Teorías de campos conformes no racionales formuladas en variedades de topología no trivial

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RESUMEN

En esta tesis estudiamos una familia de teorías conformes no-racionales en dos dimensiones formuladas sobre superficies de Riemann de topología no-trivial. Más precisamente, nos dedicamos en superficies con bordes a y superficies cerradas con número de género mayor que uno. Este conjunto de teorías corresponden a una generalización de otras teorías de campos conformes importantes, tales como la teoría de campos de Liouville y el modelo de Wess-Zumino-Novikov-Witten, los cuales tienen importantes aplicaciones en materia condensada y teorías de cuerdas. Calculamos valores de expectación de campos primarios en el disco y funciones de correlación en el toro, relacionando estas cantidades con observables de la teoría de Liouville. El desarrollo de estos cálculos se realizó empleando tanto el formalismo de integral funcional como el formalismo del gas de Coulomb, siempre chequeando la consistencia de ambos métodos. También discutimos dos aplicaciones para nuestros resultados: su uso en el estudio de D-branas en teoría de cuerdas en dos y tres dimensiones, y su uso como herramienta para realizar cálculos en teorías superconformes en cuatro dimensiones con supersimetría $\mathcal{N} = 2$. Los resultados de esta tesis están basados en trabajos [1, 2] del autor.

Palabras claves: teorías de campos conformes, teoría de campos de Liouville, topología.

Non-rational conformal field theories formulated in varieties of non-trivial topology

ABSTRACT

In this thesis we study a family of non-rational conformal field theories in two dimensions formulated on Riemann surfaces of non-trivial topology. More precisely, we study surfaces with boundaries and closed manifolds with genus number grater than zero. This set of theories generalizes important conformal theories, such as Liouville Field Theory and the Wess-Zumino-Novikov-Witten model, which have important applications in condensed matter and string theory. We compute expectation values for primary operators in the disk and correlation functions in the torus, relating these quantities with observables of the Liouville theory. The computations are done both in the path integral formalism and in the Coulomb Gas formalism, then checking their consistency. We also discuss two applications for ours results: their use in the study of D-branes in string theory in two and three dimensions, and their use as a tool to perform computations in four dimensional super-conformal field theories with $\mathcal{N} = 2$ supersymmetry. The results of this thesis are based on works [1, 2] of the author.

Keywords: conformal field theory, Liouville field theory, topology.

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Contents

I

1 Introduction

Besides being a profound topic in mathematical physics that permits to explore fundamental properties of constructive quantum field theory in general, two-dimensional conformal field theory is also one of the most useful tools in high-energy physics applications and it also has applications to condensed matter physics. Regarding high-energy physics, two-dimensional conformal field theory enters in the world-sheet formulation of string theory, in realizations of AdS/CFT correspondence, in the description of four-dimensional $\mathcal{N} = 2$ gauge theories, and in many other scenarios. In this thesis we will be concerned with a special family of conformal theories and we will comment on its applications.

Among the conformal models one can define there is one of a very special type: the non-rational conformal field theories. These are the theories that contain a continuous part in the spectrum of primary states, and are usually associated to geometric realizations that involve non-compact spaces. These models are substantially more complex than the rational ones, and are those that are important for most of the high-energy physics applications. A good example is the standard world-sheet (super)string theory formulation, which corresponds to a two-dimensional field theory on $D = 26$ (resp. $D = 10$) non-compact directions. The models we will study in this thesis are non-rational ones.

Of particular importance in string theory applications is the study of the two-dimensional non-rational conformal field theories on Riemann surfaces with boundaries and with handles. While the former allows for the description of D-brane states of the theory, the latter are well-known to represent next-to-leading contributions in the string coupling g_s expansion. In this thesis we will study theories formulated in Riemann surfaces both with boundaries and with handles. More precisely, in this thesis we will study a new class of non-rational conformal field theories that was originally conjectured to exist by Ribault [3]. In 2008, Ribault proposed a form for the n-point correlation functions of a new type of theories in the case of the simplest manifold, i.e. in the sphere. The way Ribault did so was by writing the correlation functions for these new theories in terms of correlation functions of the Liouville theory, which is the best understood non-rational model. Here, we continue the task initiated by Ribault's work. We consider the theories on manifolds with non-trivial topology, considering the inclusion of both boundaries and handles. Our main result is that of showing that there is a consistent way of defining the same kind of theories that Ribault proposed for the sphere but both for the disk and for the torus geometries. For these two Riemann surfaces we compute correlation functions explicitly both in the path integral approach and in the free fields approach. We find exact agreement between the two formalisms and we show that the results obtained are in accordance with the conjectured conformal structure. Our results generalize both the Liouville and the Wess-Zumino-Witten correlation functions, which are actually special cases of the formulas we obtain. We also discuss extensions of Ribault's construction to the case in which Liouville theory is replaced by higher-rank Toda conformal field theory. The attempts to generalize the so-called WZW-Liouville correspondence to the case of higher-rank Wess-Zumino-Witten model in the path integral approach are commented. We also comment on possible applications to string theory and gauge theory, specially in the description of non-fundamental surface operators in the so-called $\mathcal{N} = 2^*$ super Yang-Mills theory.

1.1 Outline

In chapter two we will introduce the CFTs in two dimensions. We will define the concepts of two dimensional local conformal symmetry, Virasoro algebra, central charge, primary field and radial cuantization. We will present the correlation functions of these theories and we will expose what is known about their functional form. Some examples will be given: the free boson, the free boson with a background charge and the beta-gamma system. There will also be given some examples of non-rational CFTs: the Liouville field theory and the Wess-Zumino-Witten[1] (WZW) model. Finally, we present the relation found by Ribault and Teschner [15] between $sl(2)$ level $k = b^{-2} + 2$ WZW model correlation functions and Liouville field theory correlation functions with central charge $c = 1 + 6(b + b^{-1})^2$, and we will present the new family of non-rational CFTs presented by Ribault in [3]; the formers are the main ingredient in our discussion.

In chapter three, the shorter of the thesis, we will briefly review the formulation of the new family of non-rational CFTs on the sphere, and we will comment the special cases corresponding to the parameter $m = 0, 1, b^2$.

In chapter four we will perform computation with the new family of non-rational CFT in the disk. We will present the boundary conditions, the symmetries and the relation with the Liouville field theory in the disk. We will calculate in particular the one-point function in the disk. The particular cases with $m = 0, 1, b^2$ will be commented. But first we will give a brief introduction to general aspects of CFT in the disk geometry.

In chapter five we will perform calculations in the torus. As for the case of the disk, we will present the boundary conditions, the symmetries, and the relation with the Liouville field theory in the torus. We will make comments about general correlation functions and we will compute in particular the one-point function in the torus. The particular cases with $m = 0, 1, b^2$ will be commented. First, we will give a brief introduction to CFT in the torus.

In chapter six we postulate a new family of non-rational CFTs that is itself a generalization of the family of theories presented by Ribault in [3]. We show that these new theories are related to the $sl(n)$ conformal Toda field theory in a similar way than Ribault's theories are related to Liouville field theory. Finally we show our attempt to achieve a lagrangian for $sl(3)$ WZW

[1]This model is often referred to as Wess-Zumino-Novikov-Witten model (WZNW) because of Novikov's contribution in 1984.

model by extending one of these theories and relating its correlation functions to correlation functions of $sl(3)$ Toda field theory.

In chapter seven we will discuss some applications to these new non-rational CFTs and the relations between their correlation functions and Liouville field theory correlation functions. And finally, in chapter eight we give the conclusions of the thesis.

II

2 Conformal Field Theories

In this chapter we introduce conformal field theories (CFTs) in two dimensions, emphasizing the details of the non-rational models we will be involved with. We start with basics aspects on CFTs; we define the concepts of two-dimensional local conformal symmetry, the Virasoro algebra, the central charge, the primary field, the radial quantization. We present the correlation functions of these theories and we summarize what is known about their functional form. Some illustrative examples are be discussed: the free boson, the free boson with a background charge, and the beta-gamma system. We also describe prototypical examples of non-rational CFTs: the Liouville field theory and the WZW model. Finally, we present the relation found by Ribault and Teschner [15] between $sl(2)$ *WZW model correlation functions and Liouville field theory correlation functions and we will present the new family of non-rational CFTs presented by Ribault in [3], which are the central subject in our study.*

2.1 Introduction to 2D Conformal Field Theories

CFTs in 2-dimensions are remarkable because the structure of every n-point correlation function, is almost completely given by only requiring conformal invariance.

In the next section we will give the main characteristics that define a conformal theory. For a better understanding of these theories it is necessary to start their study on an arbitrary dimension d. In the following section we will specialize in $d = 2$.

2.1.1 Conformal field theories in d-dimensions

In this section we will consider a d dimensional space-time with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ and signature (p, q). The conformal group is defined as a subgroup of the coordinate transformations that leaves the metric tensor invariant up to a scale [4]

$$g_{\mu\nu}(x) \;\rightarrow\; g'_{\mu\nu}(x') \;=\; \Omega(x) g_{\mu\nu}(x). \tag{1}$$

Every conformal transformation is locally equivalent to a rotation and a dilatation, that also leave the metric invariant up to a scale. Under a conformal transformation every angle between two crossing curves will not be affected. The Poincaré group is a subgroup of the conformal transformations corresponding to $\Omega(x) = 1$.

It is trivial to see that in 1-dimensional space every coordinate transformation is conformal since the metric is just a number and there are no angles. The 2-dimensional case is special and will be considered in the next section.

With the intention of realizing whether a theory is conformally invariant or not let us study what conformal transformations do to the action. Under a infinitesimal general transformation of coordinates $x^\mu \rightarrow x'^\mu + \epsilon^\mu(x)$ the change in the action is

$$\delta S \;=\; \int d^d x\, T^{\mu\nu}\, \partial_\mu \epsilon_\nu \;=\; \frac{1}{2} \int d^d x\, T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{2}$$

where we supposed than the energy-momentum tensor $T^{\mu\nu}$ is symmetric. The transformation also induce a variation of the metric tensor which is

$$g_{\mu\nu} \;\rightarrow\; g_{\mu\nu} \;-\; (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{3}$$

If we consider a conformal transformation, the change of variables must satisfy the requirement (1) and that implies

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \;=\; f(x)\, g_{\mu\nu} \tag{4}$$

where the factor $f(x)$ is determined by taking the trace on both sides of last equation

$$f(x) \;=\; \frac{2}{d}\, \partial_d \epsilon^d. \tag{5}$$

Then, the change in the action due to a conformal transformation of variables is

$$\delta S \;=\; \frac{1}{d}\int d^dx\, T^{\mu}_{\;\;\mu}\,\partial_\rho \epsilon^\rho \tag{6}$$

Then a theory with a traceless energy-momentum tensor is conformally invariant at the classical level. Most of the theories with scale invariance have already traceless energy-momentum tensors or have tensors that can be made traceless. Even though there is no general proof, this property holds in $d = 2$ theories and it is accepted that every such theory with scale invariance is conformally invariant at the classical level.

The finite coordinate transformation in $d > 2$ that satisfy (1) are

$$\begin{aligned}
x \to x'^\mu &= x^\mu + a^\mu \\
x \to x'^\mu &= \Lambda^\mu_\nu\, x^\nu \qquad\qquad (\Lambda^\mu_\nu \in SO(d+1,1)) \\
x \to x'^\mu &= \lambda\, x^\mu \\
x \to x'^\mu &= \frac{x^\mu - \; b^\mu\, x^2}{1 - 2b\, x + b^2\, x^2}.
\end{aligned} \tag{7}$$

The first and second correspond to Poincaré group transformations. The third is a dilatation and the fourth are the named special conformal transformations. All these transformations form the conformal group which is isomorphic to the group $SO(d+1,1)$.

There are certain functions that are invariant under transformations (7) which are important in the construction of n-point correlation functions. Invariance under translations and rotations enforce these functions to depend on the relative distance between pairs of different points $|x_i - x_j|$. Scale invariance allows these functions to be only quotients between these distances $\frac{|x_i - x_j|}{|x_k - x_l|}$. And invariance under special conformal transformations imposes these functions to depend on the following expressions [5]

$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \qquad\qquad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|} \tag{8}$$

called anharmonic ratios or cross-ratios.

Conformal invariance at the quantum level not only implies invariance of the action but also invariance of the measure of path integrals. That means that the expectation value of the trace of the energy-momentum tensor must be zero.

In the following we show how conformal invariance is imposed on n-point correlation functions.

Under a conformal transformation $x \to x'$ a field $\phi(x)$ of zero spin transform like

$$\phi(x) \;\to\; \phi'(x') \;=\; \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x) \tag{9}$$

where Δ is the conformal dimension of ϕ and $\left|\frac{\partial x'}{\partial x}\right|$ is the jacobian corresponding to the coordinates transformation. Fields transforming this way are named cuasi-primaries.

Correlation functions of a theory that is covariant under the transformation (9) must satisfy

$$\langle \phi_1(x_1)...\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} ... \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d} \langle \phi_1(x_1')...\phi_n(x_n')\rangle. \tag{10}$$

The expectation value $\langle \mathcal{O}\rangle$ is defined as $\langle 0|\mathcal{O}|0\rangle$ where $|0\rangle$ is the vacuum expectation value. If there exist a lagrangian for the theory, then $\langle \mathcal{O}\rangle = \int \mathcal{D}\phi e^{-S}\mathcal{O}$, where S is the euclidean action. The vacuum $|0\rangle$ must be invariant under the conformal group.

Due to (10) a 2-point correlation function invariant under translations, rotations and dilatations must be

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}} \tag{11}$$

if $\Delta_1 = \Delta_2 = \Delta$, or must be 0 if $\Delta_1 \neq \Delta_2$. C_{12} is a constant that depends on the normalization of the fields.

The 3-point functions are as follows

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{12}$$

But $n \geq 4$-functions are not completely determined by conformal invariance. We know them up to a factor depending on the cross ratios (8). For example 4-point functions are

$$\langle \phi_1(x_1)...\phi_4(x_4)\rangle = f\left(\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \;,\; \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}\right) \prod_{i<j}^{4} |x_i - x_j|^{\Delta/3-\Delta_i-\Delta_j} \tag{13}$$

where $\Delta = \sum_{i=1}^{4} \Delta_i$ and f is an arbitrary function of cross ratios.

2.1.2 Conformal symmetry in 2-dimensions

Conformal invariance in $d = 2$ is more interesting since it implies more and stronger restrictions to correlation functions. In the following we introduce the main points.

Considering coordinates z^0 and z^1 on the plane, the variation of the metric tensor under the coordinate transformation $z^\mu \to w^\mu(x)$ is

$$g^{\mu\nu} \;\to\; \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta}. \tag{14}$$

The condition (1) for the transformation to be conformal implies $g'^{\mu\nu}(w) \propto g^{\mu\nu}(z)$ so

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 \;=\; \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2, \qquad \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} \;=\; 0. \tag{15}$$

And these conditions are equivalent to the Cauchy-Riemann equations for holomorphic and antiholomorphic functions

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}, \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}, \qquad \text{or} \qquad \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1}. \tag{16}$$

It is clear then that in $d = 2$ the group of conformal transformations is infinite and is equal to the group of analytic functions [4, 5]. It is worthy the use of complex coordinates $z = z^0 + iz^1$ and $\bar{z} = z^0 - iz^1$ and express the transformations as

$$z \to w(z) \qquad \textit{and} \qquad \bar{z} \to \bar{w}(\bar{z}). \tag{17}$$

Every analytic map on the complex plane is known to be conformal and to preserve the angles. The conformal algebra in $d = 2$ is then infinite dimensional as can be seen from an infinitesimal transformation $z' \to z + \epsilon(z)$ with $\epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}$ on a spinless and dimensionless field

$$\begin{aligned} \phi'(z', \bar{z}') \;=\; \phi(z, \bar{z}) \;&= \phi(z', \bar{z}') \;-\; \epsilon(z')\,\partial'\phi(z', \bar{z}') \;-\; \bar{\epsilon}(\bar{z}')\,\bar{\partial}'\phi(z', \bar{z}') \;= \\ &= \sum_n \left[c_c\,\ell_n\,\phi(z', \bar{z}') \;+\; \bar{c}_n\,\bar{\ell}_n\,\phi(z', \bar{z}')\right]. \end{aligned} \tag{18}$$

The coefficients c_n are then the coefficients of the Laurent expansion around $z = 0$ of the function $\epsilon(z)$ and we have introduced the generators

$$\ell_n \;=\; -z^{n+1}\partial_z \qquad \bar{\ell}_n \;=\; -\bar{z}^{n+1}\partial_{\bar{z}} \qquad (n \in \mathbb{Z}). \tag{19}$$

It is easy to prove that these generators satisfy the following commutation relations

$$[\ell_m, \ell_n] \;=\; (m-n)\ell_{m+n}, \qquad [\bar{\ell}_m, \bar{\ell}_n] \;=\; (m-n)\bar{\ell}_{m+n}, \qquad [\ell_m, \bar{\ell}_n] \;=\; 0. \tag{20}$$

The holomorphic and antiholomorphic parts commute, therefore the algebra is a direct sum of two isomorphic subalgebras. The algebra (20) is named de-Witt algebra. From now on we will ignore the antiholomorphic part for simplicity.

There are only three generators (19) that are globally defined on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$. This set of conformal transformations correspond to the special conformal group with associated subalgebra given by $\ell_{-1}, \ell_0, \ell_1$. It can be seen that ℓ_{-1} generate the translations, ℓ_1 generate the special conformal transformations and $\ell_0 + \bar{\ell}_0$ generate the dilatations while $\ell_0 - \bar{\ell}_0$ generate the rotations on the real plane. The eigenvalue of the operator ℓ_0 ($\bar{\ell}_0$) is called holomorphic (antiholomorphic) conformal dimension h ($\bar{h}$). The conformal dimension (or weight) Δ and the spin s are respectively the eigenvalues $\Delta = h + \bar{h}$ and $s = h - \bar{h}$.

The finite global conformal transformations can be written as follows

$$f(z) \;=\; \frac{az+b}{cz+d} \qquad \text{with} \quad ad - bc \;=\; 1 \tag{21}$$

where a, b, c, d are complex numbers. These are the three global holomorphic and the three global antiholomorphic conformal transformations and form the group $SL(2,\mathbb{C})/\mathbb{Z}_2 \approx SO(3,1)$ in correspondence with the conformal transformations in $d > 2$ explained in the previous section, where there are only global conformal transformations.

As explained in previous section scale invariance in $d = 2$ is equivalent to conformal invariance at the classical level. But for a theory to be conformally invariant at the quantum level the integration measure of path integrals must be also invariant. Even more, if the theory is to be used in string theory, conformal invariance must be preserved integrating over every two-dimensional manifold.

Considering conformally invariant theory at the quantum level we analyze the restrictions to correlation functions. Under a conformal map $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$ there are certain fields that transform like

$$\phi(z, \bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \phi(w(z), \bar{w}(\bar{z})). \tag{22}$$

This property under conformal transformations defines the primary fields ϕ of conformal dimension $(h, \bar{h})$. Fields not transforming this way are named secondaries. A primary field is always a quasi-primary field because it satisfies (9) under global conformal transformations. A secondary field can be or not a quasi-primary field.

Conformal invariance forces a n-point functions of n primary fields to transforms as follows

$$\langle \phi_1(w_1, \bar{w}_1) \ldots \phi_n(w_n, \bar{w}_n) \rangle \;=\; \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i} \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle . \tag{23}$$

This relation fixes the form of 2 and 3-point functions. In contrast to previous section, primary fields can be spinful. The spin value is incorporated in the difference $h_i - \bar{h}_i$. Therefore 2-point functions are

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle \;=\; \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \tag{24}$$

if $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$. In other case it is zero. The sum of the spins within the correlation function must be zero. 3-point functions are

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3) \rangle \;=\; C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \tag{25}$$

where $z_{ij} = z_i - z_j$. Constants C_{12} and C_{123} can also be determined due to conformal invariance.

4-point functions are not completely fixed. In $d = 2$ there are only three independent cross-ratios invariant under global conformal transformations. They can be written as

$$\eta \;=\; \frac{z_{12} z_{34}}{z_{13} z_{24}}, \qquad 1 - \eta \;=\; \frac{z_{14} z_{23}}{z_{13} z_{24}}, \qquad \frac{\eta}{1 - \eta} \;=\; \frac{z_{12} z_{34}}{z_{14} z_{23}}. \tag{26}$$

Therefore, the functional form of a 4-point function is

$$\langle \phi_1(x_1) \dots \phi(x_4) \rangle = f(\eta, \bar{\eta}) \prod_{i<j}^{4} z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j} \tag{27}$$

where $h = \sum_{i=1}^{4} h_i$ and $\bar{h} = \sum_{i=1}^{4} \bar{h}_i$.

2.1.3 Radial quantization

Conformal invariance allows us to parameterize the theory in different ways. We can consider it either on a plane or on a cylinder with coordinates $\sigma \in [0, 2\pi]$ and $\tau \in [-\infty, +\infty]$. The first one is a Lorentzian variety $\mathbb{R}^2$ and the second one is an Euclidean variety $\mathbb{R} \times U(1)$.

To consider the theory defined on a cylinder presents many advantages. One of them is the possibility of develop the named radial quantization. We will describe it now.

The first step is to perform a Wick rotation $\sigma^{\pm} = \tau \pm \sigma \rightarrow -i(\tau \pm i\sigma)$ where τ and σ are two space-time coordinates. Next step is to define complex coordinates on the cylinder

$$\begin{aligned} z' &= \tau - i\sigma \\ \bar{z}' &= \tau + i\sigma. \end{aligned} \tag{28}$$

Via this modification a left-moving or right-moving field in two-dimension Minkowsky space becomes a field depending on holomorphic or anti-holomorphic coordinates in two-dimension Euclidean space. The last step is to map the cylinder into the complex plane with the following conformal transformation

$$\begin{aligned} z &= e^{z'} = e^{\tau - i\sigma} \\ \bar{z} &= e^{\bar{z}'} = e^{\tau + i\sigma}. \end{aligned} \tag{29}$$

The infinite past and future in the cylinder ($\sigma = \mp\infty$) correspond to the points $|z| = 0, \infty$ in the plane. Equal time lines in the cylinder ($\sigma = cte$) correspond to circles with center at the origin in the plane, and the time inversion in the cylinder ($\sigma \rightarrow -\sigma$) correspond to the map $z \rightarrow 1/\bar{z}$ in the complex plane.

It is important to realize that dilatations in the complex plane correspond to temporal translations in the cylinder. Consequently the generator of dilatations in the plane can be thought as the hamiltonian of the system and the Hilbert space is built form concentric circles. This method of defining a quantum theory in the plane is then called radial quantification.

2.1.4 The stress-tensor and Virasoro algebra

Noether's theorem establishes that local transformations of coordinates are generated by charges built from the energy-momentum tensor $T_{\mu\nu}$, and in $d = 2$ every such transformation is a conformal transformation. In this case the energy-momentum tensor is not only symmetric but also traceless and it results in a two component tensor that can be written as follows

$$T(z) \equiv T_{zz}(z) \qquad T(\bar{z}) \equiv \overline{T}_{\bar{z}\bar{z}}(\bar{z}) \tag{30}$$

where the holomorphic and anti-holomorphic part are separated. The operator $T(z)$ is related to the trace of the energy-momentum tensor. The following expansion of the energy-momentum tensor

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad \overline{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n \tag{31}$$

is important since the modes L_n do generate the local conformal transformations at the quantum level in equivalence to what the generators (19) do at the classical level. They satisfy the famous Virasoro algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ \left[L_n, \bar{L}_m\right] &= 0 \\ \left[\bar{L}_n, \bar{L}_m\right] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \end{aligned} \tag{32}$$

where the parameter c is the central charge. The integrability, and consequently the condition of resolubility of the theory, is satisfied due to the existence of an infinite group of generators of the symmetry.

The correlation functions can have singularities when the positions of two or more fields coincide. The operator product expansion (OPE) is the representation of a product of two operators inserted in points z and w given by a finite sum of terms, each being a single operator well defined as $z \to w$, multiplied by a function of $z - w$ [4, 5].

The OPE between the energy-momentum tensor and a primary field ϕ of conformal dimension h is

$$T(z)\phi(w,\bar{w}) \;=\; \frac{h}{(z-w)^2}\phi(w,\bar{w}) \,+\, \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \,+\, \ldots \tag{33}$$

and similarly for the anti-holomorphic part with h substituted by $\bar{h}$ and w and $\bar{w}$ interchanged. The points $\ldots$ express that there regular terms ignored in the right side of the equation.

One last advantage of radial quantization to be mentioned is the relation that it establishes between commutators and OPEs. Let us consider two operators A and B that are integrals over space at fixed time of the fields $a(z)$ and $b(z)$ respectively

$$A \;=\; \oint a(z)dz \;, \qquad B \;=\oint b(z)dz \;, \tag{34}$$

where the contours of integration are circles centered around the origin. If we perform an equal time commutator between A and B we can express it in terms of both integrations. This calculation imposes the circles to be alternative one infinitesimally bigger than the other. Operating with these contours we end up with the following

$$[A,B] \;=\; \oint_0 dw \oint_w dz\, a(z)b(w). \tag{35}$$

This way, only the term in $1/(z-w)$ of the OPE between $a(z)$ and $b(z)$ contributes to the commutator, by the theorem of residues. Therefore, OPEs establishes equal time commutators.

An example of the above are the modes of the energy-momentum tensor

$$L_n \;=\; \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z). \tag{36}$$

Via the OPE between the energy-momentum tensor and itself we can deduce the Virasoro algebra (33).

2.1.5 The central charge

There are fields not satisfying the transformation law (22) under conformal transformations. An example is a derivative of a primary field which transform in a more cumbersome way. They are called secondary fields.

Another example is the energy-momentum tensor. Its OPE with itself is

$$T(z)T(w) \;=\; \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \ldots \tag{37}$$

where the factor present in the pole of order 4 is half the central charge c of the theory. This parameter cannot be determined by symmetry considerations. It is imposed by the behavior at short distances [4] and it represent somehow an extensive measure of the number of degrees of freedom of the system. It is also related with the *soft* breaking of conformal symmetry under the introduction of a macroscopic scale via a local conformal transformation (it is trivial to see that restricting to just the global conformal group corresponding to $n = -1, 0, 1$ in (33) the central charge does not appear). Therefore c express the Casimir effect when there are scales. We will mention now two examples of how this interpretation is made manifest:

The first one is the map between the theory in the plane and the theory in a cylinder of length L. This is achieved by the transformation

$$z \longrightarrow w \;=\; \frac{L}{2\pi}\log z. \tag{38}$$

The energy-momentum tensor in the cylinder T_{cil} is related to the energy-momentum tensor in the plane T_{pl} as follows

$$T_{cil}(w) \;=\; \left(\frac{2\pi}{L}\right)^2\left[T_{pl}(z)z^2 - \frac{c}{24}\right]. \tag{39}$$

Therefore if the energy density in vacuum $\langle T_{pl}\rangle$ is zero in the plane (as is supposed because conformal invariance makes the trace of the energy-momentum tensor to be zero at the quantum level), its value in the cylinder is

$$\langle T_{cil}\rangle \;=\; -\frac{c\pi^2}{6L^2}. \tag{40}$$

The Casimir energy is due to the scale imposed by the parameter L and is zero as $L \to \infty$.

The second example is the formulation of a conformal theory over a two-dimensional variety. The curvature introduces a macroscopic scale into the system and makes the vacuum expectation value of the trace of the energy-momentum tensor to be non-zero

$$\langle T^{\mu}_{\mu} \rangle = \frac{c\mathcal{R}(x)}{24\pi} \tag{41}$$

but proportional to $\mathcal{R}$ which is the Ricci scalar of curvature.

2.2 Examples of Conformal Field Theories

2.2.1 The free boson

The first and simplest example of CFT is the free boson, also named gaussian model. The action of this theory is

$$S = \frac{1}{2\pi} \int d^2z \, \partial\phi \, \bar{\partial}\phi \tag{42}$$

where we used the integration weight $2idz \wedge d\bar{z}$. The propagator of the field ϕ can be separated into an holomorphic and an antiholomorphic part.

$$\langle \phi(z,\bar{z})\phi(w,\bar{w}) \rangle = -\log(z-w) - \log(\bar{z}-\bar{w}). \tag{43}$$

It is important to realize that the field ϕ is not a primary field, in fact it is not even a conformal field because its propagator does not follow a potential law. But the fields $\partial\phi$ and $\bar{\partial}\phi$ are primary fields and the equations of motion say that they are holomorphic and antiholomorphic respectively. Their OPEs are

$$\begin{aligned}
\partial\phi(z)\partial\phi(w) &= -\frac{1}{(z-w)^2} + \ldots \\
\bar{\partial}\phi(\bar{z})\bar{\partial}\phi(\bar{w}) &= -\frac{1}{(\bar{z}-\bar{w})^2} + \ldots \\
\partial\phi(z)\bar{\partial}\phi(\bar{w}) &= 0
\end{aligned} \tag{44}$$

The energy-momentum tensor of the holomorphic part of the quantum theory is the normally ordered operator $T(z) =: \partial\phi(z)\partial\phi(w) :$ and its OPE with the primary field $\partial\phi(w)$ is

$$T(z)\partial\phi(w) \;=\; \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)} + \ldots \tag{45}$$

so we deduce from (33) that the field $\partial\phi(z)$ is a primary field of conformal weight $(1,0)$. Repeating these calculations for the antiholomorphic part we deduce that the field $\bar{\partial}\phi$ is a primary field of conformal weight $(0,1)$.

The Wick theorem allows us to calculate the OPE between the energy-momentum tensor and itself and it ends up as follows

$$T(z)T(w) \;=\; \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \ldots \tag{46}$$

From the first term on the right hand side we see that the energy-momentum tensor is not a primary field and that the central charge of the theory is $c = 1$. The same can be said about the antiholomorphic part of the theory.

There exists an infinite set of primary fields in this theory. They are the vertex operators

$$V_\alpha(z,\bar{z}) \;= : e^{i\alpha\phi(z,\bar{z})} : \tag{47}$$

Their OPE with the energy-momentum tensor is

$$T(z)V_\alpha(w,\bar{w}) \;=\; \frac{\alpha^2 V_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial V_\alpha(w,\bar{w})}{z-w} + \ldots \tag{48}$$

so their conformal weight is (α^2, α^2). The OPEs between vertex operators are

$$\begin{aligned} V_\alpha(z,\bar{z})\, V_\beta(w,\bar{w}) \;&=\; |z-w|^{2\alpha\beta} V_{\alpha+\beta} + \ldots \\ V_\alpha(z,\bar{z})\, V_{-\alpha}(w,\bar{w}) \;&=\; |z-w|^{-2\alpha^2} + \ldots \end{aligned} \tag{49}$$

The normal ordering of the vertex operators imposes that the two-point correlation function $\langle V_\alpha(z,\bar{z}) V_\beta(w,\bar{w})\rangle$ is not zero only if $\beta = -\alpha$. In general, the correlation function of a string of vertex operators V_{α_i} vanishes unless the sum of all the α_i is equal to zero. This can be understood as a charge conservation in the correlation functions. The theory has $U(1)$ symmetry corresponding to the invariance upon the translation $\phi \to \phi + a$. The field ϕ is a phase.

The n-point correlation function of vertex operators is

$$\langle V_{\alpha_1}(z_1, \bar{z}_1 V_{\alpha_2}(z_2, \bar{z}_2)...V_{\alpha_n}(z_n, \bar{z}_n)\rangle \;=\; \prod_{i<j} |z_i - z_j|^{4\alpha_i\alpha_j} \tag{50}$$

provided the condition

$$\sum_{i=1}^{n} \alpha_i \;=\; 0 \tag{51}$$

is satisfied.

2.2.2 Free boson with a background charge

An interesting deformation of the free boson is achieved by coupling it to the scalar curvature $\mathcal{R}$ of the manifold on which the theory is defined. This is expressed in the action in the following way

$$S \;=\; \frac{1}{2\pi}\int d^2z\,\sqrt{g}\,\left(\partial\phi\bar{\partial}\phi \,+\, 2\gamma\phi\mathcal{R}\right) \tag{52}$$

where γ is a constant. The new term breaks the $U(1)$ symmetry of the free boson and the variation of the action upon the translation $\phi \to \phi + a$ is

$$\delta S \;=\; \frac{\gamma a}{\pi}\int d^2z\sqrt{g}\mathcal{R}. \tag{53}$$

The above expression is a topological invariant and its value is given by the Gauss-Bonnet theorem

$$\int d^2z\sqrt{g}\mathcal{R} \;=\; 8\pi(1 - \textit{genus number}) \tag{54}$$

where the genus number of the manifold is considered. This modification does not change the form of the correlation functions of n vertex operators (50) but the condition (51) changes to

$$\sum_{i=1}^{n} \alpha_i \;=\; 2\alpha_0. \tag{55}$$

where $\gamma = \sqrt{2}\alpha_0$. When the complex plane is used for formulating the theory on the sphere the scalar curvature of the sphere is considered zero at any point but the infinite. Therefore the coupling to the scalar curvature is associated to the presence of a background charge $-2\alpha_0$ placed at the infinite.

This change gives us a hole new theory with energy-momentum tensor

$$T(z) \;=: \partial\phi\partial\phi : +i\sqrt{2}\alpha_0\partial^2\phi \tag{56}$$

and central charge

$$c \;=\; 1 - 24\alpha_0^2 \tag{57}$$

The field $\partial\phi$ is no longer a primary field while the field V_α still is, and its conformal weight is now

$$h_\alpha = \alpha^2 - 2\alpha_0\alpha. \tag{58}$$

This dimension is invariant under the transformation $\alpha \to 2\alpha_0 - \alpha$, so the vertex operators V_α and $V_{2\alpha_0-\alpha}$ share the same dimension. This means that physical operators correlations can be represented in different ways. For example the 2-point function $\langle V_\alpha V_\alpha\rangle$ is equivalent to $\langle V_\alpha V_{2\alpha_0-\alpha}\rangle$ but only the second one satisfies the condition (55). Even more, every correlation function can be calculated by inserting *screening operators*. These operators are nonlocal operators with conformal weight zero but nonzero charge. There are two of them

$$Q_\pm \;=\; \oint dz\, Z_\pm(z) \;=\; \oint dz\, e^{i\sqrt{2}\alpha_\pm\phi(z)} \tag{59}$$

where $\alpha_\pm \,=\, \alpha_0 \pm \sqrt{\alpha_0^2+1}$ are the charges. These operators allows us to calculate correlation functions of vertex operators that in principle do not satisfy condition (55). It is possible whenever there are two integers r and s, representing the number of screenings operators of each class to be introduced into the correlation function as follows

$$\langle V_1(z_1)\, V_2(z_2)...V_n(z_n)\, Q_-^r\, Q_+^s\rangle \tag{60}$$

that satisfy the condition

$$\sum_{i=1}^{n}\alpha_i + r\alpha_- + s\alpha_+ \;=\; 2\alpha_0. \tag{61}$$

This formalism allows us to calculate correlation functions of several theories as we will see all throughout our work. It is based in the calculation of the correlation functions of vertex operators using the expression (50) which reminds the exponential of the electric potential energy between n point charges of strength $2\alpha_i$ in two dimensions. Hence the formalism is named Coulomb gas formalism.

2.2.3 Beta-gamma system

The beta-gamma system is a CFT built with two commutative fields β and γ with the following action

$$S \ = \ \frac{1}{2\pi}\int d^2z\,\beta\bar{\partial}\gamma. \tag{62}$$

The equations of motion are

$$\bar{\partial}\gamma(z) \ = \ 0\ , \qquad \bar{\partial}\beta(z) \ = \ 0. \tag{63}$$

Both fields are holomorphic henceforth. The energy-momentum tensor is

$$T \ = :(\partial\beta)\,\gamma: \, -\lambda\,\partial\,(:\beta\gamma:) \tag{64}$$

where λ is a parameter or the theory. With respect to this tensor the fields γ and β are primary fields and their conformal weights are

$$h_\gamma \ = \ 1-\lambda\ , \qquad h_\beta \ = \ \lambda. \tag{65}$$

This theory is defined in one chirality, it is all holomorphic as $\overline{T}=0$. A hole antiholomorphic theory can be defined in the same way.

The OPEs between the fields are

$$\beta(z_1)\gamma(z_2) \ \sim \ \frac{1}{z_1-z_2}\ , \qquad \gamma(z_1)\beta(z_2) \ \sim \ -\frac{1}{z_1-z_2} \tag{66}$$

and using the Wick theorem to calculate the OPE between two energy-momentum tensors we arrive to the central charge of the theory

$$c \ = \ 3(2\lambda-1)^2-1. \tag{67}$$

All throughout our work we will use $\lambda = 2$.

2.2.4 Liouville field theory: the simplest non-rational CFT

The Liouville field theory is a non-rational CFT. This means that it has a continuum set of primary fields and therefore a continuum spectrum. It is probably the simplest of this type of theories, so it can be considered a kind of prototype for the development of techniques of calculations for other non-rational CFTs.

The action of Liouville field theory is

$$S_L \;=\; \frac{1}{2\pi}\int d^2z\left((\partial\varphi)^2 \,-\, \frac{Q_L}{4}\mathcal{R}\varphi \,+\, \mu e^{2b\varphi}\right). \tag{68}$$

It is the action of a boson with a background charge and exponential interaction term. It depends on three parameters, μ, Q_L and b. The first one is a real positive parameter named cosmological constant. The second one is the background charge. The third one is determined next.

The propagator of the free Liouville field is

$$\langle\varphi(z)\varphi(w)\rangle \;=\; -\frac{1}{2}\log(z-w). \tag{69}$$

The energy-momentum tensor is obtained by varying the action of the free theory without the interaction term with respect to the metric and it ends up to be

$$T_L \;=\; -\partial\varphi\partial\varphi \,+\, Q_L\partial^2\varphi \tag{70}$$

and the central charge is $c_L = 1+6Q_L^2$ corresponding to the boson with the background charge. The same happens with the vertex operators $V_\alpha(z)$ (47), where α is a complex number. The subset $\alpha \in \frac{Q}{2}+i\mathbb{R}$ is special because the fields $V_\alpha(z)$ acting on the vacuum $|0\rangle$ to build normalized states of delta functions [6]. These operators are

$$V_\alpha(z) \;= : e^{2\alpha\varphi(z)} : \tag{71}$$

with conformal dimension

$$h_\alpha \;=\; \alpha(Q_L-\alpha). \tag{72}$$

The last expression is invariant under the reflection $\alpha \to Q_L - \alpha$ and this suggest that both states created by these vertex operators are equal up to a reflection coefficient.

For the theory to be conformally invariant the background charge must take the value

$$Q_L = \left(b + b^{-1}\right) \tag{73}$$

in order to fix the conformal weight of the interaction term $h_b = 1$ and make it marginal. Its OPE with the energy-momentum tensor inside a correlation function is then a total derivative. As the energy-momentum tensor is the generator of conformal transformations this means that the interaction term does not break conformal invariance.

Liouville field theory is also invariant under the parameter change $b \to \frac{1}{b}$, even though this is not evident at the lagrangian level. Every correlation is not altered by this variation.

Every theory is characterized by its n-point functions. In Liouville field theory these functions are

$$\Omega_L = \langle V_{\alpha_n}(z_n) \dots V_{\alpha_1}(z_1) \rangle \equiv \int \mathcal{D}\varphi e^{-S_L} \prod_{i=1}^{n} V_{\alpha_i}(z_i). \tag{74}$$

Dorn y Otto [7] used the named Coulomb gas formalism in order to express the path integral computation of (74) as a free theory perturbed by additional operators $\mu e^{\sqrt{2}b\varphi}$. Liouville correlation functions are then [8]

$$\begin{aligned} \Omega_L &= (\sqrt{2}b)^{-1} \mu^s \Gamma(-s) \delta \left(s + b^{-1} \sum_{i=1}^{N} \alpha_i - b^{-1} Q_L \right) \times \\ &\times \int \prod_{r=1}^{s} d^2 w_r \left\langle \prod_{i=1}^{n} e^{2\alpha_i \varphi(z_i)} \prod_{r=1}^{s} e^{2b\varphi(w_r)} \right\rangle \end{aligned} \tag{75}$$

where the factor $\Gamma(-s)$ comes from the field φ zero mode integration [8, 9]. The condition for non-zero correlation functions is then obtained and s refers to the number of screening operators necessary to satisfy the charge conservation, considering the background charge. This condition is

$$\sum_{\mu=1}^{N} \alpha_\mu + sb = Q_L. \tag{76}$$

The inclusion of these screening operators $S_+ = \int d^z e^{2b\varphi(w_r)}$ comes from the expansion in Taylor series of the exponential of the interaction term. From this infinite expansion survives only the term with s screening operators. This method is the same as the one used by Dotsenko and Fateev for the minimal models in [10].

It is important to realize than the function (74) is defined for the interacting theory while the function (75) is defined for the free theory, corresponding to $\mu = 0$.

One last characteristic of Liouville field theory to be mentioned is the existence of degenerate fields among the vertex operators defined in (71). Any such operator corresponds to the parameter α taking one of the following values:

$$\alpha \;=\; -\frac{pb}{2} - \frac{q}{2b} \qquad\qquad \text{with} \;\; p, q \;=\; 0, 1, 2, \ldots \tag{77}$$

The degeneracy of these operators is made manifest with respect to the conformal symmetry algebra. This means that from these fields and under the operation of the generators of the algebra (33), we can get null fields. Proceeding this way inside any correlation function with a degenerate vertex operator we get a linear differential equation. Each degenerate field has its own linear differential equation associated. This represents a practical tool for solving correlation functions.

2.2.5 The Wess-Zumino-Novikov-Witten model

This model constitute a recipe for the construction of a CFT with ie-algebraic symmetry. It is built from a semi-simple group G of dimension D and natural metric

$$ds^2 \;=\; \frac{1}{2} Tr\left[(g^{-1}dg)^2\right] \tag{78}$$

where g represents the elements of the group. The trace $Tr[...]$ is taken with respect the representation indexes.

The Wess-Zumino-Novikov-Witten model[2] (WZNW) action is

$$
\begin{aligned}
S_{WZNW} &= k\int_{\Sigma} d\sigma d\tau \delta^{\alpha\beta} Tr\left[(g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)\right] + \\
&+\frac{2}{3}k\int_V d^3x \varepsilon^{ijk} Tr\left[(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)\right]
\end{aligned}
\tag{79}
$$

where V is 3-dimensional variety bounded by the 2-dimensional variety Σ, i.e. $\partial V \equiv \Sigma$. Whenever the coordinates σ and τ for the cylinder can be mapped into the sphere through a stereographic projection, then V corresponds to B^3.

The first term in (79) does not guarantee conformal invariance at the quantum level. The second term, named Wess-Zumino term, is added in order to achieve this invariance [4].

Working on the complex plane with coordinates z and $\bar{z}$ the theory is a CFT with the additional symmetry

$$
g(z,\bar{z}) \rightarrow \Omega(z) g(z,\bar{z}) \bar{\Omega}^{-1}(\bar{z}) \tag{80}
$$

where Ω is an arbitrary element of the group G. This symmetry is wider than the isomorphic group. The associated currents to these symmetry transformations are

$$
J = J_a T^a = -\frac{k}{2}(\partial g) g^{-1} \tag{81}
$$

$$
\bar{J} = \bar{J}_a T^a = -\frac{k}{2} g^{-1}(\bar{\partial} g) \tag{82}
$$

where the operators T^a generate the Lie algebra of the group G and they obey the following relations

$$
\left[T^a, T^b\right] = f^{ab}_c T^c \tag{83}
$$

with the structure constants of the group f^{ab}_c.

Due to conformal invariance it is possible to write the operator J as a Laurent series

$$
J^a(z) = \sum_{n=-\infty}^{\infty} J^a_n z^{-n-1} \tag{84}
$$

[2] Often referred to as WZW, without mentioning Novikov's name; cf. chapter 2.

and the same happens with $\bar{J}$ in terms of $\bar{z}$. Multiplying per z^n and integrating all around $z = 0$ the Fourier modes of the currents are obtained and they are

$$J_n^a = \frac{1}{2\pi i} \oint J^a(z)\, z^n dz. \tag{85}$$

From the commutation relations of these modes, the k-level Kac-Moody algebra $\hat{g}(k)$ arises. This is

$$[J_n^a, J_m^b] = i f^{abc} J_{n+m}^c + n \frac{k}{2} g^{ab} \delta_{n+m,0} \tag{86}$$

where g^{ab} is the Cartan-Killing metric defined by

$$f_c^{ad} f_d^{cb} = Q\, g^{ab} \tag{87}$$

where Q is a number related to the Casimir C of the group through the equality

$$C = \frac{2}{Q} g_{ab} J^a J^b. \tag{88}$$

The original group algebra is found in the zero modes $n = m = 0$ of (86). The same happens with the Casimir operator since the zero mode of (88) is the usual quadratic Casimir.

As we said, the algebra (86) is wider than conformal algebra, and the last can be obtained by defining the Virasoro generators as

$$L_n = \frac{g_{ab}}{(Q+k)} \sum_m : J_{-m}^a J_{n+m}^b : \tag{89}$$

satisfying the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{kD}{12(k+Q)} \delta_{n+m,0} (m^3 - m) \tag{90}$$

with central charge

$$c = \frac{kD}{k+Q} \tag{91}$$

where D represents the number of generators of the algebra associated to the group G.

The Virasoro algebra generators can be identified with the Fourier zero modes of the energy-momentum tensor of the conformal theory. This method is called Sugawara construction:

$$T(z) = \frac{g_{ab}}{(Q+k)} : J^a(z) J^b(z) : . \tag{92}$$

The commutation relations between Kac-Moody and Virasoro generators are

$$[L_n, J^a_m] \;=\; -mJ^a_{m+n}. \tag{93}$$

The main point of Sugawara construction is that it gives a systematic method for constructing the algebra from where the conformal symmetry of the WZNW model arises. All we need is to specify the group which we want to work with.

From now on we will study the particular group $G = SL(2,\mathbb{R})$. This model is important for string theory because it describes the propagation of a string on AdS_3 space-time.

The $SL(2,\mathbb{R})$ Lie group is semi-simple, non-compact and composed of 2-dimensional square matrices with real coefficients and determinant equal to 1. The parameters Q and c takes the values $Q = -2$ y $c = \frac{3k}{k-2}$.

It is useful to define the complex basis of currents as

$$J^{\pm} \;=\; J^1 \pm J^2. \tag{94}$$

The algebra is then written as

$$\begin{aligned} \left[J^3, J^-\right] &= -J^- \\ \left[J^3, J^+\right] &= J^+ \\ \left[J^-, J^+\right] &= 2J^3 \end{aligned} \tag{95}$$

and the Casimir operator is given by

$$\hat{C} \;=\; J^3J^3 - \frac{1}{2}\left(J^-J^+ + J^+J^-\right). \tag{96}$$

A generic element g of the group can be parameterized as follows

$$g \;=\; l\begin{pmatrix} e^{it}\cosh\rho & e^{-it}\sinh\rho \\ e^{it}\sinh\rho & e^{-it}\cosh\rho \end{pmatrix} \tag{97}$$

and, in terms of these parameters the natural metric (78) is

$$ds^2 \;=\; -\cosh\rho dt^2 \,+\, d\rho^2 \,+\, \sinh\rho d\theta^2. \tag{98}$$

The last metric corresponds to the lorentzian anti-de Sitter 3-dimensional space-time. The WZNW model built from this algebra will describe the propagation of strings on AdS_3. It is easy to relate it to the WZNW model describing the propagation of strings on the euclidean version of $SL(2,\mathbb{R})$, the coset $H_3^+ = SL(2,\mathbb{C})/SU(2)$. They are both related by a Wick rotation and they are somehow similar. The WZNW model on H_3^+ is named the H_3^+ model.

The variety H_3^+ is built from the group $G = SL(2,\mathbb{C})$ modulo $H = SU(2)$. A generic element of this coset can be parameterized as

$$g = \begin{pmatrix} e^{\phi} & \gamma e^{\phi} \\ \overline{\gamma} e^{\phi} & \gamma\overline{\gamma} e^{\phi} + e^{-\phi} \end{pmatrix} \tag{99}$$

and the corresponding natural metric

$$ds^2 = d\phi^2 + e^{2\phi} d\gamma d\overline{\gamma} \tag{100}$$

is equivalent to the euclidean version of the metric of AdS_3 space-time. Its boundary corresponds to the region $\phi = \infty$. Therefore, the action (79) is

$$S = k \int d^2 z \left(\partial\phi\bar{\partial}\phi + \bar{\partial}\gamma\partial\bar{\gamma} e^{2\phi} \right) . \tag{101}$$

There exist another action for this theory which is more convenient. It is the Wakimoto free-field representation [11] and correspond to a representation of the $SL(2,\mathbb{R})$ current algebra in terms of three free fields in two dimensions. They are a free boson and a free $\beta\gamma$ system. The new action is

$$S = \frac{k}{8\pi} \int d^2 z \left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta} e^{-2\phi} \right) \tag{102}$$

where the factor k is due to the quantum effects ($k \sim \frac{-1}{\alpha' \Lambda}$) and curvature effects ($G_{\mu\nu} \sim k$).

Since there is no kinetic term for the fields β and $\bar{\beta}$, it is easy to reobtain the original action (101) from (102) by integrating out these fields. However, whether the integration measure is invariant under the affine symmetry or not, is a sensible question. In order to this not to be a problem a regularization factor must be introduced [12, 13]. Moreover, is convenient to redefine the field ϕ by the following transformation

$$\phi \rightarrow \sqrt{\frac{2}{k-2}} \phi \tag{103}$$

so as to rewrite the action as

$$S[\lambda] \;=\; \frac{1}{4\pi}\int d^2z \left(\partial\phi\bar{\partial}\phi \,+\, \beta\bar{\partial}\gamma \,+\, \bar{\beta}\partial\bar{\gamma} \,-\, \frac{b}{2\sqrt{2}}\mathcal{R}\phi \,+\, 4\pi\lambda\beta\bar{\beta}e^{\sqrt{2}b\phi}\right) \tag{104}$$

where the term $\frac{b}{2\sqrt{2}}\mathcal{R}\phi$ appears because of the quantum effects and acts as a background charge. The parameter b is related to the WZNW model level through $b^{-2} = k-2$. The coupling constant λ can be fixed by shifting the zero mode of the field ϕ. And finally, it is easy to prove that the interaction term $\beta\bar{\beta}e^{\sqrt{2}b\phi}$ is marginal with respect to the energy-momentum tensor because it has conformal weight $(h,\bar{h}) = (1,1)$.

The Kac-Moody algebra (86) generating the affine symmetry of this theory is

$$\begin{aligned}
\left[J^3_m, J^3_n\right] &= -\frac{k}{2}\delta_{n+m,0} \\
\left[J^3_m, J^\pm_n\right] &= \pm J^\pm_{m+n} \\
\left[J^+_m, J^-_n\right] &= -2J^3_{m+n} + km\delta_{n+m,0}
\end{aligned} \tag{105}$$

and can be written in terms of the free fields ϕ, β and γ by using the Wakimoto representation [11] as follows

$$\begin{aligned}
J^-(z) &= \beta(z) \\
J^3(z) &= \beta(z)\gamma(z) \,+\, \frac{\alpha_+}{2}\partial\phi(z) \\
J^+(z) &= \beta(z)\gamma^2(z) \,+\, \alpha_+\gamma(z)\partial\phi(z) \,+\, k\partial\gamma(z).
\end{aligned} \tag{106}$$

The OPEs between these fields are

$$\begin{aligned}
\phi(z)\phi(w) &\sim -2\log(z-w) \\
\gamma(z)\beta(w) &\sim \frac{1}{(z-w)}.
\end{aligned} \tag{107}$$

By means of the Wick theorem it is possible to calculate the OPEs between the currents as

$$\begin{aligned}
J^+(z)J^-(w) &= \frac{k}{(z-w)^2} \,-\, \frac{2}{(z-w)}J^3(w) \,+\, \ldots \qquad (108) \\
J^3(z)J^\pm(w) &= \pm\frac{1}{(z-w)}J^\pm(w) \,+\, \ldots \\
J^3(z)J^3(w) &= \frac{-k/2}{(z-w)^2} \,+\, \ldots
\end{aligned}$$

and they agree with the commutative relations (86).

Using the Sugawara construction the energy-momentum tensor is

$$T(z) \;=\; \beta(z)\partial\gamma(z) \;-\; \frac{1}{2}(\partial\phi(z))^2 - \frac{1}{\alpha_+}\partial^2\phi(z). \tag{109}$$

The central charge is

$$c(H_3^+) \;=\; \frac{3k}{k-2} \;=\; 3+6b^2 \;=\; 2+(1+6b^2). \tag{110}$$

It is related to the central charge of Liouville theory coupled to a free boson.

The vertex operators of the theory can be written in terms of different bases. We are interested in the named μ-basis in which the vertex operators are

$$\Phi_j(\mu|z) \;=\; |\mu|^{2j+2} e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} e^{\sqrt{2}b(j+1)\phi(z,\bar{z})}. \tag{111}$$

These operators are primary fields and their conformal weight is

$$h_j \;=\; -\frac{j(j+1)}{k-2}. \tag{112}$$

The OPEs between the currents and these operators are

$$J^a(w)\Phi_j(\mu|z) \;\sim\; \frac{1}{w-z}D^a\Phi_j(\mu|z) \tag{113}$$

where the operators D^a are defined as follows

$$D^- \;=\; \mu \quad , \quad D^0 \;=\; -\mu\partial_\mu \quad , \quad D^+ \;=\; \mu\partial_\mu^2 - \frac{j(j+1)}{\mu}. \tag{114}$$

and constitute themselves a representation of the algebra. Under their action, the vertex operators in the μ-basis are also a representation of the algebra too.

2.3 Relation between H_3^+ and Liouville field theory

Based on a work of Stoyanosky [14], Ribault and Teschner [15] found an identity in the sphere relating N-point function in Liouville field theory with $(2N-2)$-point function in H_3^+ with

$N-2$ first level degenerate operators. Before introducing it we recall both actions and vertex operators:

Liouville field theory:

The action is

$$S_L = \frac{1}{2\pi}\int d^2z\left((\partial\varphi)^2 + \frac{Q_L}{4}\mathcal{R}\varphi + \mu e^{2b\varphi}\right) \tag{115}$$

where $Q_L = (b + b^{-1})$.

The vertex operators are

$$V^L_\alpha(z) = e^{2\alpha\varphi(z)} \tag{116}$$

with conformal dimension $h_L = \alpha(Q_L - \alpha)$. In particular there are degenerate fields corresponding to $\alpha = -\frac{pb}{2} - \frac{q}{2b}$ with p and q positive integers.

H_3^+ WZW model:

Its action can be written as

$$S_{H_3^+} = \frac{1}{2\pi}\int d^2z\left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{Q_{H_3^+}}{4}\mathcal{R}\phi - b^2\beta\bar{\beta}e^{2b\phi}\right) \tag{117}$$

where $Q_{H_3^+} = b$.

The affine primary operators with spin j and isospin μ are

$$\Phi^{H_3^+}_j(\mu|z) = |\mu|^{2(j+1)}e^{\mu\alpha-\bar{\mu}\bar{\alpha}}e^{2b(j+1)\phi} \tag{118}$$

with conformal dimension $h^{H_3^+}_j = -b^2 j(j+1)$

Now we introduce the relation found in [15]

$$\left\langle\prod_{i=1}^N V^{H_3^+}_{j_i}(\mu_i|z_i)\right\rangle_{H_3^+} = \frac{\pi}{2}(-\pi)^{-n}b\,\delta^{(2)}\left(\sum_{i=1}^N\mu_i\right)|u|^2|\Theta_n|^{\frac{1}{4b^2}}\left\langle\prod_{i=1}^N V^L_{\alpha_i}(z_i)\prod_{t=1}^{N-2}V^L_{-\frac{1}{2b}}(y_t)\right\rangle_L \tag{119}$$

where

$$|\Theta_N(z_1,\cdots,z_N|y_1,\cdots,y_{N-2})| = \frac{\prod_{r<s\le N}^N(z_r-z_s)^2\prod_{t<l\le N-2}^N(y_t-y_l)^2}{\prod_{r=1}^N\prod_{t=1}^{N-2}(z_r-y_t)^2} \tag{120}$$

and

$$\alpha_i \;=\; b(j_i+1)+\frac{1}{2b}. \qquad (121)$$

The parameters u and y_t are related to the the parameters μ_i via the Sklyanin's change of variables

$$\sum_{i=1}^{N}\frac{\mu_i}{t-z_i} \;=\; u\frac{\prod_{j=1}^{N-2}(t-y_j)}{\prod_{i=1}^{N}(t-z_i)}, \qquad (122)$$

$$u \;=\; \sum\nolimits_{i=1}^{N}\mu_i z_i. \qquad (123)$$

This relation can be obtained both by comparing the Belavin-Polyakov-Zamolodchikov equations in Liouville field theory with the Knizhnik-Zamolodchikov equations in H_3^+ [15], and by path integration [16]. Even more, this relation can be generalized for varieties of genus number greater than zero [16] and varieties with boundaries [17].

2.4 A new family of non-rational Conformal Field Theories

Having introduced all the ingredients necessary to introduce the CFT we will be concerned with, let us write down its action as given by [3]

$$S_{(m,b)} \;=\; \frac{1}{2\pi}\int_{\Gamma} d^2z\, g^{1/2}\left(\partial\phi\bar{\partial}\phi+\beta\bar{\partial}\gamma+\bar{\beta}\partial\bar{\gamma}+\frac{Q_{(m,b)}}{4}\mathcal{R}\phi+b^2(-\beta\bar{\beta})^m\, e^{2b\phi}\right). \qquad (124)$$

Let us call $\mathcal{T}_{(m,b)}$ the theory defined by the action (124). Then, we notice that the case $\mathcal{T}_{(0,b)}$ corresponds to Liouville field theory coupled to a free beta-gamma system. On the other hand, the case $\mathcal{T}_{\left(1,1/\sqrt{k-2}\right)}$ corresponds to the H_3^+=SL$(2,\mathbb{C})$/SU(2) WZNW theory with level $k=b^{-2}+2$ written in the Wakimoto free-field representation [11]. Besides, $\mathcal{T}_{\left(k-2,\sqrt{k-2}\right)}$ also yields the H_3^+ WZNW theory, with level $k=b^{+2}+2$; see [18]. In fact, the action above can be regarded as a generalization of these well-known non-rational CFTs.

The free stress-tensor associated to (124) is given by

$$T_{(m,b)}(z) \;=\; -\beta(z)\partial\gamma(z)-(\partial\phi(z))^2+Q_{(m,b)}\,\partial^2\phi(z) \qquad (125)$$

and by its anti-holomorphic counterpart $\overline{T}_{(m,b)}(\bar{z})$. This yields the central charge of the theory

$$c_{(m,b)} \;=\; 1+6Q^2_{(m,b)}. \qquad (126)$$

The conformal dimension of the field $e^{2\alpha\phi}$ with respect to (125) is $(h_\alpha, \bar{h}_\alpha) = (\alpha(Q_{(m,b)} - \alpha), \alpha(Q_{(m,b)} - \alpha))$, while the conformal dimensions of β and γ are $(1,0)$ and $(0,0)$, respectively. Therefore, the last term in the (124) is marginal with respect to the stress-tensor (125) if $h_b = \bar{h}_b = b\,(Q_{(m,b)} - b) = 1 - m$, yielding the relation

$$Q_{(m,b)} \; = \; b + \frac{1-m}{b}. \tag{127}$$

The theory with $m = 1$ (and the theory with $m = b^2$) corresponds to the SL$(2,\mathbb{C})$/SU(2) WZNW model, which exhibits $\widehat{sl}(2)_k \times \widehat{sl}(2)_k$ affine Kac-Moody symmetry. This symmetry is generated by the Kac-Moody current algebra

$$J^-(z) \;=\; \beta(z) \tag{128}$$

$$J^3(z) \;=\; \beta(z)\gamma(z) \,+\, b^{-1}\partial\phi(z) \tag{129}$$

$$J^+(z) \;=\; \beta(z)\gamma^2(z) \,+\, 2b^{-1}\gamma(z)\partial\phi(z) \,-\, \left(b^{-2}+2\right)\partial\gamma(z) \tag{130}$$

together with the anti-holomorphic counterparts $\bar{J}^{3,\pm}(\bar{z})$, where $b^{-2} = k - 2$. In contrast, the theory (124) for generic m exhibits only the algebra generated only by the two currents $J^3(z)$ and $J^-(z)$. This algebra is the named Borel subalgebra of $sl(2)$ and its currents are

$$J^-(z) \;=\; \beta(z), \qquad\qquad J^3(z) \;=\; \beta(z)\gamma(z) \,+\, \frac{m}{b}\partial\phi(z) \tag{131}$$

and by the pair of anti-holomorphic analogues. Currents (131) obey the following OPE

$$J^-(z)J^3(w) \;\simeq\; \frac{J^-(w)}{(z-w)} + ... \qquad J^3(z)J^3(w) \;\simeq\; -\frac{(1+m^2b^{-2}/2)}{(z-w)^2} + ... \qquad J^-(z)J^-(w) \;\simeq\; ... \tag{132}$$

where the ellipses stand for regular terms that are omitted; and this realizes the Lie brackets for the modes $J^{3,-}_n = \frac{1}{2\pi i}\int dz J^{3,-}(z) z^{-n-1}$. The spectrum of the theory would be constructed in terms of primary states with respect to these currents. Vertex operators creating such states are of the form

$$\Phi_j(\mu|z) \;=\; |\rho(z)|^{2h_j}\, |\mu|^{2m(j+1)}\, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\, e^{2b(j+1)\phi(z,\bar{z})} \tag{133}$$

whose holomorphic and anti-holomorphic conformal dimensions are given by

$$h_j \;=\; \bar{h}_j \;= (-b^2 j + 1 - m)(j+1), \tag{134}$$

and where μ is a complex variable. The spectrum of normalizable states of the theory, which is ultimately expressed by the values that j takes, is to be determined. The dependence on $\rho(z)$ in (133) was introduced because throughout this thesis we will work in the conformal gauge where

$$ds^2 = |\rho(z)|^2 dz d\bar{z}, \tag{135}$$

and such dependence is the one required for $\Phi_j(\mu|z)$ to transform as a primary $(h_j, \overline{h}_j)$-dimension operator under conformal transformations. For short, below we will sometimes take $\rho(z) = 1$.

III

3 The theory on the sphere topology

In this chapter we will briefly study the new family of non-rational CFTs on the sphere and we will comment the special cases corresponding to $m = 0, 1, b^2$.

3.1 Conformal field theory on the sphere

A CFT on the sphere topology is performed working over all the complex plane. We can connect every point of the plane with a point of the sphere via the stereographic map. Associating the origin and the infinite with opposite points of the sphere. This way we can make use of the radial quantization.

3.2 Genus-zero correlation functions from Liouville theory

We will begin with the theory (124) on sphere so we will calculate the genus-zero correlation functions. As shown in [3] they are defined in terms of Liouville correlation functions. This is a generalization of the Ribault-Teschner relation [15] commented in chapter 2. Namely

$$\begin{aligned} \left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{\mathcal{T}_{(m,b)}} &= \delta^{(2)}\left(\sum_{\nu=1}^{N} \mu_\nu\right) |u|^{2m(1+b^{-2}(1-m))} \left|\Theta_N^{g=0}(y_i, z_\nu)\right|^{\frac{m^2}{4b^2}} \times \\ &\times \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V_{-\frac{m}{2b}}(y_i) \right\rangle_{\mathrm{L}} \end{aligned} \tag{136}$$

where the Liouville correlation functions is

$$\left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V_{-\frac{m}{2b}}(y_i) \right\rangle_{\mathrm{L}} = \int \mathcal{D}\varphi \, e^{-S_{\mathrm{L}}[\varphi]} \prod_{\nu=1}^{N} e^{2\alpha_\nu \varphi(z_\nu)} \prod_{i=1}^{N-2} e^{-\frac{m}{b}\varphi(y_i)}$$

with the following values for the parameters α_ν

$$\alpha_\nu \;=\; b(j_\nu + 1 + b^{-2}/2).$$

As shown in [3] correlation functions of $\mathcal{T}_{(m,b)}$ theories are defined in terms of Liouville correlation functions. This is a generalization of the Ribault-Teschner relation [15] between H_3^+=SL(2, $\mathbb{C}$)/SU(2) WZNW theory and Liouville field theory. It is important to notice that the function

$$\left|\Theta_N^{g=0}(y_i, z_\nu)\right| \;=\; \prod_{\mu<\nu}^{N} |z_\mu - z_\nu|^2 \prod_{i<j}^{N} |y_i - y_j|^2 \prod_{\mu=1}^{N} \prod_{i=1}^{N-2} |z_\mu - y_i|^{-2}. \tag{137}$$

can be represented as a correlation function of a free boson $X(z)$ with background charge $\widehat{Q} = im/b$ as presented in chapter 2. That is,

$$\left|\Theta_N^{g=0}(y_i, z_\nu)\right|^{\frac{m^2}{4b^2}} \;=\; \left\langle \prod_{\nu=1}^{N} e^{i\frac{m}{b}X(z_\nu)} \prod_{i=1}^{N-2} e^{-i\frac{m}{b}X(y_i)} \right\rangle_X .$$

Then, it is correct to say that correlation functions of the theory defined by action (124) are actually given by correlation functions of a theory composed by Liouville theory times a CFT with central charge $c = 1 - 6m^2 b^{-2}$.

3.3 Particular cases

In the following we mention the three particular cases corresponding to $m = 0, 1, b^{-2}$, where the Ribault-Teschner relation becomes trivial or connects H_3^+ WZNW model with Liouville field theory.

3.3.1 Case with $m = 0$

For $m = 0$ the interaction term in the action (124) disappears and the theory turns into two disconnected theories. A Liouville field theory with central charge $c = 1+6Q^2$, where $Q = b+b^{-1}$ and a beta-gamma system. The last one just establishes the condition $\sum_{\nu=1}^{N} \mu_\nu$ on the right side of (136). As $m = 0$ the rest of the right side is the N-points correlation function of Liouville field theory with no insertion of degenerated vertex operators.

3.3.2 Case with $m = 1$

For $m = 1$ the relation (136) is the original relation found by Ribault and Teschner in [15]. The action (124) corresponds to the H_3^+ WZW model with central charge $c = 3k/(k-2)$, where $k = b^{-2} + 2$. It establishes that N-point correlation functions of this model on the sphere are related to $(2N-2)$-point correlation functions of Liouville field theory with $(N-2)$ degenerated vertex operators also on the sphere. These degenerated vertex operators correspond to $\alpha = -\frac{m}{2b}$.

3.3.3 Case with $m = b^2$

For $m = b^2$ the action (124) also corresponds H_3^+ WZW model dual to the case $m = 1$ as can be seen in [18]. The central charge is the same $c = 3k/(k-2)$ but the parameter k changes to $k = b^{+2} + 2$. It is trivial to realize that the interaction has conformal weight $h = 1$ and is marginal. The relation (136) is the same Ribault-Teschner relation between N-point correlation functions of this dual model on the sphere and $(2N-2)$-point correlation functions of Liouville field theory with $N-2$ degenerated vertex operators on the sphere. These degenerated vertex operators correspond to $\alpha = -\frac{mb}{2}$.

IV

4 The theory on the disk topology

In this chapter we perform some computation for the Ribault's new family of non-rational CFT in the disk. As we did in the previous chapter for the case of the sphere, we present the boundary conditions, the symmetries, and the relation with the Liouville field theory observables in the disk. We calculate in particular the one-point function in the disk. The particular cases with $m = 0, 1, b^2$ *are discussed. First, we give a brief introduction to general aspects of CFT in the disk geometry.*

4.1 Conformal field theory on the disk

A CFT on the disk topology is performed by working over the upper half complex plane. By use of conformal invariance this can be mapped to a disk, matching both infinite extremes of the real axis.

A theory defined on the upper half plane is conformally invariant only if the conformal transformations keep the boundary and the boundary conditions invariant. Among the infinitesimal conformal transformations of the form $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$, the ones mapping the real axis onto itself satisfy the condition $\epsilon(z) = \bar{\epsilon}(\bar{z})$. This is a strong constraint that eliminates half of the conformal generators. The holomorphic and anti-holomorphic sectors of the theory are no longer independent. This invariance can be improved by regarding the dependence of the correlation functions on anti-holomorphic coordinates $\bar{z}_i$ on the upper half plane as a dependence on holomorphic coordinates $z_i^* = \bar{z}_i$ on the lower half plane. This is a mirror image of the system on the lower half plane, via a parity transformation. In going from the upper to the lower half plane, vector and tensor fields change their holomorphic indices into anti-holomorphic indices and vice versa. Thus $T(z^*) = \overline{T}(z)$, $\overline{T}(z^*) = T(z)$, and so on. Such an extension is compatible with the boundary conditions only if $\overline{T} = T$ on the real axis.

As for the boundary conditions on a scaling field ϕ, invariance under conformal transformations requires them to be homogeneous, for instance as follows:

$$\phi|_{\mathbb{R}} = 0 , \qquad \phi|_{\mathbb{R}} = \infty , \qquad \frac{\partial \phi}{dn}|_{\mathbb{R}} = 0 \tag{138}$$

where the first condition is named Dirichlet boundary condition and the third one is the Newman boundary condition.

It is then possible to consider the theory on the upper half plane as a theory on the infinite plane with the condition that every field inserted in a correlation function in the point z_i on the upper half plane has a mirror image corresponding to an opposite chirality field on the point z_i^* on the lower half plane. For example we consider the two-point function of two free bosons ϕ with Newman boundary conditions; namely on the boundary $\partial\Gamma$ we demand $\partial\phi - \bar{\partial}\phi$ to vanish. The two-point function on the upper half plane is then equivalent to a four-point function of four chiral fields in the whole plane as follows:

$$\langle \phi(z, \bar{z})\phi(w, \bar{w}) \rangle = \langle \phi(z)\phi(z^*)\phi(w)\phi(w^*) \rangle = -\log|z - w||z - \bar{w}|. \tag{139}$$

Proceeding in the path integral formalism we can consider a lagrangian with a boundary term. This term and the boundary conditions it establishes must be invariant under the preserving conformal transformations.

A theory with an affine symmetry on the upper half plane must preserve half of this symmetry in an equivalent way. Then it must be formulated in the same way when a boundary is considered, keeping half the generators of this symmetry.

4.2 The new family of non-rational theories in the disk

4.2.1 Boundary action and boundary conditions

In order to study the theory on the disk we add a boundary term to the bulk action (124). This boundary action is

$$S_{\text{boundary}} = \frac{1}{2\pi}\int_{\partial\Gamma} dx\, g^{1/4}\left(Q_m\mathcal{K}\phi + \frac{i}{2}\beta\left(\gamma+\bar{\gamma}-\xi\,\beta^{m-1}\,e^{b\phi}\right)\right), \tag{140}$$

where ξ is an arbitrary constant and $\mathcal{K}$ is the scalar curvature of the boundary. This is the natural generalization of the boundary action proposed in [17]. Since the field β is not supposed to be positive, we assume here that $m \in \mathbb{Z}$. This is a limitation of both our path integral calculation and our free field calculation, and this is why the expression we get is valid for $m \in \mathbb{Z}$.

We will consider the theory on the disk. In turn, conformal invariance permits to chose Γ as being the upper half plane $\text{Im}(z) = y \geq 0$, with $z = x + iy$, and, consequently, the boundary is given by the real line $\text{Re}(z) = x$. On the disk geometry, we will consider the case of maximally symmetric boundary conditions (see (143)-(145) below). To implement this, first we integrate by parts the γ-β terms of (140). Then, integrating the resulting expression over the semi-infinite line $\text{Re}(y) > 0$, we see that two pieces $-\frac{i}{2}\beta\gamma|_{y=0}$ and $+\frac{i}{2}\bar{\beta}\bar{\gamma}|_{y=0}$ come to cancel the contribution $\beta(\gamma+\bar{\gamma})$ in (140). The total action then takes the form

$$\begin{aligned} S_{(m,b)} = & \frac{1}{2\pi}\int_{\Gamma} d^2z\, g^{1/2}\left(\partial\phi\bar{\partial}\phi - \gamma\bar{\partial}\beta - \bar{\gamma}\partial\bar{\beta} + \frac{Q_{(m,b)}}{4}\mathcal{R}\phi + b^2(-\beta\bar{\beta})^m\, e^{2b\phi}\right) + \\ & + \frac{1}{2\pi}\int_{\partial\Gamma} dx\, g^{1/4}\left(Q_{(m,b)}\mathcal{K}\phi - \frac{i\xi}{2}\beta^m\, e^{b\phi}\right). \end{aligned} \tag{141}$$

It is convenient to introduce bulk and boundary coupling constants (λ, λ_B) to control the strength of the interacting terms. This is achieved by shifting the zero-mode of ϕ as follows $\phi \to \phi + \frac{1}{2b}\log(\lambda/b^2)$ and redefining $\xi = 2ib\lambda_B/\sqrt{\lambda}$. This makes the interaction term in the action to take the form $\lambda\int_\Gamma(-\beta\bar{\beta})^m e^{2b\phi} + \lambda_B\int_{\partial\Gamma}\beta^m e^{b\phi}$, which is useful to perform the free field calculation as in this way one has access to a perturbative treatment of the screening effects. The coupling ξ takes imaginary values for the boundary action to be real.

Varying the action (141) in the boundary and imposing the condition $\delta(\beta+\bar{\beta})_{|z=\bar{z}}=0$, one finds

$$\delta S|_{\partial\Gamma} \;=\; \frac{i}{4\pi}\int dx\Big[\left((-\partial+\bar{\partial})\phi-\xi\, b\,\beta^{m}\, e^{b\phi}\right)\delta\phi+\left(\gamma+\bar{\gamma}-\xi\, m\,\beta^{m-1}\, e^{b\phi}\right)\delta\beta\Big]. \tag{142}$$

From this, one reads the gluing conditions in $z=\bar{z}=x$, which have to be

$$\beta(x)+\bar{\beta}(x)\;_{|z=\bar{z}} \;=\; 0 \tag{143}$$

$$\gamma(x)+\bar{\gamma}(x)\;_{|z=\bar{z}} \;=\; \xi\, m\,\beta^{m-1}(x)\, e^{b\phi(x)} \tag{144}$$

$$(-\partial+\bar{\partial})\,\phi(x)\;_{|z=\bar{z}} \;=\; \xi\, b\,\beta^{m}\,(x)e^{b\phi(x)}. \tag{145}$$

And these conditions correspond to

$$J^{-}(x)+\bar{J}^{-}(x)\;_{|z=\bar{z}} \;=\; 0, \tag{146}$$

$$J^{3}(x)-\bar{J}^{3}(x)\;_{|z=\bar{z}} \;=\; 0. \tag{147}$$

While (146) is evidently satisfied, condition (147) can be checked taking into account the conditions (143)-(145).

The other condition to be considered in the boundary is

$$T\,(x)-\overline{T}(x)\;_{|z=\bar{z}} \;=\; 0 \tag{148}$$

which guarantees boundary conformal symmetry. This classical analysis suggests that with these boundary conditions one obtains a theory that preserves the conformal symmetry generated by $J^{-}(z)$, $J^{3}(z)$, and $T(z)$. The free field computation ultimately helps to prove this explicitly.

Now we have discussed the boundary conditions and proposed the form of the boundary action, we are ready to undertake the calculation of the one-point function.

4.2.2 Path integral computation

The one-point function we are interested in is the vacuum expectation value $\langle\Phi_j(\mu|z)\rangle_{\mathcal{T}_{(m,b)}}$ of one bulk vertex operator (133) in the disk geometry. This is given by

$$\Omega_j^{(m,b)}(z) \;:=\; \left\langle\Phi^{j}(\mu|z)\right\rangle_{\mathcal{T}_{(m,b)}} \;=\; \int\mathcal{D}\phi\mathcal{D}^2\beta\mathcal{D}^2\gamma\;\, e^{-S_{(m,b)}}\,|\rho(z)|^{2h_j}\,|\mu|^{2m(j+1)}\, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\, e^{2b(j+1)\phi(z,\bar{z})}, \tag{149}$$

with $\mathcal{D}^2\beta = \mathcal{D}\beta\mathcal{D}\bar{\beta}$, and $\mathcal{D}^2\gamma = \mathcal{D}\gamma\mathcal{D}\bar{\gamma}$, and imposing $\beta + \bar{\beta} = 0$ on the line $z = \bar{z}$ as boundary condition. The action $S_{(m,b)}$ in (149) is given by (141).

Integrating over the fields γ and $\bar{\gamma}$ we get

$$\int \mathcal{D}\gamma \, e^{\frac{1}{2\pi}\int d^2w \gamma\bar{\partial}\beta} \, e^{\mu\gamma(z)} \;=\; \delta\left(\frac{1}{2\pi}\,\bar{\partial}\beta(w) + \mu\,\delta^{(2)}(w-z)\right) \tag{150}$$

and, respectively,

$$\int \mathcal{D}\bar{\gamma} \, e^{\frac{1}{2\pi}\int d^2\bar{w} \bar{\gamma}\partial\bar{\beta}} \, e^{\bar{\mu}\bar{\gamma}(\bar{z})} \;=\; \delta\left(\frac{1}{2\pi}\,\partial\bar{\beta}(\bar{w}) - \bar{\mu}\,\delta^{(2)}(\bar{w}-\bar{z})\right). \tag{151}$$

Using that $\bar{\partial}\left(\frac{1}{z}\right) = \partial\left(\frac{1}{\bar{z}}\right) = 2\pi\delta^{(2)}(z)$ it can be written

$$\frac{1}{2\pi}\,\bar{\partial}\beta(w) + \frac{\mu}{2\pi}\,\bar{\partial}\left(\frac{1}{w-z}\right) \;=\; 0, \tag{152}$$

$$\frac{1}{2\pi}\,\partial\bar{\beta}(\bar{w}) - \frac{\bar{\mu}}{2\pi}\,\partial\left(\frac{1}{\bar{w}-\bar{z}}\right) \;=\; 0. \tag{153}$$

Then, in the boundary we impose

$$\beta + \bar{\beta}_{\;|z=\bar{z}} \;=\; 0. \tag{154}$$

Now, we integrate over β and $\bar{\beta}$. Considering the conditions above, we get a non-vanishing solution only if $\mu + \bar{\mu} = 0$. The solution for β and $\bar{\beta}$ are thus given by

$$\beta_0(w) \;=\; -\frac{\mu}{w-z} - \frac{\bar{\mu}}{w-\bar{z}} \;=\; \frac{-\mu\,(z-\bar{z})}{(w-z)(w-\bar{z})}, \tag{155}$$

$$\bar{\beta}_0(\bar{w}) \;=\; \frac{\mu}{\bar{w}-z} + \frac{\bar{\mu}}{\bar{w}-\bar{z}} \;=\; \frac{-\bar{\mu}\,(z-\bar{z})}{(\bar{w}-z)(\bar{w}-\bar{z})}. \tag{156}$$

Now, we follow the analysis of [16] closely. We consider the exact differential $\rho(w)\beta_0(w)$, whose expression can be given in terms of its poles, up to a global factor u. Then, evaluating β the function to be computed can be seen to take the form

$$\Omega_j^{(m,b)}(z) \;=\; \int \mathcal{D}\phi \exp - \left(\frac{1}{2\pi}\int d^2w\left(\partial\phi\bar{\partial}\phi \,+\, b^2|u|^{2m}|w-z|^{-2m}|w-\bar{z}|^{-2m}\,|\rho(w)|^{-2m}\,e^{2b\phi}\right)\right) \times$$
$$\exp\left(-\frac{i\xi}{4\pi}\int d\tau\, u^m(\tau-z)^{-m}\,(\tau-\bar{z})^{-m}\,\rho(w)^{-m}\,e^{b\phi}\right)|\rho(z)|^{2h_j}\,|\mu|^{2m(j+1)}\,e^{2b(j+1)\phi(z,\bar{z})}\delta^{(2)}(\mu+\bar{\mu}). \tag{157}$$

Now, consider the following change of variables (i.e. field redefinition)

$$\phi(w) \;\rightarrow\; \phi(w) - \frac{m}{b}\log|u|, \tag{158}$$

which consequently gives $e^{b\phi(w)} \rightarrow e^{b\phi(w)}\,|u|^{-m}$. This induces a change in the linear "dilaton" term $-\frac{1}{8\pi}\int_\Gamma d^2w\, g^{1/2}\, Q_{(m,b)}\mathcal{R}\phi \,-\, \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}Q_{(m,b)}\mathcal{K}\phi$, which now takes the form

$$\begin{aligned} &-\frac{1}{8\pi}\int d^2w\, g^{1/2}\, Q_{(m,b)}\mathcal{R}\phi \,-\, \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}Q_{(m,b)}\mathcal{K}\phi \,+ \\ &+\log|u|m\left(1+\frac{1-m}{b^2}\right)\left[\frac{1}{8\pi}\int_\Gamma d^2w\, g^{1/2}\mathcal{R} + \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}\mathcal{K}\right]. \end{aligned} \tag{159}$$

Using the Gauss-Bonnet theorem, which states that the Euler characteristic of the disk is

$$\chi(\Gamma) \;=\; \frac{1}{8\pi}\int_\Gamma d^2w\, g^{1/2}\mathcal{R} + \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}\mathcal{K} \;=\; 1, \tag{160}$$

we find

$$\begin{aligned} \Omega_j^{(m,b)}(z) \;=\; &\int \mathcal{D}\phi \exp -\left(\frac{1}{2\pi}\int d^2w\left(\partial\phi\bar\partial\phi + b^2|w-z|^{-2m}\,|w-\bar z|^{-2m}|\rho(z)|^{-2m}e^{2b\phi}\right)\right)\times \\ &\times\exp\left(-\frac{i\xi}{4\pi}\left(\frac{u}{|u|}\right)^m\int d\tau(\tau-z)^{-m}\,(\tau-\bar z)^{-m}\,\rho^{-m}(\tau)e^{b\phi}\right)\times \\ &\times\delta^{(2)}(\mu+\bar\mu)|\rho(z)|^{2h_j}\,|\mu|^{2m(j+1)}\,|u|^{-m(2j+1+b^{-2}(m-1))}e^{2b(j+1)\phi(z,\bar z)}. \end{aligned} \tag{161}$$

Now, let us perform a second change of variables,

$$\phi(w,\bar w) \;=\; \varphi(w,\bar w) + \frac{m}{2b}\left(\log|w-z|^2 + \log|w-\bar z|^2 + \log|\rho(w)|^2\right), \tag{162}$$

which amounts to say

$$e^{b\phi} \;=\; e^{b\varphi}\,|w-z|^m\,|w-\bar z|^m\,|\rho(w)|^m \tag{163}$$

and

$$\partial\bar\partial\phi(w,\bar w) \;=\; \partial\bar\partial\varphi(w,\bar w) + \frac{m\pi}{b}\delta^{(2)}\left(|w-z|\right) + \frac{m\pi}{b}\delta^{(2)}(|w-\bar z|) + \frac{m}{2b}\partial\bar\partial\log|\rho(w)|^2. \tag{164}$$

In the bulk action this change produces a transformation in the kinetic term $-\frac{1}{2\pi}\int_\Gamma d^2w\partial\phi\bar{\partial}\phi$, which becomes

$$\frac{1}{2\pi}\int_\Gamma d^2w\,\varphi\partial\bar{\partial}\varphi + \frac{m}{2b}\int_\Gamma d^2w\,\varphi\,\delta^{(2)}(|w-z|) + \frac{m}{2b}\int_\Gamma d^2w\,\varphi\,\delta^{(2)}(|w-\bar{z}|) +$$
$$+\frac{m}{4\pi b}\int_\Gamma d^2w\,\varphi\,\partial\bar{\partial}\log|\rho(w)|^2 + \frac{m^2}{4\pi b^2}\int_\Gamma d^2w\left(\log|w-z|^2 + \log|w-\bar{z}|^2 + \log|\rho(w)|^2\right)\times$$
$$\times\left(\frac{b}{m}\partial\bar{\partial}\varphi + \pi\delta^{(2)}(|w-z)| + \pi\delta^{(2)}(|w-\bar{z}|) + \frac{1}{2}\,\partial\bar{\partial}\log|\rho(w)|^2\right). \tag{165}$$

Using the regularization [16]

$$\lim_{w\to z}\log|w-z|^2 \;\equiv\; \log|\rho(z)|^2, \tag{166}$$

being

$$ds^2 \;=\; |\rho(z)|^2\,dzd\bar{z}, \qquad \sqrt{g}\mathcal{R} \;=\; -4\,\partial\bar{\partial}\log|\rho(z)|^2, \tag{167}$$

one finds that the right hand side of (165) takes the form

$$-\frac{1}{2\pi}\int_\Gamma d^2w\,\partial\varphi\bar{\partial}\varphi \;-\; \frac{1}{2\pi}\int_\Gamma d^2w\,\sqrt{g}\frac{m}{4b}\mathcal{R}\varphi \;+\; \frac{m}{b}\varphi(z) + \frac{m^2}{2b^2}\log|z-\bar{z}| + \frac{m^2}{4b^2}\log|\rho(z)|^2. \tag{168}$$

The boundary action also suffers a change. Recalling how to write the extrinsic curvature $\mathcal{K}$ in terms of $\partial_x^2\log|\rho(\tau)|^2$, we find that, under the change (162), the boundary action changes as follows

$$\delta S_{\text{boundary}} \;=\; \frac{m}{2\pi b}\int_{\partial\Gamma} g^{1/4}\mathcal{K}\phi. \tag{169}$$

In turn, the changes above induce a modification in the value of the background change $Q_{(m,b)}$, shifting it as follows

$$Q_{(m,b)} \;\to\; Q_{(m,b)} + \frac{m}{b} \;=\; b + \frac{1-m}{b} + \frac{m}{b} \;=\; b + \frac{1}{b} \;=\; Q_{(0,b)} \tag{170}$$

which gives the Liouville background charge $Q_L = b + 1/b$ as the result.

With all this, the one-point function we are trying to compute takes the form

$$\begin{aligned}\Omega_j^{(m,b)}(z) &= \delta^2(\mu+\bar\mu)|u|^{m(1+\frac{1-m}{b^2})}\,|z-\bar z|^{\frac{m^2}{2b^2}}\int \mathcal{D}\varphi\exp\left[-\left(\frac{1}{2\pi}\int_\Gamma d^2w\left(\partial\varphi\bar\partial\varphi+b^2e^{2b\varphi}\right)\right)\right]\times\\ &\times\exp\left[-\frac{i\xi}{4\pi}\left(\frac{u}{|u|}\right)^m\int_{\partial\Gamma}d\tau\left(\frac{|\tau-z|}{\tau-z}\,\frac{|\tau-\bar z|}{\tau-\bar z}\,\frac{|\rho(z)|}{\rho(z)}\right)^m e^{b\varphi}\right]e^{2b(j+1)}\,e^{\frac{m}{b}\varphi(z)}\times\\ &\times|\mu|^{2m(j+1)}\,|z-\bar z|^{2m(j+1)}\,|u|^{-2m(j+1)}|\rho(z)|^{2h_j+\frac{m^2}{2b^2}}.\end{aligned} \tag{171}$$

Then, taking into account (166),

$$e^{2b(j+1)\varphi(z,\bar z)}\,e^{\frac{m}{b}\varphi(z,\bar z)} = e^{\left(2b(j+1)+\frac{m}{b}\right)\varphi(z,\bar z)}\,|\rho(z)|^{2m(j+1)}, \tag{172}$$

what amounts to extract the pole in the coincidence limit of the two operators $e^{2b(j+1)\varphi(z,\bar z)}$ and $e^{\frac{m}{b}\varphi(w,\overline{w})}$. Then we find

$$\begin{aligned}\Omega_j^{(m,b)}(z) =\ & \delta^2(\mu+\bar\mu)|\mu|^{mb^{-2}(b^2+1-m)}(2y)^{mb^{-2}(b^2+1-m/2)}\int\mathcal{D}\varphi\exp\left[-\frac{1}{2\pi}\int_\Gamma d^2w\left(\partial\varphi\bar\partial\varphi+b^2\,e^{2b\varphi}\right)\right]\\ &\times\exp\left[-\frac{i\xi}{4\pi}(\mathrm{sgn}(\mathrm{Im}\mu))^m\int_{\partial\Gamma}d\tau\,e^{b\varphi}\right]|\rho(z)|^{2h_j+m(b^2+1-m)/b^2}\,e^{(2b(j+1)+m/b)\varphi(z,\bar z)}\end{aligned} \tag{173}$$

where $\mathrm{sgn}(\mathrm{Im}\mu)$ is the sign of the imaginary part of μ. Notice that this expression is well defined for $m\in\mathbb{Z}$, as in that case the quantity $(\mathrm{sgn}(\mathrm{Im}\mu))^m$ is real. Remarkably, the expression on the right hand side of (173) corresponds to the disk one-point function in Liouville field theory multiplied by a factor $\delta^2(\mu+\bar\mu)|\mu|^{mb^{-2}(b^2+1-m)}(2y)^{mb^{-2}(b^2+1-m/2)}$, provided one agrees on identifying the Liouville parameters α, b, μ_L and μ_B as follows

$$\mu_L = \frac{b^2}{2\pi}=\frac{\lambda}{2\pi}, \tag{174}$$

$$\mu_B = \frac{i\xi}{4\pi}(\mathrm{sgn}(\mathrm{Im}\mu))^m=\frac{\lambda_B}{4\pi}, \tag{175}$$

$$\alpha = b(j+1)+\frac{m}{2b}, \tag{176}$$

$$h_\alpha = \alpha(Q_L-\alpha) = h_j+\frac{m}{2b^2}(b^2+1-\frac{m}{2}), \tag{177}$$

where we are using the standard notation; see for instance [6, 28]. This is to say

$$\left\langle\Phi_j(\mu|z)\right\rangle_{\mathcal{T}_{(m,b)}} = \delta^2(\mu+\bar\mu)|\mu|^{m(1+\frac{1-m}{b^2})}\,(2y)^{m(1+\frac{1}{b^2}-\frac{m}{b^2})}\left\langle V_\alpha(z)\right\rangle_{\mathrm{L}} \tag{178}$$

where $V_\alpha(z) = e^{2\alpha\varphi(z,\bar{z})}$. This trick, which is exactly the one used in [17] to solve the case $m = 1$, leads us to obtain the explicit expression for $\Omega_j^{(m,b)}$ in terms of the Liouville one-point function $\Omega_{j+m/2b^2}^{(m=0,b)}$. In fact, the expression for the disk one-point function of a bulk operator in Liouville theory is actually known [28, 49]; it reads

$$\left\langle V_\alpha(z) \right\rangle_{\mathrm{L}} = |z-\bar{z}|^{-2h_\alpha} \frac{2\Omega_0}{b} \left(\pi\mu \frac{\Gamma(b^2)}{\Gamma(1-b^2)} \right)^{\frac{Q-2\alpha}{2b}} \cosh(2\pi s(2\alpha - Q))\, \Gamma(2b\alpha - b^2)\, \Gamma\left(\frac{2\alpha - Q}{b} \right) \tag{179}$$

where the parameter s obeys

$$\cosh(2\pi bs) = \frac{\mu_B}{\sqrt{\mu_L}} \sqrt{\sin(\pi b^2)} \tag{180}$$

and where Ω_0 is an irrelevant overall factor we determine below.

Thus, by following the trick in [17] (see also [74, 75, 77]), we managed to calculate the expectation value $\Omega_j^{(m,b)}$ of a bulk operator in the theory (124) on the disk by reducing such calculation to that of the observable $\Omega_{j=\frac{\alpha}{b}-\frac{m}{2b^2}-1}^{(m=0,b)}$ in Liouville field theory. The final result for $m \in \mathbb{Z}$ reads

$$\begin{aligned}\Omega_j^{(m,b)} &= \frac{2}{b}\Omega_0\, \delta(\mu+\bar{\mu})|\mu|^{m(1+\frac{1-m}{b^2})}\, |z-\bar{z}|^{-2h_j} \left(\pi \frac{\Gamma(1-b^2)}{\Gamma(b^2+1)} \right)^{j+\frac{1}{2}-\frac{1-m}{2b^2}} \Gamma\left(2j+1-\frac{1-m}{b^2} \right) \times \\ &\quad \times\Gamma\left(b^2(2j+1)+m\right) \cosh\left(\left(r - i\frac{m\pi}{2}(\mathrm{sign}(\mathrm{Im}\mu))\right) \left(2j+1-\frac{1-m}{b^2} \right) \right)\end{aligned} \tag{181}$$

with $h_j = -b^2 j(j+1) + (j+1)(1-m)$ and where we introduced the parameter

$$r = 2\pi bs + i\frac{m\pi}{2}\mathrm{sign}(\mathrm{Im}\mu). \tag{182}$$

This boundary parameter r is analogous to the one introduced in [24] for the WZNW theory $m = 1$. Here we are involved with the case of $m \in \mathbb{Z}$, for which the Lagrangian representation (141) makes sense, and then the boundary parameter r is related to ξ as follows

$$\cosh(r - i\frac{m\pi}{2}\mathrm{sign}(\mathrm{Im}\mu)) = i\,\xi\, (\mathrm{sgn}(\mathrm{Im}\mu))^m \sqrt{\frac{\sin(\pi b^2)}{8\pi b^2}}. \tag{183}$$

Again, as it happens in Liouville theory, relation (183) exhibits a symmetry under the shift $r \to r + 2\pi i$, while definition (182) does not. Consequently, it is (182) (and not (183)) the expression that has be considered as the definition of r in terms of the Liouville parameter s.

We already mentioned in the Introduction that expression (178) fulfils several non-trivial consistency checks; for example, the reflection equation (226). Actually, (178), together with (182), could be regarded as the conjecture for the bulk one-point function of the CFTs are thought to exist.

In the next section we will reobtain the result (181) using the free field calculation. As the path integral derivation, the free field derivation will be also valid in the case of m being an integer number.

4.2.3 Free field computation

In this section, we compute the bulk one-point function in a different way. We will consider the free field approach, which has proven to be a useful method to calculate correlation functions in this type of non-rational models on the sphere [12, 25, 26].

We consider Newman boundary conditions for the fields; namely on the boundary $\partial\Gamma$ we demand $\partial\phi - \bar{\partial}\phi$, $\beta + \bar{\beta}$ and $\gamma + \bar{\gamma}$ to vanish. This is consistent with the boundary conditions (143)-(145) in the perturbative free field approximation (where $\xi = 0$ is considered to compute the correlation functions and the boundary interaction term is realized by additional insertions perturbatively). For such boundary conditions on the geometry of the disk, the non-vanishing correlation functions turn out to be

$$\langle \phi(z,\bar{z})\phi(w,\bar{w}) \rangle = -\log|z-w||z-\bar{w}| \ , \tag{184}$$

$$\langle \beta(z)\gamma(w) \rangle = -\frac{1}{z-w}, \qquad \langle \bar{\beta}(\bar{z})\bar{\gamma}(\bar{w}) \rangle = -\frac{1}{\bar{z}-\bar{w}}, \tag{185}$$

$$\langle \bar{\beta}(\bar{z})\gamma(w) \rangle = \frac{1}{\bar{z}-w}, \qquad \langle \beta(z)\bar{\gamma}(\bar{w}) \rangle = \frac{1}{z-\bar{w}} \tag{186}$$

where it can be seen the mixing between holomorphic and anti-holomorphic modes due to the presence of the boundary conditions.

Now, it must be verified that the boundary term we added to the action actually corresponds to a theory that preserves the symmetry generated by $J^-(z)$, $J^3(z)$ and $T(z)$. Checking this at the first order in λ_B amounts to check that the following expectation values vanish,

$$\left\langle \left(J^-(z) + \bar{J}^-(\bar{z})\right) \int_{\partial\Gamma} d\tau \beta^m(\tau) e^{b\phi(\tau)} \ldots \right\rangle_{|z=\bar{z}} = 0, \tag{187}$$

$$\left\langle \left(J^3(z) - \bar{J}^3(\bar{z})\right) \int_{\partial\Gamma} d\tau \beta^m(\tau) e^{b\phi(\tau)} \ldots \right\rangle_{|z=\bar{z}} = 0. \tag{188}$$

Once again, this is completely analogous to the analysis done in [17] for $m = 1$. However, in contrast to the case of the WZNW theory, where the condition $J^+(z) + \bar{J}^+(\bar{z}) = 0$ is known to hold as well [17], in the case of generic m it must be imposed the following

$$\left\langle \left(T(z) - \overline{T}(\bar{z})\right) \int_{\partial\Gamma} d\tau \beta^m(\tau) e^{b\phi(\tau)} \ldots \right\rangle_{|z=\bar{z}} = 0 \tag{189}$$

explicitly. To verify that conditions (187)-(189) are obeyed, first we compute the following OPE

$$J^-(z)\beta^m(\tau)e^{b\phi(\tau)} \sim 0 \tag{190}$$

$$\bar{J}^-(\bar{z})\beta^m(\tau)e^{b\phi(\tau)} \sim 0 \tag{191}$$

and

$$J^3(z)\beta^m(\tau)e^{b\phi(\tau)} \sim 0 \tag{192}$$

$$\bar{J}^3(\bar{z})\beta^m(\tau)e^{b\phi(\tau)} \sim 0 \tag{193}$$

where we considered that in the boundary $\bar{\beta}(x) = -\beta(x)$, and where the symbol ~ 0 means that the singular terms vanish when evaluating in the boundary.

Next, it must be imposed the condition (189). To do so, we have to compute the OPE

$$T(z)\beta^m(\tau)e^{b\phi(\tau)} \sim \partial_\tau \left(\frac{\beta^m(\tau)e^{b\phi(\tau)}}{z-\tau} \right) - \frac{ib\partial_\sigma\phi(\tau)\beta^m(\tau)e^{b\phi(\tau)}}{z-\tau} + \ldots \tag{194}$$

where, again, the symbol $\sim$ means that the equivalence holds up to regular terms and exact differentials when evaluating in the boundary. Analogously, we have the anti-holomorphic part

$$\overline{T}(\bar{z})\beta^m(\tau)e^{b\phi(\tau)} \sim \partial_\tau \left(\frac{\beta^m(\tau)e^{b\phi(\tau)}}{\bar{z}-\tau} \right) - \frac{ib\partial_\sigma\phi(\tau)\beta^m(\tau)e^{b\phi(\tau)}}{\bar{z}-\tau} + \ldots \tag{195}$$

In the boundary we find that (194) and (195) contribute with the same piece and then we verify that (189) is actually satisfied.

Then, it can be computed the bulk one-point function using the free field approach. This observable is given by

$$\Omega_j^{(m,b)} = \int \mathcal{D}\phi \mathcal{D}^2\beta \mathcal{D}^2\gamma \, \exp\Big[-\frac{1}{2\pi}\int_\Gamma \partial\phi\bar{\partial}\phi + \frac{1}{2\pi}\int_\Gamma \gamma\bar{\partial}\beta + \frac{1}{2\pi}\int_\Gamma \bar{\gamma}\partial\bar{\beta} - \frac{Q_{(m,b)}}{8\pi}\int_\Gamma g^{1/2}\mathcal{R}\phi - \\ - \frac{b^2}{2\pi}\int_\Gamma (-\beta\bar{\beta})^m e^{2b\phi} - \frac{Q_{(m,b)}}{2\pi}\int_{\partial\Gamma} g^{1/4}\mathcal{K}\phi + \frac{i\xi}{4\pi}\int_{\partial\Gamma}\beta^m e^{b\phi}\Big] |\mu|^{2m(j+1)} \, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \, e^{2b(j+1)\phi(z,\bar{z})}. \tag{196}$$

The notation we use is such that $z = x + iy$ and $w = \tau + i\sigma$.

Splitting the field ϕ in its zero-mode ϕ_0 and its fluctuations $\phi' = \phi - \phi_0$, and recalling that the Gauss-Bonnet contribution gives

$$\frac{1}{8\pi}\int_\Gamma g^{1/2}\mathcal{R}\; Q_{(m,b)}\phi_0 + \frac{1}{2\pi}\int_{\partial\Gamma} g^{1/4}\mathcal{K}\; Q_{(m,b)}\phi_0 \; = \; Q_{(m,b)}\phi_0, \tag{197}$$

we find

$$\Omega_j^{(m,b)} = \int \mathcal{D}\phi' d\phi_0 \mathcal{D}^2\beta\mathcal{D}^2\gamma \, e^{-S'_{|\lambda=0}} \exp\left[\frac{-b^2}{2\pi} e^{2b\phi_0}\int_\Gamma (-\beta\bar{\beta})^m e^{2b\phi'}\right] \exp\left[\frac{i\xi}{4\pi} e^{b\phi_0}\int_{\partial\Gamma}\beta^m e^{b\phi'}\right] \times \\ \times e^{2b(j+1)\phi_0} \, e^{-Q_{(m,b)}\phi_0} \, |\mu|^{2m(j+1)} \, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \, e^{2b(j+1)\phi'(z,\bar{z})} \tag{198}$$

where $S'_{|\lambda=0}$ means the free action, i.e. the action (141) with $\lambda = \lambda_B = 0$ evaluated on the field fluctuations ϕ'.

Then, it must be integrated over the zero-mode ϕ_0. The interaction terms in the actions give the following contribution to the integrand

$$\exp\left(\frac{-b^2}{2\pi} e^{2b\phi_0}\int_\Gamma (-\beta\bar{\beta})^m e^{2b\phi'} + \frac{i\xi}{4\pi} e^{b\phi_0}\int_{\partial\Gamma}\beta^m e^{b\phi'}\right), \tag{199}$$

while the vertex operator itself contributes with an exponential $e^{(2b(j+1)-Q_m)\phi_0}$. There is also a contribution from the background charge. For the disk geometry, both bulk and boundary screening operators are present. Standard techniques in the free field calculation yield the following expression for the residues of resonant correlation functions,

$$\underset{2j+1+\frac{m-1}{b^2}=-n}{\text{Res}\ \Omega_j^{(m,b)}} = \frac{1}{2b}\,|\mu|^{2m(j+1)} \sum_{\substack{p,l=0\\2p+l=n}}^{\infty} \frac{1}{p!l!} \prod_{i=1}^{\infty} \int_{\Gamma} d^2w_i \prod_{k=1}^{\infty} \int_{\partial\Gamma} d\tau_k \Big\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \times$$
$$\times\, e^{2b(j+1)\phi(z,\bar{z})} \prod_{i=1}^{p} \frac{(-b^2)}{2\pi}(-\beta\bar{\beta})^m\, e^{2b\phi(w_i,\bar{w}_i)} \prod_{k=1}^{l} \frac{i\xi}{4\pi}\beta^m\, e^{b\phi(x_k)} \Big\rangle \tag{200}$$

where, as mentioned, now both bulk and boundary screening operators, $\int_\Gamma(-\beta\bar{\beta})^m e^{2b\phi}$ and $\int_{\partial\Gamma}\beta^m e^{b\phi}$, appear. As shown above, the integration over the zero-mode ϕ_0 gives a charge conservation condition that demands to insert a precise amount of screening operators for the correlation functions not to vanish; p of these screening operators are to be inserted in the bulk, while l of them in the boundary, with $2p+l=n$. The precise relation is

$$2b(j+1) + 2\,b\,p + b\,l \;=\; Q_{(m,b)} \;=\; b + \frac{1-m}{b}; \tag{201}$$

that is $n \;=\; 2\,p + l \;= -2j - 1 + (1-m)/b^2$.

The correlation function then factorizes out in two parts: the part that depends on ϕ, and the contribution of the γ-β ghost system. The Coulomb gas calculation of the ϕ contribution yields

$$\Big\langle e^{2(j+1)\phi(iy)} \prod_{i=1}^{p} e^{2b\phi(w_i,\bar{w}_i)} \prod_{k=1}^{l} e^{b\phi(\tau_k)} \Big\rangle \;=\; \left[\prod_{k=1}^{l}(y^2+\tau_k^2)\prod_{i=1}^{p}|y^2+w_i^2|^2\right]^{-2b^2(j+1)} \times$$
$$\times\, |2y|^{-2b^2(j+1)^2} \left[\prod_{i,k}|w_i-x_k|^2 \prod_{i<i'}|w_i-w_{i'}|^2 \prod_{i,i'}|w_i-\bar{w}_{i'}| \prod_{k<k'}|x_k-x_{k'}|\right]^{-2b^2} \tag{202}$$

On the other hand, for working out the ghost contribution γ-β it is convenient first to consider the OPE

$$\Big\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\beta(w)\Big\rangle \;=\; \frac{\mu}{w-z} + \frac{\bar{\mu}}{w-\bar{z}} \;=\; \frac{\mu\,(z-\bar{z})}{(w-z)(w-\bar{z})}. \tag{203}$$

This implies that the γ-β correlation function takes the form

$$\Big\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \prod_{i=1}^{p}\left(-\beta(w_i)\bar{\beta}(\bar{w}_i)\right)^m \Big\rangle \;=\; \mu^{2pm}\,(2iy)^{2pm} \prod_{i=1}^{p} \frac{1}{|y^2+w_i^2|^{2m}} \tag{204}$$

and

$$\left\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \prod_{k=1}^{l} \beta^m(\tau_k) \right\rangle \; = \; \mu^{lm}\,(2iy)^{lm} \prod_{k=1}^{l} \frac{1}{(y^2+\tau_k)^m}. \tag{205}$$

The full correlation function is then given by

$$\begin{aligned} &\left\langle e^{\mu\gamma(iy)-\bar{\mu}\bar{\gamma}(-iy)} \prod_{i=1}^{p} \frac{(-b^2)}{2\pi}(-\beta(w_i)\bar{\beta}(\bar{w}_i))^m \prod_{k=1}^{l} \frac{i\xi}{4\pi}\beta(\tau_k)^m \right\rangle \; = \\ &= \; 2\pi\,\delta(\mu+\bar{\mu}) \left(\frac{b^2}{2\pi}\right)^p \left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^l |2u|^{nm}\,|\mu|^{nm} \prod_{i=1}^{p} \frac{1}{|y^2+w_i^2|^{2m}} \prod_{k=1}^{l} \frac{1}{(y^2+\tau_k^2)^m} \end{aligned} \tag{206}$$

Returning to the form of the resonant correlation functions, for which p and l are integer numbers, the residues of these observables take the form

$$\begin{aligned} \underset{2j+1+\frac{m-1}{b^2}=-n}{\mathrm{Res}}\,\Omega^{(m,b)} \; &= \; \frac{\pi}{b} \sum_{p=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{p!l!}\delta_{2p+l-n,0}(-1)^p \left(\frac{b^2}{2\pi}\right)^p \left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^l \times \\ &\times \int \prod_{i=1}^{p} \frac{d^2w_i}{|y^2+w_i^2|^{4b^2(j+1)+2m}} \int \prod_{k=1}^{l} \frac{dx_k}{(y^2+x_k^2)^{b^2(j+1)+m}} \Bigg[\prod_{i,k} |w_i-x_k|^2 \prod_{i<i'} |w_i-w_{i'}|^2 \times \\ &\times \prod_{i,i'} |w_i-\bar{w}_{i'}| \prod_{k<k'} |x_k-x_{k'}| \Bigg]^{-2b^2} \delta(\mu+\bar{\mu})|\mu|^{m\left(1+\frac{1-m}{b^2}\right)}\,|2y|^{-2b^2(j+1)^2+nm}. \end{aligned} \tag{207}$$

To solve this we use the following integral formula (see Ref. [17] for the computation in the case $m=1$)

$$\begin{aligned} Y_{n,p}(a) \; &= \; \frac{1}{p!(n-2p)!} \int \prod_{i=1}^{p} \frac{d^2w_i}{|y^2+w_i^2|^{2a}} \int \prod_{k=1}^{l} \frac{dx_k}{(y^2+x_k^2)^a} \Bigg[\prod_{i,k} |w_i-x_k|^2 \prod_{i<i'} |w_i-w_{i'}|^2 \times \\ &\times \prod_{i,i'} |w_i-\bar{w}_{i'}| \prod_{k<k'} |x_k-x_{k'}| \Bigg]^{-2b^2} \end{aligned} \tag{208}$$

with $a=2b^2(j+1)+m=1+b^2-b^2n$. The solution of this multiple Selberg-type integral is given by

$$Y_{n,p}(a) \; = \; |z-\bar{z}|^{n\left(1-2a-(n-1)b^2\right)} \left(\frac{2\pi}{\Gamma(1-b^2)}\right)^n \frac{2^{-2p}}{n!(\sin(\pi b^2))^p}\, I_n(a)\, J_{n,p}(a), \tag{209}$$

where

$$I_n(a) \;=\; \prod_{i=0}^{n-1} \frac{\Gamma\left(1-(i-1)b^2\right)\Gamma\left(2a-1+(n-1+i)b^2\right)}{\Gamma^2\left(a+ib^2\right)} \tag{210}$$

and

$$\begin{aligned} J_{n,p}(a) \;&=\; \sum_{i=0}^{p}(-1)^i \frac{\Gamma(n-p-i+1)}{\Gamma(p-i+1)\Gamma(n-2p+1)} \frac{\sin(\pi b^2(n-2i+1))}{\sin(\pi b^2(n-i+1))} \times \\ &\times \prod_{r=0}^{i-1} \frac{\sin(\pi b^2(n-r))\sin(\pi a+\pi b^2(n-r))}{\sin(\pi b^2(r+1))\sin(\pi a+\pi b^2 r)}. \end{aligned} \tag{211}$$

For the case of our interest, $a = 1+b^2(1-n)$, and the function $I_n(a)$ simplifies substantially, taking the value $I_n(a) \;=\; \Gamma\left(1-b^2 n\right)$. The sum over $J_{n,p}(a)$ also simplifies notably; see below.

Then, we find the expression

$$\begin{aligned} \underset{2j+1+\frac{m-1}{b^2}=-n}{\text{Res}}\, \Omega^{(m,b)} \;&=\; \frac{\pi}{b}\,\delta(\mu+\bar{\mu})|\mu|^{m\left(1+\frac{1-m}{b^2}\right)}\,|z-\bar{z}|^{-2h_j}\left(\frac{2\pi}{\Gamma(1-b^2)}\right)^n \frac{\Gamma(1-b^2 n)}{n!} \times \\ &\times \sum_{\substack{p,l=0 \\ 2p+l=n}}^{\infty}(-1)^p\left(\frac{b^2}{2\pi}\right)^p\left(\frac{i\xi}{4\pi}\left(-\text{sgn}(\text{Im}\mu)\right)^m\right)^l \frac{2^{-2p}}{(\sin(\pi b^2))^p}\,J_{n,p}(a) \end{aligned} \tag{212}$$

with $h_j \;=\; b^2 j(j+1)-(j+1)(1-m)$.

It can be simplified the expression above further. Following [17], we find

$$\sum_{p=0}^{|n/2|}(-1)^p\left(2\cosh(2\pi bs)\right)^{n-2p} J_{n,p}(1+b^2(1-n)) \;=\; \cosh(2\pi nbs), \tag{213}$$

where $2\pi bs = r - i(m\pi/2)\text{sgn}(\text{Im}\mu)$ and where we replaced $a = 1+b^2-b^2 n$. According to the notation introduced in (183), we write

$$\xi \;=\; i(-\text{sgn}(\text{Im}\mu))^m\sqrt{\frac{8\pi b^2}{\sin(\pi b^2)}}\,\cosh(r-i\frac{m\pi}{2}\text{sgn}(\text{Im}\mu)), \tag{214}$$

and this yields

$$\left(\frac{i\xi}{4\pi}\left(-\text{sgn}(\text{Im}\mu)\right)^m\right)^{n-2p} \;=\; \left(-\frac{1}{4\pi}\sqrt{\frac{8\pi b^2}{\sin(\pi b^2)}}\right)^{n-2p}(\cosh(r-i\frac{m\pi}{2}\text{sgn}(\text{Im}\mu)))^{n-2p}. \tag{215}$$

Replacing this into the sum in (212) we obtain

$$\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\delta_{2p+l-n,0}(-1)^p\left(\frac{b^2}{2\pi}\right)^p\left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^l\frac{2^{-2p}}{(\sin(\pi b^2))^p}J_{n,p}(a) \;=\; \tag{216}$$
$$= \;(-1)^n\,2^{-\frac{3}{2}n}\,\pi^{-n}\left(\Gamma(1-b^2)\Gamma(1+b^2)\right)^{\frac{n}{2}}\cosh(n(r-i\frac{m\pi}{2}\mathrm{sgn}(\mathrm{Im}\mu))),$$

which can be proven using, in particular, the relation $\pi/\sin(\pi b^2)=\Gamma(1-b^2)\Gamma(b^2)$.

Putting all the pieces together, the final result for the resonant correlation functions reads

$$\operatorname*{Res}_{2j+1+\frac{m-1}{b^2}=-n}\Omega^{(m,b)} \;=\; 2^{-\frac{n}{2}}\frac{\pi}{b}\,\delta(\mu+\bar{\mu})|\mu|^{m\left(1+\frac{1-m}{b^2}\right)}|z-\bar{z}|^{-2h_j}\frac{(-1)^n}{n!}\left(\frac{\Gamma(1+b^2)}{\Gamma(1-b^2)}\right)^{\frac{n}{2}}\times \tag{217}$$
$$\times\,\Gamma(1-b^2n)\,\cosh(n(r-i\frac{m\pi}{2}\mathrm{sgn}(\mathrm{Im}\mu))).$$

This can be rewritten by recalling that for $n\in\mathbb{Z}_{\geq0}$ it holds

$$\lim_{\varepsilon\to0}\varepsilon\Gamma(\varepsilon-n)\;=\;(-1)^n\frac{1}{\Gamma(n+1)}, \tag{218}$$

which allows to replace the factor $(-1)^n/n!$ in (217) as follows

$$\frac{(-1)^n}{\Gamma(n+1)}\;\to\;\Gamma(2j+1+(m-1)b^{-2}). \tag{219}$$

were we used $(1-m)b^{-2}-2j-1=n$. Written in terms of j, b, and m, the final result reads

$$\Omega_j{}^{(m,b)} \;=\; \left(\frac{\pi}{2}\right)^{\frac{n}{2}}\frac{\pi}{b}\;\delta(\mu+\bar{\mu})|\mu|^{m(1+\frac{1-m}{b^2})}|z-\bar{z}|^{-2h_j}\left(\pi\,\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{(2j+1+(m-1)b^{-2})/2}\times$$
$$\times\,\Gamma\left(2j+1-\frac{1-m}{b^2}\right)\Gamma\left(b^2(2j+1)+m\right)\cosh\left((r-i\frac{m\pi}{2}\mathrm{sgn}(\mathrm{Im}\mu))\left(2j+1-\frac{1-m}{b^2}\right)\right) \tag{220}$$

with $h_j=-b^2j(j+1)+(j+1)(1-m)$ and $\cosh(r-i\frac{m\pi}{2}\mathrm{sgn}(\mathrm{Im}\mu))=i\xi(\mathrm{sgn}(\mathrm{Im}\mu))^m\sqrt{\sin(\pi b^2)/8\pi b^2}$. And it is shown that expression (220) coincides with the path integral result (181) if we chose the numerical global coefficient in (181) to be $\Omega_0=(\pi/2)^{n/2}$. Therefore, we find exact agreement between the path integral calculation and the free field calculation. This manifestly shows that free field approach turns out to be a useful method to find the general expression of correlation functions in non-rational CFTs. The path integral method actually reduced the problem to that of computing an observable of Liouville theory, which may be done by different methods, e.g. the bootstrap method. The relation existing between correlation functions of the conformal theories $\mathcal{T}_{(m,b)}$ defined by the action (141) and correlation functions of Liouville theory is a generalization of the so called H_3^+ WZNW-Liouville correspondence [14, 15, 16, 27]. The free field approach, on the other hand, gave a quite direct computation of the disk one-point function.

4.3 Analysis of the one-point function in the disk

Let us first analyze some special cases of the general result (220). The first particular example we may consider is clearly the theory for $m = 0$, namely $\mathcal{T}_{(m=0,b)}$. In this case, one-point function (220) trivially reduces to Liouville disk one-point function (179). Normalized states $|\alpha\rangle$ of the theory are created by the action of exponential vertex operators on the vacuum, $\lim_{z\to 0} e^{2\alpha\varphi(z)}\,|0\rangle = |\alpha\rangle$, having momentum $\alpha = Q + iP$, with $P \in \mathbb{R}$ and $Q_L = b + b^{-1}$; see for instance [6, 28, 29]. Then, Liouville one-point function reads

$$\left\langle V_\alpha(z) \right\rangle_{\mathrm{L}} = |z-\bar{z}|^{-2h_\alpha} \left(\pi \frac{\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{\frac{-iP}{b}} \frac{\cos(4\pi sP)}{iP}\,\Gamma(1+2ibP)\,\Gamma\left(1+\frac{2iP}{b}\right) \tag{221}$$

where s given by $\cosh(2\pi bs) = \sqrt{\sin(\pi b^2)\mu_B^2/\mu_L}$, and where we have fixed the Liouville "cosmological constant" μ_L to a specific value.

Next, we have the case $m = 1$, which corresponds to the H_+^3 WZNW theory with level k, $\mathcal{T}_{\left(m=1,b=(k-2)^{-1/2}\right)}$. In this case, the observable $\Omega_j^{(m=1,b=(k-2)^{-1/2})} = \langle\Phi_j(\mu|z)\rangle_{\mathrm{WZNW}}$ represents AdS_2 branes in Euclidean AdS_3 space. This is

$$\begin{aligned}\left\langle \Phi_j(\mu|z) \right\rangle_{\mathrm{WZNW}} &= \pi\sqrt{k-2}\,\delta(\mu+\bar{\mu})\;|\mu|\;\;|z-\overline{z}|^{-2h_j} \left(\pi\frac{\Gamma(1+\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}\right)^{ip/2} \times \\ &\times \Gamma\left(ip\right)\,\Gamma\left(1+\frac{ip}{k-2}\right)\,\cosh\left(irp+\frac{\pi}{2}p(\mathrm{sign}(\mathrm{Im}\mu))\right). \end{aligned} \tag{222}$$

where $j = -1/2 + ip$ belongs to the continuous representation, with $p \in \mathbb{R}$, and where the parameter r is that introduced in Ref. [24]. The string coupling constant $g_s^2 = e^{-2\langle\vartheta\rangle\chi} = e^{-\chi/\sqrt{2k-4}}$ was also fixed to a specific value to absorb Ω_0 (here, $\langle\vartheta\rangle$ represents the expectation value of the dilaton field, and χ is the Euler characteristic of the world-sheet manifold.) In terms of fields ϕ, γ and γ we used to describe the theory (124), AdS_3 metric is written in Poincaré coordinates as $ds^2 = l^2 \left(d\phi^2 + e^{2\phi} d\gamma d\overline{\gamma} \right)$, where l is the "radius" of the space. When formulating string theory on this background, the level of the WZNW theory relates to the string tension as follows $k = l^2/\alpha'$. Branes in Lorentzian and Euclidean AdS_3 space were extensively studied in the literature; see for instance [17, 24, 31, 32, 34, 58, 60, 76] and references therein.

The case $m = b^2$ also yields the H_+^3 WZNW theory with level $k = b^{+2} + 2$. The fact that the WZNW theory is doubly represented in the family $\{\mathcal{T}_{(m,b)}\}$ is associated to Langlands duality [18]. In the free field approach this is related to the existence of a "second" screening operator. In this context, free field calculations using the Wakimoto representation with $m > 1$ were already discussed in [25]. In fact, the computation we performed in this thesis can be regarded as a generalization of the one in [25] to the geometry of the disk and to the case where m is not necessarily equal to $b^{+2} + 2$. Furthermore, one could feel tempted to go a step further and conjecture that a relation similar to (178)-(182) also holds for one-point functions in the theory $\mathcal{T}_{(m,b)}$ with arbitrary (not necessarily integer) value of m.

Other case of interest is the theory for $m = 2$. In this case, correlation functions on the sphere were shown to obey third order differential equations that are associated to existence of singular vectors in the modulo [3]. This raises the question as to whether the existence of singular vectors could be used to compute observables in the theory with boundaries by means of the bootstrap approach or some variation of it. It could be also interesting to attempt to compute observables in the presence of a boundary by using a free field representation similar to that proposed in [36] to describe the $m = 1$ theory. Such free field representation amounts to describe the H_+^3 WZNW theory as a $c < 1$ perturbed CFT coupled to Liouville theory, resorting to the H_+^3 WZNW-Liouville correspondence (see also [37]).

Now, let us analyze here some properties of the one-point function for $m \in \mathbb{Z}$ we have computed in (220). Important information is obtained from studying how $\Omega_j^{(m,b)}$ transforms under certain changes in the set of quantum numbers (m, b, j) that leave the conformal dimension

h_j unchanged. Looking at this gives important information about the symmetries of the theory. But, first, a few words on the scaling properties of the one-point function (220): The reason why we previously said that the precise value of the overall factor Ω_0 in (179) and (181) was "irrelevant" was that, by shifting the zero-mode of the field ϕ one easily introduces a Knizhnik-Polyakov-Zamolodchikov scaling λ in the bulk expectation value, and this yields

$$\Omega_j^{(m,b)} \ \rightarrow \ \lambda^{(2j+1+(m-1)b^{-2})/2} \ \Omega_j^{(m,b)}. \tag{223}$$

Then, since Ω_0 also goes as a power $n/2 = -(2j+1+(m-1)b^{-2})/2$, its value can be absorbed and conventionally fixed to any (positive value). In turn, we prefer to write (220) by replacing

$$\Omega_0 \left(\pi \, \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)} \right)^{(2j+1+(m-1)b^{-2})/2} \quad \rightarrow \quad \left(\lambda\pi \, \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)} \right)^{(2j+1+(m-1)b^{-2})/2} . \tag{224}$$

Now, we are ready to study the reflection properties of (220). Using properties of the Γ-function, it is easy to verify that the following relation holds

$$\Omega_j^{(m,b)} \Omega_{-1-j-\frac{m-1}{b^2}}^{(m,b)} \ = \ R_j^{(m,b)}, \tag{225}$$

with

$$R_j^{(m,b)} \ = \ \frac{1}{b^2} \left(\lambda\pi \, \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)} \right)^{2j+1+\frac{m-1}{b^2}} \frac{\gamma(2j+1+(m-1)b^{-2})}{\gamma(-(2j+1)b^2-(m-1))}, \tag{226}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. It is remarkable that the reflection coefficient $R_j^{(m,b)}$, which is given by the two-point function on the sphere, arises in this expression. This generalizes what happens in Liouville field theory and in the H_3^+ WZNW model, and this is related to the fact that one eventually associates fields Φ_j and fields $R_j^{(m,b)} \Phi_{-1-j-b^{-2}(m-1)}$.

Other functional property of the one-point function that is interesting to analyze is how it behaves under duality transformation $b \rightarrow 1/b$. Actually, one can show that (220) obeys

$$\Omega_j^{(m,b)} \ = \ \Omega_{b^2(j+1-b^{-2})}^{(mb^{-2},b^{-1})}, \tag{227}$$

provided the KPZ scaling parameter λ' associated to the function on the right hand side relates to that of the function on the left hand side through

$$\left(\lambda'\pi \, \frac{\Gamma(1-b^{-2})}{\Gamma(1+b^{-2})} \right)^{b^2} \ = \ \lambda\pi \, \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}. \tag{228}$$

Last, let us mention another interesting problem that involves the theories $\mathcal{T}_{(m,b)}$ formulated on closed Riemann surfaces and that is in some sense related to the path integral techniques we discussed here. This is the problem of trying to use the path integral approach developed in [16] to define higher genus correlation functions for $m \in \mathbb{Z}$, relating higher genus correlation functions in $\mathcal{T}_{(m,b)}$ to higher genus correlation functions in Liouville theory. Correlation functions on closed genus-g n-punctured Riemann surfaces in the theories $\mathcal{T}_{(m,b)}$ could be relevant to describe higher m-monodromy operators in $\mathcal{N} = 2$ four-dimensional superconformal field theories, according to the recently proposed Alday-Gaiotto-Tachikawa (AGT) conjecture of references [38, 39, 40]; this is because n-point functions in $\mathcal{T}_{(m,b)}$ are in correspondence with $(2n + 2g - 2)$-point functions in Liouville theory including $2g - 2 + n$ degenerate fields $V_{a=-\frac{m}{2b}}$. We will investigate the case $g = 1$ in the next chapter of this thesis. Studying the relevance of the theories defined in [3] for the AGT construction is matter of future investigation.

V

5 The theory on the torus topology

In this chapter we perform computation for the Ribault's new family of non-rational CFT in the torus. As we did in the previous chapters for the cases of the sphere and the disk, we first present the boundary conditions, the symmetries, and the relation with the Liouville field theory observables in the torus. We make some comments about general correlation functions and in particular we compute the one-point function in the torus explicitly. The particular cases with $m = 0, 1, b^2$ *are discussed. First, we give a brief introduction to CFT in the torus.*

5.1 Conformal field theory on the torus

A CFT on the torus topology is formulated by working over all the complex plane and imposing periodicity conditions. A torus can be defined by specifying two linearly independent lattice vectors on the plane and identifying points that differ by an integer combination of these vectors. On the complex plane these lattice vectors con be represented by two complex numbers ω_1 and ω_2, which are called the periods of the lattice. Naturally, the properties of CFTs defined on a torus do not depend on the overall scale of the lattice, nor on the absolute orientation of the lattice vectors. The relevant parameter is the ratio $\tau = \omega_2/\omega_1$, named modular parameter. In this way the torus can be represented by the complex plane with periodic conditions under translations

$$w \to w + 1 \qquad w \to w + \tau. \tag{229}$$

The n-point correlation functions are then calculated as an ∞-point functions in the whole plane because every field inserted in the torus is like infinite identical fields inserted in a lattice of modular parameter τ in the plane. The functions depend on the infinite relative distances between these infinite points. The functional dependence of correlation functions will be given in term of the θ-function to be introduced in (238). This function allows us to write correlation functions invariant under the translations expressed in (229). This invariance is named modular invariance.

5.2 The new family of non-rational theories in the torus

5.2.1 Periodicity conditions

Here, we specialize in the study of the theory (124) on the topology of the torus. The torus can be represented by the complex plane with periodic conditions under translations $w \to w + 1$ and $w \to w + \tau$. The complex variable $\tau = \tau_1 + i\tau_2$ is the modular parameter of the torus. It is also necessary to introduce an additional parameter λ by the following non twisted periodicity conditions for β, γ and ϕ; namely

$$\beta(w + p + q\tau) = e^{2\pi iq\lambda}\beta(w), \tag{230}$$

$$\gamma(w + p + q\tau) = e^{-2\pi iq\lambda}\gamma(w), \tag{231}$$

$$\phi(w + p + q\tau\,,\, \bar{w} + p + q\bar{\tau}) = \phi(w, \bar{w}) + \frac{2\pi mq\mathrm{Im}\lambda}{b}, \tag{232}$$

where $\mathrm{Im}\lambda = \lambda_2$ stands for the imaginary part of the twist parameter $\lambda = \lambda_1 + i\lambda_2$, and p and q are two arbitrary integer numbers. We are free to choose conditions (230)-(232) even though for generic λ the fields turn out to be multivalued; this is because the action does remain univalued. For $m = 0$ the field ϕ must be periodic, but it acquires more freedom in the case $m \neq 0$, and such is parameterized by λ, which labels different twist sectors. For $m = 1$, λ is identified with the Benard parameter that appears in the Bernard-Knizhnik-Zamolodchikov modular differential equation [41]; see [16] for the discussion on the case $m = 1$. For $m > 1$ and $m \neq b^2$ the theory does not exhibit the full $\mathrm{sl}(2)_k$ affine symmetry. Nevertheless, λ still may be introduced as the action is left invariant under the transformations (230)-(232).

The next step is decomposing ϕ into the solitonic zero-mode part, denoted ϕ_{c}, and a doubly periodic fluctuation ϕ_{f}; namely

$$\phi\ (w,\bar{w}) \ = \ \phi_{\mathrm{c}}(w,\bar{w})+\phi_{\mathrm{f}}(w,\bar{w}), \qquad \text{with} \qquad \phi_{\mathrm{c}}(w,\bar{w}) \ = \ \frac{2\pi m}{b}\frac{\mathrm{Im}(\lambda)\mathrm{Im}(w)}{\mathrm{Im}(\tau)}. \tag{233}$$

This solitonic configuration, together with $\beta = 0$ and $\gamma = 0$, represents the only solution to the classical equations of motion coming from the action (124) that satisfies the required periodic boundary conditions. On the other hand, the piece ϕ_{f} is periodic under $w \to w+1$ and $w \to w+\tau$.

We are now on the torus, and thus only the fluctuations ϕ_{f} couple to the scalar curvature in the linear dilaton term

$$\frac{Q_m}{8\pi}\int_{\Gamma} d^2w\,\sqrt{g}\mathcal{R}\phi_{\mathrm{f}}. \tag{234}$$

Although one considers the flat metric on the genus one surface, this term is ultimately important to keep track of the background charge contribution. It can be restored wherever writing the action (124) on a generic surface is needed.

5.3 Path integral and genus-one correlation functions

The quantities we are interested to compute are the correlation functions

$$\left\langle \prod_{\nu=1}^{N}\Phi_{j_\nu}(\mu_\nu|z_\nu)\right\rangle_{(\lambda,\tau)} \ = \ \int \mathcal{D}\phi\mathcal{D}^2\beta\mathcal{D}^2\gamma\, e^{-S_m[\phi,\gamma,\beta]}\prod_{\nu=1}^{N}\Phi_{j_\nu}(\mu_\nu|z_\nu) \tag{235}$$

where the subscript (λ,τ) one the left hand side stands to emphasizing that the functional measure $\int\mathcal{D}\phi\mathcal{D}^2\beta\mathcal{D}^2\gamma$ depends on the modular and the twist parameters.

The integration over the fields γ and $\bar{\gamma}$ yields a δ-function that fix the conditions

$$\bar{\partial}\beta(w) \ = \ 2\pi\sum_{\nu=1}^{N}\mu_\nu\,\delta(w-z_\nu), \tag{236}$$

$$\partial\bar{\beta}(\bar{w}) \ = \ -2\pi\sum_{\nu=1}^{N}\bar{\mu}_\nu\,\delta(\bar{w}-\bar{z}_\nu). \tag{237}$$

To integrate these equations it is convenient to introduce the θ-function

$$\theta(z|\tau) \;=\; -\sum_{n\in Z} e^{i\pi(n+\frac{1}{2})^2\tau+2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}. \tag{238}$$

We will call this function simply $\theta(z)$ or θ-function whenever the modular parameter τ does not change. Function (238) obeys the periodic condition

$$\theta(z+p+q\tau|\tau) \;=\; (-1)^{p-q}\, e^{-i\pi q(2z+q\tau)}\theta(z|\tau). \tag{239}$$

This property permits to build up from $\theta(z|\tau)$ a new function $\sigma_\lambda(z|\tau)$ which happens to have a single pole and the same periodicity condition that we asked for β. Namely,

$$\sigma_\lambda(z,w|\tau) \;=\; \frac{\theta(\lambda-(z-w)|\tau)\theta\prime(0|\tau)}{\theta(z-w|\tau)\theta(\lambda|\tau)}. \tag{240}$$

In fact, we have

$$\sigma_\lambda(z+p+q\tau,w|\tau) \;=\; e^{2\pi iq\lambda}\sigma_\lambda(z,w|\tau).$$

Then, we can use these modular functions to integrate (236)-(237). The integration of these equations is unique as long as the twist parameter λ does not vanish. We have

$$\beta(w) \;=\; \sum_{\nu=1}^{N}\mu_\nu\sigma_\lambda(w,z_\nu|\tau) \;=\; u\,\frac{\prod_{i=1}^{N}\theta(w-y_i|\tau)}{\prod_{\nu=1}^{N}\theta(w-z_\nu|\tau)} \;\equiv\; u\,\mathcal{X}_1(y_i,z_\nu,w|\tau), \tag{241}$$

where u is an overall factor.

β is a meromorphic differential, and depends on $N+1$ parameter; N of these parameters are the variables μ_ν and the other parameter is λ. We can parameterize β in terms of u and N parameters y_i by defining the following set of $N+1$ implicit equations

$$u^{-1}\mu_\nu \;=\; \frac{\prod_{i=1}^{N}\theta(z_\nu-y_i|\tau)}{\theta'(0|\tau)\prod_{\mu\neq\nu,\mu=1}^{N}\theta(z_\nu-z_\mu|\tau)}, \qquad \lambda \;=\; \sum_{i=1}^{N}y_i-\sum_{\nu=1}^{N}z_\nu \tag{242}$$

with $\theta'(0|\tau) \;=\; -2\pi i\sum_{n\in Z}(n+1/2)e^{i\pi(n+1/2)^2\tau+\pi i(n+1/2)}$. This is exactly what is done in [16] in the case $m=1$. Equation (242) comes from the computation of the residue of the function β at the pole $w=z_\nu$. Then, the integration over γ and $\bar\gamma$ leads to the following δ-function

$$\delta^{(2)}(\bar\partial\beta(w)-2\pi\sum_{\nu=1}^{N}\mu_\nu\delta^2(w-z_\nu)) \;=\; |\det\partial_\lambda|^{-2}\,\delta^{(2)}\big(\beta(w)-u\mathcal{X}_1(y_i,z_\nu;w)\big). \tag{243}$$

The factor $|\det \partial_\lambda|^{-2}$ is the Jacobian corresponding to the change of variables from $\partial\beta$ to β. Then one can integrate over β and $\bar{\beta}$ and finally find

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \frac{1}{|\det \partial_\lambda|^2} \int \mathcal{D}\phi\, e^{-S_{\text{eff}}[\phi,\mathcal{X}_1]} \prod_{\nu=1}^{N} |\mu_\nu|^{2m(j_\nu+1)}\, e^{2b(j_\nu+1)\phi(z_\nu)} \tag{244}$$

with the effective action

$$S_{\text{eff}}[\phi,\mathcal{X}_1] = \frac{1}{2\pi} \int d^2w \left(\partial\phi\bar{\partial}\phi + b^2|u|^{2m}|\mathcal{X}_1|^{2m} e^{2b\phi}\right),$$

where the function

$$|\mathcal{X}_1|^{2m} = \frac{\prod_{i=1}^{N} |\theta(w-y_i|\tau)|^{2m}}{\prod_{\nu=1}^{N} |\theta(w-z_\nu|\tau)|^{2m}} \tag{245}$$

stands in the action. Performing the change of variables

$$\phi(w,\bar{w}) \to \phi(w,\bar{w}) - \frac{m}{b}\log|u| \tag{246}$$

and, using (242), we find

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \frac{1}{|\det \partial_\lambda|^2} \int \mathcal{D}\phi\, e^{-S_{eff}}$$
$$\times \prod_{\nu=1}^{N} \left(e^{2b(j_\nu+1)\phi(z_\nu)} \prod_{\mu\neq\nu}^{N} \left| \frac{\theta(z_\nu - z_\mu|\tau)}{\theta'(0|\tau)} \right|^{-2m(j_\nu+1)} \prod_{i=1}^{N} \left| \frac{\theta(z_\nu - y_i|\tau)}{\theta'(0|\tau)} \right|^{2m(j_\nu+1)} \right) \tag{247}$$

with $s = -\sum_\nu^N j_\nu - N$. Then, it is convenient to perform a second change of variables; we define

$$\varphi(w,\bar{w}) = \phi(w,\bar{w}) + \frac{m}{2b}\left(\sum_{i=1}^{N} \log|\theta(w-y_i)|^2 - \sum_{\nu=1}^{N} \log|\theta(w-z_\nu)|^2 \right); \tag{248}$$

that is,

$$e^{2b\varphi(w,\bar{w})} = e^{2b\phi(w,\bar{w})} \frac{|\theta(w-y_i|\tau)|^{2m}}{|\theta(w-z_\nu|\tau)|^{2m}}. \tag{249}$$

Then, one realizes that the new field φ is periodic under the shifting by $p+q\tau$ (with p, $q \in \mathbb{Z}$) even when the original field ϕ and the θ-functions were not. It is also convenient to replace the functions θ by a new function F defined as follows

$$F(z-w|\tau) = e^{-2\pi \frac{(\text{Im}(z-w))^2}{\text{Im}(\tau)}} \left| \frac{\theta(z-w|\tau)}{\theta'(0|\tau)} \right|^2, \tag{250}$$

which does satisfy

$$F(z+p+q\tau-w|\tau) \;=\; F(z-w|\tau). \tag{251}$$

This is standard in CFT calculation on the torus and it exactly parallels the treatment of [16] for the case $m=1$.

So we can rewrite the relation between φ and ϕ in terms of single valued variables, the fluctuation field $\phi_{\rm f}=\phi+\phi_{\rm c}$ and the function F. Namely,

$$\varphi(w) \;=\; \phi_{\rm f}(w) + \frac{m}{2b}\sum_{i=1}^{N}\log F(w-y_i|\tau) - \frac{m}{2b}\sum_{\nu=1}^{N}\log F(w-z_\nu|\tau) + \Delta \tag{252}$$

where

$$\Delta \;=\; \frac{\pi m}{{\rm Im}(\tau) b}\left(\sum_{i=1}^{N}{\rm Im}(y_i)^2-\sum_{\nu=1}^{N}{\rm Im}(z_\nu)^2\right). \tag{253}$$

The kinetic term thus changes as follows

$$\frac{1}{2\pi}\int_\Gamma d^2w\,\partial\phi\bar\partial\phi \;=\; \frac{1}{2\pi}\int_\Gamma d^2w\,\partial\phi_{\rm f}\bar\partial\phi_{\rm f} + \frac{\pi\, m^2{\rm Im}(\lambda)^2}{{\rm Im}(\tau)\, b^2} \tag{254}$$

where we used that the volume is $\int d^2w=\int 2dxdy=2{\rm Im}(\tau)=2\tau_2$.

Considering that

$$\log F(z,w|\tau) \;=\; \frac{\pi}{2\tau_2}(z-\bar z-w+\bar w)^2 + \log|\theta(z-w|\tau)|^2 - \log|\theta^{'}(0|\tau)|^2, \tag{255}$$

and

$$\bar\partial\partial\log F(z,w|\tau) \;=\; -\frac{\pi}{\tau_2} + 2\pi\,\delta^{(2)}(z-w), \tag{256}$$

it is easy to check that

$$\partial\bar\partial\varphi(w,\bar w) \;=\; \partial\bar\partial\phi_{\rm f}(w,\bar w) + \frac{m\pi}{b}\sum_{i=1}^{N}\delta^{(2)}(w-y_i) - \frac{m\pi}{b}\sum_{\nu=1}^{N}\delta^{(2)}(w-z_\nu). \tag{257}$$

Then, we have

$$\frac{1}{2\pi}\int_\Gamma d^2w\,\partial\phi_{\rm f}\bar\partial\phi_{\rm f} \;=\; -\frac{1}{2\pi}\int_\Gamma d^2w\,\varphi\partial\bar\partial\varphi + \frac{m}{b}\sum_{i=1}^{N}\varphi(y_i) - \frac{m}{b}\sum_{\nu=1}^{N}\varphi(z_\nu). \tag{258}$$

Finally, considering that $e^{2b(j+1)\phi(z\nu)} = e^{2b(j+1)\phi_f}\, e^{4\pi m \mathrm{Im}(\lambda)\mathrm{Im}(z_\nu)/\mathrm{Im}(\tau)}$ where

$$e^{2b(j+1)\phi_{\rm f}(z_\nu)} = e^{2b(j+1)\varphi(z_\nu)} \prod_{i=1}^{N} F(z_\nu - y_i|\tau)^{-m(j+1)} \prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{N} F(z_\mu - z_\nu|\tau)^{m(j+1)}\, e^{-2b(j+1)\Delta}, \qquad (259)$$

and, after some manipulation, it ends up as follows

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \frac{e^{-\pi\, m^2 \mathrm{Im}(\lambda)^2/\mathrm{Im}(\tau) b^2}}{|\det \partial|^2_\lambda} \left|\Theta_N^{g=1}(y_i, z_\nu|\tau)\right|^{\frac{m^2}{4b^2}} \times \\ \times \int \mathcal{D}\varphi\, e^{-S_{\rm L}[\varphi]} \prod_{\nu=1}^{N} e^{2\alpha_\nu\varphi(z_\nu)} \prod_{i=1}^{N} e^{-\frac{m}{b}\varphi(y_i)} \qquad (260)$$

where we find the Liouville field theory action

$$S_{\rm L}[\varphi] = \frac{1}{2\pi}\int_{\mathcal{C}} d^2w \left(\partial\varphi\bar{\partial}\varphi + \frac{1}{4}(b+b^{-1})\sqrt{g}\mathcal{R}\varphi + b^2 e^{2b\varphi}\right)$$

and

$$\left|\Theta_N^{g=1}(y_i, z_\nu|\tau)\right| = \prod_{\mu<\nu}^{N} F(z_\mu - z_\nu|\tau)^2 \prod_{i<j}^{N} F(y_i - y_j|\tau)^2 \prod_{\mu,i=1}^{N} F(z_\mu - y_i|\tau)^{-2}. \qquad (261)$$

On the left hand side of (260) we already see the Liouville correlation functions to appear. In order to normalize the correlation functions we have to consider the partition function

$$Z_{(m,b)}(\lambda|\tau) = \frac{1}{\sqrt{\tau_2}\,|\theta(\lambda|\tau)|^2}\, e^{-\pi\, m^2 (\mathrm{Im}\lambda)^2/\mathrm{Im}(\tau) b^2}. \qquad (262)$$

In particular, $Z_{(m=1,b)}$ corresponds to the partition function of H_+^3 WZW model [42]. The case $Z_{(m=0,b)}$ is, of course, the partition function of Liouville theory , Z_L, times the contribution of the free γ-β ghost system. Equation (243) imposes β to be a constant, which has be zero for $\lambda \neq 0$; so the integration yields just $|\det\partial_\lambda|^{-2}$. In the case $\lambda = 0$, β may take any value and the integration diverges.

Collecting all the ingredients we arrive to the genus-one generalization of the Ribault formula of [3] for the case of the torus. Namely,

$$\frac{1}{Z_{(m,b)}} \left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \frac{\left|\Theta_N^{g=1}(y_i, z_\nu|\tau)\right|^{\frac{m^2}{4b^2}}}{Z_{\rm L}} \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N} V_{-\frac{m}{2b}}(y_i) \right\rangle_{\rm L}, \qquad (263)$$

where
$$\left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N} V_{-\frac{m}{2b}}(y_i) \right\rangle_{\mathrm{L}} = \int \mathcal{D}\varphi \, e^{-S_{\mathrm{L}}[\varphi]} \prod_{\nu=1}^{N} e^{2\alpha_\nu \varphi(z_\nu)} \prod_{i=1}^{N} e^{-\frac{m}{b}\varphi(y_i)}$$

with $V_{\alpha_\nu}(z_\nu) = e^{2\alpha_\nu\varphi(z_\nu)}$ and $\alpha_\nu = b\left(j_\nu + 1 + \frac{mb^{-2}}{2}\right)$. The prefactor $\Theta_N^{g=1}(y_i, z_\nu|\tau)$ is given by (261).

Notice that here, for genus $g = 1$, we also find that function $\left|\Theta_N^{g=1}(y_i, z_\nu|\tau)\right|^{\frac{m^2}{4b^2}}$ can be written as the expectation value of operators $\langle \prod_{\nu=1}^{p} e^{i\frac{m}{b}X(z_\nu)} \prod_{i=1}^{p} e^{-i\frac{m}{b}X(y_i)} \rangle_X$.

Relation (263) is valid for all values $m \in \mathbb{Z}_{\geq 0}$. It generalizes the genus-zero results of [3] to genus-one, which has been accomplished by straightforwardly adopting the analysis of [16] to the generic case $m \in \mathbb{Z}_{\geq 1}$. As a consequence, now we have (263), which relates the torus N-point function of the theory defined by action (124) to Liouville $2N$-point functions.

5.4 Coulomb gas integral representation on the torus

Now, we are in a position to perform a Coulomb gas computation of the correlation functions on the torus. This amounts to calculate expectation values of vertices (133) in the theory (124) in the free field approximation, and then invoke an appropriate analytic continuation.

Consider again the action (124)

$$S_{(m,b)}[\phi, \gamma, \beta; \kappa] = \frac{1}{2\pi} \int_{\mathcal{C}} d^2w \, g^{1/2} \left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{1}{4} Q_{(m,b)} \mathcal{R}\phi + \kappa b^2 (-\beta\bar{\beta})^m e^{2b\phi} \right) \tag{264}$$

and the vertex operators

$$\Phi_j(\mu|z) = |\mu|^{2m(j+1)} e^{\mu\gamma(z) - \bar{\mu}\bar{\gamma}(\bar{z})} e^{2b(j+1)\phi(z,\bar{z})} \tag{265}$$

but now notice that in (264) we have introduced a coupling constant κ, which permits to keep track of the Knizhnik-Polyakov-Zamolodchikov scaling of the correlation functions. Introducing κ is effectively achieved by shifting the zero-mode of ϕ appropriately, and its value is ultimately associated to the Liouville cosmological constant μ in the corresponding Liouville correlation functions.

Consider the correlation functions

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \int \mathcal{D}\phi \mathcal{D}^2\beta \mathcal{D}^2\gamma \, e^{-S_{(m,b)}[\phi\gamma\beta;\kappa]} \prod_{\nu=1}^{N} |\mu_\nu|^{2m(j_\nu+1)} \, e^{\mu_\nu \gamma(z_\nu) - \bar{\mu}_\nu \bar{\gamma}(\bar{z}_\nu)} \, e^{2b(j_\nu+1)\phi(z_\nu,\bar{z}_\nu)}. \tag{266}$$

The twisted boundary conditions for the field ϕ mean that it must be decomposed by the sum of a constant zero-mode ϕ_0, a twisted zero-mode ϕ_{sol}, and the single valued fluctuation field ϕ_{f}; recall (233). In order to perform a free field computation, first we split the field ϕ in its constant zero-mode ϕ_0 and its fluctuations $\phi' = \phi - \phi_0$. The twisted zero-mode does not couple to the linear dilaton while the constant zero-mode would do so; nevertheless, due to the Gauss-Bonnet theorem, it has no effective contribution. Then, after splitting the contributions to ϕ, we have

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle = \int_{\mathbb{R}} d\phi_0 \int \mathcal{D}\phi' \mathcal{D}^2\beta \mathcal{D}^2\gamma \, e^{-S_{(m,b)}[\phi,\gamma,\beta;\kappa=0]} \exp\left(\frac{-\kappa b^2}{2\pi} e^{2b\phi_0} \int_{\mathcal{C}} d^2w (-\beta\bar{\beta})^m \, e^{2b\phi'} \right)$$
$$\times \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi_0} \, |\mu_\nu|^{2m(j_\nu+1)} \, e^{\mu_\nu \gamma(z_\nu) - \bar{\mu}_\nu \bar{\gamma}(\bar{z}_\nu)} \, e^{2b(j_\nu+1)\phi'(z_\nu,\bar{z}_\nu)} \quad . \tag{267}$$

Integrating the zero-mode ϕ_0 is done by following standard procedure, and we finally arrives to the following expression

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle = \frac{\Gamma(-s)}{2b} \left(\frac{-b^2\kappa}{2\pi} \right)^s \int \mathcal{D}\phi' \mathcal{D}^2\beta \mathcal{D}^2\gamma \, e^{-S_f} \times$$
$$\times \prod_{\nu=1}^{N} |\mu_\nu|^{2m(j_\nu+1)} \, e^{\mu_\nu \gamma(z_\nu) - \bar{\mu}_\nu \bar{\gamma}(\bar{z}_\nu)} \, e^{2b(j_\nu+1)\phi'(z_\nu,\bar{z}_\nu)} \int_{\mathcal{C}} \prod_{k=1}^{s} d^2w_k \prod_{k=1}^{s} \left(-\beta(w_k)\bar{\beta}(\bar{w}_k) \right)^m \, e^{2b\phi'(w_k,\bar{w}_k)}. \tag{268}$$

where $s = -N - \sum_{\nu=1}^{N} j_\nu$. This can be written as follows

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \frac{\Gamma(-s)}{2b} \left(\frac{-b^2\kappa}{2\pi} \right)^s \prod_{\nu=1}^{N} |\mu_\nu|^{2m(j_\nu+1)} \int_{\mathcal{C}} \prod_{k=1}^{s} d^2w_k \times$$
$$\left\langle \prod_{\nu=1}^{N} e^{\mu_\nu \gamma(z_\nu)} \prod_{k=1}^{s} (-\beta(w_k))^m \right\rangle \left\langle \prod_{\nu=1}^{N} e^{-\bar{\mu}_\nu \bar{\gamma}(\bar{z}_\nu)} \prod_{k=1}^{s} \left(\bar{\beta}(\bar{w}_k) \right)^m \right\rangle \left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi'(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi'(w_k,\bar{w}_k)} \right\rangle \tag{269}$$

where the $(N+s)$-point correlation functions on the right hand side are defined by the free theory ($\kappa = 0$).

The factor $\Gamma(-s)$, which stands from the integration over ϕ_0, develops single poles for $s \in \mathbb{Z}_{\geq 0}$. Then we extract the form of the *resonant* correlation functions from the residues of such divergent contributions. This way the computation is

$$\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \frac{\Gamma(-s)}{2b} \left(\frac{-b^2\kappa}{2\pi}\right)^s \prod_{\nu=1}^{N} |\mu_\nu|^{2m(j_\nu+1)} \int_{\mathcal{C}} \prod_{k=1}^{s} d^2w_k \times$$
$$\left\langle \prod_{\nu=1}^{N} e^{\mu_\nu \gamma(z_\nu)} \prod_{k=1}^{s} (\beta(w_k))^m \right\rangle \left\langle \prod_{\nu=1}^{N} e^{-\bar{\mu}_\nu \bar{\gamma}(\bar{z}_\nu)} \prod_{k=1}^{s} \left(-\bar{\beta}(\bar{w}_k)\right)^m \right\rangle \left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi^{'}(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi^{'}(w_k,\bar{w}_k)} \right\rangle \tag{270}$$

Correlation functions for non-integer s can be defined by analytic continuation of the resonant ones. Of special importance are the correlation functions that correspond to $j_\nu = -\frac{1}{2} + i\mathbb{R}$, which in the Liouville side correspond to normalizable states with $\alpha_\nu = \frac{Q}{2} + i\mathbb{R}$.

Recalling that the field $\phi^{'}$ can be split into the twisted zero-mode ϕ_{c} and the single valued fluctuation field ϕ_f and the former does not coupled to the linear dilaton, it is easy to notice that

$$\left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi^{'}(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi^{'}(w_k,\bar{w}_k)} \right\rangle = \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi_{\mathrm{c}}(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi_{\mathrm{c}}(w_k,\bar{w}_k)} \times$$
$$\times \left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi_{\mathrm{f}}(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi_{\mathrm{f}}(w_k,\bar{w}_k)} \right\rangle, \tag{271}$$

and, resorting to the definition in (250), these free-field correlation functions for the field ϕ_{f} are written as follows

$$\left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi_{\mathrm{f}}(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi_{\mathrm{f}}(w_k,\bar{w}_k)} \right\rangle = \prod_{\nu<\mu}^{N} F(z_\nu - z_\mu|\tau)^{-2b^2(j_\nu+1)(j_\mu+1)} \times$$
$$\times \prod_{\nu=1}^{N} \prod_{k=1}^{s} F(z_\nu - w_k|\tau)^{-2b^2(j_\nu+1)} \prod_{k<l}^{s} F(w_k - w_l|\tau)^{-2b^2}. \tag{272}$$

On the other hand, the γ-β contribution is

$$\begin{aligned} \left\langle \prod_{\nu=1}^{N} e^{\mu_\nu \gamma(z_\nu)} \prod_{k=1}^{s} (\beta(w_k))^m \right\rangle \left\langle \prod_{\nu=1}^{N} e^{-\bar{\mu}_\nu \bar{\gamma}(\bar{z}_\nu)} \prod_{k=1}^{s} \left(-\bar{\beta}(\bar{w}_k)\right)^m \right\rangle &= \\ = |u|^{2sm} \prod_{k=1}^{s} e^{2b\Delta - 2bm\phi_c(w_k,\bar{w}_k)} \left[\frac{\prod_{i=1}^{N} F(y_i - w_k|\tau)^m}{\prod_{\nu=1}^{N} F(z_\nu - w_k|\tau)^m} \right], \end{aligned} \tag{273}$$

recall $S_1 = \frac{2\pi}{\tau_2} \sum_{i=1}^{N} (\mathrm{Im} y_i)^2 - \frac{2\pi}{\tau_2} \sum_{\nu=1}^{N} (\mathrm{Im} z_\nu)^2$.

One finally finds

$$\begin{aligned} &\left\langle \prod_{\nu=1}^{N} \Phi_{j_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\lambda,\tau)} = \\ &= \frac{\Gamma(-s)}{2b} \left(\frac{-b^2\kappa}{2\pi} \right)^s |\theta'(0)|^{2sm} \prod_{\nu=1}^{N} \left(\prod_{i=1}^{N} |\theta(z_\nu - y_i|\tau)|^{2m(j_\nu+1)} \prod_{\mu\neq\nu}^{N} |\theta(z_\nu - z_\mu|\tau)|^{-2m(j_\nu+1)} \right) \times \\ &\times \int_{\mathcal{C}} \prod_{k=1}^{s} d^2 w_k \prod_{k=1}^{s} e^{2b\Delta - 2bm\phi_{\mathrm{sol}}(w_k,\bar{w}_k)} \left[\frac{\prod_{i=1}^{N} F(y_i - w_k|\tau)^m}{\prod_{\nu=1}^{N} F(z_\nu - w_k|\tau)^m} \right] \left\langle \prod_{\nu=1}^{p} e^{2b(j_\nu+1)\phi'(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi'(w_k,\bar{w}_k)} \right\rangle. \end{aligned} \tag{274}$$

This follows from the analysis of [16] applied to the case $m > 1$.

5.5 Torus one-point function

Now, let us focus on the 1-point function, which is the one we are concerned with here. According to what we proved in (263), the torus 1-point function $\langle \Phi_j(\mu|z) \rangle$ would correspond to the Liouville 2-point function $\left\langle e^{2\alpha\phi} e^{-m\phi/b} \right\rangle$ on the torus, which, as we will see in Chapter 7, has an interpretation from the $\mathcal{N} = 2^*$ gauge theory point of view. This is why we pay special attention to it.

For the one-point function, we have

$$\begin{aligned} \langle \Phi_j(\mu|z) \rangle_{(\lambda,\tau)} &= \frac{\Gamma(j+1)}{2b} \left(\frac{-b^2}{2\pi} \right)^{-1-j} |\mu|^{2m(j+1)} \int_{\mathcal{C}} \prod_{k=1}^{s} d^2 w_k \times \\ &\times \left\langle e^{\mu\gamma(z)} \prod_{k=1}^{s} (\beta(w_k))^m \right\rangle \left\langle e^{-\bar{\mu}\bar{\gamma}(\bar{z})} \prod_{k=1}^{s} \left(-\bar{\beta}(\bar{w}_k)\right)^m \right\rangle \left\langle e^{2b(j+1)\phi'(z,\bar{z})} \prod_{k=1}^{s} e^{2b\phi'(w_k,\bar{w}_k)} \right\rangle \end{aligned} \tag{275}$$

where $s = -1 - j$ and we set $\kappa = 1$. The conformal Killing group of the torus allows us to fix the position of one operator, leaving an integration over the position of only $s-1$ of them. This is analogous to the two-point function on the sphere.

$$
\begin{aligned}
\langle \Phi_j(\mu|z) \rangle_{(\lambda,\tau)} &= \frac{\Gamma(j+1)}{2b} \left(\frac{-b^2}{2\pi} \right)^{-1-j} |\mu|^{2m(j+1)} \int \prod_{k=1}^{s-1} d^2 w_k \left\langle e^{\mu\gamma(z)} \beta^m(0) \prod_{k=1}^{s-1} \beta^m(w_k) \right\rangle \times \\
\times & \left\langle (-1)^m e^{-\bar{\mu}\bar{\gamma}(\bar{z})} \bar{\beta}^m(0) \prod_{k=1}^{s-1} \bar{\beta}^m(\bar{w}_k) \right\rangle \left\langle e^{2b(j+1)\phi'(z,\bar{z})} e^{2b\phi'(0)} \prod_{k=1}^{s-1} e^{2b\phi'(w_k,\bar{w}_k)} \right\rangle
\end{aligned}
\tag{276}
$$

In particular

$$
\begin{aligned}
& \left\langle e^{\mu\gamma(z)} \beta^m(0) \prod_{k=1}^{s-1} \beta^m(w_k) \right\rangle \left\langle (-1)^m e^{-\bar{\mu}\bar{\gamma}(\bar{z})} \bar{\beta}^m(0) \prod_{k=1}^{s-1} \bar{\beta}^m(\bar{w}_k) \right\rangle = \\
&= |u|^{2sm}\, e^{2b\Delta - 2bm\phi_c(0)} \frac{F(y|\tau)^m}{F(z|\tau)^m} \prod_{k=1}^{s-1} e^{mS_1 - 2bm\phi_c(w_k,\bar{w}_k)} \frac{F(y-w_k|\tau)^m}{F(z-w_k|\tau)^m}
\end{aligned}
\tag{277}
$$

and

$$
\begin{aligned}
\left\langle e^{2b(j+1)\phi'(z,\bar{z})} e^{2b\phi'(0)} \prod_{k=1}^{s-1} e^{2b\phi'(w_k,\bar{w}_k)} \right\rangle &= e^{2b(j+1)\phi_c(z,\bar{z})}\, e^{2b\phi_c(0)} \prod_{k=1}^{s-1} e^{2b\phi_c(w_k,\bar{w}_k)} \times \\
\times\ F(z|\tau)^{-2b^2(j+1)} \prod_{k=1}^{s-1} F(z-w_k|\tau)^{-2b^2(j+1)} & \prod_{k=1}^{s-1} F(w_k|\tau)^{-2b^2} \prod_{k<l}^{s-1} F(w_k-w_l|\tau)^{-2b^2}.
\end{aligned}
\tag{278}
$$

In this case the relations between the old parameters λ and μ and the new ones y and u are

$$\lambda = y - z, \tag{279}$$

$$\mu = \frac{u\,\theta(z-y|\tau)}{\theta'(0|\tau)}. \tag{280}$$

So we finally find the following integral expression

$$\langle \Phi_j(\mu|z) \rangle_{(\lambda,\tau)} \;=\; \frac{\Gamma(j+1)}{2b} \left(\frac{-b^2}{2\pi} \right)^{-1-j} e^{\frac{2\pi}{\tau_2} m(j+1)(\mathrm{Im}(z-y))^2}\, e^{-(j+1)2b\Delta}\, e^{2b(1-m)\phi_c(0)}\, e^{2b(j+1)\phi_c(z,\bar{z})}$$
$$\times\, F(z-y|\tau)^{m(j+1)}\, F(y|\tau)^m\, F(z|\tau)^{-2b^2(j+1)-m} \int_{\mathcal{C}} \prod_{k=1}^{j} d^2 w_k \prod_{k=1}^{j} \Big[e^{-2bm\phi_c(w_k,\bar{w}_k)}\, F(y-w_k|\tau)^m$$
$$\times\; F(z-w_k|\tau)^{-2b^2(j+1)-m}\, F(w_k|\tau)^{-2b^2} \prod_{l>k}^{j} F(w_k - w_l|\tau)^{-2b^2} \Big]. \tag{281}$$

This, when combined with the prefactor (261), gives an integral representation for the Liouville observable $\left\langle e^{2\alpha\phi} e^{-m\phi/b} \right\rangle$ on the torus. This quantity will be relevant for applications of our results to gauge theories in four dimensions. We will comment one of these applications in chapter 7.

5.6 An action for $b = 1$

Before concluding, let us make a remark about the action (124) in the limit $b = 1$, which is relevant for the gauge theory applications: Consider the following screening operator

$$V \;=\; \kappa(-\beta\bar{\beta})^m e^{2b\phi} \,+\, \widetilde{\kappa}(-\beta\bar{\beta})^{mb^{-2}} e^{2b^{-1}\phi}. \tag{282}$$

The relation between κ and $\widetilde{\kappa}$ is achieved by performing the global transformation

$$\phi \;\to\; \phi + \phi_0, \qquad \gamma \;\to\; \gamma e^{-\phi_0}, \qquad \beta \;\to\; \beta e^{\phi_0}. \tag{283}$$

These transformations on V yield

$$\frac{\delta\kappa}{\kappa} \;=\; b^2\, \frac{\delta\widetilde{\kappa}}{\widetilde{\kappa}}. \tag{284}$$

And integrating it we end up with

$$\widetilde{\kappa} \;=\; f(b^{-2})\, \kappa^{b^{-2}}, \tag{285}$$

where, as calculated in [25] for the sphere, the function $f_1(b^{-2})$ is

$$f(b^{-2}) \;=\; \frac{1}{\pi\gamma(b^{-2})}\big(\pi\kappa\gamma(b^2)\big)^{b^{-2}}. \tag{286}$$

Then, we have

$$V \;=\; \kappa(-\beta\bar{\beta})^m\Bigg[e^{2b\phi} \;+\; \big(\pi\kappa\gamma(b^2)\big)^{b^{-2}-1}\frac{\gamma(b^2)}{\gamma(b^{-2})}(-\beta\bar{\beta})^{m(b^{-2}-1)}e^{2b^{-1}\phi}\Bigg]. \tag{287}$$

Now, in order to study the limit $c \to 25$ and $b \to \infty$, we define

$$\epsilon \;\equiv\; 1 - b^{-2} \tag{288}$$

and

$$\kappa_{\rm ren} \;\equiv\; \kappa\gamma(b^2). \tag{289}$$

So, up to first order in ϵ, the operator is written as

$$V \;\sim\; \kappa_{\rm ren}(-\beta\bar{\beta})e^{2\phi}\Bigg[-2\phi - \log(\pi\kappa_{\rm ren}) - m\log(-\beta\bar{\beta})\Bigg]. \tag{290}$$

After the integration of the fields γ and β this potential in the lagrangian is changed to

$$V \;=\; \kappa_{\rm ren}|u|^{2m}|\mathcal{X}_0|^{2m}e^{2\phi}\Bigg[-2\phi - \log(\pi\kappa_{\rm ren}) - 2m\log(|u||\mathcal{X}_0|)\Bigg]. \tag{291}$$

where the function u is

$$u^{-1}\mu_\nu \;=\; \frac{\prod_{i=1}^{N-2}(z_\nu - y_i)}{\prod_{\mu\neq\nu,\mu=1}^{N}(z_\nu - z_\mu)} \tag{292}$$

and corresponds to the function (242) for the sphere. In the same way, $\mathcal{X}_0$ stands for the function

$$|\mathcal{X}_0|^{2m} \;=\; \frac{\prod_{i=1}^{N-2}|(w - y_i)|^{2m}}{\prod_{\nu=1}^{N}|(w - z_\nu)|^{2m}} \tag{293}$$

and corresponds to the function $\mathcal{X}_1$ (245) for the sphere.

It is easy to see that performing the following changes of variables

$$\phi \;\to\; \phi \;-\; m\log|u| \tag{294}$$

$$\phi \;\to\; \phi \;-\; m\log|\mathcal{X}_0| \tag{295}$$

we end up with the potential

$$V = \kappa_{\rm ren} e^{2\phi}\Bigg[-2\phi - \log(\pi\kappa_{\rm ren})\Bigg], \qquad (296)$$

that corresponds to the potential for the Liouville theory in the same limit.

Taking into account the path integral computation for the relation between the family of non-rational CFTs found by Ribault and the Liouville theory, it is straightforward to realize that considering this operator and performing these steps, the rest of the calculation is not changed. So in this limit this relation keeps on working.

We conjecture that the same happens in the torus. All the calculations will be equal but the functions $f_1(b^2)$ and $\mathcal{X}_0$. The first one is valid for the sphere and is calculated through the two point function. It should exist an equivalent one for the torus and it should be the same for every non-rational CFT of the family and for the Liouville theory. Meanwhile the function $\mathcal{X}_0$ must be replaced by $\mathcal{X}_1$.

VI

6 Extensions

In this chapter we postulate a new family of non-rational CFTs that is itself a generalization of the family of theories presented by Ribault in [3]. We show that these new theories are related to the sl(n) conformal Toda field theory in a similar way than Ribault's theories are related to Liouville field theory. Finally we show our attempt to achieve a lagrangian for sl(3) WZW model by extending one of these theories and relating its correlation functions to correlation functions of sl(3) Toda field theory.

6.1 $sl(n)$ conformal Toda field theory

The $sl(n)$ conformal Toda theory [43] is a theory whose degrees of freedom live in the $(n-1)$-dimensional space of the roots of $sl(n)$ algebra. There are $(n-1)$ simple roots e_1, e_2, ..., $e_{(n-1)}$ in this Lie algebra and its Cartan matrix is

$$K_{i,j} \;=\; (e_i, e_j) \;=\; \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \cdots & \cdots \\ 0 & -1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & -1 & 2 \end{pmatrix} \tag{297}$$

where the expression (α, β) corresponds to the scalar product.

The Weyl vector ρ of this algebra is such that $(e_k, \rho) = 1$ for every simple root.

The lagrangians that represent the $sl(n)$ Toda CFT contain $n-1$ boson fields as follows

$$S_T \;=\; \frac{1}{2\pi}\int d^2z\Big((\partial\varphi,\bar\partial\varphi)+\frac{(Q_T,\varphi)}{4}\mathcal{R}+\mu\sum_{k=1}^{n-1}e^{\sqrt{2}b(e_k,\varphi)}\Big) \tag{298}$$

where

$$Q_T \;=\; \sqrt{2}\Big(b+\frac{1}{b}\Big)\rho. \tag{299}$$

For every integer value n a different theory is defined. It is important to realize that the case $n=2$ is Liouville field theory. It is the first member of this family of theories and corresponds to the theory living on the one-dimensional space of roots of $sl(2)$.

Total normalization of the actions (298) is chosen in such a way, that

$$\varphi_i(z,\bar z)\varphi_j(w,\bar w) \;=\; -\delta_{ij}\log|z-w|^2 \;+\;\ldots \qquad \text{when} \;\; (z-w)\to\infty. \tag{300}$$

These theories present $(n-1)$ holomorphic and $(n-1)$ antiholomorphic conserved currents that generate the $\mathbf{W_n}\times\overline{\mathbf{W_n}}$ algebra, which contain Virasoro algebra as subalgebra. These currents are $\mathbf{W}^k$ and $\overline{\mathbf{W}}^{\mathbf{k}}$ for $\mathbf{k}=1,2,3,\ldots,n$. The index $\mathbf{k}$ is equal to the spin of the current. In [44] it is shown the Miura transformation that allows us to write these currents in functions of the field φ:

$$\prod_{i=0}^{n-1}(q\partial+(\varrho_{n-i},\partial\varphi)) \;=\; \sum_{k=0}^{n}\mathbf{W^{n-k}}(z)(q\partial)^k \tag{301}$$

where

$$q \;=\; b+\frac{1}{b} \tag{302}$$

and the vectors ϱ_k are the weights of the first fundamental representation π_1 of the Lie algebra $sl(n)$ with the highest weight ω_1

$$\varrho_k \;=\; \omega_1\;+\;e_1\;+\;e_2\ldots e_{n-1}. \tag{303}$$

The first and second currents are trivial, $\mathbf{W^0}\;=\;1$ and $\mathbf{W^1}\;=\;0$. The third one is the energy-momentum tensor

$$\mathbf{W^2}\;=\;T_T\;=\;-\frac{1}{2}(\partial\varphi)^2+(Q_T,\varphi)\,. \tag{304}$$

As special case we consider the case $n = 3$ corresponding to $sl(3)$ Toda theory and we introduce the projection of the field φ on the highest weights ω_1 and ω_2 as follows

$$\varphi_k = (\varphi, \omega_k). \tag{305}$$

In terms of these fields the current $\mathbf{W^2}$ in (304) has the form

$$T_T = (\partial\varphi_1)^2 + (\partial\varphi_2)^2 - \partial\varphi_1\partial\varphi_2 - \partial^2\varphi_1 - \partial^2\varphi_2 \tag{306}$$

and the fourth current is

$$\begin{aligned} \mathbf{W^3} = & \left(\partial\varphi_1(\partial\varphi_2)^2 + \partial\varphi_1\partial^2\varphi_1 - \frac{1}{2}\partial\varphi_1\partial^2\varphi_2 - \frac{1}{2}\partial^3\varphi_1\right) - \\ & - \left(\partial\varphi_2(\partial\varphi_1)^2 + \partial\varphi_2\partial^2\varphi_2 - \frac{1}{2}\partial\varphi_2\partial^2\varphi_1 - \frac{1}{2}\partial^3\varphi_2\right). \end{aligned} \tag{307}$$

The central charge for these conformal theories is

$$c_T = n - 1 + 12Q_T^2 = (n-1)\left(1 + n(n+1)\left(b + \frac{1}{b}\right)^2\right). \tag{308}$$

The basic objects of conformal Toda field theory are the following primary exponential fields parameterized by a $(n-1)$-component vector parameter α

$$V_\alpha^T = e^{2(\alpha,\varphi)} \tag{309}$$

with conformal weight

$$h_\alpha = (\alpha, Q_T - \alpha)\,. \tag{310}$$

Among the vertex operators there are a infinite discrete group of degenerate operators parameterized by two highest weight as follows

$$\alpha = -\frac{b\,\omega_1}{\sqrt{2}} - \frac{\omega_2}{\sqrt{2}\,b}\,. \tag{311}$$

6.2 A new family of non-rational conformal field theories

As an extension of the conformal Toda theories and as a generalization of the family of theories presented by Ribault, we propose the following action

$$\begin{aligned} S_{(\{M_k\},b)} &= \frac{1}{2\pi}\int d^2w\Bigg((\partial\phi,\bar{\partial}\phi) + \sum_{k=1}^{n-1}\left(\beta_k\bar{\partial}\gamma_k + \bar{\beta}_k\partial\bar{\gamma}_k\right) + \frac{(Q_{(\{M_k\},b)},\phi)}{4}\mathcal{R} + \\ &+ b^2\sum_{k=1}^{n-1}(-\beta_k\bar{\beta}_k)^{M_k}\, e^{\sqrt{2}b(e_k,\phi)}\Bigg) \end{aligned} \tag{312}$$

depending on a $(n-1)$-component boson field and $n-1$ beta-gamma systems. The theory is parameterized by $n-1$ numbers M_k and b. The background charge is $Q_{(\{M_k\},b)} = \sqrt{2}\left(b+\frac{1}{b}\right)\rho - \sum_{k=1}^{n-1}\frac{\sqrt{2}M_k}{b}\omega_k$.

Considering the free fields the conformal weight of an operator $e^{2(\alpha,\phi)}$ is $(\alpha, 2Q-\alpha)$. It is straightforward to see that the interaction part is composed by marginal operators of conformal weight equal to one.

The central charge is

$$c_{(\{M_k\},b)} = 2(n-1) + 12Q^2_{(\{M_k\},b)}. \tag{313}$$

All along this section we will consider

$$ds^2 = |f(z)|^2 dz\, d\bar{z}, \tag{314}$$

$$\sqrt{g}\mathcal{R} = -4\partial\bar{\partial}\log|f(z)|^2. \tag{315}$$

In the last two equations we wrote the function $f(z)$ instead of $\rho(z)$ as in (135). All throughout this section we will keep this notation in order to distinguish it from the $sl(n)$ Weyl vector.

We propose the following primary fields

$$\Phi_\alpha(\{\mu_k\}|z_\nu) = \prod_{k=1}^{n-1}|\mu_k|^{\frac{2M_k}{b}(\alpha,\omega_k)}\, e^{\sum_{k=1}^{n-1}(\mu_k\gamma_k - \bar{\mu}_k\bar{\gamma}_k)}\, e^{\sqrt{2}(\alpha,\phi)} \tag{316}$$

with

$$\alpha = \sum_{k=1}^{n-1}\sqrt{2}b(j_k+1)\omega_k. \tag{317}$$

The quantities we are interested to compute are the correlation functions

$$\Omega\left(\{\mu_k^\nu\}|z_\nu\right) \;=\; \left\langle \prod_{\nu=1}^{N} \Phi_{\alpha_\nu}(\{\mu_k^\nu\}|z_\nu) \right\rangle_{(\{M_k\},b)} \;=\; \int \mathcal{D}\phi \prod_{k=1}^{n-1} \mathcal{D}^2\beta_k \mathcal{D}^2\gamma_k \, e^{-S_{\{M_k\}}} \prod_{\nu=1}^{N} V_{\alpha_\nu}(\{\mu_k^\nu\}|z_\nu). \tag{318}$$

The integration over the fields γ_k and $\bar{\gamma}_k$ yields a δ-function that fix the conditions

$$\bar{\partial}\beta_k(w) \;=\; 2\pi \sum_{\nu=1}^{N} \mu_k^\nu \, \delta^2(w - z_\nu) \tag{319}$$

$$\partial\bar{\beta}_k(\bar{w}) \;=\; -2\pi \sum_{\nu=1}^{N} \bar{\mu}_k^\nu \, \delta^2(\bar{w} - \bar{z}_\nu). \tag{320}$$

These $2(n-1)$ equations have solution only if $\sum_{\nu=1}^{N} \mu_k^\nu = 0$ for every k. In order to write them we consider

$$\bar{\partial}\left(\frac{1}{z}\right) \;=\; \partial\left(\frac{1}{\bar{z}}\right) \;=\; 2\pi \, \delta^2(z). \tag{321}$$

The most general solutions can be written as following

$$\beta_k(w) \;=\; \sum_{\nu=1}^{N} \frac{\mu_k^\nu}{w - z_\nu} \;=\; u_k \, \frac{\prod_{i=1}^{N-2}(w - y_i^k)}{\prod_{j=1}^{N}(w - z_\nu)} \;\equiv\; u\, \mathcal{X}_0(y_i^k, z_\nu; w), \tag{322}$$

$$\mu_k^\nu \;=\; u_k \, \frac{\prod_{i=1}^{N-2}(z_\nu - y_i^k)}{\prod_{\mu\neq\nu}^{N}(z_\mu - z_\nu)}. \tag{323}$$

It all ends up with the following Dirac deltas for the integration of the fields β_k.

$$\delta^2\left(\bar{\partial}\beta_k(w) \,-\, 2\pi\sum_{\nu=1}^{N} \mu_k^\nu \delta^2(w - z_\nu)\right) \;=\; \delta^2\left(\sum_{\nu=1}^{N} \mu_k^\nu\right) \, \delta^2\left(\beta_k \,-\, u_k\, \mathcal{X}_0(y_i^k, z_\nu; w)\right). \tag{324}$$

Now we have

$$\begin{aligned}\Omega\left(\{\mu_k^\nu\}|z_\nu\right) \;=\; &\int \mathcal{D}\phi \, \exp\Bigg(-\frac{1}{2\pi} \int \Bigg((\partial\phi, \bar{\partial}\phi) + \frac{(Q_{(\{M_k\},b)}, \phi)}{4}\mathcal{R} + \\ &+ b^2 \sum_{k=1}^{n-1} |u_k|^{2M_k} \mathcal{X}_0^{2M_k} \sum_{k=1}^{n-1} e^{\sqrt{2}b(e_k,\phi)} \Bigg)\Bigg) \prod_{\nu=1}^{N}\prod_{k=1}^{n-1} |\mu_k^\nu|^{\frac{2M_k}{b}(\alpha_\nu,\omega_k)} \, e^{\sum_{k=1}^{n-1}\sqrt{2}(\alpha_k^\nu,\phi)}.\end{aligned} \tag{325}$$

In order to integrate the field ϕ we perform two change of variables. The first one is

$$\begin{aligned}
\phi &\to \phi - \sum_{k=1}^{n-1} \omega_k \frac{\sqrt{2}M_k}{b} \log|u_k|, \\
e^{\sqrt{2}b(e_k,\phi)} &\to e^{\sqrt{2}b(e_k,\phi)}|u_k|^{-2M_k}, \\
e^{\sqrt{2}(\alpha,\phi)} &\to e^{\sqrt{2}(\alpha,\phi)} \prod_{k=1}^{n-1} |u_k|^{-\frac{2M_k}{b}(\alpha,\omega_k)}, \\
(Q_{(\{M_k\},b)},\phi) &\to (Q_{(\{M_k\},b)},\phi) - \sum_{k=1}^{n-1} \frac{2M_k}{b}\left(b + \frac{1}{b} - \sum_{j=1}^{n-1}\frac{M_j}{b}(\omega_j,\omega_k)\right),
\end{aligned} \tag{326}$$

arriving to

$$\begin{aligned}
\Omega\left(\{\mu_k^\nu\}|z_\nu\right) = & \prod_{k=1}^{n-1} |u_k|^{\frac{2M_k}{b}\left(b+\frac{1}{b}-\sum_{j=1}^{n-1}\frac{M_j}{b}(\omega_j,\omega_k)\right)} \int \mathcal{D}\phi \, \exp\Bigg(-\frac{1}{2\pi}\int \Bigg((\partial\phi,\bar\partial\phi) + \frac{(Q_{(\{M_k\},b)},\phi)}{4}\mathcal{R} \\
& + b^2 \sum_{k=1}^{n-1} \mathcal{X}_0^{2M_k} e^{\sqrt{2}b(e_k,\phi)}\Bigg)\Bigg) \prod_{\nu=1}^{N}\prod_{k=1}^{n-1} |\mu_k^\nu|^{\frac{M_k}{b}(\alpha_\nu,\omega_k)} \, |u_k|^{-\frac{\sqrt{2}M_k}{b}(\alpha_\nu^M,\omega_k)} e^{\sqrt{2}(\alpha_\nu^M,\phi)}.
\end{aligned} \tag{327}$$

And the second change of variables is

$$\phi \to \varphi - \sum_{k=1}^{n-1} \omega_k \frac{M_k}{\sqrt{2}b}\left(\sum_{i=1}^{N-2}\log|w-y_i^k|^2 - \sum_{\nu=1}^{N}\log|w-z_\nu|^2 - \log|f(w)|^2\right). \tag{328}$$

This change produces the following modifications

$$\partial\bar\partial\phi \to \partial\bar\partial\varphi - \sum_{k=1}^{n-1}\omega_k\left(\frac{\sqrt{2}\pi M_k}{b}\left(\sum_{i=1}^{N-2}\delta^2(w-y_i^k) - \sum_{\nu=1}^{N}\delta^2(w-z_\nu)\right) - \frac{M_k}{\sqrt{2}b}\partial\bar\partial\log|f(w)|^2\right),$$

$$\begin{aligned}
e^{\sqrt{2}b(e_k,\phi(w))} &\to e^{\sqrt{2}b(e_k,\varphi(w))}\frac{\prod_{\nu=1}^{N}|w-z_\nu|^{2M_k}}{\prod_{i=1}^{N-2}|w-y_i^k|^{2M_k}}|f(w)|^{2M_k}, \\
e^{\sqrt{2}(\alpha_\nu,\phi(z_\nu))} &\to e^{\sqrt{2}(\alpha_\nu,\varphi(z_\nu))}\prod_{j-1}^{n-1}\prod_{k-1}^{n-1}\frac{\prod_{\mu\neq\nu}^{N}|z_\nu-z_\mu|^{2(j_j+1)M_k(\omega_j,\omega_k)}}{\prod_{i=1}^{N-2}|z_\nu-y_i^k|^{2(j_j+1)M_k(\omega_j,\omega_k)}},
\end{aligned}$$

$$
\begin{aligned}
(\phi,\partial\bar{\partial}\phi) \;\rightarrow\; & (\varphi,\partial\bar{\partial}\varphi) \;-\; \sum_{k=1}^{n-1}(\varphi,\omega_k)\frac{\sqrt{2}\pi M_k}{b}\left(\sum_{i=1}^{N-2}\delta^2(w-y_i^k)-\sum_{\nu=1}^{N}\delta^2(w-z_\nu)\right)+\\
&+\sum_{k=1}^{n-1}(\varphi,\omega_k)\frac{M_k}{\sqrt{2}b}\partial\bar{\partial}|f(w)|^2-\\
&-\sum_{k=1}^{n-1}(\omega_k,\partial\bar{\partial}\varphi)\frac{M_k}{\sqrt{2}b}\left(\sum_{i=1}^{N-2}\log|w-y_i^k|^2-\sum_{\nu=1}^{N}\log|w-z_\nu|^2-\log|f(w)|^2\right)+\\
&+\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}(\omega_k,\omega_j)\frac{\pi M_k M_j}{b^2}\left(\sum_{i=1}^{N-2}\log|w-y_i^k|^2-\sum_{\nu=1}^{N}\log|w-z_\nu|^2-\log|f(w)|^2\right)\times\\
&\times\left(\sum_{i=1}^{N-2}\delta^2(w-y_i^k)-\sum_{\nu=1}^{N}\delta^2(w-z_\nu)\right)-\\
&-\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}(\omega_k,\omega_j)\frac{M_k M_j}{2b^2}\left(\sum_{i=1}^{N-2}\log|w-y_i^k|^2-\sum_{\nu=1}^{N}\log|w-z_\nu|^2-\log|f(w)|^2\right)\partial\bar{\partial}\log|f(w)|^2.
\end{aligned}
\tag{329}
$$

Considering the regularization $\lim_{w\to z}\log|w-z|^2 \equiv -\log|f(z)|^2$ and replacing the last equation into the integral $S = -\frac{1}{2\pi}\int\ldots$ we obtain the following changes in the action and the following contributions to the correlation function:

From the second and fourth term:

$$
-\sum_{k=1}^{n-1}\sum_{i=1}^{N-2}\left(\varphi(y_i^k),\omega_k\right)\frac{\sqrt{2}M_k}{b} \;\rightarrow\; \prod_{k=1}^{n-1}\prod_{i=1}^{N-2} e^{-\frac{\sqrt{2}M_k}{b}(\omega_k,\varphi(y_i^k))}. \tag{330}
$$

From the third and fourth term:

$$
\frac{1}{8\pi}\int\sum_{k=1}^{n-1}(\varphi,\omega_k)\frac{\sqrt{2}M_k}{b}\mathcal{R} \;\rightarrow\; Q_{(\{M_k\},b)} \rightarrow Q_T \;=\; \sqrt{2}\left(b+\frac{1}{b}\right)\rho. \tag{331}
$$

From the fourth term:

$$
\sum_{k=1}^{n-1}\sum_{\nu=1}^{N}(\varphi(z_\nu),\omega_k)\frac{\sqrt{2}M_k}{b} \;\rightarrow\; \prod_{k=1}^{n-1}\prod_{\nu=1}^{N} e^{\frac{\sqrt{2}M_k}{b}(\omega_k,\varphi(z_\nu))}. \tag{332}
$$

From the sixth term:

$$
\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}(\omega_k,\omega_j)\frac{M_kM_j}{2b^2}\left(\sum_{i=1}^{N-2}\log|f(y_i^k)|^2-\sum_{\nu=1}^{N}\log|f(z_\nu)|^2\right) \;\rightarrow\; \prod_{k=1}^{n-1}\prod_{j=1}^{n-1}\prod_{i=1}^{N-2}|f(y_i^k)|^2\prod_{\nu=1}^{N}|f(z_\nu)|^2. \tag{333}
$$

From the fifth term:

$$-\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}(\omega_k,\omega_j)\frac{M_kM_j}{2b^2}\left(\sum_{i<i'}^{N-2}\log|y_{i'}^j-y_i^k|^2-\sum_{i=1}^{N-2}\sum_{\nu=1}^{N}\log|z_\nu-y_i^k|^2+\sum_{\mu<\nu}^{N}\log|z_\nu-z_\mu|^2\right)$$
$$\to\prod_{k=1}^{n-1}\prod_{j=1}^{n-1}\left(\Theta_N^{g=0}(y_i,z_\nu)\right)^{\frac{M_kM_j}{2b^2}(\omega_k,\omega_j)} \tag{334}$$

with

$$\left|\Theta_N^{g=0}(y_i,z_\nu)\right| = \prod_{\mu<\nu}^{N}|z_\mu-z_\nu|^2\prod_{i=1}^{N-2}\prod_{\nu=1}^{N}|z_\nu-y_i^k|^{-2}\prod_{i<i'}^{n-1}|y_{i'}^j-y_i^k|^2. \tag{335}$$

The last modification it must be considered is

$$\frac{(Q_{(\{M_k\},b)},\phi)}{4}\mathcal{R} \;\to\; \frac{(Q_{(\{M_k\},b)},\varphi)}{4}\mathcal{R}-$$
$$-\sum_{k=1}^{n-1}\frac{(Q_{(\{M_k\},b)},\omega_k)}{4}\mathcal{R}\frac{M_k}{\sqrt{2}b}\left(\sum_{i=1}^{N-2}\log|w-y_i^k|^2-\sum_{\nu=1}^{N}\log|w-z_\nu|^2-\log|f(w)|^2\right) \tag{336}$$

where

$$(Q_{(\{M_k\},b)},\omega_k) = \sqrt{2}\left(b+\frac{1}{b}-\sum_{j=1}^{n-1}\frac{M_j}{b}(\omega_j,\omega_k)\right). \tag{337}$$

Then, as $\mathcal{R}=-\frac{4}{\sqrt{g}}\partial\bar{\partial}\log|f(w)|^2$, we can use the method of parts twice in order to get the next contribution for the correlation function

$$\prod_{k=1}^{n-1}\left(\prod_{i=1}^{N-2}|f(y_i^k)|^{M_k\left(1+\frac{1}{b^2}\right)-\sum_{j=1}^{n-1}\frac{M_jM_k}{b^2}(\omega_j,\omega_k)}\prod_{\nu=1}^{N}|f(z_\nu)|^{-M_k\left(1+\frac{1}{b^2}\right)+\sum_{j=1}^{n-1}\frac{M_jM_k}{b^2}(\omega_j,\omega_k)}\right). \tag{338}$$

Finally we take into account

$$e^{\sqrt{2}(\alpha_\nu,\varphi(z_\nu))}e^{\sum_{k=1}^{n-1}\frac{\sqrt{2}M_k}{b}(\omega_k,\varphi(z_\nu))} = |f(z_\nu)|^{\sqrt{2}(j_\nu+1)M_k(\omega_j,\omega_k)}e^{\sqrt{2}\left(\alpha_\nu+\sum_{k=1}^{n-1}\frac{M_k}{2b},\varphi(z_\nu)\right)}. \tag{339}$$

We arrive to the relation between N-point correlation functions for this new theories and $(N+(N-2)(n-1))$-point function for Toda theory that consists in a generalization for the relation found by Ribault and Teschner.

$$\left\langle \prod_{\nu=1}^{N} \Phi_{\alpha_\nu}(\mu_\nu|z_\nu) \right\rangle_{(\{M_k\},b)} = \prod_{k=1}^{n-1} \delta\left(\sum_{\nu=1}^{N} \mu_k^\nu\right) \prod_{k=1}^{n-1} |u_k|^{M_k\left(1+\frac{1}{b^2}-\sum_{j=1}^{n-1}\frac{M_j}{b^2}(\omega_k,\omega_j)\right)} \times$$
$$\times \prod_{k=1}^{n-1}\prod_{j=1}^{n-1} \left(\Theta_N^{g=0}(y_i,z_\nu)\right)^{\frac{M_k M_j}{2b^2}(\omega_k,\omega_j)} \left\langle \prod_{\nu=1}^{N} V^T_{\alpha_\nu-\sum_{k=1}^{n-1}\frac{M_k}{\sqrt{2}b}\omega_k}(z_\nu) \prod_{k=1}^{n-1}\prod_{i=1}^{N-2} V^T_{-\frac{M_k}{\sqrt{2}b}\omega_k}(y_i^k) \right\rangle_T . \tag{340}$$

The fields $V^T_{-\frac{M_k}{\sqrt{2}b}\omega_k}(y_i^k)$ are degenerate fields of Toda theory.

This relation makes us suggest that these new theories are in principle solvable.

6.2.1 Symmetries of these new theories

The theories (312) presented in the previous section are a generalization of the theories (124) presented by Ribault, therefore it is thinkable that the symmetries of these new theories are also a generalization of the symmetries of the previous ones (132). These generalizations corresponds to $n-1$ Borel subalgebras, one for each simple root of the $sl(n)$ algebra. We will show in detail the simplest case corresponding to $n=3$:

The currents of the two Borel subalgebras of $sl(3)$ algebra are

$$\begin{aligned} J_1^+ &= \beta_1 \\ J_2^+ &= \beta_2 \\ J_1^0 &= \frac{1}{b}(e_1,\partial\phi) + \frac{2\gamma_1\beta_1}{M_1} - \frac{\gamma_2\beta_2}{M_2} \\ J_2^0 &= \frac{1}{b}(e_2,\partial\phi) - \frac{\gamma_1\beta_1}{M_1} + \frac{2\gamma_2\beta_2}{M_2} \end{aligned} \tag{341}$$

with the energy-momentum tensor

$$\mathbf{W^2} = T_{(M_1,M_2,b)} = -(\partial\phi,\partial\phi) - \beta_1\partial\gamma_1 - \beta_2\partial\gamma_2 + \left(Q_{(M_1,M_2,b)},\partial^2\phi\right). \tag{342}$$

The OPEs between the interaction term of the action (312) with each current (341) has no singular term up to a total derivation. This means that the theory is invariant under the transformations generated by these currents. It is important to mention that it is not the case with the current $\mathbf{W^3}$ presented in (307) so it is not a generator of the symmetry algebra of these theories.

The OPEs with singular terms between the currents (341) and the energy-momentum tensor are

$$\begin{aligned}
J_1^0(z)J_1^+(w) &\sim \frac{\frac{2}{M_1}J_1^+(w)}{z-w} + \dots\,, \\
J_1^0(z)J_2^+(w) &\sim \frac{-\frac{1}{M_1}J_2^+(w)}{z-w} + \dots\,, \\
J_1^0(z)J_1^0(w) &\sim \frac{-\frac{2}{b^2}-\frac{4}{M_1^2}-\frac{1}{M_2^2}}{(z-w)^2} + \dots\,, \\
J_2^0(z)J_1^+(w) &\sim \frac{-\frac{1}{M_2}J_1^+(w)}{z-w} + \dots\,, \\
J_2^0(z)J_2^+(w) &\sim \frac{\frac{2}{M_2}J_2^+(w)}{z-w} + \dots\,, \\
J_2^0(z)J_2^0(w) &\sim \frac{-\frac{2}{b^2}-\frac{1}{M_1^2}-\frac{4}{M_2^2}}{(z-w)^2} + \dots\,, \\
J_1^0(z)J_2^0(w) &\sim \frac{\frac{1}{b^2}+\frac{2}{M_1^2}+\frac{2}{M_2^2}}{(z-w)^2} + \dots\,, \\
T_{(M_1,M_2,b)}(z)J_i^{\pm,0}(w) &\sim \frac{J_i^{\pm,0}(w)}{(z-w)^2} + \frac{\partial_w J_i^{\pm,0}(w)}{z-w} + \dots
\end{aligned} \tag{343}$$

which correspond to the generalization of Borel subalgebra presented in (132). We have no prove about whether this symmetry is even bigger or not. And it is reasonable to suppose that the same happens for the case $n > 3$.

6.3 Towards an extension to $sl(3)$ affine theories

An atempt to extend the Liouville-WZW correspondence for the $sl(3)$ case, instead of $sl(2)$, was addressed by Ribault in [45]. Nevertheless, this resulted to be a highly non-trivial extension that did not work out perfectly so far. Let us describe our attempts to achieve so in our language:

It is known how to make a representation of $sl(3)$ affine algebra in terms of free fields due to Wakimoto. The idea is to introduce a certain number of free fields whose OPEs are known and build the currents of $sl(3)$ with them. Three free beta-gamma systems and a two-component boson field ϕ are needed. Their free contractions must be

$$\begin{aligned} \phi_i(z)\phi_j(w) &\sim -\delta_{i,j}\log|z-w| \\ \gamma_i(z)\beta_j(w) &\sim \frac{\delta_{i,j}}{(z-w)}. \end{aligned} \tag{344}$$

Then the eight currents that generate $sl(3)_k$ affine algebra can be written as follows

$$\begin{aligned} J_1^+ &= \beta_1 \\ J_2^+ &= \beta_2 - \gamma_1\beta_3 \\ J_3^+ &= \beta_3 \\ J_1^0 &= \frac{\sqrt{2}}{b}(e_1,\partial\phi) + 2\gamma_1\beta_1 - \gamma_2\beta_2 + \gamma_3\beta_3 \\ J_2^0 &= \frac{\sqrt{2}}{b}(e_2,\partial\phi) - \gamma_1\beta_1 + 2\gamma_2\beta_2 + \gamma_3\beta_3 \\ J_1^- &= -\frac{\sqrt{2}}{b}(e_1,\partial\phi)\gamma_1 + k\partial\gamma_1 + \gamma_3\beta_2 - \gamma_1\gamma_1\beta_1 + \gamma_1\gamma_2\beta_2 - \gamma_1\gamma_3\beta_3 \\ J_2^- &= -\frac{\sqrt{2}}{b}(e_2,\partial\phi)\gamma_2 + (k-1)\partial\gamma_2 - \gamma_3\beta_1 - \gamma_2\gamma_2\beta_2 \\ J_3^- &= -\frac{\sqrt{2}}{b}(e_1+e_2,\partial\phi)\gamma_3 + \frac{1}{b}(e_2,\partial\phi)\gamma_1\gamma_2 + k\partial\gamma_3 + (k-1)\gamma_1\partial\gamma_2 - \gamma_1\gamma_3\beta_1 - \\ &\quad - \gamma_2\gamma_3\beta_2 - \gamma_3\gamma_3\beta_3 - \gamma_1\gamma_2\gamma_2\beta_2. \end{aligned} \tag{345}$$

We propose an action with the aim of achieving a $sl(3)_k$ WZW model. This action is written in terms of the free fields used for the previous currents. The action we present is

$$\begin{aligned} S_{sl(3)} = \frac{1}{2\pi}\int d^2w \Big(& (\partial\phi,\bar\partial\phi) + \sum_{k=1}^{3}\left(\beta_k\bar\partial\gamma_k + \bar\beta_k\partial\bar\gamma_k\right) + \frac{(Q_{sl(3)},\phi)}{4}\mathcal{R} - \\ & - b^2\,(\beta_1-\gamma_2\beta_3)\left(\bar\beta_1-\bar\gamma_2\bar\beta_3\right)e^{\sqrt{2}b(e_1,\phi)} - b^2\,\beta_2\bar\beta_2\,e^{\sqrt{2}b(e_2,\phi)}\Big) \end{aligned} \tag{346}$$

with $Q_{sl(3)} = b\rho$. This corresponds to the action proposed in the previous subsection with $n = 3$ and $M_1 = M_2 = 1$, enlarged with an additional free field β_3 and γ_3 and a particular interaction term. The OPE of this interaction term with each current has no singular term up to a total derivation. This means that the theory is invariant under the transformation generated by these currents.

It is trivial to see that a Hamiltonian reduction, making J_1^+ and J_2^+ constant and making J_3^+ zero, gives the $sl(3)$-Toda field theory with symmetry W_3.

The primary fields must form a representation of $sl(3)$ algebra. We propose two different basis for them as a generalization of the $sl(2)$ algebra case. One example is the m-basis

$$V(J_1, M_1, J_2, M_2|z) \;=\; \gamma_1^{j_1-m_1}\gamma_2^{j_2-m_2}\gamma_3^{j_3-m_3}e^{2b(J_1+1)(\omega_1,\phi)+b(J_2+1)(\omega_2,\phi)} \tag{347}$$

which creates states parameterized by their position in the space of roots of $sl(3)$ algebra. This position is established in terms of parameters J_1, J_2, M_1 and M_2 as follows

$$\begin{aligned} J_1 &= \frac{1}{2}\left(2j_1 - j_2 + j_3\right) \\ J_2 &= \frac{1}{2}\left(-j_1 + 2j_2 + j_3\right) \\ M_1 &= \frac{1}{2}\left(2m_1 - m_2 + m_3\right) \\ M_2 &= \frac{1}{2}\left(-m_1 + 2m_2 + m_3\right) \end{aligned} \tag{348}$$

where the parameters J_1 and J_2 are the highest weights and M_1 and M_2 are the times the state is translated by the simple roots e_1 and e_2 respectively.

Another example of basis is the μ-basis, proposed as follows

$$\begin{aligned} V\left(J_1, J_2, \mu_1, \mu_2, \mu_3|z\right) \;=\;& |\mu_1|^{\frac{2}{b}(\alpha,\omega_1)}\,|\mu_2|^{\frac{2}{b}(\alpha,\omega_2)}\,|\mu_3|^{\frac{1}{b}(\alpha,\omega_1+\omega_2)}\; e^{\mu_1\gamma_1-\bar{\mu}_1\bar{\gamma}_1}\, e^{\mu_2\gamma_2-\bar{\mu}_2\bar{\gamma}_2}\, e^{\mu_3\gamma_3-\bar{\mu}_3\bar{\gamma}_3} \times \\ & \times\; e^{2b(J_1+1)(\omega_1,\phi)+b(J_2+1)(\omega_2,\phi)} \end{aligned} \tag{349}$$

It is in term of this μ-basis that we expect to achieve a relation between $sl(3)$ correlation functions and $sl(3)$ Toda field theory, proceeding with a path integration as used previously. Up to now we did not succeed in this point.

VII

7 Applications

While, as we discussed, the theory on the disk may have the application of describing extended objects like D-branes if a string theory interpretation for the action with $m > 1$ is eventually given, the theory on the torus may result relevant to study four-dimensional supersymmetric gauge theories. We will review these applications below.

7.1 Application to string theory

Non-rational two-dimensional CFTs have important applications in physics. Systems of condensed matter, lower-dimensional quantum gravity, and string theory are examples of this. In particular, the problem of considering non-rational models on surfaces with boundaries has direct applications to the description of D-branes in string theory on non-compact backgrounds.

The study of CFT on two-dimensional manifolds with boundaries was initiated by J. Cardy in his early work on minimal models [46, 47, 48], and more recently it was extended to non-rational models by the experts, notably by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov [49, 50]. The CFT description of branes in two-dimensional string theory, both in tachyonic and black hole backgrounds, and in three-dimensional string theory in Anti-de Sitter space (AdS), attracted the attention of the string theory community in the last ten years. The literature on string (brane) theory applications of boundary CFT is certainly vast, and we cannot afford to give a complete list of references herein. Instead, let us address the reader's attention to the lectures [51, 52, 53], to the list of renowned works [6, 24, 28, 31, 32, 34] and [54]-[78], and to the very interesting recent studies [79]-[83] on non-rational models on the disk geometry.

In this thesis, we considered the family of $c > 1$ non-rational two-dimensional CFTs recently proposed [3]. This family of theories is parameterized by two real numbers (m, b) in such a way that the corresponding central charges $c_{(b,m)}$ are given by $c_{(m,b)} = 1+6(b+b^{-1}(1-m))^2$. For this theory we explicitly computed expectation values of a vertex operator on the disk geometry. Our result generalizes the previous results for the $m = 0$ and $m = 1$ cases, which describe D-brane states in the worldsheet formulation of two-dimensional and three-dimensional string theory, in tachyonic, black holes, and Anti-de Sitter backgrounds. Whether or not the $m > 1$ disk one-point function we computed has a clear interpretation as D-brane states in string theory is an open question; nevertheless, it does represent an interesting generalization of those D-brane states.

We performed the calculation of the bulk one-point function in two different ways: First, we gave a path integral derivation. This consists of reducing the calculation of the expectation value of one bulk operator on the disk geometry to the analogous quantity in Liouville field theory, which, conveniently, is already known [49]. We closely followed the path integral techniques developed in [16]. Then, we performed a free-field calculations of the same one-point function, finding perfect agreement. As usual when dealing with the Coulomb gas representation in non-rational CFTs, an analytic extension was necessary for the free field calculation to reproduce the result for generic $m \notin \mathbb{Z}$. Remarkably, the free field approach turned out to exactly agree with the general expression, what was ultimately verified by comparing with the path integral computation. We also analyzed some functional properties of the formula we obtain, of which the one-point function in Liouville theory and in the WZNW theory are particular cases.

7.2 Applications to gauge theory

Of fundamental importance in theoretical physics is the question about non-perturbative effects in Yang-Mills theory (YM). In the last two decades, there has been important progress in this area, mainly due to our current understanding of the supersymmetric extensions of the theory. In the last few years, one of the most promising advances in the direction of understanding non-perturbative effects of superconformal YM theories has been the observation, due to Alday, Gaiotto, and Tachikawa [38], that the Nekrasov partition functions [84] of certain class of $\mathcal{N} = 2$ quiver theories in four dimensions turn out to be given by the conformal blocks of Liouville field theory. According to this, the full partition function of such gauge theories, meaning the partition function including instanton corrections, would be abstrusely encoded in the building blocks of a two-dimensional CFT whose observables we understand relatively well.

Specifically, Alday-Gaiotto-Tachikawa conjecture (AGT1) states that the n-point conformal blocks of Liouville field theory formulated on an n-punctured genus-g Riemann surface $\mathcal{C}_{g,n}$ give the Nekrasov partition function of the Gaiotto's quiver theory[3] $\mathcal{T}_{g,n}$ that is constructed, as in [85], by compactifying the six-dimensional (2,0) theory of the A_1 type on $\mathcal{C}_{g,n}$.

In this picture, the group of conformal mapping classes of the surface $\mathcal{C}_{g,n}$ coincides with the group of duality symmetries of the theory $\mathcal{T}_{g,n}$, meaning that different ways of factorizing the Liouville conformal blocks and sewing the Riemann surface correspond to different Lagrangian representations of the gauge theory. This is to say, in particular, that crossing symmetry in 2D corresponds to duality symmetry in 4D.

By construction, theories $\mathcal{T}_{g,n}$ are superconformal quivers for the gauge group SU(2), consisting of $3g-3+n$ SU(2) gauge fields with coupling constants q_i $(i = 1, 2, ...3g-3+n)$ coupled to n matter hypermultiplets with mass parameters m_a $(a = 1, 2, ...n.)$ The couplings q_i of $\mathcal{T}_{g,n}$ are in correspondence with the modular parameters τ_i (and/or cross-ratios z_i) of $\mathcal{C}_{g,n}$ [85]. In addition, the AGT dictionary [38] relates the mass parameters m_a of $\mathcal{T}_{g,n}$ to the conformal dimensions Δ_{α_a} of the n Liouville primary fields $e^{2\alpha_a\phi(z_a)}$ involved in the n-point conformal block, while also relates the $3g-3+n$ vevs in the $\mathcal{N} = 2$ SU(2) quiver to the momenta of intermediate states in the conformal block. Meant to establish a connection between the Nekrasov partition function of 4D SCFTs and observables of the 2D CFT, AGT dictionary also identifies to which parameters of Liouville field theory the Nekrasov deformation parameters ε_1 and ε_2 correspond. The central charge of Liouville field theory is given by $c = 1+6Q^2$, with $Q = b+1/b$, identifying

[3]It is important not to mistake the Gaiotto's 4D theories $\mathcal{T}_{g,n}$ with the Ribault's 2D theories $\mathcal{T}_{m,b}$.

$b^2 = \varepsilon_1/\varepsilon_2$; while the Liouville cosmological constant can be written as $\mu = \varepsilon_1\varepsilon_2$. The limit $\varepsilon_1 = \varepsilon_2 \to 0$, being totally well defined from the gauge theory point of view, on the Liouville theory side represents the limit of infinite momenta. In the Matrix Model formulation, on the other hand, this limit corresponds to $g_s \to 0$; see [90, 91].

Typical examples of this 4D/2D correspondence are the SU(2) $\mathcal{N} = 4$ SYM theory and the SU(2) $\mathcal{N} = 2$ Seiberg-Witten theory, which correspond to Liouville field theory formulated on the Riemann surfaces $\mathcal{C}_{1,0}$ and $\mathcal{C}_{0,4}$, respectively. Besides, if a single matter hypermultiplet is added to the $\mathcal{N} = 4$ theory, one gets the so-called $\mathcal{N} = 2^*$ SYM, whose partition function thus corresponds to Liouville field theory on $\mathcal{C}_{1,1}$. Is the latter theory with which we have been concerned in this thesis.

It is also interesting to mention that in the case the gauge theory happens to be defined on the four-sphere $\mathbb{S}^4$, instead of being so on $\mathbb{R}^4$, its Nekrasov partition function then corresponds to full n-point Liouville correlation functions on $\mathcal{C}_{g,n}$ with $b = 1$. This is understood as follows: The compact four-dimensional space can be thought of as being built by bringing together two $\mathbb{R}^4$ contributions and then integrating out the fields with the appropriate measure [92]. It turns out that, on the 2D CFT side, such procedure corresponds to assembling a holomorphic and anti-holomorphic copies of the conformal block and integrating over the internal momenta weighted with the product of Liouville structure constant, yielding in this way the full correlation function.

AGT conjecture was extensively studied in the last two years, and all kind of consistency checks, particular examples, generalizations, and applications were worked out. In particular, the generalization of AGT to the case of the gauge group to be $SU(n)$ with $n \geq 2$ was studied in [86], where it was shown that in such case Liouville field theory has to be replaced by the affine $sl(n)_k$ Toda field theory with level $k = n + 1/b^2$.

Soon after [38] appeared, the extension of the AGT conjecture to the case of incorporating both loop and surface operators on the gauge theory side was proposed [39, 40]. In this generalized picture, not only the partition function, but also expectation values of defects in the

4D theory happen to be described by Liouville correlation functions. For instance, in the case of having a fundamental surface operator in the gauge theory $\mathcal{T}_{g,n}$, the corresponding Liouville observable turns out to be a $(n+1)$-point correlation function whose $n+1^{\text{th}}$ additional operator is the Liouville degenerate field of conformal dimension $\Delta = -\frac{1}{4}(2+3b^{-2})$, namely the vertex $e^{-\varphi(x)/b}$. The complex variable x, which from the 2D CFT point of view represents the point on the Riemann surface where the degenerate field is inserted, on the gauge theory side it labels the quantum numbers of the surface operator. In the case of fundamental operators in the $\mathcal{N}=4$ theory, the physical interpretation of the complex x is relatively well understood; it gives two real variables that are ultimately associated to the magnetic flux and a θ-angle of a singular vortex solution that the surface operator creates. From the M-theory point of view, the surface operators correspond to configurations that are localized at the point x on $\mathcal{C}_{g,n}$, that is, at the location of the $n+1^{\text{th}}$ puncture on the Riemann surface.

More recently, it has been observed that the 2D CFT description of expectation values of defects in the gauge theory is naturally realized in terms of CFTs with affine symmetry [89]. For the case of SU(2) gauge theories, this involves CFT with $\text{sl}(2)_k$ affine Kac-Moody symmetry, with $k = \varepsilon_1/\varepsilon_2 + 2$; while CFTs with sl(N)_k symmetry are associated to SU(N) theories. In some sense, it is fair to say that, while Liouville field theory stands as the convenient language to represent the $\mathcal{N}=2$ gauge theory partition function, the expectation values of defects in such theories are more conveniently described by conformal blocks of 2D CFTs with affine symmetry; at least it seems to be the case for the simplest defects. One of the motivations of this thesis was to propose an extension of such affine CFT realization to the case of non-fundamental surface operators. We will focus on the case of $\mathcal{N}=2^*$ YM theory for SU(2), for which such an expectation value is given by the Liouville two-point function on the torus.

Our current understanding is that non-fundamental surface operators labeled by an integer number $m \geq 1$ exist in these SU(2) $\mathcal{N}=2$ gauge theories. Such an operator exhibits $m+1$ vacua, and it corresponds to a fundamental operator if $m=1$. Expectation value of a surface operator would admit a 2D CFT description in terms of a Liouville correlation function with the additional insertion of a degenerate field of conformal dimension $\Delta = -\frac{m}{2}(1 + b^{-2}(1 + \frac{m}{2}))$, namely the vertex $e^{-m\varphi(x)/b}$. Unconveniently, a purely gauge theory description of the surface operators that correspond to $m > 1$ is still missing. According to the analysis of [87], one may think that adding such operator should correspond to coupling the gauge theory to a two-dimensional sigma-model; nevertheless, the target space of such sigma-model has not yet been identified and thus, to the best of our knowledge, the problem of generalizing the analysis of [87] to the case $m > 1$ remains an open question; see also [88].

Without a complete description of defects from the gauge theory point of view, it is worthwhile studying the problem from different perspectives. With the aim of contributing to the study of non-fundamental surface operators in the 4D $\mathcal{N} = 2$ SCFTs, in future works we will draw the attention to a yet unexplored integral representation of this kind of defects based on the 2D CFT description of 4D gauge theories à la AGT. We will be concerned with the particular case of SU(2) $\mathcal{N} = 2^*$ SYM. Resorting to some CFT results, we will argue that the expectation value of a non-fundamental surface operator (labeled by an integer m) in the $\mathcal{N} = 2^*$ theory is actually given by the expectation value of a single vertex operator in a 2D CFT which has central charge $c_{(m,b)} = 3 + 6(b^{-1} + (1 - m)b)^2$, with $b^2 = \varepsilon_1/\varepsilon_2$, and being formulated on the torus.

The strategy we propose to adopt in order to study gauge theories using the results of this thesis is the following: In this thesis we revisited and generalized results about the connection between Liouville correlation functions and correlation functions in two-dimensional CFTs that exhibit a kind of affine symmetry (or part of it.) We considered these theories on the n-punctured torus and then our results could be used to, upon AGT conjecture, addressing the problem of computing expectation values of non-fundamental surface operators in the SU(2) $\mathcal{N} = 2^*$ SYM, which corresponds to $n = 1$. Such an operator, as said, is labeled by an integer number m, and its expectation value would be associated to the Liouville 2-point function $\left\langle e^{2\alpha\varphi} e^{-m\varphi/b} \right\rangle$ on the torus. We have seen that such expectation value can also be identified with the expectation value of a single vertex operator $\langle \Phi_h \rangle$ in a 2D CFT with the central charge $c_{(b,m)}$ given above. Then, since each member of this m-parameterized family of CFTs admits a Coulomb gas representation, it provides us with an integral representation of non-fundamental defects in the gauge theory. In the case $m = 1$, which corresponds to fundamental surface operators, the 2D CFT is identified with the $sl(2)_k$ affine theory, suggesting a connection with the work [89]. Coulomb gas like integral representations were already considered in the literature in the context of the $\mathcal{N} = 2^*$ case (see for instance [93, 94, 95, 96] and references therein and thereof), but the one considered here would be, in principle, of a different sort. The representation we propose is based on a close relation that exists between Liouville correlation function and Wess-Zumino-Witten (WZW) correlation functions: As we discussed in the thesis, the Liouville $(2N+2g-2)$-point correlation functions on a genus-g Riemann surface that involve the insertion of $2g-2+N$ degenerate fields $e^{-\varphi/b}$ coincide with the N-point functions of the $\mathrm{sl}(2)_k$ WZW theory, provided certain specific dictionary is given between the quantum numbers of both theories. Along the same line, it was shown in [3] for the case $g = 0$ that Liouville $(2N-2)$-point correlation functions with $N-2$ degenerate fields $e^{-m\varphi/b}$, being m an arbitrary positive integer, seem to give the N-point correlation functions of certain CFT with central charge $c_{(m,b)} = 3+6(b^{-1}+(1-m)b)^2$. This CFT, if it exists for generic $m \in \mathbb{Z}_{>1}$, would coincide with the $\mathrm{sl}(2)_k$ WZW theory for $m = 1$ for $b^2 = (k-2)^{-1}$ (and with its Langlands dual $\mathrm{sl}(2)_{k^L}$ for $m = b^2 = k^L - 2$.) Such m-labeled family of CFTs was conjectured to exist in reference [3] and was extensively studied and substantially extended herein; we are arguing now that it has a concise application to describe obervables in 4D gauge theories: If concerned with the $\mathcal{N} = 2^*$ SYM theory, and thus with the 2D CFT on the torus, first we needed to solve a preliminary problem: we needed to undertake the task of extending the result of [3] to the case $g > 0$, meaning to extend the result of [16] to $m > 1$. This is what we did in Chapter 4, where we prove that, as expected, the torus Liouville $2n$-point functions that involve n degenerate fields $e^{-\varphi m/\sqrt{2}b}$ coincide with the torus n-point function of the m^{th} member of the family of CFTs proposed in [3]. From the CFT point of view, this result is interesting in its own right as it provides further evidence of the consistency of the construction in [3]. We also performed the free field calculation of these n-point function on the torus. The case of the one-point function, which would coincide with a Liouville two-point function is ultimately associated to the expectation value of a non-fundamental m-labeled surface operator in $\mathcal{N} = 2^*$ SYM, providing in this way a novel integral representation of such observable in the gauge theory. The integral representation, following from a Coulomb gas realization of a 2D CFT, turns out to be given in terms of modular functions made out of θ-functions.

A question remains open as to how to understand the symmetries of CFT with $m > 1$ from the 4D gauge theory point of view. Understanding this point is still an open problem.

VIII

8 Conclusions

8.1 Summary of our results

In this thesis we studied a new class of non-rational conformal field theories. These theories were originally conjectured to exist in 2008 by Ribault, who proposed a form for the N-point correlation functions in the simplest manifold, in the sphere topology. He did so by writing the correlation functions for these new theories in terms of correlation functions of the better understood Liouville theory. We continued the task initiated by Ribault considering the theories on manifolds with less simple topology, considering the inclusion of both boundaries and handles. More precisely, we proved that there is a consistent way of defining the same kind of theories that Ribault proposed both in the disk and in the torus. For these two Riemann surfaces we computed correlation functions explicitly, both resorting to the path integral and the free fields approaches. Exact agreement is found between the two formalisms. Our results consequently generalize both the Liouville and the Wess-Zumino-Witten correlation functions, which are indeed special cases (i.e. the cases $m = 0$ and $m = 1$) of our formulas. We also discussed extensions of Ribault's construction to the case Liouville theory is replaced by $sl(n)$ Toda conformal field theory with $n > 2$. The attempts to generalize the so-called WZW-Liouville correspondence to the case of $sl(3)$ WZW model in the path integral approach are commented. Below we give details about our results for the computations we performed on Riemann surfaces with boundaries and with handles, and also comment on their possible applications to string theory and gauge theory.

8.2 The computation on the disk geometry

We extended the result of reference [3] to the case of Riemann surface with boundaries. This task was actually pointed out in [3] as a further work that required to be done; so we accomplished to complete the work by considering the theory on the disk geometry. We showed that, indeed, correlation functions on the new family of non-rational CFTs defined by Ribault formulated on the disk can be written in terms of Liouville correlation functions on the same geometry. We explicitly computed the expectation value of a vertex operator on the bulk of the disk. A question that remains open is whether such observable can actually be interpreted as a D-brane state of a string theory background for generic values of m as it happens with the values $m = 0, 1, b^2$.

8.3 The computation on the torus

We have pointed out the existence of a yet unexplored integral representation that could serve to describe expectation values of defects in the $SU(2)$ $\mathcal{N} = 2^*$ super Yang-Mills. This was done by first generalizing the results of [16] to the case $m \geq 1$; namely proving that the relation between different non-rational CFTs conjectured in [3] is easily extended to genus-one. Our result is expressed in formula (263), which relates Liouville $2N$-point functions involving N degenerated fields (i.e. those fields that have null descendants at level m in the Virasoro Verma modulo) to N-point functions of the 2D CFTs defined by action (124). According to AGT conjecture, for $N = 1$ the former correlation functions are associated to observables in the $\mathcal{N} = 2^*$ gauge theory, so that a formula like (281) happens to provide a new Coulomb gas realization of the problem. Whether or not integral representation (281) actually results to be useful to address a specific question on the gauge theory side is not clear to us. Nevertheless, we found opportune to point out that such CFT description of the problem exists, as it might provide a useful tool to study non-fundamental defects in the gauge theories from a different perspective.

Something that did not avoid our attention is the reminiscence between our analysis for the case $m = 1$ and the observation made in reference [89], which is also meant to describe expectation value of surface operators in terms of the $sl(2)_k$ affine theory. Understanding the connection between our approach and the one in [89] is matter of further investigation. To this regard, let us point out that the relation between Liouville field theory and the $sl(2)_k$ affine theory we considered actually works at the level of chiral conformal blocks as well, provided a proof of it based on the Bernard-Knizhnik-Zamolodchikov equation exists [16].

The main result of this thesis can be condensed in the following corollary: We have shown that the non-rational two-dimensional conformal field theories conjectured to exist by Ribault, who in 2008 gave evidence in favor of their existence for the sphere topology, can be consistently defined in Riemann surfaces with non-trivial topology; namely, in surfaces both with boundaries and with handles. The explicit formulas we derived in this thesis both for the case of the disk and the torus geometries generalize the expectation values and correlation functions of both Liouville and Wess-Zumino-Novikov-Witten theories, which are actually particular examples of Ribault's new models.

References

[1] J. P. Babaro and G. Giribet, JHEP **1009** (2010) 077.

[2] J. P. Babaro and G. Giribet, in preparation.

[3] S. Ribault, JHEP **0805** (2008) 073.

[4] P. Di Francesco, P. Mathieu and D. Senechal, Springer, (1997).

[5] P. H. Ginsparg, Les Houches Summer School (1988) 1-168.

[6] Y. Nakayama, Int.J.Mod.Phys. **A19** (2004) 2771.

[7] H. Dorn and H. J. Otto, Phys. Lett. **B291** (1992) 39.

[8] M. Goulian and M. Li, Phys. Rev. Lett. **66** (1991) 2051.

[9] P. Di Francesco and D. Kutasov, Nucl. Phys. **B375** (1992) 119.

[10] V. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B251** (1985) 691.

[11] M. Wakimoto, Comm. Math. Phys. **104** (1986) 605.

[12] K. Hosomichi, K. Okuyama and Y. Satoh, Phys. Nucl. Phys. **B598** (2001) 451.

[13] N. Ishibashi, K. Okuyama and Y. Satoh, Nucl. Phys. **B 588** (2000) 149.

[14] A. Stoyanovsky, arXiv:math-ph/0012013.

[15] S. Ribault and J. Teschner, JHEP. **0506** (2005) 014.

[16] Y. Hikida and V. Schomerus , JHEP. **0710** (2007) 064.

[17] V. Fateev and S. Ribault, JHEP **0802** (2008) 024.

[18] G. Giribet, Y. Nakayama and L. Nicolas, Int. J. Mod. Phys. **A24** (2009) 3137.

[19] J. Teschner, arXiv:hep-th/0009138.

[20] V. Fateev, A. B. Zamolodchikov and Al. Zamolodchikov, arXiv:hep-th/0001012.

[21] S. Ribault, JHEP **0608** (2006) 015.

[22] K. Hosomichi and S. Ribault, JHEP **0701** (2007) 057.

[23] K. Hosomichi, Nucl. Phys. Proc. **171** (2007) 284.

[24] B. Ponsot, V. Schomerus and J. Teschner, JHEP **0202** (2002) 016.

[25] G. Giribet and C. Nuñez, JHEP **0106** (2001) 010.

[26] K. Becker and M. Becker, Nucl. Phys. **B418** (1994) 206.

[27] S. Ribault, JHEP **0509** (2005) 045.

[28] J. Teschner, Class. Quant. Grav. **18** (2001) R153.

[29] J. Teschner, Int. J. Mod. Phys. **A19S2** (2004) 436.

[30] S. Stanciu, JHEP **9909** (1999) 028.

[31] P. Lee, H. Ooguri, J. Park and J. Tannenhauser, Nucl. Phys. **B610** (2001) 03.

[32] C. Bachas and M. Petropoulos, JHEP **0102** (2001) 025.

[33] A. Parnachev and D. Sahakyan, JHEP **0110** (2001) 022.

[34] D. Israel, JHEP **0506** (2005) 008.

[35] S. Ribault, JHEP **0801** (2008) 004.

[36] G. Giribet, Nucl. Phys. **B737** (2006) 209.

[37] Y. Hikida and V. Schomerus, JHEP **0903** (2009) 095.

[38] L. F. Alday, D. Gaiotto and Y. Tachikawa, Lett. Math. Phys. **91** (2010) 167.

[39] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, JHEP **1001** (2010) 113.

[40] N. Drukker, J. Gomis, T. Okuda and J. Teschner, JHEP **1002** (2010) 057.

[41] T. Eguchi and H. Ooguri, Nucl. Phys. **B282** (1987) 308.

[42] K. Gawedzki, arXiv:hep-th/9110076.

[43] V. A. Fateev and A.V. Litvinov, JHEP **0711** (2007) 002.

[44] V. A. Fateev and S. L. Lukyanov, Int. J. Mod. Phys. **A3** (1988) 507.

[45] S. Ribault, JHEP **0910** (2009) 002.

[46] J. Cardy, Nucl. Phys. **B240** [FS12] (1984) 514.

[47] J. Cardy, Nucl. Phys. **B275** (1986) 200.

[48] J. Cardy, Nucl. Phys. **B 324** (1989) 581.

[49] V. Fateev, A. B. Zamolodchikov and Al. Zamolodchikov, arXiv:hep-th/0001012.

[50] A. B. Zamolodchikov and Al. Zamolodchikov, arXiv:hep-th/0101152.

[51] C. Schweigert, J. Fuchs and J. Walcher, arXiv:hep-th/0011109.

[52] V. Petkova and J-B. Zuber, arXiv:hep-th/0103007.

[53] V. Schomerus, Phys. Rept. **431** (2006) 39.

[54] J. McGreevy and H. Verlinde, JHEP **0312** (2003) 054.

[55] I. Klebanov, J. Maldacena and N. Seiberg, JHEP **0307** (2003) 045.

[56] J. McGreevy, J. Teschner and H. Verlinde, JHEP **0401** (2004) 039.

[57] S. Fredenhagen and V. Schomerus, JHEP **0505** (2005) 025.

[58] S. Stanciu, JHEP **9909** (1999) 028.

[59] A. Giveon, D. Kutasov and A. Schwimmer, Nucl. Phys. **B615** (2001) 133.

[60] A. Parnachev and D. Sahakyan, JHEP **0110** (2001) 022.

[61] Y. Hikida and Y. Sugawara, Prog. Theor. Phys. **107** (2002) 1245.

[62] P. Lee, H. Ooguri and J. Park, Nucl. Phys. **B632** (2002) 283.

[63] B. Ponsot and S. Silva, Phys. Lett. **B551** (2003) 173.

[64] M. Gutperle and A. Strominger, Phys. Rev. **D67** (2003) 126002.

[65] J. Teschner, JHEP **0404** (2004) 023.

[66] S. Ribault and V. Schomerus, JHEP **0402** (2004) 019.

[67] S. Ribault, arXiv:hep-th/0207094.

[68] S. Ribault, JHEP **0305** (2003) 003.

[69] Y. Nakayama, JHEP **0311** (2003) 017.

[70] Y. Nakayama, Soo-Jong Rey and Y. Sugawara, JHEP **0509** (2005) 020.

[71] Y. Nakayama, arXiv:hep-th/0702221.

[72] D. Israel, A. Pakman and J. Troost, Nucl. Phys. **B710** (2005) 529.

[73] D. Israel, A. Pakman and J. Troost, Nucl. Phys. **B722** (2005) 3.

[74] S. Ribault, JHEP **0608** (2006) 015.

[75] K. Hosomichi and Ribault, JHEP **0701** (2007) 057.

[76] S. Ribault, JHEP **0801** (2008) 004.

[77] K. Hosomichi, Nucl. Phys. Proc. Suppl. **171** (2007) 284.

[78] K. Hosomichi, JHEP **0806** (2008) 029.

[79] T. Creutzig and Y. Hikida, arXiv:1004.1977.

[80] T. Creutzig, arXiv:0908.1816.

[81] T. Creutzig, Nucl. Phys. **B812** (2009) 301.

[82] T. Creutzig and V. Schomerus, Nucl. Phys. **B807** (2009) 471.

[83] T. Creutzig, T. Quella and V. Schomerus, Nucl. Phys. **B792** (2008) 257.

[84] N. Nekrasov, Adv. Theor. Math. Phys. **7** (2004) 831.

[85] D. Gaiotto, arXiv:0904.2715.

[86] N. Wyllard, JHEP **0911** (2009) 002.

[87] N. Drukker, D. Gaiotto and J. Gomis, JHEP **1106** (2011) 025.

[88] T. Dimofte, S. Gukov and L. Hollands, arXiv:1006.0977.

[89] L. Alday and Y. Tachikawa, Lett. Math. Phys. **94** (2010) 87.

[90] R. Dijkgraaf and C. Vafa, arXiv:0909.2453.

[91] M. Cheng, R. Dijkgraaf and C. Vafa, JHEP **1109** (2011) 022.

[92] V. Pestun, arXiv:0712.2824.

[93] A. Mironov, A. Morozov and Sh. Shakirov, JHEP **1002** (2010) 030.

[94] A. Mironov, A. Morozov and Sh. Shakirov, J. Phys. **A44** (2011) 085401.

[95] K. Maruyoshi and F. Yagi, JHEP **1101** (2011) 042.

[96] G. Bonelli, K. Maruyoshi, A. Tanzini and F. Yagi, arXiv:1011.5417.