid J]$. Notice that when $q = 1$ this is simply the usual definition of a minor of a matrix.

We denote by $\Pi_{n,m}$ the subset of $\mathbb{N}^n$ consisting of elements $(i_1, \ldots, i_n)$ such that $1 \leq i_1 < \cdots < i_n \leq m$ endowed with the restriction of the natural product order of $\mathbb{N}^n$. It is easy to see that $\Pi_{n,m}$ is a distributive lattice of $\mathbb{N}^n$, with

$$(i_1, \ldots, i_n) \wedge (j_1, \ldots, j_n) = (\min\{i_1, j_1\}, \ldots, \min\{i_n, j_n\})$$
$$(i_1, \ldots, i_n) \vee (j_1, \ldots, j_n) = (\max\{i_1, j_1\}, \ldots, \max\{i_n, j_n\}).$$

Clearly an element $I = \{i_1 < \cdots < i_n\}$ of $\Pi_{n,m}$ determines a subset of $\{1, \ldots, m\}$ which by a slight abuse of notation we also denote by I, and hence there is a map $\Pi_{n,m} \longrightarrow \mathcal{O}_q(M_{n,m}(k))$, given by $I \mapsto [I]$.

Definition 6.1.1. Let $n, m \in \mathbb{N}$ with $n \leq m$ and let $q \in k^\times$. The *quantum grassmannian* with parameters n, m, q, denoted by $\mathcal{O}_q(G_{n,m}(k))$, is the subalgebra of $\mathcal{O}_q(M_{n,m}(k))$ generated by the $n \times n$ quantum minors of $\mathcal{O}_q(M_{n,m}(k))$, that is, by the elements $[I]$ for $I \in \Pi_{n,m}$.

The algebra $\mathcal{O}_1(G_{n,m}(k))$ is the coordinate ring of the Plücker embedding of the grassmannian parametrizing n-dimensional subspaces of k^m, see for example [BV88, chapter 1, section D].

Let $I \in \Pi_{n,m}$. We associate to this element a poset ideal $\Pi_I = \{J \in \Pi_{n,m} \mid J \not\geq I\}$ (see subsection 5.3.3 for the definition of a poset ideal). Similarly we denote by Π^I the poset ideal $\{J \in \Pi_{n,m} \mid J \not\leq I\}$. If $J \in \Pi_{n,m}$ and $I \leq J$ then we write $\Pi_I^J = \Pi_I \cup \Pi^J$. We denote by Ω_I the ideal generated in $\mathcal{O}_q(G_{n,m}(k))$ by $\{[J] \mid J \in \Pi_I\}$, and we define analogously the ideals Ω^I and Ω_I^J.

Definition 6.1.2. To each $I \in \Pi_{n,m}$ we associate the *quantum analogue of the homogeneous coordinate ring of the corresponding Schubert variety*, or quantum Schubert variety for short, defined as the quotient

$$\mathcal{O}_q(G_{n,m}(k))_I = \mathcal{O}_q(G_{n,m}(k))/\Omega_I.$$

To each $I, J \in \Pi_{m,n}$ such that $I \leq J$, we associate the *quantum analogue of the homogeneous coordinate ring on the Richardson variety* corresponding to I, J also called, to simplify, the quantum Richardson variety associated to I, J, defined as the quotient

$$\mathcal{O}_q(G_{n,m}(k))/\Omega_I^J.$$

The material in this section is a new presentation of work published in [RZ12]. This was a continuation of the work started in [LR06, LR08]. Notice that we adopt here a convention exchanging rows and columns with respect to the one used in these last two references. However, embedding all the relevant algebras in $\mathcal{O}_q(M_v(k))$ and using the transpose automorphism introduced above shows that the two different conventions lead to isomorphic algebras. Hence we are in position to use all of the results in the aforementioned papers.

6.1.2 Quantum Richardson varieties are symmetric quantum graded ASL's

For the rest of this section, our aim is to prove that $\mathcal{O}_q(G_{n,m}(k))$ is a symmetric quantum ASL over the distributive lattice $\Pi_{n,m}$ as in Definition 5.3.14. As mentioned before, the class of symmetric quantum ASL's is a subclass of quantum graded ASL's as defined in [LR06, Definition 1.1.1]. In this reference it is proved that quantum grassmannians are quantum graded ASL's. We begin with the following lemma on the structure of the lattice $\Pi_{n,m}$. Recall that the rank of an element $I \in \Pi_{n,m}$ is the maximal t such that there exists a chain $I_0 < I_1 < \ldots < I_t = I$ in $\Pi_{n,m}$. Also recall that given $I, J \in \Pi_{n,m}$ we set

$$R(I, J) = \{(K, L) \in \Pi_{n,m} \times \Pi_{n,m} \mid K \leq I \wedge J, I \vee J \leq L \text{ and } \operatorname{rk} L + \operatorname{rk} K = \operatorname{rk} I + \operatorname{rk} J\}.$$

Since we have identified the set $I \subset \{1, \ldots, m\}$ with the n-uple formed by its elements in increasing order, it makes sense to write $i \in I$ if and only if there exists t such that $i = i_t$.

Lemma 6.1.3. *Let $n, m \in \mathbb{N}$ with $n \leq m$, and let $\Pi_{n,m}$ be as above.*

1. *The rank of* $I = (i_1, \dots, i_n) \in \Pi_{n,m}$ *is* $\sum_{i \in I} i - \frac{n(n+1)}{2}$.

2. *Fix* $I, J \in \Pi_{n,m}$*, and suppose we are given* $L, K \in \Pi_{n,m}$ *such that* $L \cup K = I \cup J$, $L \cap K = I \cap J$*, and* $K \leq I \wedge J$*. Then* $\operatorname{rk} I + \operatorname{rk} J = \operatorname{rk} K + \operatorname{rk} L$ *and* $I \vee J \leq L$*, that is* $(K, L) \in R(I, J)$.

Proof. 1. We prove the result by induction on the rank of I. If $\operatorname{rk} I = 0$ then I is the unique minimal element of $\Pi_{n,m}$, namely $(1, 2, \dots, n)$, in which case the formula is clearly valid. Now suppose the formula is valid for all elements of rank less than $\operatorname{rk} I$, where $I = (i_1, \dots, i_n)$. Since $\operatorname{rk} I > 0$, there must exist $1 \leq t \leq n$ such that $i_t > t$. Clearly $I - e_t \in \Pi_{n,m}$, and furthermore $I - e_t < I$, with no element of the lattice between them. Hence $\operatorname{rk} I = \operatorname{rk}(I - e_t) + 1 = \left(\sum_{i \in I} i\right) - l + \frac{l(l+1)}{2} + 1 = \sum_{i \in I} i - \frac{l(l+1)}{2}$.

2. Suppose K, L are as in the statement. Then

$$\begin{aligned} \operatorname{rk} I + \operatorname{rk} J &= l(l+1) + \sum_{i \in I} i + \sum_{j \in J} i = l(l+1) + \sum_{i \in I \cap J} i + \sum_{i \in I \cup J} i \\ &= l(l+1) + \sum_{i \in K \cap L} i + \sum_{i \in K \cup L} i = l(l+1) + \sum_{i \in L} i + \sum_{i \in K} i \\ &= \operatorname{rk} K + \operatorname{rk} L. \end{aligned}$$

Notice that $I \wedge J$ and $I \vee J$ comply with the hypothesis of the statement, that is $(I \wedge J) \cap (I \vee J) = I \cap J$ and $(I \wedge J) \cup (I \vee J) = I \cup J$, so the hypothesis is equivalent to the same statement with I and J replaced by $I \wedge J$ and $I \vee J$ respectively, and since $R(I, J) = R(I \wedge J, I \vee J)$, we can restrict ourselves to the case where $I \leq J$.

□

Recall that to each element $I \in \Pi_{n,m}$ we may associate the $n \times n$ quantum minor $[I]$ in $\mathcal{O}_q(G_{n,m}(k))$. We will often identify an element of $\Pi_{n,m}$ with its image in $\mathcal{O}_q(G_{n,m}(k))$. By definition of $\mathcal{O}_q(G_{n,m}(k))$, the set $\{[I],\ I \in \Pi_{n,m}\}$ is a set of generators of the k-algebra $\mathcal{O}_q(G_{n,m}(k))$ since any $n \times n$ quantum minor of $\mathcal{O}_q(M_{n,m}(k))$ equals $[I]$ for some $I \in \Pi_{n,m}$. Recall, further, that $\mathcal{O}_q(G_{n,m}(k))$ has an $\mathbb{N}$-grading with respect to which the elements $[I]$, $I \in \Pi_{n,m}$, are homogeneous of degree 1. Recall that a *standard monomial* on $\mathcal{O}_q(G_{n,m}(k))$ is a product of the form $[I_1][I_2] \dots [I_t]$ with $I_1 \leq I_2 \leq \dots \leq I_t$.

In [LR06, section 3] it is proved that $\mathcal{O}_q(G_{n,m}(k))$ is a quantum graded ASL on $\Pi_{n,m}$. More precisely, the following is proved:

1. The assignation $I \in \Pi_{n,m} \mapsto [I] \in \mathcal{O}_q(G_{n,m}(k))$ is injective.

2. Standard monomials on $\Pi_{n,m}$ form a basis of the k-vector space $\mathcal{O}_q(G_{n,m}(k))$.

3. For any $(I, J) \in \Pi_{m,n} \times \Pi_{m,n}$, there exists a relation

$$[I][J] - q^{f_{I,J}}[J][I] = \sum_{(K,L)} c_{K,L}^{I,J}[K][L],$$

where $f_{I,J} \in \mathbb{Z}$ and where the sum extends over pairs (K, L) of elements of $\Pi_{m,n}$ such that $K \leq L$ and $K < I, J$ and where, for such a pair, $d_{K,L}^{I,J} \in k$.

4. For any pair of incomparable elements $I, J \in \Pi_{n,m}$, there exists a (necessarily unique) relation

$$[I][J] = \sum_{(K,L)} d_{K,L}^{I,J}[K][L],$$

where the sum extends over pairs (K, L) of elements of $\Pi_{m,n}$ such that $K \leq L$ and $K < I, J$ and where, for such a pair, $c_{K,L}^{I,J} \in k$;

Thus in order to prove that $\mathcal{O}_q(G_{n,m}(k))$ is a symmetric quantum graded ASL we need only prove that the standard monomials appearing in the commuting and straightening relations comply with Definition 5.3.14.

Proposition 6.1.4. *Let* $I, J \in \Pi_{n,m}$.

1. *If* $K, L \in \Pi_{n,m}$ *are such that* $c_{K,L}^{I,J} \neq 0$ *then* $K \leq I, J \leq L$.
2. *If* I, J *are incomparable and* $K, L \in \Pi_{n,m}$ *are such that* $d_{K,L}^{I,J} \neq 0$ *then* $K \leq I, J \leq L$.
3. *If* I, J *are incomparable then* $d_{I\wedge J, I\vee J}^{I,J} \neq 0$.

Proof. We consider two gradings on the algebra $A = \mathcal{O}_q(M_{n,m}(k))$. The first is a $\mathbb{N}^m$-grading where the degree of X_{ij} is e_j for each $1 \leq i \leq n$ and $1 \leq j \leq m$; write $\deg^1 a$ for the degree of an element $a \in A$ with respect to this grading. The second is a $\mathbb{Z}$-grading, with the degree of X_{ij} given by $j - i$. We write $\deg^2 a$ for the degree of a with respect to this second grading. Notice that the relations defining A are homogeneous for both gradings, so they are well-defined.

Clearly for every $I \in \Pi_{n,m}$ the associated quantum minor $[I]$ is a homogeneous element for the first grading, and $\deg[I] = \sum_{i\in I} e_i$. On the other hand, notice that for every $\sigma \in \mathfrak{S}_n$

$$\deg^2 X_{1i_\sigma(1)}X_{2i_\sigma(2)} \dots X_{ni_\sigma(n)} = \sum_{i\in I} i - \frac{n(n+1)}{2} = \operatorname{rk} I,$$

so $\deg^2[I] = \operatorname{rk} I$ by item 1 of Lemma 6.1.3. Given any relation of the form

$$[I][J] = \sum_{(K,L)} \lambda_{K,L}^{I,J}[K][L]$$

with $K \leq L$, the linear independence of standard monomials implies that all the monomials on the right hand side are homogeneous of the same degree as $[I][J]$ for both gradings. Thus for every K, L with $\lambda_{K,L}^{I,J} \neq 0$ we get the equalities

$$\sum_{i \in I} e_i + \sum_{i \in J} e_i = \sum_{i \in K} e_i + \sum_{i \in L} e_i \qquad \text{rk}\, I + \text{rk}\, J = \text{rk}\, K + \text{rk}\, L.$$

The first equality is equivalent to the fact that $I \cap J = K \cap L$ and $I \cup J = K \cup L$. Since we also know that $K \leq I, J$, item 2 of Lemma 6.1.3 guarantees that $I, J \leq L$. This proves items 1 and 2.

Now suppose K, L are such that $d_{K,L}^{I,J} \neq 0$, and suppose $I \vee J < L$. Since $\text{rk}\, K + \text{rk}\, L = \text{rk}\, I + \text{rk}\, J = \text{rk}\, I \wedge J + \text{rk}\, I \vee J$, it must be $\text{rk}\, R < \text{rk}\, I \wedge J$, in particular $R < I \wedge J$. Thus the straightening relation for $[I]$ and $[J]$ can be written as

$$[I][J] = d_{I\wedge J, I\vee J}^{I,J}[I \wedge J][I \vee J] + \sum_{K < I\wedge J \leq I \vee J < L} d_{K,L}^{I,J}[K][L]$$

Thus in the quantum Schubert variety $\mathcal{O}_q(G_{n,m}(k))_{I\wedge J}$ we get $[I][J] = d_{I\wedge J, I\vee J}^{I,J}[I \wedge J][I \vee J]$, since all the other monomials in the right hand side are in the Schubert ideal of $I \wedge J$. By [LR08, Corollary 3.1.7], quantum Schubert varieties are domains, and so $d_{I\wedge J, I\vee J}^{I,J} \neq 0$. □

From the definitions and the last proposition we immediately deduce that quantum grassmannians are symmetric quantum graded ASL's. Using Proposition 5.3.19 we obtain the following result

Theorem 6.1.5. *Quantum grassmannians and their quantum Schubert and Richardson varieties are symmetric quantum graded ASL's.*

By Theorem 5.3.18 and Corollary 5.3.8, this implies that Richardson varieties are normal domains, and that they are always AS-Cohen-Macaulay.

6.2 Toric degenerations of quantum flag and Schubert varieties

Throughout this section we assume that k is a field of characteristic zero. We adapt the arguments from [Calo2] to the quantum setting to show that Schubert cells of arbitrary quantum flag varieties have S-bases, and that in each case S is a normal affine semigroup.

6.2.1 Quantum flag and Schubert varieties

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let G be the corresponding simply connected Lie group. We denote by P the weight lattice of $\mathfrak{g}$, and write $\{p_1, \ldots, p_n\}$

for its fundamental weights. Let $P^+ = \sum_i \mathbb{N}p_i$ be the set of dominant weights, and $\alpha_1, \ldots, \alpha_n$ the positive roots of P. Let W be the Weyl group of $\mathfrak{g}$, and $s_i \in W$ the reflection corresponding to the i-th root. Given an element $w \in W$ we denote its length by $\ell(w)$, and set N to be the length of the longest word of W. A *decomposition* of $w \in W$ is a word on the generators s_i that equals w in W. A decomposition of the longest word of W is said to be *adapted* to w if it is of the form $s_{i_1} \ldots s_{i_N}$ with $s_{i_1} \ldots s_{i_{\ell(w)}} = w$. For every element $w \in W$ there exists a decomposition of the longest word of W adapted to w, in other words the longest word of W is the maximum for weak right Bruhat order on W, see [BB05, Proposition 3.1.2].

Fix $q \in k^\times$ not a root of unity. Let $U_q(\mathfrak{g})$ be the quantum enveloping algebra of $\mathfrak{g}$ as in [Jan96, Definition 4.3]. The algebra $U_q(\mathfrak{g})$ is generated by elements $\{E_i, F_i, K_i^{\pm 1} \mid i = 1, \ldots, n\}$ where n is the rank of P. Denote by $U_q(\mathfrak{b})$, resp. $U_q(\mathfrak{n})$, the subalgebra of $U_q(\mathfrak{g})$ generated by the elements E_i, K_i, resp. E_i, for all $1 \leq i \leq n$; $U_q(\mathfrak{b}^-)$ and $U_q(\mathfrak{n}^-)$ are defined analogously, replacing each E_i by the corresponding F_i. For each $\lambda \in P^+$ there is an irreducible highest-weight representation of $U_q(\mathfrak{g})$ which we denote by $V_q(\lambda)$. Each $V_q(\lambda)$ decomposes as the direct sum of weight spaces $\bigoplus_{\mu \in \Lambda} V_q(\lambda)_\mu$, where Λ is a finite subset of P^+; the dimensions of the weight spaces are the same as in the classical case, that is, the Weyl and Demazure character formulas hold for these representations (see [Kas95, section 12.4], or [Jan96, subsection 5.15] for the Weyl formula).

Let I be a subset of the set of fundamental weights and set $\mathcal{J}(I) = \sum_{p_i \notin I} \mathbb{N}p_i$. Denote by $W_I \subset W$ the subgroup generated by reflections s_{α_i} with $p_i \in I$, and for each class in W/W_I pick a representative of smallest length. We call W^I the set of these representatives. Since the Demazure character formula holds, for each $w \in W^I$ and $\lambda \in \mathcal{J}(I)$ the vector space $V_q(\lambda)_{w\lambda}$ has dimension 1 just as in the classical case. The *Demazure module* $V_q(\lambda)_{w\lambda}$ is the $U_q(\mathfrak{b})$-submodule of $V_q(\lambda)$ generated by a vector of weight $w\lambda$ in $V_q(\lambda)$.

Given a vector space V we denote its dual space by V^*. Since $U_q(\mathfrak{g})$ is a Hopf algebra, its dual is an algebra with convolution product induced by the coproduct of $U_q(\mathfrak{g})$. There is a map $V_q(\lambda)^* \otimes V_q(\lambda) \longrightarrow U_q(\mathfrak{g})^*$ defined by sending $\xi \otimes v \in V_q(\lambda)^* \otimes V_q(\lambda)$ to the linear functional $c^\lambda_{\xi,v}$, which assigns to each $u \in U_q(\mathfrak{g})$ the scalar $c^\lambda_{\xi,v}(u) = \xi(uv)$. Functionals of type $c^\lambda_{\xi,v}$ are called *matrix coefficients*. The span of the matrix coefficients forms a subalgebra of $U_q(\mathfrak{g})^*$ denoted by $\mathcal{O}_q[G]$, the quantized algebra of coordinate functions over the group G. Given $\lambda, \mu \in P^+$ and highest weight vectors $v_\lambda \in V_q(\lambda)$ and $v_\mu \in V_q(\mu)$, the vector $v_\lambda \otimes v_\mu \in V_q(\lambda) \otimes V_q(\mu)$ is a highest weight vector that generates a $U_q(\mathfrak{g})$-submodule isomorphic to $V_q(\lambda + \mu)$, so identifying these modules the product of two matrix coefficients $c^\lambda_{\xi,v}$ and $c^\mu_{\chi,w}$ is $c^{\lambda+\mu}_{\xi\otimes\chi, v\otimes w}$, which makes sense since $v \otimes w$ is in the $U_q(\mathfrak{g})$-submodule generated by $v_\lambda \otimes v_\mu$.

We now review the definitions of quantum flag varieties. They were introduced by Soibelman in [Soĭ92] and by Lakshmibai and Reshetikhin in [LR92]. Let B be a maximal Borel subgroup of G. Then G/B is the *full flag variety* associated to G. Let

$C_q^+(\lambda)$ be the vector space of matrix coefficients of the form $c^{\lambda}_{\xi,v_\lambda}$ in $U_q(\mathfrak{g})^*$, where v_λ is some highest weight vector in $V_q(\lambda)$, and set

$$\mathcal{O}_q[G/B] = \bigoplus_{\lambda \in P^+} C_q^+(\lambda) \subset U_q(\mathfrak{g})^*.$$

This is called the *quantum full flag variety* of G. Notice that the product of two matrix coefficients in $\mathcal{O}_q[G/B]$ is again in $\mathcal{O}_Q[G/B]$. The above decomposition as a direct sum gives $\mathcal{O}_q[G/B]$ the structure of a P^+-graded algebra.

To every subset I of the set of fundamental weights corresponds a parabolic subgroup P_I, and the variety G/P_I is the corresponding generalized flag variety. To this datum we associate the P^+-graded subalgebra of $\mathcal{O}_q[G/B]$

$$\mathcal{O}_q[G/P_I] := \bigoplus_{\lambda \in \mathcal{J}(I)} C_q^+(\lambda)$$

called *quantum partial flag variety associated to* I. The case $I = \emptyset$ corresponds to the full flag variety.

Given vector spaces $V_2 \subset V_1$, we denote by $V_2^\perp$ the set of linear functionals over V_1 which are zero on V_2. For every $w \in W^I$, the vector space

$$J_w^I = \bigoplus_{\lambda \in \mathcal{J}(I)} \left\langle c^{\lambda}_{\xi,v_\lambda} \in C_q^+(\lambda) \mid \xi \in V_q(\lambda)_w^\perp \right\rangle \subset \mathcal{O}_q[G/P_I]$$

is an ideal of $\mathcal{O}_q[G/P_I]$ called the *Schubert ideal* associated to w. The quotient algebra $\mathcal{O}_q[G/P_I]_w = \mathcal{O}_q[G/P_I]/J_w^I$ is called the *quantum Schubert variety* associated to w.

6.2.2 An S-basis for quantum Schubert varieties

The aim of this subsection is to show that, assuming k is of characteristic zero and q is transcendental over $\mathbb{Q}$, all quantum Schubert varieties have an S-basis for a suitable affine semigroup S. In order to do so we work for a moment over the field $\mathbb{Q}(v)$, where v is an indeterminate over $\mathbb{Q}$, and consider the $\mathbb{Q}(v)$-algebra $U = U_v(\mathfrak{g})$. The general case will follow by extension of scalars.

We denote by U^+ and U^- the algebras $U_v(\mathfrak{n}^+)$ and $U_v(\mathfrak{n}^-)$, respectively. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}(v)$. The algebra U has an $\mathcal{A}$-form, defined equivalently in [Lus10, paragraph 3.1.13] and [Jan96, section 11.1], which we denote by $U_{\mathcal{A}}$. This is a graded subring of U, and the algebras U^+ and U^- also have $\mathcal{A}$ forms which we denote by $U^+_{\mathcal{A}}$ and $U^-_{\mathcal{A}}$, respectively. By definition $U = \mathbb{Q}(v) \otimes_{\mathcal{A}} U_{\mathcal{A}}$, and analogous results hold for $U^+_{\mathcal{A}}$ and $U^-_{\mathcal{A}}$.

The algebra $U^-_{\mathcal{A}}$ has an homogeneous $\mathcal{A}$-basis $\mathcal{B}$, called the *canonical basis* of U^- which was discovered and studied independently by Lusztig and Kashiwara. For a general overview of the theory we refer to [Kas95], and for proofs and details to

[Lus10, section II] and [Jan96, chapters 9 - 11]. The $\mathcal{A}$-form $U^-_{\mathcal{A}}$ is compatible with the coalgebra structure of U, in the sense that $\Delta(U^-_{\mathcal{A}}) \subset U^-_{\mathcal{A}} \otimes_{\mathcal{A}} U^-_{\mathcal{A}}$, where Δ is the comultiplication of U (see [Lus10, Proposition 14.2.6 (a)]). Furthermore the highest weight modules $V_v(\lambda)$ have corresponding $\mathcal{A}$-forms, which we denote by $V_{\mathcal{A}}(\lambda)$. All this $\mathcal{A}$-forms are compatible with the weight decompositions of the original objects.

Theorem 6.2.1. *Let $\lambda \in P^+$, let I be a subset of the fundamental weights and let $w \in W^I$. There exists $\mathcal{B}_\lambda \subset \mathcal{B}$ such that $\mathcal{B}_\lambda v_\lambda \subset V_v(\lambda)$ is a basis of weight vectors of $V_v(\lambda)$, and furthermore, there exists a subset $\mathcal{B}_w \subset \mathcal{B}$ such that $(\mathcal{B}_w \cap \mathcal{B}_\lambda)v_\lambda$ is a basis of the Demazure module $V_q(\lambda)_w$.*

Proof. See [Kas93, Theorem 3.2.5]. Remark 3.2.6 of the same reference states that this decomposition induces a decomposition of the corresponding $\mathcal{A}$-forms. □

Littelman proved in [Lit98, Proposition 1.5, a)] that for every decomposition w_0 of the longest word of W there is a parametrization of the canonical basis $\mathcal{B}$ by a set $\mathcal{S}_{w_0} \subset \mathbb{N}^N$, where N is the length of the longest word of W. For each $s \in \mathcal{S}_{w_0}$ let b_s denote the corresponding element in the canonical basis. Thus if λ is a dominant weight then there exists a finite set $S_{\lambda,w_0} \subset S_{w_0}$ such that $\mathcal{B}_\lambda = \{b_s \mid s \in S_{\lambda,w_0}\}$.

Let $w \in W^I$, and let w_0 be a decomposition of the longest word of W adapted to w. Following [Cal02] we set

$$\begin{aligned}
\tilde{\mathcal{S}}_{w_0} &= \{(s,\lambda) \mid s \in S_\lambda\} \subset \mathbb{N}^N \times P^+ \cong \mathbb{N}^{N+n},\\
\tilde{\mathcal{S}}^w_{w_0} &:= \{(s,\lambda) \in \tilde{\mathcal{S}}_{w_0} \mid b_s \in \mathcal{B}_\lambda \cap \mathcal{B}_w\},\\
\tilde{\mathcal{S}}^w_{w_0,I} &:= \{(s,\lambda) \in \tilde{S}^w_{w_0} \mid \lambda \in \mathcal{J}(I)\} = \tilde{S}^w_{w_0} \cap \mathcal{J}(I).
\end{aligned}$$

We set on these groups the total order induced by pulling back the lexicographic order of $\mathbb{N}^{N+n}$. By abuse of notation, we denote this order by $\leq_{\mathsf{lex}}$.

Lemma 6.2.2. *The sets $\tilde{\mathcal{S}}_{w_0}, \tilde{\mathcal{S}}^w_{w_0}$ and $\tilde{\mathcal{S}}^w_{w_0,I}$ are normal affine semigroups.*

Proof. For $\tilde{\mathcal{S}}_{w_0}$ and $\tilde{\mathcal{S}}^w_{w_0}$ see [Cal02, Theorem 2.2 and Theorem 2.4] respectively.

Now, $\tilde{\mathcal{S}}^w_{w_0,I}$ is by definition $\tilde{S}^w_{w_0} \cap \mathbb{N}^n \times J(I)$. Let $D = \mathbb{R}_+\tilde{S}^w_{w_0}$ and let G be the enveloping group of $\mathbb{N}^n \times J(I)$. By item 2 of Lemma 5.1.2, $\tilde{S}^I_{w_0}$ is equal to $D \cap \mathbb{Z}^n \times P$, and $G \cap D$ is a normal affine semigroup. Now every point of $G \cap D$ lies in $\tilde{S}^w_{w_0}$, and hence in $\mathbb{N}^n \times P^+$. Thus $G \cap D = G \cap (\mathbb{N}^n \times P^+) \cap D = (\mathbb{N}^n \times J(I)) \cap \tilde{S}^w_{w_0} = \tilde{S}^w_{w_0,I}$. □

Let λ be a dominant integral weight. By Theorem 6.2.1, the set $\mathcal{B}_\lambda v_\lambda$ is a basis of $V_v(\lambda)$, and hence it has a dual basis $\mathcal{B}^*_\lambda$. For every $s \in S_\lambda$ there is an element $b^*_{s,\lambda} \in \mathcal{B}^*_\lambda$ defined as the only functional that sends bv_λ to δ_{b,b_s} for all $b \in B$. Each linear functional $b^*_{s,\lambda}$ induces a matrix coefficient which by abuse of notation we will

also denote by $b^*_{s,\lambda}$. We denote by $C^+_{\mathcal{A}}$ the sub $\mathcal{A}$-module of $C^+_v(\lambda)$ generated by these matrix coefficients, and set

$$\mathcal{O}_{\mathcal{A}}[G/B] = \bigoplus_{\lambda \in P^+} C^+_{\mathcal{A}}(\lambda).$$

Clearly $\mathcal{O}_{\mathcal{A}}[G/B] \subset \mathcal{O}_q[G/B] \subset U_q(\mathfrak{g})^*$. The product of $\mathcal{O}_q[G/B]$ is induced by the product of this last algebra, which in turn is induced by the coproduct of $U_q(\mathfrak{g})$. Since this $U_{\mathcal{A}}$ is stable by this coproduct, $\mathcal{O}_{\mathcal{A}}[G/B]$ is a sub $\mathcal{A}$-algebra of $\mathcal{O}_q[G/B]$.

Let λ be a dominant integral weight, let I be a finite set of fundamental weights and let $w \in W^I$. Then setting

$$\mathcal{B}^*_{\lambda,w} = \{b^*_{s,\lambda} \mid (s,\lambda) \in \tilde{S}^w_{w_{0,I}}\}$$
$$\mathcal{B}^\perp_{\lambda,w} = \{b_{s,\lambda} \in B^*_\lambda \cap V_v(\lambda)^\perp_w\}$$

we see that $\mathcal{B}_\lambda = \mathcal{B}^*_{\lambda,w} \sqcup \mathcal{B}^\perp_{\lambda,w}$, and $\mathcal{B}^\perp_{\lambda,w}$ is a basis of $V_v(\lambda)^\perp_w$, while the restriction of the functionals of $\mathcal{B}^*_{\lambda,w}$ to $V_v(\lambda)_w$ are a basis of $V_v(\lambda)^*_w$. Hence it is possible to define $\mathcal{A}$-forms of arbitrary quantum Schubert varieties, and by definition the matrix coefficients $b^*_{s,\lambda}$ with $(s,\lambda) \in \tilde{S}^w_{w_{0,I}}$ form an $\mathcal{A}$-basis of the $\mathcal{A}$-algebra $\mathcal{O}_{\mathcal{A}}[G/P_I]_w$. We keep abusing notation and denote the image of $b^*_{s,\lambda}$ in this quotient by the same symbol.

Proposition 6.2.3. *Let I be a finite set of fundamental weights and let $w \in W^I$. Then for every $(s,\lambda);(s',\lambda') \in \tilde{S}^w_{w_{0,I}}$ and every $(s'',\lambda'') <_{\mathsf{lex}} (s+s',\lambda+\lambda')$ there exist $c^{s''}_{s,s'} \in \mathcal{A}$ and $c = c((s,\lambda),(s',\lambda') \in \mathbb{Z}$ such hat*

$$b^*_{(s,\lambda)} b^*_{(s',\lambda')} = v^c b^*_{(s+s',\lambda+\lambda')} + \sum_{s'' <_{\mathsf{lex}} s+s'} c^{s''}_{s,s'} b^*_{(s'',\lambda+\lambda')}. \tag{6.1}$$

Proof. Let $k = \mathbb{C}(t^{1/d})$, where d is the length of the longest root of $\mathfrak{g}$. Then there is an obvious map $\mathbb{Q}(v) \longrightarrow k$ sending v to t. This induces an obvious morphism of Hopf algebras $U_v(\mathfrak{g}) \hookrightarrow U_t(\mathfrak{g})$. The canonical basis $\mathcal{B}$ of $U_v(\mathfrak{g})$ maps to a basis of $U_t(\mathfrak{g})$, and since Demazure modules are preserved by extension of scalars, Theorem 6.2.1 still holds over $U_t(\mathfrak{g})$ (see the reference given there). Thus there is an obvious injective $\mathcal{A}$-linear morphism from $C^+_{\mathcal{A}}(\lambda)$ to $C^+_t(\lambda)$, and hence an injective $\mathcal{A}$-linear morphism $\mathcal{O}_{\mathcal{A}}[G/B] \longrightarrow \mathcal{O}_t[G/B]$; this morphism is multiplicative since the extensions $U_{\mathcal{A}} \hookrightarrow U_v(\mathfrak{g}) \hookrightarrow U_t(\mathfrak{g})$ are morphisms of coalgebras. Thus it is enough to establish that the formula holds in $\mathcal{O}_t[G/B]$, which is done in [Calo2, Proposition 2.1]. □

We immediately obtain the following result.

Corollary 6.2.4. *Let I be a subset of the fundamental weights and let $w \in W^I$. Then the Schubert variety $\mathcal{O}_q[G/P_I]_w$ has a homogeneous $\tilde{S}^w_{w_{0,I}}$-basis.*

By Proposition 5.3.12 and Corollary 5.3.8, this implies that quantum Schubert varieties are normal domains, and that they are always AS-Cohen-Macaulay.

Appendix A

On the evaluation morphism

In this appendix we clarify some technical questions on the evaluation morphisms introduced in chapter 4. Throughout this appendix A is a noetherian connected $\mathbb{Z}^{r+1}$-graded algebra.

Let us review some of the pertinent definitions. Recall that given two complexes of A-modules $N^\bullet$ and $M^\bullet$, the n-th component of the complex $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet)$ is given by

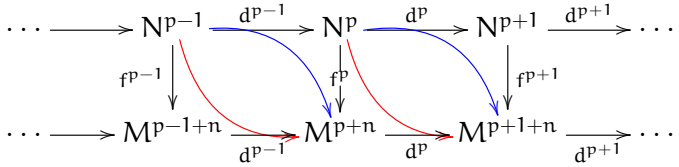
$$\prod_{p\in\mathbb{Z}} \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^p, M^{p+n}),$$

so an element in this component is a family of morphisms $f = (f^p)_{p\in\mathbb{Z}}$, with $f^p \in \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^p, M^{p+n})$ for all $p \in \mathbb{Z}$; we say that f is of degree n.

By definition the differential of $f = (f^p)_{p\in\mathbb{Z}}$ of degree n is the collection of morphisms whose p-th member is given by

$$d^n(f)^p = (-1)^{n+1} f^{p+1} \circ d_{N^\bullet}^p + d_{M^\bullet}^{p+n} \circ f^p$$

that is, given a family of morphisms of degree n, its differential is a family of morphisms of degree $n+1$ given by the sum (if n is odd) or the difference (if n is even) of the red and blue arrows in the following diagram



Fix $q \in \mathbb{Z}$ and $x \in N^q$. To each $n \in \mathbb{Z}$ and each $f = (f^p)_{p\in\mathbb{Z}} \in \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet)^n$ we can associate an element of M^{q+n}, namely $f^q(x)$; thus evaluation at x can be thought

of as a function from $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet)^n$ to M^{n+q}, which is by definition an element in the q-th component of the complex

$$\underline{\mathrm{Hom}}_k^{\mathbb{Z}^{r+1}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet), M^\bullet).$$

If $M^\bullet$ is a complex of A^e-modules then $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet)$ is a complex of A°-modules, and evaluation at x is compatible with the A°-module structure, so in fact lies in the q-th component of

$$\underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet), M^\bullet).$$

Let us denote by ev_x^n the function that assigns to each family of morphisms $(f^p)_{p\in\mathbb{Z}}$ of degree n the element $f^q(x) \in M^{n+q}$.

Lemma A.0.5. *Let $M^\bullet$ be a complex of A^e-modules and $N^\bullet$ be a complex of A-modules. Then there exists a morphism of complexes of A-modules*

$$\tau(N^\bullet, M^\bullet) : N^\bullet \longrightarrow \underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet), M^\bullet)$$

that sends each $x \in N^q$ to $((-1)^{nq}\,\mathrm{ev}_x^n)_{n\in\mathbb{Z}}$.

Proof. We write τ instead of $\tau(N^\bullet, M^\bullet)$ to alleviate notation. The function τ is well defined since we have already seen that ev_x^n sends families of morphisms of degree n to elements of M^{n+q}, so the collection $((-1)^{nq}\,\mathrm{ev}_x^n)_{n\in\mathbb{Z}}$ belongs to the q-th component of the target complex. It is only left to see that τ is a morphism complexes, i.e. that the differential of the family $((-1)^{qn}\,\mathrm{ev}_x^n)_{n\in\mathbb{Z}}$ is given by $((-1)^{(q+1)n}\,\mathrm{ev}_{d(x)}^n)_{n\in\mathbb{Z}}$.

Write δ for the differential of the complex $\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet)$. By definition,

$$d^q(\tau(x))^n = (-1)^{q+1}(-1)^{q(n+1)}\,\mathrm{ev}_x^{n+1} \circ \delta^n + (-1)^{qn} d_{M^\bullet}^{n+q} \circ \mathrm{ev}_x^n\,.$$

We now apply this to a family of morphisms $f = (f^p)_{p\in\mathbb{Z}}$ of degree q and obtain

$$\begin{aligned}
&(-1)^{qn+1}\,\mathrm{ev}_x^{n+1} \circ \delta^n(f) + (-1)^{qn} d_{M^\bullet}^{n+q} \circ \mathrm{ev}_x^n(f) \\
&\qquad = (-1)^{qn+1}\delta^n(f)^q(x) + (-1)^{qn} d_{M^\bullet}^{n+q}(f^q(x)) \\
&\qquad = (-1)^{qn+1}[(-1)^{n+1} f^{q+1} d_{N^\bullet}^q(x) + d_{M^\bullet}^{n+q}(f^q(x))] + (-1)^{qn} d_{M^\bullet}^{q+n}(f^q(x)) \\
&\qquad = (-1)^{(q+1)n} f^{q+1}(d_{N^\bullet}^q(x)) + [(-1)^{qn+1} + (-1)^{qn}] d_{M^\bullet}^{q+n}(f^q(x)) \\
&\qquad = (-1)^{(q+1)n} f^{q+1}(d_{N^\bullet}^q(x)) = (-1)^{(q+1)n}\,\mathrm{ev}_{d(x)}^n(f).
\end{aligned}$$

Thus $d(\tau(x))^n(f) = \tau(d(x))^n(f)$, which is precisely what we set out to prove. □

Given a morphism of complexes $f : N^\bullet \longrightarrow \tilde{N}^\bullet$ it is routine to prove that the following diagram commutes

$$\begin{array}{ccc}
N^\bullet & \xrightarrow{\tau(N^\bullet, M^\bullet)} & \underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(N^\bullet, M^\bullet), M^\bullet) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (f^*)^*} \\
\tilde{N}^\bullet & \xrightarrow{\tau(\tilde{N}^\bullet, M)} & \underline{\mathrm{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(\tilde{N}^\bullet, M^\bullet), M^\bullet).
\end{array}$$

so transformation τ is natural in the first variable. In section 4.2 we introduced the contravariant functors

$$D = \mathcal{R}\,\underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}(-, R^\bullet) : \mathcal{D}^b_{fg}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A) \longrightarrow \mathcal{D}^b_{fg}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^\circ),$$
$$D^\circ = \mathcal{R}\,\underline{\mathsf{Hom}}_{A^\circ}^{\mathbb{Z}^{r+1}}(-, R^\bullet) : \mathcal{D}^b_{fg}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^\circ) \longrightarrow \mathcal{D}^b_{fg}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A).$$

were $R^\bullet$ is a $\mathbb{Z}^{r+1}$-graded dualizing complex over A. Fixing an injective resolution $R^\bullet \longrightarrow I^\bullet$, the map $\tau(M^\bullet, I^\bullet)$ induces a morphism $\tau(M^\bullet) : M^\bullet \longrightarrow D^\circ D(M^\bullet)$ which is natural in $M^\bullet$ by the previous observation.

Let $\varphi : \mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}$ be a group morphism and assume that A is φ-connected. Given an object $M^\bullet$ of $\mathcal{D}^b_{fg}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A)$, Proposition 4.1.8 guarantees that $\varphi_!^{A^\circ}(D(M))$ is isomorphic to $D(\varphi_!^A(M^\bullet))$. Since $D(M^\bullet)$ is also bounded and has finitely generated cohomology groups, item 1 of Proposition 4.2.2 implies $\varphi_!^A(D^\circ D(M))$ is isomorphic to $D^\circ D(\varphi_!^A(M))$. The following result was used to prove the third item of Proposition 4.2.3.

Proposition A.0.6. *The following diagram commutes,*

$$\begin{array}{ccc} \varphi_!^A(M^\bullet) & \xrightarrow{\varphi_!^A(\tau(M^\bullet))} & \varphi_!^A(D^\circ D(M^\bullet)) \\ & \searrow^{\tau(\varphi_!^A(M^\bullet))} & \downarrow \cong \\ & & D^\circ D(\varphi_!^A(M^\bullet)) \end{array}$$

where the vertical isomorphism is induced by the natural transformation from Proposition 4.1.8.

Proof. We replace $R^\bullet$ with a resolution $I^\bullet$ in $D^+(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A^e)$ consisting of modules which are both left and right $\mathbb{Z}^{r+1}$-graded injective modules. Since the natural transformation $\varphi_!^{A^e}(I) \Rightarrow \varphi_*^{A^e}(I)$ is a quasi-isomorphism when I is a φ-finite module, item 1 of Proposition 4.1.2 implies that the same holds if we replace I by a bounded complex with finitely generated (and in particular φ-finite) cohomology modules, so $\varphi_!^{A^e}(R^\bullet) \longrightarrow \varphi_*^{A^e}(R^\bullet)$ is a quasi-isomorphism, and since $\varphi_*^{A^e}$ is exact, the composition $\varphi_!^{A^e}(R^\bullet) \longrightarrow \varphi_*^{A^e}(R^\bullet) \longrightarrow \varphi_*^{A^e}(I^\bullet)$ is also a quasi-isomorphism. Using the fact that φ_*^A sends injective modules to injective modules (see Proposition 2.2.9) and that $\varphi_*^{A^e}$ commutes with the functors Λ and P in the obvious sense, we see that the components of $\varphi_*^{A^e}(I^\bullet)$ are injective as left $\mathbb{Z}$-graded A-modules. We also fix a projective resolution $P^\bullet \longrightarrow M^\bullet$; since $\varphi_!^A$ preserves projectives, the map $\varphi_!^A(P^\bullet) \longrightarrow \varphi_!^A(M^\bullet)$ is a projective resolution.

Consider the following commutative diagram in the category $\mathcal{D}(\mathsf{Mod}^{\mathbb{Z}^{r+1}} A)$,

$$\begin{array}{ccccc} \varphi_!^{A^\circ}\,\underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}(P^\bullet, I^\bullet) & \xrightarrow{F} & \underline{\mathsf{Hom}}_A^{\mathbb{Z}}(\varphi_!^A(P^\bullet), \varphi_!^{A^e}(I^\bullet)) & \xrightarrow{\cong} & \underline{\mathsf{Hom}}_A^{\mathbb{Z}}(\varphi_!^A(P^\bullet), \varphi_*^{A^e}(I^\bullet)) \\ \cong \uparrow & & \uparrow & & \uparrow \cong \\ \varphi_!^{A^\circ}\,\underline{\mathsf{Hom}}_A^{\mathbb{Z}^{r+1}}(M^\bullet, I^\bullet) & \xrightarrow{F} & \underline{\mathsf{Hom}}_A^{\mathbb{Z}}(\varphi_!^A(M^\bullet), \varphi_!^{A^e}(I^\bullet)) & \longrightarrow & \underline{\mathsf{Hom}}_A^{\mathbb{Z}}(\varphi_!^A(M^\bullet), \varphi_*^{A^e}(I^\bullet)) \end{array}$$

where the unlabeled maps are induced by the natural inclusion $\varphi_!^{A^e}(I^\bullet) \longrightarrow \varphi_*^{A^e}(I^\bullet)$. Notice that the two complexes in the first column are quasi-isomorphic to the complex $\varphi_!^{A^\circ}(D(M^\bullet))$ and that the map connecting them labeled with $\cong$ is just the natural identification between them, so it is a quasi-isomorphism. The remaining complexes in the first row and the last one in the bottom row are all isomorphic to $D(\varphi_!^A(M^\bullet))$ and the maps between them are also natural identifications. Finally, the maps labeled F are quasi-isomorphisms by Proposition 4.1.8. Notice in particular that the first map in the top row is the morphism through which we identify $\varphi_!^{A^\circ}(D(M^\bullet))$ and $D(\varphi_!^A(M^\bullet))$; thus this map can be identified in the derived category with the map given by the composition of the two maps in the bottom row, which we call G.

It follows from the definitions that the diagram

$$\begin{array}{ccc}
 & & \varphi_!^A \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(M^\bullet, I^\bullet), I^\bullet) \cong \varphi_!^A(D^\circ D(M^\bullet)) \\
 & \overset{\varphi_!^A(\tau(M^\bullet))}{\nearrow} & \Big\downarrow G \\
\varphi_!^A(M) & & \underline{\mathrm{Hom}}_A^{\mathbb{Z}}(\varphi_!^{A^\circ} \underline{\mathrm{Hom}}_A^{\mathbb{Z}^{r+1}}(M^\bullet, I^\bullet), \varphi_*^{A^e}(I^\bullet)) \cong D^\circ(\varphi_!^{A^\circ}(D(M^\bullet))) \\
 & \underset{\tau(\varphi_!^A(M^\bullet))}{\searrow} & \Big\uparrow G \\
 & & \underline{\mathrm{Hom}}_A^{\mathbb{Z}}(\underline{\mathrm{Hom}}_A^{\mathbb{Z}}(\varphi_!^A(M^\bullet), \varphi_*^{A^e}(I^\bullet)), \varphi_*^{A^e}(I^\bullet)) \cong D^\circ D(\varphi_!^A(M^\bullet))
\end{array}$$

is commutative, and this proves the desired result. □

Appendix B

On the AS-Gorenstein condition

Let G be a group and let A and B be G-graded rings, with A (left and right) noetherian. We denote by M a G-graded left A-module and by N a G-graded $B-A$-bimodule. In order to simplify notation we write $\underline{\mathrm{Hom}}_A$ for $\underline{\mathrm{Hom}}_A^G$ and $\underline{\mathrm{Ext}}_A^i$ for its i-th derived functor. Also we write $\underline{\mathrm{Tor}}_i$ for the i-th derived functor of the tensor product of G-graded modules.

There exists a morphism of G-graded B-modules, natural in both variables

$$\begin{aligned}\eta(N,M) : N \otimes_A M &\longrightarrow \underline{\mathrm{Hom}}_{A^\circ}(\underline{\mathrm{Hom}}_A(M,A),N)\\ n \otimes_A m &\longmapsto (\varphi \in \underline{\mathrm{Hom}}_A(M,A) \mapsto n\varphi(m)).\end{aligned}$$

Lemma B.0.7. *Suppose M is finitely generated and N is injective as right A-module. Then $\eta(N,M)$ is an isomorphism.*

Proof. If M is free then $\eta(N,M)$ simply identifies $N \otimes_A A^n$ with $\underline{\mathrm{Hom}}_{A^\circ}(A^n,N) \cong N^n$ as $B-A$-bimodules, so the result is clear for free modules. For the general case we fix a finite presentation $A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$, and get morphisms between the exact complexes

$$\begin{array}{ccccccc} N \otimes_A A^m & \longrightarrow & N \otimes_A A^m & \longrightarrow & N \otimes_A M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta(N,M)} & & \\ N^m & \longrightarrow & N^n & \longrightarrow & \underline{\mathrm{Hom}}_{A^\circ}(\underline{\mathrm{Hom}}_{A^\circ}(M,A),N) & \longrightarrow & 0. \end{array}$$

Notice that in the complex below we have identified $\underline{\mathrm{Hom}}_{A^\circ}(\underline{\mathrm{Hom}}_A(A^n,A),N)$ with N^n; this complex is exact because N is injective as a right A-module. Thus $\eta(N,M)$ is an isomorphism by the five lemma. □

This implies the existence of a spectral sequence, which is a graded version of Ischebeck's spectral sequence, introduced in [Isc69].

Theorem B.o.8. *Let* A, M *and* N *be as in the preamble. Suppose that* M *is finitely generated and* N *has finite injective dimension* d *as right* A*-module. Then there exists a convergent spectral sequence of* G*-graded* B*-modules*

$$E_2^{p,q} = \underline{\mathrm{Ext}}^p_{A^\circ}(\underline{\mathrm{Ext}}^{-q}_A(M,A),N) \Rightarrow \underline{\mathrm{Tor}}^A_{-p-q}(N,M) \qquad\qquad p,-q>0$$

Proof. Let $N \longrightarrow I^\bullet$ be a resolution of N by injective $B-A$-bimodules. Then each $I^\bullet$ is injective as right A-module, and setting

$$J^\bullet = \begin{cases} I^\bullet & \text{for } \bullet < d; \\ \ker(I^d \longrightarrow I^{d+1}) & \text{for } \bullet = d; \\ 0 & \text{for } \bullet > d; \end{cases}$$

with the obvious differentials, $N \longrightarrow J^\bullet$ is a finite resolution of N by $B-A$-bimodules which are injective as right A-modules.

Let $P^\bullet \longrightarrow M$ be a resolution of M by finitely generated projective G-graded A-modules. We consider the complex $J^\bullet \otimes_A P^\bullet$, which can be identified with the complex $\underline{\mathrm{Hom}}_{A^\circ}(\underline{\mathrm{Hom}}_A(P^\bullet,A),J^\bullet)$ through the natural transformation η. Writing $\hat{P}^\bullet = \underline{\mathrm{Hom}}_A(P^\bullet,A)$ we get an isomorphism of double complexes $J^\bullet \otimes_A P^\bullet \cong \underline{\mathrm{Hom}}_{A^\circ}(\hat{P}^\bullet,J^\bullet)$. Let ${}'E$ be the spectral sequence obtained by filtering $J^\bullet \otimes_A P^\bullet$ by rows and ${}''E$ the one obtained by filtering $\underline{\mathrm{Hom}}_{A^\circ}(\hat{P}^\bullet,J^\bullet)$ by columns. Since both complexes are isomorphic, ${}''E$ is isomorphic to the spectral sequence defined by filtering $J^\bullet \otimes_A P^\bullet$ by columns.

The first page of ${}'E$ is obtained by taking the homology of the rows of $J^\bullet \otimes_A P^\bullet$, i.e. fixing $q \in \mathbb{N}$ and looking at the homology of $J^\bullet \otimes P^{-q}$. Since P^{-q} is a G-graded free A-module, the rows are exact for $\bullet > 0$, so the columns of ${}'E_1^{p,q}$ are zero, except for $p=0$ where we get ${}'E_1^{0,q} = H^0(J^\bullet \otimes_A P^{-q}) = H^0(J^\bullet) \otimes_A P^{-q} = N \otimes_A P^{-q}$, and the differentials are induced by those of $P^\bullet$. Now taking homology on the columns we see that

$${}'E_2^{p,q} \cong \begin{cases} \underline{\mathrm{Tor}}^A_{-q}(N,M) & \text{for } p=0; \\ 0 & \text{for } p \neq 0. \end{cases}$$

Hence the spectral sequence degenerates at page 2, and so

$$H^i\left(\mathrm{Tot}^\oplus(J^\bullet \otimes_A P^\bullet)\right) \cong \underline{\mathrm{Tor}}^A_{-i}(N,M).$$

On the other hand, page 1 of ${}''E$ is given by

$${}''E_1^{p,q} \cong H^q(\underline{\mathrm{Hom}}_{A^\circ}(\hat{P}^\bullet,J^p)) \cong \underline{\mathrm{Hom}}_{A^\circ}(H^{-q}(\hat{P}^\bullet),J^p) \cong \underline{\mathrm{Hom}}_{A^\circ}(\underline{\mathrm{Ext}}^{-q}_A(M,A),J^p),$$

and the differentials are those induced by $J^\bullet$, so

$${}''E_2^{p,q} \cong H^p({}''E_1^{\bullet,q}) \cong \underline{\mathrm{Ext}}^p_{A^\circ}(\underline{\mathrm{Ext}}^{-q}_A(M,A),N).$$

Notice that if $p > d$ then $E_2^{p,q} = 0$, so the spectral sequence converges to

$$H^{p+q}(\mathrm{Tot}^{\oplus}(\underline{\mathrm{Hom}}_{A^\circ}(\hat{P}^\bullet, J^\bullet)) \cong H^{p+q}(\mathrm{Tot}^{\oplus}(J^\bullet \otimes_A P^\bullet)) \cong \underline{\mathrm{Tor}}^A_{-p-q}(N, M).$$

□

Taking $N = A$ we obtain the following result.

Corollary B.0.9. *Suppose A has finite injective dimension as graded right A-module. Then there exists a convergent spectral sequence*

$$E_2^{p,q} : \underline{\mathrm{Ext}}^p_{A^\circ}(\underline{\mathrm{Ext}}^{-q}_A(M, A), A) \Rightarrow \mathbb{H}^{p+q}(M) = \begin{cases} M & \text{if } p+q = 0 \\ 0 & \text{else.} \end{cases} \qquad p, -q > 0.$$

Suppose now that A is a noetherian connected $\mathbb{Z}^{r+1}$-graded algebra.

Corollary B.0.10. *If A is left and right AS-Gorenstein, then it is AS-Gorenstein.*

Proof. Let d, e be the left and right injective dimensions of A, respectively. The hypothesis implies the existence of graded isomorphisms

$$\underline{\mathrm{Ext}}^e_{A^\circ}(\underline{\mathrm{Ext}}^d_A(k, A), A) \cong \underline{\mathrm{Ext}}^e_{A^\circ}(k[l], A) \cong k[r-l]$$

where r and l are the right and left Gorenstein shifts, respectively. Hence the second page of the graded Ischebeck spectral sequence with $M = k$ and $N = A$ looks like

$$E_2^{p,q} = \underline{\mathrm{Ext}}^p_{A^\circ}(\underline{\mathrm{Ext}}^{-q}_{A^\circ}(k, A), A) \cong \begin{cases} k[r-l] & \text{if } p = e, -q = d; \\ 0 & \text{otherwise.} \end{cases}$$

Evidently this converges to $\mathbb{H}^{p+q-e+d}(k[l-r])$, and by Corollary B.0.9 it also converges to $\mathbb{H}^{p+q}(k)$. This can only happe if $e = d$ and $l = r$. □

There is a finer result due to Zhang (see [Zha97, Theorem 0.3]), which states that if A has property χ as left or right module over itself, then it is AS-Gorenstein if and only if it has finite injective dimension, further proof that an algebra with property χ behaves much like a commutative algebra.

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