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UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Familias Playas de Foliaciones Algebraicas

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Familias Playas de Foliaciones Algebraicas

En lo que sigue el autor desarrolla una teoría para determinar la compatibilidad de la noción de familias de foliaciones algebraicas singulares definidas a través de distribuciones involutivas de campos vectoriales, o a través de ideales diferenciales de formas. Se definen, usando construcciones algebro-geométricas, espacios de módulos para familias de ideales diferenciales y para familias de distribuciones involutivas, con tales construcciones se recuperan, en el caso algebraico, los espacios de módulos construídos por Gomez-Mont y Pourcin. Usando el enfoque algebro-geométrico, se puede mostrar que los espacios de distribuciones involutivas $\mathrm{Inv}^P(X)$ y de ideales diferenciales $\mathrm{iPf}^Q(X)$ son, de hecho, birracionales, ampliando así resultados obtenidos por Pourcin al respecto. También se expone una caracterización de abiertos de $\mathrm{Inv}^P(X)$ y $\mathrm{iPf}^Q(X)$ donde el funtor $\mathcal{H}om(-, \mathcal{O}_X)$ define un isomorfismo entre los dos espacios, estos abiertos se caracterizan por los tipos de singularidades de las foliaciones. Los resultados mostrados aquí generalizan los previamente obtenidos por Cukierman y Pereira en [FCJVP08] a foliaciones definidas sobre variedades proyectivas regulares cualesquiera.

Palabras Clave: Haces coherentes, Familias Playas, Foliaciones Algebraicas, Espacios de Moduli, Singularidades Kupka.

Flat Families of Algebraic Foliations

The author develops a theory in order to establish compatibility between the related notions of families of singular algebraic foliations given by involutive distributions of vector fields, and that given by differential ideales of forms. Using algebro-geometric constructions, moduli spaces for families of differential ideals and families of involutive distributions are defined, with these constructions we recover, in the algebraic case, moduli spaces as defined by Gomez-Mont and Pourcin. With the algebro-geometric approach we can establish birationallity between the moduli spaces $\mathrm{Inv}^P(X)$ of involutive distribution and $\mathrm{iPf}^Q(X)$ of differential ideals, thus generalizing Pourcin previous results. A characterization of open sub-spaces of $\mathrm{Inv}^P(X)$ and $\mathrm{iPf}^Q(X)$ where an isomorphism is defined by the functor $\mathcal{H}om(-, \mathcal{O}_X)$ is presented, this characterization is in terms of the singularities of the foliations. The results of this work generalize previous ones by Cukierman and Pereira in [FCJVP08] to foliations over regular projective varieties.

Keywords: Coherent Sheaves, Flat Families, Algebraic Foliations, Moduli Spaces, Kupka Singularities.

Índice general

INTRODUCCIÓN

Desde sus orígenes en el *Analysis situs* de Poincaré, la teoría geométrica de ecuaciones diferenciales en variedades se centró en la cuestión de la *estabilidad* de diversas características geométricas o topológicas del retrato de fases de las ecuaciones.

Más especificamente, si consideramos, por ejemplo, el sistema lineal

$$\begin{pmatrix} \dfrac{df}{dt} \\ \dfrac{dg}{dt} \end{pmatrix} = A \cdot \begin{pmatrix} f \\ g \end{pmatrix},$$

donde $A = -Id$, tenemos el sistema

$$\begin{cases} \dfrac{df}{dt} = -f \\ \dfrac{dg}{dt} = -g \end{cases}$$

y sabemos que en este caso el punto $(0,0)$ constituye un punto atractor en el diagrama de fases. Más aún, si perturbamos los coeficientes de la matriz A por una cantidad pequeña, es decir si establecemos $A' = A + \epsilon M$ con $\epsilon \sim 0$, sabemos que la matriz A' va a seguir teniendo dos autovalores negativos y por lo tanto el sistema

$$\begin{pmatrix} \dfrac{df}{dt} \\ \dfrac{dg}{dt} \end{pmatrix} = A' \cdot \begin{pmatrix} f \\ g \end{pmatrix},$$

va a seguir teniendo un atractor en el punto $(0,0)$.

Cuando se consideran sistemas no lineales de ecuaciones la situación cambia. Lo que tenemos ahora es un sistema de la forma

$$\begin{cases} \dfrac{df}{dt} = F_0(f(t), g(t)) \\ \dfrac{dg}{dt} = G_0(f(t), g(t)) \end{cases}, \tag{0.1}$$

con F_0 y G_0 funciones analíticas, por ejemplo. En este caso la existencia de un atractor en el diagrama de fases podría no ser una característica estable por perturbaciones. Con esto se quiere decir lo siguiente: podrían existir funciónes de tres variables $F(x, y, s)$ y $G(x, y, s)$ tales que $F(x, y, 0) = F_0(x, y)$, $G(x, y, 0) = G_0(x, y)$ y tales que, para todo $\epsilon > 0$ existe s, con $|s| < \epsilon$ tal que el sistema

$$\begin{cases} \dfrac{df}{dt} = F(f(t), g(t), s) \\ \dfrac{dg}{dt} = G(f(t), g(t), s) \end{cases}$$

(donde s permanece fijo) no tiene puntos atractores, o más aún, tal que no tiene puntos singulares (ver [Dem00]).

Para poder plantear este problema con mayor generalidad conviene plantearlo en términos más geométricos. En definitiva el sistema (0.1) tiene por soluciones $(f(t), g(t))$ a las curvas integrales del campo de vectores

$$X_0 = F_0(x, y)\frac{\partial}{\partial x} + G_0(x, y)\frac{\partial}{\partial y}.$$

Las cuestiones planteadas anteriormente entonces pueden formularse en términos de la geometría de la foliación singular definida por el campo X_0. En particular los puntos singulares del retrato de fase son los puntos (x, y) tales que $X_0|_{(x,y)} = 0$ y las cuestiones relacionadas a la estabilidad de los puntos singulares se traducen en preguntas sobre familias de campos $X_s = F(x, y, s)\frac{\partial}{\partial x} + G(x, y, s)\frac{\partial}{\partial y}$ dependiendo de un parámetro s.

Más en general, podemos estudiar familias de foliaciones singulares de cualquier dimensión, en este caso estaríamos generalizando el estudio geométrico de sistemas de ecuaciones del tipo

$$\begin{cases} \dfrac{\partial f_1}{\partial t_1} = F^{11}(f_1(t_1, \ldots, t_d), \ldots, f_n(t_1, \ldots, t_d)) \\ \quad\vdots \\ \dfrac{\partial f_1}{\partial t_d} = F^{1d}(f_1(t_1, \ldots, t_d), \ldots, f_n(t_1, \ldots, t_d)) \\ \quad\vdots \\ \dfrac{\partial f_n}{\partial t_1} = F^{n1}(f_1(t_1, \ldots, t_d), \ldots, f_n(t_1, \ldots, t_d)) \\ \quad\vdots \\ \dfrac{\partial f_n}{\partial t_d} = F^{nd}(f_1(t_1, \ldots, t_d), \ldots, f_n(t_1, \ldots, t_d)), \end{cases} \tag{0.2}$$

cuyas soluciones $(f_1, \ldots, f_d)$ son variedades integrales de la distribución Δ de sub-espacios del tangente definida por $\Delta(p) =< X^1|_p, \ldots, X^d|_p >\subset T_p\mathbb{C}^n$, donde

$$X^i = \sum_j F^{ji}\frac{\partial}{\partial x_j}.$$

Una condición necesaria y suficiente para que el sistema (0.2) tenga soluciones locales (o equivalentemente para que la distribución Δ tenga variedades integrales maximales) es la condición de Frobenius, que expresada en términos del sistema (0.2) son condiciones sobre las F^{ij} para asegurar que $\frac{\partial f_k}{\partial t_l \partial t_m} = \frac{\partial f_k}{\partial t_m \partial t_l}$. En términos de los campos X^i la condición de Frobenius pide que el corchete de Lie de dos campos que definen la distribución cumpla $[X^i, X^j]|_p \in \Delta(p),\ \forall p$ (ver [War83, cap 2]).

Quisieramos ahora estudiar las propiedades de familias de distribuciones $\Delta_s =< X_s^1, \dots, X_s^d >$ que cumplan la propiedad de Frobenius (que llamamos distribuciones *involutivas*) parametrizadas por un parámetro s. Es claro en este caso que tenemos que tomar recaudos si queremos que propiedades se mantengan entre miembros de una misma famila. Por ejemplo, si tomamos una familia completamente arbitraria de distribuciones de dimensión 2, $\Delta_s =< X_s^1, X_s^2 >$ bien podría suceder que para cierto s_0 pase que $X_{s_0}^1 = X_{s_0}^2$, en particular la dimensión de la distribución Δ_{s_0} es menor que la de Δ_s para $s \neq s_0$ general. Para evitar esta y otras situaciones debemos pedir condiciones extras a la distribución generada por campos X_s^i. Para describir estas condiciones vale observar que, si denotamos $\mathcal{O}_{\mathbb{C}^n}$ al haz de funciones holomorfas en $\mathbb{C}^n$ los campos X^i generan un sub-haz $\mathcal{O}_{\mathbb{C}^n}(X^i) \subset T\mathbb{C}^n$ del haz de campos tangentes.

Desde el punto de vista puramente geométrico estamos estudiando subhaces $T\mathcal{F} \subset TM$ del haz de campos de vectores en una variedad holomorfa M que cumplen la condición de Frobenius, es decir, tales que $[T\mathcal{F}, T\mathcal{F}] \subseteq T\mathcal{F}$. Desde este punto de vista las familias de distribuciones parametrizadas por un parámetro s son sub-haces $T\mathcal{F}_S \subset T_S(M \times S)$ del haz $T_S(M \times S)$ de campos tangentes a las fibras de la proyección $M \times S \to S$, que cumplen la condición de Frobenius. Aquí, la distribución Δ_s, para $s \in S$ está dada por el pull-back $i_s^*(T\mathcal{F}_S)$ de $T\mathcal{F}_S$ por la inclusión $i_s : M \cong M \times \{s\} \hookrightarrow M \times S$.

Desde este punto de vista está bien entendida cuál tiene que ser la condición a pedir para asegurar la “continuidad”de las familias de distribuciones en el caso holomorfo/algebraico. Dado el sub-haz $T\mathcal{F}_S \subseteq T_S(M \times S)$ la condición a pedir es que en la sucesión exacta corta

$$0 \to T\mathcal{F}_S \to T_S(M \times S) \to Q_S \to 0$$

el haz Q_S sea *playo* sobre el espacio de parámetros S, cuando esto sucede decimos que la familia es playa.

A partir de estos tecnicismos Gomez-Mont pudo, en [GM88], establecer propiedades importantes de foliaciones por curvas generalizando los resultados fundacionales de Ilyashenko y, más aún, demostrar que, en el caso en que M sea compacta, existe un espacio de parámetros universal para familias de distribuciones involutivas, es decir que existe un espacio Inv_M y una familia playa $T\mathcal{F} \subseteq T_{Inv_M}(M \times Inv_M)$ de distribuciones involutivas tales que, para cualquier otra familia $T\mathcal{F}_S \subseteq T_S(M \times S)$, existe un único morfismo $f : S \to Inv_M$ tal que $T\mathcal{F}_S = (f \times id_M)^*(T\mathcal{F})$.

Por otra parte, otra manera de describir una distribución Δ es como el núcleo de un conjunto de 1-formas diferenciales $\omega_1, \ldots, \omega_q$, es decir $\Delta(p) = \{v \in T_pM : \ (\omega_1)_p(v) = \cdots = (\omega_q)_p(v) = 0\}$. En este caso Δ tiene *codimensión* q. La condición de integrabilidad de Frobenius se expresa, en estos términos como $d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_q = 0 \ \forall 1 \leq i \leq q$ ([War83, cap. 3]).

Esta descripción de la distribución es conveniente cuando la codimensión de la distribución es pequeña. Por ejemplo cuando la distribución tiene codimensión 1 está dada por los ceros de una forma ω y es integrables si $\omega \wedge d\omega = 0$. En este caso si queremos estudiar familias de foliaciones podemos estudiar simplemente familias de 1-formas $\omega_s = \sum_i f_i(x_1, \ldots, x_n, s)dx_i$. Con este enfoque Cerveau, Lins-Neto, Camacho, Calvo-Andrade, Cukierman, Pereira y otros autores han conseguido resultados de clasificación de foliaciones de codimensión 1 en variedades algebraicas (ver [ALN07] y referencias ahí dentro). Más precisamente, puede probarse que, así como existe un espacio de parámetros universal para familias $T\mathcal{F} \subseteq TM$, también existe, si M es compacta, un espacio de parámetros universal iPf_M para familias de formas integrables. Uno de los problemas principales en la teoría de foliaciones holomorfas consiste en clasificar las componentes irreducibles de este espacio en el caso $M = \mathbb{P}^n(\mathbb{C})$, fueron resultados en este sentido que describieron los autores mencionados.

Surge entonces la cuestión de relacionar estas dos nociones de familias de foliaciones. Pourcin plantea en [Pou87] este problema y construye los espacios Inv_M y iPf_M para M una variedad holomorfa compacta. En [Pou88] observa que, en general, estos dos espacios no son isomorfos (lo que equivale a decir que las dos nociones de familias de foliaciones dadas por campos y formas no son equivalentes). Lo que sucede en general es que, si $T\mathcal{F}_S \subseteq T_S(M \times S)$ es una familia playa de campos, la familia de formas $I^1_S \subseteq \Omega^1_{M\times S|S}$ que anulan a $T\mathcal{F}_S$ no es en general una familia playa; o sea que, aunque la familia de campos cumplan con la noción de continuidad, la familia de 1-formas que anulan a esos campos no tiene por qué cumplir la condición de continuidad. Pourcin demuestra que, sin embargo, las dos nociones son equivalentes si la familia es de foliaciones *no singulares*. De cualquier manera el estudio de familias de foliaciones singulares tiene interés intrínseco, como ya vimos, y, además en muchas variedades holomorfas (por ejemplo $\mathbb{P}^n(\mathbb{C})$) todas las foliaciones poseen singularidades.

En esta tesis se avanza sobre las cuestiones planteadas en [Pou87] y [Pou88] en el caso en que M sea una variedad algebraica y las foliaciones estén dadas por distribuciones algebraicas. En primer lugar, en el caso en que M sea una variedad proyectiva, se da una construcción distinta de los espacios Inv_M y iPf_M aprovechando la estructura natural de variedad algebraica que tienen estos espacios. También se demuestra que las componentes irreducibles de estos dos espacios están en biyección natural y son *birracionalmente equivalentes*. Más aún, se demuestra que la noción de familia de foliaciones dada por campos o formas es equivalente siempre y cuando el *sub-esquema singular* $\mathrm{sing}(\mathcal{F})$ de la familia de foliaciones sea playo sobre el espacio de parámetros S. El resultado más específico de la tesis es el teorema 4.5.10, que da un criterio para que el lugar singular de una familia de foliaciones sea playo sobre la base; a partir de este teorema se obtiene como corolario el resultado principal de [FCJVP08] (es decir que el teorema 4.5.10 puede considerarse como una generalización de [FCJVP08, Theorem 1] al caso de una variedad algebraica proyectiva en general).

Pasamos ahora a describir el contenido de los capítulos de la tesis.

El capítulo 1 contiene preeliminares de geometría algebraica, resultados que van a usarse más adelante en el texto y cuyo contenido está enmarcado en la teoría general de esquemas. Este capítulo comprende una resumen sobre la construcción y las propiedades del esquema $\mathrm{Quot}(\mathrm{X}, \mathscr{F})$, que parametriza cocientes de un haz dado $\mathscr{F}$. También contiene una exposición sobre el criterio valuativo de playitud, tal como aparece en [Gro65, 11.8], que es un resultado usado intensivamente a lo largo de las demostraciones. Por último hay en este capítulo una exposición sobre el lema de Nakayama para funtores semi-exactos en el sentido de [GBAO72].

En el capítulo 2 se encuentran las definiciones clásicas de foliaciones y foliaciones singulares, así como una pequeña disquisición acerca de la equivalencia de las distintas descripciones de una foliación singular. La sección 2.4 es una exposición de resultados concernientes a las singularidades de una foliación. Entre otras cosas contiene una demostración de una versión "relativa" del clásico teorema de Kupka [dM77, Fundamental Lemma], cuya demostración es esencialmente la que aparece en [dM77] con ligeras modificaciones. También se incluye en este capítulo un resumen acerca de tres familias clásicas de foliaciones en $\mathbb{P}^n(\mathbb{C})$ que representan ejemplos motivadores para este trabajo, estas son las foliaciones logarítmicas, pull-back y racionales. Se incluye una clasificación de las singularidades de estos tipos de foliaciones, esta clasificación va a ser relevante en cuanto establece que el principal teorema del Capítulo 4 aplica en particular a estas familias.

En el capítulo 3 se desarrollan resultados originales sobre la relación entre la playitud de un haz y la playitud del haz dual. Son resultados que van a tener directa injerencia sobre el caso de foliaciones.

En el capítulo 4 se da la construcción del espacio de moduli de ditribuciones involutivas, así como la construcción del espacio de moduli de (ideales de) formas integrables, a través de los esquemas $\mathrm{Quot}(TX)$ y $\mathrm{Quot}(\Omega^1_X)$. Se usan resultados sobre haces coherentes así como el lema de Nakayama para funtores semi-exactos de 1.5 para demostrar la equivalencia birracional de las componentes irreducibles de estos dos espacios. A partir del teorema de Kupka relativo y el estudio de singularidades de Reeb se puede demostrar también el teorema 4.5.10. El teorema 4.5.10 junto con un estudio de foliaciones con haz tangente escindido nos dan, a partir de un teorema de Hilbert, el resultado principal de [FCJVP08] como corolario.

1. ALGEBRAIC GEOMETRIC PRELIMINARIES

1.1. *Assorted generalities*

Here we gather general results in algebraic geometry that will be usefull in the rest of the following work.

1.1.1. *Reflexive sheaves and Serre's property S_2*

Property S_2 can be viewed as an algebraic analog of Hartog's theorem on complex holomorphic functions. For this reason, it will be extermely usefull to us, for it'll allow us to conclude global statements on sheaves that a priori holds for the restriction of this sheaves to (suitably large) open sets.

Definition 1.1.1. A module M over a ring R satisfies Serre's condition S_k if and only if

$$\operatorname{depth} M_{\mathfrak{P}} \geq \operatorname{mín}(k, \dim M_{\mathfrak{P}}).$$

for all $\mathfrak{P} \in \operatorname{Spec}(R)$, where $\dim M_{\mathfrak{P}}$ is the Krull dimension of the support of $M_{\mathfrak{P}}$.

Proposition 1.1.2. *Let X be a noetherian scheme and $\mathscr{F}$ a torsion-free coherent sheaf with property S_2. Let $Y \subset X$ be a closed subset of codimension ≥ 2. Then the restriction map $\rho : \Gamma(X, \mathscr{F}) \to \Gamma(X \setminus Y, \mathscr{F})$ is an isomorphism.*

Demostración. As $\mathscr{F}$ is torsion-free, the restriction map ρ is injective, otherwise any section in the kernel would be anihilated by an $f \in \mathcal{O}_X$ such that $f|Y = 0$. Now let's take a section $s \in \Gamma(X \setminus Y, \mathscr{F})$ and suppose s does not extend to a global section. Define the subsheaf $J_s \subseteq \mathcal{O}_X$ as

$$J_s(U) = \{f \in \mathcal{O}_X(U) \colon fs \in \operatorname{Im}(\rho|_{U\setminus Y})\}.$$

Note that J_s is a sheaf of ideals. Suppose $J_s \neq \mathcal{O}_X$ and denote by $Z \subseteq X$ the subscheme defined by J_s. As $s \in \Gamma(X \setminus Y, \mathscr{F})$, and the codimension of Y is ≥ 2, then $\operatorname{codim}(Z) \geq 2$. So let $\mathfrak{P} \in Z$ be a point whose closure have codimension bigger than 2. Then, as $\mathscr{F}$ is torsion free, $\dim \mathscr{F}_{\mathfrak{P}} \geq 2$. So by property S_2, $\operatorname{depth}\mathscr{F}_{\mathfrak{P}} \geq 2$. Lets take then an $\mathscr{F}$-regular sequence (f_1, f_2) in $\mathcal{O}_{X,\mathfrak{P}}$. Since f_1 and f_2 are part of a regular sequence, they belong to $\mathfrak{P}$.

Then some power of f_1 is in $J_{s,\mathfrak{P}}$, respectively f_2. Then there are $a, b \in \mathscr{F}_{\mathfrak{P}}$ such that

$$f_1^n s = a, \qquad f_2^m s = b,$$

for minimal $n, m \in \mathbb{N}$ such that $f_1^n, f_2^m \in J_{s,\mathfrak{P}}$. So $f_2^m a = f_1^n b$. Then $f_2^m a \equiv 0 \mod (f_1)$. As (f_1, f_2) is a regular sequence this means $a \in f_1 \cdot \mathscr{F}_{\mathfrak{P}}$. But then $f_1^{n-1} \in J_s$ contradicting the minimality of n. Then $Z = \emptyset$ and $s \in \mathrm{Im}(\rho)$. □

Corollary 1.1.3. *Let X be a noetherian scheme and $\mathscr{F}$ a torsion-free coherent sheaf with property S_2. Let $Y \subset X$ be a closed subset of codimension ≥ 2. Denote $U = X \setminus Y$ and $j : U \to X$ the inclusion. Then $\mathscr{F} = j_*(\mathscr{F}|_U)$.*

This corollary motivates the following definition due to Grothendieck [Gro65, 5.10].

Definition 1.1.4. Let X be a noetherian scheme and $\mathscr{F}$ a coherent sheaf. If for each closed subset $Y \subset X$ of codimension ≥ 2, with $U = X \setminus Y$ and $j : U \to X$ the inclusion, the natural map

$$\rho_U : \mathscr{F} = j_*(\mathscr{F}|_U)$$

is an epimorphism we say $\mathscr{F}$ is $Z^{(2)}$*-closed*, if it is an isomorphism we say it is $Z^{(2)}$*-pure*.

With this definitions we can state a partial reciprocal statement of the above corollary

Proposition 1.1.5 ([Gro65, 5.10.14])**.** *Let $\mathscr{F}$ be a coherent sheaf on X with support equal to X. For $\mathscr{F}$ to have S_2 is equivalent to $\mathscr{F}$ being $Z^{(2)}$-pure and having no associated primes of codimension 1.*

Proof. [Gro65, 5.10.14] □

Next we characterize sheaves with property S_2 as reflexive sheaves. This will come handy as so many of the sheaves we'll deal with will be reflexive.

Proposition 1.1.6. *Let X be a quasi-projective integral scheme. A coherent sheaf $\mathscr{F}$ is reflexive if and only if it can be included in an exact sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0,$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion-free.

Proof. Suppose $\mathscr{F}$ is reflexive. Take $\mathscr{F}^\vee$ the dual sheaf of $\mathscr{F}$. As X is quasi-projective then there is an exact sequence

$$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F}^\vee \to 0,$$

such that $\mathscr{L}_i$ is locally free. Dualizing we get an exact sequence

$$0 \to \mathscr{F}^{\vee\vee} \cong \mathscr{F} \to \mathscr{L}_0^{\vee} \xrightarrow{\delta} \mathscr{L}^{\vee}.$$

Then set $\mathscr{E} = \mathscr{L}^{\vee}$, is locally free because X is integral, and $\mathscr{G} = \mathrm{Im}(\delta) \subseteq \mathscr{L}^{\vee}$, is torsion-free, being a subsheaf of a locally free sheaf.

Conversely suppose that there is an exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0,$$

with $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion-free. Then $\mathscr{F}$ is torsion free so the natural map

$$\mathscr{F} \to \mathscr{F}^{\vee\vee}$$

is injective. But, as $\mathscr{E}$ is reflexive, we also have a map $\mathscr{F}^{\vee\vee} \to \mathscr{E}$. It is injective, for is generically injective and $\mathscr{F}^{\vee\vee}$ is torsion-free. Then $\mathscr{F}^{\vee\vee}/\mathscr{F}$ is a subsheaf of $\mathscr{G}$ as well as a torsion sheaf. Then $\mathscr{F} = \mathscr{F}^{\vee\vee}$, for $\mathscr{G}$ is torsion-free. $\square$

Corollary 1.1.7. *Under the above circumstances, the dual of a coherent sheaf is allways reflexive.*

Proof. Take an exact sequence

$$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0,$$

with $\mathscr{L}_i$ locally free. Dualize it to obtain an exact sequence

$$0 \to \mathscr{F}^{\vee} \to \mathscr{L}_0^{\vee} \to \mathscr{L}_1^{\vee}$$

and take $\mathscr{G}$ the image of the second arrow inside $\mathscr{L}_1^{\vee}$. Then $\mathscr{L}_0^{\vee}$ is locally free and $\mathscr{G}$ is torsion free. $\square$

Proposition 1.1.8 ([Har94, Theorem 1.9])**.** *Let X be a noetherian normal integral scheme, and $\mathscr{F}$ a coherent sheaf on X. Then, if $\mathscr{F}$ is reflexive, it has property S_2.*

Proof. The statement being local, we can assume X is quasi-projective. Given a reflexive sheaf $\mathscr{F}$, we take an exact sequence

$$0 \to \mathscr{F} \to \mathscr{L} \to \mathscr{G} \to 0$$

with $\mathscr{L}$ locally free and $\mathscr{G}$ torsion-free. Since X is normal, $\mathcal{O}_X$ satisfies property S_2 (this is [Gro65, 5.8.6]), and so does $\mathscr{L}$, being locally free. Let $\mathfrak{P}$ be a point of dimension ≥ 2. Then $\mathrm{depth}\mathscr{L}_{\mathfrak{P}} \geq 2$ by S_2, and as $\mathscr{G}$ is torsion-free, $\mathrm{depth}\mathscr{G}_{\mathfrak{P}} \geq 1$. This in turn implies $\mathrm{depth}\mathscr{F}_{\mathfrak{P}} \geq 2$. $\square$

1.1.2. Support of a sheaf, zeros of a section

In the course of this work some spaces will appear naturally as *the zero locus* of a function (or more generally a section of a given sheaf). While such notion may be intuitive and feel somehow natural, is important for us to define a distinctive scheme structure on such loci. The interest in this details arise as a scheme Y might be flat over a base S but its reduced structure Y_{red} might not.

Recall that, given a module M over a ring R we define the annihilator ideal $\mathrm{Ann}(M)$ as the set $\{x \in R \colon xm = 0,\ \forall m \in M\}$.

Definition 1.1.9. Let $\mathscr{F}$ be a quasi-coherent sheaf on a scheme X. We define the *support* of $\mathscr{F}$, $\mathrm{supp}(\mathscr{F})$ as the closed sub-scheme defined by the ideal sheaf given locally by

$$\mathcal{I}(\mathscr{F})_x := \mathrm{Ann}(\mathscr{F}_x) \subset \mathcal{O}_{X,x}.$$

note that, as taking annihilator ideal commutes with localization, this notion is well defined.

We have the following usefull characterization of the support of a sheaf in terms of a universal property:

Proposition 1.1.10. *The support of a sheaf $\mathscr{F}$ represents the functor*

$$\begin{array}{rcl} \mathrm{S}_{\mathscr{F}} : Sch & \longrightarrow & Sets \\ T & \mapsto & \{f \in \hom(T, X) \colon \mathrm{Ann}(f^*\mathscr{F}) = 0 \subset \mathcal{O}_T\}. \end{array}$$

Proof. A morphism $f : T \to X$ factorizes through $\mathrm{supp}(\mathscr{F})$ if and only if the map

$$f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_T$$

factorizes through $f^{-1}(\mathcal{O}_X/\mathrm{Ann}(\mathscr{F}))$. But this happens if and only if $f^{-1}(\mathrm{Ann}(\mathscr{F})) = 0$.

On the other hand we have the equality

$$\mathrm{Ann}(f^*\mathscr{F}) = \mathcal{O}_T \cdot f^{-1}(\mathrm{Ann}(\mathscr{F})),$$

indeed we may check this in every localization at any point $p \in T$, so if $t \in \mathrm{Ann}(f^*\mathscr{F})_p$ in particular t annihilates every element of the form $m \otimes 1 \in \mathscr{F}_x$, so $t = f^{-1}(x)t'$ where $x \in \mathrm{Ann}(\mathscr{F})_{f(p)}$.

So $\mathrm{Ann}(f^*\mathscr{F}) = 0$ if and only if $f^{-1}(\mathrm{Ann}(\mathscr{F})) = 0$ and we are done. $\square$

In other words we just proved that $\mathrm{supp}(\mathscr{F})$ is the universal scheme with the property that $f^*\mathscr{F}$ is not a torsion module. This simple observation will be very usefull when discussing the scheme structure on the singular set of a foliation.

A special case of support of a sheaf is the scheme theoretic image of a morphism.

Definition 1.1.11. The *scheme theoretic image* of a morphism $f : X \to Y$ is the sub-scheme $\operatorname{supp}(f_*\mathcal{O}_X) \subseteq Y$.

Now we turn our attention to sections and their zeros. So let X be a scheme and $\mathscr{E}$ a locally free sheaf. Remember that having a global section $s \in \Gamma(X, \mathscr{E})$ is the same as having a morphism (that, by abuse of notation, we also call s)

$$s : \mathcal{O}_X \longrightarrow \mathscr{E}.$$

Now, $s : \mathcal{O}_X \to \mathscr{E}$ defines a dual morphism

$$s^\vee : \mathscr{E}^\vee \longrightarrow \mathcal{O}_X^\vee = \mathcal{O}_X.$$

Definition 1.1.12. We define the *zero scheme* $Z(s)$ of the section s as the closed sub-scheme of X defined by the ideal sheaf $\operatorname{Im}(s^\vee) \subseteq \mathcal{O}_X$.

Note that, with this definition, the zero scheme of a non-zero section might be X. Indeed, if $s \in \mathscr{E}$ is a torsion element, then $s^\vee = 0$, and so $Z(s) = X$. We will, however, apply this definition in the better behaved situation where $\mathcal{O}_X$ (and therefore $\mathscr{E}$) is torsion-free.

Remark 1.1.13. Take the case $X = \mathbb{P}^n_k$ and $\mathscr{E} = \mathcal{O}_{\mathbb{P}^n_k}(n)$. A global section s is a homogeneous polynomial $s = F(x_0, \ldots, x_n)$. The pairing

$$\mathcal{O}_{\mathbb{P}^n}(-n) \otimes \mathcal{O}_{\mathbb{P}^n}(n) \to \mathcal{O}_{\mathbb{P}^n}$$

is given locally by multiplication of rational functions. Therefore $\operatorname{Im}(s^\vee)(U) = \mathcal{O}_{\mathbb{P}^n}(U) \cdot (F)$. So the scheme $Z(s)$ is actually the scheme theoretic zero locus $(F(x_0, \ldots, x_n) = 0) \subseteq \mathbb{P}^n_k$.

Proposition 1.1.14. *Let $\mathscr{E}$ be a locally free sheaf on X and $s \in \Gamma(X, \mathscr{E})$ a global section. The scheme $Z(s)$ represents the functor*

$$\begin{aligned} \mathrm{Z}_s : Sch &\longrightarrow Sets \\ T &\mapsto \{f \in \hom(T, X) \colon s \otimes 1 = 0 \in \Gamma(T, f^*\mathscr{E})\}. \end{aligned}$$

Proof. A morphism $f : T \to X$ factorizes through $Z(s)$ if and only if the map

$$(\mathscr{E}^\vee) \otimes \mathcal{O}_T \xrightarrow{s^\vee \otimes 1} \mathcal{O}_T$$

is identically 0. Beign locally free we have

$$\mathscr{E}^\vee \otimes \mathcal{O}_T = \mathscr{H}om(\mathscr{E}, \mathcal{O}_X) \otimes \mathcal{O}_T \cong \mathscr{H}om(\mathscr{E} \otimes \mathcal{O}_T, \mathcal{O}_T).$$

So then we have

$$(f^*\mathscr{E})^\vee \xrightarrow{s^\vee \otimes 1} \mathcal{O}_T$$

is identically 0, as $f^*\mathscr{E}$ is locally free over T, this means $s \otimes 1 = 0 \in \Gamma(T, f^*\mathscr{E})$. $\square$

1.2. Flattening stratification

As the main problem in this thesis will be to establish the flatness of a given family of sheaves, we will need a standard tool to aid us, this is the flattening stratification introduced by Mumford in [Mum66].

Theorem 1.2.1. *Let S be a noetherian scheme, and $X \to S$ a projective morphism. Let $\mathscr{F}$ be a coherent sheaf on X over S. Then the set I of Hilbert polynomials of restrictions of $\mathscr{F}$ to fibers of $X \to S$ is a finite set. Moreover, for each $P \in I$ there exist a locally closed subscheme S_P of S, such that the following conditions are satisfied.*

(i) Point-set: *The underlying set $|S_P|$ of S_P consists of all points $s \in S$ where the Hilbert polynomial of the restriction of $\mathscr{F}$ to X_s is f. In particular, the subsets $|S_f| \subset |S|$ are disjoint, and their set-theoretic union is $|S|$.*

(ii) Universal property: *Let $S' = \coprod S_P$ be the coproduct of the S_P, and let $i : S' \to S$ be the morphism induced by the inclusions $S_P \hookrightarrow S$. Then the sheaf $i^*(\mathscr{F})$ on $X_{S'}$ is flat over S'. Moreover, $i : S' \to S$ has the universal property that for any morphism $\varphi : T \to S$ the pullback $\varphi^*(\mathscr{F})$ on X_T is flat over T if and only if φ factors through $i : S' \to S$. The subscheme S_P is uniquely determined by the polynomial P.*

(iii) Closure of strata: *Let the set I of Hilbert polynomials be given a total ordering, defined by putting $P < Q$ whenever $P(n) < Q(n)$ for all $n \gg 0$. Then the closure in S of the subset $|S_P|$ is contained in the union of all $|S_Q|$ where $P \leq Q$.*

For a general proof of this statement we refer to [FLG'Ll'+05, Section 5.4.2]. Here, however, we will present a proof for the special case $X = S$, which will be more frequently used.

Proof of the case $X = S$. For any $s \in S$, the **fiber** $\mathscr{F}|_s$ of $\mathscr{F}$ over s will mean the pull-back of $\mathcal{F}$ to the subscheme $\mathrm{Spec}\kappa(s)$, where $\kappa(s)$ is the residue field at s, i.e.: $\mathscr{F}|_s = \mathscr{F} \otimes_{\mathcal{O}_{S,s}} \kappa(s)$. The Hilbert polynomial of the restriction of $\mathscr{F}$ to the fiber over s is the degree 0 polynomial $e \in \mathbb{Q}[\lambda]$, where $e = \dim_{\kappa(s)} \mathscr{F}|_s$.

By Nakayama lemma, any basis of $\mathscr{F}|_s$ prolongs to a neighbourhood U of s to give a set of generators for $\mathscr{F}|_U$. Repeating this argument, we see that there exists a smaller neighbourhood V of s in which there is a right-exact sequence

$$\mathcal{O}_V^{\oplus m} \xrightarrow{\psi} \mathcal{O}_V^{\oplus e} \xrightarrow{\phi} \mathscr{F} \to 0$$

Let $I_e \subset \mathcal{O}_V$ be the ideal sheaf formed by the entries of the $e \times m$ matrix $(\psi_{i,j})$ of the homomorphism $\mathcal{O}_V^{\oplus m} \xrightarrow{\psi} \mathcal{O}_V^{\oplus e}$. Let V_e be the closed subscheme of V defined by I_e. For any morphism of schemes $f : T \to V$, the pull-back sequence

$$\mathcal{O}_T^{\oplus m} \xrightarrow{f^*\psi} \mathcal{O}_T^{\oplus e} \xrightarrow{f^*\phi} f^*\mathscr{F} \to 0$$

is exact, by right-exactness of tensor products. Hence the pull-back $f^*\mathscr{F}$ is a locally free $\mathcal{O}_T$-module of rank e if and only if $f^*\psi = 0$, that is, f factors via the subscheme $V_e \hookrightarrow V$ defined by the vanishing of all entries $\psi_{i,j}$. Thus we have proved assertions **(i)** and **(ii)** of the theorem.

As the rank of the matrix $(\psi_{i,j})$ is lower semi-continuous, it follows that the function e is upper semi-continuous, which proves the assertion **(iii)** of the theorem, completing its proof when $X = S$. $\square$

Remark 1.2.2. Note that, while property **(i)** on the point-set structure of the stratification is a more or less direct consequence of the openness of flatness ([GR03, IV, Téorème 6.10]), establishing the correct *scheme structure* of the strata S_P is a little subtler, in general the reduce structure won't do the trick.

Example 1.2.3. Probably the simplest non-trivial flattening stratification one can think of is the following: Set $k[\epsilon] = k(x)/(x^2)$ and let $X = S = \mathrm{Spec}(k[\epsilon])$. We take $\mathscr{F} = k = k[\epsilon]/(\epsilon)$ consider as a $k[\epsilon]$-module. Then the stratification consist of only one stratum, that is the closed immersion $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(k[\epsilon])$.

1.3. *The scheme* $\mathrm{Quot}(X, \mathscr{F})$

The $\mathrm{Quot}(X, \mathscr{F})$ scheme, representing the flat quotients of the sheaf $\mathscr{F}$, is of great importance for algebraic geometry in general, for it allows to construct other moduli schemes. Such will be the case here too. Indeed, ahead we'll construct two moduli schemes related to families of foliations $\mathrm{iPf}(X)$ and $\mathrm{Inv}(X)$. This will be realized as closed sub-schemes of certain Quot schemes.

1.3.1. *The* $\mathfrak{Quot}_S(X, \mathscr{F})$ *functor*

In all generality, the $\mathfrak{Quot}_S(X, \mathscr{F})$ functor might be defined for a scheme X over S and a coherent sheaf $\mathscr{F}$ on X, no further assumtpions are required. Let Sch/S the category of schemes over S. We define the functor

$$\mathfrak{Quot}_S(X, \mathscr{F}) : Sch/S \longrightarrow Sets,$$

the following way, to an object $g : T \to S$ of Sch/S associate the set

$$\mathfrak{Quot}_S(X, \mathscr{F})(T) = \{0 \to \mathscr{K} \to g^*\mathscr{F} \to \mathscr{Q} \to 0 \colon \mathscr{Q} \in Coh(X_T) \text{ is flat over } T\}.$$

and to an arrow $\kappa : T \to T'$ over S, the function

$$\begin{array}{rcl} \mathfrak{Quot}_S(X, \mathscr{F})(\kappa) : \mathfrak{Quot}_S(X, \mathscr{F})(T') & \longrightarrow & \mathfrak{Quot}_S(X, \mathscr{F})(T) \\ (0 \to \mathscr{K} \to g^*\mathscr{F} \to \mathscr{Q} \to 0) & \mapsto & (0 \to \kappa^*\mathscr{K} \to g'^*\mathscr{F} \to \kappa^*\mathscr{Q} \to 0). \end{array}$$

Note that, as flatness is stable by base change, the functor is well-defined.

1.3.2. Representability of $\mathfrak{Quot}$ functors

The question then is: Under which conditions on $X \to S$ and $\mathscr{F}$ is the functor $\mathfrak{Quot}_S(X, \mathscr{F})$ representable?
The answer is provided by Grothendieck's theorem.

To state Grothendieck's theorem on the representability of $\mathfrak{Quot}$ first we recall a couple of definitions.

Remember that a morphism $p : X \to S$ is said to be *projective* if there exist a *relatively ample* line bundle $\mathscr{L}$ i.e.: a line bundle such that, for every coherent sheaf $\mathscr{G}$ there exist $n >> 0$ such that the natural map

$$p^* p_*(\mathscr{L}^{\otimes n} \otimes \mathscr{G}) \to \mathscr{L}^{\otimes n} \otimes \mathscr{G}$$

is an epimorphism.

We also recall the following. Let be a coherent sheaf $\mathscr{F}$ on a scheme X of finite type and proper over a field k, of dimension n, take also a line bundle $\mathscr{L}$. Then we define the *Hilbert polynomial* Φ of $\mathscr{F}$ with respect to $\mathscr{L}$ as

$$\Phi(m) = \sum_{i=0}^{n} (-1)^i \dim_k (H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m})).$$

Observe that properness of X assures finite dimensionality of $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m})$.

Then, when we have a projective morphism $X \to S$ with relative ample sheaf $\mathscr{L}$, we can decompose $\mathfrak{Quot}_S(X, \mathscr{F})$ in the sub-functors $\mathfrak{Quot}_S^{P,\mathscr{L}}(X, \mathscr{F})$, where $P(m) \in \mathbb{Q}[m]$ is a polynomial and

$$\mathfrak{Quot}_S^{P,\mathscr{L}}(X, \mathscr{F})(T) = \left\{ \begin{array}{l} 0 \to \mathscr{K} \to g^*\mathscr{F} \to \mathscr{Q} \to 0 \text{ s.t.: } \mathscr{Q} \in Coh(X_T) \\ \text{is flat over } T \text{ and the Hilbert polynomial of } \mathscr{Q}_t \\ \text{with respect to } \mathscr{L}_t \text{ is } P, \text{ for all } t \in T \text{ closed.} \end{array} \right\}.$$

Clearly $\mathfrak{Quot}_S(X, \mathscr{F}) = \coprod_{P \in \mathbb{Q}[m]} \mathfrak{Quot}_S^{P,\mathscr{L}}(X, \mathscr{F})$. Grothendieck's theorem can then be expressed as

Theorem 1.3.1. *Let X and S be noetherian schemes, $X \to S$ a finite type projective morphism with relative ample sheaf $\mathscr{L}$, and $\mathscr{F}$ a coherent sheaf on X. Then the functors $\mathfrak{Quot}_S^{P,\mathscr{L}}(X, \mathscr{F})$ are representable by schemes* $\mathrm{Quot}_S^{P,\mathscr{L}}(X, \mathscr{F})$ *o finite type and projective over S.*

We won't give here the proof of the above theorem, for which we refer to [FLG'Ll'+05, chapter 5]. We will, however, otuline the steps of the proof in the case $S = \mathrm{Spec}k$. From now on, when we talk about projective schemes over a point $\mathrm{Spec}(k)$, we'll often leave implicit the election of an ample line bundle, therefore we'll only write $\mathrm{Quot}^P(X, \mathscr{F})$ or even $\mathrm{Quot}^P(\mathscr{F})$ (when no confusion is likely to arise) when we talk about quot schemes.

We now briefly comment on the construction of $\mathrm{Quot}^P(X, \mathscr{F})$.

I) **Castelnuovo-Mumford regularity and Boundedness.** Following Grothendieck, we say a certain set of sheaves $(\mathscr{E}_i)_{i\in I}$ over X is *bounded* if there exist a scheme T and a sheaf $\mathscr{G}$ on $X\times T$ such that, for all $i\in I$, there is a morphism $\kappa_i : \mathrm{Spec}(k) \to T$ such that $\mathscr{E}_i \cong (id_X \times \kappa_i)^*(\mathscr{G})$. Note that, as we're only considering a particular *set* of sheaves, and morphism from $\mathrm{Spec}(k)$ to T, the scheme T in the above definition need not to be a moduli space of sheaves.

So the statement is that, given a polynomial $P(m) \in \mathbb{Q}[m]$ the set of sheaves $\mathscr{Q}$ such that there is an epimorphism $\mathscr{F} \to \mathscr{Q} \to 0$ and such that $\mathscr{Q}$ have Hilbert polynomial P is bounded. After [Mum66] this is usually proved by mean of Castelnuovo-Mumford regularity of a sheaf $\mathscr{Q}$, defined as the minimal $m \in \mathbb{N}$ such that

$$H^i(X, \mathscr{Q}(m-i)) = 0, \quad \forall i > 0.$$

The key lemmas here are the following two assertions, whose proofs can be found in [BGI71, exp. XIII]

Lemma 1.3.2. *Let $(\mathscr{E}_i)_{i\in I}$ be a set of sheaves such that every $\mathscr{E}_i$ have the same Hilbert polynomial and such that there exist a sheaf $\mathscr{G}$ with epimorphisms $\mathscr{G} \to \mathscr{E}_i \to 0$ for all $i \in I$. Then the Castelnuovo-Mumford regularity of the sheaves $\mathscr{E}_i$ is bounded from above.*

Lemma 1.3.3. *If $\mathscr{E}$ have C-M regularity m then for every $n \geq m$, $\mathscr{E}(n)$ is generated by global sections.*

II) $\mathfrak{Quot}^P(X, \mathscr{F})$ **is a sub-functor of a Grassmann functor.** Last two lemmas tell us that we can take $n \in \mathbb{N}$ bigger than the C-M regularity of every quotient $\mathscr{Q}$ of $\mathscr{F}$ with Hilbert polynomial P. We can take n to be also bigger than the C-M regularity of $\mathscr{F}$ itself. So, fixed a suitable n, now we can look at the epimorphism

$$H^0(\mathscr{F}(n)) \to H^0(\mathscr{Q}(n)) \to 0.$$

It's not too hard to show that, as both $\mathscr{F}(n)$ and $\mathscr{Q}(n)$ are generated by global sections, this epimorphism determines the sheaf epimorphism $\mathscr{F} \to \mathscr{Q}$ i.e.: if $\mathscr{F} \to \mathscr{Q}'$ is another such epimorphism, then the linear maps $H^0(\mathscr{F}(n)) \to H^0(\mathscr{Q}(n))$ and $H^0(\mathscr{F}(n)) \to H^0(\mathscr{Q}(n))$ will have different kernel. It is also not so hard to see, from the basic properties of Castelnuovo-Mumford regularity, that for all such $\mathscr{Q}$ one must have $\dim_k(H^0(\mathscr{Q}(n))) = P(n)$. From this one can conclude after a few technicalities on flatness that the correspondence

$$(\mathscr{F} \to \mathscr{Q} \to 0) \longmapsto \big(H^0(\mathscr{F}(n)) \to H^0(\mathscr{Q}(n)) \to 0\big)$$

defines a natural transformation from $\mathfrak{Quot}^P(X, \mathscr{F})$ to the functor represented by $\mathbb{G}^{P(n)}(H^0(\mathscr{F}(n)))$, the Grassmannian of $P(n)$-dimensional quotient vector spaces of the vector space $H^0(\mathscr{F}(n))$.

III) $\mathfrak{Quot}^P(X, \mathscr{F})$ **is a *closed* sub-functor of** $\mathbb{G}^{P(n)}(H^0(\mathscr{F}(n)))$ Given a functor $G : Sch/k \to Sets$ and a sub-functor $F \subseteq G$ we say that F is a *closed* sub-functor of G if and only if for each $T \in Sch/k$ there is a closed sub-scheme $i : Z_F \hookrightarrow T$ and a natural map such that

$$F(T) = \mathrm{Im}(G(i)) \subseteq G(T).$$

It is immediate that, if G is representable, F is also representable and the inclusion $F \subset G$ is respresented by a closed inclusion of schemes. Grothendieck's way to prove representability of $\mathbb{G}^{P(n)}(H^0(\mathscr{F}(n)))$ is to show that a finite number of rank conditions on maps $H^0(\mathcal{O}_X(j)) \otimes H^0(\mathscr{F}(n)) \to H^0(\mathscr{F}(n+j))$ determines if a quotient $H^0(\mathscr{F}(n)) \to V \to 0$ comes from a sheaf quotient $\mathscr{F} \to \mathscr{Q} \to 0$ such that $\mathscr{Q}$ have Hilbert polynomial P.

From the details of the proof is posible, in principle, to give an explicit immersion of $\mathrm{Quot}^P(\mathscr{F})$ into a grassmannian variety. In particular is posible to give an explicit ample line bundle on $\mathrm{Quot}^P(\mathscr{F})$, it will be the restriction of the Plücker line bundle on $\mathbb{G}^{P(n)}(H^0(\mathscr{F}(n)))$ (the Plücker line bundle is the line bundle determining the Plücker embedding of the grassmannian into a projective space), so the fiber of this line bundle over a point $[\mathscr{Q}] \in \mathrm{Quot}^P(\mathscr{F})$ will be naturally isomorphic to $\wedge^{P(n)} H^0(\mathscr{Q})$.

1.4. Valuative criterion for flatness

Here we'll recall a suitable special case of the valuative criterion for flatness of [Gro65, 11.8] and draw some corollaries from it.

Theorem 1.4.1 (Valuative criterion for flatness)**.** *Let $f : X \to Y$ a finite presentation morphism between noetherian schemes, $\mathscr{F}$ a coherent sheaf over X, $x \in X$ a point, and $y = f(x)$. Suppose Y is reduced. Then $\mathscr{F}_x$ is f-flat if and only if for every discrete valuation ring A' and every morphism $\mathcal{O}_{Y,y} \to A'$ the following holds:*
Taking the pull-back diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' = \mathrm{Spec}(A') & \longrightarrow & Y \end{array}$$

the $\mathcal{O}_{X'}$-module $\mathscr{F}' = \mathscr{F} \otimes \mathcal{O}_{Y'}$ is f'-flat at every point $x' \in X'$ lying over x.

Grothendieck's valuative criterion for flatness is a very powerfull method and will be used throughout this work. Its usefullness rellies on the following criterion for flatness on a DVR, whose proof can be found, for instance, in [GR03, IV.1.3.2].

Proposition 1.4.2. *Let A be a domain such that every maximal $\mathscr{M} \subset A$ is principal (e.g.: A a Dedekind domain) . Then an A-module M is flat if and only if it is torsion-free.*

Remark 1.4.3. Theorem 1.4.1 requires the base scheme Y to be *reduced.* It's easy to see that the criterion fails if this hypothesis doesn't hold.

Let Y be $\mathrm{Spec}(k[T]/(T^2))$ and X the closed subcheme $X = \mathrm{Spec}(k) \hookrightarrow Y$, take $\mathscr{F} = \mathcal{O}_X$. Then for every domain A and every morphism $\phi : k[T]/(T^2) \to A$ we have $\phi(\overline{x}) = 0$.
So $\phi^\sharp : \mathrm{Spec}(k[T]/(T^2)) \to \mathrm{Spec}(A)$ factorizes through $X = \mathrm{Spec}(k) \hookrightarrow Y$.
Then $\mathcal{O}_X \otimes_\phi \mathcal{O}_{\mathrm{Spec}(A)} \cong \mathcal{O}_{\mathrm{Spec}(A)}$ is flat.

Nevertheless, a closed immersion other than the identity is never flat.

It would be desirable, however, to have a flatness criterion working with posible non-reduced base schemes. Such schemes would be fundamental to develop a deformation theory for foliations. Moreover, we have no reason to believe that the moduli schemes of foliations appearing below will, in general, be reduced . So now we'll patch-up this situation and develop a criterion for flatness not demanding our base schemes to be reduced. They will have to be essentially of finite type over an algebraically closed field, anyway.

First we'll need a little

Lemma 1.4.4. *Let A be a ring of finite type over an algebraically closed field k, $\mathcal{M}$ a maximal ideal in A, and $f \in \mathcal{M}^n \setminus \mathcal{M}^{n+1}$. Then there is a morphism $\psi : A \to k[T]/(T^{n+1})$ such that $\psi^{-1}((T)) = \mathcal{M}$ and $\psi(f) \neq 0$.*

Proof. Set a presentation $A \cong k[y_1, \dots, y_r]/I$. By the Nullstelensatz we can assume $\mathcal{M} = (x_1, \dots, x_r)$, where x_i is the class of $y_i \mathrm{mod} I$. Write the class of f in $\mathcal{M}^n/\mathcal{M}^{n+1}$ as

$$\overline{f} = \sum_{|\alpha|=n} a_\alpha \overline{x}^\alpha \in \mathcal{M}^n/\mathcal{M}^{n+1},$$

where $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\overline{x}^\alpha = (\overline{x_1}^{\alpha_1}, \dots, \overline{x_r}^{\alpha_r})$.

As $f \notin \mathcal{M}^{n+1}$, the polynomial $q(y_1, \dots, y_r) := \sum_{|\alpha|=n} a_\alpha y^\alpha$ is not in I. Now, k being algebraically closed there is an r-tuple $(\lambda_1, \dots, \lambda_r) \in k^r$ such that $p(\lambda_1, ..., \lambda_r) = 0$ for every $p \in I$ and $q(\lambda_1, ..., \lambda_r) \neq 0$.

Finally we can define $\psi : A \to k[T]/(T^{n+1})$ as follows:

$$\psi(x_i) = \lambda_i T.$$

The morphism is well defined because $p(\lambda_1, ..., \lambda_r) = 0$ for every $p \in I$, moreover $\psi^{-1}(T) = \mathcal{M}$, and $\psi(f) = q(\lambda_1, ..., \lambda_r)T^n \neq 0$. $\square$

Proposition 1.4.5. *Let $f : X \to Y$ a projective morphism between schemes of finite type over an algebraically closed field, $\mathscr{F}$ a coherent sheaf over X, $x \in X$ a point, and $y = f(x)$. Then $\mathscr{F}_x$ is f-flat if and only if the following conditions hold:*

1. *For every discrete valuation ring A' and every morphism $\mathcal{O}_{Y,y} \to A'$ the following holds:*
   Taking the pull-back diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' = \operatorname{Spec}(A') \arrow[r] & Y \end{tikzcd}$$

   the $\mathcal{O}_{X'}$-module $\mathscr{F}' = \mathscr{F} \otimes \mathcal{O}_{Y'}$ is f'-flat at every point $x' \in X'$ lying over x.

2. *For every $n \in \mathbb{N}$ and every morphism $\mathcal{O}_{Y,y} \to k[T]/(T^{n+1})$, if we take the diagram analogous to the one above (with $k[T]/(T^{n+1})$ instead of A') then the $\mathcal{O}_{X'}$-module $\mathscr{F}' = \mathscr{F} \otimes \mathcal{O}_{Y'}$ is f'-flat at every point $x' \in X'$ lying over x.*

Proof. Clearly conditions 1 and 2 are necessary. Suppose then that 1 and 2 are satisfied.

Take the flattening stratification (see Theorem 1.2.1) of Y with respect to $\mathscr{F}$, $Y = \coprod_P Y_P$. As condition 1 is satisfied for $\mathscr{F}$ over Y, so is satisfied for $\iota^*\mathscr{F}$ over Y_{red}, where $\iota : Y_{\text{red}} \to Y$ is the closed immersion of the reduced structure. Then, by Theorem 1.4.1, $\iota^*\mathscr{F}$ is flat over Y_{red}, so by the universal property of the flattening stratification there is a factorization

$$\begin{tikzcd} Y_{\text{red}} \arrow[r, "\iota"] \arrow[d] & Y \\ \coprod_P Y_P \arrow[ur] & \end{tikzcd} .$$

As Y_{red} and Y share the same underlying topological set, the above factorization is telling us that the flattening factorization consist on a single stratum Y_P and that $Y_{\text{red}} \to Y_P$ is a closed immersion.

Assume, by way of contradiction, $Y_P \subsetneq Y$, then there is an affine open sub-scheme $U \subseteq Y$ such that $V = Y_P \cap U \neq U$. Now take the coordinate rings $k[U]$ and $k[V]$ and the morphism between them induced by the inclusion $\phi : k[U] \twoheadrightarrow k[V]$. Let's take $f \in k[U]$ such that $\phi(f) = 0$. By Lemma 1.4.4 there exists, for some $n \in \mathbb{N}$, a morphism $\psi : k[U] \to k[T]/(T^{n+1})$ such that $\psi(f) \neq 0$, so ψ doesn't factorize through ϕ.

On the other hand, let $Z = \operatorname{Spec}(k[T]/(T^{n+1}))$ and $g : Z \to Y$ be the morphism induced by ψ, as condition 2 is satisfied, the pull-back $g^*\mathscr{F}$ is flat over $Z = \operatorname{Spec}(k[T]/(T^{n+1}))$. So, by the universal property of Theorem 1.2.1, g factorizes as

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & Y \\ Y_P \arrow[ur] & \end{tikzcd} ,$$

contradicting the statement of the above paragraph, thus proving the proposition. □

Note that the hypothesis of this property on X and Y (aside from reducedness) are quite stronger than the ones of the original theorem of Grothendieck, such is the price we have paid to allow a criterion for possibly non-reduced schemes. The price paid is ok with us anyway, considering that we'll work mostly with schemes of finite type over $\mathbb{C}$.

Next we provide a criterion for a $k[T]/(T^{n+1})$-module to be flat.

Proposition 1.4.6. *Let $A = k[T]/(T^{n+1})$ and M an A-module. Then M is flat if and only if for every $m \in M$ such that $T^n \cdot m = 0$ there exist $m' \in M$ such that $m = T \cdot m'$.*

Proof. Flatness of M is equivalent to the injectivity of the map $M \otimes I \to M$ for every ideal $I \subset A$ (see e.g.:[GR03, IV.1]). In this case there are finitely many ideals:

$$\mathcal{M} = (T),\ \mathcal{M}^2, \dots,\ \mathcal{M}^n.$$

If M is flat is easy to see the second condition in our statement hold.

Suppose that for every $m \in M$ such that $T^n \cdot m = 0$ there exist $m' \in M$ such that $m = T \cdot m'$. Let $a \in M \otimes \mathcal{M}^{n-i}$ be in the kernel of $M \otimes \mathcal{M}^{n-i} \to M$. When $i = 0$, we have $a = m \otimes T^n$, and m is such that $T^n \cdot m = 0$ so, by hypothesis, $m = T \cdot m'$ and then $m \otimes T^n = n \otimes T^{n+1} = 0$.

When $i > 0$, we have $a = \sum_{j=n-i}^{i} m_j \otimes T^j$, so $T^i \cdot a = m_{n-i} \otimes T^n \in M \otimes \mathcal{M}^n$. By hypothesis, $m_{n-i} = T \cdot m'$. So $a \in M \otimes \mathcal{M}^{n-i+1}$ and we are done by induction. □

The following will be usefull in the study of foliations of codimension greater than 1.

Proposition 1.4.7. *Let $p : X \to S$ a projective morphism between schemes of finite type over an algebraically closed field k, $f : S \to Y$ another morphism, with Y of finite type over k, and $\mathscr{F}$ a coherent sheaf over X. Take a stratification $\coprod_i S_i \subseteq S$ of S such that $\mathscr{F}|_{S_i} := \mathscr{F} \otimes_S \mathcal{O}_{S_i}$ is flat for all i. If the composition $\coprod_i S_i \hookrightarrow S \xrightarrow{f} Y$ is a flat morphism, then $\mathscr{F}$ is flat over Y.*

Proof. Invoking Proposition 1.4.5 we can, after applying base change, reduce to the case where Y is either the spectrum of a DVR or $Y = \operatorname{Spec}(k[T]/(T^{n+1}))$.

(i) Case $Y = \operatorname{Spec}(A)$ with A a DVR. Suppose there is, for some point $x \in X$ a section $s \in \mathscr{F}_x$ that is of torsion over A. Consider $Z = \operatorname{supp}_S(s) \subseteq S$ the support of s over S, that is the support of s as an element of $\mathscr{F}_x$ considered as an $\mathcal{O}_{S,p(x)}$-module. Now take any stratum S_i and suppose $Z \cap S_i \neq \emptyset$. Then there is a section of the pullback $\mathscr{F}_{S_i}$ that is of torsion over A. But $\mathscr{F}_{S_i}$ is flat over S_i which is in turn flat over A, so $\mathscr{F}_{S_i}$ is flat and $Z \cap S_i$ must be empty for every stratum S_i, i.e.: $s = 0$.

(ii) Case $Y = \mathrm{Spec}(k[T]/(T^{n+1}))$**.** One can essentially repeat the argument above, now taking the section s to be such that $T^n s = 0$ but $s \notin T \cdot \mathscr{F}_x$. □

Corollary 1.4.8. *Take the flattening stratification $\coprod_P S_P \subseteq S$, of S with respect to $\mathscr{F}$. If the composition $\coprod_P S_P \hookrightarrow S \xrightarrow{f} Y$ is a flat morphism, then $\mathscr{F}$ is flat over Y.*

1.5. Nakayama's Lemma for semi-exact functors

In [GBAO72] Bergman and Ogus set up a general theory to deal with base-change problems. A base-change problem is usually a question of the type: Given a morphism $f : X \to Y$ and a functor $F_f : Coh(X) \to Coh(Y)$, and suppose we have an $\mathcal{O}_X$-module $\mathscr{G}$ and a pull-back

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd} .$$

How does the modules

$$F_f(\mathscr{G}) \otimes \mathcal{O}_{Y'}, \quad \text{and} \quad F_{f'}(\mathscr{G}_{X'})$$

compares?

An example of a base-change theorem is Grothendieck base-change for the functor $R^i f_*$. As possed in [Mum08] is the following statement

Theorem 1.5.1 (Base-change for cohomology)**.** *Let $f : X \to Y$ be a proper morphism of noetherian schemes, with $Y = \mathrm{Spec}(A)$ an affine scheme, and $\mathscr{F}$ a coherent sheaf on X, flat over Y. Then, if B is a flat A-algebra,*

$$H^p(X \times_Y \mathrm{Spec}(B), \mathscr{F} \otimes_A B) \cong H^p(X, \mathscr{F}) \otimes_A B.$$

Proof. This is [Mum08, II.5, Corollary 5]. □

In general Bergman and Ogus consider the following data as a setting for base-change problems. Let R and T be rings, $f : R \to T$ a morphism between them and F a functor $F : R-\mathrm{mod} \to T-\mathrm{mod}$ between the corresponding categories of finitely generated modules. We assume further that F is f-linear, in the sense that for any two R-modules M and N, the map

$$\hom_R(M, N) \to \hom_T(F(M), F(N))$$

is a morphism of R-modules, with the R-module structure in $\hom_T(F(M), F(N))$ induced by the ring morphism f.

Remember that a middle-exact functor is one that takes exact sequences

$$0 \to M \to N \to P \to 0$$

and return exact sequences

$$F(M) \to F(N) \to F(P)$$

with no 0 in the sides. So with this in mind we can state Nakayama's lemma for middle-exact functors.

Theorem 1.5.2 (Nakayama's lemma for middle-exact functors)**.** *With notation as above, suppose R noetherian, if F is middle-exact and, for all maximal ideals $\mathcal{M}$ of T, we have $F(R/f^{-1}(\mathcal{M})) = 0$, then $F = 0$.*

Proof. [GBAO72, Theorem 2.2] □

In particular when we set R to be local and noetherian, $T = R$, $f = id_R$ and $F(M) = N \otimes_R M$ for a fixed R-module N, we retrieve the classical Nakayama's lemma albeit only for noetherian rings.

Note that, every time we have an f-linear functor F, we also have a natural comparison morphism

$$t_F : M \otimes_R F(R) \to F(M).$$

Indeed, as we have the natural identifications

$$\begin{aligned}\hom_T(M \otimes_R F(R), F(M)) &\cong \hom_R(M, \hom_T(F(R), F(M))) \cong \\ &\cong \hom_R(\hom_R(R, M), \hom_T(F(R), F(M))),\end{aligned}$$

we obtain t_F as the element in $\hom_T(M \otimes_R F(R), F(M))$ that correspond to the R-module morphism

$$\begin{array}{rcl}\hom_R(R, M) & \longrightarrow & \hom_T(F(R), F(M)) \\ \phi & \mapsto & F(\phi)\end{array}$$

via the above isomorphisms.

Base-change problems in this setting consist in studying when is t_F an isomorphism. Theorem 1.5.2 can then be used to study this problems.

Proposition 1.5.3. *As above F is an f-linear middle-exact functor $F : R-\mathrm{mod} \to T-\mathrm{mod}$, with R noetherian, then the following conditions are equivalent:*

1. *F is isomorphic to $- \otimes_R F(R)$ (i.e.: the morphism t_F above is an isomorphism).*

2. *The natural map $F(R) \to F(R/f^{-1}(\mathcal{M}))$ is surjective for all maximal ideals $\mathcal{M} \subset T$.*

3. *F is right exact.*

Proof. [GBAO72, Theorem 4.1] □

Proposition 1.5.3 have important consequences. One of them concerns cohomological δ-functors. Indeed, to have a δ-functor $F^\bullet$ means having a sequence $(F^i)_{i\in\mathbb{Z}}$ of functors such that for every short exact sequence $0 \to M \to N \to P \to 0$ one have an exact sequence

$$\dots \to F^i(M) \to F^i(N) \to F^i(P) \to F^{i+1}(M) \to F^{i+1}(N) \to \dots$$

In particular each functor F^i is middle exact, so we can apply Theorem 1.5.2 to them and draw important conclusions.

Corollary 1.5.4. *Let $F^\bullet$ an f-linear δ-functor. Then for any q the following are equivalent.*

1. *For every maximal ideal $\mathcal{M} \subset T$ the natural map $F^q(R) \to F^q(R/f^{-1}(\mathcal{M}))$ is surjective.*

2. *For all M in $R-\mathrm{mod}$ the natural map $F^q(R) \otimes_R M \to F^q(M)$ is an isomorphism.*

3. *F^q is right-exact.*

4. *F^{q+1} is left-exact.*

If F extends to a functor $F : R-\mathrm{Mod} \to T-\mathrm{Mod}$ between the categories of (not necesarilly finitely generated) modules, and F commutes with direct limits then the above conditions are eqvuivalent to

• *For all M in $R-\mathrm{Mod}$ the natural map $F^q(R) \otimes_R M \to F^q(M)$ is an isomorphism.*

Proof. It follows at once applying Proposition 1.5.3 to the sequence

$$\dots \to F^i(f^{-1}\mathcal{M}) \to F^i(R) \to F^i(R/f^{-1}(\mathcal{M})) \to F^{i+1}(f^{-1}\mathcal{M}) \to F^{i+1}(R) \to \dots$$

□

Let us denote by $\mathrm{P}(q)$ the equivalent conditions (1)-(4) of the above corollary. Since flatness of M is the necessary and sufficient condition for $M \otimes -$ to be left-exact we get:

Proposition 1.5.5. *Let $f : R \to T$ and F as in the above corollary. Then* *(a) If $P(q+1)$ holds then $P(q)$ holds if and only if $F^{q+1}(R)$ is a flat R-module.*

(b) If $P(q+1)$ holds, and $F^{q'}(R)$ is a flat R-module for all $q' \leq q+1$, then $P(q')$ holds for all $q' \leq q+1$.

(c) If for all maximal ideals $\mathcal{M} \subset T$, $F^{q+1}(R/f^{-1}\mathcal{M}) = 0$, then $F^{q+1} = 0$ and $P(q)$ holds.

We now apply this results to the following situation. Let $p : X \to S$ be a flat morphism between noetherian schemes, let $\mathscr{G}$ be a coherent sheaf on X. Consider the following functors

$$\begin{array}{rcl} E^q : Coh(S) & \longrightarrow & Coh(X) \\ \mathscr{F} & \mapsto & \mathcal{E}xt^q_X(\mathscr{G}, p^*\mathscr{F}). \end{array}$$

Note that the E^q are δ-functors, indeed as X is flat over S, then $\mathcal{E}xt^q_X(\mathscr{G}, p^*\mathscr{F})$ is the q-th derived functor of $E^0 = \mathcal{H}om(\mathscr{G}, p^*\mathscr{F})$. We can then localize at a point $x \in X$, say $p(x) = s \in S$, and study the localized functors

$$\mathscr{F}_s \mapsto \mathcal{E}xt^q_X(\mathscr{G}, p^*\mathscr{F})_s,$$

between categories of finitely generated modules.

Then, in this situation, Proposition 1.5.5 gives us

Proposition 1.5.6 (Property of exchange for local Ext's)**.** *Let $E^q : Coh(S) \longrightarrow Coh(X)$ be the above functors, and take points $s \in S$, $x \in X_s$ assume that the base-change map to the fiber*

$$t^q_E(k(s)) : \mathcal{E}xt^q_X(\mathscr{G}, \mathcal{O}_X) \otimes k(s) \to \mathcal{E}xt^q_{X_s}(\mathscr{G} \otimes k(s), \mathcal{O}_{X_s}),$$

is surjective at x. Then the following statements are equivalent,

I*) The morphism $t^{q-1}_E(k(s))$ is surjective at x.*

II*) Locally around x, $E^{q-1}(\mathscr{F}) \cong \mathcal{E}xt^q_X(\mathscr{G}, \mathcal{O}_X) \otimes \mathscr{F}$.*

III*) Locally around x, $\mathcal{E}xt^q_X(\mathscr{G}, \mathcal{O}_X)$ is flat over S.*

Proof. Just apply Proposition 1.5.5 to the localization of the functors E^q around x. Note, however, that this proof is different than the original one in [ASK, Theorem 1.9]. □

2. PRELIMINARIES ON FOLIATIONS

In this section we work with varieties over $\mathbb{C}$. We'll sometimes abuse notation and denote the same way a vector bundle and its sheaf of sections.

2.1. Distributions

Definition 2.1.1. Let $p : E \to X$ be a vector bundle, we define the associated *Grassmann bundle* of dimension d, $\mathbb{G}_d(E)$, the following way:

Let $X = \bigcup_i U_i$ be a trvializing covering of E, $p^{-1}(U_i) \cong U_i \times \mathbb{C}^n$, with transition functions

$$\begin{array}{rcl} \phi_{ij} : U_i \times \mathbb{C}^n & \longrightarrow & U_j \times \mathbb{C}^n \\ (x, v) & \mapsto & (x, \varphi_{ij}(x, v)), \end{array}$$

being $\varphi_{ij}(x, v)$ linear in v.

Define $\pi : \mathbb{G}_d(E) \to X$ as a bundle having $\{U_i\}$ as trvializing covering, with fibers $\pi^{-1}(U_i) \cong U_i \times \mathbb{G}_d(\mathbb{C}^n)$, and transition functions

$$\begin{aligned} \Phi_{ij} : U_i \times \mathbb{G}_d(\mathbb{C}^n) \to U_j \times \mathbb{G}_d(\mathbb{C}^n) \\ (x, [S]) \mapsto (x, [\varphi_{ij}(S)]), \end{aligned}$$

where $S \subseteq \mathbb{C}^n$ is a dimension d sub-space, and $[S] \in \mathbb{G}_d(\mathbb{C}^n)$ is the point of the grassmannian represented by S . It's routine to verify that the definition doesn't depend on the choice of a trivialization.

Remark 2.1.2. A global section s of $\mathbb{G}_d(E)$ determines a sub-bundle $F \hookrightarrow E$ of dimension d. A trivializing covering of F is given by a refinement V_j of the covering U_i. The refinment is such that the image of

$$s : V_j \to p^{-1}(V_j) \cong V_j \times \mathbb{G}_d(\mathbb{C}^n)$$

is contained in $V_j \times W$, where W is some afine coordinate open subset of $\mathbb{G}_d(\mathbb{C}^n)$. Transition functions of F are, of course, induced by those of E.

Definition 2.1.3. A *singular distribution* of dimension d on a variety X is a rational section (i.e.: a section defined on an open sub-space of X) of the bundle $\mathbb{G}_d(TX)$. The closed set $Z = X \setminus U$ is called the *singular set* of the distribution.

Remark 2.1.4. A singular distribution then determines a sub-bundle

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & TU \arrow[r] \arrow[d] & TX \arrow[d] \\ U \arrow[r, equal] & U \arrow[r, hook] & X. \end{tikzcd}$$

If X is a regular variety is in particular normal. Then the shaf $\mathcal{O}_X$ have Serre's property S_2 (see [Gro65, 5.8 – 5.10]), and, being locally free, so does (the sheaf associated to) TX. If moreover $\operatorname{codim}(X \setminus U) > 2$ then, by property S_2, every section of $\Gamma(U, F)$ extends to a global section of TX. So in this way we get a sub-sheaf

$$\tilde{F} \hookrightarrow TX$$

such that it's restriction to U is F, in particular, $\tilde{F}|_U = F$ is a sub-locally free sheaf of $TX|_U = TU$.

Reciprocally, having a subsheaf $F \subset TX$ such that its restriction to an open set U is a sub-vector bundle of rank d gives rise to a rational section s of $\mathbb{G}_d(TX)$. Indeed, take the short exact sequence

$$0 \to F \to TX \to Q \to 0.$$

When restricted to U this is a s.e.s. of locally free sheaves, so $\operatorname{Tor}_1(Q, k(x)) = 0$ for every $x \in U$. Then $F \otimes k(x) \subseteq TX \otimes k(x) = T_xX$ is a d-dimensional sub-space. So we can define

$$s(x) = [F \otimes k(x)] \in \mathbb{G}_d(T_xX)$$

for every $x \in U$.

Remark 2.1.5. When X is regular of dimension n we can also give a distribution with a rational section t of $\mathbb{G}_{n-d}(T^*X)$. Indeed, we can associate to t a sub-sheaf $I \subseteq \Omega^1_X$ that is a sub-vector bundle of rank $q = n - d$ when restricted to an open set U. To I we can associate the sheaf $F \subset TX$ of vector fields that annihilate I . Then F will be a rank d sub-bundle of TX when restricted to U. So F will give, as above, a rational section of $\mathbb{G}_d(TX)$.

As we allways have an inclusion

$$\begin{array}{rcl} \mathbb{G}_q(T^*X) & \hookrightarrow & \mathbb{P}(\wedge^q T^*X) \\ V & \longmapsto & [\wedge^q V], \end{array}$$

then a rational section t of $\mathbb{G}_q(T^*X)$ gives rise to a rational section of $\mathbb{P}(\wedge^q T^*X)$. If $I \subset \Omega^1_X$ is the sub-sheaf associated to t, then we have, associated to the rational section of $\mathbb{G}_1(\wedge^q T^*X)$ the sheaf $\wedge^q I$.

Now having a rational section, say defined over $U \subseteq X$, of $\mathbb{P}(\wedge^q T^*X)$ is like having, for a covering $U = \cup V_i$, q-forms $\omega_j \in \Omega^q_X(V_j)$ such that

$\omega_j = g_{ij}\omega_i$ on $V_i \cap V_j$, with $g_{ij} : V_i \cap V_j \to \mathbb{C}^*$. The set of functions (g_{ij}) give a cocycle defining a line bundle $\mathcal{L}$ over U. If $\operatorname{codim}(X \setminus U) > 2$ then the g_{ij} extends to X and so does $\mathcal{L}$. So we have that a rational section of $\mathbb{P}(\wedge^q T^* X)$ gives a global section of $\Omega^q_X \otimes \mathcal{L}$.

2.2. Foliations

Here we present the basics of the classical theory of holomorphic foliations following [ALN07]. Although throughout this work we work mostly with distributions satisfying Frobenius condition, and we keep integration of distributions and manipulation with leaves to a minimum, it's important to remark that the geometric core of the results stated here are consequences of Frobenius theorem, notably Kupka's theorem. Othewise we would just be developing a theory of sub-sheaves of TX with some extra arbitrary condition.

It's worthwhile to mention that here we concentrate mostly on the geometric characterizations of foliations and its singularities. We won't mention anything about the dynamic of foliations, for instance, holonomy of a leaf or ergodicity of a foliation are concepts that won't even be defined. Nevertheless we remark their importance for, as was mentioned in the introduction, it was the dynamics of differential equations that originally motivated the study of foliations. For a treatment of this and more subjects on the dynamics of holomorphic foliations we defer to [GMOB89] and [GM88].

Having concluded this section's rant we can begin.

Definition 2.2.1. A *regular complex holomorphic foliation* $\mathcal{F}$, of dimension k over a complex regular variety X of dimension n $(1 \leq k \leq n)$, is a decomposition $X = \bigcup_{i \in I} L_i$ of X in connected holomorphic sub-varieties of dimension k, called *leaves* of $\mathcal{F}$, such that the following conditions hold:

1. For all $x \in X$ there is a unique leaf L_x of $\mathcal{F}$ containing x. If $y \in L_x$, then $L_x = L_y$.
2. For all $x \in X$ there is a local holomorphic chart (U, ϕ) of X, with $x \in U$, such that $\phi : U \to V_k \times V_{n-k}$, where V_k and V_{n-k} are open sets of $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$ respectively. For all $(z, w) \in V_k \times V_{n-k}$ the k-dimensional subvariety of U, $\phi^{-1}(V_k \times \{w\})$, is an open set of L_y, where $y = \phi^{-1}(z, w)$.

We'll also say that $\mathcal{F}$ is a foliation of codimension $n-k$. We'll say that the chart (U, ϕ) of the definition is an *adapted* chart to the foliation $\mathcal{F}$, we'll also call it a *trivializing* chart.

Remark 2.2.2. From the definition follows the fact that, given two trivializing charts (U_1, ϕ_1) and (U_2, ϕ_2), the coordinate change

$$\Phi_{12} : \phi_1(U_1 \cap U_2) \subseteq V_k \times V_{n-k} \longrightarrow \phi_2(U_1 \cap U_2) \subseteq V_k \times V_{n-k}$$

has the expression

$$\Phi_{12}(z,w) = (f(z,w), g(w)).$$

Indeed, as ϕ_1 and ϕ_2 map leaves of $\mathcal{F}$ to sets $V_k \times \{w\}$, then Φ_{12} must preserve such sets i.e.: must map $V_k \times \{w\}$ to $V_k \times \{w'\}$. Hence Φ_{12} must be shaped as above.

Remark 2.2.3. We stress the fact that the leaves L_i on the definition of foliation need not to be *immersed* sub-manifolds, meaning that the topology in a leaf L isn't necesarilly induced by the inclusion map $i : L \to X$. In other words, we only ask for the inclusion map $i : L \to X$ to be holomorphic, injective, and such that the differential $di_x : T_xL \to T_{i(x)}X$ is injective for all x.

Remark 2.2.4. A k-dimensional regular foliation $\mathcal{F}$ induces a rank k vector bundle over X. This bundle is naturally embedded in the tangent bundle TX.

Indeed, cover X by adapted charts $(U_i, \phi_i)_{i \in I}$, with coordinate change maps $\Phi_{ij} = (f_{ij}(z,w), g_{ij}(w))$. Define the bundle with trivialization $p_i : U_i \times \mathbb{C}^k \to U_i$ and coordinate changes

$$\begin{array}{rcl} \varphi_{ij} : U_{ij} \times \mathbb{C}^k & \longrightarrow & U_{ij} \times \mathbb{C}^k \\ (x,v) & \mapsto & (x, df_{ij}|_{\phi_i(x)}(v)). \end{array}$$

The bundle thus defined doesn't depend on the choice of adapted charts and, by looking at its coordinate changes, it's clear that is a sub-bundle of the tangent bundle.

Definition 2.2.5. The bundle defined in Remark 2.2.4 will be called *tangent bundle* to the foliation, and denoted $T\mathcal{F}$.

Note that, for any $x \in X$ the inclusion $i : L \to X$ of the leaf passing through x gives a canonical isomorphism $T_xL \cong (T\mathcal{F})_x$ with the fiber of $T\mathcal{F}$ above x (which, stressing the analogy with the tangent bundle of a manifold, we'll also denote $T_x\mathcal{F}$).

Associated with any short exact sequence of vector bundles

$$0 \to E \to T \to Q \to 0$$

we have the dual exact sequence

$$0 \to Q^\vee \to T^\vee \to E^\vee \to 0.$$

Here $Q^\vee$ is the annihilator of E in $T^\vee$. Frobenius theorem can be seen as a criterion for a sub-bundle $E \subseteq TX$ to be the tangent bundle of a foliation, in terms of conditions on E or $Q^\vee$.

Theorem 2.2.6 (Frobenius). *Let E be a rank k bundle, and $0 \to E \to TX \to Q \to 0$ be an exact sequence, denote $I = Q^\vee$. The following are equivalent:*

1. *For every two sections v and v' of $E \subseteq TX$, the lie bracket $[v, v']$ is also a section of E. (We say that E is* involutive*.)*

2. *$I \subseteq \Omega^1_X$ is locally generated by forms $\eta_1, \ldots, \eta_{n-k}$ such that*

$$d\eta_i \wedge \bigwedge_{i=1}^{n-k} \eta_i = 0.$$

   (We say that I is integrable*.)*

3. *There is a foliation $\mathcal{F}$ such that $T\mathcal{F} = E$ as sub-bundles of TX.*

Proof. See [Mal72, chapter II.5]. □

Definition 2.2.7. A *singular foliation* on X is a foliation $\mathcal{F}$ defined on an open set U of X. The closed set $X \setminus U$ is called the *singular locus* of the foliation and noted $\operatorname{sing}(\mathcal{F})$.

Remark 2.2.8. Let $\mathcal{F}$ be a singular foliation of dimension k and $\mathcal{F}|_U$ its restriction to the maximal open set where is non-singular. As discused in Remark 2.1.4, the tangent bundle $T\mathcal{F}|_U$ defines a singular distribution and, if $\operatorname{codim}(\operatorname{sing}(\mathcal{F})) > 1$, a unique coherent sub-sheaf $T\mathcal{F} \subseteq TX$.

Likewise, set $I(\mathcal{F}|_U)$ the annihilator of $T\mathcal{F}|_U$ in Ω^1_U. Then, as in Remark 2.1.5, $I(\mathcal{F}|_U)$ defines a section of $\Omega^{n-k}_X \otimes \mathcal{L}$, and, if $\operatorname{codim}(\operatorname{sing}(\mathcal{F})) > 1$, a unique coherent sub-sheaf $I(\mathcal{F}) \subseteq \Omega^1_X$.

Note further that, as the restriction $T\mathcal{F}|_U$ is involutive, and involutiveness is a closed condition, so $T\mathcal{F}$ is an involutive sub-sheaf, meaning that the $\mathbb{C}$-linear map

$$T\mathcal{F} \otimes_{\mathbb{C}} T\mathcal{F} \xrightarrow{[-,-]} TX$$

have image in $T\mathcal{F}$.

Likewise $I(\mathcal{F})$ is integrable, meaning that the sheaf of $\mathbb{C}$-modules $dI(\mathcal{F}) \wedge \bigwedge^{n-k} I(\mathcal{F})$ is zero.

2.3. Plücker relations

As is usually much easier to manipulate a single q-form than a subsheaf $I \subset \Omega^1_X$ of generic rank q or a subsheaf $F \subset TX$ of generic rank d (well, at least when $d > 1$) we'll deal quite a lot with q-forms when working with codimension q foliations. There's some price to pay though, subsheafs

$F \subset TX$ have lots of well understood invariants, and those invariants may not be so easy to trace back to the corresponding q-form.

Anyway, as our (twisted) q-forms comes from rational sections of $\mathbb{P}(\wedge^q T^*X)$ that are in the image of the Plücker map

$$\mathbb{G}_q(T^*X) \hookrightarrow \mathbb{P}(\wedge^q T^*X),$$

we'll need to characterize this image, i.e.: give equations determining $\mathbb{G}_q(T^*X)$ as a sub-variety of $\mathbb{P}(\wedge^q T^*X)$.

For this we follow [GH94, I.5] and first determine, for a vector bundle E, and a global section of it $\Lambda \in \wedge^q E(X)$, the minimal sub-bundle $F \subseteq E$ such that Λ is in the image of

$$\wedge^q F \to \wedge^q E.$$

If $\mathrm{rank}(F) = l$, then $l \geq q$ with equality holding if and only if Λ is locally decomposable, i.e. for each $x \in X$ there is a neighborhood $U \ni x$ and local sections $v_1, \dots, v_q \in E(U)$ such that, in U,

$$\Lambda = v_1 \wedge \cdots \wedge v_q.$$

First we state a well known lemma (see e.g.: [Har77, Exercise II.5.16]).

Lemma 2.3.1. *Let $0 \to M \to P \to N \to 0$ an exact sequence of A-modules. Then for any p there is a filtration*

$$\wedge^p P = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^p \supseteq F^{p+1} = 0$$

with quotients

$$F^i/F^{i+1} \cong \wedge^i M \otimes \wedge^{p-i} N$$

for each i.

Proof. We simply set

$$F^i = A \cdot (m_1 \wedge \cdots \wedge m_i \wedge x_1 \wedge \cdots \wedge x_{p-i} \ s.t.: \ m_i \in M).$$

Observe that this sub-modules verifiy the required conditions. □

Definition 2.3.2. We denote by $E^\vee$ the dual sheaf of E, for a local section $v^* \in E^\vee(U)$, we define the *contraction operator*

$$\iota_{v^*} : \wedge^q E \to \wedge^{q-1} E$$

by

$$< \iota_{v^*}(\Lambda), \Theta >=< \Lambda, v^* \wedge \Theta >$$

for all $\Theta \in \wedge^{q-1} E^\vee$.

Lemma 2.3.3. *Let $\Lambda \in \wedge^q E(U)$ be a local section. We associate to Λ the sub-sheaves*

$$\Lambda^\perp = < v^* \in E^\vee(U) \quad s.t.\text{: } \iota_{v^*}\Lambda = 0 >$$

and

$$W = \mathrm{Ann}(\Lambda^\perp) \subseteq E(U).$$

Then W is the minimal sub-sheaf of $E|_U$ such that Λ is in the image of $\wedge^q W \to \wedge^q E|_U$.

Proof. Replacing X by U we can assume that every section is global. We have the short exact sequence

$$0 \to W \to E \to N \to 0.$$

It's clear from the definitions that $N \subseteq (\Lambda^\perp)^\vee$. Lets take on $\wedge^q E$ the filtration $(F^i)_{i=0}^{q+1}$ given by Lemma 2.3.1. We want to show that $\Lambda \in \wedge^q W(X)$. Set p the minimum number such that $\Lambda \in F^p(X) \setminus F^{p+1}(X)$. Suppose $p < q$ and take $[\Lambda]$ the image of Λ in

$$[\Lambda] \in F^p/F^{p+1}(X) \cong \wedge^p W \otimes \wedge^{q-p} N(X).$$

As $N \subseteq (\Lambda^\perp)^\vee$, we can see $[\Lambda]$ as an operator

$$[\Lambda] : \wedge^{q-p}(\Lambda^\perp) \to \wedge^p W.$$

But then from the definition of $\Lambda^\perp$ we have that $[\Lambda] = 0$.

On the other hand, if T is a sub-sheaf such that Λ is in the image of $\wedge^q T \to \wedge^q E$, then $T \supseteq \mathrm{Ann}(\Lambda^\perp)$. □

Lemma 2.3.4. *Let's define another sheaf W' by*

$$W'(U) = \{w \in W(U) s.t.\text{: } w \wedge \Lambda = 0\}.$$

Then Λ is locally decomposable if and only if $W' = W$.

Proof. If Λ is locally decomposable, clearly $W' = W$. Conversely, if Λ is not decomposable so that $\dim W > q$, then, since the pairing $\wedge^q W \otimes \wedge^{l-q} W \to \wedge^l W$ is nondegenerate, we deduce that $W' \neq W$. □

Now we extend the contraction operator to sections of $\wedge^p E^\vee$ in the only sensible way. If $\Xi \in \wedge^p E$ is a local section we define

$$\begin{array}{rcl} \iota_\Xi : \wedge^q E & \to & \wedge^{q-p} E \\ < \iota_\Xi \Lambda, v^* > & = & < \Lambda, \Xi \wedge v^* > \end{array}$$

for all local sections v^* of $\wedge^{q-p} E^\vee$.

Proposition 2.3.5 (Plücker relations)**.** *The section $\Lambda \in \wedge^q E(X)$ is locally decomposable if and only if*

$$\iota_\Xi(\Lambda) \wedge \Lambda = 0 \tag{2.1}$$

for every local section Ξ of $\wedge^{q-1}E^\vee$.

Proof. We may characterize W as being the image of

$$\wedge^{q-1}E^\vee \to E$$

under the map

$$\Xi \mapsto \iota_\Xi \Lambda, \qquad \Xi \in \wedge^{q-1}E \vee (U).$$

Then the condition $W' = W$ is equivalent to eq. (2.1). □

2.4. *Singularities of codimension-1 foliations*

2.4.1. *Kupka singularities*

Studying the structural stability of foliations of codimension 1 Kupka (in [Kup64]) made a most remarkable discovery. Namely, he found that *integrable forms* (i.e.: forms giving rise to involutive distributions) can have, in general, a bigger zero locus than generic forms.

More specifically, given a form ω such that $\omega \wedge d\omega = 0$ and such that there's a point $x \in X$ such that

$$\omega_x = 0 \qquad d\omega_x \neq 0,$$

then the singular set of ω (the set of zeros of ω) around x form a codimension 2 sub-manifold. Moreover, the condition $\omega_x = 0 \qquad d\omega_x \neq 0$ is an *open condition* in the space of integrable 1-forms (endowed with a natural topology).

Let us note that a generic 1-form (as any generic section of a bundle E with $\operatorname{rank}(E) = \dim(X)$) have isolated zeros. In particular Kupka's theorem says that the closed sub-space of 1-forms with non-isolated zeros intersects the closed sub-space of integrable forms non-transversaly.

Now we make use of the calculations made with Plücker relations and use them to give a slightly simplified version of Medeiros'proof of Kupka's theorem ([dM77]).

Proposition 2.4.1. *Let X be a regular variety over $\mathbb{C}$ and $\omega \in \Omega^1_X(U)$ an integrable 1-form defined in a neighborhood U of a point $x \in X$. Then $d\omega$ is locally decomposable.*

Proof. We have to prove that $d\omega$ verifies Plücker relations (eq. (2.1) in Proposition 2.3.5). This amounts to make two observations. The first is that, as ω is integrable, we have

$$\omega \wedge d\omega = 0 \Longrightarrow d(\omega \wedge d\omega) = d\omega \wedge d\omega = 0.$$

The second is that contraction with respect to a vector field is a degree -1 derivation in the algebra of exterior differential forms, i.e.:

$$\iota_v(\eta \wedge \tau) = \iota_v(\eta) \wedge \tau - \eta \wedge \iota_v(\tau).$$

In particular, for every vector field v

$$\iota_v d\omega \wedge d\omega = \frac{1}{2}\iota_v(d\omega \wedge d\omega) = 0.$$

So eq. (2.1) is satisfied and $d\omega$ is locally decomposable. □

So now we have a locally decomposable 2-form, moreover this form is closed, in particular is integrable. If $d\omega_x \neq 0$, then the codimension 2 foliation defined by $d\omega$ is non-singular in a neighborhood of x. We will now have a closer look at the leaves of this foliation and relate them with the foliation defined by ω.

Lemma 2.4.2. *Suppose that $d\omega_x \neq 0$. Consider $\mathcal{G}$ the codimension 2 foliation defined by $d\omega$. In the neighborhood V of $x \in X$ where $\mathcal{G}$ is non-singular we have the following The leaves of $\mathcal{G}$ are integral manifolds of ω (i.e.: if $L \to X$ is a leaf of $\mathcal{G}$ then $\omega|_L = 0$).*

Proof. We know that, if v is a vector field tangent to $\mathcal{G}$, $\iota_v(d\omega) = 0$. On the other hand we have

$$0 = \iota_v(\omega \wedge d\omega) = \iota_v(\omega)d\omega.$$

Since $d\omega \neq 0$, then $\iota_v(\omega) = 0$ and we are done □

Lemma 2.4.3. *With the same hipothesis as Lemma 2.4.2. Let v be a vector field tangent to $\mathcal{G}$. Then the Lie derivative of ω with respect to v is zero.*

Proof. The Lie derivative of ω with respect to v is given by Cartan formula

$$L_v(\omega) = \iota_v(d\omega) + d(\iota_v\omega).$$

By definition of $\mathcal{G}$, $\iota_v(d\omega) = 0$, and by Lemma 2.4.2, $\iota_v(\omega) = 0$. Then $L_v(\omega) = 0$. □

Lemma 2.4.4. *Same hipothesis as Lemma 2.4.2 and 2.4.3, then* $\operatorname{sing}(\omega)$ *is saturated by leaves of $\mathcal{G}$ (i.e.: take $y \in V$ a zero of ω, and L the leaf of $\mathcal{G}$ going through y. Then the inclusion $L \to V$ factorizes through* $\operatorname{sing}(\omega)$*).*

Proof. Let $y \in \mathrm{sing}(\omega)$ and L the leaf of $\mathcal{G}$ going through y. Let $p \in L$ and v a vector field tangent to $\mathcal{G}$ such that the orbit $\gamma(t)$ of v going through y joins y with p. As $L_v(\omega) = 0$ by Lemma 2.4.3, then

$$\frac{d}{dt}(\gamma^*\omega)(t) = L_v(\omega)|_{\gamma(t)} = 0.$$

And as $\omega_{\gamma(0)} = 0$, so $\omega|_{\gamma(t)} \equiv 0$. Then $\gamma(t) \in \mathrm{sing}(\omega)$ for all t, so $L \subset \mathrm{sing}(\omega)$. □

Remark 2.4.5. In particular, if $\omega_x = 0$ and $d\omega_x \neq 0$, last lemma is telling us that, locally around x, $\mathrm{sing}(\omega)$ is a codimension 2 sub-variety. This is quite remarkable if we consider that generic 1-forms in a domain $V \subseteq \mathbb{C}^n$ have isolated zeros. Even more so if we observe that the condition $d\omega_x \neq 0$ is an open one.

So, for instance, lets take the space $\Omega^1[\mathbb{C}^n]_d$ of algebraic 1-forms in $\mathbb{C}^n$ whose coeficients are degree d polynomials. It has a natural structure of affine space. A generic member $\eta \in \Omega^1[\mathbb{C}^n]_d$ have isolated zeros. But if we restrict ourselves to the closed algebraic sub-variety Int of 1-forms ω such that $\omega \wedge d\omega = 0$, then there is an open (non-void) subvariety U of Int such that a generic member of U have non-isolated zeros. Namely, every 1-form such that there exist $x \in \mathbb{C}^n$ with $\omega_x = 0$ and $d\omega_x \neq 0$ is in U.

Theorem 2.4.6 (Kupka)**.** *Let X be a regular variety over $\mathbb{C}$ and $\omega \in \Omega^1_X(U)$ an integrable 1-form defined in a neighborhood U of a point $x \in X$. Suppose further that $d\omega_x \neq 0$. Then there is an analytical coordinated neighborhood V of x with coordinate functions $y_1, \ldots, y_n$ such that, in those coordinates, ω can be written*

$$\omega = F(y_1, y_2)dy_1 + G(y_1, y_2)dy_2.$$

In more intrinsic terms, $\omega|_V$ is the pull-back of an integrable 1-form in $\mathbb{C}^2$.

Proof. By Frobenius theorem we can take a coordinate neighborhood $(V, (y_1, \ldots, y_n))$ such that the leaves of the foliation $\mathcal{G}$ defined by $d\omega$ are the submanifolds $y_1 = ct.,\ y_2 = ct.$

Now, in this coordinate system, the vector fields

$$v_j = \frac{\partial}{\partial y_j}, \quad 3 \leq j \leq n,$$

are tangent to $\mathcal{G}$. By Lemma 2.4.3 we have $L_{v_3}\omega = L_{v_4}\omega = \cdots = L_{v_n}\omega = 0$, so in this coordinate system

$$\omega = \sum_i F_i(y_1, y_2)dy_i.$$

On the other hand, by Lemma 2.4.2, $\iota_{v_j}\omega = 0$, for all $3 \leq j \leq n$. Then, in the above expresion of ω, $F_3 = F_4 = \cdots = F_n = 0$. □

Remark 2.4.7. Note that, aside from last theorem, we've never made use of integration, Frobenius theorem or analytical coordinates in any of the previous lemmas. So, in the case X is a regular algebraic variety and ω an integrable algebraic 1-form, the conclusions of lemmas 2.4.2, 2.4.3 and 2.4.4 and Proposition 2.4.1 hold *Zariski locally.* On the other hand, the coordinate neighborhood $(V,(y_1,\dots,y_n))$ in Kupka's theorem will not, in general, be a Zariski neighborhood, nor will the y_i be algebraic morphisms.

Although we won't use it later, we'll include now a proof of Medeiros/Kupka's theorem for p-forms, which is an analogous version of Kupka's theorem for integrable p-forms. The guiding lines are the same as in the 1-form case, and, as before, we can take advantage of Plücker relations to replace some cumbersome manipulations in coordinates with intrinsic cumbersome manipulations.

Before we proceed we need to make a small obsevation.

Remark 2.4.8. Let $\Xi \in \wedge^p TX$, we have defined above the contraction operator

$$\iota_\Xi : \Omega^q_X \to \Omega^{q-p}_X.$$

Note that, in the case $\Xi = v_1 \wedge \cdots \wedge v_p$, this is just the composition

$$\iota_\Xi = \iota_{v_1} \circ \cdots \circ \iota_{v_p}.$$

Proposition 2.4.9. *Let X be a regular variety over $\mathbb{C}$ and $\omega \in \Omega^p_X(U)$ a locally decomposable and integrable p-form defined in a neighborhood U of a point $x \in X$. Then $d\omega$ is also locally decomposable.*

Proof. Take a local decomposition $\omega = \eta_1 \wedge \cdots \wedge \eta_p$. Integrability of ω means we have

$$\eta_i \wedge d\omega = 0, \quad \forall 1 \leq i \leq p. \tag{2.2}$$

We want to check Plücker relations, which are in this case $\iota_\Xi(d\omega) \wedge d\omega = 0$ for all $\Xi \in \wedge^p TX$. It is sufficient to check this for every fundamental tensor $\Xi = v_1 \wedge \cdots \wedge v_p$. So we compute

$$d\omega = d(\eta_1 \wedge \cdots \wedge \eta_p) = \sum_i \left((-1)^i d\eta_i \wedge \bigwedge_{j \neq i} \eta_j \right).$$

And in this case we have

$$\iota_{v_1 \wedge \cdots \wedge v_p} d\omega = \sum \chi_{ij} \iota_{v_i}(d\eta_j) + \sum \phi_k \eta_k,$$

where χ_{ij} and ϕ_k are functions. By eq. (2.2) is enough to show

$$\iota_{v_i}(d\eta_j) \wedge d\omega = 0.$$

To show this equation holds is enough to verify it for every point $z \in U \setminus \operatorname{sing}(\omega)$. As ω is integrable, in the dense open set $U \setminus \operatorname{sing}(\omega)$ we have that

$$d\eta_j = \sum f^j_{kl}\eta_k \wedge \eta_l,$$

where f^j_{kl} are holomorphic functions. Then

$$\iota_{v_i}(d\eta_j) \wedge d\omega = \left(\sum \iota_{v_i}(f^j_{kl}\eta_k \wedge \eta_l)\right) \wedge d\omega =$$
$$= \sum f^j_{kl}\ \iota_{v_i}\eta_k \wedge \eta_l \wedge d\omega + \sum f^j_{kl}\ \eta_k \wedge \iota_{v_i}\eta_l \wedge d\omega.$$

So, again by eq. (2.2), we are done. □

Lemma 2.4.10. *Suppose that $d\omega_x \neq 0$. Consider $\mathcal{G}$ the codimension $p+1$ foliation defined by $d\omega$. In the neighborhood V of $x \in X$ where $\mathcal{G}$ is non-singular we have the following The leaves of $\mathcal{G}$ are integral manifolds of ω.*

Proof. Analogously to Lemma 2.4.2, for v tangent to $\mathcal{G}$, and with the notation of the last proposition we have

$$0 = \iota_v(\eta \wedge d\omega) = \iota_v(\eta) \wedge d\omega.$$

Since $d\omega \neq 0$ we conclude. □

Lemma 2.4.11. *With the same hipothesis as last lemma. Let v be a vector field tangent to $\mathcal{G}$. Then the Lie derivative of ω with respect to v is zero.*

Proof. Proceed exactly as in Lemma 2.4.3. □

Lemma 2.4.12. *Same hipothesis as Lemma 2.4.10 and 2.4.11, then* $\operatorname{sing}(\omega)$ *is saturated by leaves of $\mathcal{G}$.*

Proof. Follows from Lemma 2.4.11 (c.f.: proof of Lemma 2.4.3) . □

Theorem 2.4.13 (Kupka's theorem for p-forms ([dM77]))**.** *Let X be a regular variety over $\mathbb{C}$ and $\omega \in \Omega^p_X(U)$ an integrable p-form defined in a neighborhood U of a point $x \in X$. Suppose further that $d\omega_x \neq 0$. Then there is an analytical coordinated neighborhood V of x with coordinate functions $y_1, \ldots, y_n$ such that, in those coordinates, ω can be written*

$$\omega = F_1(y_1, y_2, \ldots, y_{p+1})dy_1 + F_2(y_1, y_2, \ldots, y_{p+1})dy_2 + \cdots + F_{p+1}(y_1, y_2, \ldots, y_{p+1})dy_{p+1}.$$

In more intrinsic terms, $\omega|_V$ is the pull-back of an integrable p-form in $\mathbb{C}^{p+1}$.

Proof. Use Lemma 2.4.10, Lemma 2.4.11,and Lemma 2.4.12 to proceed as in the proof of Theorem 2.4.6. □

2.4.2. Reeb singularities

We have commented before that Kupka singularities might be viewed as an unexpected phenomenon among 1-forms, ocurring suddenly in the case where the 1-form ω is integrable. On the other side it may be the case that a 1-form ω have the type of zeros one would expect in a generic 1-form or, more generally, in a generic section of a rank-n vector bundle over a variety of dimension n. Reeb singularities are exactly that, although not every integrable 1-form have Reeb singularities.

Definition 2.4.14. Let X be an n-dimensional regular variety and $\omega \in \Omega^1_X$ an integrable 1-form. We say that a point $p \in X$ is a *Reeb singularity* of ω if $\omega_p = 0$ and, moreover, there is an analytical neighbourhood U of p, with local coordinates x_i such that in U, ω can be written as

$$\omega = \sum_{i=1}^{n} f_i(x) dx_i,$$

where the f_i are such that $df_1|_p, \dots, df_n|_p$ are linearly independent elements of the vector space $T^*_p X$.

Remark 2.4.15. In particular a Reeb singularity is an isolated singularity. Indeed, the zeros of ω are given in U by the equations $(f_1(x) = 0, \dots, f_n(x) = 0)$. Now the tangent space of the (analytical) scheme defined by the ideal $(f_1, \dots, f_n)$ on p is given by

$$\{v \in T_p X \colon \ df_i(v) = 0, \ 1 \geq i \geq n\}.$$

As the df_i's are linearly independent this tangent space is zero dimensional so the ideal $(f_1, \dots, f_n)$ actually defines a reduced 0-dimensional scheme on X.

Remark 2.4.16. At a Reeb singularity one necessarily have $d\omega_p = 0$. Otherwise p would be a Kupka singularity and p woudn't be an isolated zero of ω.

2.5. Codimension 1 foliations on $\mathbb{P}^n(\mathbb{C})$

In this section we gather results mostly about foliations defined on $\mathbb{P}^n(\mathbb{C})$. This results are the main motivation for the present work.

To deal with foliations on $\mathbb{P}^n(\mathbb{C})$ we'll use global twisted 1-forms, and global twisted vector fields. Indeed, a foliation is, in general, given by, either a sub-sheaf $T\mathcal{F} \subseteq T\mathbb{P}^n(\mathbb{C})$ or a sub-sheaf $I(\mathcal{F}) \subseteq \Omega^1_{\mathbb{P}^n(\mathbb{C})}$. So we'll be considering their associated graded modules

$$\bigoplus_i H^0(\mathbb{P}^n(\mathbb{C}), T\mathcal{F}(i)) \subseteq \bigoplus_i H^0(\mathbb{P}^n(\mathbb{C}), T\mathbb{P}^n(\mathbb{C})(i))$$

and

$$\bigoplus_i H^0(\mathbb{P}^n(\mathbb{C}), I(\mathcal{F}(i))) \subseteq \bigoplus_i H^0(\mathbb{P}^n(\mathbb{C}), \Omega^1_{\mathbb{P}^n(\mathbb{C})}(i))$$

over the homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$. To describe sheaves in $\mathbb{P}^n$ is sufficient to determine their associated graded modules from a big enough degree on. So we'll usually describe foliations over $\mathbb{P}^n(\mathbb{C})$ by a vector space $H^0(\mathbb{P}^n(\mathbb{C}), T\mathcal{F}(m)) \subseteq H^0(\mathbb{P}^n, T\mathbb{P}^n(m))$ containing generators of $\bigoplus_{i\geq m} H^0(\mathbb{P}^n(\mathbb{C}), T\mathcal{F}(i))$, or the analogous for $H^0(\mathbb{P}^n(\mathbb{C}), I(\mathcal{F})(m))$. We'll need then a suitable description of this global sections, for this we have Euler's exact sequence.

2.5.1. *Euler's short exact sequence*

Remember (from [GH94] for instance) that, given a vector space V and a line $l \subset V$ in it, we can naturally identify the tangent vector space to $\mathbb{P}(V)$ on a point $[l] \in \mathbb{P}(V)$ as $\hom_{\mathbb{C}}(l, V/l)$. Moreover, this identification being natural, we have the isomorphism of sheaves

$$T\mathbb{P}(V) \cong \mathcal{H}om(\mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)} \otimes_{\mathbb{C}} V/\mathcal{O}_{\mathbb{P}(V)}(-1)).$$

Note that $\mathcal{O}_{\mathbb{P}(V)}(-1)$ is the sheaf associated to the canonical line bundle over $\mathbb{P}(V)$, and that $\mathcal{O}_{\mathbb{P}(V)} \otimes_{\mathbb{C}} V$ is the (sheaf associated to the) trivial vector bundle with fiber V, so the fiber of $\mathcal{H}om(\mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)} \otimes_{\mathbb{C}} V/\mathcal{O}_{\mathbb{P}(V)}(-1))$ on a point $[l]$ is indeed $\hom_{\mathbb{C}}(l, V/l)$.

On the other hand we have the canonical short exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \otimes_{\mathbb{C}} V \to \mathcal{O}_{\mathbb{P}(V)} \otimes_{\mathbb{C}} V/\mathcal{O}_{\mathbb{P}(V)}(-1) \to 0.$$

So, as $\mathcal{O}_{\mathbb{P}(V)}(-1)$ is locally free, applying the functor $\mathcal{H}om(\mathcal{O}_{\mathbb{P}(V)}(-1), -)$ gives the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(1) \otimes_{\mathbb{C}} V \to T\mathbb{P}(V) \to 0.$$

This is the Euler exact sequence for the tangent sheaf.

Dualizing Euler's exact sequence for the tangent sheaf we get the sequence

$$0 \to \Omega^1_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(-1) \otimes_{\mathbb{C}} V^{\vee} \to \mathcal{O}_{\mathbb{P}(V)} \to 0.$$

Which is Euler's short exact sequence for the cotangent sheaf.

Now, lets fix coordinates $(x_0, \ldots, x_n)$ in V. So we are identifying $V \cong \mathbb{C}^{n+1}$.

Applying the global section Euler's sequence for the tangent sheaf becomes

$$\begin{array}{rcl} 0 \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \to & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}) \to & H^0(\mathbb{P}^n, T\mathbb{P}^n) \to 0 \\ 0 \to \mathbb{C} \to & (\mathbb{C}[x_o, \ldots, x_1]_{=1})^{n+1} \to & H^0(\mathbb{P}^n, T\mathbb{P}^n) \to 0, \end{array}$$

where $\mathbb{C}[x_o, \ldots, x_1]_{=1}$ is the space of homogeneous polynomials of degree 1.

Now fixing an affine open set $\mathbb{A}^n = U_i \subset \mathbb{P}^n$ we identify $\mathcal{O}_{\mathbb{P}^n}(U_i)$ with rational forms in $\mathbb{C}[x_0, \ldots, x_n, \frac{1}{x_i}]$ of degree 0. Likewise $T\mathbb{P}^n(U_i)$ is identified with derivations of this rational forms, so $T\mathbb{P}^n(U_i)$ is generated as an $\mathcal{O}_{\mathbb{P}^n}(U_i)$-module by $x_i \frac{\partial}{\partial x_j}$, $0 \geq i, j \geq n$. This gives a realization of the epimorphism $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T\mathbb{P}^n$ in U_i,

$$\begin{array}{rcl} (\mathbb{C}[x_0, \ldots, x_n, \dfrac{1}{x_i}]_{=1})^{n+1} & \longrightarrow & T\mathbb{P}^n(U_i) \\ (l_0, \ldots, l_n) & \mapsto & \displaystyle\sum_j l_j \frac{\partial}{\partial x_j}. \end{array}$$

We know the kernel of this map is the image of $\mathcal{O}_{\mathbb{P}^n}$ so is generated by a single element. On the other hand Euler's lemma tells us that if f is a homogeneous polynomial of degree d then

$$\sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f(x_0, \ldots, x_n),$$

so the kernel of $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T\mathbb{P}^n$ in Euler's spectral sequence is, under the above identification, generated by the $n+1$-tuple $(x_0, \ldots, x_1)$. Note that the above identification glues well throughout the different U_i's, so it carries over to global sections. Moreover tensoring Euler's sequence with $\mathcal{O}_{\mathbb{P}^n}(m)$ and taking global sections we have

$$\begin{array}{lcl} 0 \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m+1)^{\oplus n+1}) \to & H^0(\mathbb{P}^n, T\mathbb{P}^n(m)) \to 0 \\ 0 \to \mathbb{C}[x_o, \ldots, x_1]_{=m} \xrightarrow{\cdot(x_0, \ldots, x_n)} & (\mathbb{C}[x_o, \ldots, x_1]_{=m+1})^{n+1} \to & H^0(\mathbb{P}^n, T\mathbb{P}^n(m)) \to 0. \end{array}$$

Henceforth when we speak of degree m sections of the tangent sheaf of $\mathbb{P}^n$ we'll represent them as equivalence classes of polynomial vector fields

$$v \in H^0(\mathbb{P}^n, T\mathbb{P}^n(m)), \qquad v = \sum_{i=0}^{n} F_i \frac{\partial}{\partial x_i},$$

where $F_i \in \mathbb{C}[x_0, \ldots, x_n]_{=m+1}$, such that

$$R = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$$

is equivalent to 0.

Applying duality we have that in Euler's sequence for the cotangent sheaf says that a degree m global section of the cotangent sheaf may be identified with a polynomial 1-form

$$\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}), \qquad \omega = \sum_{i=0}^{n} G_i dx_i$$

where $G_i \in \mathbb{C}[x_0, \ldots, x_n]_{=m-1}$, such that $\iota_R(\omega) = 0$, where, as before, $R = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$.

Note that, by Euler's lemma, the condition that each one of the G_i above must be homogenous of degree $m-1$ may be stated as

$$L_R(\omega) = (m-1)\omega$$

so we can say that projective global forms of degree m on $\mathbb{P}^n$ are polynomial forms in $n+1$ variables such that $\iota_R(\omega) = 0$ and $L_R(\omega) = (m-1)\omega$.

2.5.2. *Singular set*

So now when we talk about a codimension 1 foliation on $\mathbb{P}^n$ we know we can describe it with a 1-form $\omega = \sum_{i=0}^n f_i(x)dx_i$ with f_i a degree d homogeneous polynomial, such that $\omega \wedge d\omega = 0$ and such that $i_R(\omega) = 0$. In this setting the singular set of the foliation determined by ω is the set of common zeros of $f_0, \ldots, f_n$. We have the following proposition.

Proposition 2.5.1. *Let $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(m))$ be an integrable 1-form of degree m. Then the singular set of the foliation $\mathcal{F}$ it represents have at least one component of codimension ≤ 2.*

Proof. Set $\omega = \sum_{i=0}^n f_i(x)dx_i$. As was said before the singular set $\mathrm{sing}(\mathcal{F})$ is given by $\omega_x = 0$ i.e.: by the variety defined by the ideal $(f_0, \ldots, f_n) \subset \mathbb{C}[x_0, \ldots, x_n]$. As the codimension of a scheme and its reduced structure is the same, we may deal directly with the ring $S = \mathbb{C}[x_0, \ldots, x_n]/(f_0, \ldots, f_n)$. To this ring we can apply the Koszul complex $K_\bullet(f_0, \ldots, f_n)$,

$$0 \to \mathbb{C}[x_0, \ldots, x_n] \to \cdots \to \bigwedge^p \mathbb{C}[x_0, \ldots, x_n] \xrightarrow{-\wedge(f_0,\ldots,f_n)} \bigwedge^{p+1} \mathbb{C}[x_0, \ldots, x_n] \to \ldots$$
$$\cdots \to (\mathbb{C}[x_0, \ldots, x_n])^{n+1} \xrightarrow{<-.(f_0,\ldots,f_n)>} \mathbb{C}[x_0, \ldots, x_n] \to S \to 0$$

Note that we can identify this complex with $K_\bullet(\omega)$:

$$0 \to \mathcal{O}_{\mathbb{A}^{n+1}} \xrightarrow{-\wedge\omega} \Omega^1_{\mathbb{A}^{n+1}} \xrightarrow{-\wedge\omega} \Omega^2_{\mathbb{A}^{n+1}} \to \ldots$$
$$\ldots \xrightarrow{-\wedge\omega} \Omega^n_{\mathbb{A}^{n+1}} \xrightarrow{-\wedge\omega} \Omega^{n+1}_{\mathbb{A}^{n+1}} \to S \to 0.$$

The $n-1$-th homology group $H_{n-1}(K_\bullet(\omega))$ is non-trivial. Indeed, as ω is integrable, $d\omega$ is a cycle in $K_{n-1}(\omega)$. Suppose $d\omega$ is a border in $K_{n-1}(\omega)$ (i.e.: there exist $\eta \in K_n(\omega)$ such that $d\omega = \omega \wedge \eta$), then $d\omega = \omega \wedge \eta$ for some polynomial 1-form η. Then we would have

$$(m-1)\omega = L_R\omega = d\iota_R(\omega) + \iota_R(d\omega) = \iota_R(d\omega) =$$
$$= \iota_R(\omega \wedge \eta) = -\omega \wedge \iota_R(\eta),$$

then $\iota_R(\eta) = 1 - m$ which is impossible because of degree considerations on the coefficients of η.

As $H_{n-1}(K_\bullet(\omega)) \neq 0$, this implies ([Ser02, chap. IV] or [Eis95, chap.17]) that a maximal S-regular sequence in $\mathbb{C}[x_0, \ldots, x_n]$ has length ≤ 2, so S has codimension ≤ 2 in $\mathbb{C}[x_0, \ldots, x_n]$, and so has $\operatorname{sing}(\mathcal{F}) \subset \mathbb{P}^n$. $\square$

2.5.3. *Rational foliations*

Any dominant rational map $\phi : X \to Y$ gives rise to a singular foliation where the leaves are $f^{-1}(y)_{\text{reg}}$, that is, the regular open set of the pre-image of any point $y \in Y$. In particular we can consider rational maps

$$\begin{array}{ccc} \mathbb{P}^n(\mathbb{C}) & \longrightarrow & \mathbb{P}^1(\mathbb{C}), \\ (x_0 : \cdots : x_n) & \mapsto & (F(x_0, \ldots, x_n) : G(x_0, \ldots, x_n)) \end{array}$$

where F and G are two homogeneous polynomials such that $\deg F = \deg G = d$. The foliation this map defines is also determined by the form $\omega = FdG - GdF \in H^0(\mathbb{P}^n(\mathbb{C}), \Omega^1_{\mathbb{P}^n(\mathbb{C})}(2d))$. Note that this foliation may be identyfied with the *pencil* of degree d hypersurfaces $\{(\mu F + \lambda G)\ st\colon (\mu : \lambda) \in \mathbb{P}^1\}$.

The singular set

The singular set of this foliation, the points x such that $\omega_x = 0$, have in general two distinctive types. On one side there are the points corresponding to the *base locus* of the pencil, this are the zeros of the ideal $(F, G) \subset \mathbb{C}[x_0, \ldots, x_n]$; on the other side we have the singularities of the members of the pencil.

Indeed, suppose x is a singular point, if x is in neither of the above mentioned sets, then there is a member $(\mu F + \lambda G)$ of the pencil such that $x \in V((\mu F + \lambda G))$ and is a regular point. So $V((\mu F + \lambda G))_{\text{reg}}$ is a leaf of the foliation passing through x, but then x woudn't be singular. It is easy to see that every point of the above two types is singular.

$V((F, G)) \subset \operatorname{sing}(\mathcal{F})$ is a codimension 2 subvariety, for generic choices of F and G this subvariety will be regular and irreducible. The regular points of the variety $V((F, G))$ defined by the ideal (F, G) are Kupka singularities of the foliation.

Indeed, $d\omega = 2dF \wedge dG$, and the coefficients of $dF \wedge dG$ are the 2×2 minors of the jacobian matrix $(\frac{\partial F}{\partial x_i} | \frac{\partial G}{\partial x_i})_{0 \leq i \leq n}$. If x is a regular point of $V((F, G))$, then $\omega_x = 0$ and $d\omega_x \neq 0$.

Conversely, suppose x is a Kupka singularity of the foliation, as $\omega_x = 0$ then, if x is not on $V((F, G))$, there must be a point $(\mu, \lambda) \in \mathbb{P}^1$ such that x is a singular point in the hypersurface $\mu F + \lambda G = 0$. But then $(\mu dF + \lambda dG)_x = 0$, so $d\omega_x = 0$ as well.

On the other hand one have the singularities of the members of the pencil. If the polynomials F and G are generic, the pencil generated by F and G will be a *Lefschetz pencil*, meaning that the singularities of members of the pencil are isolated and, moreover, if $x \in V(\mu F + \lambda G)$ is a singular point, then $\operatorname{hess}(\mu F + \lambda G)_x \neq 0$, where $\operatorname{hess}(H)$ is the determinant of the Hessian matrix $(\frac{\partial^2 H}{\partial x_i \partial x_j})_{0 \leq i,j \leq n}$ (see [GH94, chap. 4.2]).

In the case where the map $(F : G) : \mathbb{P}^n \to \mathbb{P}^1$ defines a Lefschetz pencil, then the singularities of the members of the pencil are Reeb type singularities.

Indeed, we may assume without loss of generality that $x \in V((G))$ is an isolated singularity. Say $\omega = FdG - GdF = \sum f_i dx_i$, as x is isolated $d\omega_x = 0$. We have to prove that the df_i are linearly independent at x. The coefficients f_i of ω are

$$\frac{\partial F}{\partial x_i} G + F \frac{\partial G}{\partial x_i}$$

So df_i is

$$\sum \left(\frac{\partial^2 F}{\partial x_i \partial x_j} G + \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} - \frac{\partial G}{\partial x_i} \frac{\partial F}{\partial x_j} + F \frac{\partial^2 G}{\partial x_j \partial x_i} \right) dx_j.$$

As $x \in V((G))$ then $G(x) = 0$, so the leftmost summand between parenthesis vanishes when evaluated in x. The term $\frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} - \frac{\partial G}{\partial x_i} \frac{\partial F}{\partial x_j}$ is a coefficient in $d\omega$ so it also vanishes at x. So df_i, $0 \leq i \leq n$ are columns in a multiple of the Hessian matrix of G, it is a nonzero multiple because x is not on the base locus of the pencil. Then the df_i's are linearlly independent and x is a Reeb singularity.

Summing up we can say that in the generic situation where F and G form a Lefschetz pencil the singular set of $FdG - GdF$ is of the form $\overline{\mathcal{K}} \cup \mathcal{R}$. Where $\overline{\mathcal{K}}$ is the codimension 2 sub-variety given by the closure of the Kupka singularities, and $\mathcal{R}$ is the discrete set of Reeb singularities.

Families.

The set of rational foliations have a natural structure of algebraic variety, namely if we associate to every pair (F, G) of polynomials the integrable form $FdG - GdF$ we have for every point of $\mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_{=m}) \times \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_{=m})$ an associated foliation. More precisely, we have the subscheme

$$\mathcal{F}ol(n, 2m) = \{\omega \in \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2m))) \colon \omega \wedge d\omega = 0\} \subset \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2m))),$$

(although we will call it differently bellow, for now we stick to the more customary notation of [ALN07]). In this projective scheme we have a subscheme isomorphic to $\mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_{=m}) \times \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_{=m})$ corresponding to rational foliations. We'll see bellow that this can be better stated by saying that rational foliations determine a subscheme of the moduli space of integrable Pfaff systems. For now we'll just state the following important result whose proof can be found in [ALN07, 3.2]

Theorem 2.5.2. *The space of rational foliations is an irreducible component of the scheme $\mathcal{F}ol(n, 2m)$.*

Proof. As we said before a proof can be found in [ALN07, 3.2], a different proof can be found in [CPV09]. □

2.5.4. Logarithmic foliations

Definition 2.5.3. A *logarithmic form* on a regular variety X is a local section ω of Ω^1_X defined on the complement of a hypersurface $Y = Y_1 \cup \cdots \cup Y_k$ such that ω can be written locally as

$$\omega = \eta + \sum \lambda_i \frac{df_i}{f_i},$$

where η is holomorphic, $\lambda_i \in \mathbb{C}$, and $Y_i = V((f_i))$. We say this ω is *logarithmic with poles on* Y or that ω is meromorphic with logarithmic singularities in Y.

Logarithmic forms with poles in a fixed hypersurface Y form a coherent sheaf $\Omega^1_X(\log Y)$. We will need to state certain properties of this sheaves in order to establish that certain features hold generically for logarithmic foliations. The following notions will be important here. Anyway, as this notions won't appear away from this section, we won't get much into them and use facts related to them as “black boxes”.

Definition 2.5.4. A divisor Y in a regular variety X over $\mathbb{C}$ is a *normal crossing divisor* (n.c.d. for short) if for each $x \in Y$ there is an analytical neighbourhood $U \subset X$ containing x and local analytical coordinates $x_1, \ldots, x_n$ in X such that $Y = V((\prod_{i=1}^k x_i))$ for some $1 \leq k \leq n$.

Definition 2.5.5. A vector bundle E on a variety X is *positive* if it admits a hermitian metric h such that its associated curvature form $\omega_h \in \Omega^2_X \otimes \mathcal{E}nd(E)(X)$ defines a definite positive operator $\omega_h(v, -)$ in E_x for each $x \in X$, and every $v \in T_xX$.

The following result that will be usefull to us can be found in [Del70].

Proposition 2.5.6. *If Y is a normal crossing divisor then the sheaf $\Omega^1_X(\log Y)$ of logarithmic forms with poles along Y is a* positive locally free sheaf *of rank equal to the dimension of X.*

Proof. [Del70, chap. 3] □

When $X = \mathbb{P}^n$ a logarithmic form can be expressed using Euler's sequence for the cotangent. Giving that a logarithmic form ω is a form defined in $U = \mathbb{P}^n \setminus Y$, we apply the sequence

$$0 \to \Omega^1_{\mathbb{P}^n}(U) \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}(U) \xrightarrow{\iota_R} \mathcal{O}_{\mathbb{P}^n}(U) \to 0.$$

Using Euler's sequence and the local expression of ω we see that we can write ω as

$$\omega = \sum_{i=0}^{n} \lambda_i \frac{dF_i}{F_i},$$

where $Y_i = (F_i = 0)$, and $\sum \lambda_i \deg(F_i) = 0$.

A *logarithmic foliation* is a foliation defined by a global tiwsted form that is a multiple of a logarithmic form. That is a form $\omega \in \Omega^1_{\mathbb{P}^n}(d)$,

$$\omega = (\prod_{i=1}^{k} F_i) \sum_{i=0}^{k} \lambda_i \frac{dF_i}{F_i},$$

such that $\sum \deg(F_i) = d$ and $\sum_i \lambda_i \deg(F_i)$.

Singularities.

Again, as in the case of rational foliations, singularities of logarithmic foliations are divided in two distinctive parts. On one hand we have the zeros of the form that correspond to the points x where $F_i(x) = F_j(x) = 0$ for $i \neq j$. On the other hand we have the zeros of the logarithmic form $\sum_{i=0}^{k} \lambda_i \frac{dF_i}{F_i}$. There is a third kind of singularity appearing only in degenerate cases, when some of the irreducible components Y_k of the divisor Y are singular, the singular set of this components are also singularities of the foliation. Indeed, if $x \in \mathbb{P}^n \setminus Y$ is a singularity of the foliation, then as $\prod F_i(x) \neq 0$, x is also a zero of the logarithmic form $\sum_{i=0}^{k} \lambda_i \frac{dF_i}{F_i}$. If $x \in Y$, say $x \in Y_1$, then

$$\omega_x = (\prod_{j \neq 1} F_j(x)) \lambda_1 (dY_1)_x$$

so if x is a regular point of Y_1 and $x \notin Y_k$ for $k \neq 1$ then x is a regular point of the foliation.

$Y_i \cap Y_j = V((F_i, F_j))$ is a codimension 2 subvariety. For generic choices of $F_1, \ldots, F_k$ the divisor Y will have normal crossings. Suppose then Y is a n.c.d. and take x a regular point of $\in Y_i \cap Y_j$ and that $x \notin Y_l$ for $l \neq i, j$. Then in an analytical neighbourhood U of x there are coordinates $(x_1, \ldots, x_n)$ such that $Y \cap U = Y_i \cap Y_j \cap U = V((x_1 \cdot x_2))$. Then, in U $d\omega_x$ is written as

$$d\omega_x = (\lambda_i - \lambda_j)(\prod_{l \neq i,j} F_l(x)) dx_1 \wedge dx_2.$$

So, for generic choices of $\lambda_1,\dots,\lambda_k$, x is a Kupka singularity.

If Y is a normal crossing divisor and x is a zero of the logarithmic form $\sum_{i=0}^n \lambda_i \frac{dF_i}{F_i}$ then for generic choices of $\lambda_1,\dots,\lambda_k$, x will be a Reeb singularity. Indeed this follows at once from the following theorem

Theorem 2.5.7. *Let E be a rank k positive vector bundle on a regular variety X. The zero scheme $Z(s)$ of a generic section s is a regular subvariety of codimension k.*

Summing up we have that the singular set of a generic logarithmic foliation is a union $\overline{\mathcal{K}} \cup \mathcal{R}$, where $\mathcal{K}$ are Kupka singularities and $\mathcal{R}$ are Reeb singularities.

Families.

The set of logarithmic foliations have a natural structure of algebraic variety. Namely, if we call the vector spaces $V_i = \mathbb{C}[x_0,\dots,x_i]_{=d_i}$ $(1 \le i \le k)$ and $W = \{(\lambda_1,\dots,\lambda_k) st : \sum \lambda_i d_i = 0\}$ then we have the map

$$\begin{array}{ccc} \mathbb{P}(V_1)\times\cdots\times\mathbb{P}(V_k)\times\mathbb{P}(W) & \longrightarrow & \mathbb{P}(H^0(\mathbb{P}^n,\Omega^1_{\mathbb{P}^n}(d)) \\ ([F_1],\dots,[F_k],(\lambda_1:\cdots:\lambda_k) & \mapsto & \left[(\prod_{i=1}^k F_i)\sum_{i=0}^k \lambda_i \dfrac{dF_i}{F_i}\right]. \end{array}$$

The image of this map is clearly inside the scheme of integrable forms. The following result of Omegar Calvo-Andrade says that its image is an irreducible component

Theorem 2.5.8. *The space of logarithmic foliations is an irreducible component of the scheme $\mathcal{F}ol(n,d)$ where $d = \sum d_i$.*

Proof. We refer to [ALN07, 3.3] for a proof of this fact. □

2.5.5. Pull-back foliations.

Every (twisted) form η in $\mathbb{P}^2$ is integrable, as defines a 1 dimensional distribution, which is allways involutive. Therefore for a linear projection $\phi : \mathbb{P}^n \to \mathbb{P}^2$ we can take the pull-back $\omega = \phi^*\eta$ and it will be an integrable form as well. Foliations defined by such forms are called pull-back foliations.

Tangent sheaf.

As described in [FCJVP08], one can give a characterization of pull-back foliations in terms of their tangent sheaves.

Proposition 2.5.9 ([FCJVP08]). *A foliation $\mathcal{F}$ on $\mathbb{P}^n$ is a pull-back foliation if and only its tangent sheaf decomposes as*

$$T\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(d) \oplus \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(-1).$$

Proof. Assume $\mathcal{F}$ is given by a form $\omega = \phi^*\eta$ for some linear projection $\phi : \mathbb{P}^n \to \mathbb{P}^2$ and some $\eta \in \Omega^1_{\mathbb{P}^2}(m)$. We may take homogeneous coordinates in $\mathbb{P}^n$ so that $\phi((x_0 : \cdots : x_n)) = (x_0 : x_1 : x_2)$. Then the vector fields $\frac{\partial}{\partial x_i} \in H^0(T\mathbb{P}^n(-1))$ are tangent to the foliation for $3 \leq i \leq n$. Then we have

$$\mathcal{O}_{\mathbb{P}^n} \cdot (\frac{\partial}{\partial x_3}, \ldots, \frac{\partial}{\partial x_n}) \cong \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^2}(-1) \subset T\mathcal{F}.$$

The quotient sheaf $\mathcal{G} = T\mathcal{F}/\bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^2}(-1)$ is generically of rank 1. Suppose $\mathcal{G}$ is not locally free, then there is a line $L \subset \mathbb{P}^n$ such that $\mathcal{G}|_L$ have torsion. Take a constants $c_1, \ldots, c_{n-2}$ in such way that the plane $\Pi = \{(x_0 : x_1 : x_2 : c_1 : \cdots : c_{n-2})\}$ intersects the line L. Then $\mathcal{G}|_\Pi$ is not locally free. On the other hand $\mathcal{G}|_\Pi$ is generated by a vector field $v \in T\mathbb{P}^2(d)$ (with the obvious identification $\Pi = \mathbb{P}^2$) such that $\eta(v) = 0$, so $\mathcal{G}|_\Pi$ is locally free, and so must be $\mathcal{G}$. Now as there are no non-trivial extensions between line bundles in projective space $T\mathcal{F}$ must split as in the statement of the proposition.

Conversely suppose $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(d) \oplus \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(-1)$. Then we can take coordinates such that

$$\bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{O}_{\mathbb{P}^n} \cdot (\frac{\partial}{\partial x_3}, \ldots, \frac{\partial}{\partial x_n}) \subset T\mathcal{F}.$$

With this coordinates, we may take a plane $\Pi = \{(x_0 : x_1 : x_2 : c_1 : \cdots : c_{n-2})\}$ and take $\eta \in \Omega^1_\Pi(m)$ defining the restriction of $\mathcal{F}$ to Π, and $\phi : \mathbb{P}^n \to \Pi$ the projection onto the first coordinates. With this choices is clear that $\omega = \phi^*\eta$ as required. □

Singularities.

The singular set of a generic pull-back foliation only have codimension 2 components which are, nevertheless, of two different nature. On one side there is the base locus of the projection $\phi : \mathbb{P}^n \to \mathbb{P}^2$. On the other hand, if $\eta \in \Omega^1_{\mathbb{P}^2}(m)$ is generic, then it will only have isolated zeros $p_1, \ldots, p_r$ such that $d\eta_{p_i} \neq 0$ for all $1 \leq i \leq r$, then $\phi^{-1}(p_i)$ is a codimension 2 linear variety inside $\mathrm{sing}(\mathcal{F})$. Is easy to see, taking suitable homogeneous coordinates, that the singular set of a pull-back foliation is the closure of the Kupka points.

Families.

Pull-back foliations form an algebraic variety isomorphic to the product

$$\{\text{Linear projections } \mathbb{P}^n \to \mathbb{P}^2\} \times \mathbb{P}(H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(m))).$$

A result that appears in a number of works is the following

Theorem 2.5.10. *Pull-back foliations form an irreducible component of the space of integrable forms in* $\mathbb{P}^n$*.*

Proof. A proof of this statement can be found on [FCJVP08]. □

Definition 3.0.11. Given a short exact sequence of sheaves

$$0 \to \mathscr{G} \xrightarrow{\iota} \mathscr{F} \to \mathscr{H} \to 0,$$

we apply to it the left-exact contravariant functor $\mathscr{F} \mapsto \mathscr{F}^\vee := \mathcal{H}om_X(\mathscr{F}, \mathcal{O}_X)$ to obtain exact sequences:

$$\begin{gathered} 0 \to \mathscr{H}^\vee \to \mathscr{F}^\vee \to \mathrm{Im}(\iota^\vee) \to 0, \\ 0 \to \mathrm{Im}(\iota^\vee) \to \mathscr{G}^\vee \to \mathcal{E}xt^1_X(\mathscr{H}, \mathcal{O}_X) \to \mathcal{E}xt^1_X(\mathscr{F}, \mathcal{O}_X). \end{gathered} \tag{3.1}$$

We say that the exact sequence (3.1), is the *dual exact sequence* of $0 \to \mathscr{G} \xrightarrow{\iota} \mathscr{F} \to \mathscr{H} \to 0$.

Lemma 3.0.12. *Let $0 \to N \xrightarrow{\iota} T \xrightarrow{\pi} M \to 0$ be a short exact sequence of R-modules such that T is reflexive and M is torsion free. Then $\mathrm{Im}(\iota^\vee)^\vee = N$ and $M = \mathrm{Im}(\pi^{\vee\vee})$.*

Proof. First we take the duals in the short exact sequence to get a sequence

$$0 \to \hom_R(M, R) \xrightarrow{\pi^\vee} \hom_R(T, R) \xrightarrow{\iota^\vee} \mathrm{Im}(\iota^\vee) \to 0$$

Then we take duals one more time and, given that T is reflexive and that M is torsion-free, we get the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Im}(\iota^\vee)^\vee \arrow[r] & T^{\vee\vee} \arrow[r, "\pi^{\vee\vee}"] & M^{\vee\vee} \arrow[r] & \mathrm{ext}^1_R(\mathrm{Im}(\iota^\vee), R) \\ 0 \arrow[r] & N \arrow[r] \arrow[u] & T \arrow[r] \arrow[u, equal] & M \arrow[r] \arrow[u, hook] & 0, \end{tikzcd}$$

whose rows are exact.

Chasing arrows we readily see that the leftmost vertical arrow must be an isomorphism. Indeed, since the monomorphism $N \to T^{\vee\vee}$ factorizes as

$$N \to \mathrm{Im}(\iota^\vee)^\vee \to T^{\vee\vee},$$

the second arrow being a monomorphism, so must $N \to \mathrm{Im}(\iota^\vee)^\vee$ be. On the other hand, given $a \in \mathrm{Im}(\iota^\vee)^\vee$, we can regard it, via the inclusion, as an element in $T^{\vee\vee} = T$, so we can compute $\pi(a)$. As the canonical map $\theta : M \to M^{\vee\vee}$ is an inclusion we have that, $\theta \circ \pi(a) = \pi^{\vee\vee}(a) = 0$, then $\pi(a) = 0$, so $a \in N$. From this we have $N \cong \mathrm{Im}(\iota^\vee)^\vee$, wich implies $M = \mathrm{Im}(\pi^{\vee\vee})$. $\square$

3.1. Exterior Powers

When dealing with foliations of codimension/dimension greater than 1 is usually convenient to work with p-forms. We'll need then to compare sub-sheaves $I \subset \Omega^1_X$ with their exterior powers $\wedge^p I \subset \Omega^p_X$. In order to do that we include the following statements, valid in a wider context.
We'll concentrate on flat modules and their exterior powers. This will be important when dealing with flat families of Pfaff systems of codimension higher than 1 (see Remark 4.0.5).

Lemma 3.1.1. *Let A be a ring containing the field $\mathbb{Q}$ of rational numbers, and let M be a flat A-module. Then, for every p, $\wedge^p M$ is also flat.*

Proof. If tensoring with M is an exact functor, so is its iterate $- \otimes M \otimes \cdots \otimes M$. So $M^{\otimes p}$ is flat. As A contains $\mathbb{Q}$, there is an anti-symmetrization operator

$$M^{\otimes p} \to \wedge^p M$$

which is a retraction of the canonical inclusion $\wedge^p M \subset M^{\otimes p}$. This makes $\wedge^p M$ a direct summand of $M^{\otimes p}$, set $M^{\otimes p} = \wedge^p M \oplus R$ for some module R. As the tensor power distributes direct sums (i.e.: $(\wedge^p M \oplus R) \otimes N \cong (\wedge^p M \otimes N) \oplus (R \otimes N)$), so does their derived functors. In particular we have, for every module N,

$$0 = \mathrm{Tor}_1(M^{\otimes p}, N) = \mathrm{Tor}_1(\wedge^p M, N) \oplus \mathrm{Tor}_1(R, N).$$

So $\wedge^p M$ is flat. $\square$

Finally, we draw some conclusions regarding flat quotient. When dealing with Pfaff systems, we'll be interested in short exact sequence of the form

$$0 \to \wedge^p I \to \Omega^p_X \to \mathcal{G} \to 0,$$

arising from short exact sequences of flat modules

$$0 \to I \to \Omega^1_X \to \Omega \to 0.$$

Note that, in general $\mathcal{G} \neq \wedge^p \Omega$. Nevertheless, we can state:

Proposition 3.1.2. *Let A be a ring containing $\mathbb{Q}$. Given an exact sequence*

$$0 \to M \to P \to N \to 0$$

of flat A-modules, we have an associated exact sequence

$$0 \to \wedge^p M \to \wedge^p P \to Q \to 0.$$

Then Q is also flat.

Proof. By Lemma 2.3.1 Q inherit a filtration from $\wedge^p P$:

$$Q = \wedge^p P / \wedge^p M = \overline{F}^0 \supseteq \overline{F}^1 \supseteq \cdots \supseteq \overline{F}^p = 0,$$

with quotients

$$F^i / F^{i+1} \cong \wedge^i M \otimes \wedge^{p-i} N.$$

Then Q have a filtration all of whose quotients are flat, so Q itself is flat. $\square$

4. FAMILIES OF DISTRIBUTIONS AND PFAFF SYSTEMS

Throughout this section we will work with a smooth morphism between schemes (of finite type over a field of characteristic 0) $p : X \to S$. We will consider subsheaves of the relative tangent sheaf $T_S X$ and the relative differentials $\Omega^1_{X|S}$.

Definition 4.0.3. A *family of distributions* is a short exact sequence

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

The family is called flat if $N_{\mathcal{F}}$ is flat over the base S.
A family of distributions is called *involutive* if it's closed under Lie bracket operation, that is, if for every pair of local sections $X, Y \in T\mathcal{F}(V)$, we have $[X, Y] \in T\mathcal{F}(V)$ where $[-,-]$ is the Lie bracket in $T_S X(V)$.

Likewise, a *family of Pfaff systems* is just a s.e.s.

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0.$$

It's called flat if $\Omega^1_{\mathcal{F}}$ is flat.
We will say that a family of Pfaff systems is *integrable* if $d(I(\mathcal{F})) \wedge \bigwedge^r I(\mathcal{F}) = 0 \subset \Omega^{r+2}_{X|S}$; where $d : \Omega^j_{X|S} \to \Omega^{j+1}_{X|S}$ is the relative de Rham differential, and r is the generic rank of the sheaf $\Omega^1_{\mathcal{F}}$.

Remark 4.0.4. Observe that the relative differential $d : \Omega^j_{X|S} \to \Omega^{j+1}_{X|S}$ is not an $\mathcal{O}_X$-linear morphism. It is, however, $f^{-1}\mathcal{O}_S$-linear, so the sheaf $d(I(\mathcal{F})) \wedge \bigwedge^r I(\mathcal{F})$, whose anihilation encodes the integrability of the Pfaff system, is actually a sheaf of $f^{-1}\mathcal{O}_S$-modules.

In particular the dual to a family of distributions is a family of Pfaff systems and viceversa.

Remark 4.0.5. The dual of an involutive family of distributions is an integrable family of Pfaff systems. Reciprocally, the dual of an integrable family of Pfaff systems is a family of involutive distributions. This is just a consequence of the Cartan-Eilenberg formula for the de Rham differential of a 1-form applied to vector fields

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Indeed, as involutiveness and integrability can be checked locally over sections, we can proceed as in [War83, Prop. 2.30].

Definition 4.0.6. The *dimension* of a family of distribution is the generic rank of $T\mathcal{F}$. Likewise, the *dimension* of a family of Pfaff systems is the generic rank of $\Omega^1_{\mathcal{F}}$.

If $p : X \to S$ is moreover projective, S is connected, and the family is flat, then $T\mathcal{F}$ is a flat sheaf over S. Therefore for every $s \in S$ the Hilbert polynomial of $T\mathcal{F}_s$ is the same, and so is its generic rank (being encoded in the principal coeficient of the polynomial). The same occurs with families of Pfaff systems.

Remark 4.0.7. Frequently, in the study of foliations of codimension higher than 1, is more convenient and better adapted to calculations to work with an alternative description of foliations. Namely, one can define a codimension q foliation on a variety X as in [dM77], with a global section ω of $\Omega^q_X \otimes \mathcal{L}$ such that:

- ω is locally decomposable, i.e.: there is, for all $x \in X$ an open set such that
$$\omega = \eta_1 \wedge \cdots \wedge \eta_q,$$
with $\eta_i \in \Omega^1_X$.
- ω is integrable, i.e.: $\omega \wedge d\eta_i = 0,\ 1 \leq i \leq q$.

With this setting, studying flat families of codimension q foliations (meaning here families of integrable Pfaff systems) as in [FCJVP08] and [CPV09], parametrized by a scheme S, amounts to studying short exact sequences of flat sheaves:
$$0 \to \mathcal{L}^{-1} \to \Omega^q_{X|S} \to \mathcal{G} \to 0,$$
that are locally decomposable and integrable. By the results of section 3.1 a flat family of codimension q Pfaff systems given as a sub-sheaf of $\Omega^1_{X|S}$ give rise to a flat family in the above sense.

4.1. Universal families

Now lets take a regular projective scheme X, a polynomial $P \in \mathbb{Q}[t]$, and consider the following functor

$$\begin{array}{rcl} \mathfrak{Inv}^P(X) : Sch & \longrightarrow & Sets \\ S & \mapsto & \left\{ \begin{array}{l} \text{flat families } 0 \to T\mathcal{F} \to T_S(X \times S) \to N_{\mathcal{F}} \to 0 \\ \text{of involutive distributions such that } N_{\mathcal{F}} \text{ have} \\ \text{Hilbert polynomial } P(t). \end{array} \right\}. \end{array}$$

Say $p : X \times S \to X$ is the projection, so $T_S(X \times S) = p^*TX$. Clearly one have $\mathfrak{Inv}^P(X)$ is a sub-functor of $\mathfrak{Quot}^P(X, TX)$. We are going to show that

$\mathfrak{Inv}^P(X)$ is actually a *closed* sub-functor of $\mathfrak{Quot}^P(X, TX)$ and therefore also representable.

So take the smooth morphism given by the projection

$$p_1 : \mathrm{Quot}_P(X, TX) \times X \to \mathrm{Quot}_P(X, TX).$$

Here we are taking as base scheme $S = \mathrm{Quot}_P(X, TX)$, then on the total space $S \times X = \mathrm{Quot}_P(X, TX) \times X$ we have the natural short exact sequence

$$0 \to \mathscr{F} \to p_2^* TX = T_S(S \times X) \to \mathscr{Q} \to 0.$$

Now we consider the push-forward of this sheaves by p_1, as X is proper, this push-forwards are coherent sheaves over S. In particular we have maps of coherent sheaves over $\mathrm{Quot}^P(X, TX)$

$$p_{1*}\mathscr{F} \otimes_S p_{1*}\mathscr{F} \xrightarrow{[-,-]} p_{1*}T_S(S \times X) \to p_{1*}\mathscr{Q}$$

induced by the maps over $S \times X$. Note that while the Lie bracket on $T_S(S \times X)$ is only $p_1^{-1}\mathcal{O}_S$-linear, the map induced on the push-forwards is $\mathcal{O}_S$-linear, so is a morphism of coherent sheaves. We then also have for any $m, n \in \mathbb{Z}$ the twisted morphisms

$$p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(n) \xrightarrow{[-,-]} p_{1*}T_S(S \times X)(m+n) \to p_{1*}\mathscr{Q}(m+n).$$

Note also that, as p_1 is a projective morphism, then there exist an $n \in \mathbb{Z}$ such that for any $m \geq n$ the natural sheaves morphism over $S \times X$, $p_1^* p_{1*}(\mathscr{F})(m) \to \mathscr{F}(m)$ is an epimorphism. So if for some $f : Z \to S$ and some $m \geq n$ one have that the composition

$$f^* p_{1*}\mathscr{F}(m) \otimes_Z f^* p_{1*}\mathscr{F}(m) \xrightarrow{[-,-]} f^* p_{1*}T_S(S \times X)(2m) \to f^* p_{1*}\mathscr{Q}(2m)$$

is zero, then the map

$$(f \times id)^*\mathscr{F}(m) \otimes_{\pi_1^{-1}\mathcal{O}_Z} (f \times id)^*\mathscr{F}(m) \xrightarrow{[-,-]} T_Z(Z \times X)(2m) \to (f \times id)^*\mathscr{Q}(2m)$$

is zero as well, here $\pi_1 : Z \times X \to Z$ is the projection, which is by the way the pull-back of p_1.

Now to conclude the representability of $\mathfrak{Inv}^P(X)$ we need one important lemma.

Lemma 4.1.1. *Let S be a noetherian scheme, $p : X \to S$ a projective morphism and $\mathscr{F}$ a coherent sheaf on X. Then $\mathscr{F}$ is flat over S if and only if there exist some integer N such that for all $m \geq N$ the push-forwards $p_*\mathscr{F}(m)$ are locally free.*

Proof. The statement being local on S we can assume $S = \mathrm{Spec}(A)$ where A is a local ring. Then we can consider the graded A-module $\bigoplus_{m \geq N} M_m$, where $M_m = \Gamma(S, p_*\mathscr{F}(m))$. The sheaf $\mathscr{F}$ is isomorphic to the (projective) sheafification of the graded module M.

If every $p_*\mathscr{F}(m)$ is flat over S, so is every M_m over A and thus so is M, and therefore $\mathscr{F}$ is flat.

Conversely, if $\mathscr{F}$ is flat, then its global sections module $\bigoplus_{m\in\mathbb{Z}} \Gamma(S, p_*\mathscr{F}(m))$ will be flat from sufficiently large degree on. That is, there will be an integer N such that M is a flat A-module. So each direct summand of M will be flat as well. Then each M_m are flat and, as p is projective, finitely generated. So M_m are free, and so are $p_*\mathscr{F}(m)$. $\square$

We can then take $m \in Z$ big enough so $p_{1*}\mathscr{F}(m)$ and $p_{1*}\mathscr{Q}(2m)$ are locally free and the morphism $p_1^* p_{1*}(\mathscr{F})(m) \to \mathscr{F}(m)$ is epimorphism. Then we can regard the composition

$$p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(m) \xrightarrow{[-,-]} p_{1*}T_S(S \times X)(2m) \to p_{1*}\mathscr{Q}(2m)$$

as a global section σ of the locally free sheaf $\mathcal{H}om_S(p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(m), \mathscr{Q}(2m))$. We can then make the following definition.

Definition 4.1.2. We define the scheme $\mathrm{Inv}^P(X)$ to be the zero scheme $Z(\sigma)$ (cf.: Definition 1.1.12) of the section σ defined above.

A direct aplication of Proposition 1.1.14 to this definition together with the discusion so far immediately gives us the following.

Proposition 4.1.3. *The subscheme* $\mathrm{Inv}^P(X) \subseteq \mathrm{Quot}^P(X, TX)$ *represents the functor* $\mathfrak{Inv}^P(X)$.

Similarly we can consider the sub-functor $\mathfrak{iPf}^P(X)$ of $\mathfrak{Quot}^P(X, \Omega^1_X)$.

$$\begin{aligned} \mathfrak{iPf}^P(X) : Sch &\longrightarrow Sets \\ S &\mapsto \left\{ \begin{array}{l} \text{flat families } 0 \to I(\mathcal{F}) \to \Omega^1 X|S \to \Omega^1_{\mathcal{F}} \to 0 \text{ of} \\ \text{integrable Pfaff systems such that } \Omega^1_{\mathcal{F}} \text{ have Hil-} \\ \text{bert polynomial } P(t). \end{array} \right\}. \end{aligned}$$

Then as before we take $S = \mathrm{Quot}^P(X, \Omega^1_X)$ and consider the map

$$p_{1*}(d(\mathscr{I}) \wedge \bigwedge^r \mathscr{I})(m) \longrightarrow p_{1*}\Omega^{r+2}_{S\times X|S}(m).$$

Which is, for large enough m a morphism between locally free sheaves on S.

Definition 4.1.4. We define the scheme $\mathrm{iPf}^P(X)$ to be the zero scheme of the above morphism, viewed as a global section of the locally free sheaf $\mathcal{H}om(p_{1*}(d(\mathscr{I}) \wedge \bigwedge^r \mathscr{I})(m), p_{1*}\Omega^{r+2}_{S\times X|S}(m))$.

And then by Proposition 1.1.14 we have representability.

Proposition 4.1.5. *The subscheme* $\mathrm{iPf}^P(X) \subset \mathrm{Quot}^P(X, \Omega^1_X)$ *represents the functor* $\mathfrak{i}\mathfrak{P}\mathfrak{f}^P(X)$.

4.2. Duality

Definition 4.2.1. The *singular locus* of a family of distributions $0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$ is the (scheme theoretic) support of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$. Its points are the points where $N_{\mathcal{F}}$ fails to be a fiber bundle.

Similarly, for a family of Pfaff systems $0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$, its singular locus is $\mathrm{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X))$.

Remark 4.2.2. Call $i : T\mathcal{F} \to T_S X$ the inclusion. We have an open non-empty set U where, for every $x \in U$, $\dim(Im(i \otimes k(x)))$ is maximal. More precisely, U is the open set where $Tor^X_1(N_{\mathcal{F}}, k(x)) = 0$, which is the maximal open set such that $N_{\mathcal{F}}|_U$ is locally free, and therefore so is $T\mathcal{F}$. Then, when restricted to U, $T\mathcal{F}$ can be given locally as the subsheaf of $T_S X$ generated by k linearly independent relative vector fields, i.e.: $T\mathcal{F}$ defines a family of non-singular foliations. In U, one have that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$. Then, the underlying topological space of the singular locus of the family given by $T\mathcal{F}$ is the singular set of the foliation in a classical (topologial space) sense.

The above discussion translates verbatim to families of Pfaff systems.

Proposition 4.2.3. *Let be a family of Pfaff systems*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

such that $\Omega^1_{\mathcal{F}}$ *is torsion-free. Its singular locus and the singular locus of the dual family*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$$

are the same sub-scheme of X. *We denote this sub-scheme by* $\mathrm{sing}(\mathcal{F})$

Proof. We are going to show that the immersions $Y_1 := \mathrm{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)) \subseteq X$ and $Y_2 := \mathrm{supp}(\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)) \subseteq X$ represent the same sub-functor of $\mathrm{Hom}(-, X)$, thus proving the proposition.

First note that, if $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$, then $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) = 0$.
Indeed, if $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$, $N_{\mathcal{F}}$ is locally free and then so is $T\mathcal{F}$. Moreover, since $\Omega^1_{\mathcal{F}}$ is torsion free, we can dualize the short exact sequence $0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$ and, by lemma 3.0.12, obtain the equality

$\Omega^1_{\mathcal{F}} = T\mathcal{F}^\vee$.
So $\Omega^1_{\mathcal{F}}$ is locally free and $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) = 0$.

Now, given a quasi-coherent sheaf $\mathscr{G}$ of X, its support $\operatorname{supp}(\mathscr{G}) \subseteq X$ represents the following sub-functor of $\operatorname{Hom}(-, X)$:

$$T \longmapsto \{f : T \to X \quad \text{s.t.: } f^*\mathscr{G} \text{ is not a torsion sheaf}\} \subseteq \operatorname{Hom}(T, X).$$

So, let's take a morphism $f : T \to Y_1 \subseteq X$.

(I) $f : T \to Y_1$ is an immersion: Suppose $f^*\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is a torsion sheaf.
Then there's a point $t \in T$ such that

$$\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes k(t) = 0.$$

By Nakayama's lemma this implies that there's an open subset $U \subseteq X$ containing t such that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$. This in turn implies $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$ contradicting the fact that $t \in T \subseteq Y_1$. Then $T \subseteq Y_2$.
Similarly one prove that if $T \subseteq Y_2$ then $T \subseteq Y_1$.

(II) General case: Taking the scheme theoretic image of f we can reduce to the above case where T is a sub-scheme of X.

□

4.3. *The codimension* 1 *case*

We now treat the case of families of codimension 1 foliations. From now on we'll suppose that $X \to S$ is a smooth morphism.

Definition 4.3.1. A family of involutive distributions

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0,$$

is of *codimension* 1 iff $N_{\mathcal{F}}$ is a sheaf of generic rank 1.

Likewise a family of Pfaff systems

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0,$$

is of codimension 1 if the sheaf $I(\mathcal{F})$ have generic rank 1.

Lemma 4.3.2. *Let be a family of codimension* 1 *Pfaff systems*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0,$$

over an integral scheme X, such that $\Omega^1_{\mathcal{F}}$ is torsion-free. Then $I(\mathcal{F})$ is a line-bundle over X.

Proof. If $\Omega^1_{\mathcal{F}}$ is torsion-free, by Lemma 3.0.12 we have $I(\mathcal{F}) \cong N^\vee_{\mathcal{F}}$. In particular $I(\mathcal{F})$ is the dual of a sheaf, then is reflexive and observes property S_2. Write $I = I(\mathcal{F})$ and consider now the sheaf $I^\vee \otimes I$ together with the canonical morphism

$$I^\vee \otimes I \to \mathcal{O}_X.$$

The generic rank of $I^\vee \otimes I$ is 1. As I is reflexive, $I^\vee \otimes I$ is self-dual. So the canonical morphism above induces the dual morphism $\mathcal{O}_X \to I^\vee \otimes I$. The composition

$$\mathcal{O}_X \to I^\vee \otimes I \to \mathcal{O}_X$$

must be invertible, otherwise the image of $I^\vee \otimes I$ in $\mathcal{O}_X$ would be a torsion sub-sheaf. Then I is an invertible sheaf. □

Proposition 4.3.3. *In the case of codimension* 1 *Pfaff systems, if* $\Omega^1_{\mathcal{F}}$ *is torsion-free over* X *and the inclusion* $I(\mathcal{F}) \to \Omega^1_{X|S}$ *is nowhere trivial on* S *(meaning that* $I(\mathcal{F}) \otimes \mathcal{O}_T \to \Omega^1_{X|S} \otimes \mathcal{O}_T$ *is never the zero morphism for any* $T \to S$*) then the family is automatically flat.*

Proof. Indeed, $\Omega^1_{\mathcal{F}}$ being torsion free implies that the rank-1 sheaf $I(\mathcal{F})$ must be a line bundle. Then if we take any morphism $f : T \to S$ and take pull-backs we'll have an exact sequence

$$0 \to \mathrm{Tor}^S_1(\Omega^1_{\mathcal{F}}, T) \to f^*I(\mathcal{F}) \to f^*\Omega^1_{X|S} \to f^*\Omega^1 \to 0.$$

Now, as $I(\mathcal{F})$ is a line bundle, the cokernel $f^*I(\mathcal{F})/\mathrm{Tor}^S_1(\Omega^1_{\mathcal{F}}, T)$ must be a torsion sheaf over X_T. But, X being smooth over S, the annihilator $f^*\Omega^1_{X|S}$ is of the form $p^*(J)$, with $J \subset \mathcal{O}_T$, so $f^*I(\mathcal{F}) \to f^*\Omega^1_{X|S}$ must be the zero morphism when restricted to $\mathcal{O}_T/J$, contradicting the nowhere triviality assumption. □

Remark 4.3.4. In the codimension 1 case, we can calculate the $\mathrm{sing}(\mathcal{F})$ by noting that $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$ is the cokernel in the exact sequence

$$T_S X \to I(\mathcal{F})^\vee \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \to 0.$$

We can then tensor the sequence by $I(\mathcal{F})$ and obtain

$$(\Omega^1_{X|S} \otimes I(\mathcal{F}))^\vee \to \mathcal{O}_X \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes I(\mathcal{F}) \to 0.$$

Now, $I(\mathcal{F})$ being a line bundle, the support of $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$ and that of $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes I(\mathcal{F})$ is exactly the same. Note then that, in the second exact sequence, the cokernel is the scheme theoretic zero locus of the twisted 1-form given by

$$\mathcal{O}_X \xrightarrow{\omega} \Omega^1_{X|S} \otimes I(\mathcal{F})^\vee$$

as defined in section 1.1.2. So, if we have a family of codimension 1 Pfaff systems given locally by a twisted form

$$\omega = \sum_{i=1}^{n} f_i(x) dx_i$$

then $\operatorname{sing}(\mathcal{F})$ is the scheme defined by the ideal $(f_1, \ldots, f_n)$.

The above proposition and remark tell us that our definition of flat family for Pfaff systems of codimension 1 is esentially the same as the one used in the now classical works of Lins-Neto, Cerveau, et. al.

Theorem 4.3.5. *Given two families*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$
$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

Of codimension 1*, dual to each other, such that $N_{\mathcal{F}}$ is torsion free (or, equivalently, such that $\Omega^1_{\mathcal{F}}$ is torsion free). And such that* $\operatorname{sing}(\mathcal{F})$ *is flat over S.*
Then one of the families is flat if and only if its dual family is also flat.

Proof. Let $\Sigma = \operatorname{sing}(\mathcal{F})$.

Let's suppose first that the family

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

is flat. We have to prove that $N_{\mathcal{F}}$ is also flat.To do this we note that applying the functor $\mathcal{H}om_X(-, \mathcal{O}_X)$ to the family of distributions not only gives us the family of Pfaff systems but also the exact sequence

$$0 \to N_{\mathcal{F}} \to I(\mathcal{F})^\vee \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \to 0.$$

Being $\Omega^1_{\mathcal{F}}$ torsion-free, $I(\mathcal{F})$ must be a line bundle, and so must $I(\mathcal{F})^\vee$, let's call $I(\mathcal{F})^\vee = \mathcal{L}$ to ease the notation. Now $\mathcal{L}$ have a $N_{\mathcal{F}}$ as a subsheaf generically of rank 1, so $N_{\mathcal{F}} = \mathcal{I} \cdot \mathcal{L}$ for some Ideal sheaf $\mathcal{I}$. Then $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \cong \mathcal{L} \otimes \mathcal{O}_X/\mathcal{I}$. As $\Sigma = \operatorname{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X))$ one necesarilly have $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \cong \mathcal{L}_\Sigma$. Then $\mathcal{L}_\Sigma$, being a locally free sheaf over Σ wich is flat over S, is flat over S. Therefore, as $\mathcal{L}$ is also flat over S, flatness for $N_{\mathcal{F}}$ follows.

Let's suppose now that the family

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$$

is flat. We have to prove that $\Omega^1_{\mathcal{F}}$ is also flat. By the above proposition it's enough to show that the morphism $I(\mathcal{F}) \xrightarrow{\iota} \Omega^1_{X|S}$ is nowhere zero. Suppose there is $T \to S$ such that $\iota_T = 0$. Take an open set $U \subset X$ where $\Omega^1_{\mathcal{F}}$ is locally free. In that open set we can apply base change with respect to the functor $\mathcal{H}om_X(-, \mathcal{O}_X)$ ([ASK] or [GBAO72]) so, restricting everything to U we have $(\iota_T)^\vee \cong (\iota^\vee)_T$. But, in U, $\iota^\vee$ is the morphism $T_S X \to N_{\mathcal{F}}$ and so it cannot become the zero morphism under any base change. □

Corollary 4.3.6. *Every irreducible component of the scheme* Inv_P *is birationally equivalent to an irreducible component of* iPf_P.

4.4. The arbitrary codimension case

To give an analogous theorem to 4.3.5 in arbitrary codimension we'll have to deal with finer invariants than the singular locus of the foliation. In the scheme X we'll consider a stratification naturally associated with $\mathcal{F}$. This stratification have been already studied and described by Suwa in [Suw88]. To deal with flatness issues we have to provide a scheme structure to Suwa's stratification, this will be a particular case of flattening stratification. Before going into that, we begin with some generalities. Remember that we are working over a smooth morphism $X \to S$.

Lemma 4.4.1. *Let* $X \to S$ *be a smooth morphism,* $\mathscr{F}$ *a coherent sheaf on* X *that is* $Z^{(2)}$*-closed. Then, for any* $s \in S$*, the sheaf* $\mathscr{F}_s = \mathscr{F} \otimes k(s)$ *is* $Z^{(2)}$*-closed over* X_s.

Proof. We have to show that for every $U \subset X_s$ such that $\operatorname{codim}(X \setminus U) \geq 2$ the restriction

$$\mathscr{F}_s \xrightarrow{\rho_U} \mathscr{F}_s|_U$$

is surjective. As the formal completion $\widehat{\mathcal{O}}_{X_s,x}$ of $\mathcal{O}_{X_s}$ with respect to any closed point x is faithfully flat [GR03, IV.3.2], we can check surjectivity of ρ_U by looking at every formal completion. As $X \to S$ is smooth, formally around a point x we have $\mathcal{O}_X \cong \mathcal{O}_S \otimes_k k[z_1, \ldots, z_d]$ so we can take an open subset $V \subseteq X$, to be $V = U \times S$. Then, with this choice of V, we have an epimorphism

$$\widehat{\mathscr{F}}|_V \to \widehat{\mathscr{F}_s}|_U \to 0.$$

Then we have a diagram with exact rows and columns

$$\begin{tikzcd}
\widehat{\mathscr{F}} \arrow[r] \arrow[d, "\rho_V"] & \widehat{\mathscr{F}_s} \arrow[r] \arrow[d, "\rho_U"] & 0 . \\
\widehat{\mathscr{F}}|_V \arrow[r] \arrow[d] & \widehat{\mathscr{F}_s}|_U \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

So ρ_U must be an epimorphism as well.

$\square$

Lemma 4.4.2. *Suppose X is a normal scheme. Let be a family of distributions*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

If $\operatorname{codim}(\operatorname{sing}(\mathcal{F})) \geq 2$ *then, for every map $T \to S$, one have*

$$\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_T \cong \mathcal{H}om_{X_T}(T\mathcal{F}_T, \mathcal{O}_T).$$

The analogous statement is true for $I(\mathcal{F})^\vee$ in a flat family of Pfaff systems.

Proof. By Theorem 1.5.2 and Proposition 1.5.3 we only have to prove that, for every closed point $s \in S$, the natural map

$$\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes k(s) \to \mathcal{H}om_{X_s}(T\mathcal{F} \otimes k(s), \mathcal{O}_X \otimes k(s))$$

is surjective. Being the dual of some sheaves, both $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X)$ and $\mathcal{H}om_{X_s}(T\mathcal{F} \otimes k(s), \mathcal{O}_X \otimes k(s))$ possess property S_2 (Proposition 1.1.8), and so are $Z^{(2)}$-closed, and so is $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes k(s)$ by the above lemma.

Let $U = X \setminus \operatorname{sing}\mathcal{F}$ and $j : U \hookrightarrow X$ the inclusion. As $T\mathcal{F}|_U$ is locally free over U, so is $T\mathcal{F}^\vee|_U$. Then, in U, we have

$$\mathcal{E}xt^1(T\mathcal{F}|_U, \mathcal{O}_X|_U \otimes_S \mathcal{G}) = 0,$$

for every $\mathcal{G} \in Coh(S)$. Then from Proposition 1.5.6 we get surjectivity on

$$\mathcal{H}om_X(T\mathcal{F}|_U, \mathcal{O}_X|_U) \otimes k(s) \to \mathcal{H}om_{X_s}(T\mathcal{F}|_U \otimes k(s), \mathcal{O}_X|_U \otimes k(s)).$$

But, as $\operatorname{codim}(\operatorname{sing}(\mathcal{F})) > 1$ and both sheaves are S_2, then surjectivity holds in all of X_s. $\square$

Lemma 4.4.3. *As above, suppose X is normal. Given a flat family*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

Such that $\operatorname{codim}(\operatorname{sing}(\mathcal{F})) \geq 2$. *Suppose further that the flattening stratification of X over $T\mathcal{F}$ is flat over S (c.f.: Proposition 1.4.7). Then $T\mathcal{F}^\vee$ is also a flat $\mathcal{O}_S$-module.*

The analogous statement is true for $I(\mathcal{F})^\vee$ in a flat family of Pfaff systems.

Proof. The proof works exactly the same for distributions or Pfaff systems mutatis mutandi.

Take $\coprod_P X_P$ the flattening stratification of X with respect to $T\mathcal{F}$. The restriction $T\mathcal{F}_{X_P}$ (being coherent and flat over X_P) is locally free over X_P, then so is its dual $\mathcal{H}om_{X_P}(T\mathcal{F}_{X_P}, \mathcal{O}_{X_P})$. By Lemma 4.4.2, in each stratum X_P we have the isomorphism

$$\mathcal{H}om_{X_P}(T\mathcal{F}_{X_P}, \mathcal{O}_{X_P}) \cong \mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_{X_P} = T\mathcal{F}^\vee \otimes \mathcal{O}_{X_P}.$$

So $T\mathcal{F}^\vee$ is flat when restricted to the filtration $\coprod_P X_P$, which is in turn flat over S. Then, by Theorem 1.2.1, $T\mathcal{F}^\vee$ is flat over S. □

Definition 4.4.4. For a family of distributions consider the flattening stratification

$$\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \subseteq X$$

of X with respect to $T\mathcal{F} \oplus N_\mathcal{F}$. We call this the *Suwa stratification* of X with respect to $T\mathcal{F}$.

Remark 4.4.5. Note that the flattening stratification of $T\mathcal{F} \oplus N_\mathcal{F}$ is the (scheme theoretic) intersection of the flattening stratification of $T\mathcal{F}$ with that of $N_\mathcal{F}$. This is because $(T\mathcal{F} \oplus N_\mathcal{F}) \otimes \mathcal{O}_Y$ is flat if and only if both $T\mathcal{F} \otimes \mathcal{O}_Y$ and $N_\mathcal{F} \otimes \mathcal{O}_Y$ are.

This tells us, in particular, that each stratum is indexed by two natural numbers r and k such that

$$x \in X_{r,k} \Longleftrightarrow \dim(T\mathcal{F} \otimes k(x)) = r \text{ and } \dim(N_\mathcal{F} \otimes k(x)) = k.$$

In [Suw88], Suwa studied a related stratification associated to a foliation. Given a distribution on a complex manifold M, $D \subset TM$ he defines the strata $M^{(l)}$ as

$$M^{(l)} = \{x \in M \ s.t.\colon D_x \subset T_xM \text{ is a sub-space of dimension } l\ \}.$$

Here D is spanned pointwise by vector fields $v_1, \ldots, v_r$, and $D_x =< v_i(x) >$. Clearly if D is of generic rank r the open stratum is $M^{(r)}$.

Note that, in the setting of distribution as sub-sheafs $i : T\mathcal{F} \hookrightarrow TX$ of the tangent sheaf of a variety, the vector space $T_x\mathcal{F}$ is actually the image of the map

$$T\mathcal{F} \otimes k(x) \xrightarrow{i \otimes k(x)} TX \otimes k(x),$$

whose kernel is $\mathrm{Tor}_1^X(N_\mathcal{F}, k(x))$. Moreover we have the exact sequence

$$0 \to T_x\mathcal{F} = \mathrm{Im}(i \otimes k(x)) \to TX \otimes k(x) \to N_\mathcal{F} \otimes k(x) \to 0.$$

In particular, in a variety X of dimension n, if $\dim(T_x\mathcal{F}) = l$ then $\dim(N_\mathcal{F} \otimes k(x)) = n - l$. So what we call Suwa stratification of X is actually a refinement of the stratification studied in [Suw88].

Our main motivation for defining this refinement of the stratification of [Suw88] is the following result.

Theorem 4.4.6. *Let be a flat family*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0,$$

parametrized by a normal scheme S of finite type over an algebraically closed field, such that $N_{\mathcal{F}}$ is torsion free and $\operatorname{codim}(\operatorname{sing}(\mathcal{F})) \geq 2$. *Suppose each stratum $X_{r,k}$ of the Suwa stratification is flat over S. Then the dual family*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

is also flat over S. Moreover, for each point $s \in S$ we have

$$I(\mathcal{F})_s = (N_{\mathcal{F}s})^\vee,$$

in other terms "the dual family is the family of the duals".

Proof. Considering the exact sequence

$$0 \to \Omega^1_{\mathcal{F}} \to T\mathcal{F}^\vee \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \to 0.$$

Is clear that to prove flatness of the dual family it's enough to show $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is flat over S.

Also, by Lemma 4.4.2, we have for every $s \in S$ the diagram with exact rows and columns,

$$\begin{tikzcd}
\mathcal{H}om(I(\mathcal{F}), \mathcal{O}_X) \otimes k(s) \arrow[r] \arrow[d] & \mathcal{H}om(I(\mathcal{F})_s, \mathcal{O}_{X_s}) \arrow[r] \arrow[d] & 0 . \\
\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \arrow[r] \arrow[d] & \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}} \otimes k(s), \mathcal{O}_{X_s}) \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

So $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}} \otimes k(s), \mathcal{O}_{X_s})$ is surjective for every $s \in S$ so by Proposition 1.5.5 the exchange property is valid for $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$ (c.f. section 1.5). If moreover $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is flat over S, then, by Proposition 1.5.5,

$$I(\mathcal{F})_s = \mathcal{H}om_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \cong \mathcal{H}om_X(N_{\mathcal{F}s}, \mathcal{O}_{X_s}) = (N_{\mathcal{F}s})^\vee.$$

By Proposition 1.4.7 is enough to show the restriction of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ to every Suwa stratum is flat over S. So let $Y \subseteq X$ be a Suwa stratum, if we can show that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes \mathcal{O}_Y$ is locally free then we're set. By hypothesis, one have the isomorphism $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_Y \cong \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y)$. So we can express $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes \mathcal{O}_Y$ as the cokernel in the $\mathcal{O}_Y$-modules exact sequence

$$\mathcal{H}om_Y(TX_Y, \mathcal{O}_Y) \to \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y) \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_Y \to 0.$$

So, localizing in a point $y \in Y$, we can realize the local $\mathcal{O}_{Y,y}$-module $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ as the set of maps $T\mathcal{F}_{Y,y} \to \mathcal{O}_{Y,y}$ modulo the ones that factorizes as

$$\begin{tikzcd} T\mathcal{F}_{Y,y} \arrow[r, dashed] \arrow[d] & \mathcal{O}_{Y,y} \\ TX_{Y,y} \arrow[ur] & \end{tikzcd} .$$

To study $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ this way, note that we have the following exact sequence.

$$0 \to \mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \to T\mathcal{F}_{Y,y} \to TX_{Y,y} \to (N_{\mathcal{F}})_{Y,y} \to 0.$$

Wich we split into two short exact sequences,

$$0 \to \mathcal{K} \to \quad TX_{Y,y} \quad \to (N_{\mathcal{F}})_{Y,y} \to 0 \quad \text{and} \tag{4.1}$$
$$0 \to \mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \to \quad T\mathcal{F}_{Y,y} \quad \to \mathcal{K} \to 0. \tag{4.2}$$

Now, as Y is a Suwa stratum, then $\mathcal{Q}_{\mathcal{Y}}$ and $T\mathcal{F}_Y$ are flat over Y, and coherent, so they are locally free. As a consequence, short exact sequence (4.1) splits, so $TX_{Y,y} \cong (N_{\mathcal{F}})_{Y,y} \oplus \mathcal{K}$. So

$$\mathcal{H}om_Y(TX_Y, \mathcal{O}_Y)_y \cong \mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \oplus \mathcal{H}om_Y((N_{\mathcal{F}})_Y, \mathcal{O}_Y)_y$$

and we get $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ as the cokernel in

$$\mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y)_y \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y} \to 0. \tag{4.3}$$

Being $(N_{\mathcal{F}})_Y$ and TX_Y locally free over Y, so is $\mathcal{K}$. Then short exact sequence (4.2) splits, so $T\mathcal{F}_{Y,y} \cong \mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \oplus \mathcal{K}$. Also, as $T\mathcal{F}_Y$ and $\mathcal{K}$ are locally free over Y, so is $\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_Y)$. Sequence (4.3) now reads

$$\mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{H}om_Y(\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}), \mathcal{O}_Y)_y \oplus \mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y} \to 0.$$

So we have

$$\mathcal{H}om_Y(\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}), \mathcal{O}_Y)_y \cong \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}.$$

Now, as $\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y})$ is locally free over Y, so is its dual. In other words, we just proved $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_Y$ is locally free over Y, which settles the theorem. □

Remark 4.4.7. During the proof of last statement we have actually obtained this stronger result:

Proposition 4.4.8. $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ *is flat over Suwa's stratification.*

In particular, if $\coprod X_Q$ denotes the flattening stratification of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$, there is a morphism

$$\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \to \coprod_{Q} X_Q.$$

Now, by the construction of flattening stratification, $\coprod X_Q$ consist of an open stratum U such that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$, and closed strata whose closure is $\operatorname{sing}(\mathcal{F})$. So the morphism $\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \to \coprod_Q X_Q$ actually defines a stratification of $\operatorname{sing}(\mathcal{F})$.

4.5. Singularities

Theorem 4.3.5 gives a condition for a flat family of integrable Pfaff systems to give rise to a flat family of involutive distributions in terms of the flatness of the singular locus. We have then to be able to decide when can we apply the theorem. More precisely, say

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

is a flat family of codimension 1 integrable Pfaff systems, and let $s \in S$. How do we know when $\operatorname{sing}(\mathcal{F})$ is flat around s? In this section we adress this question and give a sufficient condition for $\operatorname{sing}(\mathcal{F})$ to be flat at s in terms of the classification of singular points of the Pfaff system $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$.

From now on, we will only consider Pfaff systems such that $\Omega^1_{\mathcal{F}}$ is torsion-free.

Remember that, if we have a Pfaff system of codimension 1, $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$, such that $\Omega^1_{\mathcal{F}_s}$ is torsion-free, we can consider, locally on X, that is given by a single 1-form ω and that is integrable iff $\omega \wedge d\omega = 0$.

Remark 4.5.1. Note that Kupka singularities and Reeb singularities are singularities in the sense of 4.2.1 i.e.: they are points in $\operatorname{sing}(\mathcal{F})$.

We now give a version for families of the fundamental result of Kupka.

Proposition 4.5.2. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems of codimension 1*, and let* $\Sigma = \operatorname{sing}(\mathcal{F}) \subset X$*. Let* $s \in S$*, and* $x \in \Sigma_s$ *be such that* x *is a Kupka singularity of* $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$*. Then, locally around* x $I(\mathcal{F})$ *can be given by a* relative 1*-form* $\omega(z,s) \in \Omega^1_{X|S}$ *such that*

$$\omega = f_1(z,s)dz_1 + f_2(z,s)dz_2,$$

i.e.: ω is locally the pull-back of a relative form $\eta \in \Omega^1_{Y|S}$ where $Y \to S$ is of relative dimension 2*.*

The proof is esentially the same as the proof of the classical Kupka theorem, as in Theorem 2.4.6. One only needs to note that every ingredient there can be generalized to a relative setup.

For this we note that, as $p : X \to S$ is a smooth morphism, the relative tangent sheaf $T_S X$ is locally free and is the dual sheaf of the locally free sheaf $\Omega^1_{X|S}$. We note also that, if $v \in T_S X(U)$, and $\omega \in \Omega^1_{X|S}(U)$, the relative Lie derivative $L_v(\omega)$ is well defined by Cartan's formula

$$L^S_v = d_S \iota_v(\omega) + \iota_v(d_S \omega),$$

where $\iota_v(\omega) =< v, \omega >$ is the pairing of dual spaces (and by extension also the map $\Omega^q_{X|S} \to \Omega^{q-1}_{X|S}$ determined by v), and d_S is the *relative* de Rham differential. Also $\Omega^q_{X|S} = \wedge^q \Omega^1_{X|S}$.

Finally we observe that, if $p : X \to S$ is of relative dimension d and X is regular over $\mathbb{C}$ of total dimension n, a family of integrable Pfaff systems gives rise to a foliation on X whose leaves are tangent to the fibers of p. Indeed, we can pull-back the subsheaf $I(\mathcal{F}) \subset \Omega^1 X|S$ by the natural epimorphism

$$f^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X|S} \to 0,$$

and get $J = I(\mathcal{F}) + f^*\Omega^1_S \subset \Omega^1_X$, wich is an integrable Pfaff system in X, determining a foliation $\hat{\mathcal{F}}$. As $f^*\Omega^1_S \subset J$, the leaves of the foliation $\hat{\mathcal{F}}$ are contained in the fibers X_s of p.

In the general case, where p is smooth but S and X need not to be regular over $\mathbb{C}$, Frobenius theorem still gives foliations $\mathcal{F}_s$ in each fiber X_s. Indeed, as $p : X \to S$ is smooth, each fiber X_s is regular over $\mathbb{C}$ and, $\Omega^1_{\mathcal{F}}$ being flat, $I(\mathcal{F})_s \subset \Omega^1_{X_s}$ is an integrable Pfaff system on X_s.

Proposition 4.5.3. *Let $p : X \to S$ a smooth morphism over $\mathbb{C}$ and*

$$0 \to I(\mathcal{F}) \to \Omega^1 X|S \to \Omega^1_{\mathcal{F}} \to 0$$

a codimension 1 *flat family of Pfaff systems. Let $\omega \in \Omega^1_{X|S}(U)$ be an integrable* 1*-form such that $I(\mathcal{F})(U) = (\omega)$ in a neighborhood U of a point $x \in X$. Then $d\omega$ is locally decomposable.*

Proof. As $T_S X = (\Omega^1 X|S)^\vee$, and $\Omega^q_{X|S} = \wedge^q \Omega^1_{X|S}$ we can apply Plücker relations to determine if $d\omega$ is locally decomposable and proceed as in Proposition 2.4.1. □

Lemma 4.5.4. *Suppose that $d\omega_x \neq 0$. Consider $\mathcal{G}_s$ the codimension* 2 *foliations defined by $d\omega$ in X_s. In the neighborhood V of $x \in X$ where $\mathcal{G}_s$ is non-singular for every s we have the following. The leaves of $\mathcal{G}_s$ are integral manifolds of $\omega|_{X_s}$.*

Proof. We only have to prove that, for evey $v \in T_S X$ such that $\iota_v(d\omega) = 0$, then $\iota_v(\omega) = 0$. We can do this exactly as in Lemma 2.4.2. $\square$

Lemma 4.5.5. *With the same hipothesis as Lemma 4.5.4. Let v be a vector field tangent to $\mathcal{G}$. Then the* relative *Lie derivative of ω with respect to v is zero.*

Proof. Like the proof of Lemma 2.4.3. $\square$

Lemma 4.5.6. *Same hipothesis as Lemma 4.5.4 and 4.5.5, then* $\operatorname{sing}(\omega)$ *is saturated by leaves of $(\mathcal{G}_s)_{s \in S}$ (i.e.: take $y \in V$ a zero of ω such that $p(y) = s$, and L the leaf of $\mathcal{G}_s$ going through y. Then the inclusion $L \to V$ factorizes through* $\operatorname{sing}(\omega)$*).*

Proof. We can do this entirely on X_s. Then this reduces to Lemma 2.4.4. $\square$

Proof of Proposition 4.5.2. We can take an analytical neighborhood V of $x \in X$ such that $V \cong U \times D^d$ with $U \subseteq S$ an open set, D^d a complex polydisk, and

$$\begin{array}{rcl} p|V : V \cong U \times D^d & \longrightarrow & U. \\ (s, z_1, \ldots, z_d) & \mapsto & s \end{array}$$

Also, by Frobenius theorem, we can choose the coordinates z_i in such a way that

$$v_i = \frac{\partial}{\partial z_i} \in T_S X(V), \qquad 3 \leq i \leq d,$$

are tangent to $d\omega$. Then, as $L^S_{v_i}\omega = 0$ and $\iota_{v_i}\omega = 0$, we can write ω as

$$\omega = f_1(z, s)dz_1 + f_2(z, s)dz_2.$$

$\square$

Proposition 4.5.7. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0,$$

$s \in S$, and $x \in \Sigma$ be as in the above proposition. Then $\Sigma \to S$ is smooth around x.

Proof. By 4.5.2 above, we can determine Σ around x as the common zeroes of $f_1(z, s)$ and $f_2(z, s)$. The condition $\omega \wedge d\omega \neq 0$ implies Σ is *smooth* over S (remember that we are using the *relative* de Rham diferential and that means the variable s counts as a constant). $\square$

Proposition 4.5.8. *Let $0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$ and $s \in S$ be as above, and $x \in \Sigma_s$ be such that s is a* Reeb *singularity of $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$. Then $\Sigma \to S$ is étalé around x.*

Proof. The condition on x means we can actually give $I(\mathcal{F})$ locally by a relative 1-form $\omega \in \Omega^1_{X|S}$, $\omega = \sum_{i=1}^n f_i(z,s)dz_i$, with n the relative dimension of X over S and the df_i's linearlly independent on x. Then Σ is given by the equations $f_1 = \cdots = f_n = 0$ and is therefore étalé over S. □

With this two proposition we are almost in condition to state our condition for flatness of the dual family, we just need a general

Lemma 4.5.9. *Let $X \xrightarrow{p} S$ be a morphism between schemes of finite type over an algebraically closed field k. Let $U \subseteq X$ be the maximal open subscheme such that $U \xrightarrow{p} S$ is flat, and $s \in S$ a point such that X_s is without embedded components. . If $U_s \subseteq X_s$ is dense, then $U_s = X_s$.*

Proof. By Proposition 1.4.5 we must check that, for A either a discrete valuation domain or an Artin ring of the form $k[T]/(T^{n+1})$, and every arrow $\mathrm{Spec}(A) \to S$, the pull-back scheme $X_{\mathrm{Spec}}(A)$ is flat over $\mathrm{Spec}(A)$. In this way the problem reduces to the case where $S = \mathrm{Spec}(A)$.

(i) Case A DVD. In this case, A being a principal domain, flatness of X over $\mathrm{Spec}(A)$ is equivalent to the local rings $\mathcal{O}_{X,x}$ being torsion-free $A_{p(x)}$-modules for every point $x \in X$ ([GR03, IV.1.3] , so it suffices to consider the case $A_{p(x)} = A$.

Now, let $f \in \mathcal{O}_{X,x}$ and $J = \mathrm{Ann}_A(f) \subseteq A$. Suppose $J \neq (0)$ and consider $V(J) \subseteq \mathrm{Spec}(A)$, clearly $\mathrm{supp}(f) \subseteq p^{-1}(V(J)) \subseteq X$. So $U \cap \mathrm{supp}(f) = \varnothing$. But then the restriction $f|s$ of f to X_s have support disjoint with U_s. On the other hand

$$\mathrm{supp}(f|_s) = \overline{\{\mathfrak{P}_1\}} \cup \cdots \cup \overline{\{\mathfrak{P}_m\}} \subseteq X_s,$$

where $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_m\} = \mathrm{Ass}(\mathcal{O}_{X_s,x}/(f|_s)) \subseteq \mathrm{Ass}(\mathcal{O}_{X_s,x})$.
As X_s is without immersed components, the $\mathfrak{P}_i$'s are all minimal, so $X_k \cap \overline{\mathfrak{P}_i}$ is an irreducible component of X_k, but

$$U_s \cap \overline{\mathfrak{P}_i} = \varnothing.$$

Contradicting the hipotesis that U_s is dense in X_s.

(ii) Case $A = k[T]/(T^{n+1})$. Using Proposition 1.4.6 works just as the first case taking $f \in \mathcal{O}_{X,x}$ as a section such that $T^n f = 0$ but $f \notin T\mathcal{O}_{X,x}$. □

We have already said that, in a Pfaff system, Kupka singularities, if exists, form a codimension 2 sub-scheme of X. We will call $\mathcal{K}(\mathcal{F})$ this sub-scheme, and $\overline{\mathcal{K}}(\mathcal{F})$ its closure.

Theorem 4.5.10. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family, for $s \in S$ consider the Pfaff system $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$. If $\operatorname{sing}(\mathcal{F}_s)$ is without embedded components and $\operatorname{sing}(\mathcal{F}_s) = \overline{\mathcal{K}}(\mathcal{F}_s) \cup \{p_1, \dots, p_m\}$ where the p_i's are Reeb-type singularities, then $\operatorname{sing}(\mathcal{F}) \to S$ is flat in a neighborhood of $s \in S$.

Proof. Indeed, by Proposition 4.5.7, $\operatorname{sing}(\mathcal{F})$ is flat in a neighborhood of $\mathcal{K}(\mathcal{F}_s)$, and, as $\operatorname{sing}(\mathcal{F}_s)$ is without embedded components, we can apply Lemma 4.5.9 to conclude that $\operatorname{sing}(\mathcal{F})$ is flat in a neighborhood of $\overline{\mathcal{K}(\mathcal{F}_s)}$.

Lastly, from Proposition 4.5.8, follows that $\operatorname{sing}(\mathcal{F})$ is flat in a neighborhood of $\{p_1, \dots, p_m\}$. $\square$

4.6. Applications

Let $X = \mathbb{P}^n(\mathbb{C})$. It's well known that the class of sheaves $\mathscr{F}$ that splits as a direct sum of line bundles $\mathscr{F} \cong \bigoplus_i \mathcal{O}(k_i)$ have no non-trivial deformations. Indeed, as deformation theory teach us, first order deformations of $\mathscr{F}$ are parametrized by $\operatorname{Ext}^1(\mathscr{F}, \mathscr{F})$, in this case we have

$$\operatorname{Ext}^1(\mathscr{F}, \mathscr{F}) \cong \bigoplus_{i,j} \operatorname{Ext}^1(\mathcal{O}(k_i), \mathcal{O}(k_j)) \cong$$
$$\cong \bigoplus_{i,j} \operatorname{Ext}^1(\mathcal{O}, \mathcal{O}(k_j - k_i)) \cong \bigoplus_{i,j} H^1(\mathbb{P}^n, \mathcal{O}(k_j - k_i)) = 0$$

In particular, given a flat family of ditributions

$$0 \to T\mathcal{F} \to T_S(\mathbb{P}^n \times S) \to N_{\mathcal{F}} \to 0,$$

such that, for some $s \in S$, $T\mathcal{F}_s \cong \bigoplus_i \mathcal{O}(k_i)$, then the same decomposition holds true for the rest of the members of the family.

When we deal with codimension 1 foliations it's more common, however, to work with Pfaff systems or, more concretely, with integrable twisted 1-forms $\omega \in \Omega^1_{\mathbb{P}^n}(d)$, $\omega \wedge d\omega = 0$ (see [ALN07]). It's then than the following question emerged: Given a form $\omega \in \Omega^1_{\mathbb{P}^n}(d)$ such that the vector fields that anihilate ω generate a split sheaf (i.e.: a sheaf that decomposes as direct sum of line bundles), will the same feature hold for every deformation of ω? Such question was adressed by Cukierman and Pereira in [FCJVP08]. Here we use our results to recover the theorem of Cukierman-Pereira as a special case.

4.6.1. Codimension 1 Foliations with split tangent sheaf on $\mathbb{P}^n(\mathbb{C})$

As was observed before, every time we have a codimension 1 Pfaff system

$$0 \to I(\mathcal{F}) \to \Omega^1_X \to \Omega^1_{\mathcal{F}} \to 0$$

such that $\Omega^1_{\mathcal{F}}$ is torsion-free, then $I(\mathcal{F})$ must be a line bundle. In the case $X = \mathbb{P}^n(\mathbb{C})$, then $I(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ for some $d \in \mathbb{Z}$. Is then equivalent to give a Pfaff system and to give a morphism $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \Omega^1_{\mathbb{P}^n}$ wich is in turn equivalent to $0 \to \mathcal{O}_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}(d)$ that is, to give a global section ω of the sheaf $\Omega^1_{\mathbb{P}^n}(d)$.

We can explicitly write such a global section as

$$\omega = \sum_{i=0}^{n} f_i(x_0, \ldots, x_n)dx_i$$

with f_i a homogeneous polynomial of degree $d-1$ and such that $\sum_i x_i f_i = 0$.

Such an expression gives rise to a foliation with split tangent sheaf if and only if there are $n-1$ polynomial vector fields

$$\begin{aligned} X_1 &= g_1^0 \frac{\partial}{\partial x_0} + \cdots + g_1^n \frac{\partial}{\partial x_n}, \\ &\vdots \\ X_{n-1} &= g_{n-1}^0 \frac{\partial}{\partial x_0} + \cdots + g_{n-1}^n \frac{\partial}{\partial x_n}; \end{aligned}$$

such that $\omega(X_i) = 0$, for all $1 \leq i \leq n-1$, moreover on a generic point the vector fields must be linearly independents.

The singular set of this foliation is given by the ideal $I = (f_0, \ldots, f_n)$. The condition $\omega(X_i) = 0$ means that the ring $\mathbb{C}[x_0, \ldots, x_n]/I$ admits a syzygy of the form

$$0 \to \mathbb{C}[x_0, ..., x_n]^{n-1} \xrightarrow{\begin{pmatrix} g_1^0 & \cdots & g_1^n \\ \vdots & \ddots & \vdots \\ g_{n-1}^0 & \cdots & g_{n-1}^n \end{pmatrix}} \mathbb{C}[x_0, \ldots, x_n]^{n+1} \xrightarrow{(f_0, \ldots, f_n)}$$
$$\xrightarrow{(f_0, \ldots, f_n)} \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}[x_0, \ldots, x_n]/I \to 0.$$

For such rings a theorem of Hilbert and Schaps tells us the following.

Theorem 4.6.1 (Hilbert, Schaps). *Let $A = k[x_0, ...x_n]/I$ be such that there is a 3-step resolution of A as above by free modules. Then the ring A is Cohen-Macaulay, in particular, is equidimensional.*

Proof. This is theorem 5.1 in [MA76] □

We thus recover the theorem of Cukierman and Pereira ([FCJVP08, Theorem 1]).

Theorem 4.6.2 ([FCJVP08]). *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{\mathbb{P}^n \times S|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of codimension 1 *integrable Pfaff systems. And suppose* $0 \to I(\mathcal{F})_s \to \Omega^1_{\mathbb{P}^n} \to \Omega^1_{\mathcal{F}} \to 0$ *define a foliation with split tangent sheaf. If* $\operatorname{sing}(\mathcal{F}_s) \setminus \overline{\mathcal{K}(\mathcal{F}_s)}$ *have codimension greater than* 2*, then every member of the family defines a split tangent sheaf foliation.*

Proof. By the above theorem $\operatorname{sing}(\mathcal{F}_s)$ is equidimensional. The singular locus of a foliation on $\mathbb{P}^n$ always have an irreducible component of codimension 2 (see [ALN07, Teorema 1.13]), if $\operatorname{sing}(\mathcal{F}_s) \setminus \overline{\mathcal{K}(\mathcal{F}_s)}$ have codimension greater than 2, then it must be empty. So $\operatorname{sing}(\mathcal{F}_s) = \overline{\mathcal{K}(\mathcal{F}_s)}$ and we can then apply Theorem 4.5.10. So the flat family

$$0 \to I(\mathcal{F}) \to \Omega^1_{\mathbb{P}^n \times S|S} \to \Omega^1_{\mathcal{F}} \to 0$$

gives rise to a flat family

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0,$$

so $T\mathcal{F}$ must be flat over S, and then $T\mathcal{F}_s$ splits, for every $s \in S$. $\square$

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