Productos tensoriales simétricos: teoría métrica, isomorfa y aplicaciones

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Resumen

Esta tesis tiene como objeto contribuir al desarrollo de la teoría métrica e isomorfa de productos tensoriales simétricos en espacios de Banach. Mostramos varios ejemplos donde la teoría de ideales de polinomios homogéneos resulta enriquecida con el uso de técnicas tensoriales.

Probamos que la extensión de Aron-Berner preserva la norma para todo ideal maximal y minimal de polinomios homogéneos. Este resultado puede interpretarse como una versión polinomial de uno de los "Cinco Lemas Básicos" de la teoría de productos tensoriales. Más aún, enunciamos y probamos análogos simétricos de dichos lemas y damos, a lo largo del texto, varias aplicaciones.

Estudiamos las cápsulas inyectivas y projectivas de una norma tensorial simétrica, analizando sus propiedades y relaciones. Describimos los ideales de polinomios maximales asociados a dichas normas en términos de ideales de composición e ideales cocientes. Examinamos las normas naturales de Grothendieck en el n-ésimo producto tensorial simétrico y mostramos que, para $n \geq 3$, hay exactamente seis de ellas, a diferencia del caso $n = 2$ donde hay cuatro.

Definimos la propiedad de Radón-Nikodým simétrica para normas s-tensoriales y mostramos, bajo ciertas hipótesis, que los ideales de polinomios maximales asociados a normas con dicha propiedad coinciden isométricamente con su núcleo minimal. Como consecuencia, probamos la existencia de ciertas estructuras en algunos ideales de polinomios clásicos (existencia de bases o la propiedad de Radon-Nikodým). Por otra parte, damos una demostración alternativa del hecho que el ideal de los polinomios integrales coincide isométricamente con el ideal de los polinomios nucleares en espacios Asplund.

Analizamos la existencia de bases incondicionales en ideales de polinomios. Para esto, estudiamos incondicionalidad en productos tensoriales simétricos. Damos un criterio sencillo para determinar si un ideal de polinomios carece de base incondicional. Utilizando dicho criterio mostramos que muchos de los ideales usuales no poseen estructura incondicional. Entre ellos, los r-integrales, r-dominados, extendibles y r-factorizables. Para muchos de estos ejemplos obtenemos incluso que la sucesión básica monomial no es incondicional.

Estudiamos la preservación de otro tipo de estructuras en el producto tensorial simétrico: la estructura de álgebra de Banach y la estructura de M-ideal. Mostramos cuáles de las normas s-tensoriales de Grothendieck preservan la estructura de álgebra. Por otra parte, probamos que la norma inyectiva simétrica destruye la estructura de M-ideal (opuesto a lo que pasa en el producto tensorial completo con la norma inyectiva). Si bien dicha estructura se pierde en el caso simétrico, mostramos que, si E es Asplund y M-ideal en F, entonces los polinomios integrales sobre E se extienden a F preservando la norma de manera única.

Palabras clave: Productos tensoriales simétricos, normas s-tensoriales, ideales de polinomios, polinomios homogéneos, estructuras en productos tensoriales.

Symmetric tensor products: metric and isomorphic theory and applications

Abstract

This thesis aims to contribute to the development of the metric and isomorphic theory of symmetric tensor products on Banach spaces. We show several examples where the theory of polynomial ideals is enriched with the use of tensor techniques.

We prove that the Aron-Berner extension preserves the norm for every maximal and minimal ideal of homogeneous polynomials. This result can be interpretated as a polynomial version of one of the “Five basic Lemmas” of the theory of tensor products. Moreover, we state and prove symmetric analogues of these lemmas and give, throughout the text, several applications.

We study the injective and projective associates of a symmetric tensor norm, analyzing their properties and relations. We describe the maximal polynomial ideals associated with these norms in terms of composition ideals and quotient ideals. We examine Grothendieck's natural norms on the n-fold symmetric tensor product and show that there are exactly six natural symmetric tensor norms for $n \geq 3$, unlike the 2-fold case in which there are four.

We define the symmetric Radón-Nikodým property for s-tensor norms and show, under certain hypothesis, that maximal polynomial ideals associated with norms with this property coincide isometrically with their minimal kernel. As a consequence, we prove the existence of certain structures on some classical polynomial ideals (existence of basis or the Radón-Nikodým property). On the other hand, we give an alternative proof of the fact that the ideal of integral polynomials coincide isometrically with the ideal of nuclear polynomials on Asplund spaces.

We analyze the existence of unconditional basis on polynomial ideals. For this, we study unconditionality on symmetric tensor products. We provide a simple criterium to check wether a polynomial ideal lacks of unconditional basis. Using this criterium, we show that many usual polynomial ideals do not have unconditional structure. Among them we have the r-integral, r-dominated, extendible and r-factorable polynomials. For many of these examples we also get that the monomial basic sequence is not unconditional.

We study the preservation of other kind of structures on the symmetric tensor product: the Banach algebra structure and the M-ideal structure. We show which of the Grothendieck's natural symmetric tensor norms preserve the algebra structure. On the other hand, we prove that the injective s-tensor norm destroys the M-ideal structure (opposite to what happens in the full tensor product with the injective norm). Even though this structure is lost in the symmetric case, we show that, if E is Asplund and M-ideal in F, every integral polynomial in E has a unique norm preserving extension to F.

Keywords: Symmetric tensor products, s-tensor norms, polynomial ideals, homogeneous polynomials, structures in tensor products.

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Contents

Introducción

Grothendieck, en su *"Résumé de la théorie métrique des produits tensoriels topologiques"* [Gro53], creó las bases de lo que luego se llamó "teoría local", y mostró la importancia del uso de productos tensoriales en la teoría de espacios de Banach e ideales de operadores. Los productos tensoriales aparecieron en el análisis funcional en la década del treinta, en trabajos de Murray, Von Neumann y Schatten (ver [Sch50]). Pero fue Grothendieck quien observó la naturaleza local de muchas de las propiedades de productos tensoriales, permitiéndole establecer una teoría de dualidad sumamente útil. Si bien hoy en día el *"Résumé"* es considerado uno de los artículos más inspiradores del análisis funcional, dicho trabajo permaneció inadvertido por muchos años. Hay dos razones que explican por qué ocurrió esto. La primera de ellas, una razón "práctica", es que el artículo fue publicado en una revista a la que no muchas bibliotecas suscribían. La otra, una razón "académica", es que el artículo era muy difícil de entender: la notación utilizada era un poco engorrosa y no contenía demostraciones (con la excepción del teorema principal, la llamada desigualdad de Grothendieck).

Recién en 1968 el *"Résumé*" de Grothendieck fue apreciado por completo. Ese año, Lindenstrauss y Pełczyński [LP68] presentaron importantes aplicaciones a la teoría de operadores absolutamente p sumantes, traduciendo aquellos resultados escritos en términos de normas tensoriales por Grothendieck en propiedades de ideales de operadores. Al mismo tiempo, una teoría general de ideales de operadores en espacios de Banach fue desarrollada por Pietsch y su escuela en Jena, sin el uso de las normas tensoriales. Nuevas ideas y definiciones se dieron, haciendo de la teoría de ideales de operadores uno de los temas centrales de estudio para los analistas funcionales. Dicho avance culminó con el libro de Pietsch *"Operator Ideals"* [Pie78], el cual era enciclopedia de lo que se sabía hasta el momento. En esa época, los investigadores generalmente preferían el lenguage de ideales de operadores al oscuro lenguaje de productos tensoriales, por lo que la primera teoría recibió más antención.

Durante los ochenta, las técnicas tensoriales resultaron más fuertes y populares. Fue el trabajo de Pisier [Pis83, Pis88] el que mostró que tener una perspectiva tensorial podría dar un panorama más claro y fortalecer la investigación. Defant y Floret emprendieron la dificultosa tarea de describir la teoría de productos tensoriales y la teoría de ideales de operadores en conjunto. Lograron llenar el vacío en la literatura y publicaron su monografía *"Tensor Norms and Operator Ideals"* [DF93]. Este libro tuvo un tremendo impacto, iniciando un período en el cual los autores utilizan indistintamente ambos lenguajes. Como dos perspectivas diferentes son siempre mejor que sólo una, hoy resulta común atacar algunos problemas usando *la manera de pensar categórica* de Pietsch o *el ciclo de ideas tensoriales* de Grothendieck. Como dicen Defant y Floret en su libro, "ambas teorías, la teoría de normas tensoriales y la de ideales de operadores, son más sencillas de entender y ricas si uno trabaja con ambas simultaneamente".

El estudio de polinomios es uno los tópicos más antiguos en matemáticas. Al principio del siglo veinte la investigación sistemática de una teoría abstracta de polinomios en espacios de dimensión infinita empezó a florecer. Dentro de los matemáticos interesados en esta área podemos mencionar a Fréchet, Gâteaux, Michal y Banach. Banach mismo sugirió la importancia de estudiar esta teoría no lineal. Incluso tuvo la intención de escribir un segundo volumen de su libro famoso [Ban32] basado, en parte, en la teoría de polinomios en espacios normados. Lamentablemente, murió en 1945 sin comenzar este proyecto.

En el libro de Dineen [Din99] se menciona que el progreso en la teoría de polinomios puede ser dividido en dos períodos. En el primer período, que empezó a mediados de los treinta, nuevos conceptos y resultados se dieron en polinomios en infinitas variables. En esos tiempos, la investigación estaba basada en el estudio de funciones holomorfas en espacios de dimensión infinita, análisis de Fourier y series de Dirichlet. La investigación en polinomios homogéneos, los cuales aparecen naturalmente cuando se estudian expansiones en series (series de Taylor) de funciones holomorfas, resultó ser crucial para la teoría de análisis complejo en espacios de dimensión infinita. El segundo período empezó en los ochenta, cuando diferentes espacios de polinomios y propiedades de ciertas clases se convirtieron en el principal objeto de estudio. Tal como en la teoría lineal, se dio la definición de ideales de polinomios. Este concepto apareció por primera vez en [Bra84, Hol86] como una adaptación de la definición de ideales de formas multilineales dada por Pietsch [Pie84]. Básicamente, los ideales de polinomios son clases de polinomios que tienen ciertas propiedades en común.

Fue Ryan quien introdujo en su tesis [Rya80] los productos tensoriales simétricos en espacios de Banach como una herramienta para estudiar polinomios (y también funciones analíticas). Es conocido que los productos tensoriales linealizan las formas multilineales. De la misma manera, los productos tensoriales simétricos linealizan polinomios homogéneos. En otras palabras, cada polinomio homogéneo de grado n definido en un espacio E puede ser visto como una funcional lineal en $\otimes^{n,s} E$ (el n-ésimo producto tensorial simétrico de E), y viceversa. La filosofía involucrada en esta perspectiva es la siguiente: identificamos polinomios con funciones más simples (funcionales) con la contrapartida que los dominios de estas funciones (producto tensoriales simétricos) resultan más complicados.

A partir del trabajo de Ryan, muchos pasos se dieron en la teoría métrica de productos tensoriales simétricos y la teoría de ideales de polinomios. Como en el caso lineal, ambas teorías (la teoría de normas tensoriales simétricas y la teoría de ideales de polinomios) influyen y contribuyen una a otra. En su ensayo [Flo97], Floret presentó los conceptos algebraicos básicos del n-ésimo producto tensorial simétrico, conjuntamente con un tratamiento de resultados métricos fundamentales de dos normas tensoriales extremas: la norma proyectiva simétrica $\pi_{n,s}$ y la norma inyectiva simétrica $\varepsilon_{n,s}$. A pesar de que algunos aspectos de la teoría de productos tensoriales simétricos y la teoría de ideales de polinomios evolucionaron continuamente en las últimas décadas, lamentablemente no hay un tratamiento general de normas tensoriales simétricas. En palabras de Floret [Flo97], "parece adecuado desarrollar una *teoría métrica* en el n-ésimo producto tensorial simétrico en el espíritu de Grothendieck".

El principal resultado de [Flo01b] afirma que toda norma s-tensorial en el n-ésimo producto tensorial simétrico de espacios normados es equivalente a la restricción al producto tensorial simétrico de una norma tensorial completa en el n-ésimo producto tensorial. Como consecuencia, gran parte de la teoría isomorfa puede ser deducida de la teoría de normas tensoriales completas. Si bien la teoría isomorfa puede ser trasladada de un contexto a otro, la teoría métrica puede ser bastante diferente. Incluso en los casos donde la teoría métrica es parecida, la simetría introduce ciertos tecnicismos. Debemos también mencionar lo siguiente: si bien la teoría de productos tensoriales de orden 2 ha sido muy estudiada, puede diferir bastante respecto de la teoría de productos tensoriales de orden n (para $n \geq 3$). Por lo que muchos de los resultados isomorfos en el n-ésimo producto tensorial simétrico (para $n \geq 3$) también suelen ser difíciles de obtener.

El propósito de esta tesis es doble: contribuir al desarrollo sistemático de la teoría métrica e isomorfa en productos tensoriales de espacios de Banach y mostrar muchos contextos en los cuales las técnicas tensoriales pueden ser aplicadas para fortalecer la teoría de ideales de polinomios. Esperamos que esta perspectiva pueda dar una visión más clara además de favorecer la investigación. También esperamos que, en un futuro, las ideas y resultados presentados puedan ser usadas en otras áreas (como por ejemplo en holomorfía infinito dimensional).

El trabajo está organizado en seis capítulos.

El Capítulo 1 está dedicado al material necesario para entender la tesis. Damos la notación, algunas definiciones básicas y explicamos la dualidad entre productos tensoriales simétricos e ideales de polinomios. Varias nociones básicas de la teoría de normas tensoriales simétricas y de la teoría de ideales de polinomios son presentadas. También describimos algunos ideales clásicos y recordamos una forma muy conocida de extender un polinomio sobre un espacio al bidual (la extensión de Aron-Berner).

En el Capítulo 2 presentamos los "Cinco Lemas Básicos" (ver Sección 13 del libro de Defant y Floret [DF93]) para el contexto de productos tensoriales simétricos. Estos son el Lema de Aproximación, el Lema de Extensión, el Lema de Inclusión, el Lema de Densidad y el Lema de Técnica Local-$\mathcal{L}_p$. Este capítulo es crucial ya que los cinco lemas básicos y sus consecuencias son utilizados en todo el texto. Si bien seguimos las líneas de [DF93], la naturaleza simétrica de los productos tensoriales introducen varias dificultades, como se puede ver, por ejemplo, en la versión simétrica del Lema de Extensión 2.1.3, cuya prueba es mucho más complicada que su versión 2-tensorial completa. Este resultado afirma que la extensión de Aron-Berner es una isometría bien definida para todo ideal de polinomios maximal. También obtenemos el mismo resultado para ideales de polinomios minimales (ver Teorema 2.2.6). Otras importantes aplicaciones a la teoría métrica de normas tensoriales simétricas e ideales de polinomios son dadas.

En el Capítulo 3 damos la definición de la cápsula inyectiva y proyectiva de una norma s-tensorial y estudiamos algunas de sus propiedades interesantes. Describimos los ideales de polinomios maximales asociados a dichas normas en términos de ideales de composición o ideales cociente (ver Teorema 3.4.4). El estudio de normas s-tensoriales naturales de orden arbitrario, en el espíritu de Grothendieck es también presentado: basándonos en [Gro53] definimos las normas naturales simétricas como aquellas que pueden ser obtenidas de la n-ésima norma s-tensorial proyectiva $\pi_{n,s}$ aplicándole un número finito de operaciones básicas (cápsula inyectiva, cápsula proyectiva, y adjunto). En el Teorema 3.12 mostramos que hay exactamente seis normas naturales simétricas para $n \geq 3$, una diferencia notable respecto del caso de orden 2 donde hay cuatro.

El objetivo del Capítulo 4 es encontrar condiciones para las cuales un ideal de polinomios maximal coincide isométricamente con su núcleo minimal. En términos de productos tensoriales simétricos, buscamos propiedades en una norma s-tensorial que aseguren la isometría $\widetilde{\otimes}_{\alpha}^{n,s} E' = \big(\widetilde{\otimes}_{\alpha'}^{n,s} E\big)'$. Para esto, introducimos la propiedad de Radon-Nikodým simétrica para normas s-tensoriales y mostramos el principal resultado del capítulo, el Teorema 4.1.2: si una norma s-tensorial proyectiva α tiene la propiedad de Radon-Nikodým simétrica (propiedad RNs), tenemos que la aplicación natural

$$\widetilde{\otimes}_{\alpha}^{n,s} E' \to \big(\widetilde{\otimes}_{\alpha'}^{n,s} E\big)'$$

es cociente para todo espacio Asplund E. Como consecuencia, si $\mathcal{Q}$ es un ideal maximal (de polinomios n-homogéneos) asociado a una norma s-tensorial proyectiva α con la propiedad RNs, entonces $\mathcal{Q}^{min}(E) = \mathcal{Q}(E)$ isométricamente (i.e., $\mathcal{Q}$ coincide isométricamente con su núcleo minimal sobre el espacio E). Esto puede ser visto como una versión simétrica del Teorema de Lewis (ver [Lew77] y [DF93, 33.3]). Con este resultado damos una demostración alternativa del isomorfismo isométrico entre polinomios integrales y nucleares en espacios Asplund y también mostramos que el ideal de polinomios extendibles coincide con su núcleo minimal para este tipo de espacios. Por ende, el espacio de polinomios extendibles en E tiene base monomial si E' tiene base. Ejemplos de normas s-tensoriales asociadas a ideales de polinomios conocidos son presentados. También relacionamos la propiedad RNs para una norma s-tensorial con la propiedad Asplund. Precisamente, si α es una norma s-tensorial proyectiva con la propiedad RNs, probamos que $\widetilde{\otimes}_{\alpha'}^{n,s} E$ es Asplund siempre y cuando E lo es.

El Capítulo 5 contiene el estudio de incondicionalidad para productos tensoriales simétricos (y completos). Examinamos cuándo una norma destruye incondicionalidad en el sentido que, para todo espacio E con base incondicional, el correspondiente producto tensorial carece de base incondicional. Damos un test simple (Teorema 5.1.5) para determinar si una norma tensorial destruye incondicionalidad o no. Con esto obtenemos que toda norma s-tensorial inyectiva y projectiva (resp. norma tensorial completa) diferente de $\varepsilon_{n,s}$ y $\pi_{n,s}$ (resp. ε_n y π_n) destruye incondicionalidad. También investigamos incondicionalidad en ideales de polinomios y formas multilineales y exhibimos varios ejemplos de ideales de polinomios $\mathcal{Q}$ tal que, para todo espacio de Banach E con base incondicional, el espacio $\mathcal{Q}(E)$ carece de la propiedad de Gordon-Lewis. Dentro de estos ideales tenemos los polinomios r-integrales, r-dominados, extendibles y r-factorizables. Para muchos de estos ejemplos mostramos que la base monomial nunca es incondicional.

En el Capítulo 6 focalizamos nuestra atención en la preservación de dos importantes estructuras para normas s-tensoriales específicas: la estructura de álgebra de Banach y la estructura de M-ideal. Basándonos en el trabajo de Carne [Car78], estudiamos cuáles son las normas s-tensoriales naturales que preservan la estructura de álgebra. En el Teorema 6.1.3 mostramos que las dos normas s-tensoriales naturales que preservan álgebras de Banach son $\pi_{n,s}$ y $\backslash/\pi_{n,s}\backslash/$. También probamos que la estructura de M-ideal es destruida por $\varepsilon_{n,s}$ para todo n. En concreto, en el Teorema 6.2.7 mostramos que, para espacios de Banach reales E y F, si E es un M-ideal en F, entonces $\widetilde{\otimes}_{\varepsilon_{n,s}}^{n,s} E$ (el producto tensorial simétrico inyectivo de E) no es M-ideal en $\widetilde{\otimes}_{\varepsilon_{n,s}}^{n,s} F$. Este resultado muestra una gran diferencia con el comportamiento de tensores completos ya que, si E es M-ideal en F, es sabido que $\widetilde{\otimes}_{\varepsilon_n}^{n} E$ (el producto tensorial inyectivo completo de E) es un M-ideal en $\widetilde{\otimes}_{\varepsilon_n}^{n} F$. Si bien la M-estructura es destruida para productos simétricos, mostramos, en el Teorema 6.2.9, que si E es espacio Asplund y M-ideal no trivial en F, entonces todo polinomio n-homogéneo integral en E tiene una única extensión a F que preserva la norma integral. También describimos explícitamente dicha extensión.

Los principales resultados de esta tesis aparecen en [CG10, CG11a, CG11b, CG11c, CG12, DGG12].

Introduction

Grothendieck, in his *“Résumé de la théorie métrique des produits tensoriels topologiques”* [Gro53], created the basis of what was later known as ‘local theory’, and exhibited the importance of the use of tensor products in the theory of Banach spaces and operator ideals. Tensor products had appeared in functional analysis since the late thirties, in works of Murray, Von Neumann and Schatten (see [Sch50]). But it was Grothendieck who realized the local nature of many properties of tensor products, and this allowed him to establish a very useful theory of duality. Although nowadays the *“Résumé”* is considered a one of the most inspiring papers in functional analysis, the article remained widely unnoticed for many years. There are two reasons that explain why this occurred. The first one, a ‘practical’ reason, is that the article was published in a journal to which not many libraries would subscribe. The other, an ‘academical’ reason, is that the article was highly difficult to understand: the notation used was a bit annoying and it did not contain proofs (with the exception of the main theorem, the so-called Grothendieck’s inequality).

It was not until 1968 when Grothendieck’s *“Résumé”* was fully appreciated. That year, Lindenstrauss and Pełczyński [LP68] presented important applications to the theory of absolutely p-summing operators, translating results written in terms of tensor norms by Grothendieck, into properties of operator ideals. By the same time, a general theory of operator ideals on the class of Banach spaces was developed by Pietsch and his school in Jena, without the use of tensor norms. Novel ideas and definitions were given, leading the theory of operator ideals as one of the central themes of study for functional analysts. The break out culminated in Pietsch’s book *“Operator Ideals”* [Pie78], which was some sort of encyclopedia of what was known so far. At that moment, researchers generally preferred the language of operator ideals to the more abstruse language of tensor products, and so the former theory received more attention.

During the eighties, tensor product techniques became stronger and more popular. It was Pisier’s work [Pis83, Pis88], which showed that having a tensor perspective would give a clearer picture and would strengthen the investigation. Defant and Floret undertook the difficult task of describing the the theory of tensor products and the the theory of operator ideals in tandem. They manage to fill the gap in the literature and published their monograph *“Tensor Norms and Operator Ideals”* [DF93]. This book had a tremendous impact, initiating a period in which authors use indistinctly both languages. Since two different perspectives are always better than just one, it is now common to attack certain problems using the *categorical way of thinking* due to Pietsch, or Grothendieck’s *cycle of ideas on tensor products*. As stated by Defant and Floret in their book, “both theories, the theory of tensor norms and of norm operator ideals, are more easily understood and also richer if one works with both simultaneously”.

The study of polynomials is one of the oldest topics in mathematics. At the beginning of the twentieth century a systematic research on an abstract theory of polynomials defined on infinite dimensional spaces was flourishing. Among the mathematicians focused on this area we can mention Fréchet, Gâteaux, Michal and Banach. Banach himself suggested the importance of studying this non-linear theory. He even intended to write a second volume of his famous book [Ban32] based, in part, on the theory polynomials on normed spaces. Unfortunately, he died in 1945 without commencing this project.

In Dineen's book [Din99] it is mentioned that the progress of theory of polynomials can be divided into two periods. In the first period, which started in the mid thirties, new concepts and results were developed on polynomials in infinite many variables. At that time, research was based on the study of holomorphic functions on infinite dimensional spaces, Fourier analysis and Dirichlet series. Research on homogeneous polynomials, which naturally appear when studying series expansions (Taylor series) of holomorphic functions, resulted to be crucial for the theory of complex analysis on infinite dimensional spaces. The second period started in the eighties, when different spaces of polynomials and properties of polynomials of certain class became the main subject of study. Such as in the linear theory, the definition of polynomial ideal showed up. This concept appeared first in [Bra84, Hol86] as an adaptation of the definition of ideals of multilinear mappings given by Pietsch [Pie84]. Loosely speaking, polynomial ideals are classes of polynomials which have certain properties in common.

It was Ryan who introduced in his thesis [Rya80] symmetric tensor products of Banach spaces as a tool for the study of polynomials (and also holomorphic mappings). It is well know that tensor products linearize multilinear forms. Likewise, the symmetric tensor product linearize homogeneous polynomials. In other words, each n-homogeneous polynomial defined on a space E can be seen as a linear function on $\otimes^{n,s}E$ (the n-fold symmetric tensor product of E), and vice versa. The philosophy involved of this perspective is the following: we identify polynomials by simpler functions (linear functionals) with the counterpart that the domains of these functions (symmetric tensor products) get more complicated.

Since the work of Ryan, many steps were given towards a metric theory of symmetric tensor products and a theory of polynomial ideals. As in the linear case, both theories (the theory of symmetric tensor norms and products and the theory of polynomial ideals) influence and contribute to each other. In his survey [Flo97], Floret presented the algebraic basics of n-fold symmetric tensor products, together with a thorough account of fundamental metric results for the two extreme tensor norms: the symmetric projective tensor norm $\pi_{n,s}$ and the symmetric injective tensor norm $\varepsilon_{n,s}$. Despite some aspects of the theory of symmetric tensor products and polynomial ideals steadily evolved in the last decades, sadly there is not such a treatise on general symmetric tensor norms. In the words of Floret [Flo97] "it seems to be adequate to develop a *metric theory* on n-th symmetric tensor products in the spirit of Grothendieck".

The main result of [Flo01b] states that every s-tensor norm on an n-symmetric tensor product of normed spaces is equivalent to the restriction to the symmetric tensor product of a tensor norm on a full n-fold tensor product. As a consequence, a large part of the isomorphic theory of norms on symmetric tensor products can be deduced from the theory of full tensor norms. Although the isomorphic theory can be translated from one context to the other, the metric theory can be quite different. Even in the cases where the metric theory is very much alike, symmetry sometimes introduces certain technicalities. Another thing should be mention: although the 2-fold tensor product theory has been widely studied, it can vary considerably with respect to the theory of tensor products of order n (for $n \geq 3$). Therefore, many of the isomorphic results on the n-th symmetric tensor product (for $n \geq 3$) are usually hard to obtain.

The purpose of this dissertation is twofold: contributing to a systematic development of the metric and isomorphic theory of symmetric tensor products of Banach spaces and showing several contexts in which tensor techniques can be applied to enroot the theory of polynomials ideals. We expect that this perspective would yield more insight and would enhance research. We also hope that, in the future, the ideas and results presented here can be used in other areas (e.g., infinite dimensional holomorphy).

The material is organized into six chapters as follows.

Chapter 1 is devoted to background material. We set some notation, give basic definitions and explain the duality between symmetric tensor products and polynomial ideals. Several basic notions on the theory of symmetric tensor norms and the theory of polynomial ideals are presented. We also describe some classical ideals and recall a well-known way of extending a polynomial defined on a Banach space into its bidual (namely, the Aron-Berner extension).

In Chapter 2 we present the "Five Basic Lemmas" (see Section 13 in Defant and Floret's book [DF93]) for the symmetric tensor product setting. They are the Approximation Lemma, the Extension Lemma, the Embedding Lemma, the Density Lemma and the $\mathcal{L}_p$-Local Technique Lemma. This chapter is crucial since the five basic lemmas and its consequences are used throughout the whole text. Although we follow the lines of [DF93], the symmetric nature of our tensor products introduces several difficulties, as we can see, for example, in the symmetric version of the Extension Lemma 2.1.3, whose proof is much more complicated than that of its full 2-fold version. This result states that the Aron-Berner extension is a well defined isometry for every maximal polynomial ideal. We also obtain the same result for minimal polynomial ideals (see Theorem 2.2.6). Other important applications to the metric theory of symmetric tensor norms and polynomial are given.

In Chapter 3 we give the definitions of the injective and projective associates of an s-tensor norm and examine some of their interesting properties. We describe the maximal polynomial ideals associated with these norms in terms of composition ideals and quotient ideals (see Theorem 3.4.4). The study of natural symmetric tensor norms of arbitrary order, in the spirit of Grothendieck's norms is given as well: based on [Gro53] we define natural symmetric tensor norms as those that can be obtained from the n-fold projective s-tensor norm $\pi_{n,s}$ by a finite number of basic operations (injective associate, projective associate, and adjoint). In Theorem 3.12 we show that there are exactly six natural symmetric tensor norms for $n \geq 3$, a noteworthy difference with the 2-fold case in which there are four.

The goal of Chapter 4 is to find conditions under which a maximal polynomial ideal coincide isometrically with its minimal kernel. In terms of symmetric tensor products, we seek for properties on an s-tensor norms ensuring the isometry $\widetilde{\otimes}_{\alpha}^{n,s} E' = \big(\widetilde{\otimes}_{\alpha'}^{n,s} E\big)'$. For this, we introduce the symmetric Radon-Nikodým property for s-tensor norms and show our main result, Theorem 4.1.2: if a projective s-tensor norm α has the symmetric Radon-Nikodým property (sRN property), we have that the natural mapping

$$\widetilde{\otimes}_{\alpha}^{n,s} E' \to \big(\widetilde{\otimes}_{\alpha'}^{n,s} E\big)'$$

is a metric surjection for every Asplund space E. As a consequence, if $\mathcal{Q}$ is the maximal ideal (of n-homogeneous polynomials) associated with a projective s-tensor norm α having the sRN property, then $\mathcal{Q}^{min}(E) = \mathcal{Q}(E)$ isometrically (i.e., $\mathcal{Q}$ coincides isometrically with its minimal kernel over the space E). This can be seen as a symmetric version of Lewis' Theorem (see

[Lew77] and [DF93, 33.3]). With this result we reprove the isometric isomorphism between integral and nuclear polynomials on Asplund spaces and also show that the ideal of extendible polynomials coincide with its minimal kernel for this type of spaces. As a consequence, the space of extendible polynomials on E has a monomial basis whenever E' has a basis. Examples of s-tensor norms associated with well known polynomial ideals which have the sRN property are presented. We also relate the sRN property of an s-tensor norm with the Asplund property. More precisely, if α is a projective s-tensor norm with the sRN property, we prove that $\widetilde{\otimes}^{n,s}_{\alpha'} E$ is Asplund whenever E is.

Chapter 5 contains the study of unconditionality for symmetric (and full) tensor products. We examine when a tensor norm destroys unconditionality in the sense that, for every Banach space E with unconditional basis, the corresponding tensor product has not unconditional basis. We provide a simple test (Theorem 5.1.5) to check wether a tensor norm destroys unconditionality or not. With this we obtain that every injective and every projective s-tensor norm (resp. full tensor norm) other than $\varepsilon_{n,s}$ and $\pi_{n,s}$ (resp. ε_n and π_n) destroys unconditionality. We also investigate unconditionality in ideals of polynomials and multilinear forms and exhibit several examples of polynomials ideals $\mathcal{Q}$ such that, for every Banach space E with unconditional basis, the space $\mathcal{Q}(E)$ lacks the Gordon-Lewis property. Among these ideals we have the r-integral, r-dominated, extendible and r-factorable polynomials. For many of these examples we also get that the monomial basic sequence is never unconditional.

In Chapter 6 we focus our attention on the preservation of two important structures for specific s-tensor norms: the Banach-algebra structure and the M-ideal structure. Based on the work of Carne [Car78], we describe which natural s-tensor norms preserve the algebra structure. In Theorem 6.1.3 we show that the two natural s-tensor norms preserving Banach algebras are $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$. We also prove that the M-ideal structure is destroyed by $\varepsilon_{n,s}$ for every n. More precisely, in Theorem 6.2.7, we show that for real Banach spaces E and F, if E is a non trivial M-ideal in F, then $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ (the injective symmetric tensor product of E) is *never* an M-ideal in $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} F$. This result shows a big difference with the behavior of full tensors since, when E is an M-ideal in F, it is known that $\widetilde{\otimes}^{n}_{\varepsilon_n} E$ (the injective full tensor product of E) is an M-ideal in $\widetilde{\otimes}^{n}_{\varepsilon_n} F$. Even though the M-structure is destroyed for symmetric tensors, we show, in Theorem 6.2.9, that if E is an Asplund space which is a non trivial M-ideal in F, then every integral n-homogeneous polynomial in E has a unique extension to F that preserves the integral norm. We also describe explicitly this unique extension.

The main results of this thesis appear in [CG10, CG11a, CG11b, CG11c, CG12, DGG12].

Chapter 1

Preliminaries

This chapter contains all the background material. Several basic notions on the theory of polynomial ideals and the theory of symmetric tensor norms are presented. We also set some notation and explain the duality between symmetric tensor products and polynomial ideals. The Arens-extension of a multilinear form and the Aron-Berner extension of a polynomial is described. For a complete discussion on the material that appears in this chapter we recommend to read the following bibliography: we refer to [Flo97, Flo01a, Flo01b, Flo02, FH02, Din99] for the theory of s-tensor norms and polynomial ideals and [Zal05] for the Arens and the Aron-Berner extensions.

A little bit of notation

Throughout the dissertation E and F will be real or complex normed spaces and the scalar field ($\mathbb{R}$ or $\mathbb{C}$) will be denoted with the letter $\mathbb{K}$. On the other hand, E' will stand for the dual space of E, $\kappa_E : E \longrightarrow E''$ will be the canonical embedding of E into its bidual, and B_E will denote the closed unit ball of E. We denote by $FIN(E)$ the class of all finite dimensional subspaces of E and denote by $COFIN(E)$ the class of all finite codimensional closed subspaces of the space E.

A surjective mapping $T : E \to F$ is called *a metric surjection or a quotient* if $\|T(x)\|_F = \inf\{\|y\|_E : T(y) = x\}$, for all $x \in E$. As usual, a mapping $I : E \to F$ is called *an isometry* if $\|Ix\|_F = \|x\|_E$ for all $x \in E$. We use the notation $\overset{1}{\twoheadrightarrow}$ and $\overset{1}{\hookrightarrow}$ to indicate a metric surjection or an isometry, respectively. We also write $E \overset{1}{=} F$ whenever E and F are isometrically isomorphic spaces (*i.e.,* there exist a surjective isometry $I : E \to F$).

For $L \in COFIN(E)$ we denote by $Q_L^E : E \overset{1}{\twoheadrightarrow} E/L$ *the canonical quotient mapping onto* E/L *.*

1.1 Polynomial ideals

An application $p : E \to \mathbb{K}$ is an n*-homogeneous polynomial* if there exist an n-linear mapping $A : E \times \overset{n}{\cdots} \times E \to F$ such that $p(x) = A(x, \ldots, x)$ for every $x \in E$. In this case we say that p is a polynomial associated with A.

Given a polynomial p there are many n-linear forms which satisfy the condition given above, but there exists only one which is symmetric (an n-linear mapping A is symmetric if $A(x_1, \dots, x_n) = A(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for every $x_1, \dots, x_n$ and every $\sigma \in \mathcal{S}_n$, the group of permutations of $\{1, \dots, n\}$). This symmetric n-linear form, denoted by $\overset{\vee}{p}$, may be obtained from p via the *polarization formula*:

$$\overset{\vee}{p}(x_1, \dots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \dots \varepsilon_n p\Big(\sum_{i=1}^{n} \varepsilon_i x_i\Big).$$

Conversely, to each symmetric n-linear form we can associate an n-homogeneous polynomial. Thus there exist a one to one and onto correspondence between n-homogeneous polynomials and n-linear symmetric forms.

Continuous n-homogeneous polynomials are exactly those bounded on the unit ball. The space of all continuous n-homogeneous polynomials on E is denoted by $\mathcal{P}^n(E)$. This class is a Banach space endowed with the norm

$$\|p\|_{\mathcal{P}^n(E)} = \sup_{\|x\| \leq 1} |p(x)|.$$

Denote by $\mathcal{L}_s(^nE)$ the space of continuous scalar valued symmetric n-linear forms on E. This space is a Banach space with the norm $\|A\| = \sup\{\|A(x_1, \dots, x_n)\| : x_1, \dots, x_n \in B_E\}$. Then the polarization formula implies that

$$\|p\| \leq \|\overset{\vee}{p}\| \leq \frac{n^n}{n!} \|p\|.$$

The simplest class of polynomials is the class of *finite type* polynomials, $\mathcal{P}^n_f(E)$. An n-homogeneous polynomial p is of finite type if there exist $x'_1, \dots, x'_r \in E'$, such that $p(x) = \sum_{j=1}^{r} (x'_j(x))^n$ for every x in E. If E is finite dimensional then every polynomial on E is of finite type. The closure of finite type polynomials in $\mathcal{P}^n(E)$ are the *approximable* polynomials. The space of approximable polynomials is denoted by $\mathcal{P}^n_{app}(E)$. In general, the class of approximable polynomials is strictly smaller than the the class of all continuous polynomials. For example, the 2-homogeneous polynomial $p(x) = \sum_{j=1}^{\infty} x_j^2$ on ℓ_2 is continuous but not approximable. It should be mentioned that there are also spaces for which these two classes coincide (e.g., c_0 [Din99, Propositions 1.59 and 2.8]).

The concept of polynomial ideals appeared for the first time in [Bra84, Hol86], as an adaptation of the definition of ideals of multilinear mappings given by Pietsch [Pie84] (and also, of course, of operator ideals).

Let us recall some definitions extracted from [Flo02]: a *normed (Banach) ideal of continuous scalar valued n-homogeneous polynomials* is a pair $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ such that:

(i) $\mathcal{Q}(E) = \mathcal{Q} \cap \mathcal{P}^n(E)$ is a linear subspace of $\mathcal{P}^n(E)$ and $\|\cdot\|_{\mathcal{Q}}$ is a norm which makes the pair $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ a normed (Banach) space.

(ii) If $T \in \mathcal{L}(E_1, E)$, $p \in \mathcal{Q}(E)$ then $p \circ T \in \mathcal{Q}(E_1)$ and

$$\|p \circ T\|_{\mathcal{Q}(E_1)} \leq \|p\|_{\mathcal{Q}(E)} \|T\|^n.$$

(iii) $z \mapsto z^n$ belongs to $\mathcal{Q}(\mathbb{K})$ and has norm 1.

It is well-known that the only scalar ideal of 1-homogeneous polynomials (that is, of linear functionals) is, for each Banach space E, equal to E'.

We now recall the definition of some classical polynomial ideals which appear in the text.

- *Continuous polynomials*, $\mathcal{P}^n$.

  The ideal of all continuous n-homogeneous polynomials, with its usual norm is a Banach ideal of homogeneous polynomials. Other polynomial ideals with the usual norm of polynomials are:

  – *Finite type polynomials*, $\mathcal{P}^n_f$, and *approximable polynomials*, $\mathcal{P}^n_{app}$, which were already defined.
  – *Weakly continuous on bounded sets polynomials*, $\mathcal{P}^n_w$.
    A polynomial $p \in \mathcal{P}^n(E)$ is weakly continuous on bounded sets if the restriction of p to any bounded set of E is continuous when the weak topology is considered on E.
  – *Weakly sequentially continuous polynomials*, $\mathcal{P}^n_{wsc}$.
    A polynomial $p \in \mathcal{P}^n(E)$ is weakly sequentially continuous if for every weakly convergent sequence $x_n \xrightarrow{w} x$ we have $p(x_n) \to p(x)$.

- *Nuclear polynomials*, $\mathcal{P}^n_N$.

  A polynomial $p \in \mathcal{P}^n(E)$ is said to be *nuclear* if it can be written as

$$p(x) = \sum_{i=1}^{\infty} \lambda_j (x'_j(x))^n,$$

  where $\lambda_j \in \mathbb{K}$, $x'_j \in E'$ for all j and $\sum_{j=1}^{\infty} |\lambda_j| \|x'_j\|^n < \infty$. The space of nuclear n-homogeneous polynomials is denoted by $\mathcal{P}^n_N(E)$. It is a Banach space when we consider the norm

$$\|p\|_{\mathcal{P}^n_N(E)} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \|x'_j\|^n \right\}$$

  where the infimum is taken over all representations of p as above.

- *Integral polynomials*, $\mathcal{P}^n_I$.

  A polynomial $p \in \mathcal{P}^n(E)$ is *integral* if there exists a regular Borel measure μ, of bounded variation on $(B_{E'}, w^*)$ such that

$$p(x) = \int_{B_{E'}} (x'(x))^n \, d\mu(x')$$

  for every $x \in E$. The space of n-homogeneous integral polynomials is denoted by $\mathcal{P}^n_I(E)$ and the integral norm of a polynomial $p \in \mathcal{P}^n_I(E)$ is defined as

$$\|P\|_{\mathcal{P}^n_I(E)} = \inf \left\{ |\mu|(B_{E'}) \right\},$$

  where the infimum is taken over all measures μ representing p.

- *Extendible polynomials*, $\mathcal{P}_e^n$.

  A polynomial $p : E \to \mathbb{K}$ is *extendible* if for any Banach space G containing E there exists $\widetilde{p} \in \mathcal{P}^n(G)$ an extension of p. We denote the space of all such polynomials by $\mathcal{P}_e^n(E)$. For $p \in \mathcal{P}_e^n(E)$, its extendible norm is given by

$$\|p\|_{\mathcal{P}_e^n(E)} = \inf\{c > 0 : \quad \text{for all } G \supset E \text{ there is an extension of } p \text{ to } G \text{ with norm} \leq c\}.$$

- *r-dominated polynomials*, $\mathcal{D}_r^n$.

  For $x_1, \ldots, x_m \in E$, we define

$$w_r\big((x_i)_{i=1}^m\big) = \sup_{x' \in B_{E'}} \left(\sum_i |x'(x_i)|^r \right)^{1/r}.$$

  A polynomial $p \in \mathcal{P}^n(E)$ is *r-dominated* (for $r \geq n$) if there exists $c > 0$ such that for every finite sequence $(x^i)_{i=1}^m \subset E$ the following holds

$$\left(\sum_{i=1}^m |p(x_i)|^{\frac{r}{n}} \right)^{\frac{n}{r}} \leq C w_r((x_i)_{i=1}^m)^n.$$

  We denote the space of all such polynomials by $\mathcal{D}_r^n(E)$. The least of such constants c is called the *r-dominated norm* and denoted by $\|p\|_{\mathcal{D}_r^n(E)}$.

- *r-factorable polynomials*, $\mathcal{L}_r^n$.

  For $n \leq r \leq \infty$, a polynomial $p \in \mathcal{P}^n(E)$ is called *r-factorable* if there is a positive measure space (Ω, μ), an operator $T \in \mathcal{L}\big(E, L_r(\mu)\big)$ and a polynomial $q \in \mathcal{P}^n(L_r(\mu))$ with $p = q \circ T$. The space of all such polynomials is denoted by $\mathcal{L}_r^n(E)$. It is a Banach space if it is endowed with the following norm

$$\|p\|_{\mathcal{L}_r^n(E)} = \inf\{\|q\|\|T\|^n \ : \ p : E \xrightarrow{T} L_r(\mu) \xrightarrow{q} \mathbb{K} \text{ as before}\}.$$

- *Positively r-factorable polynomials*, $\mathcal{J}_r^n$.

  An n-homogeneous polynomial $q : F \to \mathbb{K}$ on a Banach lattice F is called positive, if $\check{q} : F \to \mathbb{K}$ is positive, i.e., $\check{q}(f_1, \ldots, f_n) \geq 0$ for $f_1, \ldots, f_n \geq 0$. For $n \leq r \leq \infty$, a polynomial $p \in \mathcal{P}^n(E)$ is called *positively r-factorable* if there is a positive measure space (Ω, μ), an operator $T \in \mathcal{L}\big(E, L_r(\mu)\big)$ and a positive polynomial $q \in \mathcal{P}^n(L_r(\mu))$ with $p = q \circ T$. The space of all such polynomials endowed with the norm

$$\|p\|_{\mathcal{J}_r^n(E)} = \inf\{\|q\|\|T\|^n \ : \ p : E \xrightarrow{T} L_r(\mu) \xrightarrow{q} \mathbb{K} \text{ as before}\}$$

  is denoted by $\mathcal{J}_r^n(E)$.

- *r-integrable polynomials*, $\mathcal{I}_r^n$.

  If μ is a finite, positive measure on Ω and $n \leq r \leq \infty$, the n-th integrating polynomial is defined by $q_{\mu,r}^n(f) := \int f^n d\mu$. It is straightforward to see that $\|q_{\mu,r}^n\| = \mu(\Omega)^{1/s}$ where $s = (\frac{r}{n})'$. A polynomial $p \in \mathcal{P}^n(E)$ is *r-integral* if it admits a factorization

  $$p : E \xrightarrow{T} L_r(\Omega) \xrightarrow{q_{\mu,r}^n} \mathbb{K}$$

  with a finite, positive measure μ and $T \in \mathcal{L}(E, L_r(\Omega))$. We denote the space of all such polynomials by $\mathcal{I}_r^n(E)$, which is a Banach space with the norm

  $$\|p\|_{\mathcal{I}_r^n(E)} = \inf\{\|q_{\mu,r}^n\|\|T\|^n \ : \ p = q_{\mu,r}^n \circ T \text{ as before}\}.$$

We will need also, for our purposes, the following definition. A polynomial $p : \ell_2 \to \mathbb{K}$ is *Hilbert-Schmidt* if

$$\Big(\sum_{k_1,\ldots,k_n=1}^{\infty} |\overset{\vee}{p}(e_{k_1}, \ldots, e_{k_n})|^2\Big)^{1/2}$$

is finite. The space of all such polynomials will be denoted by $\mathcal{P}_{HS}^n(\ell_2)$ and it is a Banach space with the norm $\|p\|_{\mathcal{P}_{HS}^n(\ell_2)} = (\sum_{k_1,\ldots,k_n=1}^{\infty} |\overset{\vee}{p}(e_{k_1}, \ldots, e_{k_n})|^2)^{1/2}$.

1.1.1 Minimal kernel

The minimal kernel of $\mathcal{Q}$ is defined as the composition ideal $\mathcal{Q}^{min} := \mathcal{Q} \circ \overline{\mathfrak{F}}$, where $\overline{\mathfrak{F}}$ stands for the ideal of approximable operators. In other words, a polynomial p belongs to $\mathcal{Q}^{min}(E)$ if it admits a factorization

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "T"'] & & \mathbb{K} \\ & F \arrow[ur, "q"'] & \end{tikzcd}, \tag{1.1}$$

where F is a Banach space, $T : E \to F$ is an approximable operator and a polynomial q is in $\mathcal{Q}(F)$. The minimal norm of p is given by $\|p\|_{\mathcal{Q}^{min}} := \inf\{\|q\|_{\mathcal{Q}(F)}\|T\|^n\}$, where the infimum runs over all possible factorizations as in (1.1).

We have the following properties.

Proposition 1.1.1. *[Flo01a]*

- $\mathcal{Q}^{\min} \subset \mathcal{Q}$ *with* $\|\cdot\|_{\mathcal{Q}} \leq \|\cdot\|_{\mathcal{Q}^{\min}}$.
- $(\mathcal{Q}^{\min})^{\min} \overset{1}{=} \mathcal{Q}^{\min}$.
- $\mathcal{Q}^{\min}$ *is the smallest ideal of n-homogeneous polynomials such that* $\mathcal{Q}^{\min}(M) \overset{1}{=} \mathcal{Q}(M)$ *for every finite dimensional Banach space M.*
- *If E' has the metric approximation property (see Definition 2.1.1), then* $\mathcal{Q}^{\min}(E) \overset{1}{\hookrightarrow} \mathcal{Q}(E)$ *and* $\mathcal{Q}^{\min}(E) \overset{1}{=} \overline{\mathcal{P}_f^n(E)}^{\|\cdot\|_{\mathcal{Q}}}$.

A Banach polynomial ideal is said to be *minimal* if $\mathcal{Q}^{\min} \overset{1}{=} \mathcal{Q}$.

For example, the ideals of nuclear and approximable polynomials are minimal. Moreover,

$$(\mathcal{P}_I^n)^{\min} = \mathcal{P}_N^n \quad \text{and} \quad (\mathcal{P}^n)^{\min} = \mathcal{P}_{app}^n.$$

1.1.2 Maximal hull

Let $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ be an ideal of continuous scalar valued n-homogeneous polynomials and, for $p \in \mathcal{P}^n(E)$, define

$$\|p\|_{\mathcal{Q}^{max}(E)} := \sup\{\|p|_M\|_{\mathcal{Q}(M)} : M \in FIN(E)\} \in [0, \infty].$$

The maximal hull of $\mathcal{Q}$ is the ideal given by $\mathcal{Q}^{max} := \{p \in \mathcal{P}^n : \|p\|_{\mathcal{Q}^{max}} < \infty\}$.

An ideal $\mathcal{Q}$ is said to be *maximal* if $\mathcal{Q} \overset{1}{=} \mathcal{Q}^{max}$. For example, $\mathcal{P}^n, \mathcal{P}^n_I, \mathcal{P}^n_e, \mathcal{D}^n_r, \mathcal{L}^n_r$ are maximal ideals. Also,

$$(\mathcal{P}^n_N)^{\max} = \mathcal{P}^n_I \quad \text{and} \quad (\mathcal{P}^n_{app})^{\max} = \mathcal{P}^n.$$

We have the following relations.

Proposition 1.1.2. *[Flo01a]*

- $\mathcal{Q} \subset \mathcal{Q}^{\max}$ *with* $\|\cdot\|_{\mathcal{Q}^{\max}} \leq \|\cdot\|_{\mathcal{Q}}$.
- $(\mathcal{Q}^{\max})^{\max} \overset{1}{=} \mathcal{Q}^{\max}$.
- $\mathcal{Q}^{\max}$ *is the greatest ideal of* n*-homogeneous polynomials such that* $\mathcal{Q}^{\max}(M) \overset{1}{=} \mathcal{Q}(M)$ *for every finite dimensional Banach space* M.
- $(\mathcal{Q}^{\max})^{\min} \overset{1}{=} \mathcal{Q}^{\min}$ *and* $(\mathcal{Q}^{\min})^{\max} \overset{1}{=} \mathcal{Q}^{\max}$.

1.2 Tensor products and tensor norms

For a normed space E, we denote by $\otimes^n E$ the n-fold tensor product of E . For simplicity, $\otimes^n x$ will stand for the elementary tensor $x \otimes \cdots \otimes x$. The subspace of $\otimes^n E$ consisting of all tensors of the form $\sum_{j=1}^r \lambda_j \otimes^n x_j$, where λ_j is a scalar and $x_j \in E$ for all j, is called the *symmetric* n*-fold tensor product of* E and it is denoted by $\otimes^{n,s} E$. When E is a vector space over $\mathbb{C}$, the scalars are not needed in the previous expression. For simplicity, we use the complex notation, although most of our results will hold for real and complex spaces. Denote by δ_n the canonical mapping from E to $\otimes^{n,s} E$. The symmetric tensor product has the following universal property: for every n-homogeneous polynomial $p : E \to \mathbb{K}$ there exist a unique linear functional L_p such that the diagram

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "\delta_n"'] & & \mathbb{K} \\ & \otimes^{n,s} E \arrow[ur, dashed, "L_p"'] & \end{tikzcd} \tag{1.2}$$

commutes. Moreover, every linear functional on the symmetric tensor product $L : \otimes^{n,s} E \to \mathbb{K}$ defines an n-homogeneous polynomial given by $L \circ \delta_n$. So, from now on, we identify indistinctly n-homogeneous polynomials with linear functionals on the n-fold symmetric tensor product. We often write

$$\langle p, \sum_{j=1}^r \otimes^n x_j \rangle := L_p(\sum_{j=1}^r \otimes^n x_j) = \sum_{j=1}^r p(x_j).$$

If $\sigma \in \mathcal{S}_n$, then the n-linear mapping $E^n \to \otimes^n E$ defined by

$$(x_1, \dots, x_n) \mapsto x_{\sigma^{-1}(1)} \otimes \dots x_{\sigma^{-1}(n)},$$

has a linearization $\otimes^n E \to \otimes^n E$ which will be denoted by $z \mapsto z^\sigma$. For $x_1, \dots, x_n$ we define

$$x_1 \vee \dots \vee x_n := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} x_{\sigma^{-1}(1)} \otimes \dots x_{\sigma^{-1}(n)} \in \otimes^n E \tag{1.3}$$

and for $z \in \otimes^n E$ we define

$$\sigma_E^n(z) := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} z^\sigma \in \otimes^n E, \tag{1.4}$$

which is a linearization of the symmetric n-linear form $\vee : E^n \to \otimes^n E$. The symmetric tensor product $\otimes^{n,s} E$ is a complemented subspace of the n-fold tensor product $\otimes^n E$, and the projection is given precisely by σ_E^n (this mapping is referred to as *the symmetrization operator*).

Given a normed space E and a continuous operator $T\colon E \to F$, *the symmetric n-tensor power of T* (or the *tensor operator of T*) is the mapping from $\otimes^{n,s} E$ to $\otimes^{n,s} F$ defined by

$$\Big(\otimes^{n,s} T \Big)(\otimes^n x) = \otimes^n (Tx)$$

on the elementary tensors and extended by linearity.

Since $\mathcal{P}_f^n(E)$ can be canonically identified with $\otimes^{n,s} E'$, given a finite-type polynomial $p = \sum_{j=1}^r (x_j')^n \in \mathcal{P}_f^n(E)$ we say that the tensor $z := \sum_{j=1}^r \otimes^n x_j' \in \otimes^{n,s} E'$ *represents* p. Analogously, any given tensor $z = \sum_{j=1}^s \otimes^n y_j'$ always represents a finite-type polynomial (for example, the one given by $\sum_{j=1}^s (y_j')^n$).

The *symmetric projective norm*, $\pi_{n,s}$, is computed in the following way:

$$\pi_{n,s}(z) = \inf \left\{ \sum_{j=1}^r \|x_j\|^n \right\},$$

where the infimum is taken over all the representations of the tensor z of the form $\sum_{j=1}^r \otimes^n x_j$. We denote by $\otimes_{\pi_{n,s}}^{n,s} E$ the symmetric n-fold tensor product of E endowed with the norm $\pi_{n,s}$. This norm is uniquely defined by the property

$$\mathcal{P}^n(E) \stackrel{1}{=} (\otimes_{\pi_{n,s}}^{n,s} E)'.$$

On the other hand, $\otimes_{\varepsilon_{n,s}}^{n,s} E$ the symmetric n-fold tensor product of E equipped with the norm $\varepsilon_{n,s}$ (*the symmetric injective norm*) satisfies, by definition,

$$\otimes_{\varepsilon_{n,s}}^{n,s} E \stackrel{1}{\hookrightarrow} \mathcal{P}^n(E').$$

In other words, for a tensor $z \in \otimes^{n,s} E$ we have

$$\varepsilon_{n,s}(z) = \sup_{x' \in B_{E'}} \left| \sum_{j=1}^r x'(x_j)^n \right|,$$

where $\sum_{j=1}^{r} \otimes^n x_j$ is any fixed representation of z. We also get the isometric identification

$$\mathcal{P}_I^n(E) \overset{1}{=} (\otimes_{\varepsilon_{n,s}}^{n,s} E)'.$$

For a complete treatment of these two classical norms ($\varepsilon_{n,s}$ and $\pi_{n,s}$) see [Flo97].

More generally, *reasonable* symmetric tensor norms are defined as follows. We say that α is an *s-tensor norm of order* n if α assigns to each normed space E a norm $\alpha\big(\ . \ ; \otimes^{n,s} E\big)$ on the n-fold symmetric tensor product $\otimes^{n,s} E$ such that

1. $\varepsilon_{n,s} \leq \alpha \leq \pi_{n,s}$ on $\otimes^{n,s} E$.
2. $\| \otimes^{n,s} T : \otimes_\alpha^{n,s} E \to \otimes_\alpha^{n,s} F\| \leq \|T\|^n$ for each operator $T \in \mathcal{L}(E, F)$.

Condition (2) will be referred to as the *metric mapping property*. We denote by $\otimes_\alpha^{n,s} E$ the tensor product $\otimes^{n,s} E$ endowed with the norm $\alpha\big(\ . \ ; \otimes^{n,s} E\big)$, and we write $\widetilde{\otimes}_\alpha^{n,s} E$ for its completion. When not stated, we will always assume that α has order n.

An s-tensor norm α is called *finitely generated* if for every normed space E and $z \in \otimes^{n,s} E$, we have:

$$\alpha(z, \otimes^{n,s} E) = \inf\{\alpha(z, \otimes^{n,s} M) : M \in FIN(E),\ z \in \otimes^{n,s} M\}.$$

For example, π_s and ε_s are finitely generated s-tensor norms.

The norm α is called *cofinitely generated* if for every normed space E and $z \in \otimes^{n,s} E$, we have:

$$\alpha(z, \otimes^{n,s} E) = \sup\{\alpha\big((\otimes^{n,s} Q_L^E)(z), \otimes^{n,s} E/L\big) : L \in COFIN(E)\},$$

where $Q_L^E : E \overset{1}{\twoheadrightarrow} E/L$ is the canonical quotient mapping.

If α is an s-tensor norm of order n, then the *dual tensor norm* α' is defined on FIN (the class of finite dimensional spaces) by

$$\otimes_{\alpha'}^{n,s} M :\overset{1}{=} (\otimes_\alpha^{n,s} M')' \tag{1.5}$$

and on $NORM$ (the class of normed spaces) by

$$\alpha'(z, \otimes^{n,s} E) := \inf\{\alpha'(z, \otimes^{n,s} M) : z \in \otimes^{n,s} M\},$$

the infimum being taken over all of finite dimensional subspaces M of E whose symmetric tensor product contains z. By definition, α' is always finitely generated. It follows that $\pi_{n,s}' = \varepsilon_{n,s}$ and $\varepsilon_{n,s}' = \pi_{n,s}$.

Given a tensor norm α its *"finite hull"* $\overrightarrow{\alpha}$ is defined by the following way. For $z \in \otimes^{n,s} E$, we set

$$\overrightarrow{\alpha}(z, \otimes^{n,s} E) := \inf\{\alpha(z; \otimes^{n,s} M) : M \in FIN(E), z \in \otimes^{n,s} M\}.$$

Another important remark is in order: since α and α'' coincide on finite dimensional spaces we have

$$\overrightarrow{\alpha}(z; \otimes^{n,s} E) = \inf\{\alpha''(z; \otimes^{n,s} M) : M \in FIN(E), z \in \otimes^{n,s} M\} = \alpha''(z; \otimes^{n,s} E),$$

where the second equality is due to the fact that dual norms are always finitely generated. Therefore,

$$\overrightarrow{\alpha} = \alpha''; \tag{1.6}$$

and $\alpha = \alpha''$ if and only if α is finitely generated.

The *"cofinite hull"* $\overleftarrow{\alpha}$ is given by

$$\overleftarrow{\alpha}(z; \otimes^{n,s} E) := \sup\{\alpha\big((\otimes^{n,s} Q_L^E)(z); \otimes^{n,s} E/L\big) : L \in COFIN(E)\}.$$

Is not hard to see that the "finite hull" $\overrightarrow{\alpha}$ (the "cofinite hull" $\overleftarrow{\alpha}$) is the unique finitely generated s-tensor norm (cofinitely generated s-tensor norm) that coincides with α on finite dimensional spaces. By the metric mapping property, it is enough to take cofinally many M (or L) in the definitions of the finite (or cofinite) hull. Using the metric mapping property again we have

$$\overleftarrow{\alpha} \leq \alpha \leq \overrightarrow{\alpha}.$$

If $\mathcal{Q}$ is a Banach polynomial ideal, its *associated s-tensor norm* is the unique finitely generated tensor norm α satisfying

$$\mathcal{Q}(M) \overset{1}{=} \otimes_\alpha^{n,s} M,$$

for every finite dimensional space M. For example, the s-tensor norm associated with $\mathcal{P}^n$ and $\mathcal{P}^n_{app}$ is $\varepsilon_{n,s}$ and the s-tensor norm associated with $\mathcal{P}^n_I$ and $\mathcal{P}^n_N$ is $\pi_{n,s}$.

Notice that $\mathcal{Q}$, $\mathcal{Q}^{max}$ and $\mathcal{Q}^{min}$ have the same associated s-tensor norm since they coincide isometrically on finite dimensional spaces.

Since any s-tensor norm satisfies $\alpha \leq \pi_{n,s}$, we have a dense inclusion

$$\otimes_\alpha^{n,s} E \hookrightarrow \otimes_\pi^{n,s} E.$$

As a consequence, any $p \in (\otimes_\alpha^{n,s} E)'$ identifies with a continuous n-homogeneous polynomial on E. Different s-tensor norms α give rise, by this duality, to different polynomial ideals. Ideals which are of this type are exactly the maximal ones, as it is seen in the following theorem.

Theorem 1.2.1. *(***Representation Theorem for Maximal Polynomial Ideals.***) [FH02] A normed ideal of n-homogeneous polynomials $\mathcal{Q}$ is maximal if and only if*

$$\mathcal{Q}(E) \overset{1}{=} (\otimes_{\alpha'}^{n,s} E)', \tag{1.7}$$

where α is the s-tensor norm associated with $\mathcal{Q}$. The norm α' is sometimes called the predual norm of $\mathcal{Q}$.

In particular, if α is associated with a given polynomial ideal $\mathcal{Q}$ we have:

$$\mathcal{Q}^{max}(E) \overset{1}{=} (\otimes_{\alpha'}^{n,s} E)'.$$

Let $\mathcal{Q}$ be a maximal polynomial ideal with associated s-tensor norm α. The following theorem due to Floret [Flo01a, Theorem 4.2] exhibits the close relation between $\otimes_\alpha^{n,s} E'$ and $\mathcal{Q}^{min}(E)$.

Theorem 1.2.2. **(Representation Theorem for Minimal Polynomial Ideals.)** *Let $\mathcal{Q}$ be a minimal polynomial ideal with associated s-tensor α. There is a natural quotient mapping*

$$\widetilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{\twoheadrightarrow} \mathcal{Q}(E) \tag{1.8}$$

defined on $\otimes^{n,s} E'$ by the obvious rule $z = \sum_{j=1}^{r} \otimes^n x'_j \mapsto \sum_{j=1}^{r} (x'_j)^n$ (the polynomial represented by the tensor z).

In particular, if α is associated with a given polynomial ideal $\mathcal{Q}$ we have:

$$\widetilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{\twoheadrightarrow} \mathcal{Q}^{min}(E).$$

We now recall the definition of adjoint ideal [Flo01a], which is closely related with the theory of s-tensor norms. For $q \in \mathcal{P}^n(E)$ we define

$$\|q\|_{\mathcal{Q}^*(E)} := \sup\{|\langle q|_M, p\rangle| M \in FIN(E), \|p\|_{\mathcal{Q}(M')} \leq 1\} \in [0, \infty],$$

Here $\langle q|_M, p\rangle$ stands for $\langle q|_M, z\rangle$, where z is any given tensor in $\otimes^{n,s} M$ that represents the finite type polynomial $p \in \mathcal{P}^n(M')$.

The *adjoint* ideal of $\mathcal{Q}$, denoted by $\mathcal{Q}^*$, is the class of all polynomials q such that $\|q\|_{\mathcal{Q}^*} < \infty$. It is not difficult to prove that $\left(\mathcal{Q}^*, \| \ \|_{\mathcal{Q}^*}\right)$ is a maximal Banach ideal of continuous n-homogeneous polynomials. Moreover, if α is the s-tensor norm associated with the ideal $\mathcal{Q}$ then α' is the one associated with $\mathcal{Q}^*$. Therefore, we have

$$\mathcal{Q}^*(E) \overset{1}{=} (\otimes_{\alpha}^{n,s} E)'.$$

For example,

$$(\mathcal{P}^n)^* = (\mathcal{P}^n_{app})^* = \mathcal{P}^n_I \quad \text{and} \quad (\mathcal{P}^n_N)^* = (\mathcal{P}^n_I)^* = \mathcal{P}^n.$$

We denote by $\mathcal{Q}_\alpha$ the maximal Banach ideal of α-continuous n-homogeneous polynomials, that is, $\mathcal{Q}_\alpha(E) := (\otimes_{\alpha}^{n,s} E)'$. We observe that, with this notation, $\mathcal{Q}_\alpha$ is the unique maximal polynomial ideal associated with the s-tensor norm α'.

The theory of full tensor norms of order n and the theory of ideals of multilinear forms are not defined in this text since the basics are completely analogous to the theory of s-tensor norms and the theory of polynomial ideals presented. We refer to [Flo01a, FG03, FH02] and the references therein for more information on these topics. Everything we are going to use is a straightforward generalization of the case $n = 2$.

1.3 The Arens extension morphism and the Aron-Berner extension

Let $E_1, \ldots, E_n$ be normed spaces and $A : E_1 \times \cdots \times E_n \to \mathbb{K}$ be an n-linear form. There is an easy way to extend the k-th variable, E_k, to the bidual E''_k. Namely, by weak-star continuity. In other words, we define the *k-th canonical extension of A*,

$$EXT_k(A) : E_1 \times \ldots E_{k-1} \times E''_k \times E_{k+1} \times \cdots \times E_n \to \mathbb{K}$$

in the following way:

$$EXT_k(A)(x_1,\dots,x_{k-1},x_k'',x_{k+1},\dots,x_n) := \lim_{x_{k,\gamma}\overset{w^*}{\to}x_k''} T(x_1,\dots,x_{k-1},x_{k,\gamma},x_{k+1}\dots,x_n),$$

for all $x_j \in E_j$ (for $1 \le j \le n$, $j \neq k$), $x_k'' \in E_k''$, where $x_{k,\gamma} \overset{w^*}{\to} x_k''$ stands for any bounded net on E_k weak-star convergent to x_k''. We denote by EXT_k the linear operator

$$\mathcal{L}(E_1,\dots E_{k-1},E_k,E_{k+1},\dots,E_n) \to \mathcal{L}(E_1,\dots E_{k-1},E_k'',E_{k+1},\dots,E_n)$$

defined by the above formula.

The *Arens-extension morphism* EXT is the linear mapping

$$EXT : \mathcal{L}(E_1,\dots,E_n) \to \mathcal{L}(E_1'',\dots,E_n'')$$

given by $(EXT_n)\circ\cdots\circ(EXT_1)$ (we extend from the left to the right). This extension is also referred to as the *iterated canonical extension*.

Let $A : E\times\cdots\times E \to \mathbb{K}$ be a symmetric n-linear form. The Arens extension of A, $EXT(A)$, is an n-linear form on E'' which, in general, is not symmetric. Moreover, we have chosen an order to pick the variables of A, and usually, the extension obtained depends on this order. However, it has the following properties:

- If $x \in E$ and $x_1'',\dots,x_{n-1}'' \in E''$ then

$$\begin{aligned} EXT(A)(x,x_1'',\dots,x_{n-1}'') &= EXT(A)(x_1'',x,\dots,x_{n-1}'') \\ &= \dots \\ &= EXT(A)(x_1'',\dots,x_{n-1}'',x). \end{aligned}$$

- It is w^*-w^*-continuous in the n-th variable (the last variable we extended).
- $\|EXT(A)\|_{\mathcal{L}(^nE'')} = \|A\|_{\mathcal{L}(^nE)}$.
- $EXT(A)$ is separately w^*-continuous on each variable if and only if $EXT(A)$ is symmetric.

We now define a way of extending polynomials into the bidual. If $p \in \mathcal{P}^n(E)$, then *its Aron-Berner extension* [AB78] $AB(p) \in \mathcal{P}^n(E'')$ is defined as

$$AB(p)(x'') := EXT(\overset{\vee}{p})(x'',\dots,x'').$$

In order to show that some holomorphic functions defined on the unit ball of E can be extended to the ball of E'', Davie and Gamelin [DG89] proved that the Aron-Berner extension preserves the norm of the polynomial. In other words, the Aron-Berner extension morphism $AB : \mathcal{P}^n(E) \to \mathcal{P}^n(E'')$ is an isometry. Moreover, they extended Goldstine's theorem: they showed that B_E is polynomial-star dense in $B_{E''}$, that is, for each $x'' \in B_{E''}$ there exists a net $(x_\gamma)_\gamma \subset B_E$, such that $p(x_\gamma) \to AB(p)(x'')$ for every polynomial p.

Chapter 2

The Five Basic Lemmas for symmetric tensor products

In the theory of full 2-fold tensor norms, "The Five Basic Lemmas" (see Section 13 in Defant and Floret's book [DF93]) are rather simple results which turn out to be "basic for the understanding and use of tensor norms". Namely, they are the Approximation Lemma, the Extension Lemma, the Embedding Lemma, the Density Lemma and the $\mathcal{L}_p$-Local Technique Lemma. Applications of these lemmas can be seen throughout the book. We present here the analogous results for the symmetric setting. We also exhibit some applications as example of their potential. In order to obtain our five basic lemmas and their applications we follow the lines of [DF93]. Although some proofs are similar to the 2-fold case, the symmetric nature of our tensor products introduces some difficulties, as we can see, for example, in the symmetric version of the Extension Lemma 2.1.3, whose proof is much more complicated than that of its full 2-fold version.

In Section 2.1 we state and prove the five basic lemmas, together with some direct consequences. Applications to the metric theory of symmetric tensor norms and Banach polynomial ideals are given in Section 2.2.

2.1 The lemmas

Here we give in full detail the symmetric analogues to the five basic lemmas that appear in [DF93, Section 13]. Recall first the following definition.

Definition 2.1.1. A normed space E has the *λ-approximation property* if there is a net $(T_\eta)_\eta$ of finite rank operators in $\mathcal{L}(E, E)$ with norm bounded by λ such that T_η conveges to Id_E (the identity operator on E) uniformly on compact subsets of E. A given space has the *bounded approximation property* if it has the λ-approximation property for some λ. The 1-approximation property is referred to as the *metric approximation property*.

The first of the five basic lemmas states that for normed spaces with the bounded approximation property, it is enough to check dominations between s-tensor norms on finite dimensional subspaces.

Lemma 2.1.2. (Approximation Lemma.) *Let α and β be s-tensor norms and E be a normed space with the λ-approximation property and $c \geq 0$ such that*

$$\alpha \leq c\beta \ \text{ on } \ \otimes^{n,s} M,$$

for cofinally many $M \in FIN(E)$ (i.e., for every $N \in FIN(E)$ there exist a bigger finite dimensional subspace $M \supset N$ satisfying $\alpha \leq c\beta$ on $\otimes^{n,s} M$). Then

$$\alpha \leq \lambda^n c\beta \ \text{ on } \ \otimes^{n,s} E.$$

Proof. Take $(T_\eta)_\eta$ a net of finite rank operators with $\|T_\eta\| \leq \lambda$ and $T_\eta x \to x$ for all $x \in E$. Fix $z \in \otimes^{n,s} E$ and take $\varepsilon > 0$. Since the mapping $x \mapsto \otimes^n x$ is continuous from E to $\otimes_\alpha^{n,s} E$, we have $\alpha(z - T_\eta(z), \otimes^{n,s} E) < \varepsilon$ for some η large enough. If we take $M \supset T_\eta(E)$ satisfying the hypothesis of the lemma, by the metric mapping property of the s-tensor β we have

$$\begin{aligned}
\alpha(z; \otimes^{n,s} E) &\leq \alpha(z - \otimes^{n,s} T_\eta(z); \otimes^{n,s} E) + \alpha(\otimes^{n,s} T_\eta(z); \otimes^{n,s} E) \\
&\leq \varepsilon + \alpha(\otimes^{n,s} T_\eta(z); \otimes^{n,s} M) \\
&\leq \varepsilon + c\beta(\otimes^{n,s} T_\eta(z); \otimes^{n,s} M) \\
&\leq \varepsilon + c\|T_\eta : E \to M\|^n \beta(\otimes^{n,s} z; \otimes^{n,s} E) \\
&\leq \varepsilon + \lambda^n c\beta(\otimes^{n,s} z; \otimes^{n,s} E).
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we have $\alpha(z; \otimes^{n,s} E) \leq \lambda^n c\beta(z; \otimes^{n,s} E)$. $\square$

As we mention in Chapter 1, in order to show that some holomorphic functions defined on the unit ball of E can be extended to the ball of E'', Davie and Gamelin [DG89] proved that the Aron-Berner extension preserves the norm of the polynomial. If we look at the duality between polynomials and symmetric tensor products in (1.2), Davie and Gamelin's result states that for p in $(\widetilde{\otimes}_{\pi_{n,s}}^{n,s} E)'$, its Aron-Berner extension $AB(p)$ belongs to $(\widetilde{\otimes}_{\pi_{n,s}}^{n,s} E'')'$, and has the same norm as p. A natural question arises: if a polynomial p belongs to $(\widetilde{\otimes}_\alpha^{n,s} E)'$ for some s-tensor norm α, does its Aron-Berner extension $AB(p)$ belong to $(\widetilde{\otimes}_\alpha^{n,s} E'')'$? And what about their norms? The answer is given in the following result, which can be seen as a symmetric version of the Extension Lemma [DF93, 6.7.].

Lemma 2.1.3. (Extension Lemma.) *Let α be a finitely generated s-tensor norm and $p \in (\widetilde{\otimes}_\alpha^{n,s} E)'$ a polynomial. The Aron-Berner extension $AB(p)$ of p to the bidual E'' belongs to $(\widetilde{\otimes}_\alpha^{n,s} E'')'$ and*

$$\|p\|_{(\widetilde{\otimes}_\alpha^{n,s} E)'} = \|AB(p)\|_{(\widetilde{\otimes}_\alpha^{n,s} E'')'}.$$

We will postpone the proof of this lemma to the end of this section, where we treat extensions to ultrapowers.

As a consequence of the Extension Lemma 2.1.3 we also obtain a symmetric version of [DF93, Lemma 13.3], which shows that there is a natural isometric embedding from the symmetric tensor product of a Banach space and that of its bidual.

Lemma 2.1.4. (Embedding Lemma.) *If α is a finitely or cofinitely generated tensor norm, then the natural mapping*

$$\otimes^{n,s} \kappa_E : \otimes_\alpha^{n,s} E \longrightarrow \otimes_\alpha^{n,s} E''$$

is an isometry for every normed space E.

Proof. If $z \in \otimes^{n,s} E$, by the metric mapping property we have

$$\alpha(\otimes^{n,s} \kappa_E(z); \otimes^{n,s} E'') \leq \alpha(z; \otimes^{n,s} E).$$

Suppose α is finitely generated and let p a norm one polynomial in $(\otimes^{n,s}_\alpha E)'$ such that the norm $\alpha(z; \otimes^{n,s} E)$ is $\langle p, z\rangle$. Now notice that $\langle p, z\rangle = \langle AB(p), \otimes^{n,s} \kappa_E(z)\rangle$ which, by the Extension Lemma 2.1.3, is less than or equal to $\alpha(\otimes^{n,s} \kappa_E(z); \otimes^{n,s} E'')$. This shows the reverse inequality for finitely generated tensor norms.

Suppose now that α is cofinitely generated and let $L \in COFIN(E)$. Then L^{00} (the biannihilator in E'') is in $COFIN(E'')$ and the mapping

$$\kappa_{E/L} : E/L \to (E/L)'' = E''/L^{00}$$

is an isometric isomorphism. Moreover, we have $Q^{E''}_{L^{00}} \circ \kappa_E = \kappa_{E/L} \circ Q^E_L$.

Thus,

$$\begin{aligned}
\alpha(\otimes^{n,s} Q^E_L(z); \otimes^{n,s} E/L) &= \alpha(\otimes^{n,s} (\kappa_{F/L} \circ Q^E_L)(z); \otimes^{n,s} (E/L)'') \\
&= \alpha((\otimes^{n,s} Q^{E''}_{L^{00}} \circ \otimes^{n,s} \kappa_E)(z); \otimes^{n,s} E''/L^{00}) \\
&\leq \alpha(\otimes^{n,s} \kappa_E(z), \otimes^{n,s} E'').
\end{aligned}$$

If we take supremum over all $L \in COFIN(E)$ we obtain the desired inequality. □

Since E and its completion $\widetilde{E}$ have the same bidual, the Embedding Lemma 2.1.4 shows that finitely generated and cofinitely generated s-tensor norms respect dense subspaces. More precisely, we have the following.

Corollary 2.1.5. *Let α be a finitely or cofinitely generated s-tensor norm, E a normed space and $\widetilde{E}$ its completion. Then,*

$$\otimes^{n,s}_\alpha E \to \otimes^{n,s}_\alpha \widetilde{E}$$

is an isometric and dense embedding.

We obtain as a direct consequence the symmetric version of the Density lemma [DF93, Lemma 13.4.].

Lemma 2.1.6. (Density Lemma.) *Let α be a finitely or cofinitely generated tensor norm, E a normed space and E_0 a dense subspace of E. If p is an n-homogeneous continuous polynomial such that*

$$p|_{\otimes^{n,s} E_0} \in (\otimes^{n,s}_\alpha E_0)',$$

then $p \in (\otimes^{n,s}_\alpha E)'$ and $\|p\|_{(\otimes^{n,s}_\alpha E)'} = \|p\|_{(\otimes^{n,s}_\alpha E_0)'}$.

Before we state the fifth lemma, we need some definitions. For $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$ a normed space E is called an $\mathcal{L}^g_{p,\lambda}$*-space*, if for each $M \in FIN(E)$ and $\varepsilon > 0$ there are $R \in \mathcal{L}(M, \ell^m_p)$ and $S \in \mathcal{L}(\ell^m_p, E)$ for some $m \in \mathbb{N}$ factoring the embedding I^E_M:

$$\begin{tikzcd}
M \arrow[rr, hook, "I^M_E"] \arrow[dr, "R"'] & & E \\
& \ell^m_p \arrow[ur, "S"'] &
\end{tikzcd} \tag{2.1}$$

such that $\|S\|\|R\| \leq \lambda + \varepsilon$.

The space E is called an $\mathcal{L}_p^g$-*space* if it is an $\mathcal{L}_{p,\lambda}^g$-space for some $\lambda \geq 1$. Loosely speaking, $\mathcal{L}_p^g$-spaces share many properties of ℓ_p, since they locally look like ℓ_p^m. The spaces $C(K)$ and $L_\infty(\mu)$ are $\mathcal{L}_{\infty,1}^g$-spaces, while $L_p(\mu)$ are $\mathcal{L}_{p,1}^g$-spaces. For more information and properties of $\mathcal{L}_p^g$-spaces see [DF93, Section 23].

Now we state and prove our fifth basic lemma.

Lemma 2.1.7. ($\mathcal{L}_p$-Local Technique Lemma.) *Let α and β be s-tensor norms and $c \geq 0$ such that*

$$\alpha \leq c\beta \quad \text{on} \quad \otimes^{n,s} \ell_p^m,$$

for every $m \in \mathbb{N}$. If E is an $\mathcal{L}_{p,\lambda}^g$-space then

$$\alpha \leq \lambda^n c \overrightarrow{\beta} \quad \text{on} \quad \otimes^{n,s} E.$$

Proof. Fix $z \in \otimes^{n,s} E$ and $M \in FIN(E)$ such that $z \in \otimes^{n,s} M$. Thus, for the finite dimensional subspace M we take a factorization as in (2.1) with $\|R\|\|S\| \leq \lambda(1+\varepsilon)$. We therefore have

$$\begin{aligned}
\alpha(z; \otimes^{n,s} E) = \alpha(\otimes^{n,s}(S \circ R)(z), \otimes^{n,s} M) &\leq \|S\| \alpha(\otimes^{n,s} R(z), \otimes^{n,s} \ell_p^m) \\
&\leq \|S\|^n c\beta(\otimes^{n,s} R(z), \otimes^{n,s} \ell_p^m) \leq c\|S\|^n \|R\|^n \beta(z; \otimes^{n,s} M). \\
&\leq \lambda^n c(1+\varepsilon)^n \beta(z; \otimes^{n,s} M).
\end{aligned}$$

Since ε is arbitrary, taking infimum over all finite dimensional subspaces M such that $z \in \otimes^{n,s} M$, we obtain

$$\alpha \leq \lambda^n c \overrightarrow{\beta}$$

as desired. □

Extensions to Ultrapowers and the proof of the Extension Lemma

Before giving the proof of the Extension Lemma 2.1.3 we need to recall some basic properties of ultrapowers. The reader is referred to [Hei80, Kür76] for further details. Let $\mathfrak{U}$ be an ultrafilter on a set I. Whenever the limit with respect to $\mathfrak{U}$ of a family $\{a_i : i \in I\}$ exists, we denote it by $\lim_{i,\mathfrak{U}} a_i$. For a Banach space E, $(E)_\mathfrak{U}$, the *ultrapower of E respect to the filter* $\mathfrak{U}$, consists in classes of elements of the form $z = (z_i)_\mathfrak{U}$, with $z_i \in E$, for each $i \in I$, where the norm of (z_i) is uniformly bounded, and where we identify (z_i) with (y_i) if $\lim_{i,\mathfrak{U}} \|z_i - y_i\| = 0$. The space $(E)_\mathfrak{U}$ is a Banach space under the norm

$$\|(z_i)_\mathfrak{U}\| = \lim_{i,\mathfrak{U}} \|z_i\|.$$

We may consider E as a subspace of the ultrapower $(E)_\mathfrak{U}$ by means of the canonical embedding $h_E : E \hookrightarrow (E)_\mathfrak{U}$ given by $h_E x = (x_i)_\mathfrak{U}$ where $x_i = x$ for all i.

Let us now define the ultrapower of an operator. If $T : E \to F$ is a bounded linear operator, the *ultrapower operator of T associated with the ultrafilter* $\mathfrak{U}$ will be the operator from $(E)_\mathfrak{U}$ to $(F)_\mathfrak{U}$ defined according the following rule $(z_i)_\mathfrak{U} \mapsto (Tz_i)_\mathfrak{U}$. We denote this operator $(T)_\mathfrak{U}$. It can be seen that $\|(T)_\mathfrak{U}\|$ is equal to $\|T\|$.

We also need a special property of ultrapowers [Hei80, Proposition 6.1], [Kür76, Statz 4.1.].

Proposition 2.1.8. *(***Local determination of ultrapowers.***) Let E be a Banach space and $M \in FIN((E)_{\mathfrak{U}})$. For each $\mathfrak{i} \in I$ there exist an operator $R_{\mathfrak{i}} \in \mathcal{L}(M, E)$ such that*

(1) $z = (R_{\mathfrak{i}} z)_{\mathfrak{U}}$ for all $z \in M$;

(2) $\|R_{\mathfrak{i}}\| \leq 1$ for all $\mathfrak{i} \in I$ and there is an $\mathcal{U} \in \mathfrak{U}$ with $\|R_{\mathfrak{i}}\| = 1$ for all $\mathfrak{i} \in \mathcal{U}$;

(3) for all $\varepsilon > 0$ there is an $\mathcal{U}_\varepsilon \in \mathfrak{U}$ such that the inverse $R_{\mathfrak{i}}^{-1} : R_{\mathfrak{i}}(M) \to M$ exist and $\|R_{\mathfrak{i}}\| \leq 1 + \varepsilon$ for all $\mathfrak{i} \in \mathcal{U}_\varepsilon$.

We shall only use (1) and the first part of (2).

Let $(E)_{\mathfrak{U}}$ be an ultrapower of a Banach space E. For a continuous n-linear form Φ on E we define an n-linear form $\overline{\Phi}$ on $(E)_{\mathfrak{U}}$ by

$$\overline{\Phi}(z_1, \ldots, z_n) = \lim_{\mathfrak{i}_1, \mathfrak{U}} \ldots \lim_{\mathfrak{i}_n, \mathfrak{U}} \Phi(z^{(1)}_{\mathfrak{i}_1}, \ldots, z^{(n)}_{\mathfrak{i}_n}),$$

for $z_j = (z^{(j)}_{\mathfrak{i}_j})_{\mathfrak{U}} \in (E)_{\mathfrak{U}}$ $(1 \leq j \leq n)$. The n-linear form $\overline{\Phi}$ defined on $(E)_{\mathfrak{U}}$ will be referred to as the *ultra-iterated extension of* Φ. If p is an n-homogeneous continuous polynomial and A is its associated symmetric n-linear mapping, the *ultra-iterated extension*, $\overline{p}$, of p to $(E)_{\mathfrak{U}}$ is defined by

$$\overline{p}((z_{\mathfrak{i}})_{\mathfrak{U}}) := \overline{A}((z_{\mathfrak{i}})_{\mathfrak{U}}, \ldots, (z_{\mathfrak{i}})_{\mathfrak{U}}) = \lim_{\mathfrak{i}_1, \mathfrak{U}} \ldots \lim_{\mathfrak{i}_n, \mathfrak{U}} A(z_{\mathfrak{i}_1}, \ldots, z_{\mathfrak{i}_n}).$$

Theorem 2.1.9. *Let α be a finitely generated s-tensor norm and $p \in (\widetilde{\otimes}^{n,s}_\alpha E)'$ a polynomial. The ultra-iterated extension $\overline{p}$ of p to the ultrapower $(E)_{\mathfrak{U}}$ belongs to $(\widetilde{\otimes}^{n,s}_\alpha (E)_{\mathfrak{U}})'$ and*

$$\|p\|_{(\widetilde{\otimes}^{n,s}_\alpha E)'} = \|\overline{p}\|_{(\widetilde{\otimes}^{n,s}_\alpha (E)_{\mathfrak{U}})'}.$$

We need some remarks and lemmas to prove this theorem.

First, let A be the symmetric multilinear form associated with a polynomial p (i.e., $A = \overset{\vee}{p}$). For each fixed j, $1 \leq j \leq n$, $x_1, \ldots, x_{j-1} \in E$, and $z_j, z_{j+1}, \ldots z_n \in (E)_{\mathfrak{U}}$, we have

$$\overline{A}(h_E x_1, \ldots, h_E x_{j-1}, z_j, z_{j+1}, \ldots, z_n) = \lim_{i_j, \mathfrak{U}} \overline{A}(h_E x_1, \ldots, h_E x_{j-1}, h_E z^{(j)}_{\mathfrak{i}_j}, z_{j+1}, \ldots, z_n),$$

where $\overline{A}$ is the ultra-iterated extension of A to $(E)_{\mathfrak{U}}$.

Now, we imitate the procedure used by Davie and Gamelin in [DG89]. Let A be the symmetric n-linear form associated with p. We have the following lemma:

Lemma 2.1.10. *Let $M \in FIN((E)_{\mathfrak{U}})$ and $z_1, \ldots, z_r \in M$. For a given natural number m, and $\varepsilon > 0$ there exist operators $R_1, \ldots, R_m \in \mathcal{L}(M, E)$ with norm less than or equal to 1 such that*

$$\left| A(R_{i_1} z_k, \ldots, R_{i_n} z_k) - \overline{A}(z_k, \ldots, z_k) \right| < \varepsilon \tag{2.2}$$

for every $i_1, \ldots, i_n$ distinct indices between 1 and m and every $k = 1, \ldots, r$.

Proof. Since A is symmetric, in order to prove the Lemma it suffices to obtain (2.2) for $i_1 < \cdots < i_n$. We select the operator $R_1, \ldots, R_m$ inductively by the following procedure: by Proposition 2.1.8, for each $\mathfrak{i} \in I$ there exist an operator $R_\mathfrak{i} \in \mathcal{L}(M, E)$ with norm less than or equal to 1 such that $z_k = (R_\mathfrak{i} z_k)_\mathfrak{U}$.

Since $z_k = (R_\mathfrak{i} z_k)_\mathfrak{U}$ for each k, the set

$$\{\mathfrak{i} \in I \;:\; \overline{A}(h_E R_\mathfrak{i} z_k, z_k, \ldots, z_k) - \overline{A}(z_k, z_k, \ldots, z_k)\big| < \varepsilon/n\}$$

belongs to the filter $\mathfrak{U}$. Therefore, we can pick $R_1 \in \mathcal{L}(M, E)$ such that

$$\big|\overline{A}(h_E R_1 z_k, z_k, \ldots, z_k) - \overline{A}(z_k, z_k, \ldots, z_k)\big| < \varepsilon/n,$$

for every $k = 1, \ldots, r$. In a similar way we can choose R_2 such that

$$\big|\overline{A}(h_E R_2 z_k, z_k, \ldots, z_k) - \overline{A}(z_k, z_k, \ldots, z_k)\big| < \varepsilon/n,$$

and moreover,

$$\big|\overline{A}(h_E R_1 z_k, h_E R_2 z_k, z_k, \ldots, z_k) - \overline{A}(h_E R_1 z_k, z_k, \ldots, z_k)\big| < \varepsilon/n,$$

for every $k1, \ldots, r$. Proceeding in this way, we get R_l's so that

$$\big|\overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_{r-1}} z_k, h_E R_{i_r} z_k, z_k, \ldots, z_k) - \overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_{r-1}} z_k, z_k, \ldots, z_k)\big|$$

is less than ε/n, whenever $i_1 < \cdots < i_r$ and $k = 1, \ldots r$.

Then,

$$\big|\overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_n} z_k) - \overline{A}(z_k \ldots, z_k)\big|$$

is estimated by the sum of n terms

$$\big|\overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_n} z_k) - \overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_{n-1}} z_k, z_k)\big| + \ \ldots$$
$$+\big|\overline{A}(h_E R_{i_1} z_k, z_k \ldots, z_k) - \overline{A}(z_k, \ldots, z_k)\big|,$$

each smaller than ε/n, for all $k = 1 \ldots, r$. □

Proposition 2.1.11. *Let $M \in FIN((E)_\mathfrak{U})$, $z_1, \ldots, z_r \in M$, $p : E \to \mathbb{K}$ a continuous polynomial and $\varepsilon > 0$. There exist operators $(R_i)_{1 \le i \le m}$ in $\mathcal{L}(M, E)$ with norm less than or equal to 1, such that*

$$\Big|\sum_{k=1}^{r} \overline{p}(z_k) - \sum_{k=1}^{r} p(\frac{1}{m}\sum_{i=1}^{m} R_i z_k)\Big| < \varepsilon.$$

Proof. For $\varepsilon > 0$, fix m large enough and choose $R_1, \ldots, R_m$ as in the previous lemma, such that

$$\big|\overline{A}(z_k, \ldots, z_k) - A(R_{i_1} z_k, \ldots, R_{i_n} z_k)\big| < \varepsilon/2r$$

for every $i_1, \ldots, i_n$ distinct indices between 1 and m and every $k = 1, \ldots, r$. We have for $k \in \{1, \ldots, r\}$,

$$\big|\overline{p}(z_k) - p(\frac{1}{m}\sum_{i=1}^{m} R_i z_k)\big| = \big|\frac{1}{m^n}\sum_{i_1,\ldots,i_n=1}^{m} [\overline{A}(z_k, \ldots, z_k) - A(R_{i_1} z_k, \ldots, R_{i_n} z_k)]\big|$$
$$\le |\Sigma_1^k| + \le |\Sigma_2^k|,$$

where Σ_1^k is the sum over the n-tuples of non-repeated indices (which is less than $\varepsilon/2r$) and Σ_2^k is the sum over the remaining indices. It is easy to show that there are exactly $m^n - \prod_{j=0}^{n-1}(m-j)$ summands in Σ_2^k, each bounded by a constant $C > 0$ (obviously we can assume that C is independent of k), thus

$$\left|\Sigma_2^k\right| \le \frac{1}{m^n}\left(m^n - \prod_{j=0}^{n-1}(m-j)\right)C = \left[1-(1-\frac{1}{m})\dots(1-\frac{n-1}{m})\right]C.$$

Taking m sufficiently large this is less than $\varepsilon/2r$. $\square$

We can now give a proof of Theorem 2.1.9.

Proof. (*of Theorem* 2.1.9.)

Let $w \in \otimes^{n,s}M$, where $M \in FIN((E)_{\mathfrak{U}})$. Since α is finitely generated, we only have to show that

$$|\langle\overline{p}, w\rangle| \le \|p\|_{(\tilde{\otimes}_\alpha^{n,s}E)'}\ \alpha(w, \otimes^{n,s}M).$$

Now, $w = \sum_{k=1}^r \otimes^n z_k$ with $z_k \in M$. Given $\varepsilon > 0$, by Proposition 2.1.11 we can take operators $(R_i)_{1\le i\le m}$ with $\|R_i\|_{\mathcal{L}(M,E)} \le 1$ such that $\left|\sum_{k=1}^r \overline{p}(z_k) - \sum_{k=1}^r p(\frac{1}{m}\sum_{i=1}^m R_i z_k)\right| < \varepsilon$. Therefore,

$$\begin{aligned}
\left|\langle\overline{p}, w\rangle\right| &= \left|\sum_{k=1}^r \overline{p}(z_k)\right| \le \left|\sum_{k=1}^r \overline{p}(z_k) - \sum_{k=1}^r p(\frac{1}{m}\sum_{i=1}^m R_i z_k)\right| + \left|\sum_{k=1}^r p(\frac{1}{m}\sum_{1=1}^m R_i z_k)\right| \\
&\le \varepsilon + \left|\langle p, \sum_{k=1}^r \otimes^n \frac{1}{m}\sum_{i=1}^m R_i z_k\rangle\right| \\
&\le \varepsilon + \|p\|_{(\tilde{\otimes}_\alpha^{n,s}E)'}\alpha(\sum_{k=1}^r \otimes^n \frac{1}{m}\sum_{i=1}^m R_i z_k\ ;\ \otimes^{n,s}E) \\
&\le \varepsilon + \|p\|_{(\tilde{\otimes}_\alpha^{n,s}E)'}\alpha(\otimes^{n,s}R(\sum_{k=1}^r z_k)\ ;\ \otimes^{n,s}E),
\end{aligned}$$

where $R = \frac{1}{m}\sum_{i=1}^m R_i$ (note that $\|R\|_{\mathcal{L}(M,E)} \le 1$ since each $\|R_i\|_{\mathcal{L}(M,E)} \le 1$). By the metric mapping property of α and the previous inequality we get

$$\left|\langle\overline{p}, w\rangle\right| \le \varepsilon + \|p\|_{(\tilde{\otimes}_\alpha^{n,s}E)'}\ \alpha(\sum_{k=1}^r \otimes^n z_k\ ;\ \otimes^{n,s}M),$$

which ends the proof. $\square$

To prove the Extension Lemma 2.1.3 we need to construct a special ultrapower (the local ultrapower of E), so we give the details. First, we recall the

Theorem 2.1.12. **(Principle of Local Reflexivity.)** *For each $M \in FIN(E'')$, $N \in FIN(E')$ and $\varepsilon > 0$, there exists an operator $T \in \mathcal{L}(M,E)$ such that*

(1) T is an ε-isometry; that is, $(1-\varepsilon)\|x''\| \le \|T(x'')\| \le (1+\varepsilon)\|x''\|$;

(2) $T(x'') = x''$ *for every* $x'' \in M \cap E$;

(3) $x'(T(x'')) = x''(x')$ *for* $x'' \in M$ *and* $x' \in N$.

Let I be the set of all triples (M, N, ε), where M and N are finite dimensional subspaces of E'' and E' respectively and $\varepsilon > 0$. For each $\mathfrak{i} \in I$, we denote by $M_\mathfrak{i}$, $N_\mathfrak{i}$ and $\varepsilon_\mathfrak{i}$ the three elements of the triple. We define an ordering on I by setting $\mathfrak{i} < \mathfrak{j}$ if $M_\mathfrak{i} \subset M_\mathfrak{j}$, $N_\mathfrak{i} \subset N_\mathfrak{j}$ and $\varepsilon_\mathfrak{i} > \varepsilon_\mathfrak{j}$. The collection of the set of the form $B_\mathfrak{i} = \{\mathfrak{j} \in I : \mathfrak{i} \leq \mathfrak{j}\}$ form a filter base. Let $\mathfrak{U}$ be an ultrafilter on I which contains this filter base. The filter $\mathfrak{U}$ constructed here is called *a local ultrafilter for* E, and $(E)_\mathfrak{U}$ is called *a local ultrapower of* E.

Finally, let us fix, for each $\mathfrak{i} \in I$, an operator $T_\mathfrak{i} : M_\mathfrak{i} \to E$ in accordance with the Principle of Local Reflexivity. The canonical embedding of E into the ultrapower $(E)_\mathfrak{U}$ extends to a canonical embedding $J_E : E'' \to (E)_\mathfrak{U}$ defined by $J_E(x'') = (x_\mathfrak{i})$, where $x_\mathfrak{i}$ is equal to $T_\mathfrak{i}(x'')$ if $x'' \in M_\mathfrak{i}$ and 0 otherwise. In this way, $J_E(E'')$ is the range of a norm one projection defined in $(E)_\mathfrak{U}$ by the following rule

$$(x_\mathfrak{i})_\mathfrak{U} \mapsto J_E(w^* - \lim_{\mathfrak{i},\mathfrak{U}} x_\mathfrak{i})$$

(where $w^* - \lim_{\mathfrak{i},\mathfrak{U}} x_\mathfrak{i}$ stands for the weak-star limit in E'' of the collection $(x_\mathfrak{i})$).

The following proposition is due to Lindström and Ryan [LR92, Proposition 2.1], it states that the Aron-Berner extension can be recovered from the ultra-iterated extension to a local ultrapower of E:

Proposition 2.1.13. *If* $(E)_\mathfrak{U}$ *is a local ultrapower of* E, *then the restriction of* $\overline{p}$ *to the canonical image of* E'' *in* $(E)_\mathfrak{U}$ *coincides with the Aron-Berner extension of* p *to* E''.

With all this we can give a proof of the Extension Lemma 2.1.3.

Proof. (*of the Extension Lemma* 2.1.3.)

Let $(E)_\mathfrak{U}$ a local ultrapower of E and $J_E : E'' \to (E)_\mathfrak{U}$ the canonical embedding. By Proposition 2.1.13 the ultra-iterated extension to the local ultrapower of E restricted to E'' coincides with the Aron-Berner extension of p. In other words, $AB(p) = \overline{p} \circ J_E$. Hence,

$$\begin{aligned}
\|AB(p)\|_{(\otimes_\alpha^{n,s} E'')'} &= \|\overline{p} \circ J_E\|_{(\otimes_\alpha^{n,s} E'')'} \\
&\leq \|\overline{p}\|_{(\otimes_\alpha^{n,s} E_\mathfrak{U})'} \|J_E\|^n \\
&= \|p\|_{(\otimes_\alpha^{n,s} E)'}.
\end{aligned}$$

The other inequality is immediate. □

2.2 Some applications to the metric theory of symmetric tensor products and polynomial ideals

In this section we present applications of the five basic lemmas to the study of symmetric tensor norms, specifically to their metric properties. We also obtain several results concerning the theory of polynomial ideals. The first application of the lemmas that we get relates the finite hull of an s-tensor norm with its cofinite hull on $\otimes^{n,s} E$ when E has the bounded approximation property.

Proposition 2.2.1. *Let α be an s-tensor norm and E be a normed space with the λ-bounded approximation property. Then*

$$\overleftarrow{\alpha} \leq \alpha \leq \overrightarrow{\alpha} \leq \lambda^n \overleftarrow{\alpha} \quad \text{on} \quad \otimes^{n,s} E.$$

In particular, $\overleftarrow{\alpha} = \alpha = \overrightarrow{\alpha}$ on $\otimes^{n,s} E$ if E has the metric approximation property.

Proof. The result is a direct consequence of the Approximation Lemma 2.1.2 and the fact that $\overleftarrow{\alpha} = \alpha = \overrightarrow{\alpha}$ on $\otimes^{n,s} M$ for every $M \in FIN(E)$ □

This proposition together with the Embedding Lemma 2.1.3 give the following corollary, which should be compared to the Embedding Lemma 2.1.4. Note that the assumptions on the s-tensor norm α in the Embedding Lemma are now substituted by assumptions on the normed space E.

Corollary 2.2.2. *Let α be an s-tensor norm and E be a normed space with the metric approximation property. Then*

$$\otimes^{n,s} \kappa_E : \otimes^{n,s}_{\alpha} E \longrightarrow \otimes^{n,s}_{\alpha} E''$$

is an isometry.

Proof. If $z \in \otimes^{n,s} E$, by the metric mapping property

$$\alpha(\otimes^{n,s} \kappa_E z; \otimes^{n,s} E'') \leq \alpha(\otimes^{n,s} z; \otimes^{n,s} E).$$

On the other hand, since E has the metric mapping property, Proposition 2.2.1 asserts that $\alpha = \overleftarrow{\alpha}$ on $\otimes^{n,s} E$. We then have

$$\alpha(\otimes^{n,s} z; \otimes^{n,s} E) = \overleftarrow{\alpha}(\otimes^{n,s} z; \otimes^{n,s} E) = \overleftarrow{\alpha}(\otimes^{n,s} \kappa_E z; \otimes^{n,s} E'') \leq \alpha(\otimes^{n,s} \kappa_E z; \otimes^{n,s} E''),$$

where the second equality is due to the Embedding Lemma 2.1.3 applied to the cofinitely generated s-tensor norm $\overleftarrow{\alpha}$. □

From the definition of dual tensor norm, for every finite dimensional space M we always have the isometric isomorphisms

$$\otimes^{n,s}_{\alpha'} M \overset{1}{=} (\otimes^{n,s}_{\alpha} M')', \tag{2.3}$$

$$\otimes^{n,s}_{\alpha} M' \overset{1}{=} (\otimes^{n,s}_{\alpha'} M)'. \tag{2.4}$$

The next theorem and its corollary show the behavior of the mappings in (2.3) and (2.4) in the infinite dimensional framework.

Theorem 2.2.3. (Duality Theorem.) *Let α be an s-tensor norm. For every normed space E the following natural mappings are isometries:*

$$\otimes^{n,s}_{\overleftarrow{\alpha}} E \hookrightarrow (\otimes^{n,s}_{\alpha'} E')', \tag{2.5}$$

$$\otimes^{n,s}_{\overleftarrow{\alpha}} E' \hookrightarrow (\otimes^{n,s}_{\alpha'} E)'. \tag{2.6}$$

Proof. Let us prove that the first mapping is an isometry. Observe that

$$FIN(E') = \{L^0 : L \in COFIN(E)\}.$$

Now, by the duality relations for finite dimensional spaces (2.3) and (2.4), and the fact that dual norms are finitely generated we obtain

$$\begin{aligned}
\overleftarrow{\alpha}(z; \otimes^{n,s} E) &= \sup_{L \in COFIN(E)} \alpha(Q_L^E(z); \otimes^{n,s} E/L) \\
&= \sup_{L \in COFIN(E)} \sup\{\langle Q_L^E(z), u\rangle : \alpha'(u; \otimes^{n,s} L^0) \leq 1\} \\
&= \sup\{\langle Q_L^E(z), u\rangle : \overrightarrow{\alpha'}(u; \otimes^{n,s} E') \leq 1\} \\
&= \sup\{\langle Q_L^E(z), u\rangle : \alpha'(u; \otimes^{n,s} E') \leq 1\},
\end{aligned}$$

and this shows (2.5).

For the second mapping, note that the following diagram commutes

$$\begin{tikzcd}
\otimes^{n,s}_{\overleftarrow{\alpha}} E' \arrow[r, hook, "1"] \arrow[dr] & (\otimes^{n,s}_{\alpha'} E'')' & \ni AB(p) . \\
& (\otimes^{n,s}_{\alpha'} E)' \arrow[u, hook, "1"] & \ni p \arrow[u, squiggly]
\end{tikzcd} \tag{2.7}$$

Then, the Extension Lemma 2.1.3 gives the isometry $\otimes^{n,s}_{\overleftarrow{\alpha}} E' \hookrightarrow (\otimes^{n,s}_{\alpha'} E)'$, which is (2.6). $\square$

Corollary 2.2.4. *Let α be an s-tensor norm. For every normed space the mappings*

$$\otimes^{n,s}_{\alpha} E \hookrightarrow (\otimes^{n,s}_{\alpha'} E')', \tag{2.8}$$

$$\otimes^{n,s}_{\alpha} E' \hookrightarrow (\otimes^{n,s}_{\alpha'} E)' \tag{2.9}$$

are continuous and have norm one.

If E' has the metric approximation property or α is cofinitely generated, then both mappings are isometries.

If E has the metric approximation property the mapping in (2.8) is an isometry.

Proof. Since $\overleftarrow{\alpha} \leq \alpha$, continuity and that the norm of both mappings is one follow from the Duality Theorem 2.2.3. If E' has the metric approximation property, by Proposition 2.2.1, $\overleftarrow{\alpha} = \alpha$ on $\otimes^{n,s} E$ and on $\otimes^{n,s} E'$, so the conclusion follows again from the Duality Theorem. The same happens if α is cofinitely generated.

If E has the metric approximation property, by Proposition 2.2.1 $\overleftarrow{\alpha} = \alpha$ on $\otimes^{n,s} E$, we can apply the Duality Theorem to show that the mapping in (2.8) is an isometry. $\square$

The isometry (2.9) for the case of E' having the metric approximation property can also be obtained from [Flo01a, Corollary 5.2 and Proposition 7.5]. Note also that if E (resp. E') has the λ-approximation property, then the mapping (2.8) (resp. (2.9)) is an isomorphism onto its range.

We now compile some consequences of the obtained results to the theory of polynomial ideals.

A natural question in the theory of polynomials is whether a polynomial ideal is closed under the Aron-Berner extension and, also, if the ideal norm is preserved by this extension. Positive answers for both questions were obtained for particular polynomial ideals in [CZ99, Car99, Mor84] among others. However, some polynomial ideals are not closed under the Aron-Berner extension (for example, the ideal of weakly sequentially continuous polynomials). Since dual s-tensor norms are always finitely generated, we can rephrase the Extension Lemma 2.1.3 in terms of maximal polynomial ideals and give a positive answer to the question for ideals of this kind.

Theorem 2.2.5. (Extension lemma for maximal polynomial ideals.) *Let $\mathcal{Q}$ be a maximal ideal of n-homogeneous polynomials and $p \in \mathcal{Q}(E)$, then its Aron-Berner extension is in $\mathcal{Q}(E'')$ and*

$$\|p\|_{\mathcal{Q}(E)} = \|AB(p)\|_{\mathcal{Q}(E'')}.$$

Floret and Hunfeld showed in [FH02] that there is another extension to the bidual, the so called uniterated Aron-Berner extension, which is an isometry for maximal polynomial ideals. The isometry and other properties of the uniterated extension are rather easy to prove. However, this extension is hard to compute, since its definition depends on an ultrafilter. On the other hand, the Aron-Berner extension is not only easier to compute, but also has a simple characterization that allows to check if a given extension of a polynomial is actually its Aron-Berner extension [Zal90]. Moreover, the iterated nature of the Aron-Berner extension makes it more appropriate for the study of polynomials and analytic functions. The next result shows that the Aron-Berner extension is also an isometry for minimal polynomial ideals.

Theorem 2.2.6. (Extension lemma for minimal polynomial ideals.) *Let $\mathcal{Q}$ be a minimal ideal. For $p \in \mathcal{Q}(E)$, its Aron-Berner extension $AB(p)$ belongs to $\mathcal{Q}(E'')$ and*

$$\|p\|_{\mathcal{Q}(E)} = \|AB(p)\|_{\mathcal{Q}(E'')}.$$

Proof. Since $p \in \mathcal{Q}(E) \overset{1}{=} ((\mathcal{Q}^{max})^{min})(E)$ (see [Flo01a, 3.4]), given $\varepsilon > 0$ there exist a Banach space F, an approximable operator $T : E \to F$ and a polynomial $q \in \mathcal{Q}^{max}(F)$ such that $p = q \circ T$ (as in (1.1)) and with $\|q\|_{\mathcal{Q}^{max}(F)}\|T\|^n \leq \|p\|_{\mathcal{Q}(E)} + \varepsilon$.

It is not hard to see that $AB(p) = AB(q) \circ T''$ (see for example [Car99, Section 1]). By Theorem 2.2.5 we have $\|q\|_{\mathcal{Q}^{max}(F)} = \|AB(q)\|_{\mathcal{Q}^{max}(F)}$. Since T is approximable, so is T''. With this we conclude that $AB(p)$ belongs to $\mathcal{Q}(E'')$ and

$$\begin{aligned}\|AB(p)\|_{\mathcal{Q}(E'')} &\leq \|AB(q)\|_{\mathcal{Q}^{max}(F'')}\|T''\|^n \\ &= \|q\|_{\mathcal{Q}^{max}(F)}\|T\|^n \\ &\leq \|p\|_{\mathcal{Q}(E)} + \varepsilon,\end{aligned}$$

for every ε. The reverse inequality is immediate. □

The concept of holomorphy type was introduced by Nachbin in [Nac69] (see also [Din71]). The most natural holomorphy types can be seen as sequences of polynomial ideals $\mathcal{Q} = \{\mathcal{Q}_k\}_k$ ($\mathcal{Q}_k$ is an ideal of polynomials of degree k, $k = 1, 2, \dots$), where some kind of affinity between ideals of different degrees is necessary [BBJP06, CDM09]. In [CDM07], given such a sequence of polynomial ideals, an associated Fréchet space of entire functions is defined. In [Mur10, Mur12] the corresponding definition for analytic functions defined on the unit ball of a Banach is given:

Definition 2.2.7. Let $\mathcal{Q} = \{\mathcal{Q}_k\}_k$ be a sequence of polynomial ideals and E be a Banach space. The space of $\mathcal{Q}$-holomorphic functions of bounded type on B_E is defined as

$$H_{b\mathcal{Q}}(B_E) = \left\{ f \in H(B_E) \; : \; \frac{d^k f(0)}{k!} \in \mathcal{Q}_k(E) \text{ and } \lim_{k\to\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathcal{Q}_k(E)}^{1/k} < 1 \right\}.$$

Examples of this kind of spaces are: the classical space of holomorphic functions of bounded type in the ball $H_b(B_E)$, the space of nuclearly entire functions of bouded type in the ball $H_{bN}(B_E)$ (defined by Gupta and Nachbin, see [Din99, Gup70]) and the space of integral entire functions of bounded type in the ball $H_{bI}(B_E)$ (defined by Dimant, Galindo, Maestre and Zalduendo in [DGMZ04]).

An immediate consequence of our results is the following: let $\mathcal{Q} = \{\mathcal{Q}_k\}_k$ be a sequence of polynomial ideals, each $\mathcal{Q}_k$ being either maximal or minimal. If E is a Banach space, then a holomorphic function f belongs to $H_{b\mathcal{Q}}(B_E)$ if and only if its Aron-Berner extension belongs to $H_{b\mathcal{Q}}(B_{E''})$.

Lassalle and Zalduendo [LZ00] and Cabello, Castillo and Garcia [CCG00] obtained, independently, that if two Banach spaces E and F are symmetrically Arens-regular (the definition is given after the statement of Proposition 2.2.9) and E' and F' are isomorphic (resp. isometric), then $\mathcal{P}^n(E)$ and $\mathcal{P}^n(F)$ are isomorphic (resp. isometric). We will extend their result to a wider class of polynomial ideals but, before this, some definitions are necessary.

Definition 2.2.8. Given an ideal of n-homogeneous polynomials $\mathcal{Q}$ closed under the Aron-Berner extension and a continuous linear morphism $s : E' \to F'$, we can construct the following mapping $\overline{s} : \mathcal{Q}(E) \to \mathcal{Q}(F)$ given by

$$\overline{s}(p) := AB(p) \circ s' \circ \kappa_F,$$

where $\kappa_F : F \to F^{**}$ is the canonical inclusion. The mapping $\overline{s}$ is referred to as *the extension morphism* of s .

In general $\overline{s} \circ \overline{t}(p) \neq \overline{s \circ t}(p)$ (see [Zal05, Example 2.3.]), but in the presence of some symmetry the procedure is sufficiently well-behaved to produce the following result which can be found in [LZ00, Corollary 2.2].

Proposition 2.2.9. *Let $p \in \mathcal{P}^n(E)$ a polynomial, A its associated symmetric n-linear form, and suppose $s : E' \to F'$ is an isomorphism. If $EXT(A)$, the Arens extension of A, is symmetric then $\overline{s^{-1}} \circ \overline{s}(p) = p$.*

Recall that a Banach space E is called *Arens-regular* (resp. *symmetrically Arens-regular*) if all linear operators (resp. symmetric linear operators) $E \to E'$ are weakly compact (see [AGGM96] and the references therein). Reflexive spaces are obviously Arens-regular. Also the spaces c_0 and $C(K)$ (the space of continuous functions over the compact set K) have this property (see [Are51]). Another example is the Tsirelson-James space [AD97]. A classical space that do not have this property is ℓ_1 (see [AGGM96]).

We say that a polynomial ideal $\mathcal{Q}$ is *regular* [CDM12] if, for every Banach space E and every polynomial $p \in \mathcal{Q}(E)$, the Arens-extension of A (the symmetric n-linear form associated to p) is symmetric. For example, the ideal of integral polynomials $\mathcal{P}_I^n$ [CL05, Proposition 2.14], the ideal of extendible polynomials $\mathcal{P}_e^n$ [CL05, Proposition 2.15] and the ideal of weakly-continuous on bounded sets polynomials $\mathcal{P}_w^n$ are regular [AHV83]. Since the ideal of approximable polynomials $\mathcal{P}_{app}^n$ is regular, we obtain that every minimal ideal is regular. Using the regularity of the ideal of extendible polynomials it is shown in [CDM12] that every polynomial ideal associated with a projective s-tensor norm (see Chapter 3) is regular.

With the help of Theorems 2.2.5 and 2.2.6 we can extend the main results of [CCG00] and [LZ00] to a wider class of polynomial ideals.

Theorem 2.2.10. *For a pair of Banach spaces E and F and a polynomial ideal $\mathcal{Q}$ the following holds.*

(1) If $\mathcal{Q}$ is minimal and E' and F' are isomorphic (resp. isometric), then $\mathcal{Q}(E)$ and $\mathcal{Q}(F)$ are isomorphic (resp. isometric).

(2) If $\mathcal{Q}$ is maximal, E and F are symmetrically Arens-regular and E' and F' are isomorphic (resp. isometric), then $\mathcal{Q}(E)$ and $\mathcal{Q}(F)$ are isomorphic (resp. isometric). Moreover, if $\mathcal{Q}$ is also regular the hypothesis on E and F can be removed.

Proof. (1) Let $s : E' \to F'$ be an isomorphism. Since minimal ideals are regular we have, by Proposition 2.2.9 that $\overline{s} \circ \overline{s^{-1}}(p) = p$ for every polynomial $p \in \mathcal{Q}(E)$. Analogously, for every polynomial $q \in \mathcal{Q}(F)$ we have $\overline{s^{-1}} \circ \overline{s}(q) = q$. Now, by 2.2.6 we obtain

$$\|\overline{s}(p)\|_{\mathcal{Q}(F)} = \|AB(p) \circ s' \circ \kappa_F\|_{\mathcal{Q}(F)} \leq \|AB(p)\|_{\mathcal{Q}(E'')} \|s'\|^n \|\kappa_F\|^n = \|p\|_{\mathcal{Q}(E)} \|s'\|^n,$$

and the same for $\overline{s^{-1}}$ and q.

The proof of (2) is analogous. □

It is easy to see that if E and F are isomorphic, and one is Arens-regular, then so is the other (see for example [LZ00, Remark 2.2.]). Therefore the last theorem asserts that if a given space is Arens-regular, its dual determines the structure of the majority of the know classes of polynomials over itself.

The next statement is a polynomial version of the Density Lemma 2.1.6.

Lemma 2.2.11. (Density Lemma for maximal polynomial ideals.) *Let $\mathcal{Q}$ be a polynomial ideal, E a Banach space, $E_0 \subset E$ a dense subspace and $C \subset FIN(E_0)$ a cofinal subset* (i.e., *for every N in $FIN(E_0)$ there exist a bigger finite dimensional subspace M that belongs to C). Then*

$$\|p\|_{\mathcal{Q}^{max}(E)} = \sup\{\|p|_M\|_{\mathcal{Q}(M)} : M \in C\}.$$

Proof. For α the s-tensor norm associated with $\mathcal{Q}$, by the Representation Theorem for Maximal Polynomial ideals 1.2.1:

$$\mathcal{Q}^{max}(E) = (\otimes_{\alpha'}^{n,s} E)'.$$

Using the Density Lemma 2.1.6 (since α' is finitely generated) we get

$$\|p\|_{\mathcal{Q}^{max}(E)} = \|p\|_{(\otimes_{\alpha'}^{n,s} E)'} = \|p\|_{(\otimes_{\alpha'}^{n,s} E_0)'} = \|p\|_{\mathcal{Q}^{max}(E_0)}.$$

On the other hand, by the very definition of the norm in $\mathcal{Q}^{max}$, we have

$$\|p\|_{\mathcal{Q}^{max}(E_0)} = \sup\{\|p|_M\|_{\mathcal{Q}(M)} : M \in C\},$$

which ends the proof. □

From the previous Lemma we obtain the next useful result: in the case of a Banach space with a Schauder basis, a polynomial belongs to a maximal ideal if and only if the norms of the the restrictions of the polynomial to the subspaces generated by the first elements of the basis, are uniformly bounded.

Corollary 2.2.12. *Let $\mathcal{Q}$ a maximal polynomial ideal, E a Banach space with Schauder basis $(e_k)_{k=1}^{\infty}$ and M_m the finite dimensional subspace generated by the first m elements of the basis,* i.e., $M_m := [e_k : 1 \leq k \leq m]$. *A polynomial p belongs to $\mathcal{Q}(E)$ if and only if* $\sup_{m\in\mathbb{N}} \|p|_{M_m}\|_{\mathcal{Q}(M_m)} < \infty$. *Moreover,*

$$\|p\|_{\mathcal{Q}(E)} = \sup_{m\in\mathbb{N}} \|p|_{M_m}\|_{\mathcal{Q}(M_m)}.$$

Proof. Is a direct consequence the previous lemma and the fact that $C := \{M_m\}_m$ is a cofinal subset of $FIN([e_n : n \in \mathbb{N}])$. □

As a consequence of the Duality Theorem 2.2.3 we have the following.

Theorem 2.2.13. (Embedding Theorem.) *Let $\mathcal{Q}$ be the maximal polynomial ideal associated with the s-tensor norm α. Then the relations*

$$\otimes^{n,s}_{\overleftarrow{\alpha}} E \hookrightarrow \mathcal{Q}(E'),$$

$$\otimes^{n,s}_{\overleftarrow{\alpha}} E' \hookrightarrow \mathcal{Q}(E)$$

hold isometrically.

In particular, the extensions

$$H^E_\alpha : \widetilde{\otimes}^{n,s}_{\alpha} E \to (\widetilde{\otimes}^{n,s}_{\alpha'} E')' = \mathcal{Q}(E') \tag{2.10}$$

and

$$J^E_\alpha : \widetilde{\otimes}^{n,s}_{\alpha} E' \to (\widetilde{\otimes}^{n,s}_{\alpha'} E)' = \mathcal{Q}(E) \tag{2.11}$$

of $\otimes^{n,s}_{\alpha} E' \hookrightarrow \mathcal{Q}(E)$ and $\otimes^{n,s}_{\alpha} E' \to \mathcal{Q}(E)$ respectively are well defined and have norm one.

The following proposition shows how dominations between s-tensor norms translate into inclusions between maximal polynomial ideals, and vice versa.

Proposition 2.2.14. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be maximal polynomial ideals with associated tensor norms α_1 and α_2 respectively, E be a normed space and $c \geq 0$. Consider the following conditions.*

(1) $\alpha_2' \leq c\alpha_1'$ *on* $\otimes^{n,s} E$;

(2) $\mathcal{Q}_2(E) \subset \mathcal{Q}_1(E)$ *and* $\| \ \|_{\mathcal{Q}_1} \leq c \| \ \|_{\mathcal{Q}_2}$;

(3) $\overleftarrow{\alpha_1} \leq c \overleftarrow{\alpha_2}$ *on* $\otimes^{n,s} E'$.

Then,

(a) $(1) \Leftrightarrow (2) \Rightarrow (3)$;

(b) if E' has the metric approximation property then $(1), (2)$ and (3) are equivalent.

Proof. (a) The statement $(1) \Leftrightarrow (2)$ can be easily deduced from the Representation Theorem for Maximal Polynomial Ideals 1.2.1.

Let us show $(2) \Rightarrow (3)$. Let $z \in \otimes^{n,s} E'$. By the Embedding Theorem 2.2.13 we have:

$$\otimes^{n,s}_{\overleftarrow{\alpha_1}} E' \overset{1}{\hookrightarrow} \mathcal{Q}_1(E),$$

$$\otimes^{n,s}_{\overleftarrow{\alpha_2}} E' \overset{1}{\hookrightarrow} \mathcal{Q}_2(E).$$

Denote by $p \in \mathcal{P}^n(E)$ the polynomial that represents z. Thus,

$$\overleftarrow{\alpha_1}(z) = \|p\|_{\mathcal{Q}_1(E)} \leq c\|p\|_{\mathcal{Q}_2(E)} = c\overleftarrow{\alpha_2}(z).$$

(b) Since E' has the metric approximation property, so does E (see Corollary 1 in [DF93, 16.3.]). Thus, by Proposition 2.2.1, for $i = 1, 2$ we have $\overrightarrow{\alpha_i} = \alpha_i$ and $\overleftarrow{\alpha_i'} = \alpha_i'$ on $\otimes^{n,s} E'$ and $\otimes^{n,s} E$ respectively. Condition (3) states that the mapping $(**)$ in the following diagram has norm at most c.

$$\begin{array}{ccc} \otimes^{n,s}_{\alpha_1'} E = \otimes^{n,s}_{\overleftarrow{\alpha_1'}} E & \overset{1}{\hookrightarrow} & (\otimes^{n,s}_{\alpha_1''} E')' = (\otimes^{n,s}_{\overrightarrow{\alpha_1}} E')' = (\otimes^{n,s}_{\alpha_1} E')' \,, \\ \Big\downarrow {\scriptstyle (*)} & & \Big\downarrow {\scriptstyle (**)} \\ \otimes^{n,s}_{\alpha_2'} E = \otimes^{n,s}_{\overleftarrow{\alpha_2'}} E & \overset{1}{\hookrightarrow} & (\otimes^{n,s}_{\alpha_2''} E')' = (\otimes^{n,s}_{\overrightarrow{\alpha_2}} E')' = (\otimes^{n,s}_{\alpha_2} E')' \end{array} \tag{2.12}$$

Since the diagram commutes we can conclude that the mapping $(*)$ is continuous with norm $\leq c$. Therefore (3) implies (1). $\square$

The previous proposition is a main tool for translating results on s-tensor norms into results on polynomial ideals. As an example, we have the following polynomial version of the $\mathcal{L}_p$-Local Technique Lemma 2.1.7.

Theorem 2.2.15. ($\mathcal{L}_p$-Local Technique Lemma for maximal ideals.) *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be polynomial ideals with $\mathcal{Q}_1$ maximal and let $c > 0$. Consider the following assertions.*

(1) $\| \ \|_{\mathcal{Q}_1(\ell_p^m)} \leq c \| \ \|_{\mathcal{Q}_2(\ell_p^m)}$ for all $m \in \mathbb{N}$;

(2) $\mathcal{Q}_2(\ell_p) \subset \mathcal{Q}_1(\ell_p)$ and $\| \ \|_{\mathcal{Q}_1(\ell_p)} \leq c \| \ \|_{\mathcal{Q}_2(\ell_p)}$.

Then (1) and (2) are equivalent and imply that

$$\mathcal{Q}_2(E) \subset \mathcal{Q}_1(E) \text{ and } \| \ \|_{\mathcal{Q}_1(E)} \leq c\lambda^n \| \ \|_{\mathcal{Q}_2(E)}$$

for every $\mathcal{L}^g_{p,\lambda}$-space E.

Proof. Using Corollary 2.2.12 we easily obtain that (1) implies (2).
On the other hand, since the subspace spanned by the first m canonical vectors in ℓ_p is a 1-complemented subspace isometrically isomorphic to ℓ_p^m, we get that (2) implies (1) by the metric mapping property.

Let us show that (1) implies the general conclusion. Denote by α_1 and α_2 the s-tensor norms associated with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively. By (1) and the Representation Theorem for

Maximal Polynomial Ideals 1.2.1, we have $\alpha_1' \leq c\alpha_2'$ on $\otimes^{n,s}\ell_p^m$. Using the $\mathcal{L}_p$-Local Technique Lemma 2.1.7 we get $\alpha_2' \leq c\lambda^n \alpha_1'$ on $\otimes^{n,s}E$. Notice that α_2 is also associated with $(\mathcal{Q}_2)^{max}$, thus by Proposition 2.2.14 we obtain $(\mathcal{Q}_2)^{max}(E) \subset \mathcal{Q}_1(E)$ and $\| \ \|_{\mathcal{Q}_1(E)} \leq c\lambda^n \| \ \|_{(\mathcal{Q}_2)^{max}(E)}$. Since $\mathcal{Q}_2(E) \subset (\mathcal{Q}_2)^{max}(E)$ and $\| \ \|_{(\mathcal{Q}_2)^{max}(E)} \leq \| \ \|_{\mathcal{Q}_2(E)}$, we finally obtain $\mathcal{Q}_2(E) \subset \mathcal{Q}_1(E)$ with $\| \ \|_{\mathcal{Q}_1(E)} \leq c\lambda^n \| \ \|_{\mathcal{Q}_2(E)}$. $\square$

For the case $p = \infty$, ℓ_p in assertion (2) should be replaced, in principle, by c_0. Since ℓ_∞ is a $\mathcal{L}_{\infty,1}^g$-space and ℓ_∞^n is 1-complemented in ℓ_∞ for each n we therefore have: two maximal ideals coincide on c_0 if and only if they coincide on ℓ_∞. Note that every polynomial on ℓ_∞ is extendible, since ℓ_∞ is an injective Banach space. Consequently, although c_0 is not injective, we get that every polynomial on c_0 is extendible (by our previous comment). We remark that the extendibility of polynomials on c_0 is a known fact, and that it can also be obtained from the Extension Lemma 2.1.3.

Since Hilbert spaces are $\mathcal{L}_p^g$ for any $1 < p < \infty$ (see Corollary 2 in [DF93, 23.2]), we get also the following.

Corollary 2.2.16. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be polynomial ideals, $\mathcal{Q}_1$ maximal. If for some $1 < p < \infty$ we have $\mathcal{Q}_2(\ell_p) \subset \mathcal{Q}_1(\ell_p)$, then we also have $\mathcal{Q}_2(\ell_2) \subset \mathcal{Q}_1(\ell_2)$.*

As a consequence, if two maximal polynomial ideals do not coincide on ℓ_2, then they are different in every ℓ_p with $1 < p < \infty$.

Proposition 2.2.14, Theorem 2.2.15 and Corollary 2.2.16 have their analogues for minimal ideals. For Theorem 2.2.15 and Corollary 2.2.16, the hypothesis on maximality of $\mathcal{Q}_1$ should be changed for the requirement that $\mathcal{Q}_2$ be minimal.

We end this chapter with a few words about accessibility of s-tensor norms and polynomial ideals.

Definition 2.2.17. We say that an s-tensor norm α is *accessible* if

$$\overrightarrow{\alpha} = \alpha = \overleftarrow{\alpha}$$

(*i.e.*, α is finitely and cofinitely generated).

An example of an s-tensor norm of this type is $\varepsilon_{n,s}$. Moreover, we will see in Corollary 3.3.3 that every injective s-tensor norm is accessible (injectivity will be explained in the next chapter).

The definition of accessible polynomial ideals (a term coined in [Flo01a, 3.6.]) is less direct.

Definition 2.2.18. We say that a polynomial ideal $\mathcal{Q}$ is *accessible* if the following condition holds: for every normed space E, $q \in \mathcal{P}_f^n(E)$ and $\varepsilon > 0$, there is a closed finite codimensional space $L \subset E$ and $p \in \mathcal{P}^n(E/L)$ such that $q = p \circ Q_L^E$ (where Q_L^E is the canonical quotient map) and $\|p\|_{\mathcal{Q}} \leq (1+\varepsilon)\|q\|_{\mathcal{Q}}$.

One may wonder how the definition of accessibility of a polynomial ideal relates with the one for its associated s-tensor norm. The next proposition sheds some light on this question.

Proposition 2.2.19. *Let $\mathcal{Q}$ be a polynomial ideal and let α be its associated s-tensor norm. Then, α is accessible if and only if $\mathcal{Q}^{max}$ is, in which case $\mathcal{Q}$ is also accessible.*

Proof. Suppose that α is accessible, which means that α is finitely and cofinitely generated. Fix E a normed space, $q \in \mathcal{P}_f^n(E)$ and $\varepsilon > 0$. Let $z \in \otimes^{n,s} E'$ be the tensor that represents the polynomial q. Since α is cofinitely generated, by the Duality Theorem 2.2.3 and the Representation Theorem for Maximal polynomial ideals 1.2.1 we have

$$\otimes_{\alpha}^{n,s} E' \overset{1}{\hookrightarrow} (\otimes_{\alpha'}^{n,s} E)' = \mathcal{Q}^{max}(E).$$

Thus, $\alpha(z; \otimes^{n,s} E') = \|q\|_{\mathcal{Q}^{max}(E)}$. Using that α is finitely generated we can find $M \in FIN(E')$ such that $z \in \otimes^{n,s} M$ and

$$\alpha(z; \otimes^{n,s} M) \leq (1+\varepsilon)\|q\|_{\mathcal{Q}^{max}(E)}.$$

Set $L := M^0 \subset E$, identify M' with E/L and denote by p the polynomial that represents the tensor $z \in \otimes^{n,s} M$ defined in E/L. Therefore,

$$\|p\|_{\mathcal{Q}^{max}(E/L)} = \alpha(z; \otimes^{n,s} M) \leq (1+\varepsilon)\|q\|_{\mathcal{Q}^{max}(E)}$$

and obviously $q = p \circ Q_L^E$ where $Q_E^L : E \to E/L$ is the natural quotient mapping. This implies that $\mathcal{Q}^{max}$ is accessible.

For the converse we must show that $\alpha(\ \cdot\ ; \otimes^{n,s} E) = \overleftarrow{\alpha}(\ \cdot\ , \otimes^{n,s} E)$. By the Embedding Lemma 2.1.4 it is sufficient to prove that $\alpha(\ \cdot\ , \otimes^{n,s} E'') = \overleftarrow{\alpha}(\ \cdot\ , \otimes^{n,s} E'')$. Set $F := E'$ and take $z \in \otimes^{n,s} F'$ and $\varepsilon > 0$. By the Duality Theorem 2.2.3 we have

$$\otimes_{\overleftarrow{\alpha}}^{n,s} F' \overset{1}{\hookrightarrow} (\otimes_{\alpha'}^{n,s} F)' = \mathcal{Q}^{max}(F).$$

Denote by q the polynomial represented by z in $\mathcal{Q}^{max}(F)$; by hypothesis there exist a subspace $L \in COFIN(F)$ and a polynomial $p \in \mathcal{Q}^{max}(F/L)$ such that $q = p \circ Q_L^F$ with $\|p\|_{\mathcal{Q}^{max}(F/L)} \leq (1+\varepsilon)\|q\|_{\mathcal{Q}^{max}(F)}$. If w is the tensor that represents p in $\otimes^{n,s} L^0 = \otimes^{n,s}(F/L)'$, we have $(\otimes^{n,s} Q_L^F)(w) = z$. Using the metric mapping property we obtain

$$\begin{aligned} \alpha(z; \otimes^{n,s} F) &\leq \alpha(w; \otimes^{n,s} F/L) \\ &= \|p\|_{\mathcal{Q}^{max}(F/L)} \\ &\leq (1+\varepsilon)\|q\|_{\mathcal{Q}^{max}(F)} \\ &= (1+\varepsilon)\overleftarrow{\alpha}(z; \otimes^{n,s} F), \end{aligned}$$

which proves that α is accessible.

Finally, we always have $\|\cdot\|_{\mathcal{Q}^{max}} \leq \|\cdot\|_{\mathcal{Q}}$, with equality in finite dimensional spaces. The definition of accessibility then implies that, if $\mathcal{Q}^{max}$ is accessible, then so is $\mathcal{Q}$. $\square$

Note that the operator J_α^E defined in equation (2.11) can be seen as the composition operator

$$\tilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{\twoheadrightarrow} \mathcal{Q}^{min}(E) \to \mathcal{Q}(E) \overset{1}{=} (\tilde{\otimes}_{\alpha'}^{n,s} E)', \tag{2.13}$$

where the quotient mapping $\tilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{\twoheadrightarrow} \mathcal{Q}^{min}(E)$ is the one given by the Representation Theorem for Minimal Ideals 1.2.2. The mapping J_α^E will be referred to as the *natural mapping* from $\tilde{\otimes}_{\alpha}^{n,s} E'$ to $(\tilde{\otimes}_{\alpha'}^{n,s} E)'$.

As a consequence of the Embedding Theorem 2.2.13 we recover the following results of [Flo01a] stated, also, in Proposition 1.1.1.

Corollary 2.2.20. *Let $\mathcal{Q}$ be a maximal polynomial.*

(1) If $\mathcal{Q}$ is accessible or E' has the metric approximation property then

$$\mathcal{Q}^{min}(E) \overset{1}{\hookrightarrow} \mathcal{Q}(E) \ \textit{and} \ \widetilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{=} \mathcal{Q}^{min}(E) \ \textit{and they coincide with} \ \overline{\mathcal{P}_f^n(E)}^{\mathcal{Q}}.$$

(2) If E' has the bounded approximation property then $\| \ \|_{\mathcal{Q}^{min}(E)}$ and $\| \ \|_{\mathcal{Q}(E)}$ are equivalent in $\mathcal{Q}^{min}$ and

$$\widetilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{=} \mathcal{Q}^{min}(E) \ \textit{and they coincide with} \ \overline{\mathcal{P}_f^n(E)}^{\mathcal{Q}}.$$

Proof. Let α be the s-tensor norm associated with $\mathcal{Q}$. Since $\mathcal{Q}$ is accessible, by Proposition 2.2.19 we have $\alpha = \overleftarrow{\alpha}$. Now the Embedding Theorem 2.2.13 shows that the natural mapping

$$J_{\alpha}^{E} : \widetilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{\twoheadrightarrow} \mathcal{Q}^{min}(E) \to (\widetilde{\otimes}_{\alpha'}^{n,s} E)'$$

is an isometry. This implies that $\mathcal{Q}^{min}(E) \overset{1}{\hookrightarrow} \mathcal{Q}(E)$ and $\widetilde{\otimes}_{\alpha}^{n,s} E' \overset{1}{=} \mathcal{Q}^{min}(E)$. The fact that $\mathcal{Q}^{min}(E) = \overline{\mathcal{P}_f^n(E)}^{\mathcal{Q}}$ now easily follows.

If E' has the metric approximation property then $\alpha = \overleftarrow{\alpha}$ on $\otimes^{n,s} E'$ (Proposition 2.2.1) and we can reason as before.

If E' has the bounded approximation property then $\overleftarrow{\alpha}$ is equivalent to α on $\otimes^{n,s} E'$ by Proposition 2.2.1. Hence, with the help of the Embedding Theorem 2.2.13, we obtain that the mapping J_{α}^{E} is injective. Now the result follows. □

Chapter 3

Injective and projective associates of s-tensor norms

In this chapter we treat injectivity and projectivity for s-tensor norms. In Section 3.1 and Seccion 3.2 we define the injective associate and projective associate, respectively, of an s-tensor norm and study some of their interesting properties. In Section 3.3 we give some relations between the injective and projective associates of a given tensor norm. We also study the maximal polynomial ideals associated with these norms in terms of composition ideals and quotient ideals. This is contained in Section 3.4. In Section 3.5 we study natural symmetric tensor norms of arbitrary order, in the spirit of Grothendieck's norms: we define natural symmetric tensor norms as those that can be obtained from the n-fold projective s-tensor norm $\pi_{n,s}$ by a finite number of basic operations (injective associate, projective associate, and adjoint) and conclude that there are exactly six natural symmetric tensor norms for $n \geq 3$, a noteworthy difference with the 2-fold case in which there are four.

3.1 The injective associate

We say that an s-tensor norm α is *injective* if, for every normed spaces E and F and every isometric embedding $I : E \overset{1}{\hookrightarrow} F$, the tensor product operator

$$\otimes^{n,s} I : \otimes_\alpha^{n,s} E \to \otimes_\alpha^{n,s} F,$$

is also an isometric embedding. Loosely speaking, α "respects subspaces".

It is well know that, in general, s-tensor norms do not respect subspaces (if not, this will become clear later). An example of a norm that does respect subspaces is **the** injective norm $\varepsilon_{n,s}$: if E is a subspace of F and $z = \sum_{j=1}^r \otimes^{n,s} x_j \in \otimes^{n,s} E$, we have

$$\varepsilon_{n,s}(z; \otimes^{n,s} E) = \sup_{x' \in B_{E'}} \left| \sum_{j=1}^r x'(x_j)^n \right| = \sup_{x' \in B_{F'}} \left| \sum_{j=1}^r x'(x_j)^n \right| = \varepsilon_{n,s}(z; \otimes^{n,s} F),$$

where the third equality is due to the Hahn-Banch Theorem.

Note that, if $I : E \overset{1}{\hookrightarrow} F$ is an isometric embedding and α is injective we obtain also, as a consequence of the Hahn-Banach Theorem, that every α-continuous linear form on $\otimes^{n,s} E$ can be extended to an α-continuos linear form on $\otimes^{n,s} F$ with the same norm. Thus, any polynomial $p \in \mathcal{Q}_\alpha(E)$ has an *extension* $\overline{p} \in \mathcal{Q}_\alpha(F)$ such that $\overline{p} \circ I = p$ and $\|p\|_{\mathcal{Q}_\alpha(E)} = \|\overline{p}\|_{\mathcal{Q}_\alpha(F)}$. In particular, since $\varepsilon_{n,s}$-continuous polynomials are the integral ones, if E is a subspace of F every integral polynomial $p \in \mathcal{P}_I^n(E)$ can be extended to an integral polynomial $\overline{p} \in \mathcal{P}_I^n(F)$ that has the same integral norm (*i.e.,* $\|p\|_{\mathcal{P}_I^n(E)} = \|\overline{p}\|_{\mathcal{P}_I^n(F)}$).

This special property will be referred to as the *extension property*. More precisely, we have the following definition.

Definition 3.1.1. We say that $\mathcal{Q}$ has the *extension property* if whenever E is a subspace of F, then every polynomial in $\mathcal{Q}(E)$ can be extended to a polynomial in $\mathcal{Q}(F)$ with the same ideal norm.

We have seen that, if α is injective, then $\mathcal{Q}_\alpha$ has the extension property. The converse is also true, as we will see in Proposition 3.1.3.

For a normed space E, we always have the isometry $I_E : E \overset{1}{\hookrightarrow} \ell_\infty(B_{E'})$ given by

$$I_E(x) = \big(x'(x)\big)_{x' \in B_{E'}}. \tag{3.1}$$

This mapping is referred to as the *canonical embedding of* E.

Therefore, for an injective s-tensor norm α, we always have the metric injection

$$\otimes^{n,s} I_E : \otimes_\alpha^{n,s} E \overset{1}{\hookrightarrow} \otimes_\alpha^{n,s} \ell_\infty(B_{E'}). \tag{3.2}$$

An interesting fact is that equation (3.2) characterize the injectivity of α (in the sense that α satisfies equation (3.2) for every normed space E if and only if α is injective). To see this, we first recall a definition.

Definition 3.1.2. A Banach space space E is called *injective* if for every Banach space F, every subspace $G \subset F$ and every $T \in \mathcal{L}(G, E)$ there is an extension $\overline{T} \in \mathcal{L}(F, E)$ of T. The space E has the λ-*extension property* ($\lambda \geq 1$) if some extension satisfies $\|\overline{T}\| \leq \lambda \|T\|$.

It is not hard to see that $\ell_\infty(I)$ has the 1-extension property (usually called the *metric extension property*). We therefore have

Proposition 3.1.3. *For an s-tensor norm α the following conditions are equivalent.*

(a) The ideal $\mathcal{Q}_\alpha$ has the extension property;

(b) for every normed space E, the mapping

$$\otimes^{n,s} I_E : \otimes_\alpha^{n,s} E \overset{1}{\hookrightarrow} \otimes_\alpha^{n,s} \ell_\infty(B_{E'})$$

is a metric injection;

(c) the s-tensor norm α is injective.

Proof. Let us show that (a) implies (b). Fix $z \in \otimes^{n,s} E$ and take $p \in \mathcal{Q}_\alpha(E) = \big(\otimes_\alpha^{n,s} E\big)'$ with $\|p\|_{\mathcal{Q}_\alpha(E)} = 1$ such that $|\langle p, z\rangle| = \alpha(z)$. Since $\mathcal{Q}_\alpha$ has the extension property, we can extend p to a polynomial $\overline{p}$ in $\mathcal{Q}_\alpha(\ell_{\infty(B_{E'})})$ with the same norm. Therefore,

$$\alpha(\otimes^{n,s} I(z); \otimes^{n,s} \ell_{\infty(B_{E'})}) \le \alpha(z; \otimes^{n,s} E) = |\langle \overline{p}, \otimes^{n,s} I(z)\rangle| \le \alpha(\otimes^{n,s} I(z); \otimes^{n,s} \ell_{\infty(B_{E'})}).$$

To see that (b) implies (c), fix E and F two normed spaces, z an element of $\otimes^{n,s} E$ and $I : E \overset{1}{\hookrightarrow} F$ an isometric injection. Observe that the following diagram commutes

$$\begin{tikzcd} \ell_\infty(B_{E'}) \arrow[r, hook, "I_\infty"] & \ell_\infty(B_{F'}) \\ E \arrow[u, hook, "I_E"] \arrow[r, hook, "I"] & F \arrow[u, hook, "I_F"'] \end{tikzcd} , \tag{3.3}$$

where I_∞ is the isometry given by the following rule

$$(a_{x'})_{x' \in B'_E} \mapsto (a_{I'(y')})_{y' \in B'_E}.$$

Denote by J the index set given by $B_{\ell_\infty(B_{E'})'}$ and $I_{\ell_\infty(B_{E'})} : \ell_\infty(B_{E'}) \to \ell_\infty(J)$ the canonical inclusion. Since $\ell_\infty(I)$ has the metric extension property there exist a norm one mapping $\overline{I_{\ell_\infty(B_{E'})}}$ that makes the next diagram commute:

$$\begin{tikzcd} \ell_\infty(B_{F'}) \arrow[dr, dashed, "\overline{I_{\ell_\infty(B_{E'})}}"] & \\ \ell_\infty(B_{E'}) \arrow[u, hook, "I_\infty"] \arrow[r, hook, "I_{\ell_\infty(B_{E'})}"] & \ell_\infty(J) \end{tikzcd} , \tag{3.4}$$

Now,

$$\begin{aligned}
\alpha(\otimes^{n,s} I(z); \otimes^{n,s} F) &\le \alpha(z; \otimes^{n,s} E) \\
&= \alpha(\otimes^{n,s} I_E(z); \otimes^{n,s} \ell_\infty(B_{E'})) && \text{by (b)} \\
&= \alpha(\otimes^{n,s} (I_{\ell_\infty(B_{E'})} \circ I_E)(z); \otimes^{n,s} \ell_\infty(J)) && \text{by (b)} \\
&= \alpha(\otimes^{n,s} (\overline{I_{\ell_\infty(B_{E'})}} \circ I_\infty \circ I_E)(z); \otimes^{n,s} \ell_\infty(J)) && \text{by diagram (3.4)} \\
&\le \alpha(\otimes^{n,s} (I_\infty \circ I_E)(z); \otimes^{n,s} \ell_\infty(B_{F'})) && \text{by the metric mapping property} \\
&\le \alpha(\otimes^{n,s} (I_F \circ I)(z); \otimes^{n,s} \ell_\infty(B_{F'})) && \text{by diagram (3.3)} \\
&= \alpha(\otimes^{n,s} I(z); \otimes^{n,s} F) && \text{by (b).}
\end{aligned}$$

That (c) implies (a) was mentioned before (it is just the Hahn-Banach Theorem). □

For α any fixed s-tensor norm, the ideal of integral polynomials $\mathcal{P}_I^n = \mathcal{Q}_{\varepsilon_{n,s}}$ is always contained in $\mathcal{Q}_\alpha$ (since $\varepsilon_{n,s} \le \alpha$) and has the extension property. In general there is a bigger polynomial ideal that has the extension property contained in $\mathcal{Q}_\alpha$. Or, in other words, a wider class of α-continuous polynomials that can be extended to any larger space. Therefore, it is reasonable to seek for the **biggest** maximal polynomial ideal contained in $\mathcal{Q}_\alpha$ which has the extension property. In terms of tensor norms, a moment of thought shows that our search translates into finding the greatest injective s-tensor norm smaller than or equal to α. This motivates the following definition.

Definition 3.1.4. *The injective associate of* α, denoted by $/\alpha\backslash$, is the (unique) greatest injective s-tensor norm smaller than α.

This is well-defined, as seen in Theorem 3.1.5. Therefore, $\mathcal{Q}_{/\alpha\backslash}$ has to be the biggest maximal polynomial ideal contained in $\mathcal{Q}_\alpha$ which has the extension property.

Let us give an intuitive argument of how we should construct $/\alpha\backslash$. Suppose for a moment the existence of $/\alpha\backslash$, the (unique) greatest injective s-tensor norm smaller than α and fix a normed space E. Since the space $\ell_\infty(B_{E'})$ is injective, we obtain that every α-continuous polynomial in $\ell_\infty(B_{E'})$ can be extended (with the same norm) to any bigger space containing it (recall that $\ell_\infty(B_{E'})$ is always 1-complemented in a bigger space). Having in mind the idea that $\mathcal{Q}_{/\alpha\backslash}$ is the biggest class contained in $\mathcal{Q}_\alpha$ with the extension property, we should get that $\mathcal{Q}_{/\alpha\backslash}(\ell_\infty(B_{E'}))$ and $\mathcal{Q}_\alpha(\ell_\infty(B_{E'}))$ coincide (moreover, we can expect them to coincide isometrically). So, in terms of s-tensor products, we would have $\otimes^{n,s}_{/\alpha\backslash}\ell_\infty(B_{E'}) \overset{1}{=} \otimes^{n,s}_{\alpha}\ell_\infty(B_{E'})$. Now, by Proposition 3.1.3, we get the metric injection

$$\otimes^{n,s} I_E : \otimes^{n,s}_{/\alpha\backslash} E \overset{1}{\hookrightarrow} \otimes^{n,s}_{/\alpha\backslash}\ell_\infty(B_{E'}) \overset{1}{=} \otimes^{n,s}_{\alpha}\ell_\infty(B_{E'}).$$

This formula is the only ingredient we need to construct the injective associate of α as we see in the next theorem.

Theorem 3.1.5. *Let α be an s-tensor norm, there is a unique injective s-tensor norm $/\alpha\backslash \leq \alpha$ with the following property: if $\beta \leq \alpha$ is injective, then $\beta \leq /\alpha\backslash$.*

Moreover, we can explicitly define it as

$$\otimes^{n,s} I_E\colon\ \otimes^{n,s}_{/\alpha\backslash} E \overset{1}{\hookrightarrow} \otimes^{n,s}_{\alpha}\ell_\infty(B_{E'}), \tag{3.5}$$

where E is normed space and I_E is the canonical embedding (3.1).

Proof. Let γ be the s-tensor norm defined according equation (3.5). More precisely, for every normed space E and $z \in \otimes^{n,s} E$ we define

$$\gamma(z; \otimes^{n,s} E) := \alpha(\otimes^{n,s} I_E; \otimes^{n,s}\ell_\infty(B_{E'})).$$

First we have to see that γ is actually an s-tensor norm. Let us check that $\varepsilon_{n,s} \leq \gamma \leq \pi_{n,s}$. Fix E a normed space and $z \in \otimes^{n,s} E$. By the injectivity of $\varepsilon_{n,s}$ we have

$$\varepsilon_{n,s}(z; \otimes^{n,s} E) = \varepsilon_{n,s}(\otimes^{n,s} I_E(z); \otimes^{n,s}\ell_\infty(B_{E'})) \leq \alpha(\otimes^{n,s} I_E(z); \otimes^{n,s}\ell_\infty(B_{E'})) = \gamma(z; \otimes^{n,s} E).$$

On the other hand,

$$\gamma(z; \otimes^{n,s} E) = \alpha(\otimes^{n,s} I_E(z); \otimes^{n,s}\ell_\infty(B_{E'})) \leq \pi_{n,s}(\otimes^{n,s} I_E(z); \otimes^{n,s}\ell_\infty(B_{E'})) \leq \pi_{n,s}(z; \otimes^{n,s} E).$$

Now we see that, with this definition, the metric mapping property is verified. Let $T \in \mathcal{L}(E, F)$ an operator, by the metric extension property of $\ell_\infty(B_{E'})$ we have and operator $\overline{T}$ with $\|\overline{T}\| = \|T\|$ such that

$$\begin{tikzcd} \ell_\infty(B_{E'}) \arrow[r, dashed, "\overline{T}"] & \ell_\infty(B_{F'}) . \\ E \arrow[u, hook, "I_E"] \arrow[r, "T"] & F \arrow[u, hook, "I_F"'] \end{tikzcd}$$

Therefore, for $z \in \otimes^{n,s} E$,

$$\begin{aligned}\gamma(\otimes^{n,s}T(z); \otimes^{n,s}F) &= \alpha((\otimes^{n,s}I_F \circ \otimes^{n,s}T)(z); \otimes^{n,s}\ell_\infty(B_{F'}))\\ &= \alpha((\otimes^{n,s}\overline{T} \circ \otimes^{n,s}I_E)(z); \otimes^{n,s}\ell_\infty(B_{F'}))\\ &\leq \|\overline{T}\|^n \alpha(\otimes^{n,s}I_E(z); \otimes^{n,s}\ell_\infty(B_{E'}))\\ &= \|T\|^n \gamma(z; \otimes^{n,s}F)\end{aligned}$$

We have shown that γ is a well defined s-tensor norm. Now we see that γ is the unique injective s-tensor norm smaller than α with the following property: if $\beta \leq \alpha$ is injective, then $\beta \leq \gamma$.

Using the definition of γ and the fact the $\ell_\infty(B_{E'})$ is 1-complemented in any larger space we get that γ coincides (isometrically) with α in $\otimes^{n,s}\ell_\infty(B_{E'})$. Hence,

$$\otimes^{n,s}_\gamma \ell_\infty(B_{E'}) \overset{1}{=} \otimes^{n,s}_\alpha \ell_\infty(B_{E'}).$$

By definition, for every normed space E we get the metric injection

$$\otimes^{n,s} I_E : \otimes^{n,s}_\gamma E \overset{1}{\hookrightarrow} \otimes^{n,s}_\alpha \ell_\infty(B_{E'}) \overset{1}{=} \otimes^{n,s}_\gamma \ell_\infty(B_{E'}).$$

Thus, by Proposition 3.1.3 γ is injective. Let β be an injective s-tensor norm such that $\beta \leq \alpha$. By the injectivity of β,

$$\beta(z; \otimes^{n,s}E) = \beta(\otimes^{n,s}I_E(z); \otimes^{n,s}\ell_\infty(B_{E'})),$$

which is less than or equal to

$$\alpha(\otimes^{n,s}I_E(z); \otimes^{n,s}\ell_\infty(B_{E'})) = \gamma(z; \otimes^{n,s}E).$$

We have seen that $\beta \leq \gamma$, so γ has the desired property. Uniqueness becomes trivial. To follow previous notation, we define $/\alpha\backslash := \gamma$. $\square$

Note that every Banach space with the λ-extension property is λ-complemented in $\ell_\infty(B_{E'})$. As a consequence, we have the following proposition.

Proposition 3.1.6. *Let α be an s-tensor norm and E be a Banach space with the λ-extension property, then*

$$/\alpha\backslash \leq \alpha \leq \lambda^n /\alpha\backslash \text{ on } \otimes^{n,s} E.$$

In particular,

$$\alpha \overset{1}{=} /\alpha\backslash \text{ on } \otimes^{n,s} \ell_\infty(I) \qquad \text{and} \qquad \mathcal{Q}_\alpha(\ell_\infty(I)) \overset{1}{=} \mathcal{Q}_{/\alpha\backslash}(\ell_\infty(I)),$$

for every index set I.

An s-tensor norm that appears in the literature that comes from the construction given in Theorem 3.1.5 is the norm η [KR98, Car99], which is exactly $/\pi_{n,s}\backslash$. This norm is the predual s-tensor norm of the ideal of extendible polynomials $\mathcal{P}^n_e$. The fact that $\mathcal{P}^n_e$ is just $\mathcal{Q}_{/\pi_{n,s}\backslash}$ is quite reasonable since, roughly speaking, $\mathcal{Q}_{/\pi_{n,s}\backslash}$ is the biggest class of $\pi_{n,s}$-continuous polynomials that can be extended to any larger space. This should be clear with the following description.

Proposition 3.1.7. *For an s-tensor norm α and a Banach space E we have*

$$\mathcal{Q}_{/\alpha\backslash}(E) \overset{(*)}{=} \{p \in \mathcal{P}^n(E) : \text{ there exist a } C > 0 \text{ such that for every } F \supset E \text{ there is an } \alpha\text{-continous extension } \overline{p} \text{ of } p \text{ to } F, \text{ with norm } \|\overline{p}\|_{\mathcal{Q}_\alpha(F)} \leq C\} \ .$$

Moreover, the norm in $\mathcal{Q}_{/\alpha\backslash}(E)$ is given by

$$\|p\|_{\mathcal{Q}_{/\alpha\backslash}(E)} \overset{(**)}{=} \inf\{C > 0 : \text{ for every } F \supset E \text{ there is an } \alpha\text{-continous extension } \overline{p} \text{ of } p \text{ to } F, \text{ with norm } \|\overline{p}\|_{\mathcal{Q}_\alpha(F)} \leq C\}.$$

Proof. For simplicity denote by $\mathfrak{S}$ the set on the right of $(*)$ and D the number on the right of $(**)$. Let $p \in \mathcal{Q}_{/\alpha\backslash}(E)$ and F be a space containing E; since $/\alpha\backslash$ is injective, by Proposition 3.1.3, $\mathcal{Q}_{/\alpha\backslash}$ has the extension property and therefore p can be extended to a polynomial $\overline{p}$ such that $\|p\|_{\mathcal{Q}_{/\alpha\backslash}(E)} = \|\overline{p}\|_{\mathcal{Q}_{/\alpha\backslash}(F)}$. Note that, since $/\alpha\backslash \leq \alpha$, $\overline{p}$ is α-continuous. Since this holds for every $F \supset E$, we obtain the inclusion $\subset$ in $(*)$ and $D \leq \|p\|_{\mathcal{Q}_{/\alpha\backslash}(E)}$.

For the reverse inclusion, fix $p \in \mathfrak{S}$ and $\varepsilon > 0$. We can extended p to an α continuous polynomials $\overline{p}$ on $\ell_\infty(B_{E'})$ with $\|\overline{p}\|_{\mathcal{Q}_\alpha(\ell_\infty(B_{E'}))} \leq D + \varepsilon$. But, by Proposition 3.1.7,

$$\|\overline{p}\|_{\mathcal{Q}_{/\alpha\backslash}(\ell_\infty(B_{E'}))} = \|\overline{p}\|_{\mathcal{Q}_\alpha(\ell_\infty(B_{E'}))}.$$

This implies that the extension $\overline{p}$ is $/\alpha\backslash$-continuous and therefore, so does p. We therefore obtain the inclusion $\supset$ in $(*)$ and $\|p\|_{\mathcal{Q}_{/\alpha\backslash}(E)} \leq D + \varepsilon$. Since ε was arbitrary we get $(**)$. □

3.2 The projective associate

We have described injective s-tensor norms as those norms that 'respect subspaces'. Now we devote our efforts to deal with norms that 'respect quotient mappings'. An s-tensor norm α is *projective* (*or projective on $NORM$*) if, for every pair of normed spaces G and E, and every metric surjection $Q : G \overset{1}{\twoheadrightarrow} E$, the tensor product operator

$$\otimes^{n,s}Q : \otimes^{n,s}_\alpha G \to \otimes^{n,s}_\alpha E$$

is also a metric surjection. When the same conclusion holds only for Banach space we say that α is projective on BAN.

An example of a norm with this property is **the** projective norm $\pi_{n,s}$. Indeed, if $Q : G \overset{1}{\twoheadrightarrow} E$ is a metric surjection and $z \in \otimes^{n,s}E$, take $\sum_{j=1}^r \otimes^{n,s}x_j$ a representation of z such that $\sum_{j=1}^r \|x_j\|^n < \pi_{n,s}(z; \otimes^{n,s}E) + \frac{\varepsilon}{2}$. We can find vectors y_j satisfying $Q(y_j) = x_j$ and $\|y_j\|^n \leq \|x_j\|^n + \frac{\varepsilon}{2^{j+1}}$. Therefore, if $w = \sum_{j=1}^r y_j$, we see that $\otimes^{n,s}Q(w) = z$ and

$$\pi_{n,s}(w; \otimes^{n,s}E) \leq \sum_{j=1}^r \|y_j\|^n \leq \sum_{j=1}^r \|x_j\|^n + \frac{\varepsilon}{2} \leq \pi_{n,s}(z; \otimes^{n,s}E) + \varepsilon.$$

This shows that the mapping $\otimes^{n,s}Q : \otimes^{n,s}_{\pi_{n,s}} G \to \otimes^{n,s}_{\pi_{n,s}} E$ results a metric surjection.

The following proposition reveals that s-tensor norms that are projective on BAN are always finitely generated and also projective on $NORM$.

Proposition 3.2.1. *(1) If α is projective on BAN, then α is finitely generated.*

(2) If α is an s-tensor norm on $NORM$, and α is projective on BAN, then α is projective on $NORM$.

To prove this, we make use of the following useful lemmas.

Lemma 3.2.2. *Let α be an s-tensor norm. If α is finitely generated on BAN ($\alpha = \overrightarrow{\alpha}$ on $\otimes^{n,s} E$, for every Banach space E) then α is also finitely generated. Moreover,*

$$\otimes_{\alpha}^{n,s} E \overset{1}{\hookrightarrow} \otimes_{\alpha}^{n,s} \widetilde{E} \quad \text{for every normed space } E,$$

where $\widetilde{E}$ denotes the completion of E.

Proof. Let E be a normed space and $z \in \otimes^{n,s} E$; by the metric mapping property we have

$$\overrightarrow{\alpha}(z; \otimes^{n,s} \widetilde{E}) = \alpha(z; \otimes^{n,s} \widetilde{E}) \leq \alpha(z; \otimes^{n,s} E) \leq \overrightarrow{\alpha}(z; \otimes^{n,s} E).$$

Let $M \in FIN(\widetilde{E})$, such that $z \in \otimes^{n,s} M$ and

$$\alpha(z; \otimes^{n,s} M) \leq (1+\varepsilon) \overrightarrow{\alpha}(z; \otimes^{n,s} \widetilde{E}).$$

By the well know Principle of Local Reflexivity (see Theorem 2.1.12) we can find an operator $T \in \mathcal{L}(M, E)$ such that $\|T\| \leq 1 + \varepsilon$ satisfying $Tx = x$ for every $x \in M \cap E$. Thus,

$$\overrightarrow{\alpha}(z; \otimes^{n,s} E) \leq \alpha(z; \otimes^{n,s} TM) \leq (1+\varepsilon)\alpha(z; \otimes^{n,s} M) \leq (1+\varepsilon)^2 \alpha(z; \otimes^{n,s} \widetilde{E}).$$

This concludes the proof. □

The previous lemma also shows that a finitely generated s-tensor norm α defined on BAN has a unique extension to $NORM$ (which obviously result finitely generated). Now we can prove Proposition 3.2.1.

Now we state an easy lemma, which can be found in [DF93, 7.4.]

Lemma 3.2.3. *Let E and F be normed spaces, $Q \in \mathcal{L}(E, F)$ surjective, $E_0 \subset E$ dense and $Q_0 := Q|_{E_0} : E_0 \to Q(E_0)$ the surjective restriction. Then Q_0 is a metric surjection if and only if $\overline{kerQ_0} = kerQ$ and Q is a metric surjection.*

We are now ready to prove Proposition 3.2.1.

Proof. (*of Proposition 3.2.1.*)

(1) Suppose α is projective on BAN. Let E be a Banach space, consider the quotient mapping

$$Q_E : \ell_1(B_E) \twoheadrightarrow E.$$

Since $\ell_1(B_E)$ has the metric approximation property by the approximation Lemma 2.1.2 we have $\alpha = \overrightarrow{\alpha}$ on $\otimes^{n,s} \ell_1(B_E)$. Thus, for each element $z \in \otimes^{n,s} E$ and each $\varepsilon > 0$ there is an $M \in FIN(\ell_1(B_E))$ and a $w \in \otimes^{n,s} M$ with $\otimes^{n,s} Q_E(w) = z$ and

$$\alpha(w; \otimes^{n,s} M) \leq (1+\varepsilon)\alpha(z; \otimes^{n,s} E).$$

Hence,

$$\begin{aligned}\alpha(z; \otimes^{n,s} E) &\leq \overrightarrow{\alpha}(z; \otimes^{n,s} E) \\ &\leq \alpha(z; \otimes^{n,s} Q_E(M)) \\ &\leq \alpha(w; \otimes^{n,s} M) \\ &\leq (1+\varepsilon)\alpha(z; \otimes^{n,s} E).\end{aligned}$$

Since this holds for arbitrary ε, we have $\alpha = \overrightarrow{\alpha}$ on $\otimes^{n,s} E$. This shows that α is finitely generated on BAN. Now Lemma 3.2.2 applies.

(2) Take again, a metric surjection $Q : G \twoheadrightarrow E$ between normed spaces and consider

$$\widetilde{Q} : \widetilde{G} \twoheadrightarrow \widetilde{E}$$

the completion mapping, which is also a metric surjection with $Ker\ \widetilde{Q} = \overline{Ker\ Q}$ (Lemma 3.2.3). Since α is projective on BAN, using Lemma 3.2.2 we obtain the following commutative diagram:

$$\begin{array}{ccc} \otimes_{\alpha}^{n,s} G & \xhookrightarrow{\ 1\ } & \otimes_{\alpha}^{n,s} \widetilde{G} \\ \Big\downarrow \otimes^{n,s} Q & & \Big\downarrow \otimes^{n,s}\widetilde{Q} \\ \otimes_{\alpha}^{n,s} E & \xhookrightarrow{\ 1\ } & \otimes_{\alpha}^{n,s} \widetilde{E} \end{array} .$$

Now notice that $Ker(\otimes^{n,s}\widetilde{Q})$ is exactly

$$\big(Ker(\widetilde{Q}) \otimes \widetilde{E} \otimes \cdots \otimes \widetilde{E} \ +\ \widetilde{E} \otimes Ker(\widetilde{Q}) \otimes \cdots \otimes \widetilde{E} \ +\ \cdots \ +\ \widetilde{E} \otimes \cdots \otimes \widetilde{E} \otimes Ker(\widetilde{Q})\big) \cap \otimes^{n,s} \widetilde{E}.$$

In other words, we can write $Ker(\otimes^{n,s}\widetilde{Q})$ as

$$\sigma_{\widetilde{E}}^{n}\big(Ker(\widetilde{Q}) \otimes \widetilde{E} \otimes \cdots \otimes \widetilde{E} \ +\ \widetilde{E} \otimes Ker(\widetilde{Q}) \otimes \cdots \otimes \widetilde{E} \ +\ \cdots \ +\ \widetilde{E} \otimes \cdots \otimes \widetilde{E} \otimes Ker(\widetilde{Q})\big),$$

where $\sigma_{\widetilde{E}}^{n}$ is the symmetrization operator defined in Equation (1.4). Therefore,

$$\begin{aligned} Ker(\otimes^{n,s}\widetilde{Q}) &= \sigma_{\widetilde{E}}^{n}\big(Ker(\widetilde{Q}) \otimes \widetilde{E} \otimes \cdots \otimes \widetilde{E} \ +\ \cdots \ +\ \widetilde{E} \otimes \cdots \otimes \widetilde{E} \otimes Ker(\widetilde{Q})\big) \\ &= \sigma_{\widetilde{E}}^{n}\big(\overline{Ker(Q)} \otimes \overline{E} \otimes \cdots \otimes \overline{E} \ +\ \cdots \ +\ \overline{E} \otimes \cdots \otimes \overline{E} \otimes \overline{Ker(Q)}\big) \\ &\subset \overline{\sigma_{E}^{n}\big(Ker(Q) \otimes E \otimes \cdots \otimes E \ +\ \cdots \ +\ E \otimes \cdots \otimes E \otimes Ker(Q)\big)}^{\otimes^{n,s}\widetilde{E}} \\ &= \overline{Ker(\otimes^{n,s} Q)}^{\otimes^{n,s}\widetilde{E}}. \end{aligned}$$

Hence $\overline{Ker(\otimes^{n,s} Q)}^{\otimes^{n,s}\widetilde{E}} = Ker(\otimes^{n,s}\widetilde{Q})$, which by Lemma 3.2.3 concludes the proof. $\square$

Note that every s-tensor norm α is less than or equal to $\pi_{n,s}$. Since $\pi_{n,s}$ is projective, it is reasonable to search for smaller projective s-tensor norms that also dominate α. This motivates the following definition.

Definition 3.2.4. *The projective associate of* α, denoted by $\backslash\alpha/$, will be the (unique) smallest projective s-tensor norm greater than α. The next theorem shows its existence.

To prove its existence we need a definition which is dual to the extension property.

Definition 3.2.5. A Banach space E has the *lifting property or is projective* if the following holds: given a surjective mapping $Q \in \mathcal{L}(F;G)$ between Banach spaces and an operator $T \in \mathcal{L}(E;F)$ and $\varepsilon > 0$ there exist an operator $\widetilde{T}$ (*a lifting of* T) with norm $\|\widetilde{T}\| \leq (1+\varepsilon)\|T\|$ such that $T = Q \circ \widetilde{T}$, *i.e.,*

$$\begin{tikzcd} & G \arrow[d, "Q"] \\ E \arrow[ur, dashed, "\widetilde{T}"] \arrow[r, "T"] & F \end{tikzcd} .$$

An easy exercise is to show that $\ell_1(I)$ has the lifting property for every index set I. Recall that, for a Banach space E, we have a metric surjection $Q_E : \ell_1(B_E) \overset{1}{\twoheadrightarrow} E$ given by

$$Q_E\big((a_x)_{x\in B_E}\big) = \sum_{x\in B_E} a_x x. \tag{3.6}$$

This mapping is referred to as the *canonical quotient mapping of* E.

Now we are ready to prove the following.

Theorem 3.2.6. *Let α be an s-tensor norm on $NORM$, there is a unique projective s-tensor norm $\backslash\alpha/ \geq \alpha$ with the following property: if $\beta \geq \alpha$ is projective, then $\beta \geq \backslash\alpha/$. Moreover, if E is a Banach space, we can explicitly define it as*

$$\otimes^{n,s} Q_E \colon \ \otimes_{\alpha}^{n,s} \ell_1(B_E) \overset{1}{\twoheadrightarrow} \otimes_{\backslash\alpha/}^{n,s} E,$$

where $Q_E : \ell_1(B_E) \twoheadrightarrow E$ is the canonical quotient mapping defined in (3.6).

Proof. We define the projective associate first on BAN and then extend it to $NORM$. For a Banach space E, define γ on BAN by the quotient mapping

$$\otimes^{n,s} Q_E \colon \ \otimes_{\alpha}^{n,s} \ell_1(B_E) \overset{1}{\twoheadrightarrow} \otimes_{\gamma}^{n,s} E,$$

where $Q_E : \ell_1(B_E) \twoheadrightarrow E$ is the canonical quotient mapping defined in (3.6). Strictly speaking, for $z \in \otimes^{n,s} E$,

$$\gamma(z) := \inf\{\alpha(w) : \ w \in \otimes^{n,s} \ell_1(B_E), \ \otimes^{n,s} Q_E(w) = z\}.$$

Let us see that, with this definition, β is an s-tensor norm. Obviously, $\varepsilon_{n,s} \leq \alpha \leq \beta \leq \pi_{n,s}$. Fix $\varepsilon > 0$ and $T \in \mathcal{L}(E,F)$. Using the lifting property of the spaces spaces $\ell_1(I)$ we can consider an operator $\widetilde{T}$ such that the next diagram commutes

$$\begin{tikzcd} \ell_1(B_E) \arrow[r, dashed, "\widetilde{T}"] \arrow[d, "1"] & \ell_1(B_F) \arrow[d, "1"] \\ E \arrow[r, "T"] & F \end{tikzcd} ,$$

and $\|\widetilde{T}\| \leq (1+\varepsilon)\|T\|$. Hence $\| \otimes^{n,s} \widetilde{T} : \otimes_{\alpha}^{n,s} E \to \otimes_{\alpha}^{n,s} F\| \leq \|\widetilde{T}\|^n \leq (1+\varepsilon)^n \|T\|^n$. Using the definition of γ given above we also obtain,

$$\begin{aligned}
\gamma\big(\otimes^{n,s} T(z)\big) &= \inf\{\alpha(w) : \ w \in \otimes^{n,s}\ell_1(B_F),\ \otimes^{n,s}Q_F(w) = \otimes^{n,s}T(z)\} \\
&\leq \inf\{\alpha\big(\otimes^{n,s} \widetilde{T}(u)\big) : \ u \in \otimes^{n,s}\ell_1(B_E),\ \otimes^{n,s}Q_E(u) = z\} \\
&\leq (1+\varepsilon)^n \|T\|^n \inf\{\alpha\big(u\big) : \ u \in \otimes^{n,s}\ell_1(B_E),\ \otimes^{n,s}Q_E(u) = z\} \\
&= (1+\varepsilon)^n \|T\|^n \gamma\big(z\big).
\end{aligned}$$

Since ε is arbitrary small, the metric mapping property follows.

Let us now show that γ is projective (on BAN). For this, take a metric surjection $Q : E \twoheadrightarrow F$ between Banach spaces. Again, by the lifting property $\ell_1(B_F)$ we have and operator $\widetilde{Q} : \ell_1(B_F) \to \ell_1(B_E)$, with $\|\widetilde{Q}\| \leq (1+\varepsilon)$ such that $\mathcal{Q} \circ Q_E \circ \widetilde{Q} = \mathcal{Q}_F$.The fact that the tensor operator $\otimes^{n,s}Q : \otimes_{\gamma}^{n,s} E \to \otimes_{\gamma}^{n,s} F$ is a metric surjection now follows from the commutative diagram

$$\begin{tikzcd}
\otimes_{\alpha}^{n,s}\ell_1(B_E) \arrow[d, "1"] & \otimes_{\alpha}^{n,s}\ell_1(B_F) \arrow[l, dashed, "\otimes^{n,s}\widetilde{Q}"'] \arrow[d, "1"] \\
\otimes_{\gamma}^{n,s}E \arrow[r, "\otimes^{n,s}Q"] & \otimes_{\gamma}^{n,s}F
\end{tikzcd}$$

Indeed, for $z \in \otimes^{n,s}F$ we have

$$\begin{aligned}
&\inf\{\backslash\alpha(w)/ : \ w \in \otimes^{n,s}E,\ \otimes^{n,s}Q(w) = z\} \\
&\qquad \leq \inf\{\alpha(u) : \ u \in \otimes^{n,s}E,\ \big(\otimes^{n,s} Q \circ \otimes^{n,s}Q_E\big)(u) = z\} \\
&\qquad \leq (1+\varepsilon)^n \inf\{\alpha(v) : \ v \in \otimes^{n,s}\ell_1(B_F),\ \big(\otimes^{n,s} Q \circ \otimes^{n,s}Q_E \circ \otimes^{n,s}\widetilde{Q}\big)(v) = z\} \\
&\qquad = (1+\varepsilon)^n \inf\{\alpha(v) : \ v \in \otimes^{n,s}\ell_1(B_F),\ \otimes^{n,s}Q_F(v) = z\} \\
&\qquad = (1+\varepsilon)^n \backslash\alpha(z)/.
\end{aligned}$$

The other inequality $\backslash\alpha(z)/ \leq \inf\{\backslash\alpha(w)/ : \ w \in \otimes^{n,s}E,\ \otimes^{n,s}Q(w) = z\}$ is a consequence of the metric mapping property. Thus, γ is projective (on BAN).

We would like to extend the definition γ to $NORM$. Recall that, by Proposition 3.2.1 we know that γ is finitely generated on BAN. Since we already know how to compute the norm for the s-tensor product of finite dimensional spaces, we can easily extend γ to $NORM$ by the following way: for a normed spaces G and $z \in \otimes^{n,s}G$, we define

$$\backslash\alpha/\big(z; \otimes^{n,s}G\big) := \inf\{\gamma(z; \otimes^{n,s}M) : \ z \in \otimes^{n,s}M,\ M \in FIN(G)\}.$$

Note that, according to Proposition 3.2.1 and Lemma 3.2.2, this is the only way we can extend γ to $NORM$. With this definition, $\backslash\alpha/$ coincides with γ on BAN and, by Proposition 3.2.1 again, results projective on $NORM$.

Let us now show that $\backslash\alpha/$ is the unique projective s-tensor norm $\backslash\alpha/ \geq \alpha$ with the following property: if $\beta \geq \alpha$ is projective, then $\beta \geq \backslash\alpha/$. Take any projective s-tensor norm $\beta \geq \alpha$, E a Banach space and $z \in \otimes^{n,s}E$. Therefore, if $Q_E : \ell_1(B_E) \overset{1}{\twoheadrightarrow} E$ is the canonical quotient mapping,

$$\begin{aligned}
\backslash\alpha/(z;\otimes^{n,s}E) &= \gamma(z;\otimes^{n,s}E) \\
&= \inf\{\alpha(w):\ w\in\otimes^{n,s}\ell_1(B_E),\ \otimes^{n,s}Q_E(w)=z\} \\
&\leq \inf\{\beta(w):\ w\in\otimes^{n,s}\ell_1(B_E),\ \otimes^{n,s}Q_E(w)=z\} \\
&= \beta(z;\otimes^{n,s}E).
\end{aligned}$$

We have seen that $\backslash\alpha/\leq\beta$ on BAN, since both norms are finitely generated we have the same inequality on $NORM$. Uniqueness is trivial. □

The next result shows that an s-tensor norm coincides with its projective associate on the symmetric tensor product of $\ell_1(I)$, where I is any index set.

Proposition 3.2.7. *Let α be an s-tensor norm, then*

$$\alpha=\backslash\alpha/ \text{ on } \otimes^{n,s}\ell_1(I),$$

for every index set I.

Proof. Let $Q_{\ell_1(I)}:\ell_1\big(B_{\ell_1(I)}\big)\twoheadrightarrow\ell_1(I)$ the natural quotient mapping. Since $\ell_1(I)$ is projective then there is a lifting $T:\ell_1(I)\to\ell_1\big(B_{\ell_1(I)}\big)$ of $id_{\ell_1(I)}$ (*i.e.*, $Q_{\ell_1(I)}\circ T=id_{\ell_1(I)}$) having norm less than or equal to $1+\varepsilon$. Thus, by the diagram

$$\begin{tikzcd}
\otimes^{n,s}_{\alpha}\ell_1(I) \arrow[r, "\otimes^{n,s}id"] \arrow[dr, dashed, "\otimes^{n,s}T"'] & \otimes^{n,s}_{\backslash\alpha/}\ell_1(I) \\
& \otimes^{n,s}_{\alpha}\ell_1\big(B_{\ell_1(I)}\big) \arrow[u, two heads, "\otimes^{n,s}Q_{\ell_1(I)}"']
\end{tikzcd},$$

we have $\backslash\alpha/\leq(1+\varepsilon)\alpha$. Since $\alpha\leq\backslash\alpha/$ always holds, we have the desired equality. □

A particular but crucial case of Proposition 3.2.7 and Proposition 3.1.6 is obtained with I a finite set: we get for every s-tensor norm α and $m\in\mathbb{N}$,

$$\alpha=\backslash\alpha/ \text{ on } \otimes^{n,s}\ell_1^m,$$

$$\alpha=/\alpha\backslash \text{ on } \otimes^{n,s}\ell_\infty^m.$$

The previous equalities allow us to use $\mathcal{L}_p$-Local Technique Lemma 2.1.7 to give the following.

Corollary 3.2.8. *Let α an s-tensor norm*

(1) If E is $\mathcal{L}^g_{1,\lambda}$-space, then

$$\alpha\leq\backslash\alpha/\leq\lambda^n\overrightarrow{\alpha} \quad \text{on} \quad \otimes^{n,s}E.$$

(2) If E is $\mathcal{L}^g_{\infty,\lambda}$-space, then

$$\alpha\leq/\alpha\backslash\leq\lambda^n\overrightarrow{\alpha} \quad \text{on} \quad \otimes^{n,s}E.$$

3.3 Some relations between the injective and projective associates

The next result show the relation between finite hulls, cofinite hulls, projective associates, injective associates and duality.

Proposition 3.3.1. *For an s-tensor norm α we have the following relations:*

(1) $/\alpha\backslash = /\overrightarrow{\alpha}\backslash = \overrightarrow{/\alpha\backslash}$*;*

(2) $/\alpha\backslash = /\overleftarrow{\alpha}\backslash = \overleftarrow{/\alpha\backslash}$*;*

(3) $\backslash\alpha/ = \backslash\overrightarrow{\alpha}/ = \overrightarrow{\backslash\alpha/}$*;*

(4) $\backslash\alpha/ = \backslash\overleftarrow{\alpha}/$*;*

(5) $(\backslash\alpha/)' = /\alpha'\backslash$ *and* $(/\alpha\backslash)' = \backslash\alpha'/$*.*

It is important to remark that the identity $\backslash\overleftarrow{\alpha}/ = \overleftarrow{\backslash\alpha/}$ fails to hold in general. To see this, notice that $\overleftarrow{\pi_{n,s}} = \pi_{n,s}$ on $\otimes^{n,s}\ell_1^m$. Then, by Lemma 3.3.2 below we have $\backslash\overleftarrow{\pi_{n,s}}/ = \backslash\pi_{n,s}/ = \pi_{n,s}$ (since $\pi_{n,s}$ is projective). But $\pi_{n,s}$ is not cofinitely generated [Flo01a, 2.5.]. Thus,

$$\backslash\overleftarrow{\pi_{n,s}}/ = \backslash\pi_{n,s}/ = \pi_{n,s} \neq \overleftarrow{\pi_{n,s}} = \overleftarrow{\backslash\pi_{n,s}/}.$$

To prove Proposition 3.3.1 we need the following lemma.

Lemma 3.3.2. *Let α and β be s-tensor norms.*

(1) The equality $\alpha = \beta$ holds on $\otimes^{n,s}\ell_1^m$ for every $m \in \mathbb{N}$ if and only if $\backslash\alpha/ = \backslash\beta/$.

(2) The equality $\alpha = \beta$ holds on $\otimes^{n,s}\ell_\infty^m$ for every $m \in \mathbb{N}$, if and only if $/\alpha\backslash = /\beta\backslash$.

Proof. (1) Suppose that $\otimes_\alpha^{n,s}\ell_1^m \overset{1}{=} \otimes_\beta^{n,s}\ell_1^m$ for every m. If E is a Banach space and $Q_E : \ell_1(B_E) \twoheadrightarrow E$ is the canonical quotient mapping defined in equation (3.6), we have

$$\otimes^{n,s}Q_E \colon \otimes_\alpha^{n,s} \ell_1(B_E) \overset{1}{\twoheadrightarrow} \otimes_{\backslash\alpha/}^{n,s} E,$$

$$\otimes^{n,s}Q_E \colon \otimes_\beta^{n,s} \ell_1(B_E) \overset{1}{\twoheadrightarrow} \otimes_{\backslash\beta/}^{n,s} E.$$

Since $\ell_1(B_E)$ has the metric approximation property, by the $\mathcal{L}_p$-Local Technique Lemma 2.1.7 and Propositon 2.2.1 we have $\alpha = \beta$ on $\otimes^{n,s}\ell_1(B_E)$. As a consequence, we have

$$\backslash\alpha/ = \backslash\beta/ \text{ on } \otimes^{n,s} E,$$

for every Banach space E. Since these norms are finitely generated (according Lemma 3.2.1) and coincide on the tensor product of any Banach space (in particular, in the tensor product of any finite dimensional space) we have the equality.

The converse is a direct consequence of Proposition 3.2.7.

The proof in (2) is similar. Suppose $\alpha = \beta$ on $\otimes^{n,s}\ell_\infty^m$ for every m. Again by the $\mathcal{L}_p$-Local Technique Lemma 2.1.7 and Propositon 2.2.1, we have $\alpha = \beta$ on $\ell_\infty(B_{E'})$. To finish the proof we just use the isometric embeddings

$$\otimes^{n,s} I_E \colon \ \otimes^{n,s}_{/\alpha\backslash} E \overset{1}{\hookrightarrow} \otimes^{n,s}_{\alpha} \ell_\infty(B_{E'}),$$

$$\otimes^{n,s} I_E \colon \ \otimes^{n,s}_{/\beta\backslash} E \overset{1}{\hookrightarrow} \otimes^{n,s}_{\beta} \ell_\infty(B_{E'}).$$

The converse follows from Proposition 3.1.6. $\square$

Now we are ready to prove Proposition 3.3.1.

Proof. (*of Proposition 3.3.1.*)

(1) Since $\alpha = \overrightarrow{\alpha}$ on $\otimes^{n,s}\ell_\infty^m$, for every m, by the Lemma 3.3.2 we have

$$/\alpha\backslash = /(\overrightarrow{\alpha})\backslash \text{ on } \otimes^{n,s} E.$$

To prove that $/\alpha\backslash = \overrightarrow{/\alpha\backslash}$, we first note that if $z \in \otimes^{n,s} M$ with $M \in FIN(E)$, by the injectivity of $/\alpha\backslash$ we have

$$/\alpha\backslash(z; \otimes^{n,s} M) = /\alpha\backslash(z; \otimes^{n,s} E).$$

As a consequence,

$$\begin{aligned} \overrightarrow{/\alpha\backslash}(z; \otimes^{n,s} E) &= \inf\{/\alpha\backslash(z; \otimes^{n,s} M) : z \in \otimes^{n,s} M \text{ and } M \in FIN(E)\} \\ &= \inf\{/\alpha\backslash(z; \otimes^{n,s} E)\} \\ &= /\alpha\backslash(z; \otimes^{n,s} E). \end{aligned}$$

(2) Since $\alpha = \overleftarrow{\alpha}$ on $\otimes^{n,s}\ell_\infty^m$ for every m, the equality

$$/\alpha\backslash = /(\overleftarrow{\alpha})\backslash \tag{3.7}$$

on $\otimes^{n,s} E$ follows Lemma 3.3.2. On the other hand, Proposition 2.2.1 gives $\overleftarrow{/\alpha\backslash} \leq /\alpha\backslash$. To show the reverse inequality, note that

$$/\alpha\backslash = /(/\alpha\backslash)\backslash = /(\overleftarrow{/\alpha\backslash})\backslash,$$

where the second equality is just (3.7) applied to $/\alpha\backslash$. Since by definition of the injective associate we have $/\mu\backslash \leq \mu$ for every s-tensor norms μ, taking $\mu = \overleftarrow{/\alpha\backslash}$ we get $/(\overleftarrow{/\alpha\backslash})\backslash \leq \overleftarrow{/\alpha\backslash}$, which gives de desired inequality.

(3) The equality $\backslash\alpha/ = \backslash(\overrightarrow{\alpha})/$ is again a consequence of Lemma 3.3.2. On the other hand, Proposition 2.2.1 gives $\backslash\alpha/ \leq \overleftarrow{\backslash\alpha/}$. To show the reverse inequality, note that

$$\backslash\alpha/ = \backslash(\backslash\alpha/)/ = \backslash(\overleftarrow{\backslash\alpha/})/.$$

Since by definition of the projective associate we have $\mu \leq \backslash\mu/$ for every s-tensor norms μ, taking $\mu = \overleftarrow{\backslash\alpha/}$ we have the reverse inequality. Therefore, $\backslash\alpha/ = \overrightarrow{\backslash\alpha/}$ on $\otimes^{n,s} E$.

(4) Is a direct consequence of Lemma 3.3.2.

(5) Let us see first that $(\backslash\alpha/)'$ is injective. Consider an isometric embedding $E \overset{1}{\hookrightarrow} F$ and $z \in \otimes^{n,s} M$, where M is a finite dimensional subspace of E. Fix $\varepsilon > 0$, since $(\backslash\alpha/)'$ is finitely generated we can take $N \in FIN(F)$ such that $z \in \otimes^{n,s} N$ and

$$(\backslash\alpha/)'(z; \otimes^{n,s} N) \leq (\backslash\alpha/)'(z; \otimes^{n,s} F) + \varepsilon.$$

Denote by S the finite dimensional subspace of F given by $M + N$ and $i : M \to S$ the canonical inclusion. Observe that $\otimes^{n,s} i' : \otimes^{n,s}_{\backslash\alpha/} S' \overset{1}{\twoheadrightarrow} \otimes^{n,s}_{\backslash\alpha/} M'$ is a quotient mapping since the s-tensor norm $\backslash\alpha/$ is projective. Thus, its adjoint

$$(\otimes^{n,s} i')' : \big(\otimes^{n,s}_{\backslash\alpha/} M' \big)' \overset{1}{\hookrightarrow} \big(\otimes^{n,s}_{\backslash\alpha/} S' \big)',$$

is an isometric embedding. Using the definition of the dual norm on finite dimensional spaces and the right identifications, it is easy to show that the following diagram commutes

$$\begin{tikzcd}
\otimes^{n,s}_{(\backslash\alpha/)'} M \arrow[r, "\otimes^{n,s} i"] \arrow[d, equal] & \otimes^{n,s}_{(\backslash\alpha/)'} S \arrow[d, equal] \\
\big(\otimes^{n,s}_{\backslash\alpha/} M' \big)' \arrow[r, hook, "(\otimes^{n,s} i')'"] & \big(\otimes^{n,s}_{\backslash\alpha/} S' \big)'
\end{tikzcd} . \tag{3.8}$$

Therefore $\otimes^{n,s} i : \otimes^{n,s}_{(\backslash\alpha/)'} M \to \otimes^{n,s}_{(\backslash\alpha/)'} S$ is also an isometric embedding. With this, we have the equality $(\backslash\alpha/)'(z; \otimes^{n,s} M) = (\backslash\alpha/)'(z; \otimes^{n,s} S)$. Now,

$$\begin{aligned}
(\backslash\alpha/)'(z; \otimes^{n,s} E) &\leq (\backslash\alpha/)'(z; \otimes^{n,s} M) \leq (\backslash\alpha/)'(z; \otimes^{n,s} S) \\
&\leq (\backslash\alpha/)'(z; \otimes^{n,s} N) \leq (\backslash\alpha/)'(z; \otimes^{n,s} F) + \varepsilon.
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we obtain $(\backslash\alpha/)'(z; \otimes^{n,s} E) \leq (\backslash\alpha/)'(z; \otimes^{n,s} F)$. The other inequality always holds, so $(\backslash\alpha/)'$ is injective.

We now show that $(\backslash\alpha/)'$ coincides with $/\alpha'\backslash$. Note that for $m \in \mathbb{N}$,

$$\otimes^{n,s}_{(\backslash\alpha/)'} \ell_\infty^m = \big(\otimes^{n,s}_{\backslash\alpha/} \ell_1^m \big)' = \big(\otimes^{n,s}_{\alpha} \ell_1^m \big)' = \otimes^{n,s}_{\alpha'} \ell_\infty^m = \otimes^{n,s}_{/\alpha'\backslash} \ell_\infty^m.$$

Therefore, the s-tensor norms $(\backslash\alpha/)'$ and $/\alpha'\backslash$ coincide in $\otimes^{n,s} \ell_\infty^m$ for every $m \in \mathbb{N}$ and, by Lemma 3.3.2, their corresponding injective associates coincide. But both $(\backslash\alpha/)'$ and $/\alpha'\backslash$ are injective, which means that they actually are their own injective associates, therefore $(\backslash\alpha/)'$ and $/\alpha'\backslash$ are equal.

Let us finally prove that $(/\alpha\backslash)' = \backslash\alpha'/$. We already showed that $(\backslash\beta/)' = /\beta'\backslash$ for every tensor norm β. Thus, for $\beta = \alpha'$ we have $(\backslash\alpha'/)' = /\alpha''\backslash = /\overrightarrow{\alpha}\backslash = /\alpha\backslash$, where the third equality comes from (1). Thus, by duality, the fact that $\backslash\alpha'/$ is finitely generated (by (2)) and equation (1.6) we have

$$\backslash\alpha'/ = \overrightarrow{\backslash\alpha'/} = (\backslash\alpha'/)'' = \big((\backslash\alpha'/)' \big)' = (/\alpha\backslash)',$$

which is what we wanted to prove. $\square$

As a consequence of Proposition 3.3.1 we obtain the following.

Corollary 3.3.3. *Let α be an s-tensor norm. The following holds.*

(1) If α is injective then it is accessible.

(2) If α then is projective then it is finitely generated.

(3) If α is finitely or cofinitely generated then: α is injective if and only if α' is projective.

Proof. Note that (1) is a consequence of (1) and (2) of Proposition 3.3.1 and that (2) follows from (3) of Proposition 3.3.1. Let us show (3). If α is injective, we have $\alpha = /\alpha\backslash$. Thus, we can use (5) of Proposition 3.3.1 to take dual norms:

$$\alpha' = (/\alpha\backslash)' = \backslash\alpha'/.$$

Since the last s-tensor norm is projective, so is α'. Note that for this implication we have not used the fact that α is finitely or cofinitely generated.

Suppose now that α is finitely generated and α' is projective (*i.e.*, $\alpha' = \backslash\alpha'/$). Thus, by (5) in Proposition 3.3.1 we have

$$\alpha'' = (\backslash\alpha'/)' = /\alpha''\backslash.$$

Since α is finitely generated, we have $\alpha = \alpha''$, see equation (1.6). Thus, $\alpha = /\alpha\backslash$, which asserts that α is injective.

Finally, suppose that α is cofinitely generated and α' is projective. Consider an isometric embedding $i : E \overset{1}{\hookrightarrow} F$. Since α' is projective, $\otimes^{n,s} i' : \otimes^{n,s} F' \overset{1}{\twoheadrightarrow} \otimes^{n,s} E'$ is a quotient mapping and, therefore, its adjoint $(\otimes^{n,s} i')'$ is an isometry. Consider the commutative diagram

$$\begin{tikzcd}
\otimes_\alpha^{n,s} E = \otimes_{\overleftarrow{\alpha}}^{n,s} E \arrow[r, hook, "1"] \arrow[d, dashed, "\otimes^{n,s} i"] & \left(\otimes_{\alpha'}^{n,s} E'\right)' \arrow[d, "(\otimes^{n,s} i')'"] \\
\otimes_\alpha^{n,s} F = \otimes_{\overleftarrow{\alpha}}^{n,s} F \arrow[r, hook, "1"] & \left(\otimes_{\alpha'}^{n,s} F'\right)'
\end{tikzcd} \quad . \tag{3.9}$$

By the Duality Theorem 2.2.3 the horizontal arrows are isometries. This forces $\otimes^{n,s} i$ to be also an isometry, which means that α respects subspaces isometrically. $\square$

3.4 Maximal polynomial ideals associated with injective and projective associates

We now describe the maximal Banach ideal of polynomials associated with injective and projective associates of an s-tensor norm. Some notation and a couple of definitions are needed.

Definition 3.4.1. Let $(\mathcal{U}, \| \ \|_{\mathcal{U}})$ be a Banach ideal of operators. The *composition ideal* $\mathcal{Q} \circ \mathcal{U}$ is defined in the following way: a polynomial p belongs to $\mathcal{Q} \circ \mathcal{U}(E)$ if it admits a factorization

$$\begin{tikzcd}
E \arrow[rr, "p"] \arrow[dr, "T"'] & & \mathbb{K} \\
& F \arrow[ur, "q"'] &
\end{tikzcd} , \tag{3.10}$$

for some Banach space F, $T \in \mathcal{U}(E,F)$ and $q \in \mathcal{Q}(F)$. The composition quasi-norm is given by

$$\|p\|_{\mathcal{Q}\circ\mathcal{U}} := \inf\{\|q\|_{\mathcal{Q}}\|T\|_{\mathcal{U}}^n\},$$

where the infimum runs over all possible factorizations as in (3.10).

When the quasi-norm $\| \ \|_{\mathcal{Q}\circ\mathcal{U}}$ is actually a norm, $(\mathcal{Q}\circ\mathcal{U}, \| \ \|_{\mathcal{Q}\circ\mathcal{U}})$ forms a Banach ideal of continuous polynomials. All the composition ideals that will interest us in the sequel are normed.

Let $(\mathcal{U}, \| \ \|_{\mathcal{U}})$ be a Banach ideal of operators. For $p \in \mathcal{P}^n(E)$ we define

$$\|p\|_{\mathcal{Q}\circ\mathcal{U}^{-1}(E)} := \sup\{\|p\circ T\|_{\mathcal{Q}} : T \in \mathcal{U},\ \|T\|_{\mathcal{U}} \le 1,\ p\circ T \text{ is defined}\} \in [0,\infty].$$

Definition 3.4.2. The quotient ideal $\mathcal{Q}\circ\mathcal{U}^{-1}$ is defined in the following way: a polynomial p is in $\mathcal{Q}\circ\mathcal{U}^{-1}(E)$ if $\|p\|_{\mathcal{Q}\circ\mathcal{U}^{-1}(E)} < \infty$.

It is not difficult to prove that $\left(\mathcal{Q}\circ\mathcal{U}^{-1}, \| \ \|_{\mathcal{Q}\circ\mathcal{U}^{-1}}\right)$ is Banach ideal of continuous n-homogeneous polynomials with the property that $p \in \mathcal{Q}\circ\mathcal{U}^{-1}$ if and only if $p\circ T \in \mathcal{Q}$ for all $T \in \mathcal{U}$. In other words, $\mathcal{Q}\circ\mathcal{U}^{-1}$ is the largest ideal satisfying $(\mathcal{Q}\circ\mathcal{U}^{-1})\circ\mathcal{U} \subset \mathcal{Q}$.

We also need the definition of some classical operator ideals.

Definition 3.4.3. Let $p,q \in [1,+\infty]$ such that $1/p+1/q \ge 1$. An operator $T : E \to F$ is (p,q)-factorable if there are a finite measure μ, operators $R \in \mathcal{L}\big(E, L_{q'}(\mu)\big)$ and $S \in \mathcal{L}\big(L_p(\mu), F''\big)$ such that $k_F \circ T = S\circ I\circ R$,

$$\begin{array}{ccc} E & \xrightarrow{T} & F \overset{k_F}{\hookrightarrow} F'' \\ {\scriptstyle R}\downarrow & & \nearrow_{S} \\ L_{q'}(\mu) & \underset{I}{\longrightarrow} & L_p(\mu), \end{array}$$

where I and k_F are the natural inclusions. We denote the space of all such operators by $\Gamma_{p,q}(E,F)$. For $T \in \Gamma_{p,q}(E,F)$, the (p,q)-factorable norm is given by $\gamma_{p,q}(T) = \inf\{\|S\|\|I\|\|R\|\}$, where the infimum is taken over all such factorizations.

If $1/p+1/q = 1$, $\Gamma_{p,q}$ coincides isometrically with the classical ideal Γ_p of p-factorable operators [DJT95, Chapter 9]. In this section, we only use Γ_∞ and Γ_1.

The next theorem describes the maximal Banach ideal of polynomials associated with the injective and projective associates of an s-tensor norm in terms of composition and quotient ideals.

Theorem 3.4.4. *Let α be an s-tensor norm of order n. We have the following identities:*

$$\mathcal{Q}_{/\alpha\backslash} \overset{1}{=} \mathcal{Q}_\alpha \circ \Gamma_\infty \quad \textit{and} \quad \mathcal{Q}_{\backslash\alpha/} \overset{1}{=} \mathcal{Q}_\alpha \circ (\Gamma_1)^{-1}.$$

To prove this, we need a polynomial version of the Cyclic Composition Theorem [DF93, Theorem 25.4.].

Lemma 3.4.5. ***(Polynomial version of the Cyclic Composition Theorem.)*** *Let $(\mathcal{Q}_1, \| \ \|_{\mathcal{Q}_1})$, $(\mathcal{Q}_2, \| \ \|_{\mathcal{Q}_2})$ be two Banach ideals of continuous n-homogeneous polynomials and $(\mathcal{U}, \| \ \|_{\mathcal{U}})$ a Banach operator ideal with $(\mathcal{U}^{dual}, \| \ \|_{\mathcal{U}^{dual}})$ right-accessible (see [DF93, 21.2]). If*

$$\mathcal{Q}_1\circ\mathcal{U} \subset \mathcal{Q}_2,$$

with $\| \; \|_{\mathcal{Q}_2} \leq k\| \; \|_{\mathcal{Q}_1 \circ \mathcal{U}}$ *for some positive constant* k*, then we have*

$$\mathcal{Q}_2^* \circ \mathcal{U}^{dual} \subset \mathcal{Q}_1^*,$$

and $\| \; \|_{\mathcal{Q}_1^*} \leq k\| \; \|_{\mathcal{Q}_2^* \circ \mathcal{U}^{dual}}$.

Proof. Fix $q \in \mathcal{Q}_2^* \circ \mathcal{U}^{dual}(E)$, $M \in FIN(E)$ and $p \in \mathcal{Q}_1(M')$ with $\|p\|_{\mathcal{Q}_1(M')} \leq 1$. For $\varepsilon > 0$, we take $T \in \mathcal{U}^{dual}(E, F)$ and $q_1 \in \mathcal{Q}_2^*(F)$ such that $q = q_1 \circ T$ and

$$\|q_1\|_{\mathcal{Q}_2^*} \|T\|_{\mathcal{U}^{dual}}^n \leq (1 + \varepsilon)\|q\|_{\mathcal{Q}_2^* \circ \mathcal{U}^{dual}}.$$

Since $(\mathcal{U}^{dual}, \| \; \|_{\mathcal{U}^{dual}})$ is right-accessible, by definition there are $N \in FIN(F)$ and $S \in \mathcal{U}^{dual}(M, N)$ with $\|S\|_{\mathcal{U}^{dual}} \leq (1 + \varepsilon)\|T|_M\|_{\mathcal{U}^{dual}} \leq (1 + \varepsilon)\|T\|_{\mathcal{U}^{dual}}$ satisfying

$$\begin{tikzcd} M \arrow[r, "T|_M"] \arrow[dr, "S"'] & F \\ & N \arrow[u, hook, "i_N"'] \end{tikzcd} , \tag{3.11}$$

Thus, since the adjoint S' of S belongs to $\mathcal{U}(N', M')$, we have

$$\begin{aligned}
\big|\langle q|_M, p\rangle\big| &= \big|\langle q_1 \circ T|_M, p\rangle\big| = \big|\langle q_1 \circ i_N \circ S, p\rangle\big| \\
&= \big|\langle q_1 \circ i_N, p \circ S'\rangle\big| \leq \|q_1 \circ i_N\|_{\mathcal{Q}_2^*} \; \|p \circ S'\|_{\mathcal{Q}_2} \\
&\leq k\|q_1\|_{\mathcal{Q}_2^*} \; \|p \circ S'\|_{\mathcal{Q}_1 \circ \mathcal{U}} \leq k\|q_1\|_{\mathcal{Q}_2^*} \; \|p\|_{\mathcal{Q}_1} \; \|S'\|_{\mathcal{U}}^n \\
&\leq k\|q_1\|_{\mathcal{Q}_2^*} \; \|S\|_{\mathcal{U}^{dual}}^n \leq k(1 + \varepsilon)^n\|q_1\|_{\mathcal{Q}_2^*} \; \|T\|_{\mathcal{U}^{dual}}^n \\
&\leq k(1 + \varepsilon)^{n+1}\|q\|_{\mathcal{Q}_2^* \circ \mathcal{U}^{dual}}.
\end{aligned}$$

This holds for every $M \in FIN(E)$ and every $p \in \mathcal{Q}_1(M')$ with $\|p\|_{\mathcal{Q}_1(M')} \leq 1$, thus $q \in \mathcal{Q}_1^*$ and $\|q\|_{\mathcal{Q}_1^*} \leq k(1+\varepsilon)\|q\|_{\mathcal{Q}_2^* \circ \mathcal{U}^{dual}}$. Since $\varepsilon > 0$ is arbitrary we get $\|q\|_{\mathcal{Q}_1^*} \leq k\|q\|_{\mathcal{Q}_2^* \circ \mathcal{U}^{dual}}$. □

Notice that the condition of $(\mathcal{U}^{dual}, \| \; \|_{\mathcal{U}^{dual}})$ being right-accessible is fulfilled whenever $(\mathcal{U}, \| \; \|_{\mathcal{U}})$ is a maximal left-accessible Banach ideal of operators [DF93, Corollary 21.3.].

Proposition 3.4.6. *Let* $(\mathcal{Q}, \| \; \|_{\mathcal{Q}})$ *a Banach ideal of continuous* n*-homogeneous polynomials and* $(\mathcal{U}, \| \; \|_{\mathcal{U}})$ *a Banach ideal of operators. If* $\mathcal{U}$ *is maximal and accessible (or* $\mathcal{U}$ *and* $\mathcal{U}^{dual}$ *are both right-accessible), and* $\mathcal{Q} \circ \mathcal{U}$ *is a Banach ideal of continuous polynomials, then*

$$(\mathcal{Q} \circ \mathcal{U})^* \stackrel{1}{=} \mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}.$$

Proof. Lemma 3.4.5 applied to the inclusion $\mathcal{Q} \circ \mathcal{U} \subset \mathcal{Q} \circ \mathcal{U}$ implies that $(\mathcal{Q} \circ \mathcal{U})^* \circ \mathcal{U}^{dual} \subset \mathcal{Q}^*$. Therefore, $(\mathcal{Q} \circ \mathcal{U})^* \subset \mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}$ and $\| \; \|_{\mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}} \leq \| \; \|_{(\mathcal{Q} \circ \mathcal{U})^*}$.

For the reverse inclusion we proceed similarly as in proof of Lemma 3.4.5. Fix $q \in \mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}(E)$, $M \in FIN(E)$ and $p \in \mathcal{Q} \circ \mathcal{U}(M')$ with $\|p\|_{\mathcal{Q} \circ \mathcal{U}(M')} \leq 1$. For $\varepsilon > 0$, we take $T \in \mathcal{U}(M', F)$ and $p_1 \in \mathcal{Q}(F)$ such that $p = p_1 \circ T$ and $\|p_1\|_{\mathcal{Q}} \|T\|_{\mathcal{U}}^n \leq (1+\varepsilon)$. Since $(\mathcal{U}, \| \; \|_{\mathcal{U}})$ is accessible, there are $N \in FIN(F)$ and $S \in \mathcal{U}(M', N)$ with

$$\|S\|_{\mathcal{U}^{dual}} \leq (1 + \varepsilon)\|T|_M\|_{\mathcal{U}^{dual}} \leq (1 + \varepsilon)\|T\|_{\mathcal{U}}$$

satisfying $T|_M = i_N \circ S$. Note that $S' \in \mathcal{U}^{dual}$ and $\|S'\|_{\mathcal{U}^{dual}} \leq (1+\varepsilon)\|T\|_{\mathcal{U}}$. Thus, $q_{|_M} \circ (S)^* \in \mathcal{Q}^*$ and $\|q|_M \circ (S)^*\|_{\mathcal{Q}^*} \leq (1+\varepsilon)^n \|q\|_{\mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}} \|T\|_{\mathcal{U}}^n$. Now we have:

$$\begin{aligned}
\big|\langle q|_M, p\rangle\big| &= \big|\langle q|_M, p_1 \circ T\rangle\big| = \big|\langle q|_M, p_1 \circ i_N \circ S\rangle\big| \\
&\leq \big|\langle q|_M \circ S', p_1 \circ i_N\rangle\big| \leq \|q|_M \circ S'\|_{\mathcal{Q}^*}\ \|p_1 \circ i_N\|_{\mathcal{Q}} \\
&\leq (1+\varepsilon)^n \|q\|_{\mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}}\ \|p_1\|_{\mathcal{Q}}\ \|T\|_{\mathcal{U}}^n \\
&\leq (1+\varepsilon)^{n+1} \|q\|_{\mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}}.
\end{aligned}$$

This holds for every $M \in FIN(E)$, every $p \in \mathcal{Q} \circ \mathcal{U}(M')$ with $\|p\|_{\mathcal{Q} \circ \mathcal{U}(M')} \leq 1$ and every $\varepsilon > 0$. As a consequence, $q \in (\mathcal{Q} \circ \mathcal{U})^*$ and $\|q\|_{(\mathcal{Q} \circ \mathcal{U})^*} \leq \|q\|_{\mathcal{Q}^* \circ (\mathcal{U}^{dual})^{-1}}$. $\square$

Now we can prove Theorem 3.4.4.

Proof. (*of Theorem 3.4.4*)

We have already mentioned that any $p \in \mathcal{Q}_{/\alpha\backslash}(E)$ extends to a polynomial $\overline{p}$ defined on $\ell_\infty(B_{E'})$ with $\|\overline{p}\|_{\mathcal{Q}_\alpha(\ell_\infty(B_{E'}))} = \|p\|_{\mathcal{Q}_{/\alpha\backslash}(E)}$. Therefore, p belongs to $\mathcal{Q}_\alpha \circ \Gamma_\infty$ and

$$\|p\|_{\mathcal{Q}_\alpha \circ \Gamma_\infty} \leq \|\overline{p}\|_{\mathcal{Q}_\alpha(\ell_\infty(B_{E'}))} \|i\|^n = \|p\|_{\mathcal{Q}_{/\alpha\backslash}(E)}.$$

On the other hand, for $p \in \mathcal{Q}_\alpha \circ \Gamma_\infty$ and $\varepsilon > 0$ we can take $T \in \Gamma_\infty(E, F)$ and $q \in \mathcal{Q}_\alpha(F)$ such that $p = q \circ T$ and $\|q\|_{\mathcal{Q}} \|T\|_{\Gamma_\infty}^n \leq (1+\varepsilon)\|p\|_{\mathcal{Q}_\alpha \circ \Gamma_\infty}$. We choose $R \in \mathcal{L}(E, L_\infty(\mu))$ and $S \in \mathcal{L}(L_\infty(\mu)), F'')$ factoring $\kappa_F \circ T : E \to F''$ with $\|R\|\|S\| \leq (1+\varepsilon)\|T\|_{\Gamma_\infty}$. Also, since $\mathcal{Q}_\alpha$ is a maximal polynomial ideal, its Aron-Berner extension $AB(q) : F'' \to \mathbb{K}$ belongs to $\mathcal{Q}_\alpha$ and satisfy $\|AB(q)\|_{\mathcal{Q}_\alpha} = \|q\|_{\mathcal{Q}_\alpha}$. We have the following commutative diagram:

$$\begin{tikzcd}
 & E \arrow[r, "p"] \arrow[dl, "R"'] \arrow[d, "T"] & \mathbb{K}. \\
L_\infty(\mu) \arrow[dr, "S"'] & F \arrow[ur, "q"] \arrow[d, hook, "\kappa_F"] & \\
 & F'' \arrow[uur, bend right, "AB(q)"'] &
\end{tikzcd}$$

Since $AB(q) \circ S \in \mathcal{Q}_\alpha(L_\infty(\mu)) \overset{1}{=} \mathcal{Q}_{/\alpha\backslash}(L_\infty(\mu))$ we obtain

$$\begin{aligned}
\|p\|_{\mathcal{Q}_{/\alpha\backslash}} &\leq \|AB(q) \circ S\|_{\mathcal{Q}_{/\alpha\backslash}}\ \|R\|^n \\
&= \|AB(q) \circ S\|_{\mathcal{Q}_\alpha}\ \|R\|^n \\
&\leq \|AB(q)\|_{\mathcal{Q}_\alpha}\ \|S\|^n\ \|R\|^n \\
&\leq (1+\varepsilon)^n\ \|q\|_{\mathcal{Q}_\alpha}\ \|T\|_{\Gamma_\infty}^n \\
&\leq (1+\varepsilon)^{n+1} \|p\|_{\mathcal{Q}_\alpha \circ \Gamma_\infty}.
\end{aligned}$$

Thus, $\mathcal{Q}_{/\alpha\backslash} \overset{1}{=} \mathcal{Q}_\alpha \circ \Gamma_\infty$.

Now we show the second identity. First notice that $\Gamma_1 = \Gamma_\infty^{dual}$ (this follows, for example, from Corollary 3 in [DF93, 17.8.] and the information on the table in [DF93, 27.2.]). Since Γ_∞ is maximal and accessible [DF93, Theorem 21.5.] and $\mathcal{Q}_{/\alpha\backslash}$ is a Banach ideal of continuous polynomials, we can apply Proposition 3.4.6 to the equality $\mathcal{Q}_{/\alpha'\backslash} \overset{1}{=} \mathcal{Q}_{\alpha'} \circ \Gamma_\infty$ to obtain $\mathcal{Q}_{\backslash\beta/} = \mathcal{Q}_\beta \circ \Gamma_1^{-1}$ with $\|\ \|_{\mathcal{Q}_\beta \circ \Gamma_1^{-1}} = \|\ \|_{\mathcal{Q}_{\backslash\beta/}}$. $\square$

As a consequence of Theorem 3.4.4 we recover the following classical result [Flo02, Proposition 3.4].

Corollary 3.4.7. *The polynomial ideal $\mathcal{P}_e^n$ coincides with $\mathcal{L}_\infty^n$.*

3.5 Natural symmetric tensor norms

Alexsander Grothendieck's article *"Résumé de la théorie métrique des produits tensoriels topologiques"* [Gro53] is considered one of the most influential papers in functional analysis. In this masterpiece, Grothendieck invented 'local theory', and exhibited the important connection between Operator ideals and tensor products. As part of his contributions, the *Résumé* contained the list of all *natural* tensor norms. Loosely speaking, this norms come from applying basic operations to the projective norm. More precisely, Grothendieck defined natural 2-fold norms as those that can be obtained from π_2 by a finite number of the following operations: right injective associate, left injective associate, right projective associate, left projective associate and adjoint (see [DF93]). Grothendieck proved that there were at most fourteen possible natural norms, but he did not know the exact dominations among them, or if there was a possible reduction on the table of natural norms (in fact this was one of the open problems posed in the *Résumé*). This was solved, several years later, thanks to very deep ideas of Gordon and Lewis [GL74]. All this results are now classical and can be found for example in [DF93, Section 27] and [DFS08, 4.4.2.].

Our aim of to define and study natural symmetric tensor norms of arbitrary order, in the spirit of Grothendieck's norms.

Definition 3.5.1. Let α be an s-tensor norm of order n. We say that α is a natural s-tensor norm if α is obtained from $\pi_{n,s}$ with a finite number of the operations $\backslash \cdot /$, $/ \cdot \backslash$ and $'$.

For (full) tensor norms of order 2, there are exactly four natural norms that are symmetric [DF93, Section 27]. It is easy to show that the same holds for s-tensor norms of order 2 (see the proof of Theorem 3.5.2). These are $\pi_{2,s}$, $\varepsilon_{2,s}$, $/\pi_{2,s}\backslash$ and $\backslash\varepsilon_{2,s}/$, with the same dominations as in the full case. It is important to mention that, for $n = 2$, $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$, or equivalently, $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$, coincide. However, for $n \geq 3$, we have the following

Theorem 3.5.2. *For $n \geq 3$, there are exactly 6 different natural symmetric s-tensor norms. They can be arranged as it is seen in Figure 3.1 on page 58.*

Note that what we do not have in the n-fold case is the double sense of the word *natural*: at most three among the six obtained tensor norms can be considered really natural, if by natural we understand those symmetric tensor norms that naturally appear in the theory. These are the symmetric projective and injective tensor norms and (arguably) the so-called tensor norm η (or $/\pi_{n,s}\backslash$), which appears in relation to extension of polynomials. We then stress that by natural we just mean those s-tensor norms which are obtained from the projective one by the already mentioned operations.

Before we prove Theorem 3.5.2, we need some previous results and definitions. Let δ be a full tensor norm of order n. Following the definitions given in Sections 3.1 and 3.2 we say

$$
\begin{tikzcd}
 & \boxed{\dfrac{\pi_{n,s}}{\mathcal{P}_I^n}} \arrow[d] & \\
 & \boxed{\dfrac{\backslash\!/\pi_{n,s}\backslash\!/}{\mathcal{P}_I^n \circ (\Gamma_1)^{-1} \circ \Gamma_\infty}} \arrow[dl] \arrow[dr] & \\
\boxed{\dfrac{/\pi_{n,s}\backslash}{P_I \circ (\Gamma_1)^{-1}}} \arrow[dr] & & \boxed{\dfrac{\backslash\varepsilon_{n,s}/}{\mathcal{P}_e^n}} \arrow[dl] \\
 & \boxed{\dfrac{/\!\backslash\varepsilon_{n,s}/\!\backslash}{\mathcal{P}_e^n \circ (\Gamma_1)^{-1}}} \arrow[d] & \\
 & \boxed{\dfrac{\varepsilon_{n,s}}{\mathcal{P}^n}} &
\end{tikzcd}
\tag{3.12}
$$

Here $\alpha \to \beta$ means that α dominates β. There are no other dominations than those showed in the scheme. Below each tensor norm we find its associated maximal polynomial ideal.

Figure 3.1: **Natural s-tensor norms.**

that δ is *injective* if, whenever $I_i : E_i \to F_i$ are isometric embeddings between normed spaces ($i = 1 \dots n$), the tensor product operator

$$\otimes_{i=1}^n I_i : \big(\otimes_{i=1}^n E_i, \delta \big) \to \big(\otimes_{i=1}^n F_i, \delta \big),$$

is an isometric embedding. The norm δ is *projective* if, whenever $Q_i : E_i \to F_i$ are quotient mappings between normed spaces ($i = 1 \dots n$), the tensor product operator

$$\otimes_{i=1}^n Q_i : \big(\otimes_{i=1}^n E_i, \delta \big) \to \big(\otimes_{i=1}^n F_i, \delta \big),$$

is also a quotient mapping.

The *injective associate* of δ, $/\delta\backslash$, will be the (unique) greatest injective tensor norm smaller than δ. As in Theorem 3.1.5 we get,

$$\big(\otimes_{i=1}^n E_i, /\delta\backslash \big) \overset{1}{\hookrightarrow} \big(\otimes_{i=1}^n \ell_\infty(B_{E_i'}), \delta \big).$$

The *projective associate* of δ, $\backslash\delta/$, will be the (unique) smallest projective tensor norm greater than δ. As in Theorem 3.2.6, if $E_1, \dots, E_n$ are Banach spaces, we have

$$\big(\otimes_{i=1}^n \ell_1(B_{E_i}), \delta \big) \overset{1}{\twoheadrightarrow} \big(\otimes_{i=1}^n E_i, \backslash\delta/ \big),$$

We denote by $\underline{\delta}$ the *dwarfed full tensor norm of order* $n-1$ given by

$$\underline{\delta}(z, \otimes_{i=1}^{n-1} E_i) := \delta(z \otimes 1, E_1 \otimes \cdots \otimes E_{n-1} \otimes \mathbb{C}),$$

where $z \otimes 1 := \sum_{i=1}^m x_1^i \otimes \dots x_n^i \otimes 1$, for $z = \sum_{i=1}^m x_1^i \otimes \dots x_n^i$ (this definition can be seen as dual to some ideas on [BBJP06] and [CDM09]).

Lemma 3.5.3. *For any tensor norm δ, we have: $\underline{(/\delta\backslash)} = /\underline{\delta}\backslash$ and $\underline{(\backslash\delta/)} = \backslash\underline{\delta}/$. Also, if δ and ρ are full tensor norms and there exists $C > 0$ such that $\delta \le C\rho$, then $\underline{\delta} \le C\underline{\rho}$.*

Proof. Let $z \in \otimes_{i=1}^n E_i$. For the first statement, if $I_{E_i} : E_i \to \ell_\infty(B_{E_i'})$ are the canonical embeddings, we have

$$\begin{aligned} /\underline{\delta}\backslash\big(z, E_1 \otimes \dots \otimes E_{n-1}\big) &= \underline{\delta}\big(\otimes_{i=1}^n I_{E_i}(z), \ell_\infty(B_{E_1'}) \otimes \dots \otimes \ell_\infty(B_{E_{n-1}'})\big) \\ &= \delta\big(\otimes_{i=1}^n I_{E_i}(z) \otimes 1, \ell_\infty(B_{E_1'}) \otimes \dots \otimes \ell_\infty(B_{E_{n-1}'}) \otimes \mathbb{C}\big) \\ &= /\delta\backslash\big(z \otimes 1, E_1 \otimes \dots \otimes E_{n-1} \otimes \mathbb{C}\big) \\ &= \underline{(/\delta\backslash)}\big(z, E_1 \otimes \dots \otimes E_{n-1}\big). \end{aligned}$$

For the second statement we will only prove it on BAN, the details can be completed following what was done in the proof of Theorem 3.2.6. Let $E_1, \dots, E_n$ Banach spaces, if $Q_{E_i} : \ell_1(B(E_i)) \twoheadrightarrow E_i$ are the canonical quotient mappings, we get

$$\begin{aligned} &\backslash\underline{\delta}/\big(z, E_1 \otimes \dots E_{n-1}\big) \\ &= \inf_{\{t \,:\, \otimes_{i=1}^{n-1} P_i(t) = z\}} \underline{\delta}\big(t, \ell_1(B_{E_1}) \otimes \dots \otimes \ell_1(B_{E_{n-1}})\big) \\ &= \inf_{\{t \,:\, \otimes_{i=1}^{n-1} P_i(t) = z\}} \delta\big(t \otimes 1, \ell_1(B_{E_1}) \otimes \dots \otimes \ell_1(B_{E_n}) \otimes \mathbb{C}\big) \\ &= \inf_{\{t \,:\, (P_1 \otimes \dots P_{n-1} \otimes id_{\mathbb{C}})(t \otimes 1) \,=\, z \otimes 1\}} \delta\big(t \otimes 1, \ell_1(B_{E_1}) \otimes \dots \otimes \ell_1(B_{E_{n-1}}) \otimes \mathbb{C}\big) \\ &= \backslash\delta/\big(z \otimes 1, E_1 \otimes \dots \otimes E_{n-1} \otimes \mathbb{C}\big) \\ &= \underline{(\backslash\delta/)}\big(z, E_1 \otimes \dots \otimes E_{n-1}\big). \end{aligned}$$

The third statement is immediate. $\square$

If δ is a full tensor norm of order n, we denote by $\delta|_s$ *the restriction of δ to the symmetric tensor product*.

Floret in [Flo01b] showed that for every s-tensor norm α of order n there exist a full tensor norm $\Phi(\alpha)$ of order n which is equivalent to α when restricted on symmetric tensor products. In other words, there is a constant d_n depending only on n such that $d_n^{-1}\Phi(\alpha)|_s \le \alpha \le d_n\Phi(\alpha)|_s$ on $\otimes^{n,s}E$ for every normed space E. As a consequence, a large part of the isomorphic theory of norms on symmetric tensor products can be deduced from the theory of "full" tensor norms, which is usually easier to handle and has been more studied. We give some details of the construction.

Let $E_1, \dots, E_n$ be Banach spaces, denote $\ell_2^n(E_i)$ the direct sum $\bigoplus_{i=1}^n E_i$ equipped with the ℓ_2-norm. We define the mapping $W_{E_1,\dots,E_n} : \otimes_{i=1}^n E_i \to \otimes^{n,s}\ell_2^n(E_i)$ by

$$W_{E_1,\dots,E_n} : \otimes_{i=1}^n E_i \xrightarrow{\sqrt{n}I_1 \otimes \dots \otimes I_n} \otimes^n \ell_2^n(E_i) \xrightarrow{\sigma^n_{\ell_2^n(E_i)}} \otimes^{n,s}\ell_2^n(E_i)\ , \tag{3.13}$$

where $I_i : E_i \to \ell_2^n(E_i)$ $(1 \le i \le n)$ are the natural inclusion. Note that

$$W_{E_1,\dots,E_n}(x_1 \otimes \dots \otimes x_n) = \sqrt{n}(x_1, 0, \dots, 0) \vee \dots \vee (0, \dots, 0, x_n).$$

Definition 3.5.4. Let α be an s-tensor norm. The *extension of* α will be the full tensor norm of order n, $\Phi(\alpha)$, given by

$$\Phi(\alpha)(z; \otimes_{i=1}^n E_i) := K_2(\alpha)^{-1}\alpha\big(W_{E_1,\dots,E_n}(z); \otimes^{n,s}\ell_2^n(E_i)\big),$$

where $K_2(\alpha) = \sqrt{n}\,\alpha(e_1 \vee \cdots \vee e_n; \otimes^{n,s}\ell_2^n)$.

It does not matter why $K_2(\alpha)$ is included in the definition. What we need to know about $K_2(\alpha)$ is that it is just a number depending only on α satisfying

$$(\frac{n!}{n^n})^{1/2} \le K_2(\alpha) \le (\frac{n^n}{n!})^{1/2}.$$

Below we list four important properties of this extension that appear in [Flo01b].

Theorem 3.5.5. *Let α be an s-tensor norm of order n and $\Phi(\alpha)$ its extension. The following holds.*

(1) The restriction of $\Phi(\alpha)$ to the symmetric tensor product is equivalent to α. More precisely,

$$\frac{n!}{n^n}\Phi(\alpha)|_s \le (\frac{n!}{n^n})^{1/2}K_2(\alpha)\Phi(\alpha)|_s \le \alpha \le (\frac{n^n}{n!})^{1/2}K_2(\alpha)\Phi(\alpha)|_s \le \frac{n^n}{n!}\Phi(\alpha)|_s.$$

(2) If $\alpha \le C\beta$ then $K_2(\alpha)\Phi(\alpha) \le CK_2(\beta)\Phi(\beta)$.

(3) For the dual norm α' one has $\Phi(\alpha') \sim \Phi(\alpha)'$:

$$\Phi(\alpha)' \le K_2(\alpha)K_2(\alpha')\Phi(\alpha') \le n^{n/2}\Phi(\alpha)'.$$

(4) If δ is a full symmetric tensor norm of order n, then $\Phi(\delta|s) \sim \delta$:

$$\frac{1}{\sqrt{n}}\,K_2(\delta|s)\Phi(\delta|s) \le \delta \le \sqrt{n}\,K_2(\delta|s)\Phi(\delta|s).$$

For our purposes we need the following result.

Lemma 3.5.6. *Let α be an s-tensor norm of order n. Then $\Phi(/\alpha\backslash)$ and $/\Phi(\alpha)\backslash$ are equivalent s-tensor norms. Also, $\Phi(\backslash\alpha/)$ and $\backslash\Phi(\alpha)/$ are equivalent s-tensor norms.*

Proof. For simplicity, we consider the case $n = 2$, the proof of the general case being completely analogous. The definition of the injective associate gives

$$E_1 \otimes_{/\Phi(\alpha)\backslash} E_2 \overset{1}{\hookrightarrow} \ell_\infty(B_{E_1'}) \otimes_{\Phi(\alpha)} \ell_\infty(B_{E_2'}).$$

We therefore have

$$
\begin{aligned}
/\Phi(\alpha)\backslash\big(\sum_{j=1}^{r} x_j \otimes y_j\big) &= \Phi(\alpha)\big(\sum_{j=1}^{r} I_{E_1}(x_j) \otimes I_{E_2}(y_j), \ell_\infty(B_{E_1'}) \otimes \ell_\infty(B_{E_2'})\big)\\
&= \sqrt{2}K_2(\alpha)^{-1}\alpha\big(\sum_{j=1}^{r}(I_{E_1}(x_j),0)\vee(0,I_{E_2}(y_j)), \otimes^{2,s}\{\ell_\infty(B_{E_1'}) \oplus_2 \ell_\infty(B_{E_2'})\}\big)\\
&\asymp \sqrt{2}K_2(\alpha)^{-1}\alpha\big(\sum_{j=1}^{r}(I_{E_1}(x_j),0)\vee(0,I_{E_2}(y_j)), \otimes^{2,s}\{\ell_\infty(B_{E_1'}) \oplus_\infty \ell_\infty(B_{E_2'})\}\big)\\
&= \sqrt{2}K_2(\alpha)^{-1}/\alpha\backslash\big(\sum_{j=1}^{r}(I_{E_1}(x_j),0)\vee(0,I_{E_2}(y_j)), \otimes^{2,s}\{\ell_\infty(B_{E_1'}) \oplus_\infty \ell_\infty(B_{E_2'})\}\big)\\
&\asymp \sqrt{2}K_2(\alpha)^{-1}/\alpha\backslash\big(\sum_{j=1}^{r}(I_{E_1}(x_j),0)\vee(0,I_{E_2}(y_j), \otimes^{2,s}\{\ell_\infty(B_{E_1'}) \oplus_2 \ell_\infty(B_{E_2'})\}\big)\\
&= \sqrt{2}K_2(\alpha)^{-1}/\alpha\backslash\big(\sum_{j=1}^{r}(x_j,0)\vee(0,y_j), \otimes^{2,s}\{E_1 \oplus_2 E_2\}\big)\\
&= \Phi(/\alpha\backslash)(\sum_{j=1}^{r} x_j \otimes y_j),
\end{aligned}
$$

where $\asymp$ means that the two expressions are equivalent up to universal constants. The second equivalence follows from the first one by duality, since by Theorem 3.5.5 we have $\Phi(\backslash\alpha/) = \Phi((/\alpha'\backslash)') \sim \Phi(/\alpha'\backslash)' \sim /\Phi(\alpha')\backslash' = \backslash\Phi(\alpha')'/ \sim \backslash\Phi(\alpha)/$. □

As a consequence of these results we can see that no injective norm α can be equivalent to a projective norm β. Indeed, if they were equivalet, we would have $\backslash\varepsilon_{n,s}/ \leq \backslash\alpha/ \leq C_1\beta \leq C_2\alpha \leq C_2/\pi_{n,s}\backslash$. Since Φ respects inequalities (Theorem 3.5.5 (2)), an application of Lemmas 3.5.6 and 3.5.3, together with the obvious identities $\underline{\varepsilon_{n+1}} = \varepsilon_n$, $\underline{\pi_{n+1}} = \pi_n$ would give $\backslash\varepsilon_2/ \sim w_2' \leq D/\pi_2\backslash \sim w_2$, a contradiction.

Another consequence is that $\pi_{2,s}$, $\varepsilon_{2,s}$, $/\pi_{2,s}\backslash$ and $\backslash\varepsilon_{2,s}/$ are the non-equivalent natural s-tensor norms for $n = 2$. This follows from the 2-fold result (see [DF93, Chapter 27]), which states that π_2, ε_2, $/\pi_2\backslash$ and $\backslash\varepsilon_2/$ are the only natural 2-fold tensor norms that are symmetric. So Lemma 3.5.6 and the properties of Φ give our claim, as well as the following dominations: $\varepsilon_{2,s} \leq \backslash\varepsilon_{2,s}/ \leq /\pi_{2,s}\backslash \leq \pi_{2,s}$.

Now we are ready to prove Theorem 3.5.2.

Proof. (*of Theorem 3.5.2.*)

To prove that all the possible natural n-fold s-tensor norms ($n \geq 3$) are listed in (3.12), it is enough to show that $/\backslash/\pi_{n,s}\backslash/\backslash$ coincides with $/\pi_{n,s}\backslash$. From the inequality $\backslash/\pi_{n,s}\backslash/ \leq \pi_{n,s}$ we readily obtain $/\backslash/\pi_{n,s}\backslash/\backslash \leq /\pi_{n,s}\backslash$. Also, the inequality $\varepsilon_{n,s} \leq /\backslash\varepsilon_{n,s}/\backslash$ gives $\backslash\varepsilon_{n,s}/ \leq \backslash/\backslash\varepsilon_{n,s}/\backslash/$ and, by duality, we have $/\pi_{n,s}\backslash \leq /\backslash/\pi_{n,s}\backslash/\backslash$.

Now we see that the listed norms are all different. First, $/\pi_{n,s}\backslash$ and $\backslash/\pi_{n,s}\backslash/$ cannot be equivalent, since the first one is injective and the second one is projective. Analogously, $\backslash\varepsilon_{n,s}/$

is not equivalent to $/\backslash\varepsilon_{n,s}/\backslash$. Until now, everything works just as in the case $n = 2$. The difference appears when we consider the relationship between $\backslash/\pi_{n,s}\backslash/$ and $\backslash\varepsilon_{n,s}/$: we will see in Theorem 3.5.7 below that $\backslash/\pi_{n,s}\backslash/$ and $\backslash\varepsilon_{n,s}/$ cannot be equivalent on any infinite dimensional Banach space, which is much more than we need. By duality, conclude that the six listed norms in Theorem 3.5.2 are different.

It is clear that all the dominations presented in (3.12) hold, so we must show that $/\pi_{n,s}\backslash$ does not dominate $\backslash\varepsilon_{n,s}/$ nor $\backslash\varepsilon_{n,s}/$ dominates $/\pi_{n,s}\backslash$. Note that the inequality $/\pi_{n,s}\backslash \leq C\backslash\varepsilon_{n,s}/$ would imply the equivalence between $/\pi_{n,s}\backslash$ and $\varepsilon_{n,s}$ on $\otimes^{n,s}\ell_1$, which is impossible (see [CD07, Per04a, Var75]). Finally, if $/\pi_{n,s}\backslash$ dominates $\backslash\varepsilon_{n,s}/$, then we can reason as in the comments after Lemma 3.5.6 and conclude that $/\pi_2\backslash$ dominates $\backslash\varepsilon_2/$, which contradicts [DF93, Chapter 27].

The maximal polynomial ideals associated with the natural norms are easily obtained using Proposition 3.4.4 and the fact that $\mathcal{Q}_{/\alpha\backslash}$ and $\mathcal{Q}_{\backslash\beta/}$ are associated with the norms $(/\alpha\backslash)' = \backslash\alpha'/$ and $(\backslash\beta/)' = /\beta'\backslash$ respectively. $\square$

Theorem 3.5.7 below shows that there is no infinite dimensional Banach space E such that $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$ are equivalent in $\otimes^{n,s}E$ for $n \geq 3$. This means that the splitting of $\backslash\varepsilon_{n,s}/$ when passing from $n = 2$ to $n \geq 3$ is rather drastic.

Theorem 3.5.7. *For $n \geq 3$, $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$ are equivalent in $\otimes^{n,s}E$ if and only if E is finite dimensional. The same happens if $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ are equivalent on E.*

Proof. We first prove that if E is infinite dimensional, then $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ are not equivalent in $\otimes^{n,s}E$. Suppose they are. Then, we have

$$\mathcal{P}_e^n(E) = \left(\otimes^{n,s}_{/\pi_{n,s}\backslash} E\right)' = \left(\otimes^{n,s}_{/\backslash\varepsilon_{n,s}/\backslash} E\right)' = \mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(E).$$

By the open mapping theorem, there must be a constant $M > 0$ such that

$$\|p\|_{\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(E)} \leq M\|p\|_{\mathcal{P}_e^n(E)},$$

for every extendible polynomial p on E. If F is a subspace of E, any extendible polynomial on F extends to an extendible polynomial on E with the same extendible norm. Therefore, for every subspace F of E and every extendible polynomial q on F, we have

$$\|q\|_{\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(F)} \leq M\|q\|_{\mathcal{P}_e^n(F)}.$$

Since E is an infinite dimensional space, by Dvoretzky's theorem it contains $(\ell_2^k)_k$ uniformly. Then there must be a constant $C > 0$ such that for every k and every n-homogeneous polynomial q on ℓ_2^k, we have

$$\|q\|_{\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2^k)} \leq C\|q\|_{\mathcal{P}_e^n(\ell_2^k)}.$$

Since the ideal of extendible polynomials is maximal (it is dual to an s-tensor norms), by the $\mathcal{L}_p$-Local Technique Lemma for Maximal Ideals 2.2.15 we deduce that

$$\mathcal{P}_e^n(\ell_2) \subset \mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2). \tag{3.14}$$

Let us show that this is not true. Recall that we have an inclusion $\ell_2 \hookrightarrow L_1[0,1]$ (Khintchine inequalities) thus, since $/\backslash\varepsilon_{n,s}/\backslash$ is injective, each $p \in \mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2)$ can be extended to a $/\backslash\varepsilon_{n,s}/\backslash$-continuous polynomial $\widetilde{p}$ on $L_1[0,1]$. Now, by Corollary 3.2.8, $\varepsilon_{n,s}$ coincides with $\backslash\varepsilon_{n,s}/$ on $L_1[0,1]$, which dominates $/\backslash\varepsilon_{n,s}/\backslash$. Therefore, the polynomial $\widetilde{p}$ is actually $\varepsilon_{n,s}$-continuous or, in other words, integral. Since $\widetilde{p}$ extends p, this must also be integral; we have shown that $\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2)$ is contained in $\mathcal{P}_I^n(\ell_2)$. But it is shown in [CD07, Per04a, Var75] that there are always extendible non-integral polynomials on any infinite dimensional Banach space, so (3.14) cannot hold. This contradiction shows that $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ cannot be equivalent on E.

Now we show that $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$ are not equivalent in $\otimes^{n,s}E$, for any infinite dimensional Banach space E. Suppose they are. By duality, we have $\mathcal{Q}_{\backslash\varepsilon_{n,s}/} = \mathcal{Q}_{\backslash/\pi_{n,s}\backslash/}$ with equivalent norms. The polynomial ideals $\mathcal{Q}_{\backslash\varepsilon_{n,s}/}$, $\mathcal{Q}_{\backslash/\pi_{n,s}\backslash/}$ are associated with the injective norms $(\backslash\varepsilon_{n,s}/)' = /\pi_{n,s}\backslash$, and $(\backslash/\pi_{n,s}\backslash/)' = /\backslash\varepsilon_{n,s}/\backslash$ respectively. Since this norms are accessible (Corollary 3.3.3) we have, by the Embedding Theorem 2.2.13,

$$\widetilde{\otimes}^{n,s}_{/\pi_{n,s}\backslash}E' \overset{1}{\hookrightarrow} \mathcal{Q}_{\backslash\varepsilon_{n,s}/}(E), \text{ and } \widetilde{\otimes}^{n,s}_{/\backslash\varepsilon_{n,s}/\backslash}E' \overset{1}{\hookrightarrow} \mathcal{Q}_{\backslash/\pi_{n,s}\backslash/}(E).$$

But this implies that $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ are equivalent in $\otimes^{n,s}E'$, which is impossible by the already proved first statement of the theorem. $\square$

Chapter 4

The Symmetric Radon-Nikodým property for tensor norms

A result of Boyd and Ryan [BR01] and also of Carando and Dimant [CD00] implies that, for an Asplund space E, the space $\mathcal{P}_I^n(E)$ of integral polynomials is isometric to the space $\mathcal{P}_N^n(E)$ of nuclear polynomials (the isomorphism between these spaces was previously obtained by Alencar in [Ale85a, Ale85b]). In other words, if E is Asplund, the space of integral polynomials on E coincides isometrically with its minimal kernel $(\mathcal{P}_I^n)^{min}(E) = \mathcal{P}_N^n(E)$. This fact was used, for example, in [BR01, BL10, Din03] to study geometric properties of spaces of polynomials and tensor products (*e.g.,* extreme and exposed points of their unit balls), and in [BL05, BL06] to characterize isometries between spaces of polynomials and centralizers of symmetric tensor products. When the above mentioned isometry is stated as the isometric coincidence between a maximal ideal and its minimal kernel, it resembles the Lewis theorem for operator ideals and (2-fold) tensor norms (see [Lew77] and [DF93, 33.3]). The Radon-Nikodým property for tensor norms is a key ingredient for Lewis theorem.

The aim of this chapter is to find conditions under which the equality $\mathcal{Q}(E) = \mathcal{Q}^{min}(E)$ holds isometrically for a maximal polynomial ideal $\mathcal{Q}$. In terms of symmetric tensor products, we want conditions on an s-tensor norms ensuring the isometry $\widetilde{\otimes}_{\alpha}^{n,s} E' = \left(\widetilde{\otimes}_{\alpha'}^{n,s} E\right)'$. To this end, we introduce the symmetric Radon-Nikodým property for s-tensor norms and show our main result, a Lewis-type Theorem (Theorem 4.1.2): if an s-tensor norm has the symmetric Radon-Nikodým property (sRN property), we have that the natural mapping $J_{\backslash\alpha/}^{E} : \widetilde{\otimes}_{\backslash\alpha/}^{n,s} E' \to \left(\widetilde{\otimes}_{/\alpha'\backslash}^{n,s} E\right)'$ is a metric surjection for every Asplund space E (see the notation below). As a consequence, if $\mathcal{Q}$ is the maximal ideal (of n-homogeneous polynomials) associated with a projective s-tensor norm α with the sRN property, then $\mathcal{Q}^{min}(E) = \mathcal{Q}(E)$ isometrically.

As an application of this result, we reprove the isometric isomorphism between integral and nuclear polynomials on Asplund spaces. We also show that the ideal of extendible polynomials coincide with its minimal kernel for Asplund spaces, and obtain as a corollary that the space of extendible polynomials on E has a monomial basis whenever E' has a basis.

We present examples of s-tensor norms associated with well known polynomial ideals which have the sRN property. We also relate the sRN property of an s-tensor norm with the Asplund property. More precisely, we show that if α is a projective s-tensor norm with the sRN, then α' preserves the Asplund property, in the sense that $\widetilde{\otimes}_{\alpha'}^{n,s} E$ is Asplund whenever E is. As an application, we show that the space of extendible polynomials on E has the Radon-Nikodým property if and only if E is Asplund. One might be tempted to infer that a projective s-tensor norm α with the sRN property preserves the Radon-Nikodým property, but this is not the case, as can be concluded from a result by Bourgain and Pisier [BP83]. However, we show that this is true with additional assumptions on the space E.

In order to prove our main theorem, we must show an analogous result for full tensor norms, which we feel can be of independent interest.

4.1 The symmetric Radon-Nikodým property

We refresh some classical definitions. A Banach space E has the *Radon-Nikodým property* if for every finite measure μ every operator $T : L_1(\mu) \to E$ is *representable*, *i.e.,* there exists a bounded μ-measurable function $g : \Omega \to E$ with

$$Tf = \int fgd\mu \quad \text{for all } f \in L_1(\mu).$$

A Banach space E is *Asplund* if its dual E' has the Radon-Nikodým property. A simple characterization must be mentioned: a Banach space E is Asplund if and only if every separable subspace of E has separable dual. In particular, reflexive spaces or spaces that have separable duals (*e.g.,* c_0) are Asplund. For more information of these two properties (Radon-Nikodým or Asplund) and examples see [DU76].

It is well know that the Radon-Nikodým and Asplund properties permit to understand the full duality of a tensor norm π and ε, describing conditions under which $E'\widetilde{\otimes}_\pi F' = (E\widetilde{\otimes}_\varepsilon F)'$ holds. Lewis in [Lew77] obtained many results of the form $E'\widetilde{\otimes}_\delta F' = (E\widetilde{\otimes}_{\delta'} F)'$ or, in other words, results about $\mathcal{U}^{min}(E, F') = \mathcal{U}(E, F')$ (if $\mathcal{U}$ is the maximal operator ideal associated with δ).

For $\mathcal{Q}$ a maximal ideal of n-homogeneous polynomials, we want to find conditions under which the next equality holds:

$$\mathcal{Q}^{min}(E) = \mathcal{Q}(E). \tag{4.1}$$

A related question is the following: if α is the s-tensor norm of order n associated with $\mathcal{Q}$, when does the natural mapping

$$J_\alpha^E : \widetilde{\otimes}_\alpha^{n,s} E' \overset{1}{\twoheadrightarrow} \mathcal{Q}^{min}(E) \hookrightarrow \mathcal{Q}(E) \overset{1}{=} \big(\widetilde{\otimes}_{\alpha'}^{n,s} E\big)',$$

defined in (2.13) become a metric surjection? Note that, in this case, we get the equality (4.1). To give an answer to this question we need the next definition. In a sense, it is a symmetric version of the one which appears in [DF93, 33.2].

Definition 4.1.1. A finitely generated s-tensor norm α of order n has the symmetric Radon-Nikodým property (sRN property) if

$$\widetilde{\otimes}_\alpha^{n,s} \ell_1 \overset{1}{=} \big(\widetilde{\otimes}_{\alpha'}^{n,s} c_0\big)'. \tag{4.2}$$

Here equality means that canonical mapping $J_\alpha^{c_0} : \widetilde{\otimes}_\alpha^{n,s} \ell_1 \longrightarrow \big(\widetilde{\otimes}_{\alpha'}^{n,s} c_0\big)'$ is an isometric isomorphism.

Since ℓ_1 and c_0 are, respectively, $\mathcal{L}_{1,\lambda}$ and $\mathcal{L}_{\infty,\lambda}$ spaces for every $\lambda > 1$, α and $\backslash\alpha/$ coincide in $\otimes^{n,s}\ell_1$ and $(\backslash\alpha/)' = /\alpha'\backslash$ coincides with α' on $\otimes^{n,s}c_0$. As a consequence, from the very definition we obtain that an s-tensor norm α has the sRN property if and only if its projective associate $\backslash\alpha/$ has it.

Also, ℓ_1 has the metric approximation property and, by Proposition 2.2.1 and the Duality Theorem 2.2.3, the mapping J_α^E is always an isometry. Therefore, to prove that α has the sRN property we only have to check that J_α^E is surjective. Note that, for $\mathcal{Q}$ the maximal n-homogeneous polynomial ideal associated with α, (4.2) is equivalent to

$$\mathcal{Q}^{min}(c_0) = \mathcal{Q}(c_0), \tag{4.3}$$

and the isometry is automatic.

Our interest in the sRN property is the following Lewis-type theorem.

Theorem 4.1.2. *Let α be an s-tensor norm with the sRN property and E be an Asplund space. Then we have*

$$\tilde{\otimes}^{n,s}_{\backslash\alpha/}E' \overset{1}{\twoheadrightarrow} \big(\tilde{\otimes}^{n,s}_{/\alpha'\backslash}E\big)',$$

i.e., *the natural mapping $J^E_{\backslash\alpha/}$ is a metric surjection. As a consequence,*

$$(\mathcal{Q}_{/\alpha'\backslash})^{min}(E) = \mathcal{Q}_{/\alpha'\backslash}(E) \textit{ isometrically.}$$

One may wonder if the projective associate of the tensor norm α is really necessary in Theorem 4.1.2. Let us see that, in general, it cannot be avoided. For this, we use two results that are stated and proved in the next chapter. Take any injective s-tensor norm and let $\mathcal{Q}$ be the associated maximal polynomial ideal. If T is the dual of the original Tsirelson space (which is reflexive and therefore Asplund), then we can see that $\mathcal{Q}(T) \neq \mathcal{Q}^{min}(T)$. Indeed, we consider for each m, the polynomial on ℓ_2 given by $p_m = \sum_{j=1}^m (e_j')^n$, where $(e_j')_{j=1}^\infty$ stands for the canonical dual basis. We have

$$\begin{aligned} \|p_m\|_{\mathcal{Q}(\ell_2)} &= \alpha\Big(\sum_{j=1}^m \otimes^n e_j'; \otimes^{n,s}\ell_2\Big) \\ &\leq /\pi_{n,s}\backslash\Big(\sum_{j=1}^m \otimes^n e_j'; \otimes^{n,s}\ell_2\Big) \\ &\leq C/\pi_n\backslash\Big(\sum_{j=1}^m e_j' \otimes \cdots \otimes e_j'; \otimes^n\ell_2\Big) \\ &\leq C\, K\varepsilon\Big(\sum_{j=1}^m e_j' \otimes \cdots \otimes e_j'; \otimes^{n,s}\ell_2\Big) \leq C\, K, \end{aligned}$$

where the third inequality and the constant K are taken from Lemma 5.1.10 (in Chapter 5), and the fourth inequality is immediate. So we have shown that $\|p_m\|_{\mathcal{Q}(\ell_2)}$ is uniformly bounded. Since T does not contain $(\ell_2^m)_m$ nor $(\ell_\infty^m)_m$ uniformly complemented (see [CS89, pages 33 and 66]), we can conclude that $\mathcal{Q}(T)$ cannot be separable by Proposition 5.2.8. As a consequence, $\mathcal{Q}(T)$ cannot coincide with $\mathcal{Q}^{min}(T)$.

In order to prove Theorem 4.1.2, an analogous result for full tensor products (and multilinear forms) will be necessary. As a consequence, we decided to postpone the proof to next section.

Let us then present different tensor norms with the sRN. We begin with two basic examples. The following identities are simple and well known:

$$\big(\widetilde{\otimes}^{n,s}_{\pi'_{n,s}} c_0\big)' = \big(\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} c_0\big)' = \widetilde{\otimes}^{n,s}_{\pi_{n,s}} \ell_1$$

and

$$\big(\widetilde{\otimes}^{n,s}_{\varepsilon'_{n,s}} c_0\big)' = \big(\widetilde{\otimes}^{n,s}_{\pi_{n,s}} c_0\big)' = \widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} \ell_1$$

(they easily follow from the analogous identities for full tensor products, since the symmetrization operator is a continuous projection). Therefore, we have:

Example 4.1.3. *The tensor norms $\pi_{n,s}$ and $\varepsilon_{n,s}$ have the sRN property.*

It should be noted the (2-fold) tensor norm ε_2 does not have the classical Radon-Nikodým property [DF93, 33.2]. Therefore, the sRN property defined here for s-tensor norms and for full tensor norms in Section 4.2 is less restrictive than the Radon-Nikodým property for tensor norms.

In [Ale85a, Ale85b], Alencar showed that if E is Asplund, then integral and nuclear polynomials on E coincide, with equivalent norms. Later, Boyd and Ryan [BR01] and, independently, Carando and Dimant [CD00], showed that this coincidence is isometric (with a slightly more general assumption: that $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ does not contain a copy of ℓ_1). The proof of this result was based on the study of extreme point of the unit ball of $\mathcal{P}^n_I(E)$. Note that the isometry between nuclear and integral polynomials on Asplund spaces is an immediate consequence of Theorem 4.1.2 for $\alpha = \pi_{n,s}$.

Corollary 4.1.4. *If E is Asplund, then $\mathcal{P}^n_N(E) = \mathcal{P}^n_I(E)$ isometrically.*

Let α be an s-tensor norm. An important result due to Grecu and Ryan states that if E has a Schauder basis, then so does $\widetilde{\otimes}^{n,s}_{\alpha} E$. We now describe how the basis in the tensor product is constructed. We denote by $\mathbb{N}^n_d$ the set of decreasing n-multi-indices

$$\{\mathfrak{j} \in \mathbb{N}^n : j_1 \geq j_2 \geq \cdots \geq j_n\}.$$

An order is given in $\mathbb{N}^n_d$ recursively: $\mathfrak{j} < \mathfrak{h}$ if $j_1 < h_1$ or $j_1 = h_1$ and $(j_2, \ldots, j_n) < (h_2, \ldots, h_n)$ in $\mathbb{N}^{n-1}_d$. This order is usually referred to as the *square order*.

Theorem 4.1.5. *[GR05] Let E be a Banach space with Schauder basis $(e_j)_{j=1}^{\infty}$ and α be an s-tensor norm. Then $(\sigma^n_E(e_{\mathfrak{j}}))_{\mathfrak{j}\in\mathbb{N}^n_d}$ with the square order is a Schauder basis of $\widetilde{\otimes}^{n,s}_{\alpha} E$, where $e_{\mathfrak{j}} := e_{j_1} \otimes \cdots \otimes e_{j_n}$.*

The tensors $(\sigma^n_E(e_{\mathfrak{j}}))_{\mathfrak{j}\in\mathbb{N}^n_d}$ are called the *monomials* associated with the basis $(e_j)_{j=1}^{\infty}$.

If we apply Theorem 4.1.2 and Corollary 2.2.20 to $\alpha = \varepsilon_{n,s}$, we obtain for E' with the bounded approximation property

$$\mathcal{P}^n_e(E) = (\mathcal{P}^n_e)^{min}(E) = \widetilde{\otimes}^{n,s}_{\backslash \varepsilon_{n,s}/} E' \text{ isometrically},$$

where $\mathcal{P}^n_e$ stands for the ideal of extendible polynomials. Combining this with Theorem 4.1.5 we have the following.

Corollary 4.1.6. *Let E be a Banach space such that E' has a basis. Then, the monomials associated with this basis are a Schauder basis for the space of extendible polynomials $\mathcal{P}_e^n(E)$.*

We now give other examples of s-tensor norms associated with well know maximal polynomial ideals having the sRN property.

Example 4.1.7. *Let ρ_n^r be the s-tensor norm associated with $\mathcal{L}_r^n$ ($r \geq n \geq 2$). Then, ρ_n^r has the s-RN property.*

Proof. We can assume that $r < \infty$ since $\mathcal{L}_\infty^n(c_0) = \mathcal{P}_e^n(c_0)$ (see Corollary 3.4.7) which coincides with $\mathcal{P}^n(c_0)$. For $p \in \mathcal{L}_r^n(c_0)$ there is a measure space (Ω, μ), an operator $T \in \mathcal{L}\big(c_0, L_r(\mu)\big)$ and a polynomial $q \in \mathcal{P}^n\big(L_r(\mu)\big)$ with $p = q \circ T$. Since $L_r(\mu)$ is reflexive, as a direct consequence of the Schauder Theorem and the Schur property of ℓ_1 we get that T is approximable. On the other hand q is trivially in $\mathcal{L}_r^n(L_r(\mu))$. Hence p belongs to $(\mathcal{L}_r^n)^{min}(c_0)$. $\square$

Using the the ideas of the previous proof we have the following.

Example 4.1.8. *Let δ_n^r be the s-tensor norm associated with $\mathcal{J}_r^n$ ($2 \leq n \leq r < \infty$). Then, δ_n^r has the s-RN property.*

In [CDS07, Section 4], a n-fold full tensor norm $\gamma_{r'}^n$ was introduced, so that the ideal of dominated multilinear forms is dual to $\gamma_{r'}^n$. If we use the same notation for the analogous s-tensor norm, we have that $(\gamma_{r'}^n)'$ is the s-tensor norm associated with $\mathcal{D}_r^n$.

Example 4.1.9. *The s-tensor norm $(\gamma_{r'}^n)'$ associated with $\mathcal{D}_r^n$ has the s-RN property.*

Proof. By [Sch91] we know that $\mathcal{D}_r^n = \mathcal{P}^n \circ \Pi_r$, where Π_r is the ideal of r-summing operators. Thus, for $p \in \mathcal{D}_r^n(c_0)$ we have the factorization $p = q \circ T$ where $T : c_0 \longrightarrow G$ is an r-summing operator and $q : G \longrightarrow \mathbb{K}$ an n-homogeneous continuous polynomial. We may assume without loss of generality that $G = F'$ for a Banach space F (think on the Aron-Berner extension). By [DF93, Proposition 33.5] the tensor norm $(\gamma_{r',1})'$ has the Radon Nikodým property. Using this, and the identity $(\gamma^t)' = (\gamma')^t$ (which holds for every tensor norm of order two γ) we easily get:

$$\begin{aligned}\Pi_r(c_0, G) = \Pi_r(c_0, F') &= \big(c_0 \otimes_{\gamma_{1,r'}} F\big)' = \big(F \otimes_{\gamma_{r',1}} c_0\big)' = \\ &= F' \otimes_{(\gamma_{r',1})'} \ell_1 = \ell_1 \otimes_{(\gamma_{1,r'})'} F' = \ell_1 \otimes_{(\gamma_{1,r'})'} G.\end{aligned}$$

Therefore, we have proved that $\Pi_r(c_0, G) = (\Pi_r)^{min}(c_0, G)$. Now it is easy to conclude that $\mathcal{D}_r^n(c_0) = (\mathcal{D}_r^n)^{min}(c_0)$. $\square$

A natural and important question about a tensor norm is if it preserves some Banach space property. The following result shows that the symmetric Radon-Nikodým is closely related to the preservation of the Asplund property under tensor products.

Theorem 4.1.10. *Let E be Banach space and α a projective s-tensor norm with sRN property. The tensor product $\widetilde{\otimes}_{\alpha'}^{n,s} E$ is Asplund if an only if E is Asplund.*

Proof. Necessity is clear. For the converse, let S be a separable subspace of $\widetilde{\otimes}^{n,s}_{\alpha'} E$ and let us see that it has a separable dual. We can take $(x_k)_{k\in\mathbb{N}}$ a sequence of vectors in E and $F := \overline{[x_k : k \in \mathbb{N}]}$ such that S is contained in $\otimes^{n,s} F$. Since α' is injective, we have the isometric inclusion $S \overset{1}{\hookrightarrow} \widetilde{\otimes}^{n,s}_{\alpha'} F$. Now, F' is separable (since E is Asplund) and therefore, by Theorem 4.1.2, the mappping

$$\widetilde{\otimes}^{n,s}_{\alpha'} F' \longrightarrow \big(\widetilde{\otimes}^{n,s}_{\alpha'} F\big)'$$

is surjective. So, $\big(\widetilde{\otimes}^{n,s}_{\alpha'} F\big)'$ is a separable Banach space and hence is also S' (since we have a surjective mapping $\big(\widetilde{\otimes}^{n,s}_{\alpha'} F\big)' \twoheadrightarrow S'$). □

The following is an application of the previous theorem to $\alpha = \backslash\varepsilon_{n,s}/$.

Corollary 4.1.11. *For a Banach space E and $n \in \mathbb{N}$, $\mathcal{P}^n_e(E)$ has the Radon-Nikodým property if and only if E is Alplund.*

Looking at Theorem 4.1.10 a natural question arises: if α is a projective s-tensor norm with the sRN property, does $\widetilde{\otimes}^{n,s}_{\alpha} E$ have the Radon-Nikodým property whenever E has the Radon-Nikodým property? Burgain and Pisier [BP83, Corollary 2.4] presented a Banach space E with the Radon-Nikodým property such that $E \otimes_\pi E$ contains c_0 and, consequently, does not have the Radon-Nikodým property. This construction gives a negative answer to our question since the copy of c_0 in $E \otimes_\pi E$ is actually contained in the symmetric tensor product of E and $\pi_{2,s}$ (which has the sRN property) is equivalent to the restriction of π_2 to the symmetric tensor product.

However, $\widetilde{\otimes}^{n,s}_{\alpha} E$ inherits the Radon-Nikodým property of E if, in addition, E is a dual space with the bounded approximation property (this should be compared to [DU76], where an analogous result for the 2-fold projective tensor norm π is shown).

Corollary 4.1.12. *Let α be a projective s-tensor norm with the sRN property and E a dual Banach space with the bounded approximation property. Then, $\widetilde{\otimes}^{n,s}_{\alpha} E$ has the Radon-Nikodým property if and only if E does.*

Proof. Let F be a predual of E and suppose E has the Radon-Nikodým property. The space F is Asplund hence, by Theorem 4.1.10, so is $\widetilde{\otimes}^{n,s}_{\alpha'} F$. On the other hand, by Theorem 4.1.2 we have a metric surjection $\widetilde{\otimes}^{n,s}_{\alpha} E \overset{1}{\twoheadrightarrow} \big(\widetilde{\otimes}^{n,s}_{\alpha'} F\big)'$. Since $E = F'$ has the bounded approximation property, by Corollary 2.2.20, the mapping is injective. Thus, $\widetilde{\otimes}^{n,s}_{\alpha} E \overset{1}{=} \big(\widetilde{\otimes}^{n,s}_{\alpha'} F\big)'$. Hence, $\widetilde{\otimes}^{n,s}_{\alpha} E$ is the dual of an Asplund Banach space and has the Radon-Nikodým property.

The converse follows from the fact that E is complemented in $\widetilde{\otimes}^{n,s}_{\alpha} E$ (see [Bla97]). □

Any Banach space E with a boundedly complete Schauder basis $(e_j)_{j=1}^\infty$ is a dual space with the Radon-Nikodým property and the bounded approximation property. Indeed, E turns out to be the dual of the subspace F of E' spanned by the dual basic sequence $(e'_j)_{j=1}^\infty$ (which is, by the way, a shrinking basis of F). Then we have

$$\widetilde{\otimes}^{n,s}_{\alpha} E \overset{1}{=} \big(\widetilde{\otimes}^{n,s}_{\alpha'} F\big)' \tag{4.4}$$

The monomials associated to $(e_j)_{j=1}^\infty$ and to $(e'_j)_{j=1}^\infty$ with the appropriate ordering (see Theorem 4.1.5) are Schauder basis of, respectively, $\widetilde{\otimes}^{n,s}_{\alpha} E$ and $\widetilde{\otimes}^{n,s}_{\alpha'} F$. By the equality (4.4), monomials form a boundedly complete Schauder basis of $\widetilde{\otimes}^{n,s}_{\alpha} E$ and a shrinking Schauder basis of $\widetilde{\otimes}^{n,s}_{\alpha'} F$.

On the other hand, if we start with a Banach space E with a shrinking Schauder basis and take F as its dual, we are in the analogous situation with the roles of E and F interchanged. So we have

Corollary 4.1.13. *Let α be a projective s-tensor norm with the sRN property.*

(1) If E has a boundedly complete Schauder basis, then so does $\widetilde{\otimes}_{\alpha}^{n,s} E$.

(2) If E has a shrinking Schauder basis, then so does $\widetilde{\otimes}_{\alpha'}^{n,s} E$.

The corresponding statement for the 2-fold full tensor norm π was shown by Holub in [Hol71].

4.2 The sRN property for full tensor norms

In order to prove Theorem 4.1.2 we must first show an analogous result for full tensor products (see Theorem 4.2.6 below). It should be noted that, although we somehow follow some ideas of Lewis Theorem's proof in [DF93, Section 33.3], that proof is based on some factorizations of linear operators and not on properties of bilinear forms. Therefore, the weaker nature of the symmetric Radon-Nykodým property introduced in this work together with our multilinear/polynomial framework makes our proof much more complicated.

Let us first introduce the sRN property for full tensor products in the obvious way.

Definition 4.2.1. A finitely generated full tensor norm δ of order n has the symmetric Radon-Nikodým property (sRN property) if

$$(\widetilde{\otimes}_{i=1}^{n} \ell_1, \delta) = \big(\widetilde{\otimes}_{i=1}^{n} c_0, \delta'\big)'.$$

As in [DF93, Lemma 33.3.] we have the following symmetric result for ideals of multilinears form.

Proposition 4.2.2. *Let δ be a finitely generated full tensor norm of order n with the sRN property. Then,*

$$(\widetilde{\otimes}_{i=1}^{n} \ell_1(J_i), \delta) = \big(\widetilde{\otimes}_{i=1}^{n} c_0(J_i), \delta'\big)'$$

holds isometrically for all index sets $J_1, \ldots, J_n$.

Proof. Fix $J_1, \ldots, J_n$ index sets, let $\mathfrak{A}$ be the maximal multilinear ideal associated with the norm δ. We must show $\mathfrak{A}(c_0(J_1), \ldots, c_0(J_n)) = \mathfrak{A}^{min}(c_0(J_1), \ldots, c_0(J_n))$ with equal norms. For $T \in \mathfrak{A}\big(c_0(J_1), \ldots, c_0(J_n)\big)$, let us see that the set

$$L = \{(j_1, \ldots, j_n) : T(e_{j_1}, \ldots, e_{j_n}) \neq 0\} \tag{4.5}$$

is countable. If not, there exist $(j_1^k, \ldots, j_n^k)_{k \in \mathbb{N}}$ different indexes such that

$$|T(e_{j_1^k}, \ldots, e_{j_n^k})| > \varepsilon.$$

Without loss of generality we can assume that the sequence of first coordinates j_1^k has all its elements pairwise different. Passing to subsequences we can also assume that $e_{j_i^k}$ is weakly null. Since every n-linear form on c_0 is weakly continuous on bounded sets (by the Littlewood-Bogdanowicz-Pełczyński property of c_0 [Bog57, Pel57]) we have an absurd. So the set L in (4.5) is countable.

Let $\Omega_k : J_1 \times \cdots \times J_n \longrightarrow J_k$ be given by $(j_1, \dots, j_n) \mapsto j_k$, and set $L_k := \Omega_k(L) \subset J_k$. Consider the mapping $\xi_k : c_0(J_k) \to c_0(L_k)$ given by

$$(a_j)_{j \in J_k} \mapsto (a_j)_{j \in L_k}.$$

We also have the inclusion $\imath_k : c_0(L_k) \to c_0(J_k)$ defined by

$$(a_j)_{j \in L_k} \mapsto (b_j)_{j \in J_k},$$

where b_j is a_j if $j \in L_k$ and zero otherwise. Then, we can factor

$$\begin{tikzcd} c_0(J_1) \times \cdots \times c_0(J_1) \arrow[r, "T"] \arrow[d, "\xi_1 \times \cdots \times \xi_n"'] & \mathbb{K}, \\ c_0(L_1) \times \cdots \times c_0(L_n) \arrow[ur, "\overline{T}"] & \end{tikzcd}$$

where $\overline{T} := T \circ (\imath_1 \times \cdots \times \imath_n)$. Since δ has the symmetric Radon-Nikodým property we have $\mathfrak{A}(c_0(L_1), \dots, c_0(L_n)) = \mathfrak{A}^{min}(c_0(L_1), \dots, c_0(L_n))$ with equal norms. Therefore $\overline{T}$ is in $\mathfrak{A}^{min}(c_0(L_1), \dots, c_0(L_n))$ with

$$\|\overline{T}\|_{\mathfrak{A}^{min}} = \|\overline{T}\|_{\mathfrak{A}} \leq \|T\|_{\mathfrak{A}}.$$

Thus, $\overline{T}$ belongs to $\mathfrak{A}^{min}(c_0(L_1), \dots, c_0(L_n))$ which implies that T is also in $\mathfrak{A}^{min}(c_0(J_1), \dots, c_0(J_n))$. Moreover,

$$\|T\|_{\mathfrak{A}^{min}} \leq \|\overline{T}\|_{\mathfrak{A}^{min}} \|\xi_1 \times \cdots \times \xi_n\| \leq \|T\|_{\mathfrak{A}},$$

which ends the proof. □

For $A : E_1 \times \cdots \times E_n \to \mathbb{K}$ we denote by A^n the associated $(n-1)$-linear mapping $A^n : E_1 \times \cdots \times E_{n-1} \to E_n'$. Now, if $T : E_n' \to F'$ is a linear operator, then the $(n-1)$-linear form $B : E_1 \times \cdots \times E_{n-1} \to F'$ given by $T \circ A^n$ induces an n-linear form on $E_1 \times \cdots \times E_{n-1} \times F$. It is not hard to check that

$$B(e_1, \dots, e_{n-1}, f) = (EXT_n)A(e_1, \dots, e_{n-1}, T'\kappa_F(f)),$$

where $\kappa_F : F \to F''$ is the canonical inclusion mapping and EXT_n is the canonical extension of a multilinear form to the bidual in the n-th coordinate.

For every $k = 1, \dots, n$ we define an operator

$$\Psi_k : \big((\widetilde{\otimes}_{j=1}^{k-1} E_j)\widetilde{\otimes} c_0(B_{E_k'})\widetilde{\otimes}(\widetilde{\otimes}_{j=k+1}^{n} E_j), /\delta'\backslash\big)' \to \big(\widetilde{\otimes}_{i=1}^{n} E_i, /\delta'\backslash\big)',$$

by the composition $\big((\widetilde{\otimes}_{j=1}^{k-1} Id_{E_k})\widetilde{\otimes} I_{E_k} \widetilde{\otimes}(\widetilde{\otimes}_{j=k+1}^{n} Id_{E_k})\big)' \circ EXT_k$.

The next remark is easy to check.

Remark 4.2.3. *Let $E_1, \ldots, E_n$ be Banach spaces. For every k the following diagram conmutes.*

$$\begin{tikzcd}
\big((\widetilde{\bigotimes}_{j=1}^{k-1} E'_j)\widetilde{\otimes}\ell_1(B_{E'_k})\widetilde{\otimes}(\widetilde{\bigotimes}_{j=k+1}^{n} E'_j), \backslash\delta/\big) \arrow[r] \arrow[d, "(\widetilde{\otimes}_{j=1}^{k-1} Id_{E'_j})\widetilde{\otimes} Q_{E_k}\widetilde{\otimes}(\widetilde{\otimes}_{j=k+1}^{n} Id_{E'_j})"] & \big((\widetilde{\bigotimes}_{j=1}^{k-1} E_j)\widetilde{\otimes}c_0(B_{E'_k})\widetilde{\otimes}(\widetilde{\bigotimes}_{j=k+1}^{n} E_j), /\delta'\backslash\big)' \arrow[d, "\Psi_k"] \\
\big((\widetilde{\bigotimes}_{j=1}^{k-1} E'_j)\widetilde{\otimes}E'_k\widetilde{\otimes}(\widetilde{\bigotimes}_{j=k+1}^{n} E'_j), \backslash\delta/\big) \arrow[r] & \big(\widetilde{\bigotimes}_{i=1}^{n} E_i, /\delta'\backslash\big)'.
\end{tikzcd}$$

The following proposition is crucial for our purposes.

Proposition 4.2.4. *Let $E_1, \ldots, E_n$ be Banach spaces. If E_k is Asplund then Ψ_k is a metric surjection.*

To prove it we need a classical result due to Lewis and Stegall.

Theorem 4.2.5. *(The Lewis-Stegall Theorem.) If the Banach space E has the Radon-Nikodým property, then for every $T \in \mathcal{L}(L_1(\mu), E)$ and $\varepsilon > 0$ there exist an operator $S \in \mathcal{L}(L_1(\mu), \ell_1(B_E))$ with $\|S\| \leq (1+\varepsilon)$ such that the following diagram commutes*

$$\begin{tikzcd}
L_1(\mu) \arrow[rr, "T"] \arrow[dr, "S"'] & & E\ . \\
& \ell_1(B_E) \arrow[ur, "Q_E"'] &
\end{tikzcd} \tag{4.6}$$

Now we are ready to prove Proposition 4.2.4.

Proof. (*of Proposition 4.2.4.*) We prove it assuming that $k = n$ (the other cases are analogous). Notice that Ψ_n has norm less or equal to one (since EXT_n is an isometry).

Fix $A \in \big(\widetilde{\bigotimes}_{i=1}^{n} E_i, /\delta'\backslash\big)'$ and $\varepsilon > 0$ and let $\widetilde{A} \in \big((\widetilde{\bigotimes}_{i=1}^{n-1} E_i)\widetilde{\otimes}\ell_\infty(B_{E'_n}), /\delta'\backslash\big)'$ be a Hahn-Banach extension of A. Since E'_n has the Radon-Nikodým property, by the Lewis-Stegall Theorem the adjoint of the canonical inclusion $I_{E_n} : E_n \to \ell_\infty(B_{E'_n})$ factors through $\ell_1(B_{E'_n})$ via

$$\begin{tikzcd}
\ell_\infty(B_{E'_n})' \arrow[rr, "I'_{E_n}"] \arrow[dr, "S"'] & & E'_n \\
& \ell_1(B_{E'_n}) \arrow[ur, "Q_{E_n}"'] &
\end{tikzcd} \tag{4.7}$$

whith $\|S\| \leq (1+\varepsilon)$. Let $B : E_1 \times \cdots \times E_{n-1} \times c_0(B_{E'_n}) \to \mathbb{K}$ be given by the formula $B(x_1, \ldots, x_{n-1}, a) = (EXT_n)\widetilde{A}(x_1, \ldots, x_{n-1}, S' J_{c_0(B_{E'_n})}(a))$. Note that B is the n-linear form on $E_1 \times \cdots \times E_{n-1} \times c_0(B_{E'_n})$ associated with $S \circ (\widetilde{A})^n$. Using the ideal property and the fact that the extension to the bidual is an isometry we have $B \in \big((\widetilde{\bigotimes}_{i=1}^{n-1} E_i)\widetilde{\otimes}c_0(B_{E'_n}), /\delta'\backslash\big)'$ and $\|B\| \leq \|A\|(1+\varepsilon)$.

If we show that $\Psi_n(B) = A$ we are done. It is an easy exercise to prove that $I'_{E_n}(\widetilde{A})^n = A^n$. It is also easy to see that $I_{E_n}(x)(a) = Q_{E_n}(a)(x)$ for $x \in E_n$ and $a \in \ell_1(B_{E'_n})$.

Now, $\Psi_n(B) = (\widetilde{\otimes}_{i=1}^{n-1} Id_{E_i} \widetilde{\otimes} I_{E_n})' \circ (EXT_n)(B)$. Then,

$$\begin{aligned}\Psi_n(B)(x_1,\dots,x_n) &= (I_{E_n}x_n)[B(x_1,\dots,x_{n-1},\cdot)]\\ &= (I_{E_n}x_n)S(\widetilde{A})^n(x_1,\dots,x_{n-1})\\ &= Q_{E_n}S(\widetilde{A})^n(x_1,\dots,x_{n-1})(x_n)\\ &= I'_{E_n}(\widetilde{A})^n(x_1,\dots,x_{n-1})(x_n)\\ &= A^n(x_1,\dots,x_{n-1})(x_n)\\ &= A(x_1,\dots,x_n),\end{aligned}$$

which ends the proof. □

The following result is the version of Theorem 4.1.2 for full tensor products. It should be noted that it holds for tensor products of different spaces.

Theorem 4.2.6. *Let δ be a tensor norm with the sRN property and $E_1,\dots,E_n$ be Asplund spaces. Then*

$$(\widetilde{\otimes}_{i=1}^n E_i', \backslash\delta/) \overset{1}{\twoheadrightarrow} \big(\widetilde{\otimes}_{i=1}^n E_i, /\delta'\backslash\big)'.$$

In particular,

$$(\mathfrak{A}_{/\delta'\backslash})^{min}(E_1,\dots,E_n) = \mathfrak{A}_{/\delta'\backslash}(E_1,\dots,E_n),$$

where $\mathfrak{A}_{/\delta'\backslash}$ stands for the maximal ideal of $/\delta'\backslash$-continuous n-linear forms.

Proof. Take a close look at the diagram in Figure 4.1 on page 77. Using Remark 4.2.3 we know that this diagram commutes in each square. Now examine the first commutative square. Since δ has the sRN property, R_0 is a metric surjection by Proposition 4.2.2. Moreover, by Proposition 4.2.4 we get that the composition mapping $\big((\widetilde{\otimes}_{i=1}^{n-1} Id_{c_0(B_{E_i'})})\widetilde{\otimes} I_{E_n}\big)' \circ EXT_n$ is also a metric surjection. As a consequence of these two facts we get that R_1 is a metric surjection. The same argument can be applied to the second commutative square, now that we know that R_1 is metric surjection. Thus, R_2 is also a metric surjection. Reasoning like this, it follows that $R_n : (\widetilde{\otimes}_{i=1}^n E_i', \backslash\delta/) \to \big(\widetilde{\otimes}_{i=1}^n E_i, /\delta'\backslash\big)'$ is a metric surjection. □

Let us call $\Psi : \big(\widetilde{\otimes}_{i=1}^n c_0(B'_{E_i})\big)' \to \big(\widetilde{\otimes}_{i=1}^n E_i'\big)'$ the composition of the downward mappings in the right side of the last diagram. The following proposition shows how to describe the mapping Ψ more easily (this will be useful to prove the polynomial version of the last theorem).

Proposition 4.2.7. *The mapping $\Psi : \big(\widetilde{\otimes}_{i=1}^n c_0(B'_{E_i}, /\delta'\backslash)\big)' \to \big(\widetilde{\otimes}_{i=1}^n E_i, /\delta'\backslash\big)'$ is the composition mapping*

$$\big(\widetilde{\otimes}_{i=1}^n c_0(B'_{E_i}), /\delta'\backslash\big)' \overset{EXT}{\longrightarrow} \big(\widetilde{\otimes}_{i=1}^n \ell_\infty(B'_{E_i}), /\delta'\backslash\big)' \overset{(\widetilde{\otimes}_{i=1}^n I_{E_i})'}{\longrightarrow} \big(\widetilde{\otimes}_{i=1}^n E_i, /\delta'\backslash\big)',$$

where EXT stands for the iterated extension to the bidual given by $(EXT_n) \circ \cdots \circ (EXT_1)$ (we extend from the left to the right).

Proof. For the readers' sake we give a proof for the case $n = 2$. Let B a linear form in $\big(c_0(B_{E_1'})\widetilde{\otimes} c_0(B_{E_2'}), /\delta'\backslash\big)'$, then

$$\begin{aligned}
\Psi(B)(e_1, e_2) &= (id_{E_1}\widetilde{\otimes} I_{E_2})'(EXT_2)(I_{E_1}\widetilde{\otimes} Id_{c_0(B_{E_2'})})'(EXT_1)(B)(e_1, e_2) \\
&= (EXT_2)(I_{E_1}\widetilde{\otimes} Id_{c_0(B_{E_2'})})'(EXT_1)(B)(e_1, I_{E_2}(e_2)) \\
&= I_{E_2}(e_2)\big((I_{E_1}\widetilde{\otimes} Id_{c_0(B_{E_2'})})'(EXT_1)(B)(e_1, \cdot)\big) \\
&= I_{E_2}(e_2)\big(a \mapsto I_{E_1}(e_1)B(\cdot, a)\big) \\
&= I_{E_2}(e_2)\big((EXT_1)(B)(I_{E_1}(e_1), \cdot)\big) \\
&= (EXT)(B)(I_{E_1}(e_1), I_{E_2}(e_2)) \\
&= (I_{E_1}\widetilde{\otimes} I_{E_2})'(EXT)(B)(e_1, e_2),
\end{aligned}$$

which concludes the proof. □

Now, this proposition shows that the diagram

$$\begin{tikzcd}
(\widetilde{\otimes}_{i=1}^n \ell_1(B_{E_i'}), \backslash\delta/) \arrow[r, two heads] \arrow[d, "p"] & \big(\widetilde{\otimes}_{i=1}^n c_0(B_{E_i'}), /\delta'\backslash\big)' \arrow[d, "\Psi"] \\
(\widetilde{\otimes}_{i=1}^n E_i', \backslash\delta/) \arrow[r] & \big(\widetilde{\otimes}_{i=1}^n E_i, /\delta'\backslash\big)'
\end{tikzcd}$$

conmutes and, by the proof of the Theorem 4.2.6, we have that, for $E_1, \dots, E_n$ Asplund spaces, the mapping Ψ is a metric surjection.

The next remark will be very useful. It can be proved following carefully the proof of Proposition 4.2.4 and using Proposition 4.2.7.

Remark 4.2.8. *Let E be an Asplund space and $S : \ell_\infty(B_{E'})' \to \ell_1(B_{E'})$ be the operator obtained by the Lewis-Stegall Theorem with $\|S\| \leq 1 + \varepsilon$ as in diagram (4.7). Given $A \in \big(\widetilde{\otimes}_{i=1}^n E, /\delta'\backslash\big)'$, if we take a Hahn-Banach extension $\widetilde{A} \in \big(\widetilde{\otimes}_{i=1}^n \ell_\infty(B_{E'}), /\delta'\backslash\big)'$, then the linear functional $B \in \big(\widetilde{\otimes}_{i=1}^n c_0(B_{E'}), /\delta'\backslash\big)'$ given by*

$$B(a_1, \dots, a_n) := (EXT)(\widetilde{A})(S'J(a_1), \dots, S'J(a_n)), \tag{4.8}$$

satisfies $\Psi(B) = A$ and $\|B\| \leq \|A\|(1 + \varepsilon)^n$.

We end this section with the statement of the non-symmetric versions of Theorem 4.1.10, Corollary 4.1.12 and Corollary 4.1.13, which readily follow.

Theorem 4.2.9. *Let $E_1, \dots, E_n$ be Banach spaces and δ a full tensor norm with sRN. The tensor product $\big(E_1\widetilde{\otimes} \dots \widetilde{\otimes} E_n, /\delta'\backslash\big)$ is Asplund if an only if E_i is Asplund for $i = 1 \dots n$.*

Corollary 4.2.10. *Let δ be a projective full tensor norm with the sRN property and $E_1, \dots, E_n$ dual Banach spaces with the bounded approximation property. Then, $\big(E_1\widetilde{\otimes} \dots \widetilde{\otimes} E_n, \delta\big)$ has the Radon-Nikodým property if and only if each E_i does.*

Corollary 4.2.11. *Let δ be a projective full tensor norm with the sRN property and $E_1, \dots, E_n$ be Banach spaces.*

(1) If each E_i has a boundedly complete Schauder basis, then so does $\big(E_1\widetilde{\otimes} \dots \widetilde{\otimes} E_n, \delta\big)$.

(2) If each E_i has a shrinking Schauder basis, then so does $\big(E_1\widetilde{\otimes} \dots \widetilde{\otimes} E_n, \delta'\big)$.

The proof of Theorem 4.1.2

Now that we have our multilinear Lewis-type theorem, we are ready to prove Theorem 4.1.2.

Proof. (of Theorem 4.1.2)

As in the multilinear case, the next diagram commutes:

$$\begin{tikzcd} \widetilde{\otimes}^{n,s}_{\backslash\alpha/}\ell_1(B_{E'}) \arrow[r, two heads] \arrow[d, "\widetilde{\otimes}^{n,s}Q_{E'}"] & \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}c_0(B_{E'})\right)' , \arrow[d, "\Psi"] \\ \widetilde{\otimes}^{n,s}_{\backslash\alpha/}E' \arrow[r] & \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}E\right)' \end{tikzcd}$$

where Ψ is the composition mapping

$$\left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}c_0(B_{E'})\right)' \xrightarrow{AB} \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}\ell_\infty(B_{E'})\right)' \xrightarrow{(\widetilde{\otimes}^{n,s}I_E)'} \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}E\right)'.$$

Fix $p \in \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}E\right)'$. Let $\overline{p} \in \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}\ell_\infty(B_{E'})\right)'$ be a Hahn-Banach extension of p, by the Lewis-Stegall Theorem 4.2.5 we can obtain an operator S such that $\|S\| \leq 1+\varepsilon$ satisfying diagram (4.7). Since the Aron-Berner is an isometry for maximal ideals (Theorem 2.2.5) we have, as in Remark 4.2.8, that the linear functional $q \in \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}c_0(B_{E'})\right)'$ given by $q(a) := (AB)(\overline{p})(S'J_{c_0(B_{E'})}a)$ satisfies that $\Psi(q) = p$ and $\|q\|_{\left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}c_0(B_{E'})\right)'} \leq \|p\|_{\left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}E\right)'}(1+\varepsilon)^n$. Thus, Ψ is a metric surjection and, by the diagram, we easily get that $\widetilde{\otimes}^{n,s}_{\backslash\alpha/}E' \to \left(\widetilde{\otimes}^{n,s}_{/\alpha'\backslash}E\right)'$ is also a metric surjection. $\square$

$$
\begin{tikzcd}
\left(\widetilde{\bigotimes}_{i=1}^{n}\ell_1(B_{E_i'}), \backslash\delta/\right) \arrow[rr, "R_0"] \arrow[dd, "\widetilde{\bigotimes}_{i=1}^{n-1} Id_{\ell_1(B_{E_i'})}\widetilde{\otimes}P_n"] & & \left(\widetilde{\bigotimes}_{i=1}^{n}c_0(B_{E_i'}), /\delta'\backslash\right)' \arrow[d, "EXT_n"'] \\
 & & \left((\widetilde{\bigotimes}_{i=1}^{n-1}c_0(B_{E_i'}))\widetilde{\otimes}\ell_\infty(B_{E_n'}), /\delta'\backslash\right)' \arrow[d, "\left((\widetilde{\bigotimes}_{i=1}^{n-1}Id_{c_0(B_{E_i'})})\widetilde{\otimes}I_{E_n}\right)'"'] \\
((\widetilde{\bigotimes}_{i=1}^{n-1}\ell_1(B_{E_i'}))\widetilde{\otimes}E_n', \backslash\delta/) \arrow[rr, "R_1"] \arrow[dd, "(\widetilde{\bigotimes}_{i=1}^{n-2}Id_{\ell_1(B_{E_i'})})\widetilde{\otimes}P_{n-1}\widetilde{\otimes}Id_{E_n'}"] & & \left((\widetilde{\bigotimes}_{i=1}^{n-1}c_0(B_{E_i'}))\widetilde{\otimes}E_n, /\delta'\backslash\right)' \arrow[d, "EXT_{n-1}"'] \\
 & & \left((\widetilde{\bigotimes}_{i=1}^{n-2}c_0(B_{E_i'}))\widetilde{\otimes}\ell_\infty(B_{E_{n-1}'})\widetilde{\otimes}E_n, /\delta'\backslash\right)' \arrow[d, "\left((\widetilde{\bigotimes}_{i=1}^{n-1}Id_{c_0(B_{E_i'})})\widetilde{\otimes}I_{n-1}\widetilde{\otimes}Id_{E_n'}\right)'"'] \\
((\widetilde{\bigotimes}_{i=1}^{n-2}\ell_1(B_{E_i'}))\widetilde{\otimes}E_{n-1}'\widetilde{\otimes}E_n', \backslash\delta/) \arrow[rr, "R_2"] \arrow[d] & & \left((\widetilde{\bigotimes}_{i=1}^{n-2}c_0(B_{E_i'}))\widetilde{\otimes}E_{n-1}\widetilde{\otimes}E_n, /\delta'\backslash\right)' \arrow[d] \\
{} & \cdots & {} \\
\vdots & \cdots & \vdots \\
{} \arrow[d] & \cdots & {} \arrow[d] \\
(\ell_1(B_{E_1'})\widetilde{\otimes}(\widetilde{\bigotimes}_{i=2}^{n}E_i'), \backslash\delta/) \arrow[rr, "R_{n-1}"] \arrow[dd, "P_1\widetilde{\otimes}(\widetilde{\bigotimes}_{i=2}^{n}Id_{E_i'})"] & & \left(c_0(B_{E_1'})\widetilde{\otimes}(\widetilde{\bigotimes}_{i=2}^{n}E_i), /\delta'\backslash\right)' \arrow[d, "EXT_1"'] \\
 & & \left(\ell_\infty(B_{E_1'})\widetilde{\otimes}(\widetilde{\bigotimes}_{i=2}^{n}E_i), /\delta'\backslash\right)' \arrow[d, "\left(I_{E_1}\widetilde{\otimes}(\widetilde{\bigotimes}_{i=2}^{n}Id_{E_i'})\right)'"'] \\
(\widetilde{\bigotimes}_{i=1}^{n}E_i', \backslash\delta/) \arrow[rr, "R_n"] & & \left(\widetilde{\bigotimes}_{i=1}^{n}E_i, /\delta'\backslash\right)'
\end{tikzcd}
$$

Figure 4.1: **Commutative diagram used in the proof of Theorem 4.2.6**

Chapter 5

Unconditionality in tensor products and ideals of polynomials, multilinear forms and operators

There has been a great interest in the study of unconditionality in tensor products of Banach spaces and, more recently, in spaces of polynomials and multilinear forms. As a probably uncomplete reference, we can mention [DDGM01, DK05, DP08, PV04, PV05, Pis78, Sch78]. A fundamental result obtained by Schütt [Sch78] and independently by Pisier [Pis78] (with additional assumptions) simplified the study of unconditionality in tensor products: in order to know if a tensor product of Banach spaces with unconditional basis has also unconditional basis, just look at the monomials. The extension of these results to symmetric tensor norms (of any degree n) was probably motivated by the so called Dineen's problem or conjecture. In his book [Din99], Sean Dineen asked the following question: if the dual of a Banach space E has an unconditional basis, can the space of n-homogeneous polynomials have unconditional basis? He conjectured a negative answer. Defant, Diaz, Garcia and Maestre [DDGM01] developed the symmetric n-fold versions of Pisier and Schütt's work and, also, obtained asymptotic estimates of the unconditionality constants of the monomial basis for spaces ℓ_p^m. As a result, they made clear that a counterexample to Dineen's conjecture should be very hard to find. Finally, Defant and Kalton [DK05] showed that if E has unconditional basis, then the space of polynomials on E cannot have unconditional basis. Defant and Kalton's result is based on a sort of dichotomy that they managed to establish: the space of polynomials either lacks the Gordon-Lewis property or is not separable. Therefore, should the space of polynomials have a basis, this cannot be unconditional.

On the other hand, in [PV04] Perez-Garcia and Villanueva illustrated the bad behavior of many tensor norms with unconditionality. They showed, for example, than no natural tensor norm (in the sense of Grothendieck) preserve unconditionality: for any natural 2-fold tensor norm, there exists a Banach space with unconditional basis whose tensor product fails to have the Gordon-Lewis property.

In this chapter we investigate when a tensor norm (of any degree, and either on the full or on the symmetric tensor product) destroys unconditionality in the sense that, for **every** Banach space E with unconditional basis, the corresponding tensor product has not unconditional basis. We establish a simple criterion to check wether a tensor norm destroys unconditionality or not.

With this we obtain that every injective and every projective s-tensor norm (resp. full tensor norm) other than $\varepsilon_{n,s}$ and $\pi_{n,s}$ (resp. ε_n and π_n) destroys unconditionality.

We also study unconditionality in ideals of polynomials and multilinear forms. We show that there are ideals $\mathcal{Q}$ of n-homogeneous polynomials such that, for every Banach space E with unconditional basis, the space $\mathcal{Q}(E)$ lacks the Gordon-Lewis property. Among these ideals we have the r-integral, r-dominated, extendible and r-factorable polynomials. For the last three examples we even get that the monomial basic sequence is never unconditional.

We consider ideals of multilinear forms and ideals of operators, where some results have their analogous. We also see that, for $n = 2$, the only natural tensor norms that destroy unconditionality are symmetric but, for $n \geq 3$, there are non-symmetric natural tensor norms that destroy unconditionality. A new contrasting situation between the $n = 2$ and $n \geq 3$ is obtained for n-linear forms defined on the product of n different spaces: for instance (see Example 5.3.10), if $E_1, \ldots, E_n$ are Banach spaces with unconditional basis, then the space of extendible n-linear forms $\mathcal{L}_e(E_1, \ldots, E_n)$ cannot have the Gordon-Lewis property whenever $n \geq 3$, while $\mathcal{L}_e(c_0, \ell_2)$, the space of extendible bilinear forms on $c_0 \times \ell_2$, has unconditional basis.

5.1 Tensor norms that destroy unconditionality

A Schauder basis $(e_j)_{j=1}^\infty$ of a Banach space E is said to be *unconditional* if, for every $x \in E$, the representing series $\sum_{j=1}^\infty a_j e_j = x$ converges unconditionally. More precisely, for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the permutated sum $\sum_{j=1}^\infty a_{\sigma(j)} e_{\sigma(j)}$ converges (obviously to x).

An equivalent condition is the following: a Schauder basis $(e_j)_{j=1}^\infty$ of a Banach space E is unconditional if and only if there is a constant $C \geq 1$ such that for all $n \in \mathbb{N}$, all $a_1, \ldots, a_n \in \mathbb{K}$, all signs $\mu_1, \ldots, \mu_r \in \{-1, +1\}$, and all subset W of $\{1, \ldots, n\}$

$$\| \sum_{j \in W} \mu_j a_j e_j \| \leq C \| \sum_{j=1}^{r} \mu_j a_j e_j \|, \tag{5.1}$$

and in this case the best constant in the inequality is called the unconditional basis constant of $(e_j)_{j=1}^\infty$ and denoted by $\chi((e_j)_{j=1}^\infty; E)$. Moreover, if E admits an unconditional basis we can define the unconditional basis constant of E by the following way

$$\chi(E) := \inf\{\chi((e_j)_{j=1}^\infty; E) : (e_j)_{j=1}^\infty \text{ is an unconditional basis of } E\}.$$

We set $\chi(E) = \infty$ if E does not admit an unconditional basis. A basic sequence $(e_j)_{j=1}^\infty$ is called an *unconditional basic sequence* if its an unconditional basis of $\overline{[e_j : j \in \mathbb{N}]}$; we write $\chi((e_j)_{j=1}^\infty; E) = \infty$ whenever this is not the case.

A space invariant closely related to unconditionality is the Gordon-Lewis property. A Banach space E has the *Gordon-Lewis property* if every absolutely summing operator $R : E \to \ell_2$ is 1-factorable. In this case, there is a constant $C \geq 0$ such that for all $R : E \to \ell_2$,

$$\gamma_1(R) \leq C \pi_1(R),$$

and the best constant C is called the Gordon-Lewis constant of E and denoted by $gl(E)$.

It can be shown that if a Banach space E has an unconditional basis then it has the Gordon-Lewis property. Moreover,

$$gl(E) \leq \chi(E). \tag{5.2}$$

This says that the latter property is weaker than having an unconditional basis. Moreover, the Gordon-Lewis property is preserved under complementation: if F is a complemented subspace of a space E with the Gordon-Lewis property, then F has the Gordon-Lewis property (a property that is unknown for unconditional basis).

Pisier [Pis78] and Schütt [Sch78] made a deep study of unconditionality in tensor products of Banach spaces. They showed (independently) that for any full tensor norm δ on the tensor product $E \otimes F$ of two Banach spaces with unconditional basis (e_i) and (f_j), respectively, the monomials $(e_i \otimes f_j)_{i,j}$ form an unconditional basis if and only if $E \widetilde{\otimes}_\delta F$ has some unconditional basis if and only if $E \widetilde{\otimes}_\delta F$ has the Gordon Lewis property. This was generalized by Defant, Diaz, Garcia y Maestre in [DDGM01] to the symmetric tensor product.

Theorem 5.1.1. *Let E be a Banach space and $(e_j)_{j=1}^{\infty}$ a 1-unconditional basis for E. For each s-tensor norm of order n we have*

$$\chi_{mon}(\widetilde{\otimes}_{\alpha}^{n,s} E) := \chi((\sigma_E^n(e_{\mathrm{j}}))_{\mathrm{j} \in \mathbb{N}_d^n}; \widetilde{\otimes}_{\alpha}^{n,s} E) \leq c_n gl\big(\widetilde{\otimes}_{\alpha}^{n,s} E\big),$$

where $c_n \leq (\frac{n^{4n}}{n!^2})2^n$.

As a consequence of Equation (5.2) and the previous theorem it is easy to obtain the following result.

Corollary 5.1.2. *[DDGM01, Corollary 1.] Let E be a Banach space with unconditional basis $(e_j)_{j=1}^{\infty}$. Then for each s-tensor norm α of order n, the following are equivalent.*

(1) The monomials of order n with respect to $(e_j)_{j=1}^{\infty}$ form an unconditional basis of $\widetilde{\otimes}_{\alpha}^{n,s} E$;

(2) $\widetilde{\otimes}_{\alpha}^{n,s} E$ has unconditional basis;

(3) $\widetilde{\otimes}_{\alpha}^{n,s} E$ has the Gordon-Lewis property.

An interesting result due to Pérez-Garcia and Villanueva [PV04, Proposition 2.3] is that, if $\big(\widetilde{\otimes}^n c_0, \delta\big)$ has unconditional basis, then δ has to coincide (up to constants) with the injective norm ε_n on $\otimes^n c_0$. On the other hand, if the tensor product $\big(\widetilde{\otimes}^n \ell_1, \delta\big)$ has unconditional basis then δ has to be equivalent to the projective norm π_n on $\otimes^n \ell_1$ [PV04, Proposition 2.6].

A similar statement holds when considering Hilbert spaces [PV05, Theorem 2.5.]. More precisely, if $\big(\widetilde{\otimes}^n \ell_2, \delta\big)$ has unconditional basis then δ has to coincide with the Hilbert-Schmidt norm σ_n (again, up to constants) (the definition of this norm is a straightforward generalization of the classical Hilbert-Schmidt tensor norm of order 2, see [DF93]).

For our purposes we need symmetric versions of [PV04, Propositions 2.3 and 2.6] and [PV05, Theorem 2.5.]. They follow from the properties given in Theorem 3.5.5 for the extension of an s-tensor norm. We remark that the spaces $\widetilde{\otimes}_{\varepsilon_{n,s}}^{n,s} c_0$, $\widetilde{\otimes}_{\pi_{n,s}}^{n,s} \ell_1$ and $\widetilde{\otimes}_{\sigma_{n,s}}^{n,s} \ell_2$ have unconditional basis (they are isomorphic to c_0, ℓ_1 and ℓ_2 respectively).

Theorem 5.1.3. *Let α be a s-tensor norm of order n, the following assertions hold.*

(1) If $\widetilde{\otimes}_{\alpha}^{n,s} c_0$ has unconditional basis, then α is equivalent to $\varepsilon_{n,s}$ on $\otimes^{n,s} c_0$.

(2) If $\widetilde{\otimes}_{\alpha}^{n,s} \ell_1$ has unconditional basis, then α is equivalent to $\pi_{n,s}$ on $\otimes^{n,s} \ell_1$.

(3) If $\widetilde{\otimes}_{\alpha}^{n,s} \ell_2$ has unconditional basis, then α is equivalent to $\sigma_{n,s}$ on $\otimes^{n,s} \ell_2$.

Proof. Let us prove (1). First notice that $\widetilde{\otimes}_{\alpha}^{n,s} \ell_2^n(c_0), \simeq \widetilde{\otimes}_{\alpha}^{n,s} \ell_\infty^n(c_0) \simeq \widetilde{\otimes}_{\alpha}^{n,s} c_0$, has the Gordon-Lewis property. Denote by $\Phi(\alpha)$ the extension norm of α (see Definition 3.5.4). Since the space $\left(\widetilde{\otimes}^n c_0, \Phi(\alpha)\right)$ is a complemented subspace of $\widetilde{\otimes}_{\alpha}^{n,s} \ell_2^n(c_0)$ (by construction) we have, according Corollary 5.1.2, that $\left(\widetilde{\otimes}^n c_0, \Phi(\alpha)\right)$ has unconditional basis. Thanks to Pérez-Garcia and Villanueva's result [PV04, Proposition 2.3] we can conclude that $\Phi(\alpha) \sim \varepsilon_n$. Now using the fact that $\Phi(\alpha)|_s \sim \alpha$ (Theorem 3.5.5 (1)) and $\varepsilon_n|_s \sim \varepsilon_{n,s}$ we get $\alpha \sim \varepsilon_{n,s}$.

The assertions (2) and (3) follow similarly. □

It should be noted that the last assertion in Theorem 5.1.3 was already stated in [PV05, Theorem 2.].

Definition 5.1.4. We say that an s-tensor norm α *destroys unconditionality* if the tensor product $\widetilde{\otimes}_{\alpha}^{n,s} E$ does not have unconditional basis for any Banach space E with unconditional basis.

As a consequence of Theorem 5.1.3, an s-tensor norm that preserves unconditionality has to be equivalent to $\varepsilon_{n,s}$, $\sigma_{n,s}$ and $\pi_{n,s}$ in $\otimes^{n,s} c_0$, $\otimes^{n,s} \ell_2$, $\otimes^{n,s} \ell_1$ respectively. As we see in the next theorem, if none of these conditions are satisfied, we have just the opposite: α destroys unconditionality.

Theorem 5.1.5. (Destruction Test for s-tensor norms.) *Let α be an s-tensor norm of order n. The norm α destroys unconditionality if and only if α is not equivalent to $\varepsilon_{n,s}$, $\pi_{n,s}$ and $\sigma_{n,s}$ on $\otimes^{n,s} c_0$, $\otimes^{n,s} \ell_1$ and $\otimes^{n,s} \ell_2$ respectively.*

To prove this we will need a definition, a simple lemma and a result of Tzafriri. We start with the definition.

Definition 5.1.6. A Banach space E contains an uniformly complemented sequence of $(\ell_p^m)_{m=1}^\infty$ $(1 \leq p \leq \infty)$ if there exist a positive constant C such that for every $m \in \mathbb{N}$, there are operators $S_m : \ell_p^m \to E$ and $T_m : E \to \ell_p^m$ satisfying $T_m S_m = Id_{\ell_p^m}$ and $\|T_m\| \|S_m\| \leq C$.

In other words, a Banach space contains an uniformly complemented sequence of $(\ell_p^m)_{m=1}^\infty$ if, for every m, there is a complemented subspace $F_m \subset E$ isomorphic to ℓ_p^m with projection constant independent of m. We now state a simple lemma.

Lemma 5.1.7. *Let $S : F \to E$ and $T : E \to F$ be operators such that $TS = Id_F$. Then,*

$$gl(F) \leq \|T\| \|S\| gl(E).$$

Proof. Without loss of generality we can suppose that $gl(E) < \infty$. Let $R : F \to \ell_2$ be a 1-summing operator, therefore $RT : E \to \ell_2$ also is 1-summing and $\gamma_1(RT) \leq gl(E)\pi_1(RT)$. On the other hand, $R = RTS$ therefore

$$\gamma_1(R) = \gamma_1(RTS) \leq \gamma_1(RT)\|S\| \leq gl(E)\pi_1(RT)\|S\| \leq gl(E)\|T\|\|S\|\pi_1(R).$$

Hence, $gl(F) \leq \|T\| \|S\| gl(E)$ which is exactly what we want to prove. □

The follwing theorem is a deep result of Tzafriri.

Theorem 5.1.8. *[Tza74] Let E be a Banach space with unconditional basis then E contains uniformly complemented at least one of the three sequences $(\ell_p^m)_{m=1}^\infty$ with $p \in \{1, 2, \infty\}$.*

We can now prove the Destruction Test for s-tensor norms.

Proof. (*of Theorem 5.1.5.*)

It is clear that a tensor norm that destroys unconditionality cannot enjoy any of the three equivalences in the statement. Conversely, suppose that α is not equivalent to $\varepsilon_{n,s}$, $\pi_{n,s}$ and $\sigma_{n,s}$ on $\otimes^{n,s}c_0$, $\otimes^{n,s}\ell_1$ and $\otimes^{n,s}\ell_2$ respectively.

Let us see that if E is a Banach space with unconditional basis, then $\widetilde{\otimes}_\alpha^{n,s} E$ cannot have the Gordon-Lewis property. By Theorem 5.1.8 we know that E contains an uniformly complemented sequence of $(\ell_p^m)_{m=1}^\infty$ for $p = 1, 2$ or ∞. So, fixed such p, there exist a positive constant C such that for every $m \in \mathbb{N}$, there are operators $S_m : \ell_p^m \to E$ and $T_m : E \to \ell_p^m$ satisfying $T_m S_m = Id_{\ell_p^m}$ and $\|T_m\|\|S_m\| \leq C$. Now, the operators $\widetilde{\otimes}^{n,s} S_m : \otimes_\alpha^{n,s}\ell_p^m \to \widetilde{\otimes}_\alpha^{n,s} E$ and $\widetilde{\otimes}^{n,s} T_m : \widetilde{\otimes}_\alpha^{n,s} E \to \otimes_\alpha^{n,s}\ell_p^m$ satisfy $\widetilde{\otimes}^{n,s} T_m \circ \widetilde{\otimes}^{n,s} S_m = Id_{\otimes_\alpha^{n,s}\ell_p^m}$ and

$$\|\widetilde{\otimes}^{n,s} S_m : \otimes_\alpha^{n,s}\ell_p^m \to \widetilde{\otimes}_\alpha^{n,s} E\|\|\widetilde{\otimes}^{n,s} T_m : \widetilde{\otimes}_\alpha^{n,s} E \to \otimes_\alpha^{n,s}\ell_p^m\| \leq \|T_m\|^n\|S_m\|^n \leq C^n.$$

Therefore, by Lemma 5.1.7 we have $gl(\widetilde{\otimes}_\alpha^{n,s}\ell_p^m) \leq C^n gl(\widetilde{\otimes}_\alpha^{n,s} E)$, for every m. If $gl(\widetilde{\otimes}_\alpha^{n,s} E)$ is finite then, by Theorem 5.1.1

$$\chi_{mon}(\widetilde{\otimes}_\alpha^{n,s}\ell_p) = \sup_m \chi_{mon}(\otimes_\alpha^{n,s}\ell_p^m) < \infty, \text{if } p = 1 \text{ or } 2,$$

or

$$\chi_{mon}(\widetilde{\otimes}_\alpha^{n,s}c_0) = \sup_m \chi_{mon}(\otimes_\alpha^{n,s}\ell_p^m) < \infty \text{ if } p = \infty.$$

This implies that either $\widetilde{\otimes}_\alpha^{n,s}\ell_1$ or $\widetilde{\otimes}_\alpha^{n,s}\ell_2$ or $\widetilde{\otimes}_\alpha^{n,s}c_0$ has unconditional basis. Now using Theorem 5.1.3 we get that either $\alpha \sim \varepsilon_{n,s}$ on $\otimes^n c_0$, or $\alpha \sim \pi_{n,s}$ on $\otimes^n\ell_1$, or $\alpha \sim \sigma_{n,s}$ on $\otimes^n\ell_2$, which leads us to a contradiction. So, $gl(\widetilde{\otimes}_\alpha^{n,s} E)$ is infinite and the statement is proved. $\square$

Since $\pi'_{n,s} = \varepsilon_{n,s}$, $\varepsilon'_{n,s} = \pi_{n,s}$ and $\sigma'_{n,s} = \sigma_{n,s}$, as a simple consequence of the Symmetric Destruction Test 5.1.5 and Corollary 2.2.4 we have the following result.

Corollary 5.1.9. *An s-tensor norm α destroys unconditionality if and only if its dual s-tensor norm α' destroys unconditionality.*

We now show that injective or projective s-tensor norms other than $\varepsilon_{n,s}$ and $\pi_{n,s}$ destroy unconditionality. To see this we need to relate the full tensor norms $/\pi_n\backslash$ and ε_n of certain tensor on $\otimes^n\ell_2$. First, note that from [CDS06, Proposition 3.1] (and its proof), we can see that if S is a diagonal extendible multilinear form on ℓ_p ($2 \leq p \leq \infty$), then S is nuclear and

$$\|S\|_{\mathcal{N}} \leq C\|S\|_e. \tag{5.3}$$

The definition of nuclear and extendible multilinear forms can also found in [CDS06] and are analogous to the definition presented for polynomials in this text.

If T is any multilinear form on ℓ_p, we denote by $D(T)$ the multilinear form obtained from T setting to zero all the coefficients outside the diagonal (see [CDS07] for details).

Lemma 5.1.10. *Let $2 \leq p \leq \infty$. There exist a constant K such that for every sequence of scalars $a_1, \dots, a_m$,*

$$/\pi_n\backslash(\sum_{k=1}^m a_k e_k \otimes \cdots \otimes e_k, \otimes^n \ell_p^m) \leq K \varepsilon_n(\sum_{k=1}^m a_k e_k \otimes \cdots \otimes e_k, \otimes^n \ell_p^m)$$

Proof. Notice that

$$\begin{aligned}
&/\pi_n\backslash\Big(\sum_{k=1}^m a_k e_k \otimes \cdots \otimes e_k, \otimes^n \ell_p^m\Big) \\
&= \sup_{\|T\|_e \leq 1} \Big|\sum_{k=1}^m a_k T(e_k, \dots, e_k)\Big| \\
&= \sup_{\|T\|_e \leq 1} \Big|\sum_{k=1}^m a_k D(T)(e_k, \dots, e_k)\Big|, \\
&\leq \sup\left\{\Big|\sum_{k=1}^m a_k S(e_k, \dots, e_k)\Big| : S \in \mathcal{L}^n(\ell_p^m) \text{ diagonal } : \|S\|_e \leq 1\right\},
\end{aligned}$$

where the last inequality is a consequence of the inequality $\|D(T)\|_e \leq \|T\|_e$ [CDS07, Proposition 5.1.]. Now, using (5.3), we have

$$\begin{aligned}
&/\pi_n\backslash\Big(\sum_{k=1}^m a_k e_k \otimes \cdots \otimes e_k, \otimes^n \ell_p^m\Big) \\
&\leq C^{-1} \sup\left\{\Big|\sum_{k=1}^m a_k S(e_k, \dots, e_k)\Big| : S \in \mathcal{L}^n(\ell_p^m) \text{ diagonal}, \ \|S\|_{\mathcal{N}} \leq 1\right\} \\
&\leq C^{-1} \varepsilon_n\Big(\sum_{k=1}^m a_k e_k \otimes \cdots \otimes e_k, \otimes^n \ell_p^m\Big).
\end{aligned}$$

This concludes the proof. □

Now, what we are ready to show the following.

Theorem 5.1.11. *(**Destruction Theorem.**) Every injective or projective s-tensor norms other than $\varepsilon_{n,s}$ and $\pi_{n,s}$ destroy unconditionality.*

Proof. Let us see first that $/\pi_{n,s}\backslash$ destroys unconditionality. By the Destruction Test (Theorem 5.3.2) we need to show that $/\pi_{n,s}\backslash$ is not equivalent to $\varepsilon_{n,s}$, π_n and $\sigma_{n,s}$ on $\otimes^{n,s} c_0$, $\otimes^{n,s} \ell_1$ and $\otimes^{n,s} \ell_2$ respectively.

The tensor norm $/\pi_{n,s}\backslash$ is not equivalent to $\varepsilon_{n,s}$ on $\otimes^n c_0$: since $/\pi_{n,s}\backslash = \pi_{n,s}$ on $\otimes^{n,s} c_0$ (see Corollary 3.2.8), this would imply $\pi_{n,s} \sim \varepsilon_{n,s}$, which clearly false.

The tensor norm $/\pi_{n,s}\backslash$ is not equivalent to $\pi_{n,s}$ on $\otimes^n \ell_1$: if it were, every polynomial on ℓ_1 would be extendible, but this cannot happen (see, for example, [Car01, Corollary 12]). Since $/\pi_{n,s}\backslash \leq \pi_{n,s}$, this shows that

$$\|id : \otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1^m \longrightarrow \otimes^{n,s}_{\pi_{n,s}} \ell_1^m\| \to \infty, \tag{5.4}$$

as $m \to \infty$.

The tensor norm $/\pi_{n,s}\backslash$ is not equivalent to $\sigma_{n,s}$ on $\otimes^n \ell_2$: Lemma 5.1.10 states the existence of a constant K such that:

$$/\pi_n\backslash\Big(\sum_{k=1}^m e_k \otimes \cdots \otimes e_k, \otimes^n \ell_2\Big) \leq K\varepsilon_n\Big(\sum_{k=1}^m e_k \otimes \cdots \otimes e_k, \otimes^n \ell_2\Big) \leq K.$$

On the other hand,

$$\sigma_n\Big(\sum_{k=1}^m e_k \otimes \cdots \otimes e_k, \otimes^n \ell_2\Big) = m^{1/2}.$$

Since the restrictions of $/\pi_n\backslash$ and σ_n to the symmetric tensor product $\otimes^{n,s}\ell_2$ are equivalent to $/\pi_{n,s}\backslash$ and $\sigma_{n,s}$ respectively, we get that $/\pi_{n,s}\backslash$ is not equivalent to $\sigma_{n,s}$ on $\otimes^n \ell_2$. Moreover,

$$\|id : \otimes^{n,s}_{/\pi_n\backslash} \ell_2^m \longrightarrow \otimes^{n,s}_{\sigma_{n,s}} \ell_2^m\| \to \infty, \tag{5.5}$$

as $m \to \infty$, a fact that will be used below.

Thus, we have shown that $/\pi_{n,s}\backslash$ destroys unconditionality. From Equations (5.5) and (5.4), if α is an s-tensor norm that is dominated by $/\pi_{n,s}\backslash$, then it cannot be equivalent to $\pi_{n,s}$ or $\sigma_{n,s}$ on $\otimes^m \ell_1$ or $\otimes^m \ell_2$ respectively. If it is equivalent to $\varepsilon_{n,s}$ on $\otimes^{n,s} c_0$, we would have that $/\alpha\backslash$ must be equivalent to $\varepsilon_{n,s}$ (on $NORM$). Therefore, the only (up to equivalences) injective tensor norm that does not destroy unconditionality is $\varepsilon_{n,s}$. By duality, a projective s-tensor norm that is not equivalent to $\pi_{n,s}$ must destroy unconditionality. $\square$

5.2 Unconditionality in ideals of polynomials

We begin with a reformulation of the Destruction test in terms of ideals of polynomials.

Proposition 5.2.1. *If $\mathcal{Q}$ is a Banach ideal of n-homogeneous polynomials, the following are equivalent.*

(1) For any Banach space E with unconditional basis, $\mathcal{Q}(E)$ fails to have the Gordon-Lewis property.

(2) $\|Id : \mathcal{Q}(\ell_\infty^m) \to \mathcal{P}_I^n(\ell_\infty^m)\| \to \infty$, $\|Id : \mathcal{P}^n(\ell_1^m) \to \mathcal{Q}(\ell_1^m)\| \to \infty$ and $\max\big(\|Id : \mathcal{Q}(\ell_2^m) \to \mathcal{P}_{HS}^n(\ell_2^m)\|, \|Id : \mathcal{P}_{HS}^n(\ell_2^m) \to \mathcal{Q}(\ell_2^m)\|\big) \to \infty$ as $m \to \infty$.

If $\mathcal{Q}$ is maximal, this is also equivalent to

(3) $\mathcal{Q}(c_0) \neq \mathcal{P}_I^n(c_0)$, $\mathcal{Q}(\ell_1) \neq \mathcal{P}^n(\ell_1)$ and $\mathcal{Q}(\ell_2) \neq \mathcal{P}_{HS}^n(\ell_2)$.

Proof. It is clear that (1) implies any of the other statements. To see that (2) implies (1), by Tzafriri's result (Theorem 5.1.8) it is enough to see that $gl\big(\mathcal{Q}(\ell_p^m)\big) \to \infty$ as $m \to \infty$ for all these $p = 1, 2, \infty$. We can suppose $p = 1$, the other cases being completely analogous. Let α be the s-tensor norm associated with $\mathcal{Q}$. Since $\mathcal{Q}(\ell_\infty^m) = \otimes_\alpha^{n,s} \ell_1^m$, if $gl(\otimes_\alpha^{n,s} \ell_1^m)$ were uniformly bounded we would have that $\alpha \sim \pi_{n,s}$ on $\otimes^{n,s} \ell_1$ by Theorem 5.1.3. Therefore, the norms of $\mathcal{Q}(\ell_\infty^m)$ and $\mathcal{P}_I^n(\ell_\infty^m)$ would be equivalent (with constants independent of m), a contradiction.

The $\mathcal{L}_p$-Local Technique Lemma for maximal ideals 2.2.15 ensures that, if $\|Id : \mathcal{P}^n(\ell_1^m) \to \mathcal{Q}(\ell_1^m)\|$ is uniformly bounded on m and $\mathcal{Q}$ is maximal, then $\mathcal{P}^n(\ell_1) = \mathcal{Q}(\ell_1)$ and, of course, the converse is also true. The same holds for the other two conditions in (2) and (3). $\square$

For a Banach space E with unconditional basis $(e_j)_{j=1}^{\infty}$, the authors of [DK05] studied when $\mathcal{P}^n(E)$ was isomorphic to a Banach lattice. It turned out that this happens precisely when the monomials associated to the dual basis $(e_j')_{j=1}^{\infty}$ form an unconditional basic sequence. The same holds for maximal polynomial ideals as we see in the next theorem.

Proposition 5.2.2. *Let $\mathcal{Q}$ be a maximal ideal of n-homogeneous polynomials and E be a Banach space with unconditional basis $(e_j)_{j=1}^{\infty}$. The following are equivalent.*

(1) The monomials $(e_\mathbf{j}')_{\mathbf{j}\in\mathbb{N}_d^n}$ form an unconditional basic sequence in $\mathcal{Q}(E)$;

(2) $\mathcal{Q}(E)$ is isomorphic to a Banach lattice;

(3) $\mathcal{Q}(E)$ has the Gordon-Lewis property.

The proposition can be proved similarly to [DK05, Proposition 4.1] with the help of Corollary 2.2.12.

Now we present some examples of Banach polynomial ideals that destroy the Gordon-Lewis property (in the sense of the Proposition 5.2.1). An immediate consequence of Theorem 5.1.11 is the following:

Proposition 5.2.3. *If $\mathcal{Q}$ is a Banach ideal of n-homogeneous polynomials associated with injective or projective s-tensor norm different from $\varepsilon_{n,s}$ and $\pi_{n,s}$, then $\mathcal{Q}(E)$ does not have the Gordon-Lewis property for any Banach space with unconditional basis.*

As an example of the latter, we have the following.

Example 5.2.4. *Let E be a Banach space with unconditional basis, then $\mathcal{P}_e^n(E)$ does not have the Gordon-Lewis property and the monomial basic sequence is not unconditional.*

The next example shows that the ideal of r-dominated polynomials $\mathcal{D}_r^n$ lacks of unconditionality.

Example 5.2.5. *Let E be a Banach space with unconditional basis and $r \geq n$, then $\mathcal{D}_r^n(E)$ does not have the Gordon-Lewis property and the monomial basic sequence is not unconditional.*

Proof. By Proposition 5.2.1 we must show that $\mathcal{D}_r^n(\ell_1) \neq \mathcal{P}^n(\ell_1)$, $\mathcal{D}_r^n(c_0) \neq \mathcal{P}_I^n(c_0)$ and $\mathcal{D}_r^n(\ell_2) \neq \mathcal{P}_{HS}^n(\ell_2)$.

If $\mathcal{D}_r^n(\ell_1) = \mathcal{P}^n(\ell_1)$, using [CDM09, Lemma 1.5] we would have that $\mathcal{D}_r^2(\ell_1) = \mathcal{P}^2(\ell_1)$ (since $\mathcal{D}_r$ and $\mathcal{P}$ are coherent sequences of polynomial ideals [CDM09, Examples 1.9, 1.13]). In this case, we would have: $\mathcal{P}^2(\ell_1) = \mathcal{D}_r^2(\ell_1) = \mathcal{D}_2^2(\ell_1) = \mathcal{P}_e^2(\ell_1)$ (where the second equality is due to [DF93, Proposition 12.8] and the third to [DF93, Proposition 20.17]), but we already know that $\mathcal{P}_e^2(\ell_1)$ cannot be equal to $\mathcal{P}^2(\ell_1)$.

Using coherence again, it is easy to show that $\mathcal{P}_{HS}^n(\ell_2) \not\subset \mathcal{D}_r^n(\ell_2)$ (recall that Hilbert Schmidt polynomials coincide with multiple 1-summing polynomials, which form a coherent sequence of ideals [CDM09, Example 1.14]): if $\mathcal{P}_{HS}^n(\ell_2) \subset \mathcal{D}_r^n(\ell_2)$, we would have $\mathcal{P}_{HS}^2(\ell_2) \subset \mathcal{D}_r^2(\ell_2) = \mathcal{D}_2^2(\ell_2) = \mathcal{P}_e^2(\ell_2)$ (again by [DF93, Proposition 12.8, 20.17]), which is not true, for example, by (5.5) and duality.

Similarly, $\mathcal{D}_r^n(c_0) \neq \mathcal{P}_I^n(c_0)$ ($\mathcal{P}_I$ is also a coherent sequence [CDM09, Example 1.11]). $\square$

The ideal of r-integral polynomials $\mathcal{I}_r^n$ also lacks of unconditionality as we see below.

Example 5.2.6. *Let E be a Banach space with unconditional basis and $r \geq n$, then $\mathcal{I}_r^n(E)$ does not have the Gordon-Lewis property.*

Proof. As in the proof of [CDS07, Theorem 3.5] we can see that, if M is a finite dimensional space, then $(\mathcal{D}_r^n)^*(M) = \mathcal{I}_r^n(M)$. For $p \in \{1, 2, \infty\}$, we have $gl\big(\mathcal{I}_r^n(\ell_p^m)\big) = gl\big(\mathcal{D}_r^n(\ell_p^m)\big)$, which we already know by the previous example that this goes to ∞ with m. $\square$

Note that in the proofs of the previous examples we have actually shown the following limits, which we will use below:

$$\|Id : \mathcal{P}_{HS}^n(\ell_2^m) \to \mathcal{D}_r^n(\ell_2^m)\| \to \infty \tag{5.6}$$
$$\|Id : \mathcal{I}_r^n(\ell_2^m) \to \mathcal{P}_{HS}^n(\ell_2^m)\| \to \infty \tag{5.7}$$

as m goes to infinity.

Unconditionality is also destroyed by the ideal of r-factorable polynomials $\mathcal{L}_r^n$.

Example 5.2.7. *Let E be a Banach space with unconditional basis and $r \geq n$, then $\mathcal{L}_r^n(E)$ does not have the Gordon-Lewis property and the monomial basic sequence is not unconditional.*

Proof. By [CDS07, Theorem 3.5] and then [Flo02, Proposition 4.3.], we have $\mathcal{D}_r^* = \mathcal{I}_r^{max} \subset \mathcal{L}_r$ ($\mathcal{L}_r$ is maximal [Flo02, Proposition 3.1]). Therefore, using Proposition 5.2.1 and Equation (5.7), we have $\|Id : \mathcal{L}_r^n(\ell_\infty^m) \to \mathcal{P}_I^n(\ell_\infty^m)\| \to \infty$ and $\|Id : \mathcal{L}_r^n(\ell_2^m) \to \mathcal{P}_{HS}^n(\ell_2^m)\| \to \infty$. It remains to show that $\mathcal{L}_r^n(\ell_1) \neq \mathcal{P}^n(\ell_1)$. We show this first for $n = 2$. Suppose this happens, then every symmetric operator $T : \ell_1 \to \ell_\infty$ would factorize by a reflexive Banach space, then must be weakly compact, a contradiction to the fact that ℓ_1 is not symmetrically Arens regular [ACG91, Section 8]. For $n \geq 3$ we use coherence for composition ideals [CDM09, Proposition 3.3] since $\mathcal{L}_r = \mathcal{P} \circ \Gamma_r$ [Flo97, 3.5.], where Γ_r is the ideal of r-factorable operators. $\square$

In [DK05], Defant and Kalton showed that the space $\mathcal{P}^n(E)$ of all n-homogeneous polynomials cannot have unconditional basis whenever E is a Banach space with unconditional basis. However, $\mathcal{P}^n(E)$ can have the Gordon-Lewis property (for example, when $E = \ell_1$). When this happens, $\mathcal{P}^n(E)$ is not separable and therefore it has no basis at all. One may wonder if there are other ideals with that property: that never have unconditional bases but sometimes enjoy the Gordon-Lewis property. We will present such an example but first we extend the range of ideals for which [DK05, Proposition 3.2.] apply. For each m, we define $p_m \in \mathcal{P}^n(\ell_2)$ by

$$p_m = \sum_{j=1}^{m} (e_j')^n.$$

Proposition 5.2.8. *Let E be a Banach space with unconditional basis and let $\mathcal{Q}$ be a polynomial ideal such that $(\|p_m\|_{\mathcal{Q}(\ell_2)})_m$ is uniformly bounded. If $(\mathcal{Q})^{max}(E)$ is separable, then E must contain $(\ell_2^m)_{m=1}^\infty$ or $(\ell_\infty^m)_{m=1}^\infty$ uniformly complemented.*

Proof. Let $(e_k)_{k=1}^\infty$ be an unconditional basis of E. By the proof of [DK05, Proposition 3.2.] we know that if E does not contain any of the sequences $(\ell_2^m)_{m=1}^\infty$, $(\ell_\infty^m)_{m=1}^\infty$ uniformly complemented then we may extract a subsequence $(f_j)_{j=1}^\infty$ of $(e_k)_{k=1}^\infty$ such that for any $x \in F := \overline{[(f_j)]}$,

$\sum_{j=1} |f'_j(x)|^2 < \infty$ (where $(f'_j)_j$ is the corresponding subsequence of the dual basic sequence). This means that, as sequence spaces, we have a continuous inclusion $i : F \hookrightarrow \ell_2$. For $x \in F$, we define $q_m(x) = \sum_j^m f'_j(x)^n$. We have

$$\|q_m\|_{\mathcal{Q}(F)} = \|p_m \circ i\|_{\mathcal{Q}(F)} \leq \|p_m\|_{\mathcal{P}^n(\ell_2)} \|i\|^n,$$

which is bounded uniformly on m. It follows from [CDS08, Lemma 5.4] that $(\mathcal{Q})^{max}(F)$ cannot be separable. Hence, $(\mathcal{Q})^{max}(E)$ cannot be separable either, since F is a complemented subspace of E. □

The uniform bound for $(\|p_m\|_{\mathcal{Q}(\ell_2)})_m$ is necessary for the result to be true, as the following example shows.

Example 5.2.9. *Let E be the dual of the original Tsirelson's space. Since E is a reflexive Banach space with unconditional basis, by Corollary 4.1.6 we get that $\mathcal{P}_e^n(E)$ is separable. But, E does not contain either ℓ_2^m nor ℓ_∞^m uniformly complemented [CS89, Pages 33 and 66].*

Corollary 5.2.10. *Let $\mathcal{Q}$ be a maximal Banach ideal of n-homogeneous polynomials such that $(\|p_m\|_{\mathcal{Q}(\ell_2)})_m$ is uniformly bounded. Suppose also that not ever polynomial in $\mathcal{Q}^n(c_0)$ is integral. If E or its dual has unconditional basis, then $\mathcal{Q}(E)$ does not have unconditional basis.*

Proof. Suppose first that E has unconditional basis. If $\mathcal{Q}(E)$ is separable, by Proposition 5.2.8 E must contain either $(\ell_\infty^m)_{m=1}^\infty$ or $(\ell_2^m)_{m=1}^\infty$ uniformly complemented. If E contains the sequence $(\ell_\infty^m)_{m=1}^\infty$ uniformly complemented, since not every polynomial on c_0 is integral, we have

$$gl(\mathcal{Q}^n(\ell_\infty^m)) \to \infty \quad \text{as} \quad m \to \infty,$$

by the proof of Proposition 5.2.1, so $\mathcal{Q}(E)$ cannot have the Gordon-Lewis property. If E contains $(\ell_2^m)_{m=1}^\infty$ uniformly, since $(\|p_m\|_{\mathcal{Q}(\ell_2)})_m$ is uniformly bounded and $(\|p_m\|_{\mathcal{P}_{HS}^n(\ell_2)})_m = \sqrt{m}$, we can conclude that

$$gl(\mathcal{Q}^n(\ell_2^m)) \to \infty \quad \text{as} \quad m \to \infty,$$

by the proof of Theorem 5.1.5. Therefore, if E is reflexive, $\mathcal{Q}(E)$ either fails the Gordon-Lewis property or is non-separable. In any case, it has no unconditional basis.

If E' has unconditional basis and is reflexive, then E also has unconditional basis and we are in the previous case. If E' is not reflexive and has unconditional basis, then E' contains complemented copies of c_0 or ℓ_1. If it contains c_0, it also contains ℓ_∞, so E' is not separable, a contradiction. If E' contains ℓ_1 and we denote by α the s-tensor norm associated with $\mathcal{Q}$, we obtain that $\mathcal{Q}(E)$ contains the spaces $\otimes_\alpha^{n,s} \ell_1^m$ which are uniformly isomorphic to $\mathcal{Q}^n(\ell_\infty^m)$. As in the reflexive case, the Gordon-Lewis constant of $\mathcal{Q}^n(\ell_\infty^m)$ goes to infinity, so $\mathcal{Q}^n(E)$ does not have the Gordon-Lewis property. □

As a consequence of the previous corollary, we conclude that $\mathcal{P}^n(E)$ cannot have an unconditional basis for any Banach space E that has (or its dual has) unconditional basis. Since there are Banach spaces without unconditional basis whose duals have one (see for example the remark after [LT77, 1.c.12.]), this somehow extends the answer to Dineen's question in [DK05]. However, it should be stressed that our arguments strongly rely on Defant and Kalton's work.

Another consequence is the following: suppose that E' has a Schauder basis $(e'_j)_{j=1}^{\infty}$ and $\mathcal{Q}^n$ is as in the previous corollary. Then, the monomials associated with $(e'_j)_{j=1}^{\infty}$ cannot be an unconditional basis of $\mathcal{Q}^n(E)$. Indeed, should the monomials be an unconditional sequence, then $(e'_j)_{j=1}^{\infty}$ would be also unconditional, so we can apply Corollary 5.2.10.

Now we present another example of a maximal Banach ideal of polynomials which behaves just as $\mathcal{P}^n$.

Example 5.2.11. *The polynomial ideal $\mathcal{Q} = \mathcal{D}_n^n \circ \Gamma_\infty^{-1}$ never has unconditional basis but it may enjoy the Gordon-Lewis property. For $n \geq 3$, this ideal is different from $\mathcal{P}^n$.*

Here we follow the notation given in Definition 3.4.2 for quotient ideals. More precisely, a polynomial p belongs to $\mathcal{Q}(E)$ if there exists a constant $C > 0$ such that for every ∞-factorable operator $T : F \to E$ with $\gamma_\infty(T) \leq 1$, the composition $p \circ T$ is n-dominated and $\|p \circ T\|_{\mathcal{D}_n^n} \leq C$. We define

$$\|p\|_{\mathcal{Q}} := \sup\{\|p \circ T\|_{\mathcal{D}_n^n} : \gamma_\infty(T) \leq 1\},$$

where $\mathcal{D}_n^n$ is the ideal of n-dominated polynomials.

It is not hard to see that $\mathcal{Q}$ is in fact a Banach ideal of n-homogeneous polynomials. We now see that $\mathcal{Q}$ is maximal: take $p \in (\mathcal{Q})^{max}(E)$ and let us show that $p \in \mathcal{Q}(E)$, that is, $\|p \circ T\|_{\mathcal{D}_n^n} \leq C$ for every $T \in \Gamma_\infty(F, E)$ with $\gamma_\infty(T) \leq 1$. Since D_n^n is a maximal ideal, it is sufficient to prove that $\|p \circ T|_M\|_{\mathcal{D}_n^n} \leq C$ for every $M \in FIN(F)$ and T as before. But, $p \circ T|_M = p|_{Im(T|_M)} \circ T|_M$ and since $p \in (\mathcal{Q})^{max}(E)$ we have $\|p|_N\|_{\mathcal{Q}} \leq K$ for every $N \in FIN(E)$. This means that $\sup_{\gamma_\infty(T)\leq 1} \|p|_N \circ T\|_{\mathcal{D}_n^n} \leq K$ and we are done.

We also have $\mathcal{Q}(\ell_1) = \mathcal{P}^n(\ell_1)$. Indeed, take $p \in \mathcal{P}^n(\ell_1)$ and $T \in \Gamma_\infty(F, \ell_1)$ with unit norm and let us find a constant C such that $\|p \circ T\|_{\mathcal{D}_n^n} \leq C$. If $S : F \to L_\infty(\mu)$ and $R : L_\infty(\mu) \to \ell_1$ are operators which satisfy $\|S\|\|R\| \leq 2$ and $T = S \circ R$, then $p \circ T = p \circ R \circ S$. By Grothendieck's theorem [DJT95, 3.7], R is n-summing and $\pi_n(R) \leq K_G\|R\|$. Since $\mathcal{D}_n^n$ is the composition ideal $\mathcal{P}^n \circ \Pi_n$ [Sch91] we have $\|p \circ R\|_{\mathcal{D}_n^n} \leq K_G^n \|p\|\|R\|^n$. Therefore $\|p \circ T\|_{\mathcal{D}_n^n} \leq K_G^n\|p\|\|R\|^n\|S\|^n \leq (2K_G)^n\|p\|$ and we are done.

Using a similar argument it can be shown that $\mathcal{Q}(\ell_2) = \mathcal{P}^n(\ell_2)$, so the sequence $(\|p_m\|_{\mathcal{Q}(\ell_2)})_m$ is uniformly bounded. We also have $\mathcal{Q}(c_0) = \mathcal{D}_n^n(c_0) \not\supset \mathcal{P}_I^n(c_0)$.

Thus, Corollary 5.2.10 says that $\mathcal{Q}(E)$ has not unconditional basis if E or its dual has unconditional basis. On the other hand, $\mathcal{Q}(\ell_1) = \mathcal{P}^n(\ell_1)$ has the Gordon-Lewis property.

Since $\mathcal{Q}(c_0) = \mathcal{D}_n^n(c_0)$, Lemma 5.4 in [JPPV07] ensures that $\mathcal{Q}$ is different from $\mathcal{P}^n$ for $n \geq 3$ (for $n = 2$ we actually have $\mathcal{Q} = \mathcal{P}^2$).

5.3 Unconditionality for full tensor norms and multilinear ideals

We now study unconditionality for full tensor norms. We have the obvious definition.

Definition 5.3.1. We say that a full tensor norm δ **destroys unconditionality** if the tensor product $\left(\widetilde{\bigotimes}^n E, \delta\right)$ does not have unconditional basis for any Banach space E with unconditional basis.

We list the analogous versions of Theorem 5.1.5 and Theorem 5.1.11 for full tensor norms. They can be proved similarly, using the ideas of the proofs we saw and [DDGM01, Remark 1].

Theorem 5.3.2. ***Destruction Test:*** *A full tensor norm δ destroys unconditionality if and only if δ is not equivalent to ε_n, π_n and σ_n on $\otimes^n c_0$, $\otimes^n \ell_1$, $\otimes^n \ell_2$ respectively.*

Theorem 5.3.3. *Every injective or projective full tensor norms other than ε_n and π_n destroy unconditionality.*

Note that the previous result asserts that nontrivial (different from ε_n and π_n) natural **full-symmetric** tensor norms destroy unconditionality. A natural question arises: what about the other (non-symmetric) natural norms? We know that none of them preserve unconditionality, but which of them destroy it? Again, the answer will depend on n being 2 or greater.

Remark 5.3.4. For $n = 2$, $/\pi_2\backslash$ and $\backslash\varepsilon_2/$ are the only natural norms that destroy unconditionality.

Proof. We know that $/\pi_2\backslash$ and $\backslash\varepsilon_2/$ destroy unconditionality and that π and ε do not.

On the other hand, since $(/\pi_2\backslash)/ \sim d_2$ is equivalent to σ_2 in $\otimes^2\ell_2$, we have that $(/\pi_2\backslash)/$ does not destroy unconditionality and, by duality, neither does $\backslash(/\pi_2\backslash) \sim g_2$.

By [Sch90, Corollary 3.2] we know that $\Pi_1(\ell_2, \ell_2)$ has the Gordon-Lewis property. So, $\varepsilon_2/ = d_\infty$ cannot destroy unconditionality. Transposing and/or dualizing, neither do $\backslash\varepsilon_2 = g_\infty$, $\pi_2\backslash = d'_\infty$ or $/\pi_2 = g'_\infty$.

If we show that $\backslash(/\pi_2) = \backslash g'_\infty$ does not destroy unconditionality, we obtain the same conclusion for $(\pi_2\backslash)/ = d'_\infty/$, $(\varepsilon_2/)\backslash = d_\infty\backslash$ and $/(\backslash\varepsilon_2) = /g_\infty$ (again by duality and transposition). Now, since ℓ_∞ is injective, every operator from ℓ_1 to ℓ_∞ is extendible. Therefore, $/\pi_2$ and π_2 are equivalent on $\otimes^2\ell_1$, which implies also the equivalence of $\backslash(/\pi_2)$ and π_2 on $\otimes^2\ell_1$, and thus $\backslash(/\pi_2) \sim \backslash g'_\infty$ does not destroy unconditionality, which ends the proof. $\square$

We have just shown that, for $n = 2$, nontrivial symmetric tensor norms are exactly those that destroy unconditionality. Let us see that for $n \geq 3$, there are non-symmetric natural tensor norms that destroy unconditionality. We have never defined nor introduced the notation for non-symmetric natural tensor norms, but for the following examples, it is enough to say that inj_k means to take injective associate in the kth place (*e.g.,* for $n = 2$, $inj_1\delta$ is the left injective associate $/\delta$).

Example 5.3.5. *There are non-symmetric natural norms that destroy unconditionality.*

Consider $\delta = inj_2\ inj_1\pi_n$. Note that $E \otimes_{/\pi_2\backslash} E$ is isometric to a complemented subspace of $\big(\otimes^n E, \delta\big)$ for any Banach space E. Since $/\pi_2\backslash$ destroy unconditionality, it destroys the Gordon-Lewis property, and therefore so does δ.

It is not true that every natural tensor norm different from π and ε destroys unconditionality. For example, if we take $\delta = inj_1\pi_3$ we have

$$(\ell_1 \otimes \ell_1 \otimes \ell_1, \delta) \simeq (\ell_1 \otimes_{/\pi_2} \ell_1) \otimes_{\pi_2} \ell_1 \simeq \big(\otimes^3 \ell_1, \pi_3\big).$$

Hence, $\delta = inj_1\pi$ does not destroy unconditionality.

Our original motivation was the unconditionality problem for spaces of polynomials (Dineen's problem), and so it was reasonable to consider tensor products of a single space. However, the question about unconditionality in full tensor products is interesting also when different spaces are considered. Moreover, we see that in this case, there is a new difference between $n = 2$ and $n \geq 3$. First we have the following lemma.

Lemma 5.3.6. *Let δ be a 2-fold full injective norm. There exist a constant $C \geq 0$ such that $m^{1/2} \leq C gl\big(\ell_1^m \otimes_\delta \ell_2^m\big) \leq C m^{1/2}$ for every $m \in \mathbb{N}$. In particular, $gl\big(\ell_1^m \otimes_\delta \ell_2^m\big) \to \infty$, as $m \to \infty$.*

Proof. For the lower estimate, notice first that by [DF93, Exersice 31.2] $\ell_1 \otimes_{\varepsilon_2} \ell_2 \simeq \ell_1 \otimes_\delta \ell_2$ (since ℓ_1 and ℓ_2 have cotype 2). Then, $\ell_1^m \otimes_\delta \ell_2^m$ is isomorphic to $\ell_1^m \otimes_{\varepsilon_2} \ell_2^m$ with constants independent of m. So we have to estimate $gl\big(\ell_1^n \otimes_{\varepsilon_2} \ell_2^n\big)$.

First we have:

$$\|\sum_{i,j}^m e_i \otimes e_j\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m} = \sup_{a \in B_{\ell_\infty^m}, b \in B_{\ell_2^m}} |\sum_{i,j}^m a_i b_j| \geq m \sup_{b \in B_{\ell_2^m}} |\sum_j^m b_j| = m m^{1/2} = m^{3/2}.$$

We now consider the aleatory matrices

$$R : \Omega \to \ell_1^m \otimes_{\varepsilon_2} \ell_2^m \quad R(\omega) := \sum_{i,j}^m r_{i,j}(\omega) e_i \otimes e_j$$
$$G : \Omega \to \ell_1^m \otimes_{\varepsilon_2} \ell_2^m \quad G(\omega) := \sum_{i,j}^m g_{i,j}(\omega) e_i \otimes e_j,$$

where (Ω, μ) is a probability space and $r_{i,j}$'s and $g_{i,j}$'s forms a family of m^2 independent Bernoulli and Gaussian variables on Ω, respectively.

Then, for all $\omega \in \Omega$

$$\begin{aligned} m^{3/2} \leq \|\sum_{i,j}^m e_i \otimes e_j\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m} &= \|\sum_{i,j}^m r_{i,j}(\omega) r_{i,j}(\omega) e_i \otimes e_j\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m} \\ &\leq \chi((e_i \otimes e_j)_{i,j}) \|R(\omega)\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m}. \end{aligned}$$

On the other hand, we know that $\chi((e_i \otimes e_j)_{i,j}) \leq 2^3 gl(\ell_1^n \otimes_{\varepsilon_2} \ell_2^n)$ by [DDGM01, Remark 1] (which is a 'full' version of Theorem 5.1.1). Therefore, for every $\omega \in \Omega$ we have:

$$m^{3/2} \leq \|\sum_{i,j}^m e_i \otimes e_j\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m} \leq 2^3 gl(\ell_1^m \otimes_{\varepsilon_2} \ell_2^m) \|R(\omega)\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m}.$$

Integrating the last expression,

$$m^{3/2} \leq \|\sum_{i,j}^m e_i \otimes e_j\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m} \leq 2^3 gl(\ell_1^m \otimes_{\varepsilon_2} \ell_2^m) \int_\Omega \|R(\omega)\|_{\ell_1^m \otimes_{\varepsilon_2} \ell_2^m} \, d\mu(\omega).$$

Now, since Gaussian averages dominate, up to a uniform constant, Bernoulli averages ([Tom89, Page 15.], [DJT95, Proposition 12.11.]) we get:

$$\int_\Omega \|R(\omega)\|_{\ell_1^m\otimes_{\varepsilon_2}\ell_2^m}\, d\mu(\omega) \leq L \int_\Omega \|G(\omega)\|_{\ell_1^m\otimes_{\varepsilon_2}\ell_2^m}\, d\mu(\omega).$$

It is time to use Chevet inequality.

Chevet Inequality [Tom89, (43.2)]: Let E and F be Banach spaces. Fix $x_1', \dots, x_m' \in E'$ and $y_1, \dots, y_m \in F$. If $\{g_{ij}\}, \{g_i\}, \{g_j\}$ are independent Gaussian random variables in some probability space (Ω, μ). Then there is a constant b such that,

$$\int_\Omega \| \sum_{i,j=1}^m g_{ij} x_i' \otimes y_j \|_{\mathcal{L}(E,F)}\, d\mu \leq b \sup_{\|x\|_E\leq 1} \left(\sum_{i=1}^m |x_i'(x)|^2 \right)^{1/2} \int_\Omega \| \sum_{j=1}^m g_j y_j \|\, d\mu$$
$$+\, b \sup_{\|y'\|_{F'}\leq 1} \left(\sum_{j=1}^m |y'(y_j)|^2 \right)^{1/2} \int_\Omega \| \sum_{i=1}^m g_i x_i' \|\, d\mu.$$

To conclude with our estimations it remains to observe that

$$\| \sum_{i,j}^m g_{i,j}(\omega) e_i \otimes e_j \|_{\ell_1^m\otimes_{\varepsilon_2}\ell_2^m} = \| \sum_{i,j}^m g_{i,j}(\omega) e_i \otimes e_j \|_{\mathcal{L}(\ell_\infty^m, \ell_2^m)}.$$

Then,

$$\begin{aligned} \int_\Omega \|G(\omega)\|_{\ell_1^m\otimes_{\varepsilon_2}\ell_2^m}\, d\mu(\omega) &\leq b \sup_{x\in B_{\ell_\infty^m}} \left(\textstyle\sum_{i=1}^m |x_i|^2\right)^{1/2} \int_\Omega \| \textstyle\sum_{j=1}^m g_j(\omega) e_j \|_{\ell_2^m}\, d\mu(\omega) \\ &+ b \sup_{y\in B_{\ell_2^m}} \left(\textstyle\sum_{j=1}^m |y_j|^2\right)^{1/2} \int_\Omega \| \textstyle\sum_{i=1}^m g_i(\omega) e_i \|_{\ell_1^m} d\mu(\omega). \end{aligned}$$

Using [Tom89, Proposition 45.1] we have that the last member is less or equal to Dm, where D is a constant. Gathering all together we get that $m^{1/2} \leq Cgl\big(\ell_1^m \otimes_{\varepsilon_2} \ell_2^m\big)$.

The upper estimate follows from the fact that $d(\ell_1^m \otimes_{\varepsilon_2} \ell_2^m, \ell_1^m \otimes_{\varepsilon_2} \ell_\infty^m) \leq d(\ell_2^m, \ell_\infty^m) = m^{1/2}$ (the Banach-Mazur distance, see [Tom89]) together with $\chi(\ell_1^m \otimes_{\varepsilon_2} \ell_\infty^m) = 1$ (since $\ell_1^m \otimes_{\varepsilon_2} \ell_\infty^m \overset{1}{=} \ell_\infty^m(\ell_1^m)$). □

The following proposition shows that the tensor product of three or more spaces lacks the Gordon-Lewis property.

Proposition 5.3.7. *Fix $n \geq 3$ and let δ be an n-fold natural full symmetric tensor norm other than π_n or ε_n. If $E_1, \dots, E_n$ have unconditional bases, then $\big(\widetilde{\otimes}_{i=1}^n E_i, \delta\big)$ does not have the Gordon-Lewis property (nor unconditional basis).*

To prove this proposition we will need next remark which follows from the metric mapping property and the definition of the operation $\underline{(\cdot)}$ (see the comments before Lemma 3.5.3).

Remark 5.3.8. Let $E_1, \dots, E_n$ Banach spaces, $x_j \in S_{E_j}$ $(j = 3, \dots, n)$ and δ a full tensor norm of order n. Then $\big(E_1 \otimes E_2 \otimes [x_3] \otimes \cdots \otimes [x_n], \delta\big)$ is a complemented subspace of $\big(E_1 \otimes \cdots \otimes E_n, \delta\big)$ and this space is isometrically isomorphic to $\big(E_1 \otimes E_2, \widetilde{\delta}\big)$, where $\widetilde{\delta}$ is the 2-fold tensor norm which comes from applying $n-2$ times the operation $\underline{(\cdot)}$ to the norm δ.

Proof. (*of Proposition 5.3.7*)

Let $\tilde{\tilde{\delta}}$ as in Remark 5.3.8; it is easy to show that $\tilde{\tilde{\delta}}$ is the 2-fold natural analogous to δ, thus must be one of the tensor norms that appear in (5.9) below. Recall that nontrivial natural symmetric tensor norms destroy unconditionality, therefore for $p \in \{1, 2, \infty\}$

$$gl\big(\ell_p^m \otimes_{\tilde{\tilde{\delta}}} \ell_p^m\big) \tag{5.8}$$

goes to infinity as m goes to infinity.

On the other hand, we have

$$gl\big(\ell_1^m \otimes_{/\pi_2\backslash} \ell_2^m\big) \asymp gl\big(\ell_1^m \otimes_{/\backslash\varepsilon_2/\backslash} \ell_2^m\big) = gl\big(\ell_\infty^m \otimes_{\backslash/\pi_2\backslash/} \ell_2^m\big) \asymp gl\big(\ell_\infty^m \otimes_{\backslash\varepsilon_2/} \ell_2^m\big), \tag{5.9}$$

and, by the previous lemma, all go to infinity as m goes to infinity.

By Theorem 5.1.8, the spaces E_1, E_2 and E_3 must contain, respectively, uniformly complemented copies of $(\ell_{p_1}^m)_{m=1}^\infty$, $(\ell_{p_2}^m)_{m=1}^\infty$ and $(\ell_{p_3}^m)_{m=1}^\infty$, with $p_1, p_2, p_3 \in \{1, 2, \infty\}$. If p_1, p_2 and p_3 are all different, then they must be $1, 2$ and ∞ in some order. As a consequence, we can choose two of them, say p and q, such that $gl\big(\ell_p^m \otimes_{\tilde{\tilde{\delta}}} \ell_q^m\big)$ goes to infinity as in (5.9), the choice depending on the tensor norm $\tilde{\tilde{\delta}}$. If p_1, p_2 and p_3 are not all different, we choose $p = q$ as two of them that coincide. In this case, $gl\big(\ell_p^m \otimes_{\tilde{\tilde{\delta}}} \ell_q^m\big)$ goes to infinity as in (5.8).

In any case, we have two spaces, say E_1 and E_2, containing respectively ℓ_p^m's and ℓ_q^m's uniformly complemented, so that $gl\big(\ell_p^m \otimes_{\tilde{\tilde{\delta}}} \ell_q^m\big) \to \infty$. Observe that, for fixed $x_j \in S_{E_j}$ ($j = 3, \ldots, n$), the spaces $\ell_p^m \otimes_{\tilde{\tilde{\delta}}} \ell_q^m \overset{1}{=} \big(\ell_p^m \otimes \ell_q^m \otimes [x_3] \otimes \cdots \otimes [x_n], \delta\big)$ are uniformly complemented in $\big(\tilde{\otimes}_{i=1}^n E_i, \delta\big)$ by Remark 5.3.8 and the proof is complete. $\square$

With a similar proof the same result holds for δ an n-fold nontrivial injective (nontrivial projective) full tensor norm such that $\tilde{\tilde{\delta}} \not\sim \varepsilon_2$ ($\tilde{\tilde{\delta}} \not\sim \pi_2$). It is important to note that Proposition 5.3.7 is false for $n = 2$: the space $c_0 \otimes_{/\pi_2\backslash} \ell_2$ has the Gordon-Lewis property. Indeed, $c_0 \otimes_{/\pi_2\backslash} \ell_2 = c_0 \otimes_{\pi_2\backslash} \ell_2 = c_0 \otimes_{d'_\infty} \ell_2$, so if we show that there exists $C > 0$ such that $gl(\ell_\infty^m \otimes_{d'_\infty} \ell_2^m) \leq C$ for every m, we are done. We have

$$gl\big(\ell_\infty^m \otimes_{d'_\infty} \ell_2^m\big) = gl\big(\ell_1^m \otimes_{d_\infty} \ell_2^m\big) = gl\big((\ell_1^m \otimes_{d_\infty} \ell_2^m)'\big) = gl\big(\Pi_1(\ell_1^m, \ell_2^m)\big).$$

In [Sch90], I. Schütt showed that the last expression is uniformly bounded. This fact can be deduced easily in a different way. Well, by Grothendieck's Theorem [DJT95, 1.13] we have $\Pi_1(\ell_1, \ell_2) = \mathcal{L}(\ell_1, \ell_2)$, then $gl\big(\Pi_1(\ell_1^m, \ell_2^m)\big) \asymp gl\big(\mathcal{L}(\ell_1^m, \ell_2^m)\big) = gl\big(\ell_\infty^m \otimes_{\varepsilon_2} \ell_2^m\big)$. Since $\ell_\infty^m \otimes_{\varepsilon_2} \ell_2^m \overset{1}{=} \ell_\infty^m(\ell_2^m)$ and this space has a 1-unconditional basis, therefore the result follows.

We have presented examples of several polynomial ideals that lack the Gordon-Lewis property for any Banach space with unconditional basis. It is easy to obtain the same conclusions for ideals of multilinear forms on a single space. For example, Theorem 5.1.11 gives the following proposition.

Proposition 5.3.9. *Let $\mathfrak{A}^n$ be a Banach ideal of n-linear forms associated with a nontrivial injective or projective tensor norm. If E has unconditional basis, then $\mathfrak{A}^n(E)$ does not have the Gordon-Lewis property.*

From the previous result and Proposition 5.3.7 we have the following statement for the ideal of extendible n-linear forms $\mathcal{L}_e^n$.

Example 5.3.10. *(1) If E is a Banach space with unconditional basis, then $\mathcal{L}_e^n(E)$ does not have the Gordon-Lewis property ($n \geq 2$).*

(2) If $E_1, \dots, E_n$ are Banach spaces with unconditional basis ($n \geq 3$), then $\mathcal{L}_e(E_1, \dots, E_n)$ does not have the Gordon-Lewis property.

On the other hand, the comments after Proposition 5.3.7 show that we cannot expect (2) to hold for $n = 2$. Moreover, the space $\mathcal{L}_e(c_0, \ell_2)$ not only enjoys the Gordon-Lewis property, in fact it has unconditional basis: since $\backslash\varepsilon_2/$ has the Radon-Nikodým property [DF93],

$$\mathcal{L}_e(c_0, \ell_2) = (c_0 \widetilde{\otimes}_{/\pi_2\backslash} \ell_2)' = \ell_1 \widetilde{\otimes}_{\backslash\varepsilon_2/} \ell_2,$$

and therefore $\mathcal{L}_e(c_0, \ell_2)$ has a monomial basis. Since we have shown that $c_0 \otimes_{/\pi_2\backslash} \ell_2$ has the Gordon-Lewis property, this monomial basis must be unconditional.

An example that does not follow from the injective/projective result is the ideal of r-dominated multilinear forms:

Definition 5.3.11. *Let $r \geq n$, an n-linear form $T : E_1 \times \cdots \times E_n \to \mathbb{K}$ is* r-dominated *if there is a constant $C \geq 0$ such that, however we choose finitely many vector $(x_i^j)_{i=1}^m \in E_j$, we have*

$$\Big(\sum_{i=1}^m |T(x_i^1, \dots, x_i^n)|^{r/n}\Big)^{n/r} \leq C w_r\big((x_i^1)_{i=1}^m\big) \dots w_r\big((x_i^n)_{i=1}^m\big).$$

The space of all such T will be denoted $\mathfrak{D}_r(E_1, \dots, E_n)$ with the norm $\delta_r(T) = \min C$.

Since the ideal of r-dominated polynomials $\mathcal{D}_r^n(E)$ is isomorphic to a complemented subspace of $\mathcal{D}_r(E, \dots, E)$ (the ideal of r-dominated n-linear forms), from the polynomial result (Example 5.2.5) we obtain the following.

Example 5.3.12. *Let E be a Banach space with unconditional basis, then the space $\mathcal{D}_r^n(E) := \mathcal{D}_r(E, \dots, E)$ does not have the Gordon-Lewis property.*

Let us mention that, when working with different spaces, we can obtain that dominated multilinear forms behaves exactly as the extendible ones in Example 5.3.10. The case $n = 2$ follows from the coincidence between dominated and extendible bilinear forms. The case $n \geq 3$ is similar to the proof of Proposition 5.3.7, using again that for bilinear forms extendibility is equivalent to domination.

Analogously, just as in the polynomial case, the results for r-integral and r-factorable multilinear forms (with the obvious definitions) can be deduced from the r-dominated case.

We end this section with some remarks on unconditionality for certain Banach operator ideals. We have seen that unconditionality may be present in tensor products of two different spaces, even for tensor norms that destroy unconditionality. Therefore, it is reasonable to expect that, in order to obtain results of "unconditionality destruction" type for operator ideals, certain conditions on the involved spaces must be imposed.

Example 5.3.13. *Let E and F be Banach spaces with unconditional basis such that E' and F have both finite cotype, then $\Gamma_{p,q}(E, F)$ does not have the Gordon-Lewis property.*

Proof. By Theorem 5.1.8 we know that, for $r \in \{2, \infty\}$ and $s \in \{1, 2\}$, E and F contain the uniformly complemented sequences $(\ell_r^m)_{m=1}^\infty$, $(\ell_s^m)_{m=1}^\infty$ respectively. This easily implies that $\Gamma_{p,q}(E, F)$ must contain the uniformly complemented sequence $(\Gamma_{p,q}(\ell_r^m, \ell_s^m))_{m=1}^\infty$. Therefore, if show that $gl\big(\Gamma_{p,q}(\ell_r^m, \ell_s^m)\big) \to \infty$ as $m \to \infty$ we are done.

By [DF93, 17.10] we know that $(\Gamma_{p,q}, \gamma_{p,q})$ is a maximal operator ideal associated with the tensor norm $\gamma_{p,q}$ of Lapresté (see [DF93, 12.5] for definitions). Thus,

$$\Gamma_{p,q}(\ell_r^m, \ell_s^m) = \ell_{r'}^m \otimes_{\gamma_{p,q}} \ell_s^m.$$

Now by [DF93, Exercise 31.2. (a)] we have

$$gl\big(\Gamma_{p,q}(\ell_r^m, \ell_s^m)\big) = gl\big(\ell_{r'}^m \otimes_{\gamma_{p,q}} \ell_s^m\big) \asymp gl\big(\ell_{r'}^m \otimes_{/\pi_2\backslash} \ell_s^m\big)$$

which goes to infinity as $m \to \infty$ (this is a direct consequence of the proof of Proposition 5.2.1 for $\mathcal{P}_e^2$ and Lemma 5.3.6). □

In particular, for $1 < r < \infty$ and $1 \le s < \infty$ the spaces $\Gamma_{p,q}(\ell_r, \ell_s)$ and $\Gamma_{p,q}(c_0, \ell_s)$ do not have the Gordon-Lewis property. The case $r = \infty$ and $1 \le s < \infty$ can be established just following the previous proof. In fact, proceeding as above and using [Sch78, Proposition 7], something more can be stated: for $2 \le r \le \infty$ and $1 \le s \le 2$, if E and F be Banach spaces such that E contains the sequence $(\ell_r^m)_{m=1}^\infty$ uniformly complemented and F contains the sequence $(\ell_s^m)_{m=1}^\infty$ uniformly complemented, then $\Gamma_{p,q}(E, F)$ does not have the Gordon-Lewis property. Note that in this case, we do not require that E nor F have unconditional bases.

Let us now introduce a classical operator ideal.

The ideal of (p,q)-dominated operators [DF93, Section 19]: Let $p, q \in [1, +\infty]$ such that $1/p + 1/q \le 1$. An operator $T : E \to F$ is (p, q) *dominated* if for every $m \in \mathbb{N}$, $x_1, \dots, x_m \in E$ and $y_1', \dots, y_m' \in F'$ there exist a constant $C \ge 0$ such that:

$$\ell_r(< y_k', Tx_k >) \le C w_p(x_k) w_q(y_k'),$$

where $1/p + 1/q + 1/r' = 1$. We denote the space of all such operators by $\mathcal{D}_{p,q}(E, F)$ with the norm $D_{p,q}(T)$ being the minimum of these C. Equivalently, $T \in \mathcal{D}_{p,q}(E, F)$ if there are a constant $B \ge 0$ and probability measures μ and ν such that

$$| < y', Tx > | \le B\Big(\int_{B_{E'}} | < x', x > |^p \mu(dx')\Big)^{1/p}\Big(\int_{B_{F''}} | < y'', y' > |^q \nu(dy'')\Big)^{1/q},$$

holds for all $x \in E$ and $y' \in F'$, (replace the integral by $\|\ \|$ if the exponent is ∞). In this case, the (p, q)-dominated norm of T, $D_{p,q}(T)$, is the infimum of the constants B for which the previous inequality hold (see [DF93, Corollary 19.2.]). If $1/p + 1/q = 1$, $\mathcal{D}_{p,q}$ coincides isometrically with the classical ideal of p-dominated operators [DJT95, Chapter 9].

By [DF93, Sections 17 and 19] we know that the ideal of (p,q)-dominated operators [DF93, Section 19] $\mathcal{D}_{p,q}$ is the adjoint of $\Gamma_{p',q'}$, the ideal of (p', q')-factorable operators. Using the duality that this implies on finite dimensional spaces, we can deduce the following.

Example 5.3.14. *Let E and F be Banach spaces with unconditional basis such that E and F' have both finite cotype, then $\mathcal{D}_{p,q}(E, F)$ does not have the Gordon-Lewis property.*

As above, we can see that for $1 \leq r \leq 2$ and $2 \leq s \leq \infty$, if E contains the sequence $(\ell_r^m)_{m=1}^\infty$ uniformly complemented and F contains the sequence $(\ell_s^m)_{m=1}^\infty$ uniformly complemented, then $\mathcal{D}_{p,q}(E,F)$ does not have the Gordon-Lewis property.

We have, in particular, that for $1 \leq r < \infty$ and $1 \leq s \leq \infty$ the spaces $\mathcal{D}_{p,q}(\ell_r, \ell_s)$ and $\mathcal{D}_{p,q}(\ell_r, c_0)$ do not have the Gordon-Lewis property.

Let us give a procedure to obtain more examples: if $\mathcal{U}$ is a Banach operator ideal and δ is its associated tensor norm, by $\mathcal{U}^{inj\ sur}$ we denote the maximal operator ideal associated with the norm $/\delta\backslash$ [DF93, Sections 9.7 and 9.8]. Using the ideas of Example 5.3.13 and the fact that $/\delta\backslash \leq /\pi_2\backslash$, we have:

Example 5.3.15. *Let E and F be Banach spaces with unconditional basis such that E' and F have both finite cotype, then $\mathcal{U}^{inj\ sur}(E,F)$ does not have the Gordon-Lewis property.*

For example, let us consider $\mathcal{U}$ to be the ideal of (p,q)-factorable operators $\Gamma_{p,q}$. An operator T belongs to $\Gamma_{p,q}^{inj\ sur}(E,F)$ if and only if there is a constant $C \geq 0$ such that for all natural numbers $m \in \mathbb{N}$, all matrices $(a_{k,l})$, all $x_1, \dots, x_m \in E$ and all $y'_1, \dots, y'_m \in F'$

$$\Big|\sum_{k,l=1}^{m} a_{k,l} < y'_k, Tx_l > \Big| \leq C \|(a_{k,l}) : \ell_{p'}^m \to \ell_q^m\| \ell_{p'}(x_l) \ell_{q'}(y'_k).$$

In this case, $\gamma_{p,q}^{inj\ sur}(T) := \min C$ (see [DF93, Theorem 28.4]).

Chapter 6

Structures in the symmetric tensor product

In the previous chapter we have shown that the unconditionality structure is not preserved in general for the symmetric tensor product. Now we devote our efforts to study the preservation of certain structures for specific s-tensor norms. Namely, the Banach algebra structure and the M-ideal structure.

In Section 6.1, we describe which natural s-tensor norms preserve the algebra structure. Based on the work of Carne [Car78], we show that the two s-tensor norms preserving Banach algebras are $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$.

In Section 6.2 we show that the M-ideal structure is destroyed by $\varepsilon_{n,s}$ for every n. More precisely, we prove that for real Banach spaces E and F, if E is a non trivial M-ideal in F, then $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ is *never* an M-ideal in $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} F$. This result marks up a difference with the behavior of full tensors since, when E is an M-ideal in F, it is known that $\widetilde{\otimes}^{n}_{\varepsilon_n} E$ is an M-ideal in $\widetilde{\otimes}^{n}_{\varepsilon_n} F$. Even though the M-structure for symmetric tensors fails, one may wonder whether the consequence about unique norm preserving extensions holds. That is, being E a non trivial M-ideal in F, has every integral n-homogeneous polynomial in E a unique extension to F that preserves the integral norm? We give in Theorem 6.2.9 a positive answer for the case of E being an Asplund space and describe explicitly this unique extension.

6.1 s-tensor norms preserving Banach algebra structures

Carne in [Car78] showed that there are exactly four natural 2-fold tensor norms that preserve Banach algebras, two of which are symmetric: π_2 and $\backslash\varepsilon_2/$. Based on his work we describe which natural s-tensor norms preserve this structure.

If A is a Banach algebra the n-fold symmetric tensor product $\otimes^{n,s} A$ inherits a natural algebraic structure give by

$$\Big(\sum_{j=1}^{r} \otimes^n x_j\Big) \cdot \Big(\sum_{k=1}^{s} \otimes^n y_k\Big) = \sum_{j=1}^{r}\sum_{k=1}^{s} \otimes^n (x_j \cdot y_k).$$

For a given Banach algebra A we denote $m(A) : A \otimes_{\pi_2} A \to A$ the map induced by the multiplication $A \times A \to A$. The following theorem is a symmetric version of Carne [Car78, Theorem 1]. Its proof is obtained by adapting the one in [Car78] for the symmetric setting.

Theorem 6.1.1. *For an s-tensor norm α of order n the following conditions are equivalent:*

(1) If A is Banach algebra, the n-fold symmetric tensor product $\widetilde{\otimes}_\alpha^{n,s} A$ is a Banach algebra with the natural algebra structure.

(2) For all Banach spaces E and F there is a natural continuous linear map

$$f : \big(\otimes_\alpha^{n,s} E \big) \otimes_{\pi_2} \big(\otimes_\alpha^{n,s} F \big) \to \big(\otimes_\alpha^{n,s} (E \otimes_{\pi_2} F) \big)$$

with

$$f\big((\otimes^n x) \otimes (\otimes^n y) \big) = \otimes^n (x \otimes y).$$

(3) For all Banach spaces E and F there is a natural continuous map

$$g : \big(\otimes_{\alpha'}^{n,s} (E \otimes_{\varepsilon_2} F) \big) \to (\otimes_{\alpha'}^{n,s} E) \otimes_{\varepsilon_2} (\otimes_{\alpha'}^{n,s} F)$$

given by

$$g\big(\otimes^n (x \otimes y) \big) = (\otimes^n x) \otimes (\otimes^n y).$$

(4) For all Banach spaces E and F there is a natural continuous map

$$h : \otimes_{\alpha'}^{n,s} \mathcal{L}(E, F) \to \mathcal{L}(\otimes_\alpha^{n,s} E, \otimes_{\alpha'}^{n,s} F),$$

with

$$h(\otimes^n T)(\otimes^n x) = \otimes^n (Tx).$$

If one, hence all, of the above hold, then there are constants c_1, c_2, c_3, c_4 so that

(1) $\|m(\widetilde{\otimes}_\alpha^{n,s} A)\| \leq c_1 \|m(A)\|^n$.

(2) $\|f\| \leq c_2$ for all E and F.

(3) $\|g\| \leq c_3$ for all E and F.

(4) $\|h\| \leq c_4$ for all E and F.

and the least values of these four agree.

If the s-tensor norm α preserves Banach algebras, then we call the common least value of the constants in the theorem, *the Banach algebra constant* of α.

An important comment is in order: if we take $E = F$ and $T = id_E$ in (4), then we obtain $\|h(\otimes^{n,s} id_E)\| \leq c_4$. But it is plain that $h(\otimes^n id_E)$ is just $id_{\otimes^{n,s} E}$. Therefore, we have

$$\|id_{\otimes^{n,s} E} : \otimes_\alpha^{n,s} E \to \otimes_{\alpha'}^{n,s} E\| \leq c_4,$$

which means that $\alpha' \leq c_4 \alpha$. So we can state the following remark.

Remark 6.1.2. *If α is an s-tensor norm which preserves Banach algebras there is a constant n such that $\alpha' \leq k\alpha$.*

The following Theorem is the main result of this section. The proof that $\pi_{n,s}$ preserves Banach algebra is similar to one for π_2 in [Car78], and we include it for completeness.

Theorem 6.1.3. *The only natural s-tensor norms of order n which preserves Banach algebras are: $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$. Furthermore, the Banach algebra constants of both norm are exactly one.*

It follows from Theorem 3.5.2 and Remark 6.1.2 that $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$ are the only candidates among natural s-tensor norms to preserve Banach algebras.

First we prove that π_s preserves Banach algebra. By Theorem 6.1.1, it is enough to show, for any pair of Banach spaces E and F, that the mapping

$$f : \big(\otimes^{n,s}_{\pi_{n,s}} E\big) \otimes_{\pi_2} \big(\otimes^{n,s}_{\pi_{n,s}} F\big) \to \big(\otimes^{n,s}_{\pi_{n,s}} (E \otimes_{\pi_2} F)\big)$$

defined by

$$f\big((\otimes^n x) \otimes (\otimes^n y)\big) = \otimes^n(x \otimes y),$$

has norm less than or equal to 1. Fix $\varepsilon > 0$. Given $w \in \big(\otimes^{n,s} E\big) \otimes \big(\otimes^{n,s} F\big)$, we can write it as

$$w = \sum_{i=1}^{r} u_i \otimes v_i,$$

with

$$\sum_{i=1}^{r} \pi_{n,s}(u_i)\pi_{n,s}(v_i) \leq \pi_2(w)(1+\varepsilon)^{1/3}.$$

Also, for each $i = 1, \dots, r$ we write u_i and v_i as

$$u_i = \sum_{j=1}^{J(i)} \otimes^n x_j^i \in \otimes^{n,s} E, \quad v_i = \sum_{k=1}^{K(i)} \otimes^n y_k^i \in \otimes^{n,s} F,$$

with

$$\sum_{j=1}^{J(i)} \|x_j^i\|^n \leq \pi_{n,s}(u_i)(1+\varepsilon)^{1/3}, \quad \sum_{k=1}^{K(i)} \|y_k^i\|^n \leq \pi_{n,s}(v_i)(1+\varepsilon)^{1/3}.$$

We have

$$f(w) = \sum_{i=1}^{r} \sum_{\substack{1\leq j\leq J(i)\\ 1\leq k\leq K(i)}} \otimes^n(x_j^i \otimes y_k^i),$$

and then

$$
\begin{aligned}
\pi_{n,s}(f(w)) &\leq \sum_{i=1}^{r} \sum_{\substack{1\leq j\leq J(i)\\ 1\leq k\leq K(i)}} \pi_2(x_j^i \otimes y_k^i)^n \\
&= \sum_{i=1}^{r} \sum_{\substack{1\leq j\leq J(i)\\ 1\leq k\leq K(i)}} \|x_j^i\|^n \|y_k^i\|^n \\
&= \sum_{i=1}^{r} \Big(\sum_{j\leq J(i)} \|x_j^i\|^n \Big) \Big(\sum_{k\leq K(i)} \|y_k^i\|^n \Big) \\
&= \sum_{i=1}^{r} \pi_{n,s}(u_i)(1+\varepsilon)^{1/3} \pi_{n,s}(v_i)(1+\varepsilon)^{1/3} \\
&= (1+\varepsilon)^{2/3} \sum_{i=1}^{r} \pi_2(u_i)\pi_2(v_i) \leq (1+\varepsilon)\pi_2(w).
\end{aligned}
$$

From this we conclude that $\|f\| \leq 1$.

To prove that $\backslash/\pi_{n,s}\backslash/$ preserves Banach algebras we need two technical lemmas.

Lemma 6.1.4. *Let Y and Z be Banach spaces. The operator*

$$\phi : \otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}(\ell_1(B_Y), Z) \to \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash}\, \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} Z\big)$$

given by

$$\phi(\otimes^n T)(\otimes^n u) = \otimes^n Tu,$$

has norm less than or equal to 1.

Proof. The mapping

$$
\begin{array}{rcl}
\mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big) & \to & \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash}\, \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} Z\big) \\
T & \mapsto & \otimes^n T
\end{array}
$$

is an n-homogeneous polynomial, which has norm one by the metric mapping property of the norm $/\pi_{n,s}\backslash$. As a consequence, its linearization is a norm one operator from the s-tensor product $\otimes^{n,s}_{\pi_{n,s}} \mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big)$ to $\mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} Z\big)$. Since $\mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big)$ is an $\mathcal{L}_\infty$ space, by Corollary 3.2.8 we have

$$\otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big) \overset{1}{=} \otimes^{n,s}_{\pi_{n,s}} \mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big).$$

This shows that the canonical mapping

$$\otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big) \longrightarrow \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash}\, \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_\infty(B_{Z'})\big)$$

has norm 1.

On the other hand, the following diagram commutes

$$\begin{tikzcd}
\otimes^{n,s}_{/\pi_{n,s}\backslash}\mathcal{L}\big(\ell_1(B_Y),\ell_\infty(B_{Z'})\big) \arrow[r] & \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_1(B_Y),\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_\infty(B_{Z'})\big) . \\
\otimes^{n,s}_{/\pi_{n,s}\backslash}\mathcal{L}(\ell_1(B_Y),Z) \arrow[u, hook] \arrow[r, "\phi"] & \mathcal{L}(\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_1(B_Y),\otimes^{n,s}_{/\pi_{n,s}\backslash}Z) \arrow[u, hook]
\end{tikzcd}$$

Here the vertical arrows are the natural inclusion, which are actually isometries since the norm $/\pi_{n,s}\backslash$ is injective. The horizontal arrow above is the canonical mappings whose norm was shown to be one. Therefore, the norm of ϕ must be less than or equal to one. $\square$

Before we state our next lemma, we observe that linear operators from X_1 to $\mathcal{L}(X_2,X_3)$ identify (isometrically) with bilinear operators from $X_1\times X_2$ to X_3 and, consequently, with linear operators from $X_1\otimes_\pi X_2$ to X_3. The isometry is given by

$$\begin{array}{rcl}\mathcal{L}(X_1,\mathcal{L}(X_2,X_3)) & \to & \mathcal{L}(X_1\otimes_\pi X_2,X_3)\\ T & \mapsto & B_T,\end{array} \tag{6.1}$$

where $B_T(x_1\otimes x_2)=T(x_1)(x_2)$.

Lemma 6.1.5. *Let E and F be Banach spaces. The operator*

$$\rho:\big(\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_1(B_E)\big)\otimes_{\pi_2}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_1(B_F)\big)\to\otimes^{n,s}_{/\pi_{n,s}\backslash}\big(\ell_1(B_E)\otimes_{\pi_2}\ell_1(B_F)\big)$$

given by

$$\rho\big((\otimes^n u)\otimes(\otimes^n v)\big)=\otimes^n(u\otimes v),$$

has norm less than or equal to 1.

Proof. If we take $Y=F$ and $Z=\ell_1(B_E)\otimes_{\pi_2}\ell_1(B_F)$ in Lemma 6.1.4, we see that the operator

$$\phi:\otimes^{n,s}_{/\pi_{n,s}\backslash}\mathcal{L}(\ell_1(B_F),\ell_1(B_E)\otimes_{\pi_2}\ell_1(B_F))\to\mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_1(B_E),\otimes^{n,s}_{/\pi_{n,s}\backslash}(\ell_1(B_E)\otimes_{\pi_2}\ell_1(B_F))\big)$$

has norm at most 1. Also the application $J:\ell_1(B_E)\to\mathcal{L}\big(\ell_1(B_F),\ell_1(B_E)\otimes_{\pi_2}\ell_1(B_F)\big)$ defined by $Jz(w)=z\otimes w$ has norm 1. Hence, the norm of the map $\psi:=\phi\circ\otimes^{n,s}J$ between the corresponding $/\pi_{n,s}\backslash$-tensor products is at most one.

Now, with the identification given in (6.1), the operator ρ is precisely B_ψ and therefore we conclude that ρ has norm at most one. $\square$

Now we are ready to prove that $\backslash/\pi_{n,s}\backslash/$ preserves Banach algebras with Banach algebra constant 1. Again by Theorem 6.1.1, it is enough to show that, for Banach spaces E and F, the map

$$f:\big(\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/}E\big)\otimes_{\pi_2}\big(\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/}F\big)\to\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/}(E\otimes_{\pi_2}F)$$

defined by

$$f\big((\otimes^n x)\otimes(\otimes^n y)\big)=\otimes^n(x\otimes y),$$

has norm at most one. The following diagram, where the vertical arrows are the canonical quotient maps, commutes:

$$\begin{tikzcd}
\left(\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_1(B_E)\right)\otimes_{\pi_2}\left(\otimes^{n,s}_{/\pi_{n,s}\backslash}\ell_1(B_F)\right) \arrow[r, "\rho"] \arrow[d] & \left(\otimes^{n,s}_{/\pi_{n,s}\backslash}\left(\ell_1(B_E)\otimes_{\pi_2}\ell_1(B_F)\right)\right) . \arrow[d] \\
\left(\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/}E\right)\otimes_{\pi_2}\left(\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/}F\right) \arrow[r, "f"] & \left(\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/}(E\otimes_{\pi_2}F)\right)
\end{tikzcd}$$

By the previous Lemma, ρ has norm less than or equal to one, and so is the norm of f, since the other mappings are quotients. We have finished the proof of Theorem 6.1.3.

We end the section with a new perspective on the the n-fold symmetric analogue of the classical norm w_2' for $n \geq 3$:

The 2-fold tensor norms π_2 and $\backslash\varepsilon_2/$ which is equivalent w_2' (the dual of the norm associated the classical ideal Γ_2 of 2-factorable operator, see Definition 3.4.3) share two characteristic properties. The first property is that they dominate their dual tensor norm. Indeed, the inequality $\pi_2' = \varepsilon_2 \leq \pi_2$ is clear, and we see in [DF93, 27.2] that w_2 is dominated by w_2' (or, analogously, $/\pi_2\backslash$ is dominated by $\backslash\varepsilon_2/$). The second property is that both π_2 and w_2' preserve the Banach algebra structure [Car78]. These two properties are enjoyed, of course, by their corresponding 2-fold s-tensor norms. As we have already seen, the n dimensional analogue of the s-tensor norm $\backslash\varepsilon_{2,s}/$ splits into two non-equivalent ones when passing from tensor products of order 2 to tensor products of order $n \geq 3$. Namely, $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$. It is remarkable that the two mentioned properties are enjoyed only by $\backslash/\pi_{n,s}\backslash/$ and not by $\backslash\varepsilon_{n,s}/$, as seen in Theorem 3.5.2 and Theorem 6.1.3. Therefore, we could say that, in some sense, the n-fold symmetric analogue of w_2' for $n \geq 3$ should be $\backslash/\pi_{2,s}\backslash/$ rather than the simpler (and probably nicer) $\backslash\varepsilon_{2,s}/$.

6.2 Preservation of the M-ideal structure and unique norm preserving extensions

In 1972, Alfsen and Effros [AE72] introduced the notion of an M-ideal in a Banach space. Recall the following definition.

Definition 6.2.1. A closed subspace E of a Banach space F is an M-*ideal* in F if

$$F' = E^\sharp \oplus_1 E^\perp,$$

where $E^\sharp$ is a closed subspace of F' and $E^\perp$ is the annihilator of E.

Since $E^\sharp$ can be (isometrically) identified with E', it is usual to denote $F' = E' \oplus_1 E^\perp$. However, we often prefer to state explicitly the isomety $s : E' \to F'$, thus obtaining the decomposition $F' = s(E') \oplus_1 E^\perp$. The space E is said to be M-*embedded* if E is an M-ideal in its bidual E''. The presence of an M-ideal E in a Banach space F in some way expresses that the norm of F is a sort of maximum norm (hence the letter M).

A number of authors have examined M-ideal structures in tensor products, operator spaces, spaces of polynomials or Banach algebras (see e.g., D. Werner [Wer88], W. Werner [Wer87], Dimant [Dim11] and Harmand-Werner-Werner [HWW93] and the references therein).

It is well known, that if E is an M-ideal in F then the full tensor product $\widetilde{\bigotimes}^n_{\varepsilon_n} E$ is an M-ideal in $\widetilde{\bigotimes}^n_{\varepsilon_n} F$ (use [HWW93, Proposition VI.3.1], the associativity of the injective norm and the transitivity of M-ideals). Since most of the results of the theory of tensor products and tensor norms have their natural analogue in the symmetric context, one should expect that whenever E is a non trivial M-ideal in F, then $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ would be an M-ideal in $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} F$. Surprisingly, we see in Theorem 6.2.7 that, for real Banach spaces, this never happens. To prove this, we make use of a characterization of the extreme points of the unit ball of the space of integral polynomials over real Banach spaces, which is interesting in its own right. We therefore devote some time to recall definitions, known results and remarks on extreme points of the ball.

A point $x \in B_E$ is said to be a *real extreme point* whenever $\{x+\zeta y : |\zeta| \leq 1, \zeta \in \mathbb{R}\} \subset B_E$ for $y \in E$ implies $y = 0$. Analogously, a point $x \in B_E$ is said to be a *complex extreme point* whenever $\{x + \zeta y : |\zeta| \leq 1, \zeta \in \mathbb{C}\} \subset B_E$ for $y \in E$ implies $y = 0$. In complex Banach spaces, it is easy to check that every real extreme point of B_E is also a complex extreme point. The converse however is not true, since, for instance, every point of S_{ℓ_1} is a complex extreme point of B_{ℓ_1}. We denote by $Ext(B_E)$ the set of real extreme points of the ball B_E.

Ruess-Stegall [RS82], Ryan-Turett [RT98], Boyd-Ryan [BR01], Dineen [Din03] and Boyd-Lassalle [BL10] in their investigations studied the extreme points of the unit ball of the space of (integral) polynomials defined on a Banach space. In [BR01] the authors showed the following facts:

(a) For a real Banach space E, $\{\pm(x')^n : x' \in S_{E'}$ and x' attains its norm $\} \subseteq Ext(B_{\mathcal{P}^n_I(E)})$.

(b) For a real or complex Banach space E, $Ext(B_{\mathcal{P}^n_I(E)}) \subseteq \{\pm(x')^n : x' \in S_{E'}\}$ (see also [CD00]).

Time after, Boyd and Lassalle proved in [BL10] the following result: if E is a real Banach space, E' has the approximation property and $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ does not contain a copy of ℓ_1, then $Ext(B_{\mathcal{P}^n_I(E)})$ is $\{\pm(x')^n : x' \in S_{E'}\}$. In the following theorem we use the Aron-Berner extension to show that the hypotheses of their result can be removed.

Theorem 6.2.2. *For a real Banach space E, the set of real extreme points of the unit ball of $\mathcal{P}^n_I(E)$ is $\{\pm(x')^n : x' \in S_{E'}\}$.*

Proof. Let $x' \in S_{E'}$. Since x' is a norm attaining element of $S_{E'''}$, by part (a) of the the previous comment, $(x')^n$ is an extreme point of the unit ball of $\mathcal{P}^n_I(E'')$. We use the fact that $Ext(B) \cap A \subseteq Ext(A)$ whenever $A \subseteq B$. Consider the isometric inclusion

$$AB : \mathcal{P}^n_I(E) \overset{1}{\hookrightarrow} \mathcal{P}^n_I(E'')$$

given by the Aron-Berner extension morphism. It is not difficult to prove that AB maps $(x')^n$ to $(x')^n$ seen as an n-homogeneous polynomial over E'' or, more precisely, $(\kappa_{E'}(x'))^n)$. Thus,

$$Ext(B_{\mathcal{P}^n_I(E'')}) \cap B_{\mathcal{P}^n_I(E)} \subseteq Ext(B_{\mathcal{P}^n_I(E)}).$$

Finally,

$$\{\pm(x')^n : x' \in S_{E'}\} \subseteq Ext(B_{\mathcal{P}_I^n(E'')}) \cap B_{\mathcal{P}_I^n(E)} \subseteq Ext(B_{\mathcal{P}_I^n(E)}) \subseteq \{\pm(x')^n : x' \in S_{E'}\},$$

and this concludes the proof. $\square$

We now state some observations on the above result.

Remark 6.2.3. Theorem 6.2.2 is not true for complex Banach spaces. Indeed, Dineen [Din03, Proposition 4.1] proved that, if E is a complex Banach space, then $Ext(B_{\mathcal{P}_I^n(E)})$ is contained in $\{(x')^n : x'$ is a complex extreme point of $B_{E'}\}$. Let us consider E the complex space ℓ_1. It is clear that $x' = (0, 1, \dots, 1, \dots) \in \ell_\infty$ is not a complex extreme point of B_{ℓ_∞}. Hence, $(x')^n$ is not an extreme point of $B_{\mathcal{P}_I(^n\ell_1)}$.

Remark 6.2.4. Although the spaces $\mathcal{P}_I^n(E)$ and $\mathcal{L}_I(^nE)$ can be isomorphic (for example if E is stable [AF98]), they are very different from a geometric point of view since the set $Ext(B_{\mathcal{L}_I(^nE)})$ is equal to $\{x'_1 x'_2 \cdots x'_k : x'_i \in Ext(B_{E'})\}$ (see [BR01, RS82]).

Remark 6.2.5. As it will be stated in Lemma 6.2.17, in a maximal ideal of polynomials, the w^*-convergence of a bounded net is equivalent to the pointwise convergence. So, from Theorem 6.2.2, if $\mathcal{Q}$ is a maximal ideal of n-homogeneous polynomials that satisfies that, on some real Banach space E, the set of real extreme points of its unit ball is $\{\pm(x')^n : x' \in S_{E'}\}$, then we should have $\mathcal{Q}(E) = \mathcal{P}_I^n(E)$.

We now return to our main goal: exhibit that the M-ideal structure is destroyed. The last characterization of the real extreme points of the ball of integral polynomials leads us to show that for a real Banach space E, $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ is never an M-ideal in $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E''$, unless E is reflexive. As we have already said, this is a big difference with what happens in the non symmetric case where, for E an M-embedded space, it follows that the full tensor product $\widetilde{\bigotimes}^n_{\varepsilon_n} E$ is an M-ideal in $\widetilde{\bigotimes}^n_{\varepsilon_n} E''$.

Theorem 6.2.6. *If the real Banach space E is not reflexive, then $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ is not an M-ideal in $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E''$.*

Proof. Suppose that $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ is an M-ideal in $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E''$. Then, by [HWW93, Lemma I.1.5] we would have:

$$Ext(B_{\mathcal{P}_I(^nE'')}) = Ext(B_{\mathcal{P}_I^n(E)}) \cup Ext\Big(B_{(\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E)^\perp}\Big).$$

By the description of the real extreme points of integral polynomials given in Theorem 6.2.2, this equality would imply

$$Ext\Big(B_{(\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E)^\perp}\Big) = \{\pm(x''')^n : x''' \in S_{E'''} \setminus S_{E'}\}.$$

This is not possible since through the decomposition $E''' = E' \oplus E^\perp$ if we choose $x''' \in S_{E'''}$ such that $x''' = x'''_1 + x'''_2$, with $x'''_1 \in E'$, $x'''_2 \in E^\perp$, $x'''_1, x'''_2 \neq 0$, then $x''' \in S_{E'''} \setminus S_{E'}$ but $(x''')^n \notin (\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E)^\perp$. This finishes the proof. $\square$

With almost the same argument (only changing the decomposition $E''' = E' \oplus E^\perp$ to $F' = E' \oplus_1 E^\perp$) we derive the following theorem.

Theorem 6.2.7. *If E and F are real Banach spaces and E is a nontrivial M-ideal in F, then $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} E$ is not an M-ideal in $\widetilde{\otimes}^{n,s}_{\varepsilon_{n,s}} F$.*

As it is quoted in the book written by Harmand, Werner and Werner [HWW93]: "The fact that E is an M-ideal in F has a strong impact on both F and E since there are a number of important properties shared by M-ideals, but not by arbitrary subspaces". One of the interesting properties shared by M-ideals is the following: if E is an M-ideal in F then every linear functional defined in E has a *unique* norm preserving extension to a functional in F' [HWW93, Proposition I.1.12]. For E an M-ideal in F, we know that the full injective tensor product $\widetilde{\bigotimes}^n_{\varepsilon_n} E$ is an M-ideal in $\widetilde{\bigotimes}^n_{\varepsilon_n} F$. Hence, any n-linear integral form on E (being an element of the dual of $\widetilde{\bigotimes}^n_{\varepsilon_n} E$) has a unique (integral) norm preserving extension to a n-linear integral form on F.

Now that we are aware that the M-structure for symmetric tensors fails, we can wonder about a weaker property: whether the unique norm preserving extension property holds. That is, being E a non trivial M-ideal in F, has every integral n-homogeneous polynomial in E a unique extension to F that preserves the integral norm? We give in Theorem 6.2.9 a positive answer for the case of E being Asplund. Since M-embedded spaces are Asplund [HWW93, Theorem III.3.1], if E is M-embedded then the Aron-Berner extension is the unique norm preserving extension to E''.

Let E be an M-ideal in F; note that in this case the natural inclusion $s : E' \to F'$ induces, according Definition 2.2.8, a canonical isometry $\overline{s} : \mathcal{P}^n_N(E) \to \mathcal{P}^n_N(F)$ given by

$$\overline{s}(p) := AB(p) \circ s' \circ \kappa_F.$$

To be precise, if $p \in \mathcal{P}^n_N(E)$, we have

$$\begin{aligned}
\|\overline{s}(p)\|_{\mathcal{P}^n_N(F)} &= \|AB(p) \circ s' \circ \kappa_F\|_{\mathcal{P}^n_N(F)} \\
&\leq \|AB(p)\|_{\mathcal{P}^n_N(E'')} \|s'\|^n \|\kappa_F\|^n \\
&= \|p\|_{\mathcal{P}^n_N(E)} \\
&\leq \|\overline{s}(p)\|_{\mathcal{P}^n_N(F)},
\end{aligned}$$

where the second equality is due to Theorem 2.2.6 and the last inequality is clear since $\overline{s}(p)$ is actually an extension of p.

Aron, Boyd and Choi [ABC01, Proposition 7] proved that if E is an M-ideal in E'' then the Aron-Berner extension is the unique norm preserving extension from $\mathcal{P}^n_N(E)$ to $\mathcal{P}^n_N(E'')$. Their argument can be easily adapted to the situation of E being an M-ideal in F. We therefore have the following result.

Proposition 6.2.8. *Let E be an M-ideal in F and let $s : E' \to F'$ be the associated isometric inclusion. For each $p \in \mathcal{P}^n_N(E)$, $\overline{s}(p)$ is the unique norm preserving extension to $\mathcal{P}^n_N(F)$.*

We want to prove a similar statement for integral polynomials. If E is an Asplund space (which always holds when E is an M-ideal in E'') we have a positive result. In this case, nuclear and integral polynomials over E coincide isometrically as we saw in Corollary 4.1.4. So, by the previous proposition, there is only one nuclear norm preserving extension to F. But if F is not Asplund we could presumably have integral non nuclear extensions of the same integral norm. We show that this is impossible.

Theorem 6.2.9. *Let E be an Asplund space which is an M-ideal in a Banach space F and let $s : E' \to F'$ be the associated isometric inclusion. If $p \in \mathcal{P}_I^n(E)$ then the canonical extension $\overline{s}(p)$ is the unique norm preserving extension to $\mathcal{P}_I^n(F)$.*

To prove Theorem 6.2.9 we need the following lemma.

Lemma 6.2.10. *Let E be an Asplund space which is a subspace of a Banach space F and let q be a fixed polynomial in $\mathcal{P}_I^n(F)$. Given $\varepsilon > 0$ there exists $\widetilde{q} \in \mathcal{P}_N^n(F)$ such that q and $\widetilde{q}$ coincide on E and*

$$\|\widetilde{q}\|_{\mathcal{P}_N^n(F)} \leq \|q\|_{\mathcal{P}_I^n(F)} + \varepsilon.$$

Proof. Since the restriction of q to E is nuclear, we can take sequences $(x'_j)_j \subset E'$ and $(\lambda_j)_j \subset \mathbb{K}$ such that $q|_E = \sum_{j=1}^\infty \lambda_j (x'_j)^n$ and

$$\begin{aligned}\sum_{j=1}^\infty |\lambda_j| \|x'_j\|^n &\leq \|q|_E\|_{\mathcal{P}_N^n(E)} + \varepsilon \\ &= \|q|_E\|_{\mathcal{P}_I^n(E)} + \varepsilon \\ &\leq \|q\|_{\mathcal{P}_I^n(F)} + \varepsilon.\end{aligned}$$

For each j, let y'_j be a Hahn-Banach extension of x'_j to F. If we define $\widetilde{q} = \sum_{j=1}^\infty \lambda_j (y'_j)^n$, then $\widetilde{q}$ coincides with q in E and

$$\|\widetilde{q}\|_{\mathcal{P}_N^n(F)} \leq \sum_{j=1}^\infty |\lambda_j| \|y'_j\|^n = \sum_{j=1}^\infty |\lambda_j| \|x'_j\|^n \leq \|q\|_{\mathcal{P}_I^n(F)} + \varepsilon.$$

This ends the proof. □

Now we are ready to prove Theorem 6.2.9.

Proof. (*of Theorem 6.2.9.*)

The argument is modeled on the proof of [ABC01, Proposition 7]. We include all the steps for the sake of completeness.

Let $p \in \mathcal{P}_I^n(E)$ and suppose there exists a norm preserving extension $q \in \mathcal{P}_I^n(F)$ different from $\overline{s}(p)$. Pick y a norm one vector in F such that $0 < \delta = |q(y) - \overline{s}(p)(y)|$.

Note that $E \oplus [y]$ is an Asplund space since E also is. So, by Lemma 6.2.10 applied to $E \oplus [y]$, there exists $\widetilde{q} \in \mathcal{P}_N^n(F)$ such that q and $\widetilde{q}$ coincide on $E \oplus [y]$ and

$$\begin{aligned}\|\widetilde{q}\|_{\mathcal{P}_N^n(F)} &\leq \|q\|_{\mathcal{P}_I^n(F)} + \frac{\delta}{4} \\ &= \|p\|_{\mathcal{P}_I^n(E)} + \frac{\delta}{4}.\end{aligned}$$

Take a nuclear representation of $\widetilde{q} = \sum_{j=1}^\infty \lambda_j (x'_j)^n$ such that $\sum_{j=1}^\infty |\lambda_j| \|x'_j\|^n \leq \|p\|_{\mathcal{P}_I^n(E)} + \frac{\delta}{2}$. Since E is an M-ideal in F each $x'_j \in F'$ can be written as the sum of $s(x'_j|_E)$ and $(x'_j)^\perp$. Moreover, $\|x'_j\| = \|s(x'_j|_E)\| + \|(x'_j)^\perp\|$.

Recall that $\widetilde{q}$ coincides with p on E, thus, for every $x \in E$,

$$p(x) = \sum_{j=1}^{\infty} \lambda_j \big(s\big(x_j'|_E\big)(x) + (x_j')^{\perp}(x)\big)^n = \sum_{j=1}^{\infty} \lambda_j (x_j'|_E(x))^n.$$

Using this, we easily get that $\overline{s}(p) = \sum_{j=1}^{\infty} \lambda_j \Big(s\big(x_j'|_E\big)\Big)^n$. Naturally,

$$\|p\|_{\mathcal{P}_I^n(E)} = \|p\|_{\mathcal{P}_N^n(E)} = \|\overline{s}(p)\|_{\mathcal{P}_N^n(E)} \leq \sum_{j=1}^{\infty} |\lambda_j| \|x_j'|_E\|^n.$$

Now,

$$\begin{aligned}
0 < \delta &= \big|q(y) - \overline{s}(p)(y)\big| = \big|\widetilde{q}(y) - \overline{s}(p)(y)\big| \\
&\leq \left|\sum_{j=1}^{\infty} \lambda_j \big(s\big(x_j'|_E\big)(y) + (x_j')^{\perp}(y)\big)^n - \lambda_j s\big(x_j'|_E\big)(y)^n\right| \\
&\leq \sum_{j=1}^{\infty} |\lambda_j| \sum_{i=1}^{n} \binom{k}{i} \big\|s\big(x_j'|_E\big)\big\|^{k-i} \|(x_j')^{\perp}\|^i \\
&= \sum_{j=1}^{\infty} |\lambda_j| \Big(\big\|s\big(x_j'|_E\big)\big\| + \|(x_j')^{\perp}\|\Big)^n - |\lambda_j| \big\|s\big(x_j'|_E\big)\big\|^n \\
&= \sum_{j=1}^{\infty} |\lambda_j| \|x_j'\|^n - \sum_{j=1}^{\infty} |\lambda_j| \|x_j'|_E\|^n \\
&\leq \|p\|_{\mathcal{P}_I^n(E)} + \frac{\delta}{2} - \|p\|_{\mathcal{P}_I^n(E)} = \frac{\delta}{2}.
\end{aligned}$$

This is a contradiction. Thus, the result follows. □

Since M-embedded spaces are Asplund we have a neater statement in this case.

Corollary 6.2.11. *Let E be an M-ideal in E''. If $p \in \mathcal{P}_I^n(E)$ then the Aron-Berner extension $AB(p)$ is the unique norm preserving extension to $\mathcal{P}_I^n(E'')$.*

It is known that on ℓ_∞ integral and nuclear polynomials do not coincide. By the fact that c_0 is an M-ideal in ℓ_∞ and the previous corollary, we derive the following remark.

Remark 6.2.12. *Let p be a non-nuclear polynomial in $\mathcal{P}_I^n(\ell_\infty)$ then its restriction to c_0 has a strictly smaller integral norm, i.e.,*

$$\|p|_{c_0}\|_{\mathcal{P}_I^n(c_0)} < \|p\|_{\mathcal{P}_I^n(\ell_\infty)}.$$

Unique norm preserving extension for a polynomial belonging to a maximal ideal

We have shown that, if E is an M-embedded space and p is a fixed polynomial in $\mathcal{P}_I^n(E)$, then $AB(p)$ is the unique norm preserving extension to $\mathcal{P}_I^n(E'')$. Now we want to answer the following related question: let $\mathcal{Q}$ be a maximal polynomial ideal and let p be a fixed polynomial belonging to $\mathcal{Q}(E)$, under what conditions do we have a *unique* norm preserving extension of p to the bidual E''? Since the Aron-Berner extension preserves the ideal norm for maximal polynomial ideals (Theorem 2.2.5), the question can be rephrased in the following way: when is the Aron-Berner extension the *only* norm preserving extension (for a given polynomial) in $\mathcal{Q}$? We will see necessary and sufficient conditions for this to happen that are related with the continuity of the Aron-Berner extension morphism.

Godefroy gave in [God81] a characterization of norm-one functionals having unique norm preserving extensions to the bidual as the points of $S_{E'}$ where the identity is w^*- w continuous.

Proposition 6.2.13. *[HWW93, Lemma III.2.14] Let E be a Banach space and $x' \in S_{E'}$. The following are equivalent:*

(1) *x' has a unique norm preserving extension to a functional defined on E'';*

(2) *The function $Id_{B_{E'}} : (B_{E'}, w^*) \longrightarrow (B_{E'}, w)$ is continuous at x'.*

The previous proposition says that unique norm preserving extensions is related with some kind of continuity. Aron, Boyd and Choi presented in [ABC01] a polynomial version of this result.

Proposition 6.2.14. *[ABC01, Theorem 6] Let E be a Banach space such that E'' has the metric approximation property and $p \in S_{\mathcal{P}^n(E)}$. The following are equivalent:*

(1) *p has a unique norm preserving extension to $\mathcal{P}^n(E'')$;*

(2) *if $\{p_\gamma\}_\gamma \subset B_{\mathcal{P}^n(E)}$ converges pointwise to p, then $\{AB(P_\gamma)\}_\gamma$ converges pointwise to $AB(p)$ in E''.*

We are interested on having a similar characterization for unique norm preserving extensions to the bidual of polynomials belonging to a maximal polynomial ideal. In this case, obviously, the norm that we want to preserve is the ideal norm.

Theorem 6.2.15. *Let α be an s-tensor norm of order n, E be a Banach space and $p \in \mathcal{Q}_\alpha(E)$ with $\|p\|_{\mathcal{Q}_\alpha(E)} = 1$. If α is cofinitely generated or E'' has the metric approximation property, then the following conditions are equivalent:*

(1) *p has a unique norm preserving extension to $\mathcal{Q}_\alpha(E'')$;*

(2) *the morphism $AB : \big(B_{\mathcal{Q}_\alpha(E)}, \sigma(\mathcal{Q}_\alpha(E), \widetilde{\otimes}_\alpha^{n,s} E)\big) \longrightarrow \big(B_{\mathcal{Q}_\alpha(E'')}, \sigma(\mathcal{Q}_\alpha(E''), \widetilde{\otimes}_\alpha^{n,s} E'')\big)$ is continuous at p;*

(3) *if the net $\{P_\gamma\}_\gamma \subset B_{\mathcal{Q}_\alpha(E)}$ converges pointwise to p, then $\{AB(p_\gamma)\}_\gamma$ converges pointwise to $AB(p)$ in E''.*

We postpone the proof of this theorem since we need some technical tools first. Let us define a canonical application from $\widetilde{\otimes}_\alpha^{n,s} E''$ to $(\widetilde{\otimes}_\alpha^{n,s} E)''$ by

$$\begin{aligned} \Theta_\alpha^E : \widetilde{\otimes}_\alpha^{n,s} E'' &\longrightarrow (\widetilde{\otimes}_\alpha^{n,s} E)'' = \mathcal{Q}_\alpha(E)' \\ z &\longmapsto \big(p \mapsto \langle AB(p), z\rangle\big). \end{aligned}$$

Proposition 6.2.16. *Let α be an s-tensor norm and E be a Banach space. If α is cofinitely generated or E'' has the metric approximation property then the mapping*

$$\Theta_\alpha^E : \widetilde{\otimes}_\alpha^{n,s} E'' \to (\widetilde{\otimes}_\alpha^{n,s} E)''$$

is an isometric embedding.

Proof. Let us see that Θ_α^E is a norm one operator: if $z \in \widetilde{\otimes}_\alpha^{n,s} E''$ and $p \in B_{\mathcal{Q}_\alpha(E)}$ then

$$\begin{aligned}\big|\langle AB(p), z\rangle\big| &\leq \|AB(p)\|_{\mathcal{Q}_\alpha(E'')} \ \alpha(z; \widetilde{\otimes}^{n,s} E'') \\ &= \|p\|_{\mathcal{Q}_\alpha(E)} \ \alpha(z; \widetilde{\otimes}^{n,s} E'') \\ &\leq \alpha(z; \widetilde{\otimes}^{n,s} E''),\end{aligned}$$

Therefore $\|\Theta_\alpha^E(z)\| \leq \alpha(z; \widetilde{\otimes}^{n,s} E'')$.

To prove the other inequality, we first see that the following diagram commutes:

$$\begin{tikzcd} \widetilde{\otimes}_\alpha^{n,s} E'' \arrow[rr, hook, "J_\alpha^{E'}"] \arrow[dr, "\Theta_\alpha^E"'] & & (\widetilde{\otimes}_{\alpha'}^{n,s} E')' \\ & (\widetilde{\otimes}_\alpha^{n,s} E)'', \arrow[ur, "(H_\alpha^E)'"'] & \end{tikzcd}$$

where the mappings H_α^E and $J_\alpha^{E'}$ are the ones that appear in (2.10) and (2.11) respectively. Indeed, by density and linearity it is enough to prove that $J_\alpha^{E'}(\otimes^n x'')(\otimes^n x')$ is equal to $(H_\alpha^E)' \circ \Theta_\alpha^E(\otimes^n x'')(\otimes^n x')$ for every $x' \in E'$. Note that $J_\alpha^{E'}(\otimes^n x'')(\otimes^n x')$ is just $(x''(x'))^n$. On the other hand,

$$\begin{aligned}(H_\alpha^E)' \circ \Theta_\alpha^E(\otimes^n x'')(\otimes^{n,s} x') &= \Theta_\alpha^E(\otimes^n x'')\big(H_\alpha^E(\otimes^n x')\big) \\ &= \Theta_\alpha^E(\otimes^n x'')\big((x')^n\big) \\ &= \langle AB\big((x')^n\big), \otimes^n x''\rangle \\ &= \langle(\kappa_E' x')^n, \otimes^n x''\rangle \\ &= x''(x')^n.\end{aligned}$$

If $\alpha = \overleftarrow{\alpha}$ or E'' has the metric approximation property then, by Proposition 2.2.1 and the Embedding Theorem 2.2.13 we have that $J_\alpha^{E'}$ is an isometry and also that $(H_\alpha^E)'$ is a quotient mapping. Therefore,

$$\alpha(z; \widetilde{\otimes}^{n,s} E'') = \|J_\alpha^{E'}(z)\| = \|(H_\alpha^E)' \circ \Theta_\alpha^E(z)\| \leq \|\Theta_\alpha^E(z)\|.$$

We have shown that the mapping Θ_α^E is an isometry. □

To prove Theorem 6.2.15 we also need the following equivalence for the convergence of nets of polynomials. The proof is straightforward.

Lemma 6.2.17. *Suppose that the polynomial p and the net $\{p_\gamma\}_\gamma$ are contained in the unit ball of $\mathcal{Q}_\alpha(E)$. Then, the following are equivalent:*

(1) $p_\gamma(x) \to p(x)$ *for all* $x \in E$;

(2) $p_\gamma \to p$ *in the topology* $\sigma(\mathcal{Q}_\alpha(E), \widetilde{\otimes}_\alpha^{n,s} E)$;

(3) $p_\gamma \to p$ *in the topology* $\sigma(\mathcal{P}^n(E), \widetilde{\otimes}_{\pi_{n,s}}^{n,s} E)$.

We can now prove Theorem 6.2.15.

Proof. (*of Theorem 6.2.15.*)

$(1) \Rightarrow (2)$:

Let $\{p_\gamma\}_\gamma \subset B_{\mathcal{Q}_\alpha(E)}$ such that $p_\gamma \overset{w^*}{\to} p$. We want to see that $AB(p)_\gamma \overset{w^*}{\to} AB(p)$ in $\mathcal{Q}_\alpha(E'')$. By the compactness of $\left(B_{\mathcal{Q}_\alpha(E'')}, w^*\right)$, the net $\{AB(p_\gamma)\}_\gamma$ has a subnet $\{AB(p_\gamma)\}_\gamma$ w^*-convergent to a polynomial $q \in B_{\mathcal{Q}_\alpha(E'')}$.

For each $x \in E$, we have, on one hand, that $AB(p_\gamma)(x) = p_\gamma(x) \to p(x)$ and, on the other hand, that $AB(p_\gamma)(x) \to q(x)$. So, $q|_E = p$. Also, $\|q\| \leq 1 = \|p\|$ implies $\|q\|_{\mathcal{Q}_\alpha(E'')} = \|p\|_{\mathcal{Q}_\alpha(E)}$. This means that q is a norm preserving extension of p and by (1) it should be $q = AB(p)$. Since for every subnet of $\{p_\gamma\}_\gamma$ we can find a sub-subnet such that the Aron-Berner extensions are w^*-convergent to $AB(p)$, we conclude that $AB(p)_\gamma \overset{w^*}{\to} AB(p)$.

$(2) \Rightarrow (1)$:

Let $q \in \mathcal{Q}_\alpha(E'')$ be an extension of p with $\|q\|_{\mathcal{Q}_\alpha(E'')} = 1$. From Proposition 6.2.16, the mapping

$$\Theta_\alpha^E : \widetilde{\otimes}_\alpha^{n,s} E'' \longrightarrow \left(\widetilde{\otimes}_\alpha^{n,s} E\right)''$$

is an isometry. Due to this, each polynomial $q \in \mathcal{Q}_\alpha(E'') = \left(\widetilde{\otimes}_\alpha^{n,s} E''\right)'$ has a Hahn-Banach extension $\widetilde{q} \in \left(\widetilde{\otimes}_\alpha^{n,s} E\right)''' = \mathcal{Q}_\alpha(E)''$. By Goldstine's Theorem, there exist a net $\{p_\gamma\}_\gamma \subset B_{\mathcal{Q}_\alpha(E)}$ such that $p_\gamma \to \widetilde{q}$ in the topology $\sigma(\mathcal{Q}_\alpha(E)'', \mathcal{Q}_\alpha(E)')$.

Let $z \in \widetilde{\otimes}_\alpha^{n,s} E \subset \mathcal{Q}_\alpha(E)'$. So we have

$$\langle p_\gamma, z\rangle \to \langle \widetilde{q}, z\rangle = \langle q, z\rangle = \langle p, z\rangle.$$

This means that $p_\gamma \to p$ in the topology $\sigma(\mathcal{Q}_\alpha(E), \widetilde{\otimes}_\alpha^{n,s} E)$. By (2), this implies that $AB(p_\gamma) \to AB(p)$ in the topology $\sigma(\mathcal{Q}_\alpha(E''), \widetilde{\otimes}_\alpha^{n,s} E'')$.

Now, if $v \in \widetilde{\otimes}_\alpha^{n,s} E''$, it follows that

$$\langle AB(p)_\alpha, v\rangle \to \langle AB(p), v\rangle.$$

But also, since $v \in \mathcal{Q}_\alpha(E)'$,

$$\langle AB(p_\gamma), v\rangle = \langle v, p_\gamma\rangle \to \langle v, \widetilde{q}\rangle = \langle q, v\rangle.$$

Therefore, $AB(p) = q$.

The equivalence between (2) and (3) is a consequence of the Lemma 6.2.17. $\square$

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