Estructura y propiedades de espacios invariantes por traslaciones en grupos abelianos localmente compactos

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Resumen

En esta tesis se estudian los espacios invariantes por traslaciones en el contexto de grupos localmente compactos y abelianos (grupos LCA). Para un grupo LCA G y un subgrupo cerrado $H \subseteq G$, se introduce la noción de espacio H-*invariante* o espacio *invariante por traslaciones en* H.

En el caso en que H es un subgrupo discreto y numerable de G, se muestra que el concepto de función rango y las técnicas de fibración son válidos en este contexto. Combinando estas dos herramientas, se prueba una caracterización de los espacios H-invariantes en término de las fibras de sus elementos. Como consecuencia, se obtienen caracterizaciones de marcos y bases de Riesz de estos espacios, extendiendo así resultados previos y conocidos para el caso $\mathbb{R}^d$ y el reticulado $\mathbb{Z}^d$.

Por otro lado, se estudia el problema de la *extra invariancia* de los espacios H-invariantes. Los resultados obtenidos de la extra invariancia establecen condiciones necesarias y suficientes para que un espacio H-invariante sea además invariante por traslaciones en un subgrupo cerrado M de G que contiene a H. También, se prueba que dado un subgrupo cerrado M de G que contiene a H existe un espacio H-invariante V que es exactamente M-invariante. Es decir, V no es invariante por traslaciones en ningún otro subgrupo M' que contiene a M. Además, se obtienen estimaciones de los tamaños de los soportes de la transformada de Fourier de los generadores de los espacios H-invariantes en relación a su M-invariancia.

Finalmente, se investigan los subespacios de $L^2(G)$ que son invariantes por traslaciones en un subgrupo K de G y también invariantes por modulaciones en Λ, siendo Λ un subgrupo del grupo dual de G. Se prueba una caracterización de estos espacio para el caso en que K y Λ son discretos.

Palabras Claves: Espacios invariantes por traslaciones enteras; Espacios invariante por

traslaciones; grupos LCA; Funciones rango; fibras; Espacios invariantes por modulaciones y traslaciones.

Structure and properties of shift invariant spaces on locally compact abelian groups

Abstract

In this thesis we study shift invariant spaces in the context of locally compact abelian (LCA) groups. For G an LCA group and $H \subseteq G$ a closed subgroup of G we introduce the notion of *H-invariant space* or *shift invariant space under translations in H*.

In case when H is a countable discrete subgroup of G, we show that the concept of range functions and the techniques of fiberization are valid in this context. Combining these tools, we provide a characterization for H-invariant spaces in terms of the fibers of its elements. As a consequence, we prove characterizations of frames and Riesz bases of these spaces extending previous results that were known for the classical case of $\mathbb{R}^d$ and the lattice $\mathbb{Z}^d$.

On the other hand, we study the problem of *extra invariance* of H-invariant spaces. Our results of extra invariance state several necessary and sufficient conditions for an H-invariant spaces to be invariant along translations in a closed subgroup of G, M, containing H. In addition we show that for each closed subgroup M of G which contains H there exists an H-invariant space V that is exactly M-invariant. That is, V is not invariant under any other subgroup M' containing M. We also obtain estimates on the support of the Fourier transform of the generators of the H-invariant spaces, related to its M-invariance.

Lastly, we investigate the structure of those closed subspace of $L^2(G)$ which are invariant by translations along K and also invariant under modulations in Λ, begin K and Λ closed subgroups of G and the dual group of G respectively. We obtain a characterization of these spaces when K and Λ are discrete.

Key words: Shift-invariant space; Translation invariant space; LCA groups; Range function; Fibers; Shift-modulation invariant space.

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Para Andrés Lofiego:
Porque no te quedás inmóvil
al borde del camino
y porque no te salvás.

Contents

Introduction

A *shift invariant space* (SIS) is a closed subspace of $L^2(\mathbb{R})$ that is invariant under translations by integers. The Fourier transform of a shift invariant space is a closed subspace that is invariant under integer modulations (multiplications by complex exponentials of integer frequency). Spaces that are invariant under integer modulations are called *doubly invariant spaces*. Every result on doubly invariant spaces can be translated to an equivalent result in shift invariant spaces via the Fourier transform. Doubly invariant spaces have been studied in the sixties by Helson [Hel64] and also by Srinivasan [Sri64], [HS64], in the context of operators related to harmonic analysis.

Shift invariant spaces are very important in applications and the theory had a great development in the last twenty years, mainly in approximation theory, sampling, wavelets, and frames. In particular they serve as models in many problems in signal and image processing.

In order to understand the structure of doubly invariant spaces, Helson introduced the notion of *range function*. This became an essential tool in the modern development of the theory. See [dBDR94a], [dBDR94b], [RS95] and [Bow00].

Range functions characterize completely shift invariant spaces and provide a series of techniques known in the literature as *fiberization* that allow to have a different view and a deeper insight of these spaces.

Fiberization techniques are very important in the class of *finitely generated* shift invariant spaces. A key feature of these spaces is that they can be generated by the integer translations of a finite number of functions. Using range functions allows us to translate problems on finitely generated shift invariant spaces, into problems of linear algebra (i.e. finite dimensional problems).

Shift invariant spaces generalize very well to several variables where the invariance is understood to be under the lattice $\mathbb{Z}^d$.

When looking carefully at the theory it becomes apparent that it is strongly based on the additive group operation of $\mathbb{R}^d$ and the action of the subgroup $\mathbb{Z}^d$.

It is therefore interesting to see if the theory can be set in a context of general *locally compact abelian* groups (LCA groups). The locally compact abelian group framework has several advantages. First because it is important to have a valid theory for the classical groups such as $\mathbb{Z}^d$, $\mathbb{T}^d$ and $\mathbb{Z}_n$. This will be crucial particularly in applications, as in the case of the generalization of the Fourier Transform to LCA groups and also Kluvanek's theorem, where the Classical Sampling theorem is extended to this general context, (see [Klu65], [Dod07]).

On the other side, the LCA groups setting, unifies a number of different results into a general framework with a concise and elegant notation. This fact enables us to visualize hidden relationships between the different components of the theory, what, as a consequence, will translate in a deeper and better understanding of shift invariant spaces, even in the case of the real line.

In this thesis we develop the theory of shift invariant spaces on LCA groups. We begin by introducing the concept of K-invariant spaces or shift invariant spaces under translation in K, for K being a closed subgroup of a fixed LCA group G. Then, we turn our attention for the case when the translates are in an *uniform lattice* in G. That is, a discrete subgroup H of G for which the quotient G/H is compact. In this context, our emphasis will be on range functions and fiberization techniques. With these tools we will investigate the structure of H-invariant spaces. The order of the subjects follows mainly the treatment of Bownik in $\mathbb{R}^d$, [Bow00]. In [KR08] the authors study, in the context of LCA groups, *principal* shift invariant spaces, that is, shift invariant spaces generated by one single function. However they don't develop the general theory.

With the description of H-invariant spaces that we obtain, we are able to study different problems about them.

First, we are interested in a problem concerning to frames and Riesz bases on H-invariant spaces.

In $\mathbb{R}^d$, SIS are separable Hilbert spaces in themselves and the same occurs for shift invariant spaces on LCA groups.

As it is well known, each separable Hilbert space has an orthonormal basis. In addition to the mere existence, for the particular case of shift invariant spaces, it is useful to have bases with elements having a common structure. But these requirements on the basis can not always be satisfied.

Fortunately, the concept of frames provides an alternative to orthonormal bases. Working with a frame $\{f_n\}_n$ allows us to represent each element of the Hilbert space (shift-invariant spaces in our case) as $f = \sum_n c_n f_n$. In general, the scalars $\{c_n\}_n$ are not unique and the elements $\{f_n\}_n$ are not required to be orthogonal. Nevertheless, frame's definition still retains good control on the behavior on the coefficients $\{c_n\}_n$.

An important property about frames on shift invariant spaces is that they always exist and moreover, we can always find frames of a very specific type, *frames of translates*. By frames of translates we mean frames in which their elements are translations of a fixed set of functions. This particular structure, which is essential for applications, is quite compatible with the Fourier transform. As a consequence, fiberization techniques become a very well-adapted tool for studying frames of translates of shift invariant spaces (see Chapter 3).

The concrete problem concerning frames of translates for H-invariant spaces that we consider in this thesis is the following. Let $V \subseteq L^2(G)$ be the H-invariant space given by $V = \overline{\text{span}}\{T_h\varphi : \varphi \in \mathcal{A}, h \in H\}$ with $\mathcal{A}$ being a (countable) set of functions in $L^2(G)$ and with H being an uniform lattice in G. Here, T_h denotes the translation operator by h defined as $T_h f(x) = f(x-h)$ for a.e. $x \in G$ and $f \in L^2(G)$. We want to know when the set $\{T_h\varphi : \varphi \in \mathcal{A}, h \in H\}$ constitutes a frame for V. In addition, we study the analogous problem for, instead of frames, Riesz bases. These bases are an interesting generalization of orthonormal bases. Therefore, we analyze under which conditions $\{T_h\varphi : \varphi \in \mathcal{A}, h \in H\}$ is a frame or a Riesz basis for V.

Another question which is relevant for this thesis, is whether H-invariant spaces, with H being an uniform lattice in G, have the property to be invariant under any other translation than those that are in H. A limit case is when the space is invariant under translations by all $x \in G$. In this case the space is called *translation invariant*. However, there exist H-invariant spaces with some *extra* invariance that are not necessarily translation invariant. That is, there are some intermediate cases between H-invariance and translation invariance. The question is then, how can we identify them?

Recently, Hogan and Lakey defined the *discrepancy* of a shift invariant space as a way to quantify the *non-translation invariance* of the subspace, (see [HL05]). The discrepancy measures how far a unitary norm function of the subspace, can move away from it, when translated by non integers. A translation invariant space has discrepancy zero.

In another direction, Aldroubi et al, see [ACHKM10], studied shift invariant spaces of $L^2(\mathbb{R})$ that have some extra invariance. They show that if V is a shift invariant space, then its *invariance set* is a closed additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. The invariance set associated to a shift invariant space is the set M of real numbers satisfying that for each $p \in M$ the translations by p of every function in V, belongs to V. As a consequence, since every additive subgroup of $\mathbb{R}$ is either discrete or dense, there are only two possibilities left for the extra invariance. That is, either V is invariant under translations by the group $\frac{1}{n}\mathbb{Z}$, for some positive integer n (and not invariant under any bigger subgroup) or it is translation invariant. They found different characterizations, in terms of the Fourier transform, of when a shift invariant space is $\frac{1}{n}\mathbb{Z}$-invariant.

A natural question arises in this context. Are the characterizations of extra invariance that hold on the line, still valid in the context of LCA groups?

The invariance set $M \subseteq G$ associated to an H-invariant space V, that is, the set of elements of G that leave V invariant when translated by its elements, is again, as in the $\mathbb{R}$ case, a closed subgroup of G which contains H (see Proposition 4.1.1). The problem of the extra invariance can then be reformulated as finding necessary and sufficient conditions for an H-invariant space to be invariant under a closed subgroup $M \subseteq G$ containing H.

The main difference with the $\mathbb{R}$ case studied in [ACHKM10], is that the structure of the closed subgroups of G containing uniform lattices is not as simple.

The results obtained for the $\mathbb{R}$ case translate very well in the case in which the invariance set M is a discrete subgroup or when M is dense, that is $M = G$. However, there are subgroups of G that are neither discrete nor dense. So, can there exist H-invariant spaces which are M-invariant for such a subgroup M and are not translation invariant?

Our approach in this work is to study the extra invariance of H-invariant spaces on LCA groups. We were able to obtain several characterizations paralleling the 1-dimensional results. In addition our results show the existence of H-invariant spaces that are *exactly* M-invariant for every closed subgroup $M \subseteq G$ containing H. By 'exactly M-invariant' we mean that they are not invariant under any other subgroup containing M. We apply our results to obtain estimates on the size of the support of the Fourier transform of the generators of the space.

The particular case $G = \mathbb{R}^d$ can be treated in a slightly different way than H-invariant spaces, in which the general context of LCA groups can be omitted. The characterization of extra invariance of shift invariant spaces on $L^2(\mathbb{R}^d)$ with $d > 1$ is studied using an appropriated description of the closed subgroups of $\mathbb{R}^d$ that contain $\mathbb{Z}^d$. For this, we review the structure of closed subgroups of $\mathbb{R}^d$.

Finally in this work, we consider a problem related to shift-modulation invariant spaces. Shift-modulation invariant (SMI) spaces are shift invariant spaces that have the extra condition to be also invariant under some group of modulations. These shift invariant spaces with the extra assumption of modulation invariance are of particular interest and are usually known as Gabor or Weyl-Heisenberg spaces. They have been intensively studied in [Bow07], [CC01b], [CC01a], [Chr03], [Dau92], [GD04], [GD01], [Gro01].

A very deep and detailed study of the structure of shift-modulation invariant spaces of $L^2(\mathbb{R}^d)$, was given by Bownik (see [Bow07]). In that work, the author provides a characterization of SMI spaces based on fiberization techniques and range functions.

Since modulations become translations in the Fourier domain, shift-modulations invariant spaces are spaces that are shift invariant in time and frequency. As a consequence the techniques of shift invariant spaces can be applied to study the structure of SMI spaces. Having at hand a theory of SIS on LCA groups it is natural to ask whether a general theory of SMI spaces could be developed in this more general context.

We define and study the structure of SMI spaces on the context of LCA groups. First we introduce the concept of shift-modulation spaces where translations are on a closed subgroup of an LCA group G and modulations are on a closed subgroup of the dual group of G. Next we investigate the case where both, translations and modulations, are along uniform lattices of G and the dual group of G respectively, with some minor hypotheses. Using previous result for shift invariant spaces on LCA groups, we are able to develop a fiberization isometry and range functions well adapted to this more complicated structure which combines translations and modulations. Then, we prove a characterization of shift-modulation invariance spaces, extending to LCA groups the result obtained by Bownik in [Bow07] for the case of $L^2(\mathbb{R}^d)$. While some properties are a simple generalizations of the known case, there are others that do not translate easily at this very abstract context and whose validity it is not clear a priori.

Thesis outline

The rest of the thesis is organized as follows.

Chapter 1 includes a review about LCA groups background to make this thesis self-contained. Mainly, we summarize results concerning to basic facts about generalities of LCA groups , Haar measures and the Fourier transform.

In addition, we develop a fundamental tool for this work called fiberization isometry and we introduce the precise definition of K-invariant spaces on LCA groups.

In Chapter 2 we present a characterization of shift invariant spaces along uniform lattices on LCA groups. First we provide a description of shift invariant spaces generated by a single function in terms of the Fourier transform of its generator. In order to characterize general shift invariant spaces along uniform lattices, we introduces range functions. Then, we state necessary and sufficient condition for a closed subspace of $L^2(G)$ to be a shift invariant space combining range functions with fiberizatons techniques.

We devote Chapter 3 to study frames and Riesz bases of translates of shift invariant spaces on LCA groups. For an uniform lattice H in an LCA group G and $\mathcal{A} \subseteq L^2(G)$ a subset of functions, we want to determine when the set $\{T_h\varphi : \varphi \in \mathcal{A},\ h \in H\}$ is a frame or a Riez basis for $V = \overline{\text{span}}\{T_h\varphi : \varphi \in \mathcal{A}, h \in H\}$. Our analysis will be based on the results obtained in Chapter 2.

In Chapter 4 we study the problem of the extra invariance. Given a shift invariant space under an uniform lattice of G, our purpose is to determine precisely when the space is also invariant under translations on a closed subgroup of G, M, which contains the original uniform lattice. The results included in this chapter are an extension of those stated in [ACHKM10]. We want to remark that our generalization is not straightforward. Our main difficulty lies in the fact that we do not know a priori the structure of the subgroup M.

Finally, we devote Chapter 5 to investigate shift-modulation invariant spaces in the context of LCA groups. A (K, Λ) shift-modulation invariant space is a subspace of $L^2(G)$, that is invariant by translations along elements in K and modulations by elements in Λ, with K and Λ being closed subgroups of G and the dual group of G respectively. We provide a characterization of shift-modulation invariant spaces in this general context when K and Λ are uniform lattices. For getting the desired characterization, we develop fiberization techniques and suitable range functions adapted to this new structure.

Included publications

Most of the results of this thesis have been published, or submitted for publication, as research articles in different journals.

The papers included in the thesis are:

- C. Cabrelli and V. Paternostro, *Shift-Invariant Spaces on LCA Groups*, J. Funct. Anal. 258, (2010), 6, 2034–2059.
- M. Anastasio, C. Cabrelli, V. Paternostro, *Invariance of a Shift-Invariant Space in several variables*, Complex Anal. Oper. Theory 5, (2010), 4, 1031–1050.
- M. Anastasio, C. Cabrelli, V. Paternostro, *Extra invariance of shift-invariant spaces on LCA groups*, J. Math. Anal. Appl., 370, (2010), 2, 530–537.
- C. Cabrelli and V. Paternostro, *Shift-Modulation Invariant Spaces on LCA Groups*, (2011). Submitted for publications. http://arxiv.org/abs/1109.0482.

1

Preliminaries

This chapter compiles some preliminary facts and the notation that will be used in this thesis. It is our intention to make the exposition the more self-contained possible. However, we omit the proof of those result that we believe are not an essential contribution for this thesis. In each case, we indicate the corresponding literature.

The known material of this chapter can be probably found in the literature in a more general form. Here, it will be exposed in an appropriated way to our purpose.

The chapter is organized as follows. Section 1.1 is divided in three subsections. In the first one we summarize without proofs the relevant material on locally compact abelian groups. The second one contains results concerning to Haar measures. We also discuss there the existence and properties of Borel-sections which play a key role in most of the main arguments in this thesis. In the last part of Section 1.1 we present some properties of the Fourier transform on locally compact abelian groups. In Section 1.2 we introduce the concept of K-invariant spaces and we give some relevant examples. A brief summary about vector-valued functions is given in Section 1.3. Finally in Section 1.4 we develop the fiberization isometry, one of the most important tools used in this work.

1.1 Background on LCA Groups

In this section we review some basic known results from the theory of LCA groups, that we need for the remainder of the thesis. For details and proofs see [Rud62], [Fol95], [HR79], [HR70].

1.1.1 LCA Groups

Throughout this thesis, G will denote a locally compact abelian, Hausdorff group (LCA) and Γ or $\widehat{G}$ its dual group. That is,

$$\Gamma = \{\gamma : G \to \mathbb{C} : \gamma \text{ is a continuous character of } G\},$$

where a character is a function such that:

(a) $|\gamma(x)| = 1, \ \forall \ x \in G$.

(b) $\gamma(x+y) = \gamma(x)\gamma(y), \ \forall \ x, y \in G$.

Thus, characters generalize the exponential functions $\gamma_t(y) = e^{2\pi i t y}$, from the case $G = (\mathbb{R}, +)$.

Since in this context, both the algebraic and topological structures coexist, we will say that two groups G and G' are *topologically isomorphic* and we will write $G \approx G'$, if there exists a topological isomorphism from G onto G'. That is, an algebraic isomorphism which is an homeomorphism as well.

The group Γ, with the operation $(\gamma + \gamma')(x) = \gamma(x)\gamma'(x)$, is an LCA group. Moreover,

Theorem 1.1.1. *Let G be an LCA group and Γ its dual. Then, the dual group of Γ is topologically isomorphic to G.*

Therefore, every LCA group is the dual of its own dual group, with the identification

$$x \in G \leftrightarrow \phi_x \in \widehat{\Gamma},$$

where $\phi_x(\gamma) := \gamma(x)$. According to this, it is convenient to use the notation (x, γ) for the complex number $\gamma(x)$, representing the character γ applied to x or the character x applied to γ.

Note that from properties (a) and (b) of the elements of Γ, the following equalities are obtained:

$$(0, \gamma) = 1 = (x, 0) \qquad \text{and} \qquad (x, \gamma)^{-1} = (x, -\gamma) = (-x, \gamma) = \overline{(x, \gamma)},$$

$\forall \ x \in G$ and $\forall \ \gamma \in \Gamma$.

Theorem 1.1.2. *Let G be an LCA group and Γ its dual group. Then, if G is discrete, Γ is compact and if G is compact, Γ is discrete.*

As a consequence of Theorem 1.1.1 and Theorem 1.1.2, it holds that an LCA group is compact if an only if its dual is discrete.

Next we list the most basic examples that are relevant to Fourier analysis. The details are left to the reader. As usual, we identify the interval $[0, 1)$ with the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \approx \mathbb{R}/\mathbb{Z}$.

Example 1.1.3.

(I) In case that $G = (\mathbb{R}^d, +)$, the dual group Γ is also $(\mathbb{R}^d, +)$, with the identification $x \in \mathbb{R}^d \leftrightarrow \gamma_x \in \Gamma$, where $\gamma_x(y) = e^{2\pi i \langle x, y\rangle}$. (See [Rud62, Section 1.2.7])

(II) In case that $G = \mathbb{T}$ its dual group is topologically isomorphic to $\mathbb{Z}$, identifying each $k \in \mathbb{Z}$ with $\gamma_k \in \Gamma$, being $\gamma_k(\omega) = e^{2\pi i k \omega}$. This is due to, since $\mathbb{T} \approx \mathbb{R}/\mathbb{Z}$, characters of $\mathbb{R}$ can be defined from $\mathbb{T}$ to $\mathbb{C}$ under the condition of being 1-periodic. As it is point out in item (I), characters on $\mathbb{R}$ are of the form $\gamma_x(y) = e^{2\pi i x y}$ with $x \in \mathbb{R}$ and then, γ_x is 1-periodic if and only if $x \in \mathbb{Z}$.

(III) Let $G = \mathbb{Z}$. If $\gamma \in \Gamma$, then $(1, \gamma) = e^{2\pi i \alpha}$ for same $\alpha \in \mathbb{R}$. Therefore, $(k, \gamma) = e^{2\pi i \alpha k}$. Thus, the complex number $e^{2\pi i \alpha}$ identifies the character γ. This proves that Γ is $\mathbb{T}$.

(IV) Finally, in case that $G = \mathbb{Z}_n$, we can identify Γ with $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$. Indeed. According to item (III), $\widehat{\mathbb{Z}} = \{e^{2\pi i \alpha \cdot} : \mathbb{Z} \to \mathbb{C} : \alpha \in [0, 1)\}$. Since $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, characters in $\widehat{\mathbb{Z}_n}$ will be those in $\widehat{\mathbb{Z}}$ that are constant in the cosets of $\mathbb{Z}_n$. This happens exactly when $\alpha \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$. This argument also shows that the dual group of $\mathbb{Z}_n$ is $\mathbb{Z}_n$ as well.

Let us now consider $K \subseteq G$, a closed subgroup of an LCA group G. Then, the quotient G/K is a regular (T3) topological group. Moreover, with the quotient topology, G/K is an LCA group and if G is second countable, the quotient G/K is also second countable.

For an LCA group G and $K \subseteq G$ a subgroup of G, we define the subgroup K^* of Γ as follows:

$$K^* = \{\gamma \in \Gamma : (k, \gamma) = 1, \ \forall \ k \in K\}.$$

This subgroup is called the *annihilator of* K. Since each character in Γ is a continuous function on G, K^* is a closed subgroup of Γ.

The next result establishes duality relationships among the groups K, K^*, G/K and Γ/K^*.

Theorem 1.1.4. *If G is an LCA group and $K \subseteq G$ is a closed subgroup of G, then:*

(i) K^* is topologically isomorphic to the dual group of G/K, i.e: $K^* \approx \widehat{(G/K)}$.

(ii) Γ/K^* is topologically isomorphic to the dual group of K, i.e: $\Gamma/K^* \approx \widehat{K}$.

Note that item *(ii)* of the previous theorem can be obtained combining the results in item *(i)* of Theorem 1.1.4, Theorem 1.1.1 and the following lemma, the proof of which can be read in [Rud62, Lemma 2.1.3.].

Lemma 1.1.5. *Let G be an LCA group and $K \subseteq G$ a closed subgroup. If K^* is the annihilator of K, then K is the annihilator of K^*.*

Remark 1.1.6. According to Theorem 1.1.1, each element of G induces one character in $\widehat{\Gamma}$. In particular, if K is a closed subgroup of G, each $k \in K$ induces a character that has the additional property of being K^*-periodic. That is, for every $\delta \in K^*$, $(k, \gamma + \delta) = (k, \gamma)$ for all $\gamma \in \Gamma$.

In this thesis, we work with periodic functions several times. Then, we want the specify what a periodic function is. If K is a subgroup of G, we say that $f : G \longrightarrow \mathbb{C}$ is *K-periodic* if $f(x+k) = f(x)$ for all $x \in G$ and $k \in K$.

The following definition will be useful throughout this thesis. It agrees with the one given in [KK98].

Definition 1.1.7. Given G an LCA group, a *uniform lattice K in G* is a discrete subgroup of G such that the quotient group G/K is compact.

Example 1.1.8.

(I) In case that $G = \mathbb{R}^d$, subgroups of the form $K = A\mathbb{Z}^d$ with A being a invertible matrix with integer entries are uniform lattices in $\mathbb{R}^d$.

(II) When $G = \mathbb{T}$, $K = G_n$ where G_n denotes the set $\{z \in \mathbb{C} : z^n = 1\}$, is an uniform lattices in $\mathbb{T}$ for each $n \in \mathbb{N}$.

(III) Every subgroup of $\mathbb{Z}$ is a uniform lattices since all of them are of the form $m\mathbb{Z}$ for some $m \in \mathbb{N}$.

There exist LCA groups which do not contain uniform lattices. For a discussion about this, we refer to [KK98], where an example of an LCA group without a uniform lattice is given.

The next theorem points out a number of relationships which occur among G, K, Γ, K^* and their respective quotients. The properties stated in it will be crucial on the remainder of this work.

Theorem 1.1.9. *Let G be a second countable LCA group. If $K \subseteq G$ is a countable (finite or countably infinite) uniform lattice, the following properties hold.*

(1) G is separable

(2) $K \subseteq G$ is closed.

(3) G/K is second countable and metrizable.

(4) $K^* \subseteq \Gamma$, the annihilator of K, is closed, discrete and countable.

(5) $\widehat{H} \approx \Gamma/K^*$ and $(\widehat{G/K}) \approx K^*$.

(6) Γ/K^* is a compact group.

Note that in particular, this theorem states that K^* is a countable uniform lattice in Γ.

1.1.2 Haar Measure on LCA groups

On every LCA group G, there exists a *Haar measure*. That is, a non-negative, regular Borel measure m_G, which is not identically zero and *translation-invariant*. This last property means that,

$$m_G(E + x) = m_G(E)$$

for every element $x \in G$ and every Borel set $E \subseteq G$. This measure is unique up to constants, in the following sense: if m_G and m'_G are two Haar measures on G, then there exists a positive constant λ such that $m_G = \lambda m'_G$. For a thorough proof of the existence and uniqueness of Haar measures we refer the reader to [Fol95, Theorem 2.10 and Theorem 2.20].

We say that a function $f : G \longrightarrow \mathbb{C}$ is m_G*-measurable* if it is Borel-measurable respect to the Haar measure m_G on G.

Given a Haar measure m_G on an LCA group G, the integral over G is translation-invariant in the sense that,

$$\int_G f(x + y)\, dm_G(x) = \int_G f(x)\, dm_G(x)$$

for each element $y \in G$ and for each m_G-measurable function f on G.

As in the case of the Lebesgue measure, we can define the spaces $L^p(G, m_G)$, that we will denote as $L^p(G)$, in the following way

$$L^p(G) = \{f : G \to \mathbb{C} : f \text{ is measurable and } \|f\|_p^p := \int_G |f(x)|^p\, dm_G(x) < \infty\}.$$

We call $\|f\|_p$ the L^p*-norm* of f.

If G is a second countable LCA group, $L^p(G)$ is separable, for all $1 \leq p < \infty$. We will focus here on the cases $p = 1$ and $p = 2$. For $p = \infty$ the space $L^\infty(G)$ is given by

$$L^\infty(G) = \{f : G \to \mathbb{C} : f \text{ is measurable and } \|f\|_\infty := \text{esssup}_{x\in G}|f(x)| < \infty\},$$

where by $\text{esssup}_{x\in G}|f(x)|$ we mean the *essencial supremum* of f defined by $\inf\{c : |f(x)| \leq c \text{ } a.e. \text{ } x \in G\}$.

In case where G is discrete, we note $\ell^p(G)$ instead of $L^p(G)$ for all $1 \leq p \leq \infty$.

The next theorem is a generalization of a periodization argument usually applied in case $G = \mathbb{R}$ (for details see [HR70, Theorem 28.54]).

Theorem 1.1.10. *Let G be an LCA group, $K \subseteq G$ a closed subgroup and $f \in L^1(G)$. Then,*

(i) For almost every $x \in G$, the function $k \mapsto f(x + k)$ is m_K-measurable and belongs to $L^1(K)$. The function $x \mapsto \int_K f(x + k)\, dm_K(k)$ depends only on the coset $[x] = x + K$, and therefore it can be considered as a function of the quotient G/K, that is $F([x]) = \int_K f(x + k)dm_K(k)$.

(ii) The function F defined above is $m_{G/K}$-measurable and belongs to $L^1(G/K)$. Furthermore, the Haar measures m_G, m_K and $m_{G/K}$ can be chosen such that

$$\int_G f(x)\,dm_G(x) = \int_{G/K}\int_K f(x+k)\,dm_K(k)\,dm_{G/K}([x]). \tag{1.1}$$

Equality (1.1) is usually called Wiel's formula.

If G is a countable discrete group, the integral of $f \in L^1(G)$ over G, is determined by the formula

$$\int_G f(x)\,dm_G(x) = m_G(\{0\})\sum_{x\in G} f(x),$$

since, due to the translations invariance, $m_G(\{x\}) = m_G(\{0\})$, for each element $x \in G$.

Definition 1.1.11. Let G be an LCA group and $K \subseteq G$ a closed subgroup. A *section* of G/K is a set of representatives of this quotient. That is, a subset C of G containing exactly one element of each coset. Thus, each element $x \in G$ has a unique expression of the form $x = c + k$ with $c \in C$ and $k \in K$.

We will need later in the thesis to work with Borel sections. The existence of Borel sections is provided by the following lemma (see [KK98] and [FG64]).

Lemma 1.1.12. *Let G be an LCA group and K a uniform lattice in G. Then, there exists a section of the quotient G/K, which is Borel measurable.*

Moreover, there exists a section of G/K which is relatively compact and therefore with finite m_G-measure.

A section $C \subseteq G$ of G/K is in one to one correspondence with G/K by the *cross-section* map $\tau : G/K \to C$, $[x] \mapsto [x] \cap C$. Therefore, we can carry over the topological and algebraic structure of G/K to C. Moreover, if C is a Borel section, $\tau : G/K \to C$ is measurable with respect to the Borel σ-algebra in G/K and the Borel σ-algebra in G (see [FG64, Theorem 1]). Therefore, the set value function defined by $m(E) = m_{G/K}(\tau^{-1}(E))$ is well defined on Borel subsets of C. In the next lemma, we will prove that this measure m is equal to m_G up to a constant.

Lemma 1.1.13. *Let G be an LCA group, K a countable uniform lattice in G and C a Borel section of G/K. Fix m_G, m_K and $m_{G/K}$ such that the Weil's formula holds. Then, for every Borel set $E \subseteq C$*

$$m_G(E) = m_K(\{0\})m_{G/K}(\tau^{-1}(E)),$$

where τ is the cross-section map.

In particular, $m_G(C) = m_K(\{0\})m_{G/K}(G/K)$.

Proof. According to Lemma 1.1.12, there exists a relatively compact section of G/K. Let us call it C'. Therefore, if C is any other Borel section of G/K,

$$\begin{aligned} m_G(C') &= m_G(G\cap C') = m_G(\bigcup_{k\in K}(C+k)\cap C') \\ &= m_G(\bigcup_{k\in K}[(C'-k)\cap C]+k) = \sum_{k\in K} m_G([(C'-k)\cap C]+k) \\ &= \sum_{k\in K} m_G((C'-k)\cap C) = m_G(\bigcup_{k\in K}(C'-k)\cap C) \\ &= m_G(G\cap C) = m_G(C). \end{aligned}$$

Since C' has finite m_G measure, C must have finite measure as well.

Now, take $E\subseteq C$ a Borel set. Thus, $m_G(E)\le m_G(C)<\infty$. Hence, sing Theorem 1.1.10,

$$\begin{aligned} m_G(E) = \int_G \chi_E(x)\,dm_G(x) &= \int_{G/K}\int_K \chi_E(x+k)dm_K(k)\,dm_{G/K}([x]) \\ &= m_K(\{0\})\int_{G/K}\sum_{k\in K}\chi_E(x+k)\,dm_{G/K}([x]) \\ &= m_K(\{0\})\int_{G/K}\chi_{\tau^{-1}(E)}([x])\,dm_{G/K}([x]) \\ &= m_K(\{0\})m_{G/K}(\tau^{-1}(E)). \end{aligned}$$

□

Remark 1.1.14. Notice that C, together with the LCA group structure inherited by G/K through τ, has the Haar measure m. We proved that $m_G|_C$, the restriction of m_G to C, is a multiple of m. It follows that $m_G|_C$ is also a Haar measure on C.

In this thesis we will consider C as an LCA group with the structure inherited by G/K and with the Haar measure m_G.

A *trigonometric polynomial* in an LCA group G is a function of the form

$$P(x) = \sum_{j=1}^{n} a_j(x,\gamma_j),$$

where $\gamma_j\in\Gamma$ and $a_j\in\mathbb{C}$ for all $1\le j\le n$.

As a consequence of Stone-Weierstrass Theorem, the following result holds, (see [Rud62], page 24).

Lemma 1.1.15. *If G is a compact LCA group, then the trigonometric polynomials are dense in $C(G)$, where $C(G)$ is the set of all continuous complex-valued functions on G.*

Another important property of characters in compact groups is the following:

Lemma 1.1.16. *Let G be a compact LCA group and Γ its dual. Then, the characters of G verify the following orthogonality relationship:*

$$\int_G (x,\gamma)\overline{(x,\gamma')}\, dm_G(x) = m_G(G)\delta_{\gamma\gamma'},$$

for all $\gamma, \gamma' \in \Gamma$, where $\delta_{\gamma\gamma'} = 1$ if $\gamma = \gamma'$ and $\delta_{\gamma\gamma'} = 0$ if $\gamma \neq \gamma'$.

Proof. Since Γ is a group, we only need to see that $\int_G (x,\gamma)\, dm_G(x) = m_G(G)\delta_{\gamma 0}$, for all $\gamma \in \Gamma$.

Take $\gamma \in \Gamma$. If $\gamma = 0$, it is clear that $\int_G (x,\gamma)\, dm_G(x) = m_G(G)$. If $\gamma \neq 0$, there exists $x_0 \in G$ with $(x_0,\gamma) \neq 1$. Then,

$$\begin{aligned} \int_G (x,\gamma)\, dm_G(x) &= (x_0,\gamma)\int_G (x - x_0,\gamma)\, dm_G(x) \\ &= (x_0,\gamma)\int_G (x,\gamma) dm_G(x). \end{aligned}$$

Therefore $\int_G (x,\gamma)\, dm_G(x) = 0$. □

Let us now suppose that K is a uniform lattice in G. If Γ is the dual group of G and K^* is the annihilator of K, the following characterization of the characters of the group Γ/K^* will be useful to understand what follows.

For each $k \in K$, the function $\gamma \mapsto (k,\gamma)$ is constant on the cosets $[\gamma] = \gamma + K^*$. Therefore, it defines a character on Γ/K^*. Moreover, each character on Γ/K^* is of this form. Thus, this correspondence between K and the characters of Γ/K^*, which is actually a topological isomorphism, shows the dual relationship established in Theorem 1.1.4.

Furthermore, since Γ/K^* is compact, we can apply Lemma 1.1.16 to Γ/K^*. Then, for $k \in K$, we have

$$\int_{\Gamma/K^*} (k,[\gamma])\, dm_{\Gamma/K^*}([\gamma]) = \begin{cases} m_{\Gamma/K^*}(\Gamma/K^*) & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}. \tag{1.2}$$

1.1.3 The Fourier transform on LCA groups

Given a function $f \in L^1(G)$ *Fourier Transform* of f is defined as

$$\widehat{f}(\gamma) = \int_G f(x)(x,-\gamma)\, dm_G(x), \quad \gamma \in \Gamma. \tag{1.3}$$

Theorem 1.1.17. *The Fourier transform is a linear operator from $L^1(G)$ into $C_0(\Gamma)$, where $C_0(\Gamma)$ is the subspace of $C(\Gamma)$ of functions vanishing at infinite, that is, $f \in C_0(\Gamma)$ if $f \in C(\Gamma)$ and for all $\varepsilon > 0$ there exists a compact set $R \subseteq G$ with $|f(x)| < \varepsilon$ if $x \in R^c$.*

Furthermore, $\wedge : L^1(G) \to C_0(\Gamma)$ satisfies

$$\widehat{f}(\gamma) = 0\ \forall\, \gamma \in \Gamma \;\Rightarrow\; f(x) = 0 \text{ a.e. } x \in G. \tag{1.4}$$

For $\varphi \in L^1(\Gamma)$, the *inverse Fourier Transform* of φ is defined as

$$\check{\varphi}(x) = \int_\Gamma (x,\gamma)\varphi(\gamma)\, dm_\Gamma(\gamma)$$

and the function $\check{\varphi} : G \mapsto \mathbb{C}$ is continuous as well.

The Haar measure of the dual group Γ of G, can be normalized so that, for a specific class of functions, the following inversion formula holds (see [Rud62, Section 1.5]),

$$f(x) = \int_\Gamma \widehat{f}(\gamma)(x,\gamma)\, dm_\Gamma(\gamma).$$

In the case that the Haar measures m_G and m_Γ are normalized such that the inversion formula holds, the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended to a unitary operator from $L^2(G)$ onto $L^2(\Gamma)$, the so-called Plancharel transformation. We also denote this transformation by ”$\wedge$”.

Thus, the Parseval formula holds

$$\langle f, g\rangle = \int_G f(x)\overline{g(x)}\, dm_G(x) = \int_\Gamma \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}\, dm_\Gamma(\gamma) = \langle \widehat{f}, \widehat{g}\rangle,$$

for all $f, g \in L^2(G)$.

Let us now suppose that G is compact. Then Γ is discrete. Fix m_G and m_Γ in order that the inversion formula holds. Thus,

$$1 = m_\Gamma(\{0\})m_G(G). \tag{1.5}$$

Let us prove that (1.5) holds. Since $\chi_G = \frac{1}{m_G(G)}\chi_G * \chi_G$, inversion formula is valid for χ_G (see [Rud62] for details). Then

$$1 = \chi_G(x) = \int_\Gamma \widehat{\chi_G}(\gamma)(x,\gamma)\, dm_\Gamma(\gamma) = m_\Gamma(\{0\}) \sum_{\gamma\in\Gamma} \widehat{\chi_G}(\gamma)(x,\gamma). \tag{1.6}$$

Now, since $\widehat{\chi_G}(\gamma) = \int_G (x,\gamma)\, dm_G(x)$, Lemma 1.1.16 gives $\widehat{\chi_G}(\gamma) = \delta_{\gamma 0} m_G(G)$. From this fact and equation (1.6), equality (1.5) is obtained.

The following lemma is a straightforward consequence of Lemma 1.1.16 and statement (1.4).

Lemma 1.1.18. *If G is a compact LCA group and its dual Γ is countable, then the characters $\{\gamma \,:\, \gamma \in \Gamma\}$ form an orthogonal basis for $L^2(G)$.*

For an LCA group G and a countable uniform lattice K in G, we will denote by Ω_{K^*} a Borel section of Γ/K^*. In the remainder of this thesis we will identify $L^p(\Omega_{K^*})$ with the set $\{\varphi \in L^p(\Gamma) : \varphi = 0 \ a.e.\ \Gamma \setminus \Omega_{K^*}\}$ for $p = 1$ and $p = 2$, and $L^\infty(\Omega_{K^*})$ with the set $\{\varphi \in L^\infty(\Gamma) : \varphi = 0 \ a.e.\ \Gamma \setminus \Omega_{K^*}\}$.

Let us now define the functions $\eta_k : \Gamma \mapsto \mathbb{C}$, as $\eta_k(\gamma) = (k, -\gamma)\chi_{\Omega_{K^*}}(\gamma)$.

Then we have:

Proposition 1.1.19. *Let G be an LCA group and K a countable uniform lattice in G. Then, $\{\eta_k\}_{k\in K}$ is an orthogonal basis for $L^2(\Omega_{K^*})$. Moreover, if $m_\Gamma(\Omega_{K^*}) = 1$ then, $\{\eta_k\}_{k\in K}$ is an orthonormal basis for $L^2(\Omega_{K^*})$*

Proof. Since Γ/K^* is compact and equality (1.2) holds, applying Theorem 1.1.10 we obtain,

$$\begin{aligned}\int_\Gamma \eta_k(\gamma)\,dm_\Gamma(\gamma) &= \int_\Gamma (k,-\gamma)\chi_{\Omega_{K^*}}(\gamma)\,dm_\Gamma(\gamma)\\ &= m_{K^*}(\{0\})\int_{\Gamma/K^*}(k,-[\gamma])(\sum_{\delta\in K^*}\chi_{\Omega_{K^*}}(\gamma+\delta))\,dm_{\Gamma/K^*}([\gamma])\\ &= m_{K^*}(\{0\})\int_{\Gamma/K^*}(k,-[\gamma])\,dm_{\Gamma/K^*}([\gamma])\\ &= m_{K^*}(\{0\})\,m_{\Gamma/K^*}(\Gamma/K^*)\,\delta_{k0} = m_\Gamma(\Omega_{K^*})\,\delta_{k0}.\end{aligned}$$

Then, $\{\eta_k\}_{k\in K}$ is an orthogonal set in $L^2(\Omega_{K^*})$.

Let us see completeness. Each $\varphi \in L^2(\Omega_{K^*})$ induces a function φ' defined in Γ/K^* as $\varphi'([\gamma]) = \sum_{\delta\in K^*}\varphi(\gamma+\delta)$. Note that $\varphi'([\gamma]) = \frac{1}{m_{K^*}(\{0\})}\int_{K^*}\varphi(\gamma+\delta)\,dm_{K^*}(\delta)$. Then, as a consequence of Theorem 1.1.10, $\varphi'([\gamma])$ is a m_{Γ/K^*}-measurable function and we have that $\varphi' \in L^2(\Gamma/K^*)$.

Now let $\varphi \in L^2(\Omega_{K^*})$ be a function such that $\int_\Gamma \varphi(\gamma)\eta_k(\gamma)dm_\Gamma(\gamma) = 0$, for all $k \in K$. Then,

$$\begin{aligned}0 &= \int_\Gamma \varphi(\gamma)(k,-\gamma)\chi_{\Omega_{K^*}}(\gamma)\,dm_\Gamma(\gamma)\\ &= m_{K^*}(\{0\})\int_{\Gamma/K^*}\sum_{\delta\in K^*}\varphi(\gamma+\delta)(k,-\gamma+\delta)\chi_{\Omega_{K^*}}(\gamma+\delta)\,dm_{\Gamma/K^*}([\gamma])\\ &= m_{K^*}(\{0\})\int_{\Gamma/K^*}\varphi'([\gamma])(k,-[\gamma])\,dm_{\Gamma/K^*}([\gamma]),\end{aligned}$$

for all $k \in K$. Therefore, since Γ/K^* is compact, by Lemma 1.1.18 we obtain that $\varphi' = 0$. This proves that $\varphi = 0$ a.e. in Ω_{K^*}. So $\{\eta_k\}_{k\in K}$ is complete system in $L^2(\Omega_{K^*})$. □

Remark 1.1.20. As we have done in Proposition 1.1.19, we can associate to each $\varphi \in L^2(\Omega_{K^*})$, a function φ' defined on Γ/K^* as $\varphi'([\gamma]) = \sum_{\delta\in K^*}\varphi(\gamma+\delta)$. Since

$$||\varphi||^2_{L^2(\Omega_{K^*})} = m_{K^*}(\{0\})||\varphi'||^2_{L^2(\Gamma/K^*)},$$

the correspondence $\varphi \mapsto \varphi'$ is an isometric isomorphism up to a constant between $L^2(\Omega_{K^*})$ and $L^2(\Gamma/K^*)$.

Combining the above remark, Proposition 1.1.19, and the relationships established in Theorem 1.1.4, we obtain the following proposition, which will be very important on the remainder of the thesis.

Proposition 1.1.21. *Let G be an LCA group, K countable uniform lattice in G, $\Gamma = \widehat{G}$ and K^* the annihilator of K. Fix Ω_{K^*} a Borel section of Γ/K^* and choose m_K and m_{Γ/K^*} such that the inversion formula holds. Then*

$$\|a\|_{\ell^2(K)} = \frac{m_K(\{0\})^{1/2}}{m_\Gamma(\Omega_{K^*})^{1/2}} \|\sum_{k\in K} a_k \eta_k\|_{L^2(\Omega_{K^*})},$$

for each $a = \{a_k\}_{k\in K} \in \ell^2(K)$.

Proof. Let $a \in \ell^2(K)$. Thus,

$$\|a\|_{\ell^2(K)} = \|\widehat{a}\|_{L^2(\Gamma/K^*)}, \tag{1.7}$$

since $\widehat{K} \approx \Gamma/K^*$ and therefore $\wedge : K \to \Gamma/K^*$.

Take $\varphi(\gamma) = \sum_{k\in K} a_k(k, -\gamma)\chi_{\Omega_{K^*}}(\gamma)$. Then, by Proposition 1.1.19, $\varphi \in L^2(\Omega_{K^*})$. Furthermore, $\varphi'([\gamma]) = \varphi(\gamma)$, a.e. $\gamma \in \Omega_{K^*}$. So, as a consequence of Remark 1.1.20, we have

$$\|\varphi'\|^2_{L^2(\Gamma/K^*)} = \frac{1}{m_{K^*}(\{0\})}\|\varphi\|^2_{L^2(\Omega_{K^*})}. \tag{1.8}$$

Now, $\widehat{a}([\gamma]) = m_K(\{0\}) \sum_{k\in K} a_k(k, -[\gamma])$. Therefore, substituting in equations (1.7) and (1.8),

$$\|a\|_{\ell^2(K)} = \frac{m_K(\{0\})}{m_{K^*}(\{0\})^{1/2}}\|\varphi\|_{L^2(\Omega_{K^*})}.$$

Finally, since $m_\Gamma(\Omega_{K^*}) = m_{K^*}(\{0\})\, m_{\Gamma/K^*}(\Gamma/K^*)$, using (1.5) we have that

$$\frac{m_K(\{0\})}{m_{K^*}(\{0\})^{1/2}} = \frac{m_K(\{0\})^{1/2}}{m_\Gamma(\Omega_{K^*})^{1/2}},$$

which completes the proof. □

We finish this section with a result which is a consequence of statement (1.4) and Theorem 1.1.10.

Proposition 1.1.22. *Let G, K and Ω_{K^*} as in Proposition 1.1.21 and fix m_Γ, m_{K^*} and m_{Γ/K^*} such that Theorem 1.1.10 holds. If $\phi \in L^1(\Omega_{K^*})$ and $\widehat{\phi}(k) = 0$ for all $k \in K$, then $\phi(\omega) = 0$ a.e. $\omega \in \Omega_{K^*}$.*

Proof. Take $\phi \in L^1(\Omega_{K^*})$. Then, in the same way as in Remark 1.1.20, we can associate to ϕ a function ϕ' defined over Γ/K^* as $\phi'([\gamma]) = \sum_{\delta\in K^*} \phi(\gamma + \delta)$. Since $\phi \in L^1(\Omega_{K^*})$, it holds holds that $\phi' \in L^1(\Gamma/K^*)$.

Now, let $k \in K$. Then, by Theorem 1.1.10

$$\begin{aligned}
\widehat{\phi}(k) &= \int_{\Gamma} \phi(\gamma)\chi_{\Omega_{K^*}}(\gamma)(-k,\gamma)\, dm_{\Gamma}(\gamma) \\
&= \int_{\Gamma/K^*}\int_{K^*} \phi(\gamma+\delta)\chi_{\Omega_{K^*}}(\gamma+\delta)(-k,\gamma+\delta)\, dm_{K^*}(\delta)\, dm_{\Gamma/K^*}([\gamma]) \\
&= m_{K^*}(\{0\})\int_{\Gamma/K^*}(-k,[\gamma])\sum_{\delta\in K^*}\phi(\gamma+\delta)\chi_{\Omega_{K^*}}(\gamma+\delta)\, dm_{\Gamma/K^*}([\gamma]) \\
&= m_{K^*}(\{0\})\int_{\Gamma/K^*}(-k,[\gamma])\phi'([\gamma])\, dm_{\Gamma/K^*}([\gamma]) \\
&= m_{K^*}(\{0\})\widehat{\phi'}(k).
\end{aligned}$$

Therefore, $\widehat{\phi'}(k) = 0$ for all $k \in K$ and according to (1.4) we can conclude that $\phi' = 0$ a.e. $[\gamma] \in \Gamma/K^*$. Thus, $\phi = 0$ a.e. $\omega \in \Omega_{K^*}$. □

1.2 K-invariant Spaces

This section is devoted to introduce the definition of K-invariant spaces on LCA groups. This notion generalizes the usual definition of shift invariant spaces (SIS) on $\mathbb{R}^d$, where the translations are in $\mathbb{Z}^d$. The theory of SIS has been mainly developed by [dBDR94a], [dBDR94b], [Bow00], [Hel64], [RS95] in the last twenty years. It has an important role in approximation theory, wavelets, frames and it is useful to shape problems about signal processing.

For every $y \in G$ and $f \in L^2(G)$ we denote by $T_y f$ the translation of f by the element y defined as $T_y f(x) = f(x-y)$ a.e. $x \in G$.

Definition 1.2.1. Let $K \subseteq G$ be a closed subgroup of G. We say that a closed subspace $V \subseteq L^2(G)$ is *K-invariant* or *shift invariant under translations in K* if

$$f \in V \Rightarrow T_k f \in V \quad \forall\ k \in K.$$

For a subset $\mathcal{A} \subseteq L^2(G)$, we define

$$E_K(\mathcal{A}) = \{T_k\varphi : \varphi \in \mathcal{A}, k \in K\} \quad \text{and} \quad S_K(\mathcal{A}) = \overline{\operatorname{span}}\, E_K(\mathcal{A}).$$

A straightforward computation shows that the space $S_K(\mathcal{A})$ is a K-invariant space. Thus, we will call $S_K(\mathcal{A})$ the K-invariant space generated by $\mathcal{A}$. In case that $\mathcal{A} = \{\varphi\}$, we simply write $E_K(\varphi)$ and $S_K(\varphi)$ instead of $E_K(\{\varphi\})$ and $S_K(\{\varphi\})$. We call $S_K(\varphi)$ the *principal K-invariant space* generated by φ. When $\mathcal{A}$ is a finite set we say that $S_K(\mathcal{A})$ is a *finitely generated K-invariant space*.

Each K-invariant space V can be describe as $S_K(\mathcal{A})$ for some subset $\mathcal{A} \subseteq L^2(G)$. Indeed, $V = S_K(V)$.

An important fact is that, when G is a second countable LCA group, every K-invariant space V is generated by a countable (finite or countably infinite) set. That is, there exists a countable subset $\mathcal{A}$ of $L^2(G)$ such that $V = S_K(\mathcal{A})$. This fact is due to, if G is second countable, then $L^2(G)$ is a separable Hilbert space.

We now give some examples of K-invariant spaces.

Example 1.2.2.

(I) For $G = \mathbb{R}$ and $K = \mathbb{Z}$ consider the principal $\mathbb{Z}$-invariant space generated by φ where $\widehat{\varphi} = \chi_{[-\frac{1}{2},\frac{1}{2}]}$. This is the classical $\mathbb{Z}$-invariant space called the Paley-Wiener space and it is usually noted as $PW(\mathbb{R})$. The Paley-Wiener space can be described as

$$PW(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \ : \ \operatorname{supp}(\widehat{f}) \subseteq [-\frac{1}{2}, \frac{1}{2}]\}$$

and it also known as the space of bandlimited functions.

This space has an important roll for both theory and applications. For instance, it is involved in the Shannon sampling Theorem which states that each function $f \in PW(\mathbb{R})$ can be reconstructed from the set $\{f(k)\}_{k\in\mathbb{Z}}$. More precisely, if $f \in PW(\mathbb{R})$ then

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)T_k\varphi(x),$$

with the series converging uniformly on $\mathbb{R}$, as well as in $L^2(\mathbb{R})$. The Sampling Theorem was generalized to the context of LCA groups by Kluvánek. For details see [Dod07], [Klu65]. For further results concerning The Sampling Theorem on LCA groups, we also refer to [FG07], [FP03a] and [FP03b].

Another important property about the space $PW(\mathbb{R})$ is that if $\alpha \in \mathbb{R}$ and $f \in PW(\mathbb{R})$, then $T_\alpha f \in PW(\mathbb{R})$. That is, $PW(\mathbb{R})$ is invariant under any translation in $\mathbb{R}$.

(II) Consider the function $\phi = \chi_{[-\frac{1}{2},\frac{1}{2}]} \in L^2(\mathbb{R})$. Then, $S_{\mathbb{Z}}(\phi) \subsetneq L^2(\mathbb{R})$ is a $\mathbb{Z}$-invariant space and, in contrast to the above example, one can easily check that $S_{\mathbb{Z}}(\phi)$ is invariant only under translations in $\mathbb{Z}$. In other words, it holds that if $f \in S_{\mathbb{Z}}(\phi)$, $\alpha \in \mathbb{R}$ and $T_\alpha f \in S_{\mathbb{Z}}(\phi)$ then α must belong to $\mathbb{Z}$.

(III) Suppose now that $G = \mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ for a fixed $n \in \mathbb{N}$. Take $k, d \in \mathbb{N}$ such that $n = k.d$ and let $K \subseteq G$ the subgroup of G defined as $K = \{0, d, 2d, \cdots, d(n-1)\}$ (i.e. $K \approx \mathbb{Z}_k$).

If $\varphi \in L^2(\mathbb{Z}_n)$ is the function $\varphi = (1, 0, \cdots, 0) = e_0$ then,

$$S_K(\varphi) = \operatorname{span}\{e_0, e_d, e_{2d}, \cdots, e_{d(n-1)}\}$$

where for $j \in \{0, 1, 2, \cdots, n-1\}$, e_j is such that $e_j(i) = 0$ for $i \neq j$ and $e_j(j) = 1$.

1.3 Spaces of Vector-valued Functions

This section deal with some basic and known definitions and properties of vector-valued functions. It is not our purpose to get into details. We will only state here the results that will be needed in this thesis and they will be described in a way suitable for our purpose of concrete applications. For details about this theme see [DU77] and [Muj86].

The basis of this material is $(X, \sum, \lambda)$ a finite measure space and $\mathcal{H}$ a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and norm $||\cdot||_{\mathcal{H}}$.

We begin with the notion of measurability of vector-valued functions.

Definition 1.3.1.

(1) A function $F : X \to \mathcal{H}$ is called *simple* if there exist $x_1, x_2, \cdots, x_n \in \mathcal{H}$ and $E_1, E_2, \cdots, E_n \in \sum$ such that $F(\omega) = \sum_{i=1}^{n} x_i \chi_{E_i}(\omega)$.

(2) We say that a function $F : X \to \mathcal{H}$ is *strongly measurable* if there exists a sequence of simple functions $\{F_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} ||F_n(\omega) - F(\omega)||_{\mathcal{H}} = 0$ for a.e. $\omega \in X$.

(3) If $F : X \to \mathcal{H}$, we say that F is *weakly measurable* if for each $x^* \in \mathcal{H}^*$, the function x^*F from X to $\mathbb{C}$ is measurable in the usual sense. Here $\mathcal{H}^*$ denotes the dual space of $\mathcal{H}$.

The usual facts concerning to the stability of weakly and strongly measurable functions under sum, scalar multiplication and pointwise (almost everywhere) limits are valid.

As is expected, strongly measurability implies weakly measurability. Indeed. Let $F : X \to \mathcal{H}$ be a strongly measurable function. Then, there exists a sequence of simple functions $\{F_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} ||F_n(\omega) - F(\omega)||_{\mathcal{H}} = 0$ for a.e. $\omega \in X$. For each $x^* \in \mathcal{H}^*$, x^*F_n is a simple function in the usual sense for all $n \in \mathbb{N}$ and, since $|x^*F_n(\omega) - x^*F(\omega)| \leq ||x^*||_{op}||F_n(\omega) - F(\omega)||_{\mathcal{H}}$, we have that x^*F can be approximated by simple functions from X to $\mathbb{C}$. Then, x^*F is measurable in the usual sense.

The relationship between strong and weak measurability is given by the following result, known as Pettis' theorem or Pettis measurability theorem.

Theorem 1.3.2 (Pettis' Measurability Theorem)**.** *A function $F : X \to \mathcal{H}$ is strongly measurable if and only the following two conditions hold:*

(i) There exists $E \in \sum$ with $\lambda(E) = 0$ such that $F(X \setminus E)$ is a separable subset of $\mathcal{H}$.

(ii) F is weakly measurable.

Remark 1.3.3. If the Hilbert space $\mathcal{H}$ is a separable space, as a consequence of Pettis' Theorem, the concepts of weak and strong measurability agree. Therefore, in separable Hilbert spaces there is only one measurability notion. According to Riesz representation theorem we can state that, if $\mathcal{H}$ is a separable Hilbert spaces, then $F : X \to \mathcal{H}$ is measurable if and only if for each $a \in \mathcal{H}$ the function $\omega \mapsto \langle F(\omega), a\rangle_{\mathcal{H}}$ is a measurable function from X into $\mathbb{C}$.

We now state a definition that will be used later for a specific choice of X and $\mathcal{H}$.

Definition 1.3.4. Let $(X, \sum, \lambda)$ be a finite measure space and $\mathcal{H}$ a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. We define the space $L^2(X,\mathcal{H})$ as the space of all measurable functions $\Phi : X \to \mathcal{H}$ such that

$$\|\Phi\|_2^2 := \int_X \|\Phi(\omega)\|_{\mathcal{H}}^2 \, d\lambda(\omega) < \infty.$$

The space $L^2(X,\mathcal{H})$, with the inner product

$$\langle\Phi,\Psi\rangle := \int_X \langle\Phi(\omega),\Psi(\omega)\rangle_{\mathcal{H}} \, d\lambda(\omega)$$

is a complex Hilbert space.

We will also need to deal with spaces like $L^2(X,\mathcal{H})$ when X has not finite measure. In this case we have the following definition.

Definition 1.3.5. Let $(Y, \sum, \nu)$ be a σ-finite measure space. That is $Y = \bigcup_{n\in\mathbb{N}} Y_n$ with $\nu(Y_n) < +\infty$. If $\mathcal{H}$ is a separable Hilbert space, the space $L^2(Y,\mathcal{H})$ consists of all vector-valued strongly measurable functions $\Phi : Y \to \mathcal{H}$ with

$$\|\Phi\|_2^2 := \int_Y \|\Phi(\omega)\|_{\mathcal{H}}^2 \, d\nu(\omega) < \infty.$$

Here *strongly measurable* is defined in the same way as in Definition 1.3.1, item (1).

With the inner product $\langle\Phi,\Psi\rangle := \int_Y \langle\Phi(\omega),\Psi(\omega)\rangle_{\mathcal{H}} \, d\nu(\omega)$, $L^2(Y,\mathcal{H})$ is a Hilbert space.

1.4 Fiberization Isometry

In this section we develop the fiberization isometry needed for the next chapter where shift invariant spaces under uniform lattices are characterized. The treatment will follow the lines of [Bow00]. In that work, the author states the fiberization isometry for the case where $G = \mathbb{R}^d$ and $H = \mathbb{Z}^d$.

First we will fix some notation and set the assumptions that will be in force for the remainder of this section.

Assumptions 1.4.1. We will assume throughout this section that:

- G is a second countable LCA group.
- H is a countable uniform lattice on G.

We denote by Γ the dual group of G, by Δ the annihilator of H (i.e. $H^* = \Delta$), and by Ω_Δ a fixed Borel section of Γ/Δ.

In order to avoid carrying over constants through the thesis, we choose the Haar measure m_Δ such that $m_\Delta(\{0\}) = 1$. We also fix m_G and m_Γ such that the inversion formula holds. This particular choice does not affect the validity of the results included here.

Note that under our Assumptions 1.4.1, Theorem 1.1.9 applies. So we will use the properties of G, H, Γ and Δ stated in that theorem.

As we have seen in Section 1.1 and according to our hypotheses, Δ is a countable uniform lattice on Γ. Therefore, $\ell^2(\Delta)$ is a separable Hilbert space. Besides, since Ω_Δ has finite m_Γ-measure (see Lemma 1.1.13) we have the space $L^2(\Omega_\Delta, \ell^2(\Delta))$ defined according to Definition 1.3.4.

Note that for $\Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$ and $\omega \in \Omega_\Delta$

$$\|\Phi(\omega)\|_{\ell^2(\Delta)} = \Big(\sum_{\delta\in\Delta} |(\Phi(\omega))_\delta|^2\Big)^{1/2},$$

where $(\Phi(\omega))_\delta$ denotes the value of the sequence $\Phi(\omega)$ in δ. If $\Phi \in L^2(\Omega, \ell^2(\Delta))$, the sequence $\Phi(\omega)$ is the *fiber of* Φ *at* ω.

The following proposition shows that the space $L^2(\Omega_\Delta, \ell^2(\Delta))$ is isometric to $L^2(G)$.

Proposition 1.4.2. *The mapping* $\mathcal{T}_H : L^2(G) \longrightarrow L^2(\Omega_\Delta, \ell^2(\Delta))$ *defined as*

$$\mathcal{T}_H f(\omega) = \{\widehat{f}(\omega + \delta)\}_{\delta\in\Delta},$$

is an isomorphism that satisfies $\|\mathcal{T}_H f\|_2 = \|f\|_{L^2(G)}$. *Therefore we will usually refer to* $\mathcal{T}_H$ *as the fiberization isometry.*

The next periodization lemma will be necessary for the proof of Proposition 1.4.2.

Lemma 1.4.3. *Let* $g \in L^2(\Gamma)$. *Define the function* $\mathcal{G}(\omega) = \sum_{\delta\in\Delta} |g(\omega + \delta)|^2$. *Then,* $\mathcal{G} \in L^1(\Omega_\Delta)$ *and moreover*

$$\|g\|_{L^2(\Gamma)} = \|\mathcal{G}\|_{L^1(\Omega_\Delta)}.$$

Proof. Since Ω_Δ is a section of Γ/Δ, we have that $\Gamma = \bigcup_{\delta\in\Delta} \Omega_\Delta - \delta$, where the union is disjoint. Therefore,

$$\begin{aligned}
\int_\Gamma |g(\gamma)|^2 \, dm_\Gamma(\gamma) &= \sum_{\delta\in\Delta} \int_{\Omega_\Delta - \delta} |g(\omega)|^2 \, dm_\Gamma(\omega) \\
&= \sum_{\delta\in\Delta} \int_{\Omega_\Delta} |g(\omega + \delta)|^2 \, dm_\Gamma(\omega) \\
&= \int_{\Omega_\Delta} \sum_{\delta\in\Delta} |g(\omega + \delta)|^2 \, dm_\Gamma(\omega).
\end{aligned}$$

This proves that $\mathcal{G} \in L^1(\Omega_\Delta)$ and $\|g\|_{L^2(\Gamma)} = \|\mathcal{G}\|_{L^1(\Omega_\Delta)}$. ◻

Proof of Proposition 1.4.2. First we prove that $\mathcal{T}_H$ is well defined. For this we must show that, $\forall f \in L^2(G)$, the vector-valued function $\mathcal{T}_H f$ is measurable and $\|\mathcal{T}_H f\|_2 < \infty$.

According to Lemma 1.4.3, the sequence $\{\widehat{f}(\omega+\delta)\}_{\delta\in\Delta} \in \ell^2(\Delta)$, a.e. $\omega \in \Omega_\Delta$, for all $f \in L^2(G)$. Then, given $a = \{a_\delta\}_{\delta\in\Delta} \in \ell^2(\Delta)$, the product $\langle \mathcal{T}_H f(\omega), a\rangle_{\ell^2(\Delta)} = \sum_{\delta\in\Delta} \widehat{f}(\omega+\delta)a_\delta$ is finite a.e. $\omega \in \Omega_\Delta$. From here the measurability of f implies that $\omega \mapsto \langle \mathcal{T}_H f(\omega), a\rangle_{\ell^2(\Delta)}$ is a measurable function in the usual sense. This proves the measurability of $\mathcal{T}_H f$.

If $f \in L^2(G)$, as a consequence of Lemma 1.4.3, we have

$$\begin{aligned}
\|\mathcal{T}_H f\|_2^2 &= \int_{\Omega_\Delta} \|\mathcal{T}_H f(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) \\
&= \int_{\Omega_\Delta} \sum_{\delta\in\Delta} |\widehat{f}(\omega+\delta)|^2\, dm_\Gamma(\omega) \\
&= \int_\Gamma |\widehat{f}(\gamma)|^2\, dm_\Gamma(\gamma) \\
&= \int_G |f(x)|^2\, dm_G(x).
\end{aligned}$$

Thus, $\|\mathcal{T}_H f\|_2 < \infty$ and this also proves that $\|\mathcal{T}_H f\|_2 = \|f\|_{L^2(G)}$.

What is left is to show that $\mathcal{T}_H$ is onto. So, given $\Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$ let us see that there exists a function $f \in L^2(G)$ such that $\mathcal{T}_H f = \Phi$. Using that the Fourier transform is an isometric isomorphism between $L^2(G)$ and $L^2(\Gamma)$, it will be sufficient to find $g \in L^2(\Gamma)$ such that $\{g(\omega+\delta)\}_{\delta\in\Delta} = \Phi(\omega)$ a.e. $\omega \in \Omega_\Delta$ and then take $f \in L^2(G)$ such that $\widehat{f} = g$.

Given $\gamma \in \Gamma$, there exist unique $\omega \in \Omega_\Delta$ and $\delta \in \Delta$ such that $\gamma = \omega + \delta$. So, we define $g(\gamma)$ as $g(\gamma) = (\Phi(\omega))_\delta$.

Let us see that g is measurable. For this, we will prove that its real and imaginary parts are measurable. Let $\alpha \in \mathbb{R}$ and fix $\delta \in \Delta$. Then,

$$\begin{aligned}
\{\gamma\in\Gamma : Re(g(\gamma)) > \alpha\} \cap (\Omega_\Delta + \delta) &= \{\omega+\delta : \omega\in\Omega_\Delta \;\text{ y }\; Re(g(\omega+\delta)) > \alpha\} \\
&= \{\omega\in\Omega_\Delta : Re(g(\omega+\delta)) > \alpha\} + \delta \\
&= \{\omega\in\Omega_\Delta : Re((\Phi(\omega))_\delta) > \alpha\} + \delta \\
&= \{\omega\in\Omega_\Delta : Re(\langle\Phi(\omega), e_\delta\rangle) > \alpha\} + \delta,
\end{aligned}$$

where e_δ is the sequence in $\ell^2(\Delta)$ which has value one in δ place and value zero in the rest. Thus, since Φ is measurable, we have that $\{\gamma \in \Gamma : Re(g(\gamma)) > \alpha\} \cap (\Omega_\Delta + \delta)$ is a measurable set and then, so is $\{\gamma \in \Gamma : Re(g(\gamma)) > \alpha\}$.

Proceeding in the same way for the imaginary part of g, it results that g is a measurable function and it remains to be proved that g belongs to $L^2(\Gamma)$.

Once again, according to Lemma 1.4.3,

$$\begin{aligned}\int_{\Gamma}|g(\gamma)|^2\,dm_{\Gamma}(\gamma) &= \int_{\Omega_\Delta}\sum_{\delta\in\Delta}|g(\omega+\delta)|^2\,dm_{\Gamma}(\omega)\\ &= \int_{\Omega_\Delta}\sum_{\delta\in\Delta}|(\Phi(\omega)_\delta)|^2\,dm_{\Gamma}(\omega)\\ &= \int_{\Omega_\Delta}\|\Phi(\omega)\|^2_{\ell^2(\Delta)}\,dm_{\Gamma}(\omega)\\ &= \|\Phi\|_2^2 < +\infty.\end{aligned}$$

Thus, $g \in L^2(\Gamma)$ and this completes the proof. □

The mapping $\mathcal{T}_H$ will be important to study the properties of functions of $L^2(G)$ in terms of their fibers, (i.e. in terms of the fibers $\mathcal{T}_H f(\omega)$).

Since the Haar measure is translation-invariant, we can compute

$$\widehat{T_y f}(\gamma) = \int_G f(x-y)(x,-\gamma)\,dm_g(x) = \int_G f(x)(x+y,-\gamma)\,dm_g(x) = (y,-\gamma)\widehat{f}(\gamma).$$

This property combined with Remark 1.1.6 implies a very important property of $\mathcal{T}_H$ that will be crucial for what follows.

Remark 1.4.4. For $f \in L^2(G)$ and $h \in H$, it holds that

$$\mathcal{T}_H T_h f(\omega) = (h,-\omega)\mathcal{T}_H f(\omega).$$

To finish this section, we present some examples to illustrated the fiberization isometry $\mathcal{T}_H$.

Example 1.4.5.

(I) When $G = \mathbb{R}^d$ and $H = \mathbb{Z}^d$ then $\mathcal{T}_{\mathbb{Z}^d} : L^2(\mathbb{R}^d) \to L^2([0,1)^d, \ell^2(\mathbb{Z}^d))$ is given by $\mathcal{T}_H f(\omega) = \{\widehat{f}(\omega+k)\}_{k\in\mathbb{Z}^d}$. This map was introduced by Bownik in [Bow00] where the author followed an idea from Helson's book, [Hel64].

(II) In case that $G = \mathbb{Z}$ and $H = m\mathbb{Z}$ for a fixed $m \in \mathbb{N}$, we have that $\Gamma = \mathbb{T}$ and a trivial verification shows that $\Delta = \frac{1}{m}\mathbb{Z}_m = \{0, \frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}\}$. Therefore, a measurable section for the quotient $\mathbb{T}/\frac{1}{m}\mathbb{Z}_m$ is $[0, \frac{1}{m})$. Then, $\mathcal{T}_{m\mathbb{Z}} : \ell^2(\mathbb{Z}) \to L^2([0,\frac{1}{m}), \ell^2(\frac{1}{m}\mathbb{Z}_m))$ and if $a \in \ell^2(\mathbb{Z})$ it holds that

$$\mathcal{T}_{m\mathbb{Z}} a(\omega) = (\widehat{a}(\omega), \widehat{a}(\omega+\frac{1}{m}), \cdots, \widehat{a}(\omega+\frac{m-1}{m})).$$

(III) Suppose now that $G = \mathbb{T}$ and $H = \frac{1}{m}\mathbb{Z}_m = \{0, \frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}\}$. Thus, as we have seen in Example 1.1.3 item (II), $\Gamma = \mathbb{Z}$ and it can be proved that $\Delta = m\mathbb{Z}$. Thus,

$$\mathcal{T}_{\frac{1}{m}\mathbb{Z}_m} : L^2(\mathbb{T}) \to L^2(\{0, 1, \cdots, m-1\}, \ell^2(m\mathbb{Z}))$$

and for $f \in L^2(\mathbb{T})$ we obtain that for each $k \in \{0, 1, \cdots, m-1\}$, $\mathcal{T}_{\frac{1}{m}\mathbb{Z}_m} f(k)$ is the subsequence $\{a_{k_j}\}_{j\in\mathbb{Z}}$ of $\widehat{f} \in \ell^2(\mathbb{Z})$ given by $a_{k_j} = \widehat{f}_{k+mj}$.

(IV) The last example is the finite case, when $G = \mathbb{Z}_n$ and $H = \{0, k, 2k, \cdots, (d-1)k\} \approx \mathbb{Z}_d$ with $n = k.d$. According to Example 1.1.3 item (IV), we can identify Γ with $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$. Moreover, it holds that $\Delta = \{0, \frac{d}{n}, \frac{2d}{n}, \cdots, \frac{(k-1)d}{n}\} \approx \mathbb{Z}_k$ and then, we can consider $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{(d-1)}{n}\}$ as a section for the quotient Γ/Δ. Thus, for each $v \in \ell^2(\mathbb{Z}_n) \approx \mathbb{C}^n$ and $j \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{(d-1)}{n}\}$ we have

$$\mathcal{T}_H v(j) = (\widehat{v}(j), \widehat{v}(j + \frac{d}{n}), \cdots, \widehat{v}(j + \frac{(k-1)d}{n})).$$

2

Shift Invariant Spaces under Uniform Lattices in LCA Groups

The theory of shift invariant spaces in $L^2(\mathbb{R}^d)$ (SIS) has been intensive studied by [Hel64], [Bow00], [dBDR94a], [dBDR94b], [RS95] in the last twenty years. Its importance lies in the fact that SIS have many applications in numerical analysis, multiresolution analysis, wavelets theory, frames and signal processing.

Since most properties about SIS is based on the action of $\mathbb{Z}^d$ as an additive subgroup of $\mathbb{R}^d$, we are interested in knowing if the theory if shift invariant spaces in $\mathbb{R}^d$ can be extended to general LCA groups. This is precisely our purpose in this chapter.

In order to generalize the notion of shift invariant spaces in $\mathbb{R}^d$ to LCA groups, we have introduced the concept of K-invariant spaces for a closed subgroup K of G (see Section 1.2 of Chapter 1). In this chapter we will investigate the structure of H-invariant spaces, with H being a countable uniform lattice in an LCA group G.

We have divided our analysis in two cases, principal and general H-invariant spaces. We will characterize principal H-invariant spaces in terms of the Fourier transform of its generator and the treatment for general H-invariant spaces will use range functions, following the ideas stated in [Hel64] and [Bow00].

Then, throughout this chapter, H will be a countable uniform lattice in an LCA group G. The dual group of G will be denoted by Γ and Δ will be the annihilator of H. By Ω_Δ we will denote a fixed Borel section for the quotient Γ/Δ. We will also assume that Assumptions 1.4.1 are valid.

The choice of particular Haar measure in each of the groups considered in this work does not affect the validity of the results. However, different constants will appear in the formulas.

Since we have the freedom to choose the Haar measures, we fix a particular normalization in order to avoid carrying over constants and to simplify the statements of the results.

In this chapter we fix the constants of the Haar measures. Since we will use the fiberization isometry defined in Section 1.4 we fix m_Δ such that $m_\Delta(\{0\}) = 1$. Then we choose m_Γ and $m_{\Gamma/\Delta}$ in order that Theorem 1.1.10 holds for m_Γ, $m_{\Gamma/\Delta}$ and m_Δ.

The rest of the chapter is organized in the following way. In Section 2.1 we study the structure of principal H-invariant spaces. Then, in Section 2.2 we consider the general case. For analyze general H-invariant spaces we introduce in Section 2.2.1 the concept of range functions. Finally, in Section 2.2.2 we provide a characterization of H-invariant spaces in terms of fibers.

2.1 Principal H-invariant spaces

We recall that a principal H-invariant space is an H-invariant space generated by a single function. Obviously, they can be characterized as general H-invariant spaces using the result that we will prove in the next section. However, we will show here that these particular H-invariant spaces, denoted as $S_H(\varphi)$, can be described in an elegant and simply way in terms of the Fourier transform of its generator φ. This is stated in the following theorem.

Theorem 2.1.1. *Let $\varphi \in L^2(G)$. If $f \in S_H(\varphi)$, then there exists a measurebla Δ-periodic function η such that $\widehat{f} = \eta\widehat{\varphi}$.*

Conversely, if η is a measurable Δ-periodic function such that $\eta\widehat{\varphi} \in L^2(\Gamma)$, then the function f defined by $\widehat{f} = \eta\widehat{\varphi}$ belongs to $S_H(\varphi)$.

For the proof of the above theorem we need some previous lemmas and statements.

To begin with we give a description of the orthogonal complement of $S_H(\varphi)$ in terms of the fiberization isometry $\mathcal{T}_H$.

Proposition 2.1.2. *Let $\varphi \in L^2(G)$. Then, the orthogonal complement of $S_H(\varphi)$ in $L^2(G)$ is given by*

$$S_H(\varphi)^\perp = \{f \in L^2(G) : \langle \mathcal{T}_H f(\omega), \mathcal{T}_H \varphi(\omega)\rangle_{\ell^2(\Delta)} = 0 \text{ a.e. } \omega \in \Omega_\Delta\}.$$

Proof. Let $f \in S_H(\varphi)^\perp$. In particular, it holds that $\langle f, T_h\varphi\rangle_{L^2(G)} = 0$ for all $h \in H$. Then, $\langle \mathcal{T}_H f, \mathcal{T}_H T_h \varphi\rangle = 0$ for all $h \in H$.

Since $\langle \mathcal{T}_H f(\cdot), \mathcal{T}_H \varphi(\cdot)\rangle_{\ell^2(\Delta)} \in L^1(\Omega_\Delta)$ and

$$0 = \langle \mathcal{T}_H f, \mathcal{T}_H T_h \varphi\rangle = \int_{\Omega_\Delta} (-h, \omega)\langle \mathcal{T}_H f(\omega), \mathcal{T}_H \varphi(\omega)\rangle_{\ell^2(\Delta)}\, dm_\Gamma(\omega)$$

for all $h \in H$, Proposition 1.1.22 gives us $\langle \mathcal{T}_H f(\omega), \mathcal{T}_H \varphi(\omega)\rangle_{\ell^2(\Delta)} = 0$ a.e. $\omega \in \Omega_\Delta$. Therefore, $S_H(\varphi)^\perp \subseteq \{f \in L^2(G) : \langle \mathcal{T}_H f(\omega), \mathcal{T}_H \varphi(\omega)\rangle_{\ell^2(\Delta)} = 0 \text{ a.e. } \omega \in \Omega_\Delta\}$.

On the other hand, if $f \in L^2(G)$ satisfies $\langle \mathcal{T}_H f(\omega), \mathcal{T}_H \varphi(\omega)\rangle_{\ell^2(\Delta)} = 0$ a.e. $\omega \in \Omega_\Delta$, then in the same way as above we can prove that $\langle \mathcal{T}_H f, \mathcal{T}_H T_h \varphi\rangle = 0$ for all $h \in H$. Thus, using that $\mathcal{T}_H$ is an isometry, we obtain $\langle f, T_h\varphi\rangle_{L^2(G)}$ = for all $h \in H$. Therefore, since $\{T_h\varphi : h \in H\}$ is a dense set in $S_H(\varphi)$ it follows that $f \in S_H(\varphi)^\perp$ and this finishes the proof. □

We define now the m_Γ-measurable set $\mathcal{E}_\varphi \subseteq \Omega_\Delta$ as

$$\mathcal{E}_\varphi = \{\omega \in \Omega_\Delta : \|\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} \neq 0\}. \tag{2.1}$$

Then, for a $f \in L^2(G)$ we consider the function η_f defined over Ω_Δ as

$$\eta_f(\omega) = \begin{cases} \frac{\langle \mathcal{T}_H f(\omega), \mathcal{T}_H\varphi(\omega)\rangle_{\ell^2(\Delta)}}{\|\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}} & \text{if } \omega \in \mathcal{E}_\varphi \\ 0 & \text{if } \omega \in \Omega_\Delta \setminus \mathcal{E}_\varphi. \end{cases} \tag{2.2}$$

To define η_f over all Γ we extend it in a Δ-periodic way. This extension will be also denoted by η_f.

Lemma 2.1.3. *The function η_f defined in (2.2) satisfies:*

(a) $|\eta_f(\omega)|^2\|\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} \leq \|\mathcal{T}_H f(\omega)\|^2_{\ell^2(\Delta)}$ *a.e.* $\omega \in \Omega_\Delta$

(b) $\eta_f\widehat{\varphi} \in L^2(\Gamma)$. *Moreover,* $\|\eta_f\widehat{\varphi}\|_{L^2(\Gamma)} \leq \|f\|_{L^2(G)}$

Proof. Item (a) is a straightforward consequence of the Cauchy Schwartz inequality.

Let us prove now item (b). Using the Δ-periodicity of η_f and item (a), we obtain

$$\begin{aligned} \int_\Gamma |\eta_f(\gamma)\widehat{\varphi}(\gamma)|^2\, dm_\Gamma(\gamma) &= \int_{\Omega_\Delta}\sum_{\delta\in\Delta}|\eta_f(\omega+\delta)\widehat{\varphi}(\omega+\delta)|^2\, dm_\Gamma(\omega) \\ &= \int_{\Omega_\Delta}|\eta_f(\omega)|^2\|\mathcal{T}_H\varphi(\omega)\|{\ell^2(\Delta)}^2\, dm_\Gamma(\omega) \\ &\leq \int_{\Omega_\Delta}\|\mathcal{T}_H f(\omega)\|{\ell^2(\Delta)}^2\, dm_\Gamma(\omega) \\ &= \|\mathcal{T}_H f\|_2^2 = \|f\|^2_{L^2(G)}. \end{aligned}$$

Then, it holds that $\eta_f\widehat{\varphi} \in L^2(\Gamma)$ and that $\|\eta_f\widehat{\varphi}\|_{L^2(\Gamma)} \leq \|f\|_{L^2(G)}$ as we wanted to prove. □

An important consequence of Lemma 2.1.3 is that the operator $Q : L^2(\Gamma) \longrightarrow L^2(\Gamma)$ given by

$$Q\widehat{f} = \eta_f\widehat{\varphi}, \tag{2.3}$$

is well-defined and bounded.

We will denote by $\mathcal{P}$ the orthogonal projection onto $S_H(\varphi)$ and by $\widehat{\mathcal{P}}$ the orthogonal projection onto $\widehat{S_H(\varphi)}$, where the subspace $\widehat{S_H(\varphi)}$ is given by

$$\widehat{S_H(\varphi)} = \{\widehat{g} : g \in S_H(\varphi)\}.$$

An easy computation shows that

$$\widehat{\mathcal{P}f} = \widehat{\mathcal{P}}\widehat{f} \quad \text{for all } f \in L^2(G). \tag{2.4}$$

Lemma 2.1.4. *If $\mathcal{Q}$ is the operator defined in (2.3), then $\mathcal{Q} = \widehat{\mathcal{P}}$.*

Proof. We begin by proving $\mathcal{Q}\,|_{\widehat{S_H(\varphi)}} = \widehat{\mathcal{P}}\,|_{\widehat{S_H(\varphi)}}$, where $\mathcal{Q}\,|_{\widehat{S_H(\varphi)}}$ and $\widehat{\mathcal{P}}\,|_{\widehat{S_H(\varphi)}}$ are the restrictions of $\mathcal{Q}$ and $\widehat{\mathcal{P}}$ to $\widehat{S_H(\varphi)}$.

Let $h \in H$. Since $T_h\varphi \in S_H(\varphi)$, $\widehat{T_h\varphi} \in \widehat{S_H(\varphi)}$ and then we have that $\widehat{\mathcal{P}}(\widehat{T_h\varphi}) = \widehat{T_h\varphi} = (-h,\cdot)\widehat{\varphi}$.

On the other hand $\mathcal{Q}(\widehat{T_h\varphi}) = \eta_{T_h\varphi}\widehat{\varphi}$.

Note that, using Remark 1.4.4

$$\eta_{T_h\varphi}(\omega) = \begin{cases} (-h,\omega) & \text{if } \omega \in \mathcal{E}_\varphi \\ 0 & \text{if } \omega \in \Omega_\Delta \setminus \mathcal{E}_\varphi. \end{cases}$$

Therefore, since $\widehat{\varphi} = 0$ on $\left(\bigcup_{\delta\in\Delta}(\mathcal{E}_\varphi + \delta)\right)^c$, it holds that $\eta_{T_h\varphi}\widehat{\varphi} = (-h,\cdot)\widehat{\varphi}$.

Then $\widehat{\mathcal{P}}(\widehat{T_h\varphi}) = \mathcal{Q}(\widehat{T_h\varphi})$ and this is valid for all $h \in H$. Since $\mathcal{Q}$ and $\widehat{\mathcal{P}}$ are bounded operators and $\{T_h\varphi : h \in H\}$ is dense in $S_H(\varphi)$ we obtain $\mathcal{Q}\,|_{\widehat{S_H(\varphi)}} = \widehat{\mathcal{P}}\,|_{\widehat{S_H(\varphi)}}$.

We now want to show that the restrictions of $\mathcal{Q}$ and $\widehat{\mathcal{P}}$ to $\widehat{S_H(\varphi)}^\perp$ also agree.

Let $\widehat{f} \in \widehat{S_H(\varphi)}^\perp$. Thus, since $\widehat{S_H(\varphi)}^\perp = \widehat{S_H(\varphi)^\perp}$, we have that $\widehat{f} \in \widehat{S_H(\varphi)^\perp}$ and then $f \in S_H(\varphi)^\perp$.

Therefore, according to Proposition 2.1.2, $\langle \mathcal{T}_H f(\omega), \mathcal{T}_H\varphi(\omega)\rangle = 0$ a.e. $\omega \in \Omega_\Delta$. Thus, $\eta_f = 0$ a.e. $\omega \in \Omega_\Delta$ and then $\mathcal{Q}\widehat{f} = 0$. Hence $\mathcal{Q}\,|_{\widehat{S_H(\varphi)}^\perp} = \widehat{\mathcal{P}}\,|_{\widehat{S_H(\varphi)}^\perp}$.

Since $L^2(\Gamma)$ is the orthogonal sum of $\widehat{S_H(\varphi)}$ and $\widehat{S_H(\varphi)}^\perp$, that is $L^2(\Gamma) = \widehat{S_H(\varphi)} \oplus \widehat{S_H(\varphi)}^\perp$, we have that $\mathcal{Q} = \widehat{\mathcal{P}}$. □

Now, we are able to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Let $f \in S_H(\varphi)$. Then, since $\mathcal{P}f = f$, by (2.4) it holds that $\widehat{\mathcal{P}}\widehat{f} = \widehat{f}$. Therefore, according to Lemma 2.1.4, we have $\widehat{f} = \eta_f\widehat{\varphi}$.

Conversely. Let $f \in L^2(G)$ such that $\widehat{f} = \eta\widehat{\varphi}$ being η a Δ-periodic function with $\eta\widehat{\varphi} \in L^2(\Gamma)$.

Then,

$$\begin{aligned} \eta_f(\omega) &= \chi_{\mathcal{E}_\varphi}(\omega)\frac{\langle \mathcal{T}_H f(\omega), \mathcal{T}_H\varphi(\omega)\rangle_{\ell^2(\Delta)}}{\|\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}} \\ &= \chi_{\mathcal{E}_\varphi}(\omega)\frac{\eta(\omega)\langle \mathcal{T}_H \varphi(\omega), \mathcal{T}_H\varphi(\omega)\rangle_{\ell^2(\Delta)}}{\|\mathcal{T}_H\varphi(\omega)\|\ell^2(\Delta)^2} \\ &= \chi_{\mathcal{E}_\varphi}(\omega)\eta(\omega), \end{aligned}$$

for a.e. $\omega \in \Omega_\Delta$. Thus, since η and η_f are Δ-periodic functions, it holds that $\eta(\gamma) = \eta_f(\gamma)$ for a.e. $\gamma \in \bigcup_{\delta\in\Delta}(\mathcal{E}_\varphi + \delta)$.

On the other hand, $\widehat{\varphi} = 0$ on $\left(\bigcup_{\delta\in\Delta}(\mathcal{E}_\varphi + \delta)\right)^c$. Hence $\eta\widehat{\varphi} = \eta_f\widehat{\varphi}$.

Thus, $\widehat{\mathcal{P}}\widehat{f} = \widehat{f}$ and therefore $f \in S_H(\varphi)$. □

Remark 2.1.5. Observe that if $\eta_1\widehat{\varphi} = \eta_2\widehat{\varphi}$ on Γ for η_1 and η_2 Δ-periodic functions we can conclude that $\eta_1 = \eta_2$ on $\bigcup_{\delta\in\Delta}(\mathcal{E}_\varphi + \delta)$. Then, η_f in Theorem 2.1.1 is, in general, not unique.

Corollary 2.1.6. *Let $\varphi, \phi \in L^2(G)$ with $S_H(\varphi) = S_H(\phi)$. Then $\mathcal{E}_\varphi = \mathcal{E}_\phi$ up to a set of zero m_Γ-measure.*

Proof. Since $\varphi \in S_H(\phi)$, by Theorem 2.1.1 there exists η a measurable Δ-periodic function such that $\widehat{\varphi} = \eta\widehat{\phi}$. Hence, $\|\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} = |\eta(\omega)|^2\|\mathcal{T}_H\phi(\omega)\|^2_{\ell^2(\Delta)}$ for a.e. $\omega \in \Omega_\Delta$. From this we obtain $\mathcal{E}_\varphi \subseteq \mathcal{E}_\phi$ up to a set of zero m_Γ-measure. In the same way it can be proved the other inclusion. □

2.2 General H-invariant spaces

In this section we give a characterization of general H-invariant spaces. This particular description of H-invariant spaces is done using a very deep tool called range function.

The concept of range function was first introduced by Helson in [Hel64]. Then, it became an essential tool for the treatment of shift invariant spaces (see [Bow00], [dBDR94a], [dBDR94b], [RS95]).

We prove here, that using range functions, H-invariant spaces can be described in terms of the fibers of their elements. This is particularly important in the class of finitely generated H-invariant spaces, since we can translate problems about these spaces (infinite dimensional problems) into finite dimensional problems.

2.2.1 Range Functions

We start this section with the concept of range function.

Definition 2.2.1. A *range function* is a mapping

$$J : \Omega_\Delta \longrightarrow \{\text{closed subspaces of } \ell^2(\Delta)\}.$$

The subspace $J(\omega)$ is called the *fiber space* associated to ω.

For a given range function J, we associate to each $\omega \in \Omega_\Delta$ the orthogonal projection onto $J(\omega)$, $P_\omega : \ell^2(\Delta) \to J(\omega)$.

A range function J is *measurable* if for each $a \in \ell^2(\Delta)$ the function $\omega \mapsto P_\omega a$, from Ω_Δ into $\ell^2(\Delta)$, is measurable. Recall that, since $\ell^2(\Delta)$ is a separable Hilbert space, by Remark 1.3.3, we have only one measurability notion for a function $\Psi : \Omega_\Delta \to \ell^2(\Delta)$. Moreover, Remark 1.3.3 gives us that $\omega \mapsto P_\omega a$, from Ω_Δ into $\ell^2(\Delta)$, is measurable if and only if, for each $a, b \in \ell^2(\Delta)$, $\omega \mapsto \langle P_\omega a, b\rangle$ is measurable in the usual sense.

Lemma 2.2.2. *Let J be a range function . Then, J is a measurable range function if and only if for all $\Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$, the function $\omega \mapsto P_\omega(\Phi(\omega))$ is measurable.*

Proof. Suppose first that J is measurable. Then, $\omega \mapsto P_\omega a$, from Ω_Δ into $\ell^2(\Delta)$, is measurable for all $a \in \ell^2(\Delta)$. Let Φ be a simple function. That is, $\Phi(\omega) = \sum_{i=1}^n a_i\chi_{E_i}(\omega)$ with $a_i \in \ell^2(\Delta)$ and $E_i \subseteq \Omega_\Delta$ measurable sets for $i = 1, \cdots, n$. Then, $P_\omega(\Phi(\omega)) = \sum_{i=1}^n P_\omega a_i\chi_{E_i}(\omega)$. Since $\omega \mapsto P_\omega a_i$ is measurable for all $i = 1, \cdots, n$, we can conclude that $\omega \mapsto P_\omega(\Phi(\omega))$ is measurable as well.

Let us consider now $\Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$. Then, according to Definition 1.3.1, there exists $\{\Phi_n\}_{n\in\mathbb{N}} \subseteq L^2(\Omega_\Delta, \ell^2(\Delta))$ a sequence of simple functions such that $\|\Phi_n(\omega)-\Phi(\omega)\|^2_{\ell^2(\Delta)} \to 0$ when $n \to \infty$, a.e. $\omega \in \Omega_\Delta$.

Since P_ω is an orthogonal projection, we have that

$$\|P_\omega(\Phi_n(\omega)) - P_\omega(\Phi(\omega))\|^2_{\ell^2(\Delta)} \leq \|\Phi_n(\omega) - \Phi(\omega)\|^2_{\ell^2(\Delta)}$$

for a.e. $\omega \in \Omega_\Delta$. Thus, $\|P_\omega(\Phi_n(\omega)) - P_\omega(\Phi(\omega))\|^2_{\ell^2(\Delta)} \to 0$ when $n \to \infty$, a.e. $\omega \in \Omega_\Delta$. This shows that $\omega \mapsto P_\omega(\Phi(\omega))$ is an almost everywhere point-wise limit of measurable functions. Then, $\omega \mapsto P_\omega(\Phi(\omega))$ is measurable as well.

The other implication is straightforward.

□

Given a range function J (not necessarily measurable) we associated to J the subset M_J defined as

$$M_J = \{\Phi \in L^2(\Omega_\Delta, \ell^2(\Delta)) : \Phi(\omega) \in J(\omega) \text{ a.e. } \omega \in \Omega_\Delta\}.$$

Lemma 2.2.3. *The subset M_J is closed in $L^2(\Omega_\Delta, \ell^2(\Delta))$.*

Proof. Let $\{\Phi_j\}_{j\in\mathbb{N}} \subseteq M_J$ such that $\Phi_j \to \Phi$ when $j \to \infty$ in $L^2(\Omega_\Delta, \ell^2(\Delta))$. Let us consider the functions $g_j : \Omega_\Delta \to \mathbb{R}_{\geq 0}$ defined as $g_j(\omega) := \|\Phi_j(\omega) - \Phi(\omega)\|^2_{\ell^2(\Delta)}$. Then, g_j is measurable for all $j \in \mathbb{N}$ and $\forall\ \alpha > 0$ it holds that

$$m_\Gamma(\{g_j > \alpha\}) \leq \frac{1}{\alpha}\int_{\Omega_\Delta} g_j(\omega)\,dm_\Gamma(\omega) = \frac{1}{\alpha}\int_{\Omega_\Delta} \|\Phi_j(\omega) - \Phi(\omega)\|^2_{\ell^2(\Delta)}\,dm_\Gamma(\omega) \to 0,$$

when $j \to \infty$. So, $g_j \to 0$ in measure and therefore, there exists a subsequence $\{g_{j_k}\}_{k\in\mathbb{N}}$ of $\{g_j\}_{j\in\mathbb{N}}$ which goes to zero a.e. $\omega \in \Omega_\Delta$. Then, $\Phi_{j_k}(\omega) \to \Phi(\omega)$ in $\ell^2(\Delta)$ a.e. $\omega \in \Omega_\Delta$ and hence, since $\Phi_{j_k}(\omega) \in J(\omega)$ a.e. $\omega \in \Omega_\Delta$ and $J(\omega)$ is closed, $\Phi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega_\Delta$. Therefore $\Phi \in M_J$. □

The following proposition is a generalization to the context of groups of a lemma of Helson, (see [Hel64] and also [Bow00]).

Proposition 2.2.4. *Let J be a measurable range function and P_ω the associated orthogonal projections. Denote by $\mathcal{P}$ the orthogonal projection onto M_J. Then,*

$$(\mathcal{P}\Phi)(\omega) = P_\omega(\Phi(\omega)), \quad a.e.\ \omega \in \Omega_\Delta, \quad \forall\ \Phi \in L^2(\Omega_\Delta, \ell^2(\Delta)).$$

Proof. Let $Q : L^2(\Omega_\Delta, \ell^2(\Delta)) \to L^2(\Omega_\Delta, \ell^2(\Delta))$ be the linear mapping $\Phi \mapsto Q\Phi$, where

$$(Q\Phi)(\omega) := P_\omega(\Phi(\omega)).$$

We want to show that $Q = \mathcal{P}$.

Since J is a measurable range function, due to Lemma 2.2.2, $Q\Phi$ is measurable for each $\Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$. Furthermore, since P_ω is an orthogonal projection, it has norm one, and therefore

$$\begin{aligned} \|Q\Phi\|_2^2 &= \int_{\Omega_\Delta} \|(Q\Phi)(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) \\ &= \int_{\Omega_\Delta} \|P_\omega(\Phi(\omega))\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) \\ &\leq \int_{\Omega_\Delta} \|\Phi(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) = \|\Phi\|_2^2 < \infty. \end{aligned}$$

Then, Q is well defined and it has norm less or equal to 1.

From the fact that P_ω is an orthogonal projection, it follows that $Q^2 = Q$ and $Q^* = Q$, where by Q^* we denote the adjoint operator of Q. Thus, Q is also an orthogonal projection. To complete our proof we must see that $M = M_J$, where $M := Ran(Q)$.

By definition of Q, $M \subseteq M_J$.

If we suppose that M is properly included in M_J,then there exists $\Psi \in M_J$ such that $\Psi \neq 0$ and $\Psi \perp M$. Then, $\forall\ \Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$, $0 = \langle Q\Phi, \Psi\rangle$. Since $Q^* = Q$, $0 = \langle \Phi, Q\Psi\rangle$ $\forall\ \Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$.

Hence, $Q\Psi = 0$ and therefore $P_\omega(\Psi(\omega)) = 0$ a.e. $\omega \in \Omega_\Delta$. Since $\Psi \in M_J$, $\Psi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega_\Delta$, thus $P_\omega(\Psi(\omega)) = \Psi(\omega)$ a.e. $\omega \in \Omega_\Delta$. Finally, $\Psi = 0$ a.e. $\omega \in \Omega_\Delta$ and this is a contradiction. □

2.2.2 The Characterization

We now give a characterization of H-invariant spaces using range functions.

Theorem 2.2.5. *Let $V \subseteq L^2(G)$ be a closed subspace and $\mathcal{T}_H$ the map defined in Proposition 1.4.2. Then, V is H-invariant if and only if there exists a measurable range function J such that*

$$V = \{f \in L^2(G) : \ \mathcal{T}_H f(\omega) \in J(\omega) \ a.e.\ \omega \in \Omega_\Delta\}.$$

Identifying range functions which are equal almost everywhere, the correspondence between H-invariant spaces and measurable range functions is one to one and onto.

Moreover, if $V = S_H(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$, the measurable range function J associated to V is given by

$$J(\omega) = \overline{\operatorname{span}}\{\mathcal{T}_H\varphi(\omega) : \ \varphi \in \mathcal{A}\}, \quad a.e.\ \omega \in \Omega_\Delta. \tag{2.5}$$

For the proof, we need the following result.

Lemma 2.2.6. *If J and J' are two measurable range functions such that $M_J = M_{J'}$, then $J(\omega) = J'(\omega)$ a.e. $\omega \in \Omega_\Delta$. That is, J and J' are equal almost everywhere.*

Proof. Let P_ω and Q_ω be the projections associate to J and J' respectively. If $\mathcal{P}$ is the orthogonal projection onto $M_J = M_{J'}$, by Proposition 2.2.4 we have that, for each $\Phi \in L^2(\Omega, \ell^2(\Delta))$

$$(\mathcal{P}\Phi)(\omega) = P_\omega(\Phi(\omega)) \quad \text{and} \quad (\mathcal{P}\Phi)(\omega) = Q_\omega(\Phi(\omega)) \;\; \text{a.e. } \omega \in \Omega_\Delta.$$

Then, $P_\omega(\Phi(\omega)) = Q_\omega(\Phi(\omega))$ a.e. $\omega \in \Omega_\Delta$, for all $\Phi \in L^2(\Omega_\Delta, \ell^2(\Delta))$. In particular, if $e_\lambda \in \ell^2(\Delta)$ is defined by $(e_\lambda)_\delta = 1$ if $\delta = \lambda$ and $(e_\lambda)_\delta = 0$ otherwise, $P_\omega(e_\lambda) = Q_\omega(e_\lambda)$ a.e. $\omega \in \Omega_\Delta$, for all $\lambda \in \Delta$. Hence, since $\{e_\lambda\}_{\lambda\in\Delta}$ is a basis for $\ell^2(\Delta)$, it follows that $P_\omega = Q_\omega$ a.e. $\omega \in \Omega_\Delta$. Thus $J(\omega) = J'(\omega)$ a.e. $\omega \in \Omega_\Delta$. □

Proof of Theorem 2.2.5. Let us first suppose that V is H-invariant. Since $L^2(G)$ is separable, $V = S_H(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$.

We define the range function J as $J(\omega) = \overline{\text{span}}\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\}$. Note that since $\mathcal{A}$ is a countable set, J is well defined a.e. $\omega \in \Omega_\Delta$. We will prove that J satisfies:

(i) $V = \{f \in L^2(G) : \mathcal{T}_H f(\omega) \in J(\omega) \text{ a.e. } \omega \in \Omega_\Delta\}$,

(ii) J is measurable.

To show (i) it is sufficient to prove that $M = M_J$, where $M := \mathcal{T}_H V$. Let $\Phi \in M$. Then, $\mathcal{T}_H^{-1}\Phi \in V = \overline{\text{span}}\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$. Therefore, there exists a sequence $\{g_j\}_{j\in\mathbb{N}} \subseteq \text{span}\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$ such that $\mathcal{T}_H g_j := \Phi_j$ converges in $L^2(\Omega_\Delta, \ell^2(\Delta))$ to Φ, when $j \to \infty$.

Due to the definition of J and Remark 1.4.4, $\Phi_j(\omega) \in J(\omega)$ a.e. $\omega \in \Omega_\Delta$. Thus, $\Phi_j \in M_J$ for all $j \in \mathbb{N}$. Since M_J is closed and $\Phi_j \to \Phi$ when $j \to \infty$ in $L^2(\Omega_\Delta, \ell^2(\Delta))$, it follows that $\Phi \in M_J$. Then, $M \subseteq M_J$.

Let us suppose now that there exists $\Psi \in L^2(\Omega_\Delta, \ell^2(\Delta))$, such that $\Psi \neq 0$ and Ψ is orthogonal to M. Then, for each $\Phi \in M$, $\langle \Phi, \Psi \rangle = 0$. In particular, if $\Phi \in \mathcal{T}_H\mathcal{A} \subseteq \mathcal{T}_H V = M$ and $h \in H$, we have that, since $(h,.)\Phi(.) = \mathcal{T}_H(T_{-h}\mathcal{T}_H^{-1}\Phi)(.)$ and $T_{-h}\mathcal{T}_H^{-1}\Phi \in V$, $(h,.)\Phi(.) \in \mathcal{T}_H V = M$.

Therefore, as $(h,.)$ is Δ-periodic,

$$0 = \langle (h,.)\Phi(.), \Psi \rangle = \int_{\Omega_\Delta} (h,\omega)\langle \Phi(\omega), \Psi(\omega)\rangle_{\ell^2(\Delta)} dm_\Gamma(\omega).$$

Hence, by Proposition 1.1.22, $\langle \Phi(\omega), \Psi(\omega)\rangle_{\ell^2(\Delta)} = 0$ a.e. $\omega \in \Omega_\Delta$, and this holds $\forall\ \Phi \in \mathcal{T}_H\mathcal{A}$. Therefore $\Psi(\omega) \in J(\omega)^\perp$ a.e. $\omega \in \Omega_\Delta$.

Now, if M is properly included in M_J, there exists $\Psi \in M_J$, with $\Psi \neq 0$ and orthogonal to M. Hence, $\Psi(\omega) \in J(\omega)^{\perp}$ a.e. $\omega \in \Omega_\Delta$. On the other hand since $\Psi \in M_J$, $\Psi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega_\Delta$. Thus, $\Psi(\omega) = 0$ a.e. $\omega \in \Omega_\Delta$ and this is a contradiction. Thus $M = M_J$.

It remains to prove that the range function J is measurable. Let $P_\omega : \ell^2(\Delta) \to J(\omega)$ be the orthogonal projections associated to $J(\omega)$.

Let $\mathcal{I}$ be the identity mapping in $L^2(\Omega_\Delta, \ell^2(\Delta))$ and $\mathcal{P} : L^2(\Omega_\Delta, \ell^2(\Delta)) \to M$ the orthogonal projection associated to M. Then, if $\Psi \in L^2(\Omega_\Delta, \ell^2(\Delta))$, the function $(\mathcal{I} - \mathcal{P})\Psi$ is orthogonal to M and, by the above reasoning, $(\mathcal{I} - \mathcal{P})\Psi(\omega) \in J(\omega)^{\perp}$, a.e. $\omega \in \Omega_\Delta$. Then,

$$P_\omega((\mathcal{I} - \mathcal{P})\Psi(\omega)) = P_\omega(\Psi(\omega) - \mathcal{P}\Psi(\omega)) = 0$$

a.e. $\omega \in \Omega_\Delta$ and therefore $P_\omega(\Psi(\omega)) = P_\omega(\mathcal{P}\Psi(\omega)) = \mathcal{P}\Psi(\omega)$ a.e $\omega \in \Omega_\Delta$. Thus we have that the function $\omega \mapsto P_\omega(\Psi(\omega))$ from Ω_Δ to $\ell^2(\Delta)$ agrees with $\mathcal{P}\Psi$ a.e. $\omega \in \Omega_\Delta$. Consequently, $\omega \mapsto P_\omega(\Psi(\omega))$ is measurable and then, Lemma 2.2.2 implies that J is a measurable range function.

Conversely. If J is a measurable range function, let us see that the closed subspace in $L^2(G)$, defined by $V := \mathcal{T}_H^{-1} M_J$ is H-invariant. For this, let us consider $f \in V$ and $h \in H$ and let us prove that $T_h f \in V$.

Since $\mathcal{T}_H(T_h f)(\omega) = (h, -\omega)\mathcal{T}_H f(\omega)$ a.e. $\omega \in \Omega_\Delta$ and $\mathcal{T}_H f \in M_J$, we have that $(h, -\omega)\mathcal{T}_H f(\omega) \in J(\omega)$ a.e. $\omega \in \Omega_\Delta$. Then, $\mathcal{T}_H(T_h f) \in M_J$ and therefore $T_h f \in V$.

Furthermore, $V = S_H(\mathcal{A})$ for some countable set $\mathcal{A}$ of $L^2(G)$. Then, we have proved that

$$J'(\omega) = \overline{\operatorname{span}}\{\mathcal{T}_H \varphi(\omega) : \ \varphi \in \mathcal{A}\} \ \text{ a.e. } \omega \in \Omega_\Delta,$$

defines a measurable range function which satisfies $V = \mathcal{T}_H^{-1} M_{J'}$. Thus, $M_{J'} = \mathcal{T}_H V = M_J$. Since J and J' are both measurable range functions, Lemma 2.2.6 implies that $J = J'$ a.e $\omega \in \Omega_\Delta$.

This also shows that the correspondence between V and J is onto and one to one. □

3

Frames and Riesz Bases for H-invariant Spaces

The concept of frame was first introduced by Duffin and Schaeffer in [DS52] as a tool to study non-harmonic Fourier series. About 30 years after that, Daubechies, Grossmann and Meyer [DGM86] realized that frames were very useful to obtain expansions of functions of $L^2(\mathbb{R}^d)$ similar to those given by orthonormal bases. Since then, frames became a fundamental tool in harmonic and functional analysis.

On the other hand, Riesz bases showed up as a natural generalization of the concept of orthonormal bases.

We will work here with frames and Riesz sequences of $L^2(G)$ with the property that each of their elements is obtained by the translation of a fixed set of functions (frames or Riesz sequence of translates). This particular structure simplifies manipulations on the frame or Riesz sequence and makes it easier to store information about them.

The concrete problem of interest for us is to decide whether or not $\{T_h\varphi\}_{h\in H,\varphi\in\mathcal{A}}$ is a frame or a Riesz basis for the H-invariant space $S_H(\mathcal{A})$. The characterization of H-invariant spaces that we have given in Theorem 2.2.5 of Chapter 2 will allow us to describe the problem in terms of fibers. We will give necessary and sufficient conditions on the fiber set $\{\mathcal{T}_H\varphi(\omega)\}_{\varphi\in\mathcal{A}}$ for $\{T_h\varphi\}_{h\in H,\varphi\in\mathcal{A}}$ being a frame or Riesz basis for $S_H(\mathcal{A})$.

Then, we will show that every H-invariant space can be decomposed in an orthogonal sum of principal H-invariant spaces. From this result we will conclude that each H-invariant space has a frame of translations.

Throughout this chapter G, H, Γ, Δ and Ω_Δ will be as in Chapter 2 with the following normalization for the Haar measures. First we choose m_Δ and m_H such that $m_\Delta(\{0\}) = m_H(\{0\}) = 1$. Then, we fix $m_{\Gamma/\Delta}$ such the inversion formula holds between H and Γ/Δ. Therefore, by formula (1.5) it holds that $m_{\Gamma/\Delta}(\Gamma/\Delta) = 1$. Next, we set m_Γ such that Theorem 1.1.10 holds for m_Γ, $m_{\Gamma/\Delta}$ and m_Δ. Finally, we normalize m_G such that the inversion formula holds for m_Γ and m_G.

As a consequence of the normalization given above and Lemma 1.1.13, it follows that $m_\Gamma(\Omega_\Delta) = 1$.

The chapter is organized as follows. In Section 3.1 we set some known material on frames and Riesz families. In Section 3.2 we give necessary and sufficient conditions for $\{T_h\varphi\}_{h\in H,\varphi\in\mathcal{A}}$ being a frame or Riesz basis for $S_H(\mathcal{A})$ in terms of fibers. Finilly we devote Section 3.3 to show an orthogonal decomposition for H-invariant spaces and to prove the existence of frames of translates in H-invariant spaces.

3.1 General Frames and Riesz Families

In this section we summarize without proof the relevant material on frames and Riesz bases and sequences. From now on I will be a finite or countably infinite index set, $\mathcal{H}$ will denote a Hilbert space and $\{u_i\}_{i\in I}$ a sequence in $\mathcal{H}$.

Definition 3.1.1. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. The sequence $\{u_i\}_{i\in I}$ is called *Riesz basis* if $u_i = Ue_i$ for all $i \in I$ where $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded bijective operator.

It is well known that if $\{e_i\}_{i\in I}$ is a fixed orthonormal basis for $\mathcal{H}$, any other orthonormal basis for $\mathcal{H}$ is of the form $\{Ue_i\}_{i\in I}$ where $U : \mathcal{H} \longrightarrow \mathcal{H}$ is unitary operator (see [Chr03, Theorem 3.6.6]). According to this, it is clear in what sense Riesz bases generalize orthonormal bases.

The following proposition gives an equivalent condition for $\{u_i\}_{i\in I}$ being a Riesz basis. For its proof see [Chr03, Theorem 3.4.7].

Proposition 3.1.2. *The sequence $\{u_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{u_i\}_{i\in I}$ is complete in $\mathcal{H}$ and there exist positive constants A and B such that*

$$A\sum_{i\in I}|a_i|^2 \leq \|\sum_{i\in I} a_i u_i\|^2_{\mathcal{H}} \leq B\sum_{i\in I}|a_i|^2 \tag{3.1}$$

holds for all $\{a_i\}_{i\in I}$ with finite support.

A sequence $\{u_i\}_{i\in I}$ satisfying condition (3.1) for all $\{a_i\}_{i\in I}$ with finite support is called a *Riesz sequence*. Therefore, a Riesz sequence $\{u_i\}_{i\in I}$ is a Riesz basis for the subspace of $\mathcal{H}$ given by $\overline{\text{span}}\{u_i\}_{i\in I}$.

Since the set $\{\{a_i\}_{i\in I} : \{a_i\}_{i\in I}$ has finite support$\}$ is a dense in $\ell^2(I)$, if (3.1) holds for all $\{a_i\}_{i\in I}$ with finite support, then it immediately holds for all $\{c_i\}_{i\in I} \in \ell^2(I)$.

Observe that orthonormal bases are exactly those Riesz bases which satisfy condition (3.1) with $A = B = 1$. This is another way to see Riesz bases as a generalization of orthonormal bases.

Definition 3.1.3. The sequence $\{u_i\}_{i\in I}$ is a *Bessel sequence* in $\mathcal{H}$ with constant $B > 0$ if

$$\sum_{i\in I}|\langle f, u_i\rangle|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in \mathcal{H}.$$

The constant B is called a *Bessel bound* for $\{u_i\}_{i\in I}$.

The proof of the next result can be found in [Chr03, Theorem 3.2.3].

Lemma 3.1.4. *Let* $\{u_i\}_{i\in I}$ *be a sequence of* $\mathcal{H}$. *Then,* $\{u_i\}_{i\in I}$ *is a Bessel sequence with Bessel bound B if and only if*

$$T : \ell^2(I) \longrightarrow \mathcal{H}, \quad T(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i u_i$$

is a well-defined bounded operator and $\|T\|_{op} \leq \sqrt{B}$.

Definition 3.1.5. The sequence $\{u_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$ with constants A and B if

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, u_i\rangle|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in \mathcal{H}.$$

The frame $\{u_i\}_{i\in I}$ is a *tight frame* if $A = B$, and the frame $\{u_i\}_{i\in I}$ is a *Parseval frame* if $A = B = 1$.

The numbers A and B are called *frame bounds*. We say that $\{u_i\}_{i\in I}$ is a *frame sequence* if it is a frame for $\overline{\text{span}}\{u_i\}_{i\in I}$.

If $\{u_i\}_{i\in I}$ is a frame for $\mathcal{H}$, it is, in particular, a Bessel sequence. Then, by Lemma 3.1.4 the operator

$$T : \ell^2(I) \longrightarrow \mathcal{H}, \quad T(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i u_i$$

is well-defined and bounded. The operator T is called *pre-frame* or *synthesis operator*. The adjoint of the pre-frame operator is given by

$$T^* : \mathcal{H} \longrightarrow \ell^2(I), \quad T(f) = \{\langle f, u_i\rangle\}_{i\in I},$$

and it is usually called *the analysis operator*.

By composing T and T^* we obtain the bounded, invertible, self-adjoint and positive operator

$$S : \mathcal{H} \longrightarrow \mathcal{H}, \quad Sf = TT^*f = \sum_{i\in I} \langle f, u_i\rangle u_i.$$

We will call S the *frame operator*.

In [Chr03] it is proven that, if $\{u_i\}_{i\in I}$ is a frame for $\mathcal{H}$, the sequence $\{S^{-1}u_i\}_{i\in I}$ is a frame for $\mathcal{H}$ as well. Moreover, the frame operator associated to $\{S^{-1}u_i\}_{i\in I}$ is S^{-1}. This allows us to obtain the following frame decomposition

$$f = \sum_{i\in I} \langle f, S^{-1}u_i\rangle u_i = \sum_{i\in I} \langle f, u_i\rangle S^{-1}u_i, \tag{3.2}$$

where the series converges unconditionally for all $f \in \mathcal{H}$.

Despite that the frame $\{u_i\}_{i\in I}$ is a set of non-independent vectors, formula (3.2) gives a straightforward and completely explicit expansion for every vector of $\mathcal{H}$ in terms of $\{u_i\}_{i\in I}$.

3.2 Characterization of Frames and Riesz basis for H-invariant spaces

We now prove the result which characterizes when $E_H(\mathcal{A})$ is a frame for $S_H(\mathcal{A})$ in terms of the fibers $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\}$. It generalizes Theorem 2.3 of [Bow00] to the context of groups. Recall that we can associate to each H-invariant space V an unique measurable range function J which characterizes V according to Theorem 2.2.5.

Theorem 3.2.1. *Let $\mathcal{A}$ be a countable subset of $L^2(G)$, J the measurable range function associated to $S_H(\mathcal{A})$ and $A \leq B$ positive constants. Then, the following propositions are equivalent:*

(i) The set $E_H(\mathcal{A})$ is a frame for $S_H(\mathcal{A})$ with constants A and B.

(ii) For almost every $\omega \in \Omega_\Delta$, the set $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a frame for $J(\omega)$ with constants A and B.

Proof. Since $\langle f, g\rangle_{L^2(G)} = \langle \mathcal{T}_H f, \mathcal{T}_H g\rangle_{L^2(\Omega_\Delta,\ell^2(\Delta))}$, by Remark 1.4.4 we have that

$$\sum_{h\in H}\sum_{\varphi\in\mathcal{A}} |\langle T_h\varphi, f\rangle_{L^2(G)}|^2 = \sum_{h\in H}\sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H(T_h\varphi), \mathcal{T}_H f\rangle_{L^2(\Omega_\Delta,\ell^2(\Delta))}|^2$$

$$= \sum_{\varphi\in\mathcal{A}}\sum_{h\in H} \Big|\int_{\Omega_\Delta} (h,-\omega)\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)} dm_\Gamma(\omega)\Big|^2.$$

Let us define for each $\varphi \in \mathcal{A}$,

$$R(\varphi) = \sum_{h\in H} \Big|\int_{\Omega_\Delta} (h,-\omega)\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)} dm_\Gamma(\omega)\Big|^2$$

and

$$T(\varphi) = \int_{\Omega_\Delta} |\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega).$$

$(i) \Rightarrow (ii)$ If $E_H(\mathcal{A})$ is a frame for $S_H(\mathcal{A})$, in particular it holds that $\forall f \in S_H(\mathcal{A})$, $\sum_{h\in H\,\varphi\in\mathcal{A}} |\langle T_h\varphi, f\rangle|^2 < \infty$.

Then, for each $\varphi \in \mathcal{A}$, we have that $R(\varphi) < \infty$. Therefore, the sequence $\{c_h\}_{h\in H}$, with

$$c_h := \int_\Omega (h,\omega)\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)} dm_\Gamma(\omega),$$

belongs to $\ell^2(H)$.

Let us consider the function $F(\omega) := \sum_{h\in H} c_h\eta_h(\omega)$, where η_h are the functions defined in Proposition 1.1.19. Then, since $\{c_h\}_{h\in H} \in \ell^2(H)$ and $\{\eta_h\}_{h\in H}$ is an orthonormal basis of $L^2(\Omega_\Delta)$, we have that $F \in L^2(\Omega_\Delta) \subseteq L^1(\Omega_\Delta)$ (recall that $m_\Gamma(\Omega_\Delta) < \infty$).

On the other hand, the function $\psi(\omega) := \langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}$ belongs to $L^1(\Omega_\Delta)$. So, $\psi - F \in L^1(\Omega_\Delta)$ and moreover

$$\int_{\Omega_\Delta} (h, -\omega)(\psi(\omega) - F(\omega))dm_\Gamma(\omega) = c_{-h} - c_{-h} = 0$$

for all $h \in H$. Thus, Proposition 1.1.22 yields that $F = \psi$ a.e. $\omega \in \Omega_\Delta$. Therefore $\psi \in L^2(\Omega_\Delta)$ and

$$\psi(\omega) = \sum_{h\in H} c_h \eta_h(\omega),$$

a.e. $\omega \in \Omega_\Delta$.

As a consequence of Proposition 1.1.21, we obtain that $R(\varphi) = T(\varphi)$ holds for all $\varphi \in \mathcal{A}$.

We will now prove that, for almost every $\omega \in \Omega_\Delta$, $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\}$ is a frame with constants A and B for $J(\omega)$.

Let us suppose that

$$A\|P_\omega d\|^2_{\ell^2(\Delta)} \le \sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), P_\omega d\rangle|^2_{\ell^2(\Delta)} \le B\|P_\omega d\|^2_{\ell^2(\Delta)} \tag{3.3}$$

a.e. $\omega \in \Omega_\Delta$, for each $d \in \mathcal{D}$, where $\mathcal{D}$ is a dense countable subset of $\ell^2(\Delta)$ and P_ω are the orthogonal projections associated to J. Then, for each $d \in \mathcal{D}$, let $Z_d \subseteq \Omega_\Delta$ be a measurable set with $m_\Gamma(Z_d) = 0$ such that (3.3) holds for all $\omega \in \Omega_\Delta \setminus Z_d$. So the set $Z = \bigcup_{d\in\mathcal{D}} Z_d$ has null m_Γ-measure. Therefore for $\omega \in \Omega_\Delta \setminus Z$ and $a \in J(\omega)$, using a density argument it follows from (3.3) that

$$A\|a\|^2_{\ell^2(\Delta)} \le \sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), a\rangle|^2_{\ell^2(\Delta)} \le B\|a\|^2_{\ell^2(\Delta)}.$$

Thus, it is sufficient to show that (3.3) holds. For this, we will suppose that this is not so and we will prove that there exist $d_0 \in \mathcal{D}$, a measurable set $W \subseteq \Omega_\Delta$ with $m_\Gamma(W) > 0$, and $\varepsilon > 0$ such that

$$\sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), P_\omega d_0\rangle|^2_{\ell^2(\Delta)} > (B+\varepsilon)\|P_\omega d_0\|^2_{\ell^2(\Delta)}, \ \forall\ \omega \in W$$

or

$$\sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), P_\omega d_0\rangle|^2_{\ell^2(\Delta)} < (A-\varepsilon)\|P_\omega d_0\|^2_{\ell^2(\Delta)}, \ \forall\ \omega \in W.$$

So, let us take $d_0 \in \mathcal{D}$ for which (3.3) fails. Then at least one of this sets

$$\{\omega \in \Omega_\Delta : K(\omega) - B\|P_\omega d_0\|^2_{\ell^2(\Delta)} > 0\} \quad , \quad \{\omega \in \Omega_\Delta : K(\omega) - A\|P_\omega d_0\|^2_{\ell^2(\Delta)} < 0\}$$

has positive measure, where $K(\omega) := \sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), P_\omega d_0\rangle|^2_{\ell^2(\Delta)}$. Let us suppose, without loss of generality, that

$$m_\Gamma(\{\omega \in \Omega_\Delta : K(\omega) - B\|P_\omega d_0\|^2_{\ell^2(\Delta)} > 0\}) > 0.$$

Since

$$\{\omega \in \Omega_\Delta : K(\omega) - B\|P_\omega d_0\|^2_{\ell^2(\Delta)} > 0\} = \bigcup_{j\in\mathbb{N}} \{\omega \in \Omega_\Delta : K(\omega) - (B + \frac{1}{j})\|P_\omega d_0\|^2_{\ell^2(\Delta)} > 0\},$$

there exists at least one set in the union, in the right hand side of this equality, with positive measure and this proves our claim.

Then, we can suppose that

$$\sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), P_\omega d_0\rangle|^2_{\ell^2(\Delta)} > (B+\varepsilon)\|P_\omega d_0\|^2_{\ell^2(\Delta)}, \ \forall\ \omega \in W \tag{3.4}$$

holds. Now take $f \in S_H(\mathcal{A})$ such that $\mathcal{T}_H f(\omega) = \chi_W(\omega)P_\omega d_0$. Note that this is possible since, by Theorem 2.2.5, $\chi_E(\omega)P_\omega d_0$ is a measurable function.

As $E_H(\mathcal{A})$ is a frame for $S_H(\mathcal{A})$ and

$$\sum_{h\in H}\sum_{\varphi\in\mathcal{A}} |\langle T_h\varphi, f\rangle_{L^2(G)}|^2 = \sum_{\varphi\in\mathcal{A}} \int_{\Omega_\Delta} |\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega),$$

we have that

$$A\|f\|^2_{L^2(G)} \le \sum_{\varphi\in\mathcal{A}} \int_{\Omega_\Delta} |\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega) \le B\|f\|^2_{L^2(G)}. \tag{3.5}$$

Using Proposition 1.4.2, we can rewrite (3.5) as

$$A\|\mathcal{T}_H f\|^2 \le \sum_{\varphi\in\mathcal{A}} \int_{\Omega_\Delta} |\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega) \le B\|\mathcal{T}_H f\|^2. \tag{3.6}$$

Now,

$$\|\mathcal{T}_H f\|^2 = \int_{\Omega_\Delta} \chi_W(\omega)\|P_\omega d_0\|^2_{\ell^2(\Delta)} dm_\Gamma(\omega)$$

and if we integrate in (3.4) over W, we obtain

$$\sum_{\varphi\in\mathcal{A}} \int_{\Omega_\Delta} |\langle \mathcal{T}_H\varphi(\omega), \chi_W(\omega)P_\omega d_0\rangle_{\ell^2(\Delta)}|^2\, dm_\Gamma(\omega) \ge (B+\varepsilon)\|\mathcal{T}_H f\|^2.$$

This is a contradiction with inequality (3.6). Therefore, we proved inequality (3.3).

(ii) $\Rightarrow$ *(i)* If now $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\}$ is a frame for $J(\omega)$ a.e $\omega \in \Omega_\Delta$ with frame bounds A and B, we have that

$$A\|a\|^2_{\ell^2(\Delta)} \le \sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), a\rangle_{\ell^2(\Delta)}|^2 \le B\|a\|^2_{\ell^2(\Delta)}$$

for all $a \in J(\omega)$. In particular, if $f \in S_H(\mathcal{A})$, by Theorem 2.2.5, $\mathcal{T}_H f(\omega) \in J(\omega)$ a.e. $\omega \in \Omega_\Delta$ and then,

$$A\|\mathcal{T}_H f(\omega)\|^2_{\ell^2(\Delta)} \le \sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}|^2 \le B\|\mathcal{T}_H f(\omega)\|^2_{\ell^2(\Delta)} \tag{3.7}$$

a.e. $\omega \in \Omega_\Delta$.

Thus, integrating (3.7) over Ω_Δ, we obtain

$$A\|\mathcal{T}_H f\|^2 \leq \int_{\Omega_\Delta} \sum_{\varphi\in\mathcal{A}} |\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega) \leq B\|\mathcal{T}_H f\|^2. \tag{3.8}$$

So, $\langle \mathcal{T}_H\varphi(.), \mathcal{T}_H f(.)\rangle_{\ell^2(\Delta)}$ belongs to $L^2(\Omega_\Delta)$ for each $\varphi \in \mathcal{A}$ and the equality $R(\varphi) = T(\varphi)$ can be obtained in a similar way as we did before.

Finally, since $\|\mathcal{T}_H f\|_2^2 = \|f\|^2_{L^2(G)}$ and

$$\sum_{h\in H}\sum_{\varphi\in\mathcal{A}} |\langle T_h\varphi, f\rangle_{L^2(G)}|^2 = \sum_{\varphi\in\mathcal{A}} \int_{\Omega_\Delta} |\langle \mathcal{T}_H\varphi(\omega), \mathcal{T}_H f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega),$$

inequality (3.8) implies that $E_H(\mathcal{A})$ is a frame for $S_H(\mathcal{A})$ with frame bounds A and B. □

Theorem 3.2.1 reduces the problem of when $E_H(\mathcal{A})$ is a frame for $S_H(\mathcal{A})$ to when the fibers $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\}$ form a frame for $J(\omega)$. The advantage of this reduction is that, for example, when $\mathcal{A}$ is a finite set, the fiber spaces $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\}$ are finite dimensional while $S_H(\mathcal{A})$ has infinite dimension.

If $\mathcal{A} = \{\varphi\}$, Theorem 3.2.1 generalizes a known result for the case $G = \mathbb{R}^d$ to the context of groups. This is stated in the next corollary, which was proved in [KR08]. We give here a different proof.

Corollary 3.2.2. *Let $\varphi \in L^2(\Omega_\Delta)$ and $\mathcal{E}_\varphi$ defined as in (2.1). Then, the following are equivalent:*

(i) The set $E_H(\varphi)$ is a frame for $S_H(\varphi)$ with frame bounds A and B.

(ii) $A \leq \sum_{\delta\in\Delta} |\widehat{\varphi}(\omega + \delta)|^2 \leq B$, a.e. $\omega \in \mathcal{E}_\varphi$.

Proof. Let J be the measurable range function associated to $S_H(\varphi)$. Then, by Theorem 2.2.5, $J(\omega) = \text{span}\{\mathcal{T}_H\varphi(\omega)\}$ a.e $\omega \in \Omega_\Delta$. Thus, each $a \in J(\omega)$ can be written as $a = \lambda\mathcal{T}_H\varphi(\omega)$ for some $\lambda \in \mathbb{C}$.

Therefore, by Theorem 3.2.1, (i) holds if and only if, for almost every $\omega \in \Omega_\Delta$ and for all $\lambda \in \mathbb{C}$,

$$A\|\lambda\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} \leq |\lambda|^2\|\mathcal{T}_H\varphi(\omega)\|^4_{\ell^2(\Delta)} \leq B\|\lambda\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}. \tag{3.9}$$

Then, since $\|\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} = \sum_{\delta\in\Delta} |\widehat{\varphi}(\omega + \delta)|^2$, (3.9) holds if and only if

$$A \leq \sum_{\delta\in\Delta} |\widehat{\varphi}(\omega + \delta)|^2 \leq B, \quad \text{a.e.}\, \omega \in \mathcal{E}_\varphi.$$

□

For the case of Riesz basis, we have an analogue result to Theorem 3.2.1.

Theorem 3.2.3. *Let $\mathcal{A}$ be a countable subset of $L^2(G)$, J the measurable range function associated to $S_H(\mathcal{A})$ and $A \leq B$ positive constants. Then, they are equivalent:*

(i) The set $E_H(\mathcal{A})$ is a Riesz basis for $S_H(\mathcal{A})$ with constants A and B.

(ii) For almost every $\omega \in \Omega_\Delta$, the set $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a Riesz basis for $J(\omega)$ with constants A and B.

For the proof we will need the next lemma.

Lemma 3.2.4. *For each $m \in L^\infty(\Omega_\Delta)$ there exists a sequence of trigonometric polynomials $\{P_k\}_{k\in\mathbb{N}}$ such that:*

(i) $P_k(\omega) \to m(\omega)$, a.e. $\omega \in \Omega_\Delta$,

(ii) There exists $C > 0$, such that $\|P_k\|_\infty \leq C$, for all $k \in \mathbb{N}$.

Proof. By Lemma 1.1.15, taking into account Remark 1.1.14, we have that the trigonometric polynomials are dense in $C(\Omega_\Delta)$.

By Lusin's Theorem ([WZ77, Theorem 4.20]), for each $k \in \mathbb{N}$, there exists a closed set $E_k \subseteq \Omega_\Delta$ such that $m_\Gamma(\Omega_\Delta \setminus E_k) < 2^{-k}$ and $m|_{E_k}$ is a continuous function where $m|_{E_k}$ denotes the function m restricted to E_k.

Since Ω_Δ is compact, E_k is compact as well. Therefore, $m|_{E_k}$ is bounded.

Let $m_1, m_2 : E_k \to \mathbb{R}$ be continuous function such that $m|_{E_k} = m_1 + im_2$. As a consequence of Tietze's Extension Theorem (see [Mun75]), it is possible to extend m_1 and m_2, continuously to all Ω_Δ keeping their norms in $L^\infty(E_k)$. Let us call the extensions $\overline{m_1}$ and $\overline{m_2}$ and let $\overline{m_k} = \overline{m_1} + i\overline{m_2}$. Then, we have:

(1) $\overline{m_k}|_{E_k} = m|_{E_k}$,

(2) $\|\overline{m_k}\|_\infty \leq \|\overline{m_1}\|_\infty + \|\overline{m_2}\|_\infty \leq \|m_1\|_\infty + \|m_2\|_\infty \leq 2\|m\|_\infty$.

Now, by Lemma 1.1.15, there exists a trigonometric polynomial P_k such that $\|P_k - \overline{m_k}\|_\infty < 2^{-k}$. So,

(a) $|P_k(\omega) - m(\omega)| < 2^{-k}$, for all $\omega \in E_k$,

(b) $\|P_k\|_\infty \leq \|P_k - \overline{m_k}\|_\infty + \|\overline{m_k}\|_\infty \leq 2^{-k} + 2\|m\|_\infty \leq 1 + 2\|m\|_\infty$.

Repeating this argument for each $k \in \mathbb{N}$, we obtain a sequence $\{P_k\}_{k\in\mathbb{N}}$ of trigonometric polynomials and a sequence $\{E_k\}_{k\in\mathbb{N}}$ of sets, which satisfy conditions (a) and (b).

Let $E = \cup_{j=1}^{\infty} \cap_{k=j}^{\infty} E_k$. It is a straightforward to see that $m_\Gamma(\Omega_\Delta \setminus E) = 0$. Let us prove that if $\omega \in E$, $P_k(\omega) \to m(\omega)$, for $k \to \infty$. Since $\omega \in E$, there exists $k_0 \in \mathbb{N}$ for which $\omega \in E_k$, $\forall\ k \geq k_0$. Then, for all $k \geq k_0$, we obtain that $|P_k(\omega) - m(\omega)| = |P_k(\omega) - m_k(\omega)| < 2^{-k} \to 0$, when $k \to \infty$. This proves part (i) of this lemma and taking $C := 1 + 2\|m\|_\infty$ we have that (ii) holds. □

Proof of Theorem 3.2.3. Since $S_H(\mathcal{A}) = \overline{\text{span}}\, E_H(\mathcal{A})$ and, by Theorem 2.2.5, $J(\omega) = \overline{\text{span}}\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\}$, we only need to show that $E_H(\mathcal{A})$ is a Riesz sequence for $S_H(\mathcal{A})$ with constants A and B if and only if for almost every $\omega \in \Omega_\Delta$, the set $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a Riesz sequence for $J(\omega)$ with constants A and B.

For the proof of the equivalence in the theorem, we will use the following reasoning.

Let $\{a_{\varphi,h}\}_{(\varphi,h)\in\mathcal{A}\times H}$ be a sequence of finite support and let P_φ be the trigonometric polynomials defined by

$$P_\varphi(\omega) = \sum_{h\in H} a_{\varphi,h}\eta_h(\omega),$$

with $\omega \in \Omega_\Delta$ and η_h as in Proposition 1.1.19.

Note that, since $\{a_{\varphi,h}\}_{(\varphi,h)\in\mathcal{A}\times H}$ has finite support, only a finite number of the polynomials P_φ are not zero.

Now, as a consequence of Proposition 1.4.2 we have

$$\begin{aligned}
\|\sum_{(\varphi,h)\in\mathcal{A}\times H} a_{\varphi,h}T_h\varphi\|^2_{L^2(G)} &= \|\sum_{(\varphi,h)\in\mathcal{A}\times H} a_{\varphi,h}\mathcal{T}_H T_h\varphi\|^2_{L^2(\Omega_\Delta,\ell^2(\Delta))} \\
&= \int_{\Omega_\Delta} \|\sum_{(\varphi,h)\in\mathcal{A}\times H} a_{\varphi,h}(-h,\omega)\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) \\
&= \int_{\Omega_\Delta} \|\sum_{\varphi\in\mathcal{A}} P_\varphi(\omega)\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega).
\end{aligned} \tag{3.10}$$

Furthermore, by Lemma 1.1.21,

$$\sum_{h\in H} |a_{\varphi,h}|^2 = \|\{a_{\varphi,h}\}_{h\in H}\|^2_{\ell^2(H)} = \|P_\varphi\|^2_{L^2(\Omega_\Delta)},$$

and adding over $\mathcal{A}$, we obtain

$$\sum_{(\varphi,h)\in\mathcal{A}\times H} |a_{\varphi,h}|^2 = \sum_{\varphi\in\mathcal{A}} \|P_\varphi\|^2_{L^2(\Omega_\Delta)}. \tag{3.11}$$

$(ii) \Rightarrow (i)$ If we suppose that for almost every $\omega \in \Omega_\Delta$, $\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a Riesz sequence for $J(\omega)$ with constants A and B,

$$A\sum_{\varphi\in\mathcal{A}} |a_\varphi|^2 \leq \|\sum_{\varphi\in\mathcal{A}} a_\varphi\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} \leq B\sum_{\varphi\in\mathcal{A}} |a_\varphi|^2 \tag{3.12}$$

for all $\{a_\varphi\}_{\varphi\in\mathcal{A}}$ with finite support.

In particular, the above inequality holds for $\{a_\varphi\}_{\varphi\in\mathcal{A}} = \{P_\varphi(\omega)\}_{\varphi\in\mathcal{A}}$. Now, in (3.12), we can integrate over Ω_Δ with $\{a_\varphi\}_{\varphi\in\mathcal{A}} = \{P_\varphi(\omega)\}_{\varphi\in\mathcal{A}}$, in order to obtain

$$\begin{aligned}
A\sum_{\varphi\in\mathcal{A}} \|P_\varphi\|^2_{L^2(\Omega_\Delta)} &\leq \int_{\Omega_\Delta} \|\sum_{\varphi\in\mathcal{A}} P_\varphi(\omega)\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) \\
&\leq B\sum_{\varphi\in\mathcal{A}} \|P_\varphi\|^2_{L^2(\Omega_\Delta)}.
\end{aligned} \tag{3.13}$$

Using equations (3.10) and (3.11) we can rewrite (3.13) as

$$A \sum_{(\varphi,h)\in\mathcal{A}\times H} |a_{\varphi,h}|^2 \le \| \sum_{(\varphi,h)\in\mathcal{A}\times H} a_{\varphi,h} T_h\varphi\|^2_{L^2(G)} \le B \sum_{(\varphi,h)\in\mathcal{A}\times H} |a_{\varphi,h}|^2.$$

Therefore $E_H(\mathcal{A})$ is a Riesz sequence of $S_H(\mathcal{A})$ with constants A and B.

$(i) \Rightarrow (ii)$ We want to prove that, for every $a = \{a_\varphi\}_{\varphi\in\mathcal{A}} \in \ell^2(\mathcal{A})$ with finite support, we have a.e. $\omega \in \Omega_\Delta$

$$A \sum_{\varphi\in\mathcal{A}} |a_\varphi|^2 \le \| \sum_{\varphi\in\mathcal{A}} a_\varphi \mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} \le B \sum_{\varphi\in\mathcal{A}} |a_\varphi|^2. \tag{3.14}$$

Let us suppose that (3.14) fails. Then, using a similar argument as in Theorem 3.2.1, we can see that there exist $a = \{a_\varphi\}_{\varphi\in\mathcal{A}} \in \ell^2(\mathcal{A})$ with finite support, a measurable set $W \subseteq \Omega_\Delta$ with $m_\Gamma(W) > 0$ and $\varepsilon > 0$ such that

$$\| \sum_{\varphi\in\mathcal{A}} a_\varphi \mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} > (B+\varepsilon) \sum_{\varphi\in\mathcal{A}} |a_\varphi|^2, \ \forall\ \omega \in W \tag{3.15}$$

or

$$\| \sum_{\varphi\in\mathcal{A}} a_\varphi \mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)} < (A-\varepsilon) \sum_{\varphi\in\mathcal{A}} |a_\varphi|^2, \ \forall\ \omega \in W. \tag{3.16}$$

With $a = \{a_\varphi\}_{\varphi\in\mathcal{A}}$ and W, we define for each $\varphi \in \mathcal{A}$, $m_\varphi := a_\varphi \chi_W$. Thus, $m_\varphi \in L^\infty(\Omega_\Delta)$ and only finitely many of these functions are not null.

By Lemma 3.2.4, for each $\varphi \in \mathcal{A}$ there exists a trigonometric polynomial sequence $\{P^\varphi_k\}_{k\in\mathbb{N}}$ such that

(i) $P^\varphi_k \to m_\varphi$,

(ii) $\|P^\varphi_k\|_\infty \le 1 + 2\|m_\varphi\|_\infty, \ \forall\ k \in \mathbb{N}$.

Since $E_H(\mathcal{A})$ is a Riesz sequence for $S_H(\mathcal{A})$ with constants A and B,

$$\begin{aligned} A \sum_{(\varphi,h)\in\mathcal{A}\times H} |a_{\varphi,h}|^2 &\le \| \sum_{(\varphi,h)\in\mathcal{A}\times H} a_{\varphi,h} T_h\varphi\|^2_{L^2(G)} \\ &\le B \sum_{(\varphi,h)\in\mathcal{A}\times H} |a_{\varphi,h}|^2, \end{aligned}$$

for each sequence $\{a_{\varphi,h}\}_{(\varphi,h)\in\mathcal{A}\times H}$ with finite support.

Now, for each $k \in \mathbb{N}$ take $\{a_{\varphi,h}\}_{(\varphi,h)\in\mathcal{A}\times H}$ to be the sequence formed with the coefficients of the polynomials $\{P^\varphi_k\}_{\varphi\in\mathcal{A}}$.

Then, using (3.10) and (3.11), we have for each $k \in \mathbb{N}$

$$A \sum_{\varphi\in\mathcal{A}} \|P^\varphi_k\|^2_{L^2(\Omega_\Delta)} \le \int_{\Omega_\Delta} \| \sum_{\varphi\in\mathcal{A}} P^\varphi_k(\omega) \mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) \le B \sum_{\varphi\in\mathcal{A}} \|P^\varphi_k\|^2_{L^2(\Omega_\Delta)}. \tag{3.17}$$

Therefore, since $m_\Gamma(\Omega_\Delta) < \infty$ and by the Dominated Convergence Theorem ([WZ77, Theorem 5.36]), inequality (3.17) can be extended to m_φ as

$$A \sum_{\varphi\in\mathcal{A}} \|m_\varphi\|^2_{L^2(\Omega_\Delta)} \leq \int_{\Omega_\Delta} \|\sum_{\varphi\in\mathcal{A}} m_\varphi(\omega)\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}\, dm_\Gamma(\omega) \leq B \sum_{\varphi\in\mathcal{A}} \|m_\varphi\|^2_{L^2(\Omega_\Delta)}. \tag{3.18}$$

So, if (3.15) occurs, integrating over Ω_Δ we obtain

$$\int_{\Omega_\Delta} \|\sum_{\varphi\in\mathcal{A}} m_\varphi(\omega)\mathcal{T}_H\varphi(\omega)\|^2_{\ell^2(\Delta)}|^2\, dm_\Gamma(\omega) > (B+\varepsilon)\int_{\Omega_\Delta} \sum_{\varphi\in\mathcal{A}} |m_\varphi(\omega)|^2\, dm_\Gamma,$$

which contradicts inequality (3.18). We can proceed analogously if (3.16) occurs. Hence, (3.14) holds.

□

For the case of principal H-invariant spaces we have the following corollary.

Corollary 3.2.5. *Let $\varphi \in L^2(\Omega_\Delta)$. Then, the following are equivalent:*

(i) The set $E_H(\varphi)$ is a Riesz basis for $S_H(\varphi)$ with constants A and B.

(ii) $A \leq \sum_{\delta\in\Delta} |\widehat{\varphi}(\omega+\delta)|^2 \leq B$, a.e. $\omega \in \Omega_\Delta$.

Proof. The proof is a straightforward consequence of Theorem 3.2.3 and Theorem 2.2.5. □

We now want to give another characterization of when the set $E_H(\mathcal{A})$ is a frame (Riesz sequence) for $S_H(\mathcal{A})$ with constants A and B. For this we will work with the synthesis and analysis operators introduced in Section 3.1.

Let us consider a subset $\mathcal{A} = \{\varphi_i : i \in I\}$ of $L^2(G)$ where I is a countable set.

Fix $\omega \in \Omega_\Delta$. Then, we can formally define the synthesis operator associated to $\mathcal{A}$ at ω, $K_\mathcal{A}(\omega) : \ell^2(I) \to \ell^2(\Delta)$ as

$$K_\mathcal{A}(\omega)c = \sum_{i\in I} c_i \mathcal{T}_H\varphi_i(\omega),$$

and the analysis operator $K^*_\mathcal{A}(\omega)$, $K^*_\mathcal{A}(\omega) : \ell^2(\Delta) \to \ell^2(I)$ as

$$K^*_\mathcal{A}(\omega)a = (\langle \mathcal{T}_H\varphi_i(\omega), a\rangle_{\ell^2(\Delta)})_{i\in I}.$$

Recall that, as we have said in Section 3.1, $K_\mathcal{A}(\omega)$ and $K^*_\mathcal{A}(\omega)$ are well defined and bounded if and only if $\{\mathcal{T}_H\varphi_i(\omega) : i \in I\}$ is a Bessel sequence.

Definition 3.2.6. Let $\mathcal{A} = \{\varphi_i : i \in I\} \subseteq L^2(G)$ be a countable subset and $K_\mathcal{A}(\omega)$ and $K^*_\mathcal{A}(\omega)$ the synthesis and analysis operators. We define the *Gramian* of $\mathcal{A}$ at $\omega \in \Omega_\Delta$ as the operator $\mathcal{G}_\mathcal{A}(\omega) : \ell^2(I) \to \ell^2(I)$ given by $\mathcal{G}_\mathcal{A}(\omega) = K^*_\mathcal{A}(\omega)K_\mathcal{A}(\omega)$, and we also define the *dual Gramian* of $\mathcal{A}$ at $\omega \in \Omega_\Delta$ as the operator $\tilde{\mathcal{G}}_\mathcal{A}(\omega) : \ell^2(\Delta) \to \ell^2(\Delta)$ given by $\tilde{\mathcal{G}}_\mathcal{A}(\omega) = K_\mathcal{A}(\omega)K^*_\mathcal{A}(\omega)$.

Note that, when $\{\mathcal{T}_H\varphi_i(\omega) : i \in I\}$ is a frame, $\mathcal{G}_{\mathcal{A}}(\omega)$ is precisely the frame operator associated to $\{\mathcal{T}_H\varphi_i(\omega) : i \in I\}$.

The Gramian $\mathcal{G}_{\mathcal{A}}(\omega)$ can be associated with the (possible) infinite matrix

$$\mathcal{G}_{\mathcal{A}}(\omega) = \Big(\sum_{\delta\in\Delta} \hat{\varphi}_i(\omega+\delta)\overline{\hat{\varphi}_j(\omega+\delta)} \Big)_{i,j\in I}$$

since $\langle \mathcal{G}_{\mathcal{A}}(\omega)e_i, e_j\rangle = \langle \mathcal{T}_H\varphi_i(\omega), \mathcal{T}_H\varphi_j(\omega)\rangle$, where $\{e_i\}_{i\in I}$ is the standard basis for $\ell^2(I)$. In a similar way, considering the canonical basis $\{e_\delta\}_{\delta\in\Delta}$ for $\ell^2(\Delta)$, we can associate the dual Gramian $\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)$ with the matrix

$$\tilde{\mathcal{G}}_{\mathcal{A}}(\omega) = \Big(\sum_{i\in I} \hat{\varphi}_i(\omega+\delta)\overline{\hat{\varphi}_i(\omega+\delta')} \Big)_{\delta,\delta'\in\Delta}.$$

Remark 3.2.7. The operator $K_{\mathcal{A}}(\omega)$ ($K^*_{\mathcal{A}}(\omega)$) is bounded if and only if $\mathcal{G}_{\mathcal{A}}(\omega)$ ($\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)$) is bounded. In that case we have $\|K_{\mathcal{A}}(\omega)\|^2 = \|K^*_{\mathcal{A}}(\omega)\|^2 = \|\mathcal{G}_{\mathcal{A}}(\omega)\| = \|\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\|$.

Now we will give a characterization of when $E_H(\mathcal{A})$ is a frame (Riesz sequence) for $S_H(\mathcal{A})$ in terms of the Gramian $\mathcal{G}_{\mathcal{A}}(\omega)$ and the dual Gramian $\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)$.

Proposition 3.2.8. *Let $\mathcal{A} = \{\varphi_i : i \in I\} \subseteq L^2(G)$ be a countable set. Then,*

(1) The following are equivalent:

- *(a_1) $E_H(\mathcal{A})$ is a Bessel sequence with constant B.*
- *(b_1) $\operatorname{esssup}_{\omega\in\Omega_\Delta}\|\mathcal{G}_{\mathcal{A}}(\omega)\| \le B$.*
- *(c_1) $\operatorname{esssup}_{\omega\in\Omega_\Delta}\|\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\| \le B$.*

(2) The following are equivalent:

- *(a_2) $E_H(\mathcal{A})$ is a frame for $S_H(\mathcal{A})$ with constants A and B.*
- *(b_2) For almost every $\omega \in \Omega_\Delta$,*

$$A\|a\|^2 \le \langle \tilde{\mathcal{G}}_{\mathcal{A}}(\omega)a, a\rangle \le B\|a\|^2,$$

for all $a \in \operatorname{span}\{\mathcal{T}_H\varphi_i(\omega) : i \in I\}$.

- *(c_2) For almost every $\omega \in \Omega_\Delta$,*

$$\sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)) \subseteq \{0\} \cup [A, B].$$

(3) The following are equivalent:

- *(a_3) $E_H(\mathcal{A})$ is a Riesz sequence for $S_H(\mathcal{A})$ with constants A and B.*
- *(b_3) For almost every $\omega \in \Omega_\Delta$,*

$$A\|c\|^2 \le \langle \mathcal{G}_{\mathcal{A}}(\omega)c, c\rangle \le B\|c\|^2,$$

for all $c \in \ell^2(I)$.

(c_3) For almost every $\omega \in \Omega_\Delta$

$$\sigma(\mathcal{G}_{\mathcal{A}}(\omega)) \subseteq [A, B].$$

Proof. Taking into account that a version of Theorem 3.2.1 holds for Bessel sequence, item (1) follows easily from Lemma 3.1.4 and Remark 3.2.7.

In order to prove equivalences in (2), note that, for almost $\omega \in \Omega_\Delta$

$$\langle \tilde{\mathcal{G}}_{\mathcal{A}}(\omega)a, a\rangle = \langle K^*_{\mathcal{A}}(\omega)a, K^*_{\mathcal{A}}(\omega)a\rangle = \sum_{i\in I} |\langle \mathcal{T}_H\varphi_i(\omega), a\rangle|^2 \tag{3.19}$$

for all $a \in \ell^2(\Delta)$. Then, equivalence between (a_2) and (b_2) is consequence of Theorem 3.2.1.

Let us prove now that (b_2) holds in and only if (c_2) holds. Since $\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)$ is a self-adjoint operator, $\ell^2(\Delta) = Ker(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)) \oplus \overline{Ran(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega))}$. Furthermore, according to (3.19), $Ker(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)) = Ker(K^*_{\mathcal{A}}(\omega))$ and it is easy to see that $Ker(K^*_{\mathcal{A}}(\omega)) = J(\omega)^\perp$, where J is the measurable range function associated with $S_H(\mathcal{A})$. Consequently, $J(\omega) = \overline{Ran(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega))}$. We consider now $\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)} : J(\omega) \to J(\omega)$, the operator $\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)$ restricted to $J(\omega)$. Then, it can be proved that

$$\sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}) \subseteq \sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)) \qquad \text{and} \qquad \sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)) \setminus \{0\} \subseteq \sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}). \tag{3.20}$$

On the other hand, if

$$M = \sup_{\|a\|=1, a\in J(\omega)} \langle \tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}a, a\rangle \quad \text{and} \quad m = \inf_{\|a\|=1, a\in J(\omega)} \langle \tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}a, a\rangle,$$

it is well known that $\sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}) \subseteq [m, M]$. From this and (3.20) it follows that (b_2) implies (c_2).

Let us suppose now that $\sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)) \subseteq \{0\} \cup [A, B]$. Thus, by (3.20),

$$\sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}) \subseteq \{0\} \cup [A, B].$$

The statement (b_2) will be proved once we prove that $0 \notin \sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)})$.

Since $\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}$ is self-adjoint operator, every isolated point of its spectrum is an eigenvalue (this can be proved using continuous function calculus). Therefore, if $0 \in \sigma(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)})$, 0 must be an eigenvalue of $\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}$. Hence, $Ker(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)\big|_{J(\omega)}) \neq 0$ which is a contradiction because $Ker(\tilde{\mathcal{G}}_{\mathcal{A}}(\omega)) = J(\omega)^\perp$.

To prove equivalences in (3), let us observe that, for almost $\omega \in \Omega_\Delta$,

$$\langle \mathcal{G}_{\mathcal{A}}(\omega)c, c\rangle = \langle K_{\mathcal{A}}(\omega)c, K_{\mathcal{A}}(\omega)c\rangle = \|\sum_{i\in I} c_i\mathcal{T}_H\varphi_i(\omega)\|^2. \tag{3.21}$$

Thus, by Theorem 3.2.3, we have that (a_3) holds if an only if (b_3) holds.

Since $\mathcal{G}_{\mathcal{A}}(\omega)$ is a self-adjoint operator, $\sigma(\mathcal{G}_{\mathcal{A}}(\omega)) \subseteq [m', M']$ where

$$M' = \sup_{\|c\|=1} \langle \mathcal{G}_{\mathcal{A}}(\omega)c, c\rangle \quad \text{and} \quad m' = \inf_{\|c\|=1} \langle \mathcal{G}_{\mathcal{A}}(\omega)c, c\rangle.$$

Therefore, it follows that (b_3) holds if an only if (c_3) holds and this finishes the proof. □

Note that Corollary 3.2.2 and Corollary 3.2.5 can also be obtained from the previous proposition.

3.3 Decomposition of H-invariant spaces

In this section, we show that every H-invariant space can be decomposed into an orthogonal sum of principal H-invariant spaces. This can be easily obtained as a consequence of Zorn's Lemma as in [KR08]. The theorem that we present here, establishes a decomposition of H-invariant space with additional properties as in [Bow00].

We first need the following definition.

Definition 3.3.1. For an H-invariant space $V \subseteq L^2(G)$ we define the *dimension function of* V as the map $\dim_V : \Omega_\Delta \to \mathbb{N}_0$ given by $\dim_V(\omega) = \dim J(\omega)$, where J is the range function associated to V. We also define the *spectrum of* V as $s(V) = \{\omega \in \Omega_\Delta : J(\omega) \neq 0\}$.

Now we state the theorem which gives the orthogonal decomposition for H-invariant spaces. We do not include its proof since it follows readily from the $\mathbb{R}^d$ case (see [Bow00, Theorem 3.3]).

Theorem 3.3.2. *Let us suppose that V is an H-invariant space of $L^2(G)$. Then V can be decomposed as an orthogonal sum*

$$V = \bigoplus_{n\in\mathbb{N}} S_H(\varphi_n),$$

where $E_H(\varphi_n)$ is a Parseval frame for $S_H(\varphi_n)$ and $s(S_H(\varphi_{n+1})) \subseteq s(S_H(\varphi_n))$ for all $n \in \mathbb{N}$.

Moreover, $\dim_{S_H(\varphi_n)}(\omega) = \|\mathcal{T}_H\varphi_n(\omega)\|_{\ell^2(\Delta)}$ *for all $n \in \mathbb{N}$, and*

$$\dim_V(\omega) = \sum_{n\in\mathbb{N}} \|\mathcal{T}_H\varphi_n(\omega)\|_{\ell^2(\Delta)}, \qquad a.e.\ \omega \in \Omega_\Delta.$$

The next example shows that there exist H-invariant spaces without an orthonormal basis of translates. However, we will deduce from Theorem 3.3.2 that H-invariant spaces have always frames of translates.

Example 3.3.3. Let $\varphi \in L^2(\mathbb{R})$ defined via the Fourier transform as $\widehat{\varphi}(\omega) = \chi_{[0,\frac{1}{2}]}(\omega)$ and consider the $\mathbb{Z}$-invariant space generated by φ, $S_{\mathbb{Z}}(\varphi)$.

Suppose that there exists $\phi \in S_{\mathbb{Z}}(\varphi)$ with $E_{\mathbb{Z}}(\phi) = \{T_k\phi\}_{k\in\mathbb{Z}}$ being an orthonormal basis for $S_{\mathbb{Z}}(\varphi)$. Then $S_{\mathbb{Z}}(\varphi) = S_{\mathbb{Z}}(\phi)$, and by Corollary 2.1.6, $\mathcal{E}_\varphi = \mathcal{E}_\phi$ up to a set of zero measure.

On the other hand, taking into account that $\Omega_{\mathbb{Z}} = [0, 1)$, from Corollary 3.2.5 we obtain that $\mathcal{E}_\phi = [0, 1)$.

Now, it is easy to check that

$$\|\mathcal{T}_{\mathbb{Z}}\varphi(\omega)\|^2_{\ell^2(\mathbb{Z})} = \begin{cases} 1 & \text{if } \omega \in [0, \frac{1}{2}] \\ 0 & \text{if } \omega \in (\frac{1}{2}, 1). \end{cases}$$

Thus, $E_\varphi = [0, \frac{1}{2}]$ which is a contradiction. Then $S_{\mathbb{Z}}(\varphi)$ can not have an orthonormal basis of the form $\mathcal{E}_{\mathbb{Z}}(\phi) = \{T_k\phi\}_{k\in\mathbb{Z}}$ for any $\phi \in S_{\mathbb{Z}}(\varphi)$. □

We now present a result which shows that each H-invariant space has a frame of translates. This result is well-known when $G = \mathbb{R}^d$ and it was proved for the case when V is finitely generated by Ron and Shen in [RS95]. In that work they also provided a way to construct a Parseval frame for $S_{\mathbb{Z}^d}(f_1, \cdots, f_n) \subseteq L^2(\mathbb{R}^d)$ in terms of the Gramian of $\{f_1, \cdots, f_n\}$. This construction can be translated to the context of LCA groups using the material that we have developed in this thesis. Here, we will give a proof for the countably generated case, which obviously includes the finitely generated case, based on the decomposition of H-invariant spaces stated in Theorem 3.3.2.

Theorem 3.3.4. *Let V be an H-invariant spaces of $L^2(G)$. Then, there exists $\mathcal{A} \subseteq G$ a countable set such that $\{T_h\varphi : \varphi \in \mathcal{A}, , h \in H\}$ is a Parseval frame for V.*

Proof. By Theorem 3.3.2, $V = \bigoplus_{n\in\mathbb{N}} S_H(\varphi_n)$, where $E_H(\varphi_n)$ is a Parseval frame for $S_H(\varphi_n)$ for all $n \in \mathbb{N}$. We will show that $\{T_h\varphi_n : n \in \mathbb{N}, , h \in H\}$ is a Parseval frame for V.

Let $f \in V$ and write $f = \sum_{n\in\mathbb{N}} f_n$ where each f_n is the orthogonal projection of f on $S_H(\varphi_n)$. Then, since $S_H(\varphi_n)$ is orthogonal to $S_H(\varphi_m)$ if $n \neq m$, it follows that

$$\|f\|^2_{L^2(G)} = \sum_{n\in\mathbb{N}} \|f_n\|^2_{L^2(G)}. \tag{3.22}$$

On the other hand, since $E_H(\varphi_n)$ is a Parseval frame for $S_H(\varphi_n)$ for each $n \in \mathbb{N}$, we have

$$\sum_{h\in H} |\langle f, T_h\varphi_n\rangle_{L^2(G)}|^2 = \sum_{h\in H} |\langle f_n, T_h\varphi_n\rangle_{L^2(G)}|^2 = \|f_n\|^2_{L^2(G)}. \tag{3.23}$$

Thus, adding over all $n \in \mathbb{N}$ in (3.23) and using (3.22) we have

$$\|f\|^2_{L^2(G)} = \sum_{n\in\mathbb{N}} \sum_{h\in H} |\langle f, T_h\varphi_n\rangle_{L^2(G)}|^2,$$

and this proves that $\{T_h\varphi_n : n \in \mathbb{N}, , h \in H\}$ is a Parseval frame for V. □

4

Extra invariance of H-invariant spaces

If $\phi(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$, then $S_{\mathbb{Z}}(\phi) \subseteq L^2(\mathbb{R})$ is a shift invariant space (SIS) with the property of being invariant only under integer translations. In contrast to this SIS, the Paley-Wiener space of functions that are bandlimited to $[-\frac{1}{2}, \frac{1}{2}]$ (see Example 1.2.2) a is shift invariant space which is invariant under every real translation. Moreover, there exist SIS with some extra invariance that are not necessarily invariant under all real translations.

Shift invariant spaces in the real line with extra invariance have been characterized by Aldroubi et al [ACHKM10]. First, they show that if V is a SIS that is invariant under translations other than integers, it holds that either V is invariant under any real translation or V is invariant under translation on $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N}$ and it is not invariant under any bigger subgroup. Then, they give several characterizations of those SIS that are also $\frac{1}{n}\mathbb{Z}$-invariant.

In this chapter, we want to study the problem of the extra invariance in the context of general LCA groups. More precisely, if H is a countable uniform lattice in G, we investigate those H-invariant spaces of $L^2(G)$, that are invariant under a closed subgroup M of G containing H.

The difficulty here lies in the fact that we do not have an explicit structure of the subgroup M, as the authors do have in [ACHKM10] (in that work, M is of the form $\frac{1}{n}\mathbb{Z}$). However, we are able to show necessary and sufficient condition for an H-invariant space to be M-invariant.

As a consequence of our results we prove that for each closed subgroup M of G containing the lattice H, there exists an H-invariant space V that is exactly M-invariant. That is, V is not invariant under any other subgroup M' containing M. We also obtain estimates on the support of the Fourier transform of the generators of the H-invariant space, related to its M-invariance.

Here and subsequently G will be an LCA group, H a countable uniform lattice in G with Δ its annihilator, Γ the dual group of G and Ω_Δ a measurable section of the quotient Γ/Δ. We choose the Haar measure of the groups involved here as follows. We set m_Δ such that $m_\Delta(\{0\}) = 1$. Then we fix $m_{\Gamma/\Delta}$ and m_Γ in order to Weil's formula of Theorem 1.1.10 holds among $m_{\Gamma/\Delta}$, m_Γ ad m_Δ. Finally, we choose m_G such that inversion formula holds for m_G and m_Γ.

The chapter is organized in the following way. In Section 4.1 we prove that the set of parameters $x \in G$ that leave the H-invariant space invariant under translation by x forms a closed subgroup of G which contains H. Then, in Section 4.2 we study the structure of principal M-invariant spaces. Section 4.3 contains the characterizations of M-invariance for general H-invariant spaces. Some relevant applications of the results of Section 4.3 are given in Section 4.4. In Section 4.5 we show how the result about extra invariance for SIS in $L^2(\mathbb{R}^d)$ with $d > 1$ can be obtained using the structure of the closed subgroups of $\mathbb{R}^d$ that contain $\mathbb{Z}^d$.

4.1 The invariance set

Let $V \subseteq L^2(G)$ be an H-invariant space, we define the *invariance set* as

$$M = \{x \in G \,:\, T_x f \in V, \ \forall\ f \in V\}. \tag{4.1}$$

If $\mathcal{A}$ is a set of generators for V, it is easy to check that $m \in M$ if and only if $T_m\varphi \in V$ for all $\varphi \in \mathcal{A}$.

In case that $M = G$, Wiener's theorem (see [Hel64], [Sri64] and [HS64]) states that there exists a measurable set $E \subseteq \Gamma$ satisfying

$$V = \{f \in L^2(G) \,:\, \sup(\widehat{f}\,) \subseteq E\}.$$

We are interested in describing V when M is not all G. We will first study the structure of the set M.

Proposition 4.1.1. *Let V be an H-invariant space of $L^2(G)$ and let M be defined as in (4.1). Then M is a closed subgroup of G containing H.*

For the proof of this proposition we will need the following lemma. Recall that a semigroup is a nonempty set with an associative additive operation.

Lemma 4.1.2. *Let K be a closed semigroup of G containing H, then K is a group.*

Proof. Let π be the quotient map from G onto G/H. Since K is a semigroup containing H, we have that $K + H = K$, thus

$$\pi^{-1}(\pi(K)) = \bigcup_{k\in K} k + H = K + H = K. \tag{4.2}$$

This shows that $\pi(K)$ is closed in G/H and therefore compact.

By [HR79, Theorem 9.16], we have that a compact semigroup of G/H is necessarily a group, thus $\pi(K)$ is a group and consequently K is a group.

□

Proof of Proposition 4.1.1. Since V is an H-invariant space, $H \subseteq M$.

We first proceed to show that M is closed. Let $x_0 \in G$ and $\{x_\lambda\}_{\lambda\in\Lambda}$ a net in M converging to x_0. Then

$$\lim_\lambda \|T_{x_\lambda} f - T_{x_0} f\|_{L^2(G)} = 0.$$

Since V is closed, it follows that $T_{x_0} f \in V$, thus $x_0 \in M$.

It is easy to check that M is a semigroup of G, hence we conclude from Lemma 4.1.2 that M is a group.

□

4.2 The structure of principal M-invariant spaces

In this section we will use fiberization techniques and range functions for a more general setting than in Chapter 2, since the subspaces will be invariant under a closed subgroup which is not necessarily discrete.

The results from Section 1.4 of Chapter 1 and from Chapter 2 can be extended straightforwardly to the case in which the spaces are invariant under a closed subgroup M of G containing H as follows.

First, we consider the following normalization for the Haar measures m_{M^*} and m_{Γ/M^*}. Note that, since $H \subseteq M$, $M^* \subseteq \Delta$ and, in particular, M^* is discrete. Then, we fix m_{M^*} such that $m_{M^*}(\{0\}) = 1$ and m_{Γ/M^*} such that Weil's formula of Theorem 1.1.10 holds among m_Γ, m_{M^*} and m_{Γ/M^*}.

As $M^* \subseteq \Delta$ and M^* is discrete, there exists a countable section $\mathcal{N}$ of Δ/M^*. Then, the set given by

$$\Omega_{M^*} = \bigcup_{\sigma\in\mathcal{N}} \Omega_\Delta + \sigma \tag{4.3}$$

is a σ-finite measurable section for the quotient Γ/M^*. Using this section for Γ/M^* it is possible to define $L^2(\Omega_{M^*}, \ell^2(M^*))$, according to Definition 1.3.5. We can also use this section to define what a range function with respect to M is.

Definition 4.2.1. Let H be a uniform lattice on G and M a closed subgroup of G containing H. A *range function with respect to* M is any application

$$J : \Omega_{M^*} \longrightarrow \{\text{closed subspaces of } \ell^2(M^*)\}.$$

The subspace $J(\xi)$ is called the *fiber space* associated to ξ.

For a given range function with respect to M J, we associate to each $\xi \in \Omega_{M^*}$ the orthogonal projection onto $J(\xi)$, $P_\xi : \ell^2(M^*) \to J(\xi)$.

A range function with respect to M J is *measurable* if for each $a \in \ell^2(M^*)$ the function $\xi \mapsto P_\xi a$, from Ω_{M^*} into $\ell^2(M^*)$ is strongly measurable.

With the normalization of the Haar measures that we have stated, it can be proved the following proposition which compiles extensions of Proposition 1.4.2, Proposition 2.2.4 and Theorem 2.2.5.

Proposition 4.2.2.

i) The mapping $\mathcal{T}_M : L^2(G) \longrightarrow L^2(\Omega_{M^*}, \ell^2(M^*))$ defined as

$$\mathcal{T}_M f(\xi) = \{\widehat{f}(\xi + m^*)\}_{m^* \in M^*}$$

is an isomorphism that satisfies $\|\mathcal{T}_M f\|_2 = \|f\|_{L^2(G)}$.

ii) Let V be a closed subspace of $L^2(G)$. Then V is an M-invariant space if and only if there exists J a measurable range function with respect to M such that

$$V = \{f \in L^2(G) : \mathcal{T}_M f(\xi) \in J(\xi) \text{ for a.e. } \xi \in \Omega_{M^*}\}.$$

Moreover, if V is an M-invariant space generated by a countable set $\mathcal{A}$, the measurable range function with respect to M J associated with V is given by

$$J(\xi) = \overline{\text{span}}\{\mathcal{T}_M \varphi(\xi) : \varphi \in \mathcal{A}\}.$$

iii) Let V be an M-invariant space and let $\mathcal{P}_V$ and P_ξ be the orthogonal projections onto V and $J(\xi)$ respectively, where J is the measurable range function with respect to M associated to V. Then, for every $g \in L^2(G)$,

$$\mathcal{T}_M(\mathcal{P}_V g)(\xi) = P_\xi(\mathcal{T}_M g(\xi)) \quad \text{a.e. } \xi \in \Omega_{M^*}.$$

Remark 4.2.3. Since in this context coexist H-invariance and M-invariance we need to distinguish fibers with respect to H form fibers with respect to M. For this, we will refer to $\mathcal{T}_H f(\omega)$ as *H-fiber* and to $\mathcal{T}_M f(\xi)$ as *M-fiber*.

4.2.1 Principal M-invariant Spaces

We prove now the following characterization of principal M-invariant spaces. This result extends Theorem 2.1.1 to the non-discrete case.

Theorem 4.2.4. *Let $f \in L^2(G)$ and M a closed subgroup of G containing H. If $g \in S_M(f)$, then there exists an M^*-periodic function η such that $\widehat{g} = \eta \widehat{f}$.*

Conversely, if η is an M^*-periodic function such that $\eta \widehat{f} \in L^2(\Gamma)$, then the function g defined by $\widehat{g} = \eta \widehat{f}$ belongs to $S_M(f)$.

Proof. Let us call $V = S_M(f)$ and let $\mathcal{P}_V$ and P_ξ be the orthogonal projections onto V and $J(\xi)$ respectively. Given $g \in V$, we first define η_g in Ω_{M^*} as

$$\eta_g(\xi) = \begin{cases} \frac{\langle \mathcal{T}_M g(\xi), \mathcal{T}_M f(\xi) \rangle_{\ell^2(M^*)}}{\|\mathcal{T}_M f(\xi)\|^2_{\ell^2(M^*)}} & \text{if } \xi \in \tilde{\mathcal{E}}_f \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\mathcal{E}}_f$ is the set $\{\xi \in \Omega_{M^*} : \|\mathcal{T}_M f(\xi)\|^2_{\ell^2(M^*)} \neq 0\}$. Then, since $\{\Omega_{M^*} + m^*\}_{m^* \in M^*}$ forms a partition of Γ, we can extend η_g to all Γ in an M^*-periodic way.

Now, by Proposition 4.2.2 we have that

$$\mathcal{T}_M g(\xi) = \mathcal{T}_M(\mathcal{P}_V g)(\xi) = P_\xi(\mathcal{T}_M g(\xi)) = \eta_g(\xi)\mathcal{T}_M f(\xi).$$

Since η_g is an M^*-periodic function, $\widehat{g} = \eta_g \widehat{f}$ as we wanted to prove.

Conversely, if $\widehat{g} = \eta \widehat{f}$, with η an M^*-periodic function, then $\mathcal{T}_M g(\xi) = \eta(\xi)\mathcal{T}_M f(\xi)$. By Proposition 4.2.2, $g \in V$. □

Remark 4.2.5. Observe that Theorem 4.2.4 has been proved using Proposition 4.2.2. In the same way, Theorem 2.1.1 can be obtained from Theorem 2.2.5. However, in Section 2.1 we have given a proof of Theorem 2.1.1 similar to the one given for Theorem 4.2.4 which does not use the characterization of H-invariant spaces stated in Theorem 2.2.5.

4.3 Characterization of M-invariance

If $H \subseteq M \subseteq G$, where H is a countable uniform lattice in G and M is a closed subgroup of G, we are interested in describing when an H-invariant space V is also M-invariant.

Let Ω_Δ be a measurable section of Γ/Δ and $\mathcal{N}$ a countable section of Δ/M^*. For $\sigma \in \mathcal{N}$ we define the set B_σ as

$$B_\sigma = \Omega_\Delta + \sigma + M^* = \bigcup_{m^* \in M^*} (\Omega_\Delta + \sigma) + m^*. \tag{4.4}$$

Therefore, each B_σ is an M^*-periodic set. Here, by M^*-periodic set we mean a set for which its characteristic function is an M^*-periodic function.

Since Ω_Δ tiles Γ by Δ translations and $\mathcal{N}$ tiles Δ by M^* translations, it follows that $\{B_\sigma\}_{\sigma \in \mathcal{N}}$ is a partition of Γ.

In order to understand this construction we give two basic examples of the partition $\{B_\sigma\}_{\sigma \in \mathcal{N}}$.

Example 4.3.1.

(1) Let $H = \mathbb{Z}$ and $M = \frac{1}{n}\mathbb{Z} \subseteq \mathbb{R}$, then $M^* = n\mathbb{Z}$, $\Omega_\Delta = [0, 1)$ and $\mathcal{N} = \{0, \ldots, n-1\}$. Given $\sigma \in \{0, \ldots, n-1\}$, we have

$$B_\sigma = \bigcup_{m^* \in n\mathbb{Z}} ([0, 1) + \sigma) + m^* = \bigcup_{j \in \mathbb{Z}} [\sigma, \sigma + 1) + nj.$$

Figure 4.1 illustrates the partition for $n = 4$. In the picture, the black dots represent the set $\mathcal{N}$. The set B_2 is the one which appears in gray.

(2) Let $H = \mathbb{Z}^2$ and $M = \frac{1}{2}\mathbb{Z} \times \mathbb{R} \subseteq \mathbb{R}^2$, then $\Omega_\Delta = [0, 1)^2$, $M^* = 2\mathbb{Z} \times \{0\}$ and $\mathcal{N} = \{0, 1\} \times \mathbb{Z}$.

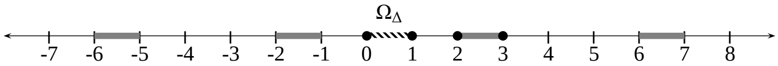

Figure 4.1: Partition of the real line for $M = \frac{1}{4}\mathbb{Z}$.

So, the sets $B_{(i,j)}$ are

$$B_{(i,j)} = \bigcup_{k\in\mathbb{Z}} ([0,1)^2 + (i,j)) + (2k,0)$$

where $(i,j) \in \mathcal{N}$. See Figure 4.2, where the sets $B_{(0,0)}$, $B_{(1,1)}$ and $B_{(-1,-1)}$ are represented by the squares painted in light gray, gray and dark gray respectively. As in the previous figure, the set $\mathcal{N}$ is represented by the black dots.

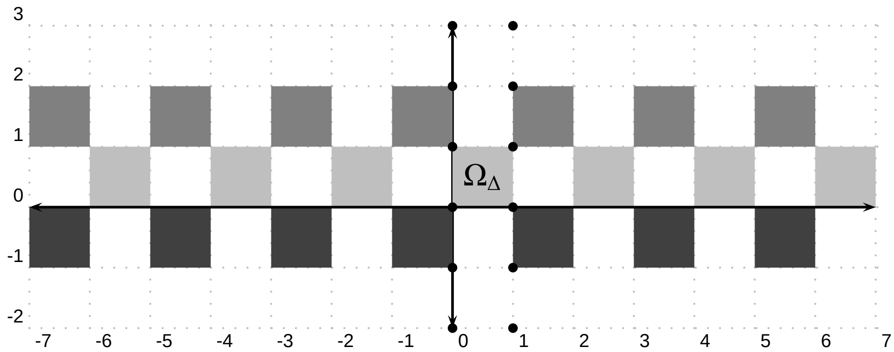

Figure 4.2: Partition of the plane for $M = \frac{1}{2}\mathbb{Z} \times \mathbb{R}$.

4.3.1 Characterization of M-invariance in terms of subspaces

Let $V \subseteq L^2(G)$ be an H-invariant space. Now, for each $\sigma \in \mathcal{N}$, we define, using the partition $\{B_\sigma\}_{\sigma\in\mathcal{N}}$, the subspaces

$$U_\sigma = \{f \in L^2(G) : \widehat{f} = \chi_{B_\sigma}\widehat{g}, \text{ with } g \in V\}. \tag{4.5}$$

The main theorem of this section characterizes the M-invariance of V in terms of the subspaces U_σ.

Theorem 4.3.2. *If $V \subseteq L^2(G)$ is an H-invariant space and M is a closed subgroup of G containing H, then the following are equivalent.*

i) V is M-invariant.

ii) $U_\sigma \subseteq V$ for all $\sigma \in \mathcal{N}$.

Moreover, in case any of these hold we have that V is the orthogonal direct sum

$$V = \dot{\bigoplus_{\sigma\in\mathcal{N}}} U_\sigma.$$

Now we state a lemma that we need to prove Theorem 4.3.2.

Lemma 4.3.3. *Let V be an H-invariant space and $\sigma \in \mathcal{N}$. Assume that the subspace U_σ defined in (4.5.15) satisfies $U_\sigma \subseteq V$. Then, U_σ is an M-invariant space and in particular is H-invariant.*

Proof. Let us prove first that U_σ is closed. Suppose that $f_j \in U_\sigma$ and $f_j \to f$ in $L^2(G)$. Since $U_\sigma \subseteq V$ and V is closed, f must be in V. Further,

$$\|\widehat{f_j} - \widehat{f}\|_2^2 = \|(\widehat{f_j} - \widehat{f})\chi_{B_\sigma}\|_2^2 + \|(\widehat{f_j} - \widehat{f})\chi_{B_\sigma^c}\|_2^2 = \|\widehat{f_j} - \widehat{f}\chi_{B_\sigma}\|_2^2 + \|\widehat{f}\chi_{B_\sigma^c}\|_2^2.$$

Since the left-hand side converges to zero, we must have that $\widehat{f}\chi_{B_\sigma^c} = 0$ a.e. $\gamma \in \Gamma$. Then, $\widehat{f} = \widehat{f}\chi_{B_\sigma}$. Consequently $f \in U_\sigma$, so U_σ is closed.

Now we show that U_σ is M-invariant. Given $m \in M$ and $f \in U_\sigma$, we will prove that $(m,\cdot)\widehat{f}(\cdot) \in \widehat{U_\sigma}$.

Since $f \in U_\sigma$, there exists $g \in V$ such that $\widehat{f} = \chi_{B_\sigma}\widehat{g}$. Hence,

$$(m,\cdot)\widehat{f}(\cdot) = (m,\cdot)(\chi_{B_\sigma}\widehat{g})(\cdot) = \chi_{B_\sigma}(\cdot)((m,\cdot)\widehat{g}(\cdot)). \tag{4.6}$$

If we were able to find an Δ-periodic function ℓ_m verifying

$$(m,\gamma) = \ell_m(\gamma) \quad \text{a.e.}\ \gamma \in B_\sigma, \tag{4.7}$$

then, we can rewrite (4.6) as

$$(m,\cdot)\widehat{f}(\cdot) = \chi_{B_\sigma}(\cdot)(\ell_m\widehat{g})(\cdot).$$

Thus, since ℓ_m is Δ-periodic, Theorem 2.1.1 gives us $\ell_m\widehat{g} \in \widehat{S_H(g)} \subseteq \widehat{V}$ and so, $(m,\cdot)\widehat{f}(\cdot) \in \widehat{U_\sigma}$.

Now we define the function ℓ_m as follows. For each $\delta \in \Delta$, set

$$\ell_m(\omega + \delta) = (m, \omega + \sigma) \quad \text{a.e.}\ \omega \in \Omega_\Delta. \tag{4.8}$$

It is clear that ℓ_m is Δ-periodic.

Since $(m,\cdot)$ is M^*-periodic,

$$(m, \omega + \sigma) = (m, \omega + \sigma + m^*) \quad \text{a.e.}\ \omega \in \Omega_\Delta,\ \forall\, m^* \in M^*.$$

Thus, (4.7) holds.

Note that, since $H \subseteq M$, the H-invariance of U_σ is a consequence of the M-invariance.

□

Proof of Theorem 4.3.2. i) ⇒ ii): Fix $\sigma \in \mathcal{N}$ and $f \in U_\sigma$. Then $\widehat{f} = \chi_{B_\sigma}\widehat{g}$ for some $g \in V$. Since χ_{B_σ} is an M^*-periodic function, by Theorem 4.2.4, we have that $f \in S_M(g) \subseteq V$, as we wanted to prove.

ii) ⇒ i): Suppose that $U_\sigma \subseteq V$ for all $\sigma \in \mathcal{N}$. Note that Lemma 4.3.3 implies that U_σ is M-invariant, and that, since the sets B_σ are disjoint, the subspaces U_σ are mutually orthogonal.

Suppose that $f \in V$. Then, since $\{B_\sigma\}_{\sigma\in\mathcal{N}}$ is a partition of Γ, it follows that $\widehat{f} = \sum_{\sigma\in\mathcal{N}} \widehat{f}\chi_{B_\sigma}$. Then $f \in \dot{\bigoplus}_{\sigma\in\mathcal{N}} U_\sigma$ and consequently, V is the orthogonal direct sum

$$V = \dot{\bigoplus_{\sigma\in\mathcal{N}}} U_\sigma.$$

As each U_σ is M-invariant, so is V. □

To finish this section we want to point out that the subspaces U_σ are not necessarily closed. To see this we need to introduce some useful notation that we will also use in the remainder of this chapter.

If $f \in L^2(G)$ and $\sigma \in \mathcal{N}$, we define the function f^σ as

$$\widehat{f^\sigma} = \widehat{f}\chi_{B_\sigma}.$$

Let $\mathcal{P}_\sigma$ be the orthogonal projection onto S_σ, where

$$S_\sigma = \{f \in L^2(G) : \operatorname{supp}(\widehat{f}) \subseteq B_\sigma\}. \tag{4.9}$$

Observe that the subspaces S_σ defined above are invariant under any translation in G. In particular they are H-invariant spaces.

It is easy to see that $f^\sigma = \mathcal{P}_\sigma f$. Then, it follows from the definition of U_σ that

$$U_\sigma = \mathcal{P}_\sigma(V) = \{f^\sigma : f \in V\}. \tag{4.10}$$

It is a general result that if M and N are closed subspaces of a Hilbert spaces $\mathcal{H}$, then, $M + N$ is closed if and only if $\mathcal{P}_{N^\perp}(M)$ is closed, where as usual, $\mathcal{P}_{N^\perp}$ denotes the orthogonal projection onto $N^\perp$ and $N^\perp$ denotes the orthogonal complement of N in $\mathcal{H}$ (see [Kat95]).

Using this result, we have that $U_\sigma = \mathcal{P}_\sigma(V)$ is closed if and only if $V + S_\sigma^\perp$ is closed.

In order to understand when $V + S_\sigma^\perp$ is closed we introduce the notion of angle between closed subspaces (for details see [Deu95]).

Let U and V be closed subspaces of a Hilbert spaces $\mathcal{H}$. The *Friedrichs angle* between U and V is the angle in $[0, \frac{\pi}{2}]$ whose cosine is

$$\mathbf{c}[U, V] := \sup\{|\langle u, v\rangle| : u \in U \ominus V, v \in V \ominus U \text{ and } \|u\| \le 1, \|v\| \le 1\}.$$

Here, $U \ominus V$ means the orthogonal complement of $U \cap V$ in U. That is $U \ominus V = U \cap (U \cap V)^\perp$.

Then, it is known that $U + V$ is closed if and only if $\mathbf{c}[U, V] < 1$.

In case when $\mathcal{H} = L^2(G)$ and U and V are H-invariant spaces the Friedrichs angle between U and V can be formulated in terms of H-fibers as follows (see [AC09, Lemma 6.8]). If J_U and J_V denote the measurable range functions associated to U and V respectively then

$$\mathbf{c}[U, V] = \operatorname{esssup}\{\mathbf{c}[J_U(\omega), J_V(\omega)] \,:\, \omega \in \Omega_\Delta\}.$$

Now we are able to give an example which shows that the spaces U_σ can be not closed.

Example 4.3.4. Let us consider the $\mathbb{Z}$-invariant space generated by $\phi(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$, $V := S_{\mathbb{Z}}(\phi)$ and let $M = \frac{1}{2}\mathbb{Z}$. We will see that the subspace $U_0 = \{f \in L^2(\mathbb{R}) : \widehat{f} = \chi_{B_0}\widehat{g},$ with $g \in V\}$, where $B_0 = [0, 1) + 2\mathbb{Z}$, is not closed. For this, we will prove that $\mathbf{c}[V, S_0^\perp] = 1$.

First, note that $S_0^\perp = S_1 = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subseteq B_1\}$, with B_1 being the set $[0, 1) + 1 + 2\mathbb{Z}$.

Now, if J_V is the measurable range function associated to V, then $J_V(\omega) = \overline{\operatorname{span}}\{\mathcal{T}_{\mathbb{Z}}\phi(w)\}$ for a.e. $\omega \in [0, 1)$. Since $\widehat{\phi}(\omega) = \frac{\sin(\pi\omega)}{\pi\omega} := sinc(\omega)$, we rewrite $J_V(\omega) = \operatorname{span}\{\{sinc(\omega + j)\}_{j\in\mathbb{Z}}\}$.

On the other hand, if J_{S_1} is the measurable range function associated to S_1, one can easily check that $J_{S_1}(\omega) = \overline{\operatorname{span}}\{e_{2k+1} \,:\, k \in \mathbb{Z}\}$.

Then, for each $\omega \in [0, 1)$ fixed, we have

$$\mathbf{c}[J_V(\omega), J_{S_1}(\omega)] = \sup_{k\in\mathbb{Z}} |\langle \mathcal{T}_{\mathbb{Z}}\phi(\omega), e_{2k+1}\rangle| = \sup_{k\in\mathbb{Z}} |sinc(\omega + 2k + 1)| \geq |sinc(\omega - 1)|. \quad (4.11)$$

Thus, taking essential supreme over $\omega \in [0, 1)$ in equation (4.11) it follows

$$\mathbf{c}[V, S_1] = 1.$$

4.3.2 Characterization of M-invariance in terms of H-fibers

In this section we will first express the conditions of Theorem 4.3.2 in terms of H-fibers. Then, we will give a useful characterization of the M-invariance for a finitely generated H-invariant space in terms of the Gramian.

As we have say in the last section, the subspaces S_σ defined in (4.9) are H-invariant spaces. Then, we will denote by P_ω^σ the orthogonal projections associated to the range function of S_σ.

Lemma 4.3.5. *If $V = S_H(\mathcal{A})$ with $\mathcal{A}$ a countable subset of $L^2(G)$, then*

$$\overline{\{\mathcal{T}_H f(\omega) \,:\, f \in U_\sigma\}} = \overline{\operatorname{span}}\{\mathcal{T}_H(\varphi^\sigma)(\omega) \,:\, \varphi \in \mathcal{A}\},$$

for a.e. $\omega \in \Omega_\Delta$.

Proof. Since $\varphi^\sigma \in U_\sigma$ for all $\varphi \in \mathcal{A}$, it holds that $\overline{\text{span}}\{\mathcal{T}_H(\varphi^\sigma)(\omega) : \varphi \in \mathcal{A}\} \subseteq \overline{\{\mathcal{T}_H f(\omega) : f \in U_\sigma\}}$.

To prove the other inclusion, observe that, since $U_\sigma = \mathcal{P}_\sigma(V)$,

$$\overline{\{\mathcal{T}_H f(\omega) : f \in U_\sigma\}} = \overline{\{\mathcal{T}_H(\mathcal{P}_\sigma f)(\omega) : f \in V\}}.$$

Now, by Proposition 4.2.2, we have $\mathcal{T}_H(\mathcal{P}_\sigma f)(\omega) = P^\sigma_\omega(\mathcal{T}_H f(\omega))$ for a.e. $\omega \in \Omega_\Delta$. Thus,

$$\overline{\{\mathcal{T}_H(\mathcal{P}_\sigma f)(\omega) : f \in V\}} = \overline{\{P^\sigma_\omega(\mathcal{T}_H f(\omega)) : f \in V\}} = \overline{P^\sigma_\omega\{\mathcal{T}_H f(\omega) : f \in V\}}$$

If J is the measurable range function associated with V as an H-invariant spaces, using (2.5), it follows that

$$\overline{P^\sigma_\omega\{\mathcal{T}_H f(\omega) : f \in V\}} \subseteq \overline{P^\sigma_\omega(\overline{\text{span}}\{\mathcal{T}_H\varphi(\omega) : \varphi \in \mathcal{A}\})} \subseteq \overline{\text{span}\{P^\sigma_\omega(\mathcal{T}_H\varphi(\omega)) : \varphi \in \mathcal{A}\}},$$

where the last inclusion is due to the continuity and linearity of P^σ_ω.

Then, using once again Proposition 4.2.2, we obtain

$$\overline{\text{span}\{P^\sigma_\omega(\mathcal{T}_H\varphi(\omega)) : \varphi \in \mathcal{A}\}} = \overline{\text{span}\{\mathcal{T}_H(\mathcal{P}_\sigma\varphi)(\omega) : \varphi \in \mathcal{A}\}}$$

which finishes the proof. □

An important thing to point out is, since $U_\sigma = \mathcal{P}_\sigma(V)$, U_σ is invariant under translations along H. Nevertheless, it is not necessarily closed (see Example 4.3.4). Then, in general, it is not an H-invariant space. On the other hand, the mapping J_{U_σ} from Ω_Δ to {closed subspaces of $\ell^2(\Delta)$} which assigns to each ω the subspace defined in Lemma 4.3.5 is a measurable range function. As a comment we want to remark that, when U_σ is an H-invariant space, the range function J_{U_σ} is precisely the measurable range function associated with U_σ through Theorem 2.2.5.

Combining Theorems 4.3.2 and 2.2.5 with Lemma 4.3.5 we obtain the following result.

Proposition 4.3.6. *Let V be an H-invariant space generated by a countable set $\mathcal{A} \subseteq L^2(G)$ and denote by J_V the measurable range function associated to V through Theorem 2.2.5. The following statements are equivalent.*

i) V is M-invariant.

ii) $\mathcal{T}_H(\varphi^\sigma)(\omega) \in J_V(\omega)$ a.e. $\omega \in \Omega_\Delta$ for all $\varphi \in \mathcal{A}$ and $\sigma \in \mathcal{N}$.

Proof. i)⇒ii): Since V is M-invariant , Theorem 4.3.2 gives $U_\sigma \subseteq V$ for all $\sigma \in \mathcal{N}$. Using (4.10), we obtain that $\varphi^\sigma \in V$ for all $\sigma \in \mathcal{N}$. Then ii) follows from Theorem 2.2.5.

ii)⇒i): By the hypothesis and Lemma 4.3.5 it follows that $\overline{\{\mathcal{T}_H f(\omega) : f \in U_\sigma\}} \subseteq J_V(\omega)$ for a.e. $\omega \in \Omega_\Delta$. Hence, using Theorem 2.2.5, $U_\sigma \subseteq V$. Thus, V is M-invariant as a consequence of Theorem 4.3.2. □

Let us now turn our attention to the finitely generated case. Let $\Phi = \{\varphi_1, \dots, \varphi_\ell\}$ be a finite collection of functions in $L^2(G)$. Then, according to Definition 3.2.6, the Gramian of Φ is the $\ell \times \ell$ matrix of Δ-periodic functions

$$\begin{aligned}[\mathcal{G}_\Phi(\omega)]_{ij} &= \left\langle \mathcal{T}_H\varphi_i(\omega), \mathcal{T}_H\varphi_j(\omega) \right\rangle \\ &= \sum_{\delta\in\Delta} \widehat{\varphi_i}(\omega+\delta)\,\overline{\widehat{\varphi_j}(\omega+\delta)} \end{aligned} \tag{4.12}$$

for $\omega \in \Omega_\Delta$.

Now we give a slightly simpler characterization of M-invariance for the finitely generated case. Here we use the notation $\dim_{U_\sigma}(\omega)$ for $\dim(\overline{\operatorname{span}}\{\mathcal{T}_H(\varphi^\sigma)(\omega) : \varphi \in \Phi\})$.

Theorem 4.3.7. *If V is an H-invariant space, finitely generated by Φ, then the following statements are equivalent.*

i) *V is M-invariant.*

ii) *For almost every $\omega \in \Omega_\Delta$, $\dim_V(\omega) = \sum_{\sigma\in\mathcal{N}} \dim_{U_\sigma}(\omega)$.*

iii) *For almost every $\omega \in \Omega_\Delta$, $\operatorname{rank}[\mathcal{G}_\Phi(\omega)] = \sum_{\sigma\in\mathcal{N}} \operatorname{rank}[\mathcal{G}_{\Phi^\sigma}(\omega)]$, where $\Phi^\sigma = \{\varphi^\sigma : \varphi \in \Phi\}$.*

For the proof of this theorem we need the following result which is a straightforward consequence of Theorem 2.2.5.

Proposition 4.3.8. *Let V_1 and V_2 be H-invariant spaces. If $V = V_1 \dot{\oplus} V_2$, and J_V, J_{V_1}, J_{V_2} denote the measurable range functions associated to V, V_1 and V_2 respectively, then*

$$J_V(\omega) = J_{V_1}(\omega) \dot{\oplus} J_{V_2}(\omega), \quad a.e.\ \omega \in \Omega_\Delta.$$

The converse of this proposition is also true, but it will not be needed.

Remark 4.3.9. Note that the fibers

$$\mathcal{T}_H(\varphi^\sigma)(\omega) = \{\chi_{B_\sigma}(\omega+\delta)\widehat{\varphi}(\omega+\delta)\}_{\delta\in\Delta}$$

can be described in a simple way as

$$\chi_{B_\sigma}(\omega+\delta)\widehat{\varphi}(\omega+\delta) = \begin{cases} \widehat{\varphi}(\omega+\delta) & \text{if } \delta \in \sigma + M^* \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, if $\sigma \neq \sigma'$, $\overline{\operatorname{span}}\{\mathcal{T}_H(\varphi^\sigma)(\omega) : \varphi \in \mathcal{A}\}$ and $\overline{\operatorname{span}}\{\mathcal{T}_H(\varphi^{\sigma'})(\omega) : \varphi \in \mathcal{A}\}$ are orthogonal subspaces for a.e. $\omega \in \Omega_\Delta$.

Proof. i) $\Rightarrow$ ii): By Theorem 4.3.2, $V = \dot{\oplus}_{\sigma\in\mathcal{N}} U_\sigma$ and U_σ is an H-invariant space for all $\sigma \in \mathcal{N}$. Then, ii) follows from Proposition 4.3.8.

ii) $\Rightarrow$ i): Let J_V be the measurable range function associated to V. Since $\{B_\sigma\}_{\sigma\in\mathcal{N}}$ is a partition of Γ, $V \subseteq \dot{\oplus}_{\sigma\in\mathcal{N}} U_\sigma$. Then, by Remarks 4.3.9 we have that

$$J_V(\omega) \subseteq \dot{\bigoplus_{\sigma\in\mathcal{N}}} \overline{\operatorname{span}}\{\mathcal{T}_H(\varphi^\sigma)(\omega) \,:\, \varphi \in \Phi\}.$$

Due to V is finitely generated, we use ii) to obtain that $J_V(\omega) = \dot{\oplus}_{\sigma\in\mathcal{N}} \overline{\operatorname{span}}\{\mathcal{T}_H(\varphi^\sigma)(\omega) \,:\, \varphi \in \Phi\}$. The proof follows as a consequence of Proposition 4.3.6.

The equivalence between ii) and iii) follows from the straightforward equality $\dim_V(\omega) = \operatorname{rank}[\mathcal{G}_\Phi(\omega)]$. □

4.4 Applications of M-invariance

In this section we estimate the size of the supports of the Fourier transforms of the generators of a finitely generated H-invariant space which is also M-invariant. Finally, given M a closed subgroup of G containing H, we will construct an H-invariant space V which is exactly M-invariant. That is, V is not invariant under any other closed subgroup of G containing H.

Theorem 4.4.1. *Let V be an H-invariant space finitely generated by the set $\{\varphi_1, \dots, \varphi_\ell\}$, and define*

$$E_j = \{\omega \in \Omega_\Delta \,:\, \dim_V(\omega) = j\}, \quad j = 0, \dots, \ell.$$

If V is M-invariant and Ω'_{M^*} is any measurable section of Γ/M^*, then

$$m_\Gamma(\{\gamma \in \Omega'_{M^*} \,:\, \widehat{\varphi_i}(\gamma) \neq 0\}) \leq \sum_{j=0}^{\ell} m_\Gamma(E_j)\, j \leq \ell,$$

for each $i = 1, \dots, \ell$.

Proof. The measurability of the sets E_j follows from the results of Helson [Hel64], e.g., see [BK06] for an argument of this type.

Fix any $i \in \{0, \dots, \ell\}$ and denote by J_{U_σ} the measurable range function associated to the H-invariant spaces U_σ. Note that, as a consequence of Remark 4.3.9, if $J_{U_\sigma}(\omega) = \{0\}$, then $\widehat{\varphi_i}(\omega + \sigma + m^*) = 0$ for all $m^* \in M^*$.

On the other hand, since $\{\Omega_\Delta + \sigma + m^*\}_{\sigma\in\mathcal{N}, m^*\in M^*}$ is a partition of Γ, if $\omega \in \Omega_\Delta$ and $\sigma \in \mathcal{N}$ are fixed, there exists a unique $m^*_{(\omega,\sigma)} \in M^*$ such that $\omega + \sigma + m^*_{(\omega,\sigma)} \in \Omega'_{M^*}$.

So,

$$\{\sigma \in \mathcal{N} \,:\, \widehat{\varphi_i}(\omega + \sigma + m^*_{(\omega,\sigma)}) \neq 0\} \subseteq \{\sigma \in \mathcal{N} \,:\, \dim_{U_\sigma}(\omega) \neq 0\}.$$

Therefore

$$\begin{aligned}\#\{\sigma\in\mathcal{N}\,:\,\widehat{\varphi_i}(\omega+\sigma+m^*_{(\omega,\sigma)})\neq 0\} &\leq \#\{\sigma\in\mathcal{N}\,:\,\dim_{U_\sigma}(\omega)\neq 0\}\\ &\leq \sum_{\sigma\in\mathcal{N}}\dim_{U_\sigma}(\omega)\\ &= \dim_V(\omega).\end{aligned}$$

Consequently, by Fubini's Theorem,

$$\begin{aligned}m_\Gamma(\{\gamma\in\Omega'_{M^*}\,:\,\widehat{\varphi_i}(\gamma)\neq 0\}) &= \sum_{\sigma\in\mathcal{N}} m_\Gamma(\{\omega\in\Omega_\Delta\,:\,\widehat{\varphi_i}(\omega+\sigma+m^*_{(\omega,\sigma)})\neq 0\})\\ &= (m_\Gamma\times\#)(\{(\omega,\sigma)\in\Omega_\Delta\times\mathcal{N}\,:\,\widehat{\varphi_i}(\omega+\sigma+m^*_{(\omega,\sigma)})\neq 0\})\\ &= \int_{\Omega_\Delta}\#\{\sigma\in\mathcal{N}\,:\,\widehat{\varphi_i}(\omega+\sigma+m^*_{(\omega,\sigma)})\neq 0\}\,dm_\Gamma(\omega)\\ &\leq \int_{\Omega_\Delta}\dim_V(\omega)\,dm_\Gamma(\omega) = \sum_{j=0}^{\ell} j\,m_\Gamma(E_j)\leq \ell.\end{aligned}$$

□

Corollary 4.4.2. *Let $\varphi\in L^2(G)$ be given. If $S_H(\varphi)$ is M-invariant for some closed subgroup M of G such that $H\subsetneq M$, then $\widehat{\varphi}$ must vanish on a set of positive m_Γ-measure.*

Furthermore, if $m_\Gamma(\Gamma)=+\infty$, $\widehat{\varphi}$ must vanish on a set of infinite m_Γ-measure.

Proof. Let Ω_{M^*} be the section of the quotient Γ/M^* defined in (4.3). Then,

$$m_\Gamma(\{\gamma\in\Gamma\,:\,\widehat{\varphi}(\gamma)=0\}) = \sum_{m^*\in M^*} m_\Gamma(\{\gamma\in\Omega_{M^*}+m^*\,:\,\widehat{\varphi}(\gamma)=0\}). \tag{4.13}$$

By Theorem 4.4.1, we have that, for each $m^*\in M^*$,

$$m_\Gamma(\{\gamma\in\Omega_{M^*}+m^*\,:\,\widehat{\varphi}(y)\neq 0\})\leq 1,$$

which implies

$$m_\Gamma(\{\gamma\in\Omega_{M^*}+m^*\,:\,\widehat{\varphi}(y)=0\})\geq \#\mathcal{N}-1.$$

Combining this with equality (4.13), we obtain

$$m_\Gamma(\{\gamma\in\Gamma\,:\,\widehat{\varphi}(\gamma)\geq \#(M^*)(\#\mathcal{N}-1). \tag{4.14}$$

Since $H\subsetneq M$, it follows that $\#\mathcal{N}>1$, so $m_\Gamma(\{\gamma\in\Gamma\,:\,\widehat{\varphi}(\gamma)=0\})>0$.

If $m_\Gamma(\Gamma)=+\infty$, then either $m_\Gamma(\Omega_{M^*})=+\infty$ or $\#M^*=+\infty$. In case that $\#M^*=+\infty$, by (4.14), $\widehat{\varphi}$ must vanish on a set of infinite m_Γ-measure. If $m_\Gamma(\Omega_{M^*})=+\infty$, since $m_\Gamma(\Omega_\Delta)=1$, it follows that $\#\mathcal{N}=+\infty$. Then, using again (4.14), we can conclude the same as before.

□

As a consequence of Theorem 4.4.1, in case that $M = G$, we obtain the following corollary.

Corollary 4.4.3. *If $\varphi \in L^2(G)$ and $S_H(\varphi)$ is G-invariant, then*

$$m_\Gamma(\mathrm{supp}(\widehat{\varphi})) \leq 1.$$

The next theorem states that there exists an M-invariant space V that is *not* invariant under any vector outside M. We will say in this case that V is *exactly* M-invariant.

Note that because of Proposition 4.1.1, an M-invariant space is exactly M-invariant if and only if it is not invariant under any closed subgroup M' containing M.

Theorem 4.4.4. *For each closed subgroup M of G containing a countable uniform lattice H, there exists an H-invariant space of $L^2(G)$ which is exactly M-invariant.*

Proof. Suppose that $0 \in \mathcal{N}$ and take $\varphi \in L^2(G)$ satisfying $\mathrm{esssup}(\widehat{\varphi}) = B_0$, where B_0 is defined as in (4.24). Let $V = S_H(\varphi)$.

Then, $U_0 = V$ and $U_\sigma = \{0\}$ for $\sigma \in \mathcal{N}$, $\sigma \neq 0$. So, as a consequence of Theorem 4.3.2, it follows that V is M-invariant.

Now, if M' is a closed subgroup such that $M \subsetneq M'$, we will show that V can not be M'-invariant.

Since $M \subseteq M'$, $(M')^* \subseteq M^*$. Consider a section C of the quotient $M^*/(M')^*$ containing the neutral element of Γ. Then, the set given by

$$\mathcal{N}' := \{\sigma + c \ : \ \sigma \in \mathcal{N},\ c \in C\},$$

is a section of $H^*/(M')^*$ and $0 \in \mathcal{N}'$.

If $\{B'_{\sigma'}\}_{\sigma' \in \mathcal{N}'}$ is the partition defined in (4.24) associated to M', for each $\sigma \in \mathcal{N}$ it holds that $\{B'_{\sigma+c}\}_{c \in C}$ is a partition of B_σ, since

$$B_\sigma = \Omega_\Delta + \sigma + M^* = \bigcup_{c \in C} \Omega_\Delta + \sigma + c + (M')^* = \bigcup_{c \in C} B'_{\sigma+c}. \tag{4.15}$$

In particular, $B'_0 \subsetneq B_0$. Moreover, the set $B_0 \setminus B'_0$ contains a measurable section of Γ/Δ which is a translation of Ω_Δ.

We will show now that $U'_0 \not\subseteq V$, where U'_0 is the subspace defined in (4.5.15) for M'. Let $g \in L^2(G)$ such that $\widehat{g} = \widehat{\varphi}\chi_{B'_0}$. Then $g \in U'_0$. Moreover, since $\mathrm{esssup}(\widehat{\varphi}) = B_0$, by (4.15), $\widehat{g} \neq 0$.

Suppose that $g \in V$, then, by Theorem 2.1.1, $\widehat{g} = \eta\widehat{\varphi}$ where η is an Δ-periodic function. Thus, η must vanish in $B_0 \setminus B'_0$. Therefore, the Δ-periodicity of η implies that $\eta(\gamma) = 0$ a.e. $\gamma \in \Gamma$. So $\widehat{g} = 0$, which is a contradiction.

This shows that $U'_0 \not\subseteq V$. Hence, V is not M'-invariant. □

4.5 Extra invariance: a particular case

We will devote this section to study the extra invariances of a shift invariant space (SIS) in $L^2(\mathbb{R}^d)$ with $d > 1$. Obviously, this case is included in the more general context of LCA groups developed in the above sections. However, we are interested in showing how the results about extra invariance of a SIS in $L^2(\mathbb{R}^d)$ can be stated using the structure of closed subgroups of $\mathbb{R}^d$.

For this, we begin by given some characterizations concerning closed subgroups of $\mathbb{R}^d$.

4.5.1 Closed subgroups of $\mathbb{R}^d$

Throughout this section we describe the additive closed subgroups of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. We first study closed subgroups of $\mathbb{R}^d$ in general.

General case

Here, we will state some basic definitions and properties of closed subgroups of $\mathbb{R}^d$, for a detailed treatment and proofs we refer the reader to [Bou74].

Definition 4.5.1. Given M a subgroup of $\mathbb{R}^d$, the *range* of M, denoted by $\mathbf{r}(M)$, is the dimension of the subspace generated by M as a real vector space.

It is known that every closed subgroup of $\mathbb{R}^d$ is either discrete or contains a subspace of at least dimension one (see [Bou74, Proposition 3]).

Definition 4.5.2. Given M a closed subgroup of $\mathbb{R}^d$, there exists a subspace W whose dimension is the largest of the dimensions of all the subspaces contained in M. We will denote by $\mathbf{d}(M)$ the dimension of W. Note that $\mathbf{d}(M)$ can be zero.

Observe that $0 \leq \mathbf{d}(M) \leq \mathbf{r}(M) \leq d$.

The next theorem establishes that every closed subgroup of $\mathbb{R}^d$ is the direct sum of a subspace and a discrete group.

Theorem 4.5.3. *Let M be a closed subgroup of $\mathbb{R}^d$ such that $\mathbf{r}(M) = r$ and $\mathbf{d}(M) = p$. Let W be the subspace contained in M as in Definition 4.5.2. Then, there exists a basis $\{u_1, \dots, u_d\}$ for $\mathbb{R}^d$ such that $\{u_1, \dots, u_r\} \subseteq M$ and $\{u_1, \dots, u_p\}$ is a basis for W. Furthermore,*

$$M = \Big\{ \sum_{i=1}^{p} t_i u_i + \sum_{j=p+1}^{r} n_j u_j \,:\, t_i \in \mathbb{R},\, n_j \in \mathbb{Z} \Big\}.$$

Corollary 4.5.4. *If M is a closed subgroup of $\mathbb{R}^d$ such that $\mathbf{r}(M) = r$ and $\mathbf{d}(M) = p$, then*

$$M \approx \mathbb{R}^p \times \mathbb{Z}^{r-p}.$$

Closed subgroups of $\mathbb{R}^d$ containing $\mathbb{Z}^d$

We are interested in closed subgroups of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. For their understanding, we identify the dual group of M, M^*, with the subgroup of $\mathbb{R}^d$ $\{x \in \mathbb{R}^d \ : \ \langle x, m\rangle \in \mathbb{Z} \quad \forall m \in M\}$. Then, in particular, $(\mathbb{Z}^d)^* = \mathbb{Z}^d$.

Now we will list some properties of the dual group.

Proposition 4.5.5. *Let M, N be subgroups of $\mathbb{R}^d$.*

i) If $N \subseteq M$, then $M^* \subseteq N^*$.

ii) If M is closed, then $\mathbf{r}(M^*) = d - \mathbf{d}(M)$ and $\mathbf{d}(M^*) = d - \mathbf{r}(M)$.

iii) $(M^*)^* = \overline{M}$.

Let K be a subgroup of $\mathbb{Z}^d$ with $\mathbf{r}(K) = q$, we will say that a set $\{v_1, \ldots, v_q\} \subseteq K$ is a *basis* for K if for every $x \in K$ there exist unique $k_1, \ldots, k_q \in \mathbb{Z}$ such that

$$x = \sum_{i=1}^{q} k_i v_i.$$

Note that $\{v_1, \ldots, v_d\} \subseteq \mathbb{Z}^d$ is a basis for $\mathbb{Z}^d$ if and only if the determinant of the matrix A which has $\{v_1, \ldots, v_d\}$ as columns is 1 or -1.

Given $B = \{v_1, \ldots, v_d\}$ a basis for $\mathbb{Z}^d$, we will call $\widetilde{B} = \{w_1, \ldots, w_d\}$ a *dual basis* for B if $\langle v_i, w_j\rangle = \delta_{i,j}$ for all $1 \leq i, j \leq d$.

If we denote by $\widetilde{A}$ the matrix with columns $\{w_1, \ldots, w_d\}$, the relation between B and $\widetilde{B}$ can be expressed in terms of matrices as $\widetilde{A} = (A^*)^{-1}$.

The closed subgroups M of $\mathbb{R}^d$ containing $\mathbb{Z}^d$, can be described with the help of the dual relations. Since $\mathbb{Z}^d \subseteq M$, we have that $M^* \subseteq \mathbb{Z}^d$. So, we need first the characterization of the subgroups of $\mathbb{Z}^d$. This is stated in the following theorem.

Theorem 4.5.6. *Let K be a subgroup of $\mathbb{Z}^d$ with $\mathbf{r}(K) = q$, then there exist a basis $\{w_1, \ldots, w_d\}$ for $\mathbb{Z}^d$ and unique integers $a_1, \ldots, a_q$ satisfying $a_{i+1} \equiv 0\,(\text{mod. } a_i)$ for all $1 \leq i \leq q-1$, such that $\{a_1 w_1, \ldots, a_q w_q\}$ is a basis for K. The integers $a_1, \ldots, a_q$ are called* invariant factors.

The proof of the previous result can be found in [Bou81].

Remark 4.5.7. Under the assumptions of the above theorem we obtain

$$\mathbb{Z}^d / K \approx \mathbb{Z}_{a_1} \times \ldots \times \mathbb{Z}_{a_q} \times \mathbb{Z}^{d-q}.$$

We are now able to characterize the closed subgroups of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. The proof of the following theorem can be found in [Bou74], but we include it here for the sake of completeness.

Theorem 4.5.8. *Let $M \subseteq \mathbb{R}^d$. The following conditions are equivalent:*

i) M is a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ and $\mathbf{d}(M) = d - q$.

ii) There exist a basis $\{v_1, \dots, v_d\}$ for $\mathbb{Z}^d$ and integers $a_1, \dots, a_q$ satisfying $a_{i+1} \equiv 0\,(mod.\ a_i)$ for all $1 \leq i \leq q-1$, such that

$$M = \Big\{ \sum_{i=1}^{q} k_i \frac{1}{a_i} v_i + \sum_{j=q+1}^{d} t_j v_j \;:\; k_i \in \mathbb{Z}, t_j \in \mathbb{R} \Big\}.$$

Furthermore, the integers q and $a_1, \dots, a_q$ are uniquely determined by M.

Proof. Suppose i) is true. Since $\mathbb{Z}^d \subseteq M$ and $\mathbf{d}(M) = d - q$, we have that $M^* \subseteq \mathbb{Z}^d$ and $\mathbf{r}(M^*) = q$. By Theorem 4.5.6, there exist invariant factors $a_1, \dots, a_q$ and $\{w_1, \dots, w_d\}$ a basis for $\mathbb{Z}^d$ such that $\{a_1 w_1, \dots, a_q w_q\}$ is a basis for M^*.

Let $\{v_1, \dots, v_d\}$ be the dual basis for $\{w_1, \dots, w_d\}$.

Since M is closed, it follows from item iii) of Proposition 4.5.5 that $M = (M^*)^*$. So, $m \in M$ if and only if

$$\langle m, a_j w_j \rangle \in \mathbb{Z} \quad \forall\ 1 \leq j \leq q. \tag{4.16}$$

As $\{v_1, \dots, v_d\}$ is a basis, given $u \in \mathbb{R}^d$, there exist $u_i \in \mathbb{R}$ such that $u = \sum_{i=1}^{d} u_i v_i$. Thus, by (4.16), $u \in M$ if and only if $u_i a_i \in \mathbb{Z}$ for all $1 \leq i \leq q$.

We finally obtain that $u \in M$ if and only if there exist $k_i \in \mathbb{Z}$ and $u_j \in \mathbb{R}$ such that

$$u = \sum_{i=1}^{q} k_i \frac{1}{a_i} v_i + \sum_{j=q+1}^{d} u_j v_j.$$

The proof of the other implication is straightforward.

The integers q and $a_1, \dots, a_q$ are uniquely determined by M since $q = d - \mathbf{d}(M)$ and $a_1, \dots, a_q$ are the invariant factors of M^*.

□

As a consequence of the proof given above we obtain the following corollary.

Corollary 4.5.9. *Let $\mathbb{Z}^d \subseteq M \subseteq \mathbb{R}^d$ be a closed subgroup with $\mathbf{d}(M) = d - q$. If $\{v_1, \dots, v_d\}$ and $a_1, \dots, a_q$ are as in Theorem 4.5.8, then*

$$M^* = \Big\{ \sum_{i=1}^{q} n_i a_i w_i \;:\; n_i \in \mathbb{Z} \Big\},$$

where $\{w_1, \dots, w_d\}$ is the dual basis of $\{v_1, \dots, v_d\}$.

Example 4.5.10. Assume that $d = 3$. If $M = \frac{1}{2}\mathbb{Z} \times \frac{1}{3}\mathbb{Z} \times \mathbb{R}$, then $v_1 = (1,1,0)$, $v_2 = (3,2,0)$ and $v_3 = (0,0,1)$ verify the conditions of Theorem 4.5.8 with the invariant factors $a_1 = 1$ and $a_2 = 6$. On the other hand $v_1' = (1,1,0)$, $v_2' = (3,2,1)$ and $v_3' = (0,0,1)$ verify the same conditions. This shows that the basis in Theorem 4.5.8 is not unique.

Remark 4.5.11. If $\{v_1, \dots, v_d\}$ and $a_1, \dots, a_q$ are as in Theorem 4.5.8, let us define the linear transformation T as

$$T : \mathbb{R}^d \to \mathbb{R}^d, \quad T(e_i) = v_i \quad \forall\, 1 \le i \le d,$$

where $\{e_1, \dots, e_d\}$ denotes the canonical basis for $\mathbb{R}^d$.

Then T is an invertible transformation that satisfies

$$M = T(\frac{1}{a_1}\mathbb{Z} \times \cdots \times \frac{1}{a_q}\mathbb{Z} \times \mathbb{R}^{d-q}).$$

If $\{w_1, \dots, w_d\}$ is the dual basis for $\{v_1, \dots, v_d\}$, the inverse of the adjoint of T is defined by

$$(T^*)^{-1} : \mathbb{R}^d \to \mathbb{R}^d, \quad (T^*)^{-1}(e_i) = w_i \quad \forall\, 1 \le i \le d.$$

By Corollary 4.5.9, it is true that

$$M^* = (T^*)^{-1}(a_1\mathbb{Z} \times \cdots \times a_q\mathbb{Z} \times \{0\}^{d-q}).$$

4.5.2 M-invariance of a SIS in $L^2(\mathbb{R}^d)$.

Let M be a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. In order to characterize the M-invariance of a SIS, we first want to give a slight idea about how Theorem 4.2.4 for $G = \mathbb{R}^d$ can be proved using the structure of M stated in the above section. For this we will follow the arguments used to prove Theorem 2.1.1. Most of the reasonings can be readily obtained from Section 2.1. Hence, we do not include their proofs. The point that deserves to be carefully explored is the extension of Proposition 2.1.2, (see Lemma 4.5.12).

We will first need some definitions and properties.

By Remark 4.5.11, there exists a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $M = T(\frac{1}{a_1}\mathbb{Z} \times \cdots \times \frac{1}{a_q}\mathbb{Z} \times \mathbb{R}^{d-q})$ and $M^* = (T^*)^{-1}(a_1\mathbb{Z} \times \cdots \times a_q\mathbb{Z} \times \{0\}^{d-q})$, where $q = d - \mathbf{d}(M)$.

We will denote by $\mathcal{D}$ the section of the quotient $\mathbb{R}^d/M^*$ defined as

$$\mathcal{D} = (T^*)^{-1}([0, a_1) \times \cdots \times [0, a_q) \times \mathbb{R}^{d-q}). \tag{4.17}$$

Therefore, $\{\mathcal{D} + m^*\}_{m^* \in M^*}$ forms a partition of $\mathbb{R}^d$.

Lemma 4.5.12. *Let $f \in L^2(\mathbb{R}^d)$, M a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ and $\mathcal{D}$ defined as in (4.17). Then,*

$$S_M(f)^\perp = \{g \in L^2(\mathbb{R}^d) : \sum_{m^* \in M^*} \widehat{f}(\omega + m^*)\overline{\widehat{g}(\omega + m^*)} = 0 \text{ a.e. } \omega \in \mathcal{D}\}.$$

Proof. Since the span of the set $\{T_m f : m \in M\}$ is dense in $S_M(f)$, we have that $g \in S_M(f)^{\perp}$ if and only if $\langle \widehat{g}, e_m \widehat{f} \rangle = 0$ for all $m \in M$, where $e_m(\omega) := e^{-2\pi i \langle \omega, m \rangle}$. As e_m is an M^*-periodic function and $\{\mathcal{D} + m^*\}_{m^* \in M^*}$ forms a partition of $\mathbb{R}^d$, using a periodization argument, we obtain that $g \in S_M(f)^{\perp}$ if and only if

$$\int_{\mathcal{D}} e_m(\omega)\Big(\sum_{m^* \in M^*} \widehat{f}(\omega + m^*)\overline{\widehat{g}(\omega + m^*)}\Big)\, d\omega = 0, \tag{4.18}$$

for all $m \in M$.

At this point, what is left to show is that if (4.18) holds then $\sum_{m^* \in M^*} \widehat{f}(\omega + m^*)\overline{\widehat{g}(\omega + m^*)} = 0$ a.e. $\omega \in \mathcal{D}$. For this, taking into account that $\sum_{m^* \in M^*} \widehat{f}(\cdot + m^*)\overline{\widehat{g}(\cdot + m^*)} \in L^1(\mathcal{D})$, it is enough to prove that if $h \in L^1(\mathcal{D})$ and $\int_{\mathcal{D}} h e_m = 0$ for all $m \in M$ then $h = 0$ a.e $\omega \in \mathcal{D}$.

We will prove the preceding property for the case $M = \mathbb{Z}^q \times \mathbb{R}^{d-q}$. The general case will follow from a change of variables using the description of M and $\mathcal{D}$ given in Remark 4.5.11 and (4.17).

Suppose now $M = \mathbb{Z}^q \times \mathbb{R}^{d-q}$, then $\mathcal{D} = [0,1)^q \times \mathbb{R}^{n-q}$. Take $h \in L^1(\mathcal{D})$, such that

$$\iint_{[0,1)^q \times \mathbb{R}^{n-q}} h(x,y) e^{-2\pi i (kx+ty)}\, dxdy = 0 \quad \forall\, k \in \mathbb{Z}^q,\ t \in \mathbb{R}^{d-q}. \tag{4.19}$$

Given $k \in \mathbb{Z}^q$, define $\alpha_k(y) := \int_{[0,1)^q} h(x,y) e^{-2\pi i k x}\, dx$ for a.e. $y \in \mathbb{R}^{d-q}$. It follows from (4.19) that

$$\int_{\mathbb{R}^{d-q}} \alpha_k(y) e^{-2\pi i t y}\, dy = 0 \quad \forall\, t \in \mathbb{R}^{d-q}. \tag{4.20}$$

Since $h \in L^1(\mathcal{D})$, by Fubini's Theorem, $\alpha_k \in L^1([0,1)^q)$. Thus, using (4.20), $\alpha_k(y) = 0$ a.e. $y \in \mathbb{R}^{d-q}$. That is

$$\int_{[0,1)^q} h(x,y) e^{-2\pi i k x}\, dx = 0 \tag{4.21}$$

for a.e. $y \in \mathbb{R}^{d-q}$. Define now $\beta_y(x) := h(x,y)$. By (4.21), for a.e. $y \in \mathbb{R}^{d-q}$ we have that $\beta_y(x) = 0$ for a.e. $x \in [0,1)^q$. Therefore, $h(x,y) = 0$ a.e. $(x,y) \in [0,1)^q \times \mathbb{R}^{d-q}$ and this completes the proof. □

Now we will give a formula for the orthogonal projection onto $S_M(f)$.

Lemma 4.5.13. *Let $\mathcal{P}$ be the orthogonal projection onto $S_M(f)$. Then, for each $g \in L^2(\mathbb{R}^d)$, we have $\widehat{\mathcal{P}g} = \eta_g \widehat{f}$, where η_g is the M^*-periodic function defined by*

$$\eta_g := \begin{cases} \dfrac{\sum_{m^* \in M^*} \widehat{f}(\omega+m^*)\overline{\widehat{g}(\omega+m^*)}}{\sum_{m^* \in M^*} \widehat{f}(\omega+m^*)\overline{\widehat{f}(\omega+m^*)}} & \text{on } \mathcal{E}_f + M^* \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{E}_f$ is the set $\{\omega \in \mathcal{D} : \sum_{m^* \in M^*} \widehat{f}(\omega + m^*)\overline{\widehat{f}(\omega + m^*)} \neq 0\}$.

With these results Theorem 4.5.14 can be proved. This theorem provides a non-discrete version of a result of [dBDR94a] (see also [dBDR94b],[RS95]).

Theorem 4.5.14. *Let $f \in L^2(\mathbb{R}^d)$ and M a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. If $g \in S_M(f)$, then there exists an M^*-periodic function η such that $\widehat{g} = \eta \widehat{f}$.*

Conversely, if η is an M^*-periodic function such that $\eta\widehat{f} \in L^2(\mathbb{R}^d)$, then the function g defined by $\widehat{g} = \eta\widehat{f}$ belongs to $S_M(f)$.

We will focus now in characterizing the M-invariance of a general SIS in $L^2(\mathbb{R}^d)$. Then, fix M a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. Let T be the linear transformation stated in Remark 4.5.11. Using the mapping T we define Ω, a the section of the quotient $\mathbb{R}^d/\mathbb{Z}^d$, as

$$\Omega = (T^*)^{-1}([0,1)^d), \tag{4.22}$$

and $\mathcal{N}$, a section of the quotient $\mathbb{Z}^d/M^*$, as

$$\mathcal{N} = (T^*)^{-1}(\{0,\dots,a_1-1\} \times \dots \times \{0,\dots,a_q-1\} \times \mathbb{Z}^{d-q}), \tag{4.23}$$

where $a_1,\dots,a_q$ are the invariant factors of M.

Hence, given $\sigma \in \mathcal{N}$ we define

$$B_\sigma = \Omega + \sigma + M^* = \bigcup_{m^* \in M^*} (\Omega + \sigma) + m^*. \tag{4.24}$$

Therefore, $\{B_\sigma\}_{\sigma\in\mathcal{N}}$ forms a partition f $\mathbb{R}^d$ and each B_σ is an M^*-periodic set. An example of this construction is given below.

Example 4.5.15. Let $M = \{k\frac{1}{3}v_1 + tv_2 : k \in \mathbb{Z} \text{ and } t \in \mathbb{R}\}$, where $v_1 = (1,0)$ and $v_2 = (-1,1)$. Then, $\{v_1, v_2\}$ satisfy conditions in Theorem 4.5.8. By Corollary 4.5.9, $M^* = \{k3w_1 : k \in \mathbb{Z}\}$, where $w_1 = (1,1)$ and $w_2 = (0,1)$. Note that the sets Ω and $\mathcal{N}$ can be expressed in terms of w_1 and w_2 as

$$\Omega = \{tw_1 + sw_2 : t, s \in [0,1)\} \quad \text{and} \quad \mathcal{N} = \{aw_1 + kw_2 : a \in \{0,1,2\}, k \in \mathbb{Z}\}.$$

This is illustrated in Figure 4.3. In this case the sets $B_{(0,0)}$, $B_{(1,0)}$ and $B_{(1,2)}$ correspond to the light gray, gray and dark gray regions respectively. The black dots represent the set $\mathcal{N}$.

Let V be a SIS. Then, using the partition $\{B_\sigma\}_{\sigma\in\mathcal{N}}$ we define for each $\sigma \in \mathcal{N}$

$$U_\sigma = \{f \in L^2(\mathbb{R}^d) : \widehat{f} = \chi_{B_\sigma}\widehat{g}, \text{ with } g \in V\}.$$

The next theorem characterizes the M-invariance of V in terms of the subspaces U_σ. The proof of this result can be done in the same way as Theorem 4.3.2

Theorem 4.5.16. *If $V \subseteq L^2(\mathbb{R}^d)$ is a SIS and M is a closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$, then the following are equivalent.*

i) V is M-invariant.

ii) $U_\sigma \subseteq V$ for all $\sigma \in \mathcal{N}$.

Moreover, in case any of the above holds, we have that V is the orthogonal direct sum

$$V = \dot{\bigoplus_{\sigma \in \mathcal{N}}} U_\sigma .$$

It is known that on the real line, the SIS generated by a function φ with compact support can only be invariant under integer translations. That is, $T_x\varphi \notin S_{\mathbb{Z}}(\varphi)$ for all $x \in \mathbb{R}\setminus\mathbb{Z}$. The following proposition extends this result to $\mathbb{R}^d$. Thus, a shift invariant spaces in $L^2(\mathbb{R}^d)$ generated by a compactly supported function is exactly $\mathbb{Z}^d$-invariant.

Proposition 4.5.17. *If a nonzero function $\varphi \in L^2(\mathbb{R}^d)$ has compact support, then $S_{\mathbb{Z}^d}(\varphi)$ is not M-invariant for any M closed subgroup of $\mathbb{R}^d$ such that $\mathbb{Z}^d \subsetneq M$. In particular,*

$$T_x\varphi \notin S_{\mathbb{Z}^d}(\varphi) \quad \forall\, x \in \mathbb{R}^d \setminus \mathbb{Z}^d. \tag{4.25}$$

Proof. The first part of the proposition is a straightforward consequence of Corollary 4.4.2 with $G = \mathbb{R}^d$ and $H = \mathbb{Z}^d$. To show (4.25), take $x \in \mathbb{R}^d \setminus \mathbb{Z}^d$ and suppose that $T_x\varphi \in S(\varphi)$. If M is the closed subgroup generated by x and $\mathbb{Z}^d$, then $S_{\mathbb{Z}^d}(\varphi)$ must be M-invariant, which is a contradiction. □

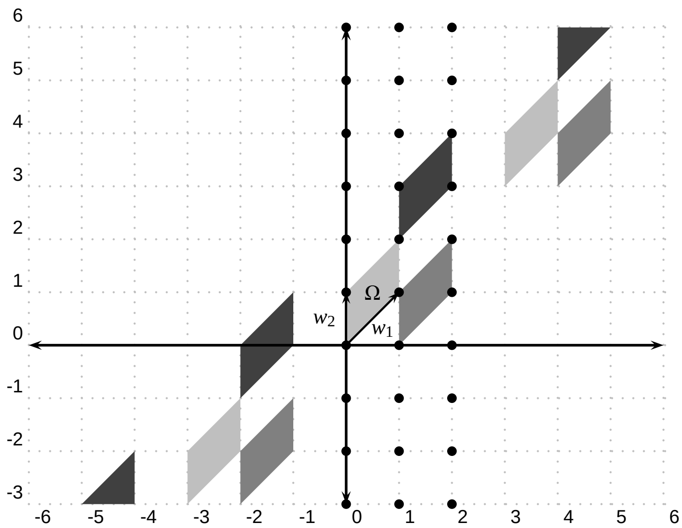

Figure 4.3: Partition for $M = \{k\frac{1}{3}(1,0) + t(-1,1) \, : \, k \in \mathbb{Z} \text{ and } t \in \mathbb{R}\}$.

5

Shift-Modulation Invariant Spaces

In this chapter, we will deal with subspaces of $L^2(G)$, with G being an LCA group, that are invariant under translations and also under modulations. For this we will first introduce the concept of shift-modulation invariant spaces under the pair (K, Λ) in $L^2(G)$, K is a closed subgroup of G and Λ is closed subgroup the dual group of G. These spaces are the extension to the LCA setting of the well-known shift-modulation invariant spaces in $L^2(\mathbb{R}^d)$, (SMI spaces). SMI spaces, also called Gabor or Weyl-Heinsenberg spaces, have been studied in [Bow07], [CC01b], [CC01a], [Chr03], [Dau92], [GD04], [GD01], [Gro01], [Fei02] among others, and they become fundamental in time-frequency analysis.

Here, we restrict our attention to shift-modulation invariant spaces under the pair (K, Λ) in $L^2(G)$ with K and Λ being uniform lattices. The aim will be get a description of these spaces in terms of the fiber of its elements. In order to obtain such characterization, we develop fiberization techniques and suitable range functions adapted to this more complicated structure which involves translations and modulations. As in the $L^2(\mathbb{R}^d)$ case, the Zak transform and its properties will be essential in our analysis.

The result that we obtain, generalizes a result concerned shift-modulation invariant spaces in $L^2(\mathbb{R}^d)$ due to Bownik [Bow07].

We organize the chapter as follows. First, we fix our work setting in Section 5.1. In Section 5.2, we first develop fiberization techniques (Section 5.2.1) and then suitable range functions well adapted to this context (Section 5.2.2). Finally, in Section 5.3, the main result of this chapter is stated and proved.

5.1 Shift-Modulation Setting

In this section, we will introduce the notion of shift-modulation invariant spaces on LCA groups and we will fix our work setting.

Definition 5.1.1. Let G be an LCA group and Γ its dual group. If $K \subseteq G$ and $\Lambda \subseteq \Gamma$ are subgroups, we will say that a closed subspace $V \subseteq L^2(G)$ is (K, Λ)-*invariant* or *shift-*

modulation invariant under (K, Λ) if

$$f \in V \Rightarrow M_\lambda T_k f \in V \quad \forall\, k \in K \text{ and } \lambda \in \Lambda,$$

where $M_\lambda f(x) = (x, \lambda) f(x)$ and $T_k f(x) = f(x - k)$.

For a subset $\mathcal{A} \subseteq L^2(G)$, define

$$E_{(K,\Lambda)}(\mathcal{A}) = \{M_\lambda T_k \varphi : \varphi \in \mathcal{A}, k \in K, \lambda \in \Lambda\}$$

and

$$S_{(K,\Lambda)}(\mathcal{A}) = \overline{\operatorname{span}}\, E_{(K,\Lambda)}(\mathcal{A}).$$

A straightforward computation shows that the space $S_{(K,\Lambda)}(\mathcal{A})$ is shift-modulation invariant under the pair (K, Λ). Then, we call $S_{(K,\Lambda)}(\mathcal{A})$ the (K, Λ)-invariant space generated by $\mathcal{A}$. Note that, when G is second countable, for every shift-modulation invariant space V, there exists a countable set of generators $\mathcal{A} \subseteq L^2(G)$ such that $V = S_{(K,\Lambda)}(\mathcal{A})$.

Here we want to characterize (F, Δ)-invariant spaces for F and Δ being uniform lattices in G and Γ respectively such that $F \cap \Delta^*$ is an uniform lattice in G.

As we have done in the shift invariant case (see Chapter 2), the characterization of shift-modulation invariant spaces will be established in terms of appropriate range functions and fiberization techniques.

Remark 5.1.2. If $K_1 \subseteq K_2$ are lattices in G, then K_2/K_1 is finite. To prove this, observe that, since $K_2^* \subseteq K_1^*$, $\widehat{K_1^*/K_2^*} \approx K_2/K_1$ due to the duality relationships stated in Theorem 1.1.4. Therefore, K_2/K_1 is both compact and discrete. Hence K_2/K_1 must be finite. This fact will be important in what follows.

We now fix our setting which will be in effect throughout the next sections.

- G is a second countable LCA group and Γ its dual group.
- F is a countable uniform lattice on G.
- Δ is a countable uniform lattice on Γ.
- $E := F \cap \Delta^*$ is a (countable) uniform lattice on G.

As a consequence of the results stated in Theorem 1.1.9 and Remark 5.1.2 we obtain that:

(a) E^* is an uniform lattice in Γ and $\Delta \subseteq E^*$.

(b) $H := \Delta^*$ is an uniform lattice in G.

(c) The quotient E^*/Δ is finite.

(d) The quotient F/E is finite.

According to these facts we can fix $\Omega_{E^*} \subseteq \Gamma$ a measurable section for the quotient Γ/E^* and $D_{E^*} \subseteq E^*$ a finite section for E^*/Δ. Then, we can construct the measurable section Ω_Δ for the quotient Γ/Δ as

$$\Omega_\Delta = \bigcup_{e \in D_{E^*}} \Omega_{E^*} + e. \tag{5.1}$$

In the same way, considering $D_F \subseteq F$ a finite section for F/E and I_{F+H} a measurable section for $G/(F+H)$, we have that

$$I_H = \bigcup_{d \in D_F} I_{F+H} - d \tag{5.2}$$

is a section for the quotient G/H.

We are able for fixing the normalization of the Haar measures of the groups considered in this chapter. As usual, this particular choice of the Haar measures does not affect the validity of the results.

First, we choose m_H such that $m_H(\{0\}) = 1$. Then we fix m_G and $m_{G/H}$ such that the Wiel's formula holds among m_H, m_G and $m_{G/H}$. Furthermore, we choose m_{Γ/E^*}, m_{E^*} in order to get $m_{E^*}(\{0\})m_{\Gamma/E^*}(\Gamma/E^*) = \frac{1}{\sharp D_{E^*}}$ where by $\sharp D_{E^*}$ we denote the cardinal of D_{E^*}. Then, we set m_Γ such that Wiel's formula of Theorem 1.1.10 holds among m_{Γ/E^*}, m_{E^*} and m_Γ.

If Ω_Δ is given by (5.1), this normalization implies that $m_\Gamma(\Omega_\Delta) = 1$. This is due to formula $m_\Gamma(\Omega_{E^*}) = m_{E^*}(\{0\})m_{\Gamma/E^*}(\Gamma/E^*)$ proved in Lemma 1.1.13.

5.2 The Fiberization Isometry and Range Functions

The goal of this section is to develop the fiberization isometry and a suitable range function required to achieve the characterization of (F, Δ)-invariant spaces that we want to prove.

The well-known property about the Fourier transform $\widehat{M_\lambda f} = T_\lambda \widehat{f}$, guarantees that a space which is invariant under modulations can be seen, via the Fourier transform, as a shift invariant space. Therefore, we can treat the shift-modulation invariant spaces as shift-invariant spaces on both sides, on time and on frequency. Then, our analysis will be strongly based on the shift invariant case. The fiberization isometry for shift invariant spaces will help us to construct the fiberization isometry for the shift-modulation setting.

5.2.1 The Isometry

Fix now $F \subseteq G$ and $\Delta \subseteq \Gamma$ countable uniform lattices verifying the conditions stated in the above section.

In order to construct the fiberization isometry, we must introduce the following isomorphisms.

First, let $\tilde{\mathcal{T}}_H : L^2(G) \longrightarrow L^2(I_H, \ell^2(H))$ be the mapping defined as

$$\tilde{\mathcal{T}}_H f(x) = \{f(x+h)\}_{h\in H}. \tag{5.3}$$

The proof that $\tilde{\mathcal{T}}_H$ is an isometric isomorphism is an straightforward adaptation of Proposition 1.4.2.

On the other hand, consider $\mathcal{T}_E : \ell^2(H) \longrightarrow L^2(\Omega_{E^*}, \ell^2(D_{E^*}))$ defined by

$$\mathcal{T}_E a(\xi) = \{\sum_{h\in H} a_h \eta_h(\xi + e)\}_{e\in D_{E^*}}, \tag{5.4}$$

where the functions η_h are as in Proposition 1.1.19 and $a = \{a_h\}_{h\in H}$.

Lemma 5.2.1. *The map $\mathcal{T}_E$ defined in (5.4) is and isometric isomorphism between $\ell^2(H)$ and $L^2(\Omega_{E^*}, \ell^2(D_{E^*}))$.*

Proof. Since D_{E^*} is an index set, we have that

$$\begin{aligned} \|\mathcal{T}_E a\|_2^2 &= \int_{\Omega_{E^*}} \sum_{e\in D_{E^*}} |\sum_{h\in H} a_h \eta_h(\xi + e)|^2 \, dm_\Gamma(\xi) \\ &= \int_{\Omega_\Delta} |\sum_{h\in H} a_h \eta_h(\omega)|^2 \, dm_\Gamma(\omega) \\ &= \|\sum_{h\in H} a_h \eta_h\|^2_{L^2(\Omega_\Delta)}. \end{aligned}$$

Now, applying Proposition 1.1.21 we obtain

$$\|\sum_{h\in H} a_h \eta_h\|^2_{L^2(\Omega_\Delta)} = \frac{m_\Gamma(\Omega_\Delta)}{m_H(\{0\})} \|a\|^2_{\ell^2(H)}.$$

Hence, by our normalization of the Haar measures, $\frac{m_\Gamma(\Omega_\Delta)}{m_H(\{0\})} = 1$ and then $\|\mathcal{T}_E a\|_2^2 = \|a\|^2_{\ell^2(H)}$.

Let $\Phi \in L^2(\Omega_{E^*}, \ell^2(D_{E^*}))$. Then Φ induces the function $\widetilde{\Phi} \in L^2(\Omega_\Delta)$ given by

$$\widetilde{\Phi}(\omega) = (\Phi(\xi))_e,$$

where $\omega = \xi + e \in \Omega_\Delta$, with $\xi \in \Omega_{E^*}$ and $e \in D_{E^*}$. Here $(\Phi(\xi))_e$ denotes the value of the sequence $\Phi(\xi)$ at e. It is easy to check that $\|\Phi\|_2 = \|\widetilde{\Phi}\|_{L^2(\Omega_\Delta)}$.

According to Proposition 1.1.19, $\{\eta_h\}_{h\in H}$ is an orthonormal basis for $L^2(\Omega_\Delta)$. Thus, $\widetilde{\Phi} = \sum_{h\in H} a_h \eta_h$ for some $a = \{a_h\}_{h\in H} \in \ell^2(H)$. From here, it follows that $\mathcal{T}_E a = \Phi$. Therefore, $\mathcal{T}_E$ is an isomorphism. □

Remark 5.2.2. Observe that E^{*_H}, the annihilator of E as a subgroup of H, is topologically isomorphic to E^*/Δ. Then, using the dual relationship stated in Theorem 1.1.4, it follows that $\widehat{H}/E^{*_H} \approx \Gamma/E^*$. This allows us to see $\mathcal{T}_E$ as a particular case of the map of Proposition 1.4.2.

The isometric isomorphism $\mathcal{T}_E$ induces another isometric isomorphism

$$\Psi_1 : L^2(I_H, \ell^2(H)) \longrightarrow L^2(I_H, L^2(\Omega_{E^*}, \ell^2(D_{E^*})))$$

defined as

$$\Psi_1(\phi)(x) = \mathcal{T}_E(\phi(x)).$$

In addition, we can identify the Hilbert space $L^2(I_H, L^2(\Omega_{E^*}, \ell^2(D_{E^*})))$ with $L^2(I_H \times \Omega_{E^*}, \ell^2(D_{E^*})))$ using the isometric isomorphism

$$\Psi_2 : L^2(I_H, L^2(\Omega_{E^*}, \ell^2(D_{E^*}))) \longrightarrow L^2(I_H \times \Omega_{E^*}, \ell^2(D_{E^*}))$$

given by

$$\Psi_2(\phi)(x, \xi) = \phi(x)(\xi).$$

Definition 5.2.3. We define $\mathcal{T} : L^2(G) \longrightarrow L^2(I_H \times \Omega_{E^*}, \ell^2(D_{E^*}))$ as

$$\mathcal{T} = \Psi_2 \circ \Psi_1 \circ \tilde{\mathcal{T}}_H.$$

This mapping $\mathcal{T}$, which is actually an isometric isomorphism and that we call *the fiberization isometry*, can be explicitly defined as

$$\mathcal{T} f(x, \xi) = \mathcal{T}_E(\widetilde{\mathcal{T}}_H f(x))(\xi) = \{\sum_{h \in H} f(x - h)(h, \xi + e)\}_{e \in D_{E^*}}. \tag{5.5}$$

For a simply way to describe $\mathcal{T}$, we recall $Z : L^2(G) \to \mathcal{Z}$ the usual Zak transform given by

$$Zf(x, \xi) = \sum_{h \in H} f(x - h)(h, \xi),$$

where $\mathcal{Z}$ is the set of all measurable functions $F : G \times \Gamma \to \mathbb{C}$ satisfying

(*a*) $F(x + h, \xi) = (h, \xi)F(x, \xi)\ \forall\, h \in H$,

(*b*) $F(x, \xi + \delta) = F(x, \xi)\ \forall\, \delta \in \Delta$ and

(*c*) $\|F\|^2 = \int_{\Omega_\Delta} \int_{I_H} |F(x, \xi)|^2\, dm_G(x)\, dm_\Gamma(\xi) < \infty$.

For further information about Zak transform we refer to [Gro01], [Wei64], [Zak67], [Jan82], [Jan88].

Then, it is clear that

$$\mathcal{T} f(x, \xi) = \{Zf(x, \xi + e)\}_{e \in D_{E^*}}.$$

The next lemma states an important property about $\mathcal{T}$ which will be useful in what follows. Its proof is a straightforwards consequence of properties (*a*), (*b*) and (*c*) formulated above.

Lemma 5.2.4. *For each $f \in L^2(G)$ the map $\mathcal{T}$ of Definition 5.2.3 satisfies*

$$\mathcal{T}(M_\delta T_y f)(x, \xi) = (x, \delta)(-z, \xi)\mathcal{T}(T_d f)(x, \xi) \quad a.e.\ (x, \xi) \in I_H \times \Omega_{E^*},$$

where $\delta \in \Delta$, $y \in F$ and $y = z + d$ with $z \in E$ and $d \in D_F$.

5.2.2 Shift-modulation Range Functions

We are now able to define range function according to (F, Δ)-invariant spaces.

Definition 5.2.5. A *shift-modulation range function* with respect to the pair (F, Δ) is a mapping

$$J : I_H \times \Omega_{E^*} \longrightarrow \{\text{subspaces of } \ell^2(D_{E^*})\},$$

satisfying the following periodicity property:

$$J(x, \xi) = J(x - d, \xi) \quad \forall\, d \in D_F \tag{5.6}$$

for a.e. $(x, \xi) \in I_{F+H} \times \Omega_{E^*}$.

For a shift-modulation range function J, we associated to each $(x, \xi) \in I_H \times \Omega_{E^*}$ the orthogonal projection onto $J(x, \xi)$, $P_{(x,\xi)} : \ell^2(D_{E^*}) \to J(x, \xi)$.

We say that a shift-modulation range function J is measurable if the function $(x, \xi) \mapsto P_{(x,\xi)}$ from $I_H \times \Omega_{E^*}$ to $\ell^2(D_{E^*})$ is measurable.

For a shift-modulation range function J (not necessarily measurable) we define the subset M_J as

$$M_J = \{\Psi \in L^2(I_H \times \Omega_{E^*}, \ell^2(D_{E^*})) : \Psi(x, \xi) \in J(x, \xi), \text{a.e.}(x, \xi) \in I_H \times \Omega_{E^*}\} \tag{5.7}$$

Remark 5.2.6. The subspace M_J defined above is a closed subspace in $L^2(I_H \times \Omega_{E^*}, \ell^2(D_{E^*}))$. For the proof of this fact see Lemma 2.2.3.

The shift-modulation invariant spaces associated to a range function

The following proposition states that if J is a given shift-modulation range function with respect to the pair (F, Δ), we can associate to J an (F, Δ)-invariant space.

Proposition 5.2.7. *Let J be a shift-modulation range function and define $V := \mathcal{T}^{-1} M_J$, where M_J is as in (5.7) and $\mathcal{T}$ is the fiberization isometry.*

Then, V is an (F, Δ)-invariant space in $L^2(G)$.

Proof. To begin with, observe that, by Remark 5.2.6 and since $\mathcal{T}$ is an isometry, $V \subseteq L^2(G)$ is a closed subspace.

Let $f \in V$, $\delta \in \Delta$ and $y \in F$. We need to show that $M_\delta T_y f \in V$.

According to Lemma 5.2.4, we have that

$$\mathcal{T}(M_\delta T_y f)(x, \xi) = (x, \delta)(-z, \xi)\mathcal{T}(T_d f)(x, \xi) \qquad \text{a.e. } (x, \xi) \in I_H \times \Omega_{E^*},$$

where $y = z + d$ with $z \in E$ and $d \in D_F$.

In particular, if $x \in I_{F+H}$ we can rewrite $\mathcal{T}(T_d f)(x, \xi)$ as $\mathcal{T} f(x - d, \xi)$. Then, since $\mathcal{T} f \in M_J$ and J satisfies (5.6), it holds that

$$\mathcal{T}(T_d f)(x, \xi) = \mathcal{T} f(x - d, \xi) \in J(x - d, \xi) = J(x, \xi),$$

for a.e. $(x,\xi) \in I_{F+H} \times \Omega_{E^*}$. Thus,

$$\mathcal{T}(M_\delta T_y f)(x,\xi) \in J(x,\xi) \quad \text{a.e. } (x,\xi) \in I_{F+H} \times \Omega_{E^*}, \tag{5.8}$$

and this is valid for all $y \in F$ and $\delta \in \Delta$.

We now want to show that (5.8) holds on $I_H \times \Omega_{E^*}$.

Let $(x,\xi) \in I_H \times \Omega_{E^*}$. By (5.2) we can set $x = x' - d$ with $x' \in I_{F+H}$ and $d \in D_F$.

If we fix $\delta \in \Delta$ and $y \in F$ then, $\mathcal{T}(M_\delta T_y f)(x,\xi) = \mathcal{T}(T_d M_\delta T_y f)(x',\xi)$.

Since $M_\lambda T_k g = (k,\lambda) T_k M_\lambda g$ for all $g \in L^2(G)$, $\lambda \in \Delta$ and $k \in F$, we have

$$\mathcal{T}(T_d M_\delta T_y f)(x',\xi) = (-d,\delta)\mathcal{T}(M_\delta T_{d+y} f)(x',\xi) \in J(x',\xi) = J(x,\xi).$$

Then, (5.8) holds on $I_H \times \Omega_{E^*}$. Therefore, $M_\delta T_y f \in V$ for all $\delta \in \Delta$ and $y \in F$. □

The range function associated to an (F,Δ)-invariant space

The characterization of shift invariant spaces under uniform lattices in G stated in Theorem 2.2.5, gives a specific way to describe the shift range function associated to each shift invariant space. Since a shift-modulation invariant spaces is a shift invariant space in time and frequency, we will use the results of Theorem 2.2.5 to construct a shift-modulation range function from a given (F,Δ)-invariant space.

Assume that $V \subseteq L^2(G)$ is an (F,Δ)-invariant space and that $V == S_{(F,\Delta)}(\mathcal{A})$ for some countable set $\mathcal{A} \subseteq L^2(G)$. We will show now, how to associated to V a shift-modulation range function.

First notice that $\widehat{V} = \{\widehat{f} : f \in V\} \subseteq L^2(\Gamma)$ is invariant under translations in Δ. Then, by Theorem 2.2.5, V can be describe as

$$V = \{f \in L^2(G) : \tilde{\mathcal{T}}_H f(x) \in J_H(x) \text{ a.e. } x \in I_H\}, \tag{5.9}$$

where $\tilde{\mathcal{T}}_H$ is the isometry defined in (5.2.1) and J_H is the shift range function associated to V given by

$$J_H : I_H \longrightarrow \{\text{closed subspaces of } \ell^2(H)\}$$

$$J_H(x) = \overline{\operatorname{span}}\{\tilde{\mathcal{T}}_H(T_y\varphi)(x) \;:\; y \in F,\ \varphi \in \mathcal{A}\}.$$

Note that this holds since V, as a space invariant under modulations in Δ, is generated by the set $\{T_y\varphi \;:\; y \in F,\ \varphi \in \mathcal{A}\}$.

Now, let us see that $J_H(x)$ is a shift invariant space under translations in E.

Since $D_F \subseteq F$ is a section for the quotient F/E, every $y \in F$ can be written in a unique way as $y = z + d$ with $z \in E$ and $d \in D_F$. Then, using that $\tilde{\mathcal{T}}_H T_z f = T_z \tilde{\mathcal{T}}_H f$ for all $z \in E$,we can rewrite $J_H(x)$

$$J_H(x) = \overline{\operatorname{span}}\{T_z \tilde{\mathcal{T}}_H(T_d\varphi)(x) \;:\; z \in E,\ d \in D_F,\ \varphi \in \mathcal{A}\}.$$

This description shows that $J_H(x)$ is a shift invariant space under translations in E generated by the set $\{\tilde{\mathcal{T}}_H(T_d\varphi)(x) \,:\, d \in D_F,\, \varphi \in \mathcal{A}\}$.

Using Theorem 2.2.5, we can characterize $J_H(x)$ for a.e. $x \in I_H$ as follows. For each $x \in I_H \setminus Z$, where Z is the exceptional zero m_G-measure set, there exists a range function $J_E^x : \Omega_{E^*} \longrightarrow \{\text{subspaces of } \ell^2(D_{E^*})\}$ such that

$$J_H(x) = \{a \in \ell^2(H) : \ \mathcal{T}_E a(\xi) \in J_E^x(\xi) \text{ a.e. } \xi \in \Omega_{E^*}\},$$

where $\mathcal{T}_E$ is the map given in (5.4).

Moreover,

$$\begin{aligned} J_E^x(\xi) &= \overline{\text{span}}\{\mathcal{T}_E(\tilde{\mathcal{T}}_H T_d\varphi(x))(\xi) \,:\, d \in D_F,\, \varphi \in \mathcal{A}\} \\ &= \overline{\text{span}}\{\mathcal{T}(T_d\varphi)(x,\xi) \,:\, d \in D_F,\, \varphi \in \mathcal{A}\} \\ &= \text{span}\{\mathcal{T}(T_d\varphi)(x,\xi) \,:\, d \in D_F,\, \varphi \in \mathcal{A}\}, \end{aligned}$$

where in the last equality we use that $\dim(\ell^2(D_{E^*})) < \infty$.

This leads to the function $J : I_H \times \Omega_{E^*} \longrightarrow \{\text{subspaces of } \ell^2(D_{E^*})\}$ defined as

$$J(x,\xi) = \text{span}\{\mathcal{T}(T_d\varphi)(x,\xi) \,:\, d \in D_F,\, \varphi \in \mathcal{A}\}, \tag{5.10}$$

for a.e. $(x,\xi) \in I_H \times \Omega_{E^*}$.

Lemma 5.2.8. *Let $\mathcal{A} \subseteq L^2(G)$ a countable set. Then, the map defined in (5.10) is a shift-modulation range function.*

Proof. We need to show that J satisfies property (5.6).

Let $d_0 \in D_F$. For each $d \in D_F$, we have that $\mathcal{T}(T_d\varphi)(x - d_0, \xi) = \mathcal{T}(T_{d+d_0}\varphi)(x,\xi)$ for a.e. $(x,\xi) \in I_{F+H} \times \Omega_{E^*}$.

Since $d + d_0 \in F$, it can be written as $d + d_0 = d' + z'$ with $d' \in D_F$ and $z' \in E$. Then, according to Lemma 5.2.4, $\mathcal{T}(T_{d+d_0}\varphi)(x,\xi) = (z',\xi)\mathcal{T}(T_{d'}\varphi)(x,\xi)$. Thus, $\mathcal{T}(T_d\varphi)(x - d_0,\xi) \in J(x,\xi)$ due to $\mathcal{T}(T_{d'}\varphi)(x,\xi) \in J(x,\xi)$.

This shows that $J(x - d_0,\xi) \subseteq J(x,\xi)$ for a.e. $(x,\xi) \in I_{F+H} \times \Omega_{E^*}$ for each $d_0 \in D_F$.

With an analogous argument, it can be proved that $J(x,\xi) \subseteq J(x - d_0,\xi)$ for a.e. $(x,\xi) \in I_{F+H} \times \Omega_{E^*}$ for each $d_0 \in D_F$. □

As we have seen in Proposition 5.2.7, each shift-modulation range function induces a (F,Δ)-invariant space. Furthermore, in the last section we associate each shift-modulation invariant space V a shift-modulation range function from a system of generators of V.

This leads to a natural question. If V is a (F,Δ)-invariant space and J the shift-modulation range function that V induces, what is the relationship between V and the (F,Δ)-invariant space induced from J?

That will be the content of the following section.

5.3 (F,Δ)-Invariant Spaces

We can now state our main result which characterizes (F,Δ)-invariant spaces in terms of the fiberization isometry and shift-modulation range functions.

Theorem 5.3.1. *Let $V \subseteq L^2(G)$ be a closed subspace and $\mathcal{T}$ the fiberization isometry of Definition 5.2.3. Then, V is an (F,Δ)-invariant space if and only if there exists a measurable shift-modulation range function $J : I_H \times \Omega_{E^*} \longrightarrow \{\text{subspaces of } \ell^2(D_{E^*})\}$ such that*

$$V = \{f \in L^2(G) : \ \mathcal{T} f(x,\xi) \in J(x,\xi) \text{ a.e. } (x,\xi) \in I_H \times \Omega_{E^*}\}.$$

Identifying shift-modulation range functions which are equal almost everywhere, the correspondence between (F,Δ)-invariant spaces and measurable shift-modulation range functions is one to one and onto.

Moreover, if $V = S_{(F,\Delta)}(\mathcal{A}) \subseteq L^2(G)$ for some countable subset $\mathcal{A}$ of $L^2(G)$, the measurable shift-modulation range function J associated to V is given by

$$J(x,\xi) = \operatorname{span}\{\mathcal{T}(T_d\varphi)(x,\xi) \ : \ d \in D_F, \ \varphi \in \mathcal{A}\},$$

a.e. $(x,\xi) \in I_H \times \Omega_{E^*}$.

For the proof of Theorem 5.3.1 we need the following previous lemma. It is an adaptation of Lemma 2.2.6.

Lemma 5.3.2. *If J and J' are two measurable shift-modulation range functions such that $M_J = M_{J'}$, where M_J and $M_{J'}$ are given by (5.7), then $J(x,\xi) = J'(x,\xi)$ a.e. $(x,\xi) \in I_H \times \Omega_{E^*}$. That is, J and J' are equal almost everywhere.*

Proof of Theorem 3.1. If V is an (F,Δ)-invariant space, then, since $L^2(G)$ is separable, it holds that $V = S_{(F,\Delta)}(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$.

Let us consider the function J defined as $J(x,\xi) = \operatorname{span}\{\mathcal{T}(T_d\varphi)(x,\xi) \ : \ d \in D_F, \ \varphi \in \mathcal{A}\}$ from $I_H \times \Omega_{E^*}$ to $\{\text{subspaces of } \ell^2(D_{E^*})\}$. As a consequence of Lemma 5.2.8, J is a shift-modulation range function. We must prove that $\mathcal{T}V = M_J$ where M_J is as in (5.7) and that J is measurable.

We will first show $\mathcal{T}V = M_J$.

Take $\delta \in \Delta$, $y \in F$ written as $y = z + d$ with $z \in E$ and $d \in D_F$, and $\varphi \in \mathcal{A}$. Then, by Lemma 5.2.4 it holds that

$$\mathcal{T}(M_\delta T_y\varphi)(x,\xi) = (x,\delta)(-z,\xi)\mathcal{T}(T_d\varphi)(x,\xi) \qquad \text{a.e. } (x,\xi) \in I_H \times \Omega_{E^*}.$$

Thus, since $\mathcal{T}(T_d\varphi)(x,\xi) \in J(x,\xi)$, we have that $\mathcal{T}(M_\delta T_y\varphi)(x,\xi) \in J(x,\xi)$ a.e. $(x,\xi) \in I_H \times \Omega_{E^*}$. Therefore,

$$\mathcal{T}(\operatorname{span}\{M_\delta T_y\varphi : \varphi \in \mathcal{A}, y \in F, \delta \in \Delta\}) \subseteq M_J.$$

Using that $\mathcal{T}$ is a continuous function and Remark 5.2.6, we can compute

$$\begin{aligned}\mathcal{T}V &= \mathcal{T}(\overline{\text{span}}\{M_\delta T_y\varphi : \varphi\in\mathcal{A}, y\in F, \delta\in\Delta\})\\ &\subseteq \overline{\mathcal{T}(\text{span}\{M_\delta T_y\varphi : \varphi\in\mathcal{A}, y\in F, \delta\in\Delta\})}\\ &\subseteq \overline{M_J} = M_J.\end{aligned}$$

Let us suppose that $\mathcal{T}V \subsetneq M_J$. Then, there exists $\Psi\in M_J\setminus\{0\}$ orthogonal to $\mathcal{T}V$. In particular, we have that $\langle\Psi, \mathcal{T}(M_\delta T_y\varphi)\rangle = 0$ for all $\varphi\in\mathcal{A}, y\in F$ and $\delta\in\Delta$.

Hence, if we write $y = z + d$ with $z\in E$ and $d\in D_F$, by Lemma 5.2.4 we obtain

$$\begin{aligned}0 &= \int_{I_H}\int_{\Omega_\Delta}\langle\Psi(x,\xi), \mathcal{T}(M_\delta T_y\varphi)(x,\xi)\rangle\, dm_\Gamma(\xi)\, dm_G(x)\\ &= \int_{I_H}\int_{\Omega_\Delta}(x,\delta)(-z,\xi)\langle\Psi(x,\xi), \mathcal{T}(T_d\varphi)(x,\xi)\rangle\, dm_\Gamma(\xi)\, dm_G(x)\\ &= \int_{I_H}\int_{\Omega_\Delta}\eta_\delta(x)\eta_{-z}(\xi)\langle\Psi(x,\xi), \mathcal{T}(T_d\varphi)(x,\xi)\rangle\, dm_\Gamma(\xi)\, dm_G(x),\end{aligned}$$

where η_δ and η_{-z} are as in Proposition 1.1.19.

If we define $\nu_{(\delta,z)}(x,\xi) := \eta_\delta(x)\eta_{-z}(\xi)$, then, using Proposition 1.1.19, it can be shown that $\{\nu_{(\delta,z)}\}_{(\delta,z)\in\Delta\times E}$ is an orthogonal basis for $L^2(I_H\times\Omega_{E^*})$. Therefore, $\langle\Psi(x,\xi), \mathcal{T}(T_d\varphi)(x,\xi)\rangle = 0$ a.e. $(x,\xi)\in I_H\times\Omega_{E^*}$ for all $d\in D_F$.

This shows that $\Psi(x,\xi)\in J(x,\xi)^\perp$ a.e. $(x,\xi)\in I_H\times\Omega_{E^*}$ and, since $\Psi\in M_J$ it must be $\Psi = 0$, wich is a contradiction. Thus $\mathcal{T}V = M_J$.

Let us prove now that J is measurable. If $\mathcal{P}$ is the orthogonal projection onto M_J, $\mathcal{I}$ is the identity mapping in $L^2(I_H\times\Omega_{E^*}, \ell^2(D_{E^*}))$ and $\Psi\in L^2(I_H\times\Omega_{E^*}, \ell^2(D_{E^*}))$ we have that $(\mathcal{P}-\mathcal{I})\Psi$ is orthogonal to M_J. Then, with the above reasoning $(\mathcal{P}-\mathcal{I})\Psi(x,\xi)\in J(x,\xi)^\perp$ for a.e. $(x,\xi)\in I_H\times\Omega_{E^*}$. Thus,

$$P_{(x,\xi)}((\mathcal{P}-\mathcal{I})\Psi(x,\xi)) = 0 \quad \text{a.e. } (x,\xi)\in I_H\times\Omega_{E^*}$$

and then, $\mathcal{P}\Psi(x,\xi) = P_{(x,\xi)}(\Psi(x,\xi))$ for a.e. $(x,\xi)\in I_H\times\Omega_{E^*}$.

If in particular $\Psi(x,\xi) = a$ for all $(x,\xi)\in I_H\times\Omega_{E^*}$, it holds that $\mathcal{P}a(x,\xi) = P_{(x,\xi)}(a)$. Therefore, since $(x,\xi)\mapsto\mathcal{P}a(x,\xi)$ is measurable, $(x,\xi)\mapsto P_{(x,\xi)}a$ is measurable as well.

Conversely. If J is a shift-modulation range function, by Proposition 5.2.7, $V := \mathcal{T}^{-1}M_J$ is an (F,Δ)-invariant space. Then, $V = S_{(F,\Delta)}(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$ and, according to Lemma 5.2.8 we can define the shift-modulation range function J' as

$$J'(x,\xi) = \text{span}\{\mathcal{T}(T_d\varphi)(x,\xi) \,:\, d\in D_F,\ \varphi\in\mathcal{A}\} \quad \text{a.e. } (x,\xi)\in I_H\times\Omega_{E^*}.$$

Thus, as we have shown, J' is measurable and $M_{J'} = \mathcal{T}V = M_J$. Then, Lemma 5.3.2 gives us $J = J'$ a.e.

This also proves that the correspondence between (F, Δ)-invariant spaces and shift-modulation measurable range functions is one to one and onto.

□

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Index