Tesis Doctoral

Sistemas cuánticos compuestos: un enfoque algebraico

Holik, Federico Hernán

2010

UNIVERSIDAD DE BUENOS AIRES

Facultad de Ciencias Exactas y Naturales

Departamento de Física

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Tesis presentada para optar al título de Doctor de la
Universidad de Buenos Aires en el área de Ciencias Físicas

Federico Hernán Holik

Director de Tesis: Graciela Domenech
Consejero de Estudios: Carmen Núñez

Lugar de Trabajo: Instituto de Astronomía y Física del Espacio

Buenos Aires, 2010

Agradecimientos

A mi directora, compañeros de trabajo, a mi novia, a los compañeros de lucha, familiares y amigos.

Dedicado a mi vieja

Sistemas cuánticos compuestos: un enfoque algebraico

Resumen

En este trabajo estudiamos dos características no clásicas de los sistemas cuánticos compuestos, a saber, el entrelazamiento y la indistinguibilidad, usando herramientas lógicas y algebraicas. Primero estudiamos a las mezclas impropias desde un punto de vista lógico y geométrico. Esto se hace extendiendo el retículo de proposiciones de von Neumann de forma tal de incluir a las mezclas impropias como átomos del nuevo retículo. Luego estudiamos la indistinguibilidad cuántica. Usamos una estructura cuántica que es una modificación de la teoría de conjuntos de Zermelo-Frenkel basada en la mecánica cuántica, a saber, la teoría de cuasiconjuntos (Q). Usando Q desarrollamos una formulación de la mecánica cuántica que no usa la identidad de primer orden en sus bases lógicas. Luego, desarrollamos un marco de retículos proposicionales para partículas idénticas. Estas construcciones responden a discusiones interesantes planteadas en la literatura.

Palabras Claves: Entrelazamiento, Lógica cuántica, Teoría de cuasiconjuntos, Indistinguibilidad cuántica, Conjunto convexo de estados.

Compound Quantum Systems: An Algebraic Approach

Abstract

In this work we study two non-classical features of quantum compound systems, namely, entanglement and indistinguishability using logical and algebraic techniques. First, we study improper mixtures from a quantum logical and geometrical point of view. This is done by extending the von Neumann lattice of propositions in order to include improper mixtures as atoms of the new lattice. Then, we study the problem of quantum non-individuality. We use a quantum structure which is a modification of Zermelo-Frenkel set-theory based on quantum mechanics, namely, Quasi-set Theory (Q). Using Q we develop a new formulation of quantum mechanics which does not uses first order identity on its logical bases. These constructions answer interesting discussions posed in the literature.

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Chapter 1

Introduction

The study of compound quantum systems presents -at least- two striking features: entanglement [14, 87, 61, 66] and indistinguishability [74, 31, 77, 76]. These radical departures from classical mechanics show themselves when quantum systems interact or are considered as joint systems. The clarification of this subject is of importance in several areas of research, such as decoherence [79] or quantum information processing [39].

Quantum entanglement was considered by Schrödinger as "the characteristic trait of Quantum Mechanics, the one that enforces its entire departure from classical lines of thoughts" [78]. He also had similar considerations regarding the problem of identical particles [76, 77]. Both features of quantum mechanics are of a great importance for the rising of new physics and the development of new technologies [39]. In this work we study quantum entanglement and quantum indistinguishability from an algebraic and quantum logical point of view.

The study of quantum mechanics using logical and algebraic tools has different motivations. On one hand, it has been used in the foundations of quantum mechanics, in a vast range of studies [15, 58, 47, 31, 64, 42, 69]. On the other hand it has produced new results in more applied fields as for example in the axiomatization of quantum logical gates in quantum computing [20]. In this thesis we study compound quantum systems using different quantum structures. Most of the material presented here is new, and is based on the work of the author during the last years.

The work is divided in two chapters. In chapter 2 we give a short overview of the quantum logical approach to quantum mechanics and study improper

mixtures from a quantum logical and geometrical point of view. This is done by extending the von Neumann lattice of propositions in order to include improper mixtures as atoms of new lattices. The content of this chapter is based mainly on the algebraic constructions that we have developed in [24, 38]. In chapter 3 we study the problem of quantum indistinguishability. This chapter is based mainly in our work developed in [21, 22, 23] and we also present some unpublished results. After an overview of the problem, we use a quantum structure which is a modification of Zermelo-Frenkel set-theory motivated on quantum mechanics, namely, Quasi-set Theory (Q)[31, 46, 47, 49]. Using Q we develop a formulation of quantum mechanics which does not use first order identity in its logical bases. As we shall see, this construction answers interesting discussions posed in the literature [59, 60, 75]. Next, we apply some of the tools developed in Chapter 2 to the indistinguishable particle case. Finally, we expose our conclusions in 4.

Chapter 2

The Quantum Logical Approach

We study mixed states from a quantum logical and geometrical point of view. We adopt the point of view that mixtures must be considered as states in their own right. We use the convex set of states in order to construct two new lattices whose atoms are all physical states: pure states and mixtures. This is done in order to overcome one of the problems which appear in the standard quantum logical formalism, namely, that for a subsystem of a larger system in an entangled state the conjunction of all actual properties of the subsystem does not yield its actual state. In fact, its state is a mixture and cannot be represented in the von Neumann lattice as a minimal property which determines all other properties as is the case for pure states or classical systems. We construct two new lattice structures. The new lattices also contain all propositions of the von Neumann lattice as elements. We argue that these extensions express in an algebraic form the fact that -alike the classical case- quantum interactions produce non trivial correlations between the systems. Finally, we study the maps which can be defined between the extended lattice of a compound system and the lattices of its subsystems.

2.1 Introduction

Non-separability of the states of quantum systems is considered with continuously growing interest in relation to quantum information theory. In fact, today entanglement is regarded not only as a feature that gives rise to inter-

esting foundational questions. It is considered also as a powerful resource for quantum information processing [39]. In this chapter we pose the problem of studying non-separability with algebraic and geometrical tools related to quantum logic (QL).

The algebraic approach to the formalization of quantum mechanics was initiated by Birkhoff and von Neumann [15], who gave it the name of "quantum logic". Although an algebraic structure, for historical reasons it has conserved its name. QL was developed mainly by Mackey [50], Jauch [42], Piron [64], Kalmbach [43, 44], Varadarajan [82, 83], Greechie [34], Gudder [36], Giuntini [33], Pták and Pulmannova [62], Beltrametti and Cassinelli [13], among others. For a complete bibliography see for example [25] and [29]. The Geneva school of QL extended this research to analysis of compound systems. The first results where obtained by Aerts and Daubechies [1, 2, 3] and Randall and Foulis [70]. For more discussion on the notion of entanglement in the Operational QL (OQL) approach, see [17].

In the tradition of the quantum logical research, a property of (or a proposition about) a quantum system is related to a closed subspace of the Hilbert space $\mathcal{H}$ of its (pure) states or, analogously, to the projector operator onto that subspace. Moreover, each projector is associated to a dichotomic question about the actuality of the property [85, pg. 247]. A physical magnitude $\mathcal{M}$ is represented by an operator M acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition $M = \sum_i a_i P_i = \sum_i a_i |a_i\rangle\langle a_i|$ are satisfied (along this work we will restrict the study to the finite dimensional case). The real numbers a_i are related to the outcomes of measurements of the magnitude $\mathcal{M}$ and projectors $|a_i\rangle\langle a_i|$ to the mentioned properties. The physical properties of the system are organized in the lattice of closed subspaces $\mathcal{L}(\mathcal{H})$ that, for the finite dimensional case, is a modular lattice [51]. In this frame, the pure state of the system is represented by the meet (i.e. the lattice infimum) of all actual properties, more on this below. A comprehensive description of QL in present terminology may be found in [80].

Mixed states represented by density operators had a secondary role in the classical treatise by von Neumann because they did not add new conceptual features to pure states. In fact, in his book, mixtures meant "statistical mixtures" of pure states [85, pg. 328], which are known in the literature as "proper mixtures" [26, Ch. 6]. They usually represent the states of realistic physical systems whose preparation is not well described by pure states. A system prepared in this way is called a "Gemenge" in [58] or a "proper

mixture" in [26]. They can be considered as ensembles of pure states weighted with different probabilities, and thus, they admit an ignorance interpretation.

But there is another way in which a system can be prepared in a mixed state, namely, by separation (see section 2.2 for technical details). If a system is a subsystem of a compound system which is in an entangled state, then its state will be a mixture. Mittelstaedt states that a state as such "...is a genuine mixed state, and the question whether it admits an 'ignorance interpretation is more complicated and requires a separate investigation" [58]. Such states are named "improper mixtures" in [26]. The distinction between proper and improper mixtures has been the source of a rich debate in the literature [26, 58, 45, 27, 53, 30]). Improper mixtures are an expression of one of the main features of quantum systems, namely entanglement. They appear in processes like measurements on some degrees of freedom of the system, and also when considering one system in a set of interacting systems. In fact, in each (non trivial) case in which a part of the system is considered, we have to deal with improper mixtures.

In classical mechanics mixtures are of a very different kind. They represent an state of ignorance of the observer, because we know in principle that the system is in a given state s of phase space. In quantum mechanics, a proper mixture could be considered as a density matrix *plus* a piece of classical information, which encodes classical probabilities for preparations of ensembles of pure states. This extra piece of classical information may have its source on imperfections of the preparation procedure, or could be produced deliberately, but the important fact is that this probabilities could be determined -at least- in principle as is the case of mixtures in classical mechanics. But for the case of improper mixtures, this information is not available, not even in principle. Also for statistical mixtures it was stated that the ignorance interpretation becomes untenable in cases of nonunique decomposability of the density operator [13, Ch. 2]. But the debate is far from finished (see for example [79]). We will not discuss these interesting issues in this work, but we will mention some consequences when it is necessary.

In the standard formulation of QL, mixtures as well as pure states are included as measures over the lattice of projections [69, Ch. 3], that is, a state s is a function:

$$s : \mathcal{L}(\mathcal{H}) \longrightarrow [0; 1]$$

such that:

1. $s(\mathbf{0}) = 0$ ($\mathbf{0}$ is the null subspace).

2. $s(P^\perp) = 1 - s(P)$

3. For any pairwise orthogonal family of projections P_j, $s(\sum_j P_j) = \sum_j s(P_j)$

In a similar way, in classical mechanics statistical distributions are represented as measures over the phase space. But while pure states can be put in a bijective correspondence to the atoms of $\mathcal{L}(\mathcal{H})$, this is not the case for mixtures of neither kind. On the contrary, the standard formulation of QL treats improper mixtures in an analogous way as classical statistical distributions. But improper mixtures have a very different physical content, and the question of their interpretation is a subtle one. After a brief review of the problem of quantum non-separability in Section 2.2, we turn in Section 2.3 to the reasons why this difference leads to a dead end when compound systems are considered from the standard quantum logical point of view. We also discuss that the physical necessity to consider mixtures indicates that the algebraic structure of the properties of compound systems should be studied in a frame that takes into account the fact that density operators are states in their own right. We show in Section 2.4 that a frame with these characteristics can be built by enlarging the scope of standard QL. We do this by constructing two lattices based on the convex set of density operators which incorporate improper mixtures as atoms. Then, we study the relationship between these lattices and the lattices of its subsystems and show how our construction overcomes the problem posed in Section 2.3.

2.2   Quantum non-separability

We briefly review here the main arguments and results of the analysis of non-separability and relate them to the frame of quantum logical research for the sake of completeness. We start by analyzing classical compound systems in order to illustrate their differences with the quantum case.

2.2.1   Classical systems

When considering in classical mechanics two systems S_1 and S_2 and their own state spaces Γ_1 and Γ_2 (or, analogously, two parts of a single system), the

state space Γ of the composite system is the cartesian product $\Gamma = \Gamma_1 \times \Gamma_2$ of the phase spaces of the individual systems, independently of the kind of interaction between both of them. The physical intuition behind this fact is that, no matter how they interact, every interesting magnitude corresponding to the parts and the whole may be written in terms of the points in phase space.

In the logical approach, classical properties are associated with subsets of the phase space, precisely with the subsets consisting of the points corresponding to those states such that, when being in them, one may say that the system has the mentioned property. Thus, subsets of Γ are good representatives of the properties of a classical system. The power set $\wp(\Gamma)$ of Γ, partially ordered by set inclusion $\subseteq$ (the implication) and equipped with set intersection $\cap$ as the meet operation, set union $\cup$ as the join operation and relative complement $'$ as the complement operation gives rise to a complete Boolean lattice $< \wp(\Gamma),\ \cap,\ \cup,\ ',\ \mathbf{0},\ \mathbf{1} >$ where $\mathbf{0}$ is the empty set $\emptyset$ and $\mathbf{1}$ is the total space Γ. According to the standard interpretation, partial order and lattice operations may be put in correspondence with the connectives *and*, *or*, *not* and the *material implication* of classical logic.

In this frame, the points $(p,q) \in \Gamma$ (pure states of a classical system) represent pieces of information that are maximal and logically complete. They are maximal because they represent the maximum of information about the system that cannot be consistently extended (any desired magnitude is a function of $(p,\ q)$) and complete in the sense that they semantically decide any property [25]. Statistical mixtures are represented by measurable functions:

$$\sigma : \Gamma \longrightarrow [0;1]$$

such that

$$\int_{\Gamma} \sigma(p,q) d^3p d^3q = 1$$

We point out that statistical mixtures are not fundamental objects in classical mechanics, in the sense that they admit an ignorance interpretation. They appear as a state of affairs in which the observer cannot access to an information which lies objectively in the system. Although the physical status of quantum improper mixtures is very different, they are treated in a similar way as classical mixtures by standard QL.

When considering two systems, it is meaningful to organize the whole set of their properties in the corresponding (Boolean) lattice built up as the

cartesian product of the individual lattices. Informally one may say that each factor lattice corresponds to the properties of each physical system. More precisely, in the category of lattices as objects and lattice morphisms as arrows, the cartesian product of lattices is the categorial product. This category is Ens, and the cartesian product is the categorial product in Ens.

2.2.2 Quantum systems

The quantum case is completely different. When two or more systems are considered together, the state space of their pure states is taken to be the tensor product of their Hilbert spaces. Given the Hilbert state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as representatives of two systems, the pure states of the compound system are given by rays in the tensor product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. But it is not true –as a naive classical analogy would suggest– that any pure state of the compound system factorizes after the interaction in pure states of the subsystems, and that they evolve with their own Hamiltonian operators [58, 6]. The mathematics behind the persistence of entanglement is the lack of a product of lattices and even posets [4, 7, 28]. A product of structures is available for weaker structures [29, Ch. 4] but those structures, though mathematically very valuable and promising, have a less direct relation with the standard formalism of quantum mechanics.

In the standard quantum logical approach, properties (or propositions regarding the quantum system) are in correspondence with closed subspaces of Hilbert space $\mathcal{H}$. The set of subspaces $\mathcal{C}(\mathcal{H})$ with the partial order defined by set inclusion $\subseteq$, intersection of subspaces $\cap$ as the lattice meet, closed linear spam of subspaces $\oplus$ as the lattice join and orthocomplementation $\neg$ as lattice complement, gives rise in the finite dimensional case to a modular lattice $\mathcal{L}(\mathcal{H}) = < \mathcal{C}(\mathcal{H}), \cap, \oplus, \neg, \mathbf{0}, \mathbf{1} >$ where $\mathbf{0}$ is the empty set $\emptyset$ and $\mathbf{1}$ is the total space $\mathcal{H}$. We will refer to this lattice as $\mathcal{L}_{v\mathcal{N}}$, the 'von Neumann lattice'.

When trying to repeat the classical procedure of taking the product of the lattices of the properties of two systems to obtain the lattice of the properties of the composite the procedure fails [5]. Mathematically, this is the expression of the fact that the category of Hilbert lattices as objects and lattice morphisms as arrows has not a categorial product because of the failure of orthocomplementation. This problem is studied for example in [2, 36]. Attempts to vary the conditions that define the product of lattices

have been made [68, 40], but in all cases it results that the Hilbert lattice factorizes only in the case in which one of the factors is a Boolean lattice or when systems have never interacted. For a complete review, see [28].

Let us briefly recall the defining properties of the tensor product of a finite collection of vector spaces in order to discuss the main features that make the difference with the classical case. Let us first define (following [16]) $\otimes \mathcal{H}_i$ as the unique vector space which satisfies the following properties:

1. for each family $\{|x_i\rangle\}$, $|x_i\rangle \in \mathcal{H}_i$, there exists an element $\otimes_i |x_i\rangle \in \otimes_i \mathcal{H}_1$ depending multilinearly on the $\{|x_i\rangle\}$. All vectors in $\otimes_i \mathcal{H}_i$ are finite linear combinations of such elements.

2. (universal property) for each multilinear mapping π of the product of the $\mathcal{H}_i$ into a vector space Y, there exists a unique linear map $\varphi : \otimes_i \mathcal{H}_i \longrightarrow Y$ such that

$$\varphi(\otimes_i |x_i\rangle) = \pi(\{|x_i\rangle\})$$

   for all $|x_i\rangle \in \mathcal{H}_i$.

3. (associativity) for each partition $\cup_k I_k$ of $\{1, \cdots, n\}$ there exists a unique isomorphism from $\otimes_i \mathcal{H}_i$ onto $\otimes_k (\otimes_{i \in I_k} \mathcal{H}_i)$ transforming $\otimes_i |x_i\rangle$ into $\otimes_k (\otimes_{i \in I_k} |x_i\rangle)$.

When the spaces $\mathcal{H}_i$ are Hilbert spaces, it is possible to define an inner product on $\otimes \mathcal{H}_i$ by extending the following definition by linearity:

$$(\otimes_i |x_i\rangle, \otimes_i |y_i\rangle) = \prod_i (|x_i\rangle, |y_i\rangle)$$

Note that as we are using Dirac notation, we may write $\langle x_i | y_i \rangle$ instead of $(|x_i\rangle, |y_i\rangle)$. The completion of $\otimes \mathcal{H}_i$ in the associated norm is the tensor product of the Hilbert spaces $\otimes_i \mathcal{H}_i$. Thus we see that the tensor product of Hilbert spaces is in essence a multilinear extension of the direct product. From a physical point of view, it is for this reason that the state of the joint system contains much more information than 'the sum' of the information contained in the states of its parts.

This feature of quantum systems may be regarded as a strange fact when using classical reasoning, but it not strange at all in a landscape where the superposition principle holds. Given two systems S_1 and S_2, if we prepare

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them independently in states $|a\rangle$ and $|b\rangle$ respectively, then we would have something like the direct product of the states of both systems $|a\rangle \times |b\rangle$ for the state of the joint system. We could perform also different preparations and obtain $|a'\rangle \times |b'\rangle$. Then, if there are no superselection rules and according to the superposition principle, it is quite natural to suppose that it is at least in principle possible to prepare the superposition state of the form $\alpha|a\rangle \otimes |b\rangle + \beta|a'\rangle \otimes |b'\rangle$, and so, we *need* $\otimes$ instead of $\times$. This last state is not a product of the states of the parties. It is for this reason that the product in quantum mechanics has to be the multilinear extension of the direct product.

Let us now briefly review the standard relationship between the states of the joint system and the states of the subsystems. If $\{|x_k^{(i)}\rangle\}$ is an orthonormal basis for $\mathcal{H}_i$, then

$$\otimes_{i=1}^{n}|x_{k_i}^{(i)}\rangle$$

forms a basis of $\otimes_i \mathcal{H}_i$. Let us focus for simplicity on the case of two systems, S_1 and S_2. If $\{|x_i^{(1)}\rangle\}$ and $\{|x_i^{(2)}\rangle\}$ are the corresponding orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_1$ respectively, then $\{|x_i^{(1)}\rangle \otimes |x_j^{(2)}\rangle\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. A general (pure) state of the composite system can be written as:

$$\rho = |\psi\rangle\langle\psi|$$

where $|\psi\rangle = \sum_{i,j} \alpha_{ij}|x_i^{(1)}\rangle \otimes |x_j^{(2)}\rangle$. And if M represents an observable, its mean value $\langle M \rangle$ is given by:

$$\text{tr}(\rho M) = \langle M \rangle$$

When observables of the form $O_1 \otimes \mathbf{1}_2$ and $\mathbf{1}_1 \otimes O_2$ (with $\mathbf{1}_1$ and $\mathbf{1}_2$ the identity operators over $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively) are considered, then partial state operators ρ_1 and ρ_2 can be defined for systems S_1 and S_2. The relation between ρ, ρ_1 and ρ_2 is given by:

$$\rho_1 = tr_2(\rho) \quad \rho_2 = tr_1(\rho)$$

where tr_i stands for the partial trace over the i degrees of freedom. It can be shown that:

$$tr_1(\rho_1 O_1 \otimes \mathbf{1}_2) = \langle O_1 \rangle$$

and that a similar equation holds for S_2. Operators of the form $O_1 \otimes \mathbf{1}_2$ and $\mathbf{1}_1 \otimes O_2$ represent magnitudes related to S_1 and S_2 respectively. When S is in

a product state $|\varphi_1\rangle \otimes |\varphi_2\rangle$, the mean value of the product operator $O_1 \otimes O_2$ will yield:

$$\text{tr}(|\varphi_1\rangle \otimes |\varphi_2\rangle\langle\varphi_1| \otimes \langle\varphi_2|O_1 \otimes O_2) = \langle O_1\rangle\langle O_2\rangle$$

reproducing statistical independence. But, as is well known, this is not the general case.

The state of the whole system $\rho = |\psi\rangle\langle\psi|$ carries the information about the correlations between S_1 and S_2. The fact that ρ_1 and ρ_2 are not pure states is an expression of the non-triviality of these correlations, that are stronger and of a different kind than those of the classical case. This radical difference expresses itself also in the violation of Bell inequalities by quantum systems [12]. These facts suggest that mixtures have to be considered as states in their own right and be given a place in the algebraic approach to the study of quantum properties.

2.2.3 The Convex Set of States of a Quantum System

From the analysis of the last section it becomes clear that for a complete description that includes compound systems it is not sufficient to consider only pure states, but we have to consider also mixtures. The standard way of doing this is by representing the states of the system by positive, Hermitian and trace one operators, (also called 'density matrices', which are connected with measures over the lattice of projections via Gleason's theorem [69]). The set of all density matrixes forms a convex set (of states), which we will denote by $\mathcal{C}$:

$$\mathcal{C} := \{\rho \in \mathcal{A} \,|\, \text{tr}(\rho) = 1, \, \rho \geq 0\} \tag{2.2.1}$$

where $\mathcal{A}$ is the set of Hermitian operators. As usual, physical observables $\mathcal{M}$ are represented by elements M of $\mathcal{A}$, the $\mathbb{R}$-vector space of Hermitian operators acting on $\mathcal{H}$:

$$\mathcal{A} := \{M \in B(\mathcal{H}) \,|\, M = M^\dagger\}$$

$B(\mathcal{H})$ stands for the algebra of bounded operators in $\mathcal{H}$. The mean value of the observable represented by the operator M when the system is in a state ρ is given by $\langle M\rangle = \text{tr}(\rho M)$.

The set P of pure states can be defined as

$$P := \{\rho \in \mathcal{C} \,|\, \rho^2 = \rho\}$$

This set is in correspondence with the rays of $\mathcal{H}$ by the usual association (using Dirac notation) $[|\psi\rangle] \longmapsto |\psi\rangle\langle\psi|$ between the elements of the projective space of $\mathcal{H}$ and the class defined by the normalized vector $|\psi\rangle$ ($|\varphi\rangle \sim |\psi\rangle \longleftrightarrow |\varphi\rangle = \lambda|\psi\rangle$, $\lambda \neq 0$). $\mathcal{C}$ is a convex set inside the hyperplane $\{\rho \in \mathcal{A} \,|\, \mathrm{tr}(\rho) = 1\}$. If $\dim_{\mathbb{C}}(\mathcal{H}) = n < \infty$, we have an $\mathbb{R}$-linear isomorphism $B(\mathcal{H}) \cong M_n(\mathbb{R}) \times M_n(\mathbb{R})$, then

$$\mathcal{A} \cong \{(R, I) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) \,|\, R^t = R,\, I^t = -I\} = S_n(\mathbb{R}) \times \wedge_n(\mathbb{R})$$

$$\mathcal{A} \cap \{\mathrm{tr}(\rho) = 1\} \cong \{(R, I) \in S_n(\mathbb{R}) \times \wedge_n(\mathbb{R}) \,|\, \mathrm{tr}(R) = 1\}$$

So the convex set $\mathcal{C}$ lies inside an $\mathbb{R}$-algebraic variety of dimension

$$\dim_{\mathbb{R}}(\{\rho \in \mathcal{A} \,|\, \mathrm{tr}(\rho) = 1\}) = n^2 - 1 \tag{2.2.2}$$

When a system S composed of subsystems S_1 and S_2 is considered, the state of S cannot be decomposed in general in a product state $\rho = \rho_1 \otimes \rho_2$. *Separable states* are those states of S which can be written as a convex combination of product states [14, 86]:

$$\rho_{Sep} = \sum_k \lambda_k \rho_k^{(1)} \otimes \rho_k^{(2)} \tag{2.2.3}$$

where $\rho_k^{(1)} \in \mathcal{C}_1$ and $\rho_k^{(2)} \in \mathcal{C}_2$, $\sum_k \lambda_k = 1$ and $\lambda_k \geq 0$. It is easy to see that this expression may be written as

$$\rho_{Sep} = \sum_{i,j} \lambda_{ij} \rho_i^{(1)} \otimes \rho_j^{(2)} \tag{2.2.4}$$

with $\sum_{i,j} \lambda_{ij} = 1$ and $\lambda_{ij} \geq 0$. We will denote $\mathcal{S}(\mathcal{H})$ the (convex) set of separable states. R. Werner called a density matrix as *classically correlated,* if it can be approximated (e.g., in the trace norm) by density matrixes of the form 2.2.4. This is because in [86] he shows that their correlations can be reproduced by a classical random generator which can be choosen as a purely classical devise. But it is important to remark that he also noticed in [86] that:

"Since there are usually very different ways of preparing the same state W, classical correlation does not mean that the state has actually been prepared in the manner described, but only that its statistical properties can be reproduced by a classical mechanism."

It is a remarkable fact that there are many states in $\mathcal{C}$ which are not separable. If the state is non-separable, it is said to be *entangled*. The estimation of the volume of $\mathcal{S}(\mathcal{H})$ is of great interest (see for example [87], [39] and [10]).

What is one of the main implications of considering mixtures as actual states? For the standard formulation of QM we have at hand what it is usually called "the superposition principle":

Principle 1. *Superposition Principle. If $|\psi_1\rangle$ and $|\psi_1\rangle$ are physical states, then $\alpha|\psi_1\rangle + \beta|\psi_1\rangle$ ($|\alpha|^2 + |\beta|^2 = 1$) will also be a physical state.*

Are there other operations which allows us to form new states up from two given states? If we accept that improper mixtures are states of a fundamental nature as much as pure states do, then, the fact that we can create new physical states *mixing* two given states, could be thought as a principle which stands besides the superposition principle:

Principle 2. *Mixing Principle. If ρ and ρ' are physical states, then $\alpha\rho + \beta\rho'$ ($\alpha + \beta = 1$, $\alpha, \beta \geq 0$) will also be a physical state.*

Mixing principle is not contained directly in the superposition principle. Mixing principle appears as a consequence of the axiom which states that to a compounded system corresponds the tensor product of Hilbert spaces. It expresses the fact that improper mixtures are physical states. We will not consider proper mixtures in this work because they do not add interesting features, we only concentrate in improper mixtures.

There is a remarkable physical consequence of all this. While for pure states there always exist "true propositions" [64], i.e., propositions for which a test will yield the answer "yes" with certainty (and a similar situation for "false propositions"), the situation is radically different for improper mixtures. If we accept that improper mixtures are states of a fundamental nature as well as pure states, then we must face the fact that there exist

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$$\mathcal{L}_{\mathcal{CM}} = \mathcal{L}_{\mathcal{CM}_1} \times \mathcal{L}_{\mathcal{CM}_2}$$

π_1 π_2

$\mathcal{L}_{\mathcal{CM}_1}$ $\mathcal{L}_{\mathcal{CM}_2}$
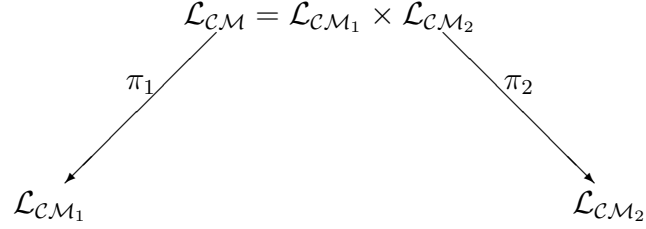
Figure 2.1: In the classical case, we can go from the state of the system to the states of the subsystems using the set-theoretical projections π_1 and π_2

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states for which no "true propositions" exist (discarding the trivial proposition represented by the Hilbert space itself). This is the case for example for the maximum uncertainty state (finite dimension), $\rho = \frac{1}{N}\mathbf{1}$.

2.3 The Problem of Representation of The States of the Subsystems in QL

In the quantum logical approach, there is a bijective correspondence between the states of the system and the atoms of the lattice $\mathcal{L}_{vN}$ of its properties: the atoms of $\mathcal{L}_{vN}$ are the pure states. The relationship between pure states $\rho_{pure} = |\psi\rangle\langle\psi|$ of the quantum system and its actual properties p is given by:

$$< \ |\psi\rangle \ >= \wedge\{p \in \mathcal{L}_{vN} \,|\, p \text{ is actual}\} \qquad (2.3.1)$$

and an equivalent relation holds for the classical case. This is an expected fact, because in $\mathcal{L}_{vN}$ states are the most elemental properties of the system, up from which all other properties are inferred. We claim that any reasonable definition of state must satisfy this property. Furthermore, the representatives of states must be atoms of the lattice, in order to grant that no other non-trivial property be more elementary. But pure states form in general a quite small subset of the border of $\mathcal{C}$ (the atoms of $\mathcal{L}_{vN}$ are in one to one

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correspondence with this subset): pure states are in a 2(N-1)-dimensional subset of the (N^2-2)-dimensional boundary of $\mathcal{C}$. And so all non-pure states are excluded from $\mathcal{L}_{vN}$. Or in the best case, they have a different status, when considered (as in the classical case) as measures over the lattice of projections.

Let us emphasize that a remarkable problem appears in standard QL, linked to the status that it gives to improper mixtures (see [8] for more discussion on this problem and a proposal for its solution different than the one presented here). Suppose that S_1 and S_2 are subsystems of a larger system S which is in a pure entangled state $|\psi\rangle$. Then we may ask which the states of its subsystems are. If we make the conjunction of all actual properties for, say S_1, we will no longer obtain an atom of $\mathcal{L}_{v\mathcal{N}_1}$ (see Theorem 18 of [5]). Instead of it, we will obtain a property which corresponds, in the non-trivial case, to a subspace of dimension strictly greater than one and does not correspond to the state of the subsystem. In fact, the state of the subsystem is the (improper) mixture given by the partial trace $\text{tr}_2(|\psi\rangle\langle\psi|)$. Thus, there is no way to obtain the actual physical state of S_1 using the $\wedge$ operation of $\mathcal{L}_{v\mathcal{N}_1}$, as it would be reasonable according to the definition of state as minimal property out of which all other properties are inferred.

To put things graphically, consider Figures 1 and 2. For the classical case, there exist set-theoretical projections π_1 and π_2 from $\mathcal{L}_{CM}$ to $\mathcal{L}_{CM1}$ and $\mathcal{L}_{CM2}$ which relate the states of the system S and the states of the subsystems S_1 and S_2. In the quantum case (Figure 2), we do not have arrows which map states of $\mathcal{L}_{vN}$ into states of $\mathcal{L}_{vNi}$ ($i = 1, 2$), simply because non-pure states are not properly included in the property lattice. Thus, the "?" arrows of Figure 2 are missing.

In spite of the fact that mixtures are also considered in classical mechanics, they pose there no fundamental problem. This is so because classical mixtures represent a lack of information that is -at least in principle- available. On the contrary, according to the orthodox interpretation of QM, information encoded in (improper) mixtures is all that exists, there is no further information available. But the orthodox quantum logical approach puts in different levels pure states and mixtures (the lattice of properties and a measure over it) as is done in the classical case. In the classical case this works, for pure states of the whole system and its subsystems can be properly linked as Figure 1 shows. But we cannot do the same in the quantum case, because subsystems are rarely found in pure states.

All of this motivates our search of algebraic structures which contain mix-
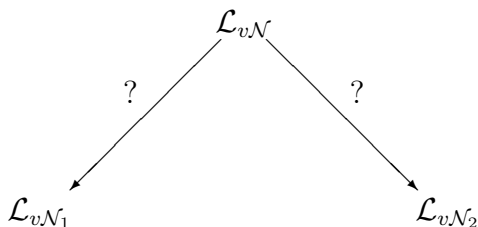
$$\mathcal{L}_{v\mathcal{N}}$$

? $\nearrow$ $\searrow$?

$$\mathcal{L}_{v\mathcal{N}_1} \qquad\qquad\qquad \mathcal{L}_{v\mathcal{N}_2}$$

Figure 2.2: We cannot apply partial traces in order to go down from $\mathcal{L}_{v\mathcal{N}}$ to $\mathcal{L}_{v\mathcal{N}_1}$,and $\mathcal{L}_{v\mathcal{N}_2}$.

tures in such a way that they may be given an equal treatment as the one given to pure states. We will show that this is possible and that such structures may be defined in a natural manner, extending (in a sense explained below) $\mathcal{L}_{vN}$ so to be compatible with the physics of compounded quantum systems. Precisely, in the following section we construct a lattice $\mathcal{L}$ that has all physical states as its atoms and whose meet operation over all actual properties of a system gives the actual physical state of that system. It also includes $\mathcal{L}_{vN}$ set theoretically, so we are able to reobtain all well known results of single isolated systems.

$\mathcal{L}$ is constructed in such a way that there exist projection functions which map *all* states (atoms) of the structure corresponding to the whole system S to the corresponding states (atoms) of its subsystems S_1 and S_2. This assignation rule is compatible with the physics of the problem, i.e., it is constructed using partial traces, which are the natural functions which map states of the larger system with the states of it subsystems. Improper mixtures are put in correspondence with atoms of $\mathcal{L}$, granting that they are the most elementary properties.

There is another important feature of $\mathcal{L}$. In $\mathcal{L}_{v\mathcal{N}}$ from two given pure states, say $|\psi_1\rangle$ and $|\psi_2\rangle$, a new state $\alpha|\psi_1\rangle + \beta|\psi_2\rangle$ may be constructed; we have at hand the superposition principle. The $\vee_{\mathcal{L}_{v\mathcal{N}}}$ operation of the von Neumann lattice is directly linked to the superposition principle: starting with two rays, the $\vee_{\mathcal{L}_{v\mathcal{N}}}$ operation yields the closed subspace formed by all linear combinations of the generators of the rays. But there is another operation available, namely we can mix states, we can perform a "mixing operation" to get $p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$. There is no place for such a thing

in $\mathcal{L}_{v\mathcal{N}}$, but it may be performed in $\mathcal{L}$. The $\vee_{\mathcal{L}}$ operation reflects the fact that we can mix states, playing an analogous role to that of $\vee_{\mathcal{L}_{v\mathcal{N}}}$ in relation to the superposition principle.

To sum up, the strategy that we follow in this work is to search for structures which contain improper mixtures in such a way that they have an equal treatment as the one given to pure states. As we will see, this is possible, and such structures can be defined in a natural way, extending (in a sense explained below) $\mathcal{L}_{v\mathcal{N}}$ and in a way which is compatible with the physics of compound quantum systems. We want to avoid the fact that actual properties of the propositional system do not determine the state of the system, understood as the state of affairs which determines its physics. We think that every reasonable notion of *physical* state in a propositional system should satisfy equation 2.3.1.

As we said above, there are improper mixtures for which all yes-no tests are uncertain. But it is important to remark that this does not imply that the system has no testable properties at all. Making quantum state tomographies we can determine the state of the system. These kind of "tests" however, are of a very different nature than that of the yes-no experiments. But the only thing that we care about is that of the reality of physical process and our capability of experimentally test this reality. We search for structures which reflect this physics in a direct way.

There are other reasons for considering structures which contains improper mixtures in a same status as that of pure states. There are a lot of studies of interest which concentrate on mixtures. For example, this is the case in quantum decoherence, quantum information processing, or the independent generalizations of quantum mechanics which emphasize the convex nature of mechanics (not necessarily equivalent to "Hilbertian" QM) [55, 56, 57]. The set of interest in these studies is $\mathcal{C}$ instead of the lattice of projections. So it seems to be adequate to study structures which include improper mixtures as well as pure states in a same level of "discourse". Such structures could provide a natural framework in which we study foundational issues related to these topics with a propositional structure which include propositions formed of statistical mixtures as well as the well known propositions defined by the yes-no experiments of the von Neumann's lattice of projections.

Let us see examples of physical situations which could be captured by propositional structures based on $\mathcal{C}$. Suppose that we have a system S_1 in a given state ρ_1 and its environment S_2. Then we may state the proposition

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"the state of affairs is such that S_1 is in state ρ_1". We note that when we look the things from the point of view of the total system $S = \text{System} + \text{Environment}$, to this proposition it corresponds a convex subset of $\mathcal{C}$ (the convex set of states of S). This is because S can be in any state such that $\text{tr}_2(\rho) = \rho_1$, and this corresponds to the convex set (see section 2.7.4) $\text{tr}_2^{-1}(\{\rho_1\})$. Similarly, we obtain the convex set $\text{tr}_2^{-1}(\{\rho_1\}) \cap \text{tr}_1^{-1}(\{\rho_2\})$ for the proposition S_1 is in state ρ_1 and S_2 is in state ρ_2. These propositions represent the ignorance that we have about the actual state of the whole system. A propositional structure which includes propositions of these kind could be useful (or more natural) for the study of fields such as quantum information.

It is important to notice that propositions such as the one represented by $\text{tr}_2^{-1}(\{\rho_1\})$ above cannot be tested by yes-no experiments in general. Notwithstanding, they represent actual states of affairs, and they can certainly be tested making measures on correlations, quantum tomographies, etc.

As another example, consider the von Neumann entropy $S(\rho) = -\text{tr}(\rho \ln(\rho))$. It has the following property of concavity [14]

Proposition 1. *If $\rho = \alpha\rho_1 + (1 - \alpha)\rho_2$, $0 \leq \alpha \leq 1$, we have*

$$S(\rho) \geq \alpha S(\rho_1) + (1 - \alpha)S(\rho_2) \tag{2.3.2}$$

Now consider the proposition "the entropy of the system is greater than S_0". To such a proposition -which has a very definite physical meaning- there corresponds a convex subset of $\mathcal{C}$. This is so because if we consider the set

$$S_{\geq S_0} = \{\rho \in \mathcal{C} \mid S(\rho) \geq S_0\} \tag{2.3.3}$$

if $\rho_1, \rho_2 \in S_{\geq S_0}$, then any convex combination $\rho = \alpha\rho_1 + (1 - \alpha)\rho_2$ -due to the concavity property- will also belong to $S_{\geq S_0}$. This example shows that there are propositions with a very clear physical meaning which correspond to subsets of $\mathcal{C}$ instead of subspaces of the Hilbert space.

We summarize below the desired properties for structures that we are searching for, in order to solve the problems posed in this section:

1 All physical states are included as atoms of the the new lattice. Atoms and physical states are in one to one correspondence.

2 A state of the system will be the conjunction of all the actual properties (i.e. elements of the structure). This means that actual properties determine univocally the state of the system.

3 There exist projection functions which map *all* states (atoms) of the structure corresponding to the whole system S, to the corresponding states (atoms) of its subsystems S_1 and S_2. This assignation rule must be compatible with the physics of the problem.

4 $\mathcal{L}_{vN}$ is set theoretically included in the new structure, in order to preserve physical properties in the standard sense.

5 Given two propositions of the structure there must exist an operation which yields a proposition which expresses the fact that we can make mixtures of states.

There is a trivial example which satisfies the conditions $1-4$ listed above, namely, the set of all subsets of $\mathcal{C}$, which we denote $\mathcal{P}(\mathcal{C})$. Doted with the conjunction (set intersection) and disjunction (set union) it is a boolean lattice. If we fix an entanglement measure, consider the proposition "the system has such amount of entanglement" or given an entropy measure, we can say "'the system has so much entropy" and so on. To such propositions we can assign elements of $\mathcal{P}(\mathcal{C})$, the set of all states which satisfy those propositions. But the boolean "or" defined by the set union hides the fact that in quantum mechanics we can make superpositions of states (superposition principle, principle 1) and that we can mix states (mixing principle, principle 2). In this work we search for structures which satisfy condition 5. For that reason, the lattice formed by $\mathcal{P}(\mathcal{C})$ (from now on $\mathcal{L}_{\mathcal{B}}$) is not of interest. It expresses the almost trivial fact that we can make propositions such as "the states of $\mathcal{C}$ which make a given function to have such a value" but it hides the radical differences between QM and CM.

We can define -at least- two structures which satisfy the above list. One of them called $\mathcal{L}$ (see section 2.4 and [24]) is in close connection with the lattice of subspaces of the space of hermitian matrixes [24]. In this work it plays the role of a technical step to reach $\mathcal{L}_{\mathcal{C}}$ (section 2.6), the lattice formed by the convex subsets of $\mathcal{C}$. We show below that -alike $\mathcal{L}_{\mathcal{B}}$- the study of these structures sheds light on the study of compound quantum systems, and provide a suitable (natural) language for them, mainly because of they sort the problems possed above. They show things that $\mathcal{L}_{vN}$ hides, or in

other words, which are not expressed clearly. For example, given two pure states $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$ we can apply the "or" operation of $\mathcal{L}_{v\mathcal{N}}$, $\vee_{\mathcal{L}_{v\mathcal{N}}}$, which yields the linear (closed) spam of $|\psi_1\rangle$ and $|\psi_2\rangle$. But we can also consider the "$\vee_{\mathcal{L}_\mathcal{C}}$" operation (see section 2.6), which yields all statistical mixtures of the form $\alpha\rho_1 + (1 - \alpha)\rho_2$. This operation is different of linear combination (quantum superposition), and is related to the -non classical- mixing of states (improper mixing). This "mixing" operation cannot be represented in QL at the level of $\mathcal{L}_{v\mathcal{N}}$ itself, i.e., it is not a lattice operation, but it has to be represented at the level of statistical mixtures (measures over $\mathcal{L}_{v\mathcal{N}}$).

It is important to notice that it is not the aim of this work to replace the von Neumann lattice by these new structures, but to stress its limitations for the problem of compound systems and to define its domain of applicability. We adopt the point of view that these constructions -including $\mathcal{L}_{v\mathcal{N}}$- yield different complementary views of quantum systems. We only stress that it is necessary to consider other structures than $\mathcal{L}_{v\mathcal{N}}$ in order to sort the problems mentioned above. In the following sections, we present $\mathcal{L}$ and $\mathcal{L}_\mathcal{C}$.

2.4 The Lattice of Density Operators

In order to construct a lattice for density operators, let us consider the pair $G(\mathcal{A}) := (\mathcal{A}, \text{tr})$ where $\mathcal{A}$ is the $\mathbb{R}$-vector space of operators over $\mathcal{H}$ and tr is the usual trace operator on $B(\mathcal{H})$, which induces the scalar product $< A, B >= tr(A \cdot B)$ $(\dim(\mathcal{H}) < \infty)$. The restriction to $\mathcal{A}$ of tr, makes $\mathcal{A}$ into an $\mathbb{R}$-Euclidean vector space. With the standard $\vee$, $\wedge$ and $\neg$ operations, $G(\mathcal{A})$ is a modular, orthocomplemented, atomic and complete lattice (not distributive, hence not a Boolean algebra).

Let $\mathcal{L}_o$ be the set of subspaces:

$$\mathcal{L}_o := \{L = S \cap \mathcal{C} \,|\, S \in G(\mathcal{A})\}$$

There are a lot of subspaces $S, S_i \in G(\mathcal{A})$ such that $S \cap \mathcal{C} = S_i \cap \mathcal{C}$, so for each $L \in \mathcal{L}_o$ we may choose the subspace with the least dimension $[S]$ as the representative element:

$$[S] := \min\{\dim_\mathbb{R}(S) \,|\, L = S \cap \mathcal{C}, \, S \in G(\mathcal{A})\}$$

Let $[S] = L$, being $S \in G(\mathcal{A})$ an element of the class L, then

$$S \cap \mathcal{C} \subseteq\, < S \cap \mathcal{C} >_{\mathbb{R}}\, \subseteq S \Rightarrow S \cap \mathcal{C} \cap \mathcal{C} \subseteq\, < S \cap \mathcal{C} >_{\mathbb{R}} \cap \mathcal{C} \subseteq S \cap \mathcal{C} \Rightarrow$$

$$< S \cap \mathcal{C} > \cap \mathcal{C} = S \cap \mathcal{C}$$

So $< S \cap \mathcal{C} >$ and S are in the same class L. Note that $< S \cap \mathcal{C} >\, \subseteq S$ and if S is the subspace with the least dimension, then $< S \cap \mathcal{C} >= S$. Also note that the representative with least dimension is unique, because if we choose S' such that $S' \cap \mathcal{C} = S \cap \mathcal{C}$, then

$$S =< S \cap \mathcal{C} >=< S' \cap \mathcal{C} >= S'$$

Finally, the representative of a class L that we choose is the unique $\mathbb{R}$-subspace $S \subseteq \mathcal{A}$ such that

$$S =< S \cap \mathcal{C} >_{\mathbb{R}}$$

We call it the *good representative*. It is important to remark that in the case of infinite dimensional Hilbert spaces we cannot define good representatives in such a way.

Let us now define $\vee$, $\wedge$ and $\neg$ operations in $\mathcal{L}_o$ as:

$$(S \cap \mathcal{C}) \wedge (T \cap \mathcal{C}) =< S \cap \mathcal{C} > \cap < T \cap \mathcal{C} > \cap \mathcal{C}$$

$$(S \cap \mathcal{C}) \vee (T \cap \mathcal{C}) = (< S \cap \mathcal{C} > + < T \cap \mathcal{C} >) \cap \mathcal{C}$$

$$\neg(S \cap \mathcal{C}) =< S \cap \mathcal{C} >^{\perp} \cap \mathcal{C}$$

They are well defined for every element of the classes $[S]$ and $[T]$. It is easy to see that $\mathcal{L} =< \mathcal{L}_o, \vee, \wedge, \mathbf{0}, \mathbf{1} >$ is a complete lattice, with $\mathbf{0} = \emptyset$ represented by the class of $G(\mathcal{A})$ whose elements are disjoint with $\mathcal{C}$ and $\mathbf{1} = \mathcal{C}$, represented by the class of $\mathcal{A}$. It is an atomic lattice: the atoms of $\mathcal{L}$ are given by the intersection of rays in $G(\mathcal{A})$ and $\mathcal{C}$. They are the sets $\{\rho\}$, with ρ a density operator.

It is important to notice that with respect to the $\neg$ operation, $\mathcal{L}$ is not an orthocomplemented lattice -alike $\mathcal{L}_{v\mathcal{N}}$- because if we take $L = \{\frac{1}{N}\mathbf{1}\} =< \frac{1}{N}\mathbf{1} > \cap \mathcal{C}$, then

$$\neg(\neg L) = \neg(< \frac{1}{N}\mathbf{1} >^{\perp} \cap \mathcal{C}) = \neg \emptyset = \mathcal{C} \neq L$$

On the other hand it is easy to show that non-contradiction holds

$$L \wedge \neg L = \mathbf{0}$$

and also contraposition

$$L_1 \leq L_2 \implies \neg L_2 \leq \neg L_1$$

Proposition 2. *If* $\dim(\mathcal{H}) < \infty$, $\mathcal{L}$ *is a modular lattice.*

Proof. To prove the modular equality

$$[S] \leq [R] \Rightarrow [S] \vee ([T] \wedge [R]) = ([S] \vee [T]) \wedge [R]$$

the key point is that

$$[S] \leq [R] \Leftrightarrow S \cap \mathcal{C} \subseteq R \cap \mathcal{C} \Rightarrow S = < S \cap \mathcal{C} > \subseteq < R \cap \mathcal{C} > = R$$

So, using $S \subseteq R$, is easy to see that $(S + (T \cap R)) \cap \mathcal{C} = ((S + T) \cap R) \cap \mathcal{C}$. $\square$

Furthermore, we can prove the following:

Proposition 3. *There is a one to one correspondence between the states of the system and the atoms of $\mathcal{L}$.*

Proof. For every $\rho \in \mathcal{C}$, we have that $< \rho > \cap \, \mathcal{C} = \{\rho\}$. This is so because the only positive matrix of trace one that is a multiple of ρ is ρ itself. Then, $\{\rho\}$ is an element of $\mathcal{L}$. Suppose that there exists L such that $\mathbf{0} \leq L \leq \{\rho\}$. If $L \neq \mathbf{0}$, we can write $L = S \cap \mathcal{C}$, with S being the good representative for the class of L. $L \leq \{\rho\}$ implies that $S \subseteq < \rho >$ and thus $S = < \rho >$, so it follows that $L = \{\rho\}$. Conversely, if L is an atom of $\mathcal{L}$, take $\rho \in L$. Define $L' = < \rho > \cap \, \mathcal{C} = \{\rho\}$. It is clear that $L' \subseteq L$ and, given that $L' \neq \mathbf{0}$, we have $L' = L$. $\square$

The last proposition shows that we can represent the states of subsystems of a larger system as elements of the lattice $\mathcal{L}$ giving them a similar status as pure states, something impossible in the standard formalism of QL and one of the desiderata in searching a structure to deal with composite systems. It is a well established fact [14], that there is a lattice isomorphism between the complemented and complete lattice of faces of the convex set $\mathcal{C}$ and $\mathcal{L}_{vN}$. As desired, $\mathcal{L}_{vN}$ is included in $\mathcal{L}$ granting to represent all the good features of standard QL in the new algebra. This is a non trivial result, and it is ensured by the following proposition and its corollary:

Proposition 4. *Every face of $\mathcal{C}$ is an element of $\mathcal{L}$.*

Proof. Let $F \subseteq \mathcal{C}$ be a face. Then there exists a $\mathbb{R}$-hyperplane H inside $\{\rho \in \mathcal{A} \,|\, \mathrm{tr}(\rho) = 1\}$ such that $F = H \cap \mathcal{C}$.

Given that $H = \{l = \alpha\}$ with $\alpha \in \mathbb{R}$ and l an $\mathbb{R}$-linear form on $\mathcal{A}$, we have that:

$$F = H \cap \mathcal{C} = H \cap \mathcal{C} \cap \{\mathrm{tr} = 1\} = \{l = \alpha, \, \mathrm{tr} = 1\} \cap \mathcal{C} =$$

$$\{l = \alpha\mathrm{tr}, \, \mathrm{tr} = 1\} \cap \mathcal{C} = \{l = \alpha\mathrm{tr}\} \cap \mathcal{C} \cap \{\mathrm{tr} = 1\} = \{l - \alpha\mathrm{tr} = 0\} \cap \mathcal{C}$$

So $\{\rho \in \mathcal{A} \,|\, l(\rho) - \alpha\mathrm{tr}(\rho) = 0\} \in G(\mathcal{A})$, and then $F \in \mathcal{L}$. $\qquad\square$

So, we can naturally embed $\mathcal{L}_{vN}$ in $\mathcal{L}$ as a poset.

Corollary 1. *The complemented and complete lattice of faces of the convex set $\mathcal{C}$ is a subposet of $\mathcal{L}$.*

Proof. We have already seen that $\mathcal{L}_{vN} \subseteq \mathcal{L}$ as sets. Moreover it is easy to see that if $F_1 \leq F_2$ in $\mathcal{L}_{vN}$ then $F_1 \leq F_2$ in $\mathcal{L}$. This is so because both orders are set theory inclusions. $\qquad\square$

The previous Corollary shows that $\mathcal{L}$ and $\mathcal{L}_{vN}$ are closely connected. Let us analyze the relationship between the operations of the two lattices in order to characterize this connection. We recall that the meet of two faces is their intersection and the join is the smallest face containing both. In $\mathcal{L}_{vN}$, the meet of two subspaces is their intersection and the join is their closed linear spam.

$\wedge$: $F_1, F_2 \in \mathcal{L}_{vN}$, then $F_1 \wedge F_2$ in $\mathcal{L}_{vN}$ is the same as in $\mathcal{L}$. So the inclusion $\mathcal{L}_{vN} \subseteq \mathcal{L}$ preserves the $\wedge$-operation.

$\vee$: In general it does not preserve the $\vee$-operation. The relation between the two operations is:

$$F_1 \vee_{\mathcal{L}} F_2 \leq F_1 \vee_{\mathcal{L}_{vN}} F_2$$

$$F_1 \leq F_2 \Rightarrow F_1 \vee_{\mathcal{L}} F_2 = F_1 \vee_{\mathcal{L}_{vN}} F_2 = F_2$$

For example, if the convex set $\mathcal{C}$ is a rectangle and F_1 and F_2 are two opposite vertices then, the face-join of them is the whole rectangle, and the $\mathcal{L}$-join is the diagonal joining them.

$\neg$: In any lattice, x is a complement to y if $x \vee y = 1$ and $x \wedge y = 0$. In general the lattice of faces of a convex set is complemented, but in the case of $\mathcal{C}$ it is orthocomplemented, that is, it has a distinguished complemented face for every face $F \subset \mathcal{C}$. Given that $\mathcal{L}_{vN} \cong \mathcal{P}(\mathcal{H})$, the lattice of projectors in $\mathcal{H}$, the $\neg$-operation in $\mathcal{L}_{vN}$ is that induced from $\mathcal{P}(\mathcal{H})$. If $F \subseteq \mathcal{C}$ is a face, there exists a unique projector $P \in \mathcal{A}$ such that

$$F = \{\rho \in \mathcal{C} \mid \mathrm{tr}(P\rho) = 0\} = \{\rho \in \mathcal{C} \mid \rho \perp P\} \Rightarrow$$

$$\neg_{\mathcal{L}_{vN}} F = \{\rho \in \mathcal{C} \mid \rho \perp 1 - P\}$$

It is easy (using eigenvalues) to see that it is well defined and that $\neg F$ is again a face. Given that $F \in \mathcal{L}$, it has a good representative $F = [S]$. Then

$$\neg_{\mathcal{L}} F = S^{\perp} \cap \mathcal{C}$$

Using this, we can prove that $\neg_{\mathcal{L}} F \leq \neg_{\mathcal{L}_{vN}} F$, because:

$$\nu \in \neg_{\mathcal{L}} F \text{ then } \nu \perp \rho, \ \rho \in F$$

and, in particular,

$$\nu \perp 1 - P \text{ then } \nu \in \neg_{\mathcal{L}_{vN}} F$$

2.4.1 Quantum Interactions Enlarge the Lattice of Properties

The results of the last section show that $\mathcal{L}$ is a quite natural extension of $\mathcal{L}_{vN}$ and satisfies that improper mixtures are in a bijective correspondence with the atoms of the lattice. This feature allows this lattice to avoid the problems which appear in the standard formulation of QL posed in [8] and also discussed in section 2.3 of this work. In the new lattice, the conjunction of all actual properties yields the physical state of the system because all states are in correspondence with atoms, which are minimal elements. From the physical point of view the necessity of an extension becomes clear from the comparison between classical and quantum compound systems. When we have a single classical system, its properties are faithfully represented by the subsets of its phase space. When another classical system is added and the compound system is considered, no enrichment of the state space of the former system is needed in order to describe its properties, *even* in the presence of interactions. No matter which the interaction may be, the cartesian product of the individual phase spaces gives all is needed to represent the compound system, and the same stands for the property lattices. But the situation is quite different in quantum mechanics. This is so because if we add a new quantum system to the first one, pure states are no longer faithful in order to describe subsystems. Interactions produce non trivial correlations, which are reflected in the presence of entangled states and violation of Bell inequalities. These non trivial correlations are behind the fact that within the standard quantum logical approach, the conjunction of all actual properties does not yield the physical state of the subsystem. Thus, besides their own properties, we need information about the non trivial *correlations* that each subsystem has with other subsystems -for example, a system with the environment- that may be regarded as new elements in the structure of properties and cannot be described otherwise. For this reason an enlargement of the lattice of properties is needed to represent improper mixtures by atoms $\{\rho\}$ in $\mathcal{L}$. We will come back to this point in Subsection 2.5.2, where we study the projections from the lattice of the compound system onto the lattices of the subsystems.

2.5 The Relationship Between $\mathcal{L}$ and $\mathcal{L}_i$

Given two systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we can construct the lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ according to the procedure of section 2.4. We can also construct $\mathcal{L}$, the lattice associated to the product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. In this section we examine their mutual relations. We study some special maps between these lattices and their properties, in order to get an insight in the characterization of compound quantum systems.

2.5.1 Separable States (Going Up)

We start defining the map:

$$\Psi : \mathcal{L}_1 \times \mathcal{L}_2 \longrightarrow \mathcal{L}$$
$$(S_1 \cap \mathcal{C}_1, S_2 \cap \mathcal{C}_2) \longrightarrow S \cap \mathcal{C}$$
$$\text{where } S = (< S_1 \cap \mathcal{C}_1 > \otimes < S_2 \cap \mathcal{C}_2 >)$$

In terms of good representatives, $\Psi([S_1], [S_2]) = [S_1 \otimes S_2]$. We can prove the following:

Proposition 5. *Fixing $[U] \in \mathcal{L}_2$ then $\mathcal{L}_1$ is isomorphic (as complete lattice) to $\mathcal{L}_1 \times [U] \subseteq \mathcal{L}$. The same is true for $\mathcal{L}_2$ and an arbitrary element of $\mathcal{L}_1$.*

Proof. Let us prove it for $\mathcal{L}_1$. Let $([S], [U]) \in \mathcal{L}_1 \times [U]$ with S a good representative for $[S]$ and U for $[U]$. When we apply Ψ we obtain the proposition $[S \otimes U] \in \mathcal{L}$, then, we can consider the image under Ψ of $\mathcal{L}_1 \times [U] \subseteq \mathcal{L}_1 \times \mathcal{L}_2$:

$$\Psi(\mathcal{L}_1 \times [U]) = \{[S \otimes U] \text{ where } S \text{ is a good representative for } [S] \in \mathcal{L}_1\}$$

From this characterization it is easy to see that Ψ is injective. If $[S \otimes U] = [T \otimes U]$ (S and T are good representatives), taking partial traces (more in Subsection 2.5.2) then $[S] = [T]$.

Moreover, $\Psi(-, [U])$ is a lattice morphism: let $[S \otimes U]$, $[T \otimes U] \in \mathcal{L}$ with S and T good representatives of $[S], [T] \in \mathcal{L}_1$. The key observation is that $S \otimes U$ and $T \otimes U$ are also good representatives (taking partial traces). Then we have:

$$[S \otimes U] \wedge [T \otimes U] = [(S \cap T) \otimes U] = \Psi([S] \wedge [T], [U])$$

$$[S \otimes U] \vee [T \otimes U] = [(S \oplus T) \otimes U] = \Psi([S] \vee [T], [U])$$

This ensures that $\mathcal{L}_1$ is a sublattice of $\mathcal{L}$. The same is true for $\mathcal{L}_2$. $\qquad \square$

Notice that we can use an arbitrary atom $\rho_2 \in \mathcal{C}_2$ instead of some $[U] \in \mathcal{L}_2$ and that the application Ψ restricted to $\mathcal{L}_1 \times \rho_2$ does not preserve the $\neg$-operation. This is so, because:

$$\Psi(\neg[S], \rho_2) = [S^{\perp} \otimes \rho_2] \subset [(S \otimes \rho_2)^{\perp}] =$$

$$= \neg[S \otimes \rho_2] = \neg\Psi([S], \rho_2)$$

The inclusion holds, because if $\rho \in [S^{\perp} \otimes \rho_2]$, then $\rho = (\Sigma\lambda_i\rho_i) \otimes \rho_2 = \Sigma\lambda_i\rho_i \otimes \rho_2$, with $\rho_i \in S^{\perp}$. It is clear that all the $\rho_i \otimes \rho_2$ are orthogonal to $S \otimes \rho_2$, and then $\rho \in (S \otimes \rho_2)^{\perp}$. In general the inclusion is strict, because we can have elements of the form $\rho_1 \otimes \rho_2'$, with $\rho_1 \in S^{\perp}$ and $\rho_2' \neq \rho_2$. Then, $\rho_1 \otimes \rho_2 \in S \otimes \rho_2$, but $\rho_1 \otimes \rho_2 \notin S^{\perp} \otimes \rho_2$. This has a clear physical meaning: in fact, when the system S_1 is isolated, its lattice of properties $\mathcal{L}_1$ is equivalent to $\mathcal{L}_1 \times \rho_2$. But when we add system S_2 we can, for example, prepare the systems independently, in such a way that the state after preparation is $\rho_1 \otimes \rho_2'$ with $\rho_1 \in S^{\perp}$ and ρ_2' an arbitrary state of S_2. Then, we see that there is much more freedom in the space of all states.

Let us study now the image of Ψ. First, we note that given $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_2$, we can define the following convex tensor product:

Definition 1. $L_1 \widetilde{\otimes} L_2 := \{\sum \lambda_{ij}\rho_i^1 \otimes \rho_j^2 \mid \rho_i^1 \in L_1, \ \rho_j^2 \in L_2, \ \sum \lambda_{ij} = 1 \ and \ \lambda_{ij} \geq 0\}$

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that is formed by all possible convex combinations of tensor products of elements of L_1 and elements of L_2, and it is again a convex set.

Proposition 6. $L_1 \widetilde{\otimes} L_2 \subseteq \Psi(L_1, L_2)$

Proof. If $\rho \in L_1 \widetilde{\otimes} L_2$, then $\rho = \sum \lambda_{ij} \rho_i^1 \otimes \rho_j^2$, with $\rho_i^1 \in L_1$, $\rho_j^2 \in L_2$, $\sum \lambda_{ij} = 1$ and $\lambda_{ij} \geq 0$. For each i, j, $\rho_i^1 \otimes \rho_j^2$ is again a positive trace one operator and so belongs to $\mathcal{C}$. It belongs to $< L_1 > \otimes < L_2 >$ because of the definition of tensor product. Then, it belongs to $\Psi(L_1, L_2)$. As $\mathcal{C}$ is convex, then $\rho \in \mathcal{C}$, because it is a convex combination of elements in $\mathcal{C}$. It is a linear combination of elements of $< L_1 > \otimes < L_2 >$ also, and so it belongs to it. This proves that $\rho \in \Psi(L_1, L_2)$. $\qquad \square$

We can also prove that:

Proposition 7. *If $L \in Im(\Psi)$, then $L \cap \mathcal{S}(\mathcal{H}) \neq \emptyset$.*

Proof. $L \in Im(\Psi)$ implies that there exist L_1 and L_2 such that $L = \Psi(L_1, L_2)$. By definition $\Psi(L_1, L_2) = (S_1 \otimes S_2) \cap \mathcal{C}$, with $L_1 = S_1 \cap \mathcal{C}_1$ and $L_2 = S_2 \cap \mathcal{C}_2$. Let $\rho_1 \in L_1$ and $\rho_2 \in L_2$. Then, $\rho_1 \otimes \rho_2 \in L$. But we have also that $\rho_1 \otimes \rho_2 \in \mathcal{S}(\mathcal{H})$, and then $L \cap \mathcal{S}(\mathcal{H}) \neq \emptyset$. $\qquad \square$

From the last proposition it follows that $Im(\Psi) \subset \mathcal{L}$, because if we take a nonseparable state $\rho \in \mathcal{C}$, then $\rho \in \mathcal{L}$, but $\rho \cap \mathcal{S}(\mathcal{H}) = \emptyset$, and so, it cannot belong to $Im(\psi)$. Note that in general $L_1 \widetilde{\otimes} L_2$ is not an element of $\mathcal{L}$.

Let us compute $\mathcal{C}_1 \widetilde{\otimes} \mathcal{C}_2$. Remember that $\mathcal{C}_1 = [\mathcal{A}_1] \in \mathcal{L}_1$ and $\mathcal{C}_2 = [\mathcal{A}_2] \in \mathcal{L}_2$:

$$\mathcal{C}_1 \widetilde{\otimes} \mathcal{C}_2 = \{\sum \lambda_{ij} \rho_i^1 \otimes \rho_j^2 \,|\, \rho_i^1 \in \mathcal{C}_1, \ \rho_j^2 \in \mathcal{C}_2, \ \sum \lambda_{ij} = 1 \text{ and } \lambda_{ij} \geq 0\}$$

So, using the definition of $\mathcal{S}(\mathcal{H})$, the set of all separable states, we have

$$\mathcal{S}(\mathcal{H}) = \mathcal{C}_1 \widetilde{\otimes} \mathcal{C}_2 \tag{2.5.1}$$

We know that $\mathcal{C}_1 \widetilde{\otimes} \mathcal{C}_2 \subset \mathcal{C}$. But it does not necessarily belong to $\mathcal{L}$. We can prove also the following propositions:

Proposition 8. *Let $L \in Im(\Psi)$ and $\rho \in L$. Then, ρ is a linear combination of product states.*

Proof. Let $L \in Im(\Psi)$. Then, there exist $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_2$ such that $\Psi(L_1, L_2) = L$. If $L_1 = S_1 \cap \mathcal{C}_1$ and $L_2 = S_2 \cap \mathcal{C}_2$, with S_1 and S_2 good representatives, we have:

$$L = (S_1 \otimes S_2) \cap \mathcal{C} \Longrightarrow \rho = \sum_{i,j} \lambda_{ij} \rho_i^1 \otimes \rho_j^2$$

$\square$

Proposition 9. *Let $\rho = \rho_1 \otimes \rho_2$, with $\rho_1 \in \mathcal{C}_1$ and $\rho_2 \in \mathcal{C}_2$. Then $\{\rho\} = \Psi(\{\rho_1\}, \{\rho_2\})$ with $\{\rho_1\} \in \mathcal{L}_1$, $\{\rho_2\} \in \mathcal{L}_2$ and $\{\rho\} \in \mathcal{L}$.*

Proof. We already know that the atoms are elements of the lattices.

$$\Psi(\{\rho_1\}, \{\rho_2\}) = (<\rho_1> \otimes <\rho_2>) \cap \mathcal{C} = <\rho_1 \otimes \rho_2> \cap \mathcal{C} = \{\rho_1 \otimes \rho_2\} = \{\rho\}$$

$\square$

Proposition 10. *Let $\rho \in \mathcal{S}(\mathcal{H})$, the set of separable states. Then, there exist $L \in \mathcal{L}$, $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_2$ such that $\rho \in L$ and $L = \Psi(L_1, L_2)$.*

Proof. If $\rho \in \mathcal{S}(\mathcal{H})$, then $\rho = \sum_{ij} \lambda_{ij} \rho_i^1 \otimes \rho_j^2$, with $\sum_{ij} \lambda_{ij} = 1$ and $\lambda_{ij} \geq 0$. Consider the subspaces $S_1 = <\rho_1^1, \rho_2^1, \cdots, \rho_k^1>$ and $S_2 = <\rho_1^2, \rho_2^2, \cdots, \rho_l^2>$.

Take $L_1 = S_1 \cap \mathcal{C}_1$ and $L_2 = S_2 \cap \mathcal{C}_2$. Let us observe first that $< S_1 \cap \mathcal{C}_1 > \subseteq S_1$. We have $\rho_i^1 \in \mathcal{C}_1$ and so $< S_1 \cap \mathcal{C}_1 > = S_1$, because S_1 is generated by the set ρ_i^1. We also have that $< S_2 \cap \mathcal{C}_2 > = S_2$. Now we can compute:

$$\Psi(L_1, L_2) = (< S_1 \cap \mathcal{C}_1 > \otimes < S_2 \cap \mathcal{C}_2 >) \cap \mathcal{C} = (S_1 \otimes S_2) \cap \mathcal{C}$$

But the set $\{\rho_i^1 \otimes \rho_j^2\}$ generates $S_1 \otimes S_2$, and then, $(S_1 \otimes S_2) \cap \mathcal{C}$ is formed by all the possible convex combinations of $\{\rho_i^1 \otimes \rho_j^2\}$. This proves that $\rho \in L$. $\square$

The above propositions show that $Im(\Psi)$ encodes information related to separable states. As a general state in S is non separable, we obtain that $Im(\Psi)$ is not equal to $\mathcal{L}$. This is a reasonable result. If we interpret $\mathcal{L}_1$ and $\mathcal{L}_2$ as encoding all the information that is available for S_1 and S_2 expressed via observables of the subsystems separately, it will never be possible to reconstruct from it alone all the information about the correlations between S_1 and S_2, which is encoded in $\mathcal{L}$. This information is available only in observables of the whole system S. From $Im(\Psi)$ it is possible to recover information about separated states only. As said above, the tensor product contains more information than that of its parties, and this is directly linked to the non existence of a satisfactory theory of tensor products of orthomodular posets and lattices compatible with physics.

2.5.2 Projections Onto $\mathcal{L}_1$ and $\mathcal{L}_2$ (Going Down)

There are other maps of interest. If the whole system is in a state ρ, using partial traces we can define states for the subsystem $\rho_1 = tr_2(\rho)$ and similarly for ρ_2. Then, we can consider the maps:

$$\text{tr}_i : \mathcal{C} \longrightarrow \mathcal{C}_j \quad | \quad \rho \longrightarrow \text{tr}_i(\rho)$$

from which we can construct the induced projections:

$$\tau_i : \mathcal{L} \longrightarrow \mathcal{L}_i \quad | \quad S \cap \mathcal{C} \longrightarrow \text{tr}_i(< S \cap \mathcal{C} >) \cap \mathcal{C}_i$$

In terms of good representatives $\tau_i([S]) = [\mathrm{tr}_i(S)]$. Then we can define the product map

$$\tau : \mathcal{L} \longrightarrow \mathcal{L}_1 \times \mathcal{L}_2 \quad | \quad L \longrightarrow (\tau_1(L), \tau_2(L))$$

We can prove the following about the image of τ_i.

Proposition 11. *The functions τ_i are surjective and preserve the $\vee$-operation. They are not injective.*

Proof. Take $L_1 \in \mathcal{L}_1$. Choose an arbitrary element of $\mathcal{C}_2$, say ρ_2. Now consider the following element of $\mathcal{L}$

$$L = <L_1 \otimes \rho_2> \cap \mathcal{C}$$

It is clear that $\tau_1(L) = L_1$, because if $\rho_1 \in L_1$, then $\mathrm{tr}(\rho_1 \otimes \rho_2) = \rho_1$. So, τ_1 is surjective. On the other hand, the arbitrariness of ρ_2 implies that it is not injective. An analogous argument follows for τ_2.

Let us see that τ_i preserves the $\vee$-operation:

$$\tau_i([S] \vee [T]) = \tau_i([S \oplus T]) = [\mathrm{tr}_i(S \oplus T)] =$$

$$= [\mathrm{tr}_i(S) \oplus \mathrm{tr}_i(T)] = [\mathrm{tr}_i(S)] \vee [\mathrm{tr}_i(T)] = \tau_i([S]) \vee \tau_i([T])$$

$\square$

Let us now consider the $\wedge$-operation. Let us compute:

$$\tau_i([S] \wedge [T]) = \tau_i([S \cap T]) = [\mathrm{tr}_i(S \cap T)] \subseteq$$
$$\subseteq [\mathrm{tr}_i(S) \cap \mathrm{tr}_i(T)] = [\mathrm{tr}_i(S)] \wedge [\mathrm{tr}_i(T)] = \tau_i([S]) \wedge \tau_i([T])$$
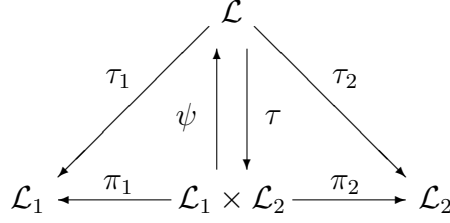
Figure 2.3: The different maps between $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 \times \mathcal{L}_2$, and $\mathcal{L}$. π_1 and π_2 are the canonical projections.

It is easy to see that $\text{tr}_i(S \cap T) \subseteq \text{tr}_i(S) \cap \text{tr}_i(T)$. This is because if $\rho \in \text{tr}_i(S \cap T)$, then $\rho = \text{tr}_i(\sigma)$, with $\sigma \in S$ and $\sigma \in T$. This means that $\rho \in \text{tr}_i(S) \cap \text{tr}_i(T)$, and so we have the inclusion of classes. But these sets are not equal in general, as the following example shows. Take $\{\rho_1 \otimes \rho_2\} \in \mathcal{L}$ and $\{\rho_1 \otimes \rho_2'\} \in \mathcal{L}$, with $\rho' \neq \rho$. It is clear that $\{\rho_1 \otimes \rho_2\} \wedge \{\rho_1 \otimes \rho_2'\} = \mathbf{0}$ and so, $\tau_1(\{\rho_1 \otimes \rho_2\} \wedge \{\rho_1 \otimes \rho_2'\}) = \mathbf{0}$. On the other hand, $\tau_1(\{\rho_1 \otimes \rho_2\}) = \{\rho_1\} = \tau_1(\{\rho_1 \otimes \rho_2'\})$, and so, $\tau_1(\{\rho_1 \otimes \rho_2\}) \wedge \tau_1(\{\rho_1 \otimes \rho_2'\}) = \{\rho_1\}$. A similar fact holds for the $\neg$-operation.

The lack of injectivity of the τ_i may be physically recognized from the fact that the state of the whole system encodes information about correlations between its parts. It is again useful to make a comparison with the classical case in order to illustrate what is happening. The same as in classical mechanics, we have atoms in $\mathcal{L}$ which are tensor products of atoms of $\mathcal{L}_1$ and $\mathcal{L}_2$. But in contrast to classical mechanics, entangled states originate atoms of $\mathcal{L}$ which cannot be expressed in such a way and thus, loosely using a topological language, we may say the fiber of the projection τ_i is much bigger than that of its classical counterpart.

It is important to note that the projection function τ cannot be properly defined within the frame of the traditional approach of QL because there was no place for improper mixtures in $\mathcal{L}_{vN}$, where they have to be defined as functions over the sublattices. On the contrary, mixtures are *elements* of the lattices $\mathcal{L}$ and $\mathcal{L}_i$, and thus we can define the projections from the lattice of the whole system to the lattices of the subsystems mapping the states of S into the corresponding states of S_i. This enables a more natural approach when compound systems are considered from a quantum logical

point of view.

It is interesting also to analyze the functions $\Psi \circ \tau$ and $\tau \circ \Psi$.

Proposition 12. $\tau \circ \Psi = Id$.

Proof. Let us see it in terms of good representatives:

$$\tau_1(\Psi([S], [T])) = \tau_1([S \otimes T]) = [\mathrm{tr}_1(S \otimes T)] = [S]$$

$$\tau_2(\Psi([S], [T])) = \tau_2([S \otimes T]) = [\mathrm{tr}_2(S \otimes T)] = [T]$$

Then $\tau(\Psi([S], [T])) = ([S], [T])$. □

It is clear, from a physical point of view, that $\Psi \circ \tau$ is not the identity function: when we take partial traces information is lost that cannot be recovered by making products of states. This can be summarized as *"going down and then going up is not the same as going up and then going down"* (another way to express quantum non-separability). We show these maps in Figure 2.5.

Let us finally make an observation about the image of Ψ. Consider the category of lattices as objects and lattice-morphisms as arrows. A bi-morphism is a morphism in each variable, and Proposition 5 ensures that Ψ is a bi-morphism. Let us define $\mathcal{I}$ as the lattice generated by $Im(\Psi)$ inside $\mathcal{L}$. Then, the following relationship holds between $\mathcal{I}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ (See Figure 2.5.2):

Proposition 13. $(\mathcal{I}, \Psi)$ *is the lattice tensor product (in categorical terms) of $\mathcal{L}_1$ and $\mathcal{L}_2$. That is, it satisfies the following universal property: for every bi-morphism of lattices $\phi : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{M}$ there exists a unique $\hat{\phi} : \mathcal{I} \to \mathcal{M}$ such that $\hat{\phi}\Psi = \phi$. And if $(\mathcal{I}', \Psi')$ is another product then they are isomorphic by a unique isomorphism.*

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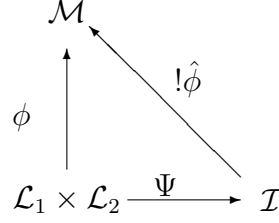
Figure 2.4: This is a commutative diagram. $(\mathcal{I}, \Psi)$ is the lattice tensor product (in categorical terms) of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof. Let $\phi : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{M}$ a bi-morphism where $\mathcal{M}$ is an arbitrary lattice. Given that $Im(\Psi)$ lattice-generates $\mathcal{I}$ we can define $\hat{\phi}$ over the elements of the form $[S \otimes T]$:

$$\hat{\phi}([S \otimes T]) := \phi([S], [T])$$

Note that it is unique by definition and $\hat{\phi}\Psi = \phi$.

The unicity of $(\mathcal{I}, \Psi)$ follows from a standard categorical argument: Given that Ψ' is a bi-morphism we have $\hat{\Psi}'\Psi = \Psi'$ because Ψ has the universal property. Given that Ψ' also has the universal property we have $Id_{\mathcal{I}'}\Psi' = \Psi'$. The same holds for Ψ, that is $\hat{\Psi}\Psi' = \Psi$ and $Id_{\mathcal{I}}\Psi = \Psi$. Note that $\hat{\Psi}', \hat{\Psi}, Id_{\mathcal{I}'}, Id_{\mathcal{I}}$ are all unique having this property. Given that $\hat{\Psi}'\hat{\Psi}\Psi' = \Psi'$ and $\hat{\Psi}\hat{\Psi}'\Psi = \Psi$ then we have:

$$\hat{\Psi}\hat{\Psi}' = Id_{\mathcal{I}} \quad \hat{\Psi}'\hat{\Psi} = Id_{\mathcal{I}'}$$

So $\mathcal{I}$ and $\mathcal{I}'$ are isomorphic by a unique isomorphism. □

2.6 The Lattice of Convex Subsets

In section 2.4 we saw that $\mathcal{L}$ is a suitable extension of $\mathcal{L}_{v\mathcal{N}}$. The elements of $\mathcal{L}$ are formed by intersections between closed subspaces and $\mathcal{C}$. Given that closed subspaces are closed sets and so is $\mathcal{C}$, they are also convex subsets of $\mathcal{C}$. We may go on further and consider all convex subsets of $\mathcal{C}$. On the other hand (because of linearity), partial trace operators preserve convexity and so they will map propositions of the system into propositions of the subsystem, as desired.

Another motivation for a further extension comes from the following analogy. If the propositions of classical mechanics are the subsets of *the set* of states (classical phase space), why cannot we consider the *convex* subsets of the *convex* set of states? It seems, after all, that convexity is an important feature of quantum mechanics (see for example [56], [57] and [55]). And as will be seen below, this "convexification" of the lattice allows for an algebraic characterization of entanglement.

Let us begin by considering the set of all convex subsets of $\mathcal{C}$:

Definition 2. $\mathcal{L}_{\mathcal{C}} := \{C \subseteq \mathcal{C} \mid C$ *is a convex subset of* $\mathcal{C}\}$

In order to give $\mathcal{L}_{\mathcal{C}}$ a lattice structure, we introduce the following operations:

Definition 3. *For all* $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}}$

$\wedge$ $C_1 \wedge C_2 := C_1 \cap C_2$

$\vee$ $C_1 \vee C_2 := conv(C_1, C_2)$. *It is again a convex set, and it is included in*

$\mathcal{C}$ *(using convexity).*

$\neg$ $\neg C := C^{\perp} \cap \mathcal{C}$

$\longrightarrow$ $C_1 \longrightarrow C_2 := C_1 \subseteq C_2$

where $conv(C_1, C_2)$ stands for the convex hull of C_1 and C_2. With the operations of definition 3, it is apparent that $(\mathcal{L}_{\mathcal{C}}; \longrightarrow)$ is a poset. If we set $\emptyset = \mathbf{0}$ and $\mathcal{C} = \mathbf{1}$, then, $(\mathcal{L}_{\mathcal{C}}; \longrightarrow; \mathbf{0}; \emptyset = \mathbf{0})$ will be a bounded poset.

Proposition 14. $(\mathcal{L}_\mathcal{C}; \longrightarrow; \wedge; \vee)$ *satisfies*

(a) $C_1 \wedge C_1 = C_1$

(b) $C_1 \wedge C_2 = C_2 \wedge C_1$

(c) $C_1 \vee C_2 = C_2 \vee C_1$

(d) $C_1 \wedge (C_2 \wedge C_3) = (C_1 \wedge C_2) \wedge C_3$

(e) $C_1 \vee (C_2 \vee C_3) = (C_1 \vee C_2) \vee C_3$

(f) $C_1 \wedge (C_1 \vee C_2) = C_1$

(g) $C_1 \vee (C_1 \wedge C_2) = C_1$

Proof. $C_1 \wedge C_1 = C_1 \cap C_1 = C_1$, so we have (a). (b), (c) and (d) are equally trivial. In order to prove e we have that

$$C_1 \vee (C_2 \vee C_3) = conv(C_1, conv(C_2, C_3))$$

Given that $conv(C_2, C_3) \subseteq conv(C_1, conv(C_2, C_3))$, then,

$$C_1, C_2, C_3 \subseteq conv(C_1, conv(C_2, C_3))$$

Using the above equation and convexity of $conv(C_1, conv(C_2, C_3))$, we have that

$$conv(C_1, C_2) \subseteq conv(C_1, conv(C_2, C_3))$$

and so, using convexity,

$$conv(conv(C_1, C_2), C_3) \subseteq conv(C_1, conv(C_2, C_3))$$

A similar argument implies the converse inclusion, and so we conclude that

$$(C_1 \vee C_2) \vee C_3 = conv(conv(C_1, C_2), C_3) = conv(C_1, conv(C_2, C_3)) = C_1 \vee (C_2 \vee C_3)$$

In order to prove (f), we have $C_1 \wedge (C_1 \vee C_2) = C_1 \cap conv(C_1, C_2)$. As $C_1 \cap conv(C_1, C_2) \subseteq C_1$ and $C_1 \subseteq conv(C_1, C_2)$, we have $C_1 = C_1 \cap conv(C_1, C_2)$, and so (f) is true. Let us finally check (g). $C_1 \vee (C_1 \wedge C_2) = conv(C_1, C_1 \cap C_2)$. This implies that $C_1, C_1 \cap C_2 \subseteq conv(C_1, C_1 \cap C_2)$. As C_1 is convex, we have $conv(C_1, C_1 \cap C_2) \subseteq C_1$, and so we have (g). $\qquad \square$

Regarding the "$\neg$" operation, if $C_1 \subseteq C_2$, then $C_2^\perp \subseteq C_1^\perp$. So $C_2^\perp \cap \mathcal{C} \subseteq C_1^\perp \cap \mathcal{C}$, and hence

$$C_1 \longrightarrow C_2 \implies \neg C_2 \longrightarrow \neg C_1 \tag{2.6.1}$$

Given that $C \cap (C^\perp \cap \mathcal{C}) = \emptyset$, we also have:

$$C \wedge (\neg C) = \mathbf{0} \tag{2.6.2}$$

and so, contraposition and non contradiction hold. But if we take the proposition $C = \{\frac{1}{N}\mathbf{1}\}$, then an easy calculation yields $\neg C = \mathbf{0}$. And then, $\neg(\neg C) = \mathbf{1}$, and thus $\neg(\neg C) \neq C$ in general. Double negation does not hold, thus, $\mathcal{L}_\mathcal{C}$ is not an ortholattice.

$\mathcal{L}_\mathcal{C}$ is a lattice which includes all convex subsets of the quantum space of states. It includes $\mathcal{L}$, and so, all quantum states (including all improper mixtures) as propositions. It is also in strong analogy with classical physics, where the lattice of propositions is formed by all measurable subsets of phase space (the space of states).

2.6.1 The Relationship Between $\mathcal{L}_{v\mathcal{N}}$, $\mathcal{L}$ and $\mathcal{L}_\mathcal{C}$

Proposition 15. $\mathcal{L}_{v\mathcal{N}} \subseteq \mathcal{L} \subseteq \mathcal{L}_\mathcal{C}$ *as posets.*

Proof. We have already seen that $\mathcal{L}_{v\mathcal{N}} \subseteq \mathcal{L}$ as sets. Moreover it is easy to see that if $F_1 \leq F_2$ in $\mathcal{L}_{v\mathcal{N}}$ then $F_1 \leq F_2$ in $\mathcal{L}$. This is so because both orders are set theoretical inclusions. Similarly, if $L_1, L_2 \in \mathcal{L}$, because intersection of convex sets yields a convex set (and closed subspaces are convex sets also), $L_1, L_2 \in \mathcal{L}_\mathcal{C}$, then we obtain set theoretical inclusion. And, again, because of both orders are set theoretical inclusions, we obtain that they are included as posets. $\qquad\square$

Regarding the $\vee$ operation, let us compare $\vee_{\mathcal{L}_{v\mathcal{N}}}$, $\vee_\mathcal{L}$ and $\vee_{\mathcal{L}_\mathcal{C}}$. If $L_1, L_2 \in \mathcal{L}$, then they are convex sets and so, $L_1, L_2 \in \mathcal{L}_\mathcal{C}$. Then we can compute

$$L_1 \vee_{\mathcal{L}_C} L_2 = conv(L_1, L_2) \tag{2.6.3}$$

On the other hand (if S_1 and S_2 are good representatives for L_1 and L_2), then:

$$L_1 \vee_\mathcal{L} L_2 = (<S_1 \cap \mathcal{C}> + <S_2 \cap \mathcal{C}>) \cap \mathcal{C} \tag{2.6.4}$$

The direct sum of the subspaces $<S_1 \cap \mathcal{C}>$ and $<S_2 \cap \mathcal{C}>$ contains as a particular case all convex combinations of elements of L_1 and L_2. So we can conclude

$$L_1 \vee_{\mathcal{L}_C} L_2 \leq L_1 \vee_\mathcal{L} L_2 \tag{2.6.5}$$

As faces of $\mathcal{C}$ can be considered as elements of $\mathcal{L}_C$ because they are convex, if F_1 and F_2 are face, we can also state

$$F_1 \vee_{\mathcal{L}_C} F_2 \leq F_1 \vee_\mathcal{L} F_2 \leq F_1 \vee_{\mathcal{L}_{v\mathcal{N}}} F_2 \tag{2.6.6}$$

Intersection of convex sets is the same as intersections of elements of $\mathcal{L}$ and so we have

$$L_1 \wedge_{\mathcal{L}_C} L_2 = L_1 \wedge_{\mathcal{L}} L_2 \tag{2.6.7}$$

and similarly

$$F_1 \vee_{\mathcal{L}_{v\mathcal{N}}} F_2 = F_1 \wedge_{\mathcal{L}_C} F_2 = F_1 \wedge_{\mathcal{L}} F_2 \tag{2.6.8}$$

What is the relationship between $\neg_{\mathcal{L}_C}$ and $\neg_{\mathcal{L}}$? Suppose that $L_1 \in \mathcal{L}$, then they are convex sets also, and so $L_1 \in \mathcal{L}_C$. Then we can compute $\neg_{\mathcal{L}_C} L_1$. We obtain:

$$\neg_{\mathcal{L}_C} L_1 = L_1^\perp \cap \mathcal{C} \tag{2.6.9}$$

On the other hand, if $L_1 = S \cap \mathcal{C}$, with S a good representative

$$\neg_{\mathcal{L}} L_1 = < S \cap \mathcal{C} >^\perp \cap \mathcal{C} \tag{2.6.10}$$

As $L_1 \subseteq < S \cap \mathcal{C} >$, then $< S \cap \mathcal{C} >^\perp \subseteq L_1^\perp$, and so

$$\neg_{\mathcal{L}} L_1 \leq \neg_{\mathcal{L}_C} L_1 \tag{2.6.11}$$

2.6.2 Interactions in QM and CM Compared

The results of sections 2.4 and 2.6 show that $\mathcal{L}$ and $\mathcal{L}_C$ are structures which satisfy conditions listed in section 2.3. Both structures extend $\mathcal{L}_{v\mathcal{N}}$ quite naturally (see proposition 15) and are adequate for the description of subsystems of a larger system because this construction takes into account improper mixtures as atoms of the lattice.

The origin of the extension of $\mathcal{L}_{v\mathcal{N}}$ becomes clear if wee make a comparison between classical and quantum compound systems. As said above, for a single classical system its properties are faithfully represented by the subsets of its phase space. When another classical system is added and the compound system is considered, no enrichment of the state space of the former system is needed in order to describe its properties, *even* in the presence of interactions. No matter which the interaction may be, the cartesian product of phase spaces is sufficient for the description of the compound system.

The situation is quite different in quantum mechanics. This is so because, if we add a new quantum system to a previously isolated one, pure states are no longer faithful in order to describe subsystems. Interactions produce non

trivial correlations, which are reflected in the presence of entangled states (and violation of Bell inequalities). Thus, besides their own properties, we need information about the non trivial *correlations* that each subsystem has with other subsystems -for example, a system with the environment- that may be regarded as new elements in the structure of properties and cannot be described otherwise.

2.7 The Relationship Between $\mathcal{L}_\mathcal{C}$ and The Tensor Product of Hilbert Spaces

In this section we study the relationship between the lattice $\mathcal{L}_\mathcal{C}$ of a system S composed of subsystems S_1 and S_2, and the lattices of its subsystems, $\mathcal{L}_{\mathcal{C}1}$ and $\mathcal{L}_{\mathcal{C}2}$ respectively. We do this by making the physical interpretation of maps which can be defined between them.

2.7.1 Separable States (Going Up)

Let us define:

Definition 4. *Given $C_1 \subseteq \mathcal{C}_1$ and $C_2 \subseteq \mathcal{C}_2$*

$$C_1 \otimes C_2 := \{\rho_1 \otimes \rho_2 \,|\, \rho_1 \in C_1, \rho_2 \in C_2\}$$

Then, we define the map:

Definition 5.

$$\Lambda : \mathcal{L}_{\mathcal{C}1} \times \mathcal{L}_{\mathcal{C}2} \longrightarrow \mathcal{L}_\mathcal{C}$$

$$(C_1, C_2) \longrightarrow conv(C_1 \otimes C_2)$$

In the rest of this work will use the following proposition (see for example [84]):

Proposition 16. *Let S be a subset of a linear space $\mathbb{L}$. Then $x \in conv(S)$ iff x is contained in a finite dimensional simplex Δ whose vertices belong to S.*

From Equation 2.5.1 and Definition 1 it should be clear that $\Lambda(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{S}(\mathcal{H})$. Definition 1 also implies that for all $C_1 \subseteq \mathcal{C}_1$ and $C_2 \subseteq \mathcal{C}_2$:

$$\Lambda(C_1, C_2) = C_1 \widetilde{\otimes} C_2 \tag{2.7.1}$$

Proposition 17. *Let $\rho = \rho_1 \otimes \rho_2$, with $\rho_1 \in \mathcal{C}_1$ and $\rho_2 \in \mathcal{C}_2$. Then $\{\rho\} = \Lambda(\{\rho_1\}, \{\rho_2\})$ with $\{\rho_1\} \in \mathcal{L}_{C1}$, $\{\rho_2\} \in \mathcal{L}_{C2}$ and $\{\rho\} \in \mathcal{C}$.*

Proof. We already know that the atoms are elements of the lattices.

$$\Lambda(\{\rho_1\}, \{\rho_2\}) = conv(\{\rho_1 \otimes \rho_2\}) = \{\rho_1 \otimes \rho_2\} = \{\rho\}$$

$\square$

Proposition 18. *Let $\rho \in \mathcal{S}(\mathcal{H})$, the set of separable states. Then, there exist $C \in \mathcal{L_C}$, $C_1 \in \mathcal{L}_{\mathcal{C}_1}$ and $C_2 \in \mathcal{L}_{\mathcal{C}_2}$ such that $\rho \in C$ and $L = \Lambda(C_1, C_2)$.*

Proof. If $\rho \in \mathcal{S}(\mathcal{H})$, then $\rho = \sum_{ij} \lambda_{ij} \rho_i^1 \otimes \rho_j^2$, with $\sum_{ij} \lambda_{ij} = 1$ and $\lambda_{ij} \geq 0$. Consider the convex sets:

$$C_1 = conv(\{\rho_1^1, \rho_2^1, \cdots, \rho_k^1\}) \quad C_2 = conv(\{\rho_1^2, \rho_2^2, \cdots, \rho_l^2\})$$

Then we define:

$$C := \Lambda(C_1, C_2) = conv(C_1 \otimes C_2)$$

Clearly, the set $\{\rho_i^1 \otimes \rho_j^2\} \subseteq C_1 \otimes C_2$, and then $\rho \in C$. $\square$

2.7.2   Projections Onto $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathcal{C}_2}$ (Going Down)

Let us now study the projections onto $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathcal{C}_2}$. From a physical point of view, it is of interest to study the partial trace operators. If the whole system is in a state ρ, using partial traces we can define states for the subsystems $\rho_1 = tr_2(\rho)$ and a similar definition for ρ_2. Then, we can consider the maps:

$$\text{tr}_i : \mathcal{C} \longrightarrow \mathcal{C}_j \quad | \quad \rho \longrightarrow \text{tr}_i(\rho) \tag{2.7.2}$$

from which we can construct the induced projections:

$$\tau_i : \mathcal{L}_{\mathcal{C}} \longrightarrow \mathcal{L}_{\mathcal{C}_i} \quad | \quad C \longrightarrow \text{tr}_i(C) \tag{2.7.3}$$

Then we can define the product map

$$\tau : \mathcal{L}_{\mathcal{C}} \longrightarrow \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} \quad | \quad C \longrightarrow (\tau_1(C), \tau_2(C)) \tag{2.7.4}$$

We use the same notation for τ and τ_i (though they are different functions) as in 2.5.2, and this should not introduce any difficulty. These maps preserve the convex structure.
We can prove the following about the image of τ_i.

Proposition 19. *The maps τ_i preserve the convex structure, i.e., they map convex sets into convex sets.*

Proof. Let $C \subseteq \mathcal{C}$ be a convex set. Let C_1 be the image of C under τ_2 (a similar argument holds for τ_1). Let us show that C_1 is convex. Let ρ_1 and ρ_1' be elements of C_1. Consider $\sigma_1 = \alpha\rho_1 + (1-\alpha)\rho_1'$, with $0 \le \alpha \le 1$. Then, there exist $\rho, \rho' \in \mathcal{C}$ such that:

$$\sigma_1 = \alpha\text{tr}_2(\rho) + (1-\alpha)\text{tr}_2(\rho') = \text{tr}_2(\alpha\rho + (1-\alpha)\rho')$$

where we have used the linearity of trace. Because of convexity of C, $\sigma :=$ $\alpha\rho + (1-\alpha)\rho' \in C$, and so, $\sigma_1 = \text{tr}_2(\sigma) \in C_1$. $\qquad \square$

Proposition 20. *The functions τ_i are surjective and preserve the $\vee$-operation.*

They are not injective.

Proof. Take the convex set $C_1 \in \mathcal{L}_{\mathcal{C}_1}$. Choose an arbitrary element of $\mathcal{C}_2$, say

ρ_2. Now consider the following element of $\mathcal{L}_\mathcal{C}$

$$C = C_1 \otimes \rho_2$$

C is convex, and so belongs to $\mathcal{L}_\mathcal{C}$, because if $\rho \otimes \rho_2, \sigma \otimes \rho_2 \in C$, then any

convex combination $\alpha\rho \otimes \rho_2 + (1 - \alpha)\sigma \otimes \rho_2 = (\alpha\rho + (1 - \alpha)\sigma)\rho_2 \in C$

(where we have used convexity of C_1). It is clear that $\tau_1(C) = C_1$, because if

$\rho_1 \in C_1$, then $\text{tr}(\rho_1 \otimes \rho_2) = \rho_1$. So, τ_1 is surjective. On the other hand, the

arbitrariness of ρ_2 implies that it is not injective. An analogous argument

follows for τ_2.

Let us see that τ_i preserves the $\vee$-operation. Let C and C' be convex subsets

of $\mathcal{C}$. We have to compute $\text{tr}_2(C \vee C') = \text{tr}_2(conv(C, C'))$. We have to show

that this is the same as $conv(\text{tr}_2(C), \text{tr}_2(C'))$. Take $x \in conv(\text{tr}_2(C), \text{tr}_2(C'))$.

Then $x = \alpha\text{tr}_2(\rho) + (1-\alpha)\text{tr}_2(\rho')$, with $\rho \in C$, $\rho' \in C'$ and $0 \leq \alpha \leq 1$. Using

linearity of trace, $x = \text{tr}_2(\alpha\rho + (1 - \alpha)\rho')$. $\alpha\rho + (1 - \alpha)\rho' \in conv(C, C')$, and

so, $x \in \text{tr}_2(conv(C, C'))$. Hence we have

$$conv(\text{tr}_2(C), \text{tr}_2(C')) \subseteq \text{tr}_2(conv(C, C'))$$

In other to prove the other inclusion, take $x \in \text{tr}_2(conv(C, C'))$. Then,
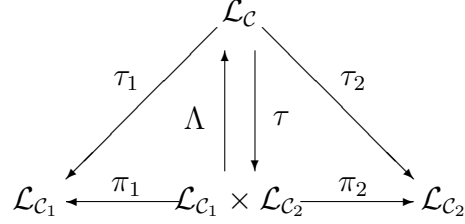
Figure 2.5: The different maps between $\mathcal{L}_{\mathcal{C}_1}$, $\mathcal{L}_{\mathcal{C}_2}$, $\mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2}$, and $\mathcal{L}_{\mathcal{C}}$

$$x = \mathrm{tr}_2(\alpha\rho + (1-\alpha)\rho') = \alpha\mathrm{tr}_2(\rho) + (1-\alpha)\mathrm{tr}_2(\rho')$$

with $\rho \in C_1$ and $\rho' \in C'$. On the other hand, $\mathrm{tr}_2(\rho) \in \mathrm{tr}_2(C)$ and $\mathrm{tr}_2(\rho') \in \mathrm{tr}_2(C')$. This proves that:

$$\mathrm{tr}_2(conv(C, C')) \subseteq conv(\mathrm{tr}_2(C), \mathrm{tr}_2(C'))$$

$\square$

Let us now consider the $\wedge$-operation. If $x \in \tau_i(C \wedge C') = \tau_i(C \cap C')$ then $x = \tau_i(\rho)$ with $\rho \in C \cap C'$. But if $\rho \in C$, then $x = \tau_i(\rho) \in \mathrm{tr}_i(C)$. As $\rho \in C'$ also, a similar argument shows that $x = \tau_i(\rho) \in \mathrm{tr}_i(C')$. Then $x \in \tau_i(C) \cap \tau_i(C')$. And so:

$$\tau_i(C \cap C') \subseteq \tau_i(C) \cap \tau_i(C')$$

which is the same as:

$$\tau_i(C \wedge C') \leq \tau_i(C) \wedge \tau_i(C')$$

But these sets are not equal in general, as the following example shows. Take $\{\rho_1 \otimes \rho_2\} \in \mathcal{L}$ and $\{\rho_1 \otimes \rho'_2\} \in \mathcal{L}$, with $\rho' \neq \rho$. It is clear that $\{\rho_1 \otimes \rho_2\} \wedge \{\rho_1 \otimes \rho'_2\} = \mathbf{0}$ and so $\tau_1(\{\rho_1 \otimes \rho_2\} \wedge \{\rho_1 \otimes \rho'_2\}) = \mathbf{0}$. On the other hand, $\tau_1(\{\rho_1 \otimes \rho_2\}) = \{\rho_1\} = \tau_1(\{\rho_1 \otimes \rho'_2\})$, and so $\tau_1(\{\rho_1 \otimes \rho_2\}) \wedge \tau_1(\{\rho_1 \otimes \rho'_2\}) = \{\rho_1\}$. A similar fact holds for the $\neg$-operation.

The last result is in strong analogy with what happens in $\mathcal{L}$, where lack of injectivity of the τ_i may be physically interpreted in the fact that the whole system has much more information than that of its parts. It is again useful to make a comparison with the classical case in order to illustrate what is happening. The same as in classical mechanics, we have atoms in $\mathcal{L}$ which are tensor products of atoms of $\mathcal{L}_1$ and $\mathcal{L}_2$. But in contrast to classical mechanics, entangled states originate atoms of $\mathcal{L}$ which cannot be expressed in such a way, and thus, the fiber of the projection τ_i is much bigger than that of its classical counterpart.

It is again an important result that the projection function τ cannot be defined properly within the frame of the traditional approaches of QL because there no place for improper mixtures in those formalisms. But in the formalism presented here they are included as *elements* of the lattices, and so we can define the projections from the lattice of the whole system to the lattices of the subsystems. This enables a more natural approach when compound systems are considered from a quantum logical point of view.

2.7.3 An Algebraic Characterization for Entanglement

We have shown that it is possible to extend $\mathcal{L}_{v\mathcal{N}}$ in order to deal with mixed states and that $\mathcal{L}$ and $\mathcal{L}_\mathcal{C}$ are possible extensions. It would be interesting to search for a characterization of entanglement within this framework. Let us see first what happens with the functions $\Lambda \circ \tau$ and $\tau \circ \Lambda$. We have:

Proposition 21. $\tau \circ \Lambda = Id.$

Proof.

$$\tau_1(\Lambda(C_1, C_2)) = \tau_1(conv(C_1 \otimes C_2)) = \mathrm{tr}_2(conv(C_1 \otimes C_2)) = C_1$$

$$\tau_2(\Lambda(C_1, C_2)) = \tau_2(conv(C_1 \otimes C_2)) = \mathrm{tr}_1(conv(C_1 \otimes C_2)) = C_2$$

Then $\tau(\Lambda(C_1, C_2)) = (C_1, C_2)$. $\qquad\qquad\square$

Again, as in [24], if we take into account physical considerations, $\Lambda \circ \tau$ is not the identity function. This is because when we take partial traces,

we face the risk of losing information which will not be recovered when we make products of states. So we obtain the same slogan as before (Section 2.5.2): *"going down and then going up is not the same as going up and then going down"*. We show these maps in Figure 2.5. How is this related to entanglement? If we restrict $\Lambda \circ \tau$ to the set of product states, then it reduces to the identity function, for if $\rho = \rho_1 \otimes \rho_2$, then:

$$\Lambda \circ \tau(\{\rho\}) = \{\rho\} \tag{2.7.5}$$

On the other hand, it should be clear that if ρ is an entangled state

$$\Lambda \circ \tau(\{\rho\}) \neq \{\rho\} \tag{2.7.6}$$

because $\Lambda \circ \tau(\{\rho\}) = \{\mathrm{Tr}_2(\rho) \otimes \mathrm{Tr}_1(\rho)\} \neq \{\rho\}$ for any entangled state. This property points in the direction of an arrow characterization of entanglement. Unfortunately, there are mixed states which are not product states, and so, entangled states are not the only ones who satisfy equation 2.7.6. What is the condition satisfied for a general mixed state? The following proposition summarizes all of this.

Proposition 22. *If ρ is a separable state, then there exists a convex set $S_\rho \subseteq \mathcal{S}(\mathcal{H})$ such that $\rho \in S_\rho$ and $\Lambda \circ \tau(S_\rho) = S_\rho$. More generally, for a convex set $C \subseteq \mathcal{S}(\mathcal{H})$, then there exists a convex set $S_C \subseteq \mathcal{S}(\mathcal{H})$ such that $\Lambda \circ \tau(S_C) = S_C$. For a product state, we can choose $S_\rho = \{\rho\}$. Any proposition $C \in \mathcal{L}_\mathcal{C}$ which has at least one non-separable state satisfies that there is no convex set S such that $C \subseteq S$ and $\Lambda \circ \tau(S) = S$.*

Proof. We have already seen above that if ρ is a product state, then $\Lambda \circ \tau(\{\rho\}) = \{\rho\}$, and so $S_\rho = \{\rho\}$. If ρ is a general separable state, then there exists $\rho_{1k} \in \mathcal{C}_1$, $\rho_{2k} \in \mathcal{C}_1$ and $\alpha_k \geq 0, \sum_{k=1}^N \alpha_k = 1$ such that $\rho = \sum_{k=1}^N \alpha_k \rho_{1k} \otimes \rho_{2k}$. Now consider the convex set (a simplex)

$$M = \{\sigma \in \mathcal{C} \,|\, \sigma = \sum_{i,j=1}^{N} \lambda_{i,j}\rho_{1i} \otimes \rho_{2j}, \lambda_{i,j} \geq 0, \sum_{i,j=1}^{N} \lambda_{i,j} = 1\} \qquad (2.7.7)$$

It is formed by all convex combinations of products of the elements which appear in the decomposition of ρ. It should be clear that $\rho \in M$. If we apply $(\mathrm{tr}_1(), \mathrm{tr}_2())$ to $\sigma \in M$, we get

$$(\mathrm{tr}_1(\sigma), \mathrm{tr}_2(\sigma)) = (\sum_{i=1}^{N}(\sum_{j=1}^{N}\lambda_{i,j})\rho_{1i}, \sum_{j=1}^{N}(\sum_{i=1}^{N}\lambda_{i,j})\rho_{2j}) = (\sum_{i=1}^{N}\mu_i\rho_{1i}, \sum_{j=1}^{N}\nu_j\rho_{2j})$$

$$(2.7.8)$$

with $\mu_i = \sum_{j=1}^{N}\lambda_{i,j}$ and $\nu_i = \sum_{i=1}^{N}\lambda_{i,j}$. Note that $\sum_{j=1}^{N}\mu_j = \sum_{j=1}^{N}\nu_j = 1$. If we now apply Λ we will obtain:

$$\Lambda((\sum_{i=1}^{N}\mu_i\rho_{1i}, \sum_{j=1}^{N}\nu_j\rho_{2j})) = \sum_{i,j=1}^{N}\mu_i\nu_j\rho_{1i} \otimes \rho_{2j} \qquad (2.7.9)$$

which is an element of M. In a similar way, we conclude that $\Lambda \circ \tau(M) \subseteq M$ (using the definition of Λ and τ). On the other hand, if $\sigma \in M$, then $\sigma = \sum_{i,j=1}^{N}\lambda_{i,j}\rho_{1i} \otimes \rho_{2j}$ (convex combination). It is important to notice that $\Lambda \circ \tau(M)$ is a convex set, because trace operators preserve convexity, and Λ is a convex hull. On the other hand $\Lambda \circ \tau(\{\rho_{1i} \otimes \rho_{2j}\}) = \{\rho_{1i} \otimes \rho_{2j}\}$ and it is also easy to see that $\{\rho_{1i} \otimes \rho_{2j}\} \subseteq \Lambda \circ \tau(M)$ for all i, j. And so, by convexity of $\Lambda \circ \tau(M)$, $\sigma \in \Lambda \circ \tau(M)$. Finally, $\Lambda \circ \tau(M) = M$ (and $\rho \in M$). Then M is the desired $S_\rho \subseteq \mathcal{S}(\mathcal{H})$.

If $C \subseteq \mathcal{S}(\mathcal{H})$, then all $\rho \in C$ are separable. $\mathcal{S}(\mathcal{H})$ is by definition, a convex set. Let us see that it is invariant under $\Lambda \circ \tau$. First of all, we know

that $\mathcal{S}(\mathcal{H})$ is formed by all possible convex combinations of products of the form $\rho_1 \otimes \rho_2$, with $\rho_1 \in \mathcal{C}_1$ and $\rho_2 \in \mathcal{C}_2$. But for each one of these tensor products, $\Lambda \circ \tau(\{\rho_1 \otimes \rho_2\}) = \{\rho_1 \otimes \rho_2\}$, and it is easy to see that they belong to $\Lambda \circ \tau(\mathcal{S}(\mathcal{H}))$. This is a convex set, thus all convex combinations of them belong to it. So we can conclude that

$$\Lambda \circ \tau(\mathcal{S}(\mathcal{H})) = \mathcal{S}(\mathcal{H}) \tag{2.7.10}$$

Now, consider $C \in \mathcal{L}_\mathcal{C}$ such that there exists $\rho \in C$, being ρ nonseparable. $\Lambda \circ \tau(S) \subseteq \mathcal{S}(\mathcal{H})$ for all $S \in \mathcal{L}_\mathcal{C}$. Then, it could never happen that there exists $S \in \mathcal{L}_\mathcal{C}$ such that $C \subseteq S$ and $\Lambda \circ \tau(S) = S$. $\qquad \square$

From the last proposition, we conclude that there is a property which the convex subsets of separable states satisfy, and convex subsets which include non-separable sates do not. This motivates the following definition.

Definition 6. *If $C \in \mathcal{L}_\mathcal{C}$, we will say that it is a separable proposition if there exists $S_C \in \mathcal{L}_\mathcal{C}$ such that $\Lambda \circ \tau(S_C) = S_C$ and $C \subseteq S_C$. Otherwise, we will say that it is a non-separable or entangled proposition.*

Another conclusion of proposition 22 is that a density matrix ρ is separable iff there exists a convex set S_ρ such that $\rho \in S_\rho$ and $\Lambda \circ \tau(S_\rho) = S_\rho$. Thus, proposition 22 also provides an entanglement criteria.

2.7.4   The Inverse τ-map

In sections 2.5.2 and 2.7.2 we defined the function $\tau = (\tau_1, \tau_2)$. Now we show that we can define lattice morphisms using the inverse map $\tau^{-1} = (\tau_1^{-1}, \tau_2^{-1})$. It is easy to show that τ_i^{-1} maps any proposition from $\mathcal{C}_i$ into a proposition of $\mathcal{C}$. This is because the pre-image of a convex set under these functions is again a convex set. If C_1 is a proposition of $\mathcal{C}$ and if $\tau_1(\rho), \tau_1(\rho') \in C_1$, it is

clear that any convex combination of ρ and ρ' will belong to $\tau_1^{-1}(C_1)$, because the partial trace is linear and C_1 is convex. Consider then the following set

$$\Omega = \{(\tau_1^{-1}(C_1), \tau_2^{-1}(C_2)) \in \mathcal{L}_\mathcal{C} \times \mathcal{L}_\mathcal{C} \mid C_1 \in \mathcal{L}_{\mathcal{C}_1} \text{ and } C_2 \in \mathcal{L}_{\mathcal{C}_2}\} \qquad (2.7.11)$$

With the elements of Ω we can construct different maps. Consider $\Upsilon :$ $\mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} \longrightarrow \mathcal{L}_\mathcal{C}$ such that $\Upsilon(C_1, C_2) = \tau_1^{-1}(C_1) \vee \tau_2^{-1}(C_2)$. We can construct also the map $\Xi : \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} \longrightarrow \mathcal{L}_\mathcal{C}$ such that $\Xi(C_1, C_2) = \tau_1^{-1}(C_1) \wedge \tau_2^{-1}(C_2)$. We can prove the following propositions:

Proposition 23. $\Xi(\mathcal{C}_1, \mathcal{C}_2) = \Upsilon(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{C}$

Proof. It is easy to show that $\tau_1^{-1}(\mathcal{C}_1) = \mathcal{C} = \tau_2^{-1}(\mathcal{C}_2)$. Then, it follows that $\Xi(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{C} \wedge \mathcal{C} = \mathcal{C}$. A similar arguments runs for Υ. $\qquad \square$

Proposition 24. *For all $X \in \mathcal{L}_C$ $X \subseteq \tau_1^{-1}(\tau_1(X))$ and for all $Y \in \mathcal{L}_{C_1}$, $\tau_1(\tau_1^{-1}(Y))$. For all $C \subseteq \mathcal{C}$ we have $C \subseteq \Xi(\tau_1(C), \tau_2(C))$*

Proof. Let $X \in \mathcal{L}_C$. Then, if $x \in X$ it follows that $\tau_1(x) \in \tau_1(X)$ and so, $X \subseteq \tau_1^{-1}(\tau_1(X))$. If $Y \in \mathcal{L}_{C_1}$ and $z \in \tau_1(\tau_1^{-1}(Y))$. Then by definition of $\tau_1^{-1}(Y)$, it follows that $z \in Y$.

Let $C \in \mathcal{L}_\mathcal{C}$. Now $\tau_1(C) = C_1 \in \mathcal{L}_{\mathcal{C}_1}$ and $\tau_2(C) = C_2 \in \mathcal{L}_{\mathcal{C}_2}$. Then, it is apparent that $C \subseteq \tau_1^{-1}(C_1)$ and $C \subseteq \tau_2^{-1}(C_2)$. And so $C \subseteq \tau_1^{-1}(C_1) \wedge \tau_2^{-1}(C_2) = \Xi(C_1, C_2)$. $\qquad \square$

Proposition 25. *For all $a, b \in \mathcal{L}_{\mathcal{C}_1}$ $\tau_1^{-1}(a \wedge b) = \tau_1^{-1}(a) \wedge \tau_1^{-1}(b)$, $\tau_1^{-1}(a \vee b) = \tau_1^{-1}(a) \vee \tau_1^{-1}(b)$. Furthermore, τ_1^{-1} is an injective function and if $a, b \in \mathcal{L}_{\mathcal{C}_1}$ and $a \subseteq b$, then $\tau_1^{-1}(a) \subseteq \tau_1^{-1}(b)$. If $\rho \neq \rho'$ then $\tau_1^{-1}(\rho) \wedge \tau_1^{-1}(\rho') = \mathbf{0}$.*

Proof. Consider the sets $\tau_1^{-1}(a \wedge b)$ and $\tau_1^{-1}(a) \wedge \tau_1^{-1}(b)$. Then, $x \in \tau_1^{-1}(a)$ and $x \in \tau_1^{-1}(b)$. If $x \in \tau_1^{-1}(a \wedge b)$, then $\tau_1(x) \in a \wedge b \subseteq a$, and we obtain also $\tau_1(x) \in a \wedge b \subseteq b$. This means that $x \in \tau_1^{-1}(a)$ and $x \in \tau_1^{-1}(b)$. So we have $\tau_1^{-1}(a \wedge b) \subseteq \tau_1^{-1}(a) \wedge \tau_1^{-1}(b)$. On the other hand, if $x \in \tau_1^{-1}(a) \wedge \tau_1^{-1}(b)$, then $x \in \tau_1^{-1}(a)$ and $x \in \tau_1^{-1}(b)$. This means that $\tau_1(x) \in a$ and $\tau_1(x) \in b$, and so, $\tau_1(x) \in a \wedge b$. This means that $x \in \tau_1^{-1}(a \wedge b)$. This concludes the proof that $\tau_1^{-1}(a \wedge b) = \tau_1^{-1}(a) \wedge \tau_1^{-1}(b)$.

If $x \in \tau_1^{-1}(a) \vee \tau_1^{-1}(b)$ then $x = \alpha\rho + \beta\rho'$, with $\tau_1(\rho) \in a$ and $\tau_1(\rho') \in b$. So $\tau_1(x) = \alpha\tau_1(\rho) + \beta\tau_1(\rho') \in a \vee b$. This means that $x \in \tau_1^{-1}(a \vee b)$, and we have $\tau_1^{-1}(a \vee b) \supseteq \tau_1^{-1}(a) \vee \tau_1^{-1}(b)$. Now, let $x \in \tau_1^{-1}(a \vee b)$. Then, $\tau_1(x) \in a \vee b$. This means that $\tau_1(x) = \alpha\rho + \beta\rho'$ (convex combination), with $\rho \in a$ and $\rho' \in b$. There exist $\sigma \in \tau_1^{-1}(a)$ and $\sigma' \in \tau_1^{-1}(b)$ such that $\tau_1(\sigma) = \rho$ and $\tau_1(\sigma') = \rho'$. Then $\tau_1(x) = \alpha\tau_1(\sigma) + \beta\tau_1(\sigma')$. $\tau_1()$ is a linear function so, the last equality implies $\tau_1(x - (\alpha\sigma + \beta\sigma')) = 0$. Then, there exists $\varsigma \in \mathbf{Ker}(\tau_1())$ such that $x = \alpha\sigma + \beta\sigma' + \varsigma$. If $\beta = 0$, then $\alpha = 1$ (convex combination), and then, $x = \sigma \in \tau_1^{-1}(a)$, and in that case $x \in \tau_1^{-1}(a) \vee \tau_1^{-1}(b)$. If $\beta \neq 0$, we can put $x = \alpha\sigma + \beta(\sigma' + \frac{1}{\beta}\varsigma)$. $\tau_1((\sigma' + \frac{1}{\beta}\varsigma)) = \tau_1(\sigma') + 0 \in b$, and so $\sigma' + \frac{1}{\beta}\varsigma \in \tau_1^{-1}(b)$. This proves that $x \in \tau_1^{-1}(a) \vee \tau_1^{-1}(b)$, and thus $\tau_1^{-1}(a \vee b) \subseteq \tau_1^{-1}(a) \vee \tau_1^{-1}(b)$

Let a and b be two propositions such that $a \neq b$. Suppose that $\tau_1^{-1}(a) = \tau_1^{-1}(b)$. If $a \neq b$, there exists $\rho_a \in a$ such that $\rho_a \notin b$. It is clear that $\tau_1^{-1}(\rho_a) \subseteq \tau_1^{-1}(a) = \tau_1^{-1}(b)$ and then, there exists $\rho \in \tau_1^{-1}(b)$ such that $\tau_1(\rho) = \rho_a$. But by definition of $\tau_1^{-1}(b)$, we would have that $\rho_a \in b$, a contradiction. Thus, we have $\tau_1^{-1}(a) \neq \tau_1^{-1}(b)$. If $a \subseteq b$, suppose that

$x \in \tau_1^{-1}(a)$. Then $\tau_1(x) \in b$, and so $x \in \tau_1^{-1}(b)$ also. If $x \in \tau_1^{-1}(\rho)$, $x \in \tau_1^{-1}(\rho')$ and $\rho \neq \rho'$, then $\rho = \tau_1(x) = \rho'$, a contradiction. $\qquad\square$

Chapter 3

Quantum Indistinguishability

In this chapter we briefly review the subject of quantum indistinguishability using algebraic tools. We concentrate on two problems. First, we show how a formulation of (non-relativistic) quantum mechanics can be done using Quasi-set Theory (Q). Our aim is to take quite seriously Heinz Post's claim that the non-individuality and the indiscernibility of quantum objects should be introduced *right at the start*, and not made a posteriori by introducing symmetry conditions [67]. Using a different mathematical framework, namely, Q, we avoid working within a label-tensor-product-Hilbert-space-formalism (LTPHSF), to use Redhead and Teller's words [71, 72], and get a more intuitive way of dealing with the formalism of quantum mechanics, although the underlying logic should be modified. We build a vector space with inner product, the Q-space, using the non-classical part of quasi-set theory, to deal with indistinguishable elements. Vectors in Q-space refer only to occupation numbers and permutation operators act as the identity operator on them, reflecting in the formalism the fact of unobservability of permutations. After that, we apply some of the constructions developed in section 2.6 for the case of indistinguishable particles. Doing so, we show that it is possible to develop a lattice theoretical formalism for the indistinguishable particle case, a subject which was not widely studied in the literature (though see for example [35]).

3.1   Quasi-set theory

We recall here some notions of quasi-set theory that will play an important role in what follows (for further details, see [31, Chap. 7]). We shall not present all the postulates and definitions of the theory, but just review the main ideas and results which interest us here. Intuitively speaking a quasi-set is a collection of indistinguishable (but not identical) objects. This of course is not a strict "definition" of a quasi-set, acting more or less as Cantor's "definition" of a set as "any collection into a whole M of definite and separate [that is, distinguishable] objects m of our intuition or our thought" (see the discussion in [31, §6.4]), giving no more than an intuitive account of the concept.

The Quasi-set theory (Q in the following) was conceived to handle collections of indistinguishable objects, and was motivated by some considerations taken from quantum physics, mainly in what respects Schrödinger's idea that the concept of identity would not be applied to elementary particles [76, pp. 17-18] (see also [31] and [54]). Of course the theory can be developed independently of any formulation of quantum mechanics, but here we shall have this motivation always in mind. The way in which Q deals with indistinguishability is by assuming that expressions like $x = y$ are not always well formed. This is expressed by saying that the concept of identity *does not apply* to the entities denoted by x and y when they "refer" to indistinguishable quantum objects. Due to the lack of sense in applying the concept of identity to certain elements, informally, a quasi-set (qset), that is, a collection involving such objects, may be such that its elements cannot be identified by names, counted, ordered, although there is a sense in saying that these collections have a cardinal (not defined by means of ordinals, as usual [37] –but see below). But Q is constructed so to keep standard mathematics intact. The theory is developed in a way that ZFU (and hence ZF [37], perhaps with the axiom of choice, ZFC) is a subtheory of Q. In other words, the theory is constructed so that it extends standard Zermelo-Frenkel with *Urelemente* (ZFU) set theory [31]; thus standard sets (of ZFU) can be viewed as particular qsets (that is, there are qsets that have all the properties of the sets of ZFU; the objects in Q corresponding to the *Urelemente* of ZFU are termed M-atoms). These objects will be called Q-sets, or just *sets* when there will be no confusion. But quasi-set theory encompasses another kind of *Urelemente*, the m-atoms, to which the standard theory of identity does not apply (that is, expressions like $x = y$ are not well formed if either x or y denote

m-atoms). Thus, we can say that Q-sets are qsets whose transitive closure [37] (defined as usual) does not contain m-atoms (in other words, they are "constructed" in the "classical" part of the theory –see Fig. 1).
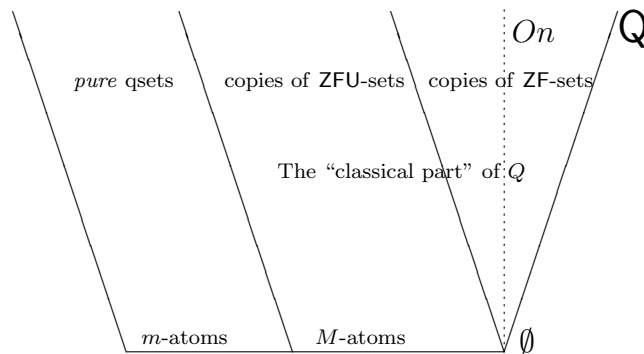

Figure 3.1: The quasi-set universe: Q is a "model" of Q.

When Q is used in connection with quantum physics, these m-atoms are thought of as representing quantum objects (henceforth, *q-objects*). The q-objets can represent 'particles', waves and whatever 'objects' sharing the property of indistinguishability of point-like elementary particles. The lack of the concept of identity for the m-atoms makes them *non-individuals* in a sense, and it is mainly (but not only) to deal with collections of m-atoms that the theory was conceived. So, Q is a theory of generalized collections of objects, involving non-individuals. For details about Q and about its historical motivations, see [31, Chap. 7].

In order to distinguish between Q-sets and q-sets that have m-atoms in their transitive closure, we write (in the metalanguage) $\{x : \varphi(x)\}$ for the former and $[x : \varphi(x)]$ for the latter. In Q, the so called 'pure' qsets have only q-objects as elements (although these elements may be not always indistinguishable from one another), and to them it is assumed that the usual notion of identity cannot be applied (that is, let us recall, $x = y$, so as its negation, $x \neq y$, are not well formed formulas if either x or y stand for q-objects). Notwithstanding, there is a primitive relation $\equiv$ of indistinguishability hav-

ing the properties of an equivalence relation, and a concept of *extensional identity*, not holding among m-atoms, is defined and has the properties of standard identity of classical set theories. More precisely, we write $x =_E y$ (x and y are extensionally identical) iff they are both qsets having the same elements (that is, $\forall z(z \in x \longleftrightarrow z \in y)$) or they are both M-atoms and belong to the same qsets (that is, $\forall z(x \in z \longleftrightarrow y \in z)$). From now on, we shall use the symbol "$=$" for the extensional equality, except when explicitly mentioned.

Since the elements of a qset may have properties (and satisfy certain formulas), they can be regarded as *indistinguishable* without turning to be *identical* (that is, being *the same* object), that is, $x \equiv y$ does not entail $x = y$. Since the relation of equality (and the concept of identity) does not apply to m-atoms, they can also be thought of as entities devoid of individuality. We remark further that if the 'property' $x = x$ (to be identical to itself, or *self-identity*, which can be defined for an object a as $I_a(x) := x = a$) is included as one of the properties of the considered objects, then the so called Principle of the Identity of Indiscernibles (PII) in the form $\forall F(F(x) \leftrightarrow F(y)) \rightarrow x = y$ is a theorem of classical second order logic, and hence there cannot be indiscernible but not identical entities (in particular, non-individuals). Thus, if self-identity is linked to the concept of non-individual, and if quantum objects are to be considered as such, these entities fail to be self-identical, and a logical framework to accommodate them is in order (see [31] for further argumentation).

3.1.1   The basic ideas of quasi-set theory

Quasi-sets are the collections obtained by applying ZFU-like (Zermelo-Frenkel plus *Urelemente*) axioms to a basic domain composed of m-atoms, M-atoms and aggregates of them. The theory still admits a primitive concept of quasi-cardinal which intuitively stands for the 'quantity' of objects in a collection. This is made so that certain quasi-sets x (in particular, those whose elements are q-objects) may have a quasi-cardinal, written $qc(x)$, but not an ordinal. It is also possible to define a translation from the language of ZFU into the language of Q in such a way so that there is a 'copy' of ZFU in Q (the 'classical' part of Q). In this copy, all the usual mathematical concepts can be defined (inclusive the concept of ordinal for the Q-sets), and (as said above) the Q-sets turn out to be those quasi-sets whose transitive closure (this con-

cept is like the usual one) does not contain m-atoms. So, we can make sense to the primitive concept of quasi-cardinal of a quasi-set x as being a cardinal defined in the 'classical' part of the theory. It is worth to mention that we have defined the quasi-cardinal for finite qsets as a derived concept [21] by modifying the axioms of Q.

To understand the basic ideas involved here, let us consider the three protons and the four neutrons in the nucleus of a ^7Li atom. As far as quantum mechanics goes, nothing distinguishes these *three* protons. If we regard these protons as forming a quasi-set, its quasi-cardinal should be 3, and there is no apparent contradiction in saying that there are also 3 subquasi-sets with 2 elements each, despite we cannot distinguish their elements, and so on. So, it is reasonable to postulate that the quasi-cardinal of the power quasi-set of x is $2^{qc(x)}$. Whether we can distinguish among these subquasi-sets is a matter which does not concern logic.

In other words, we may consistently (with the axiomatics of Q) reason as if there are three entities in our quasi-set x, but x must be regarded as a collection for which it is not possible to discern its elements as individuals. The theory does not enable us to form the corresponding singletons. The grounds for such kind of reasoning has been delineated by Dalla Chiara and Toraldo di Francia [18] as partly theoretical and partly experimental. Speaking of electrons instead of protons, they note that in the case of the helium atom we can say that there are two electrons because, *theoretically*, the appropriate wave function depends on six coordinates and thus "we can therefore say that the wave function has the same degrees of freedom as a system of two classical particles". Dalla Chiara and Toraldo di Francia have also noted that, "[e]xperimentally, we can ionize the atom (by bombardment or other means) and extract two separate electrons . . ." [18].

Of course, the electrons can be counted as two only at the moment of measurement; as soon as they interact with other electrons (in the measurement apparatus, for example) they enter into entangled states once more. It is on this basis that one can assert that there are two electrons in the helium atom or six in the 2p level of the sodium atom or (by similar considerations) three protons in the nucleus of a ^7Li atom.

An axiom of 'weak extensionality' is stated in Q, which says that those quasi-sets that have the same quantity of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are indistinguishable. This axiom has interesting consequences. As we have said, there is no space here for the details, but let us mention just

one of them which is related to the above discussion on the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta. In standard set theories (like ZF or ZFU), if $w \in x$, then of course $(x - \{w\}) \cup \{z\} = x$ iff $z = w$. That is, we can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. But in Q we can prove the following theorem, where z' (and similarly w') stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z (respectively, from w –one should not think that this element *is identical to either* z or w, for the relation of equality does not apply here; the set theoretical operations can be understood according to their usual definitions):

Theorem 3.1.1. [Unobservability of Permutations] *Let x be a finite qset such that x does not contain all indistinguishable from z, where z is an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that*

$$(x - z') \cup w' \equiv x$$

Supposing that x has n elements, then if we 'exchange' their elements z by correspondent indistinguishable elements w (set theoretically, this means performing the operation $(x - z') \cup w'$), then the resulting qset remains *indistinguishable* from the original one. In a certain sense, it is not important whether we are dealing with x or with $(x - z') \cup w'$. This of course gives a 'set-theoretical' sense to the following claim made by Roger Penrose:

> "[a]ccording to quantum mechanics, any two electrons must necessarily be completely identical [in the physicist's jargon, that is, indistinguishable], and the same holds for any two protons and for any two particles whatever, of any particular kind. This is not merely to say that there is no way of telling the particles apart; the statement is considerably stronger than that. If an electron in a person's brain were to be exchanged with an electron in a brick, then the state of the system would be *exactly the same state* as it was before, not merely indistinguishable from it! The same holds for protons and for any other kind of particle, and for the whole atoms, molecules, etc. If the entire material content of

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a person were to be exchanged with the corresponding particles in the bricks of his house then, in a strong sense, nothing would be happened whatsoever. What distinguishes the person from his house is the *pattern* of how his constituents are arranged, not the individuality of the constituents themselves" [**?**, p. 32].

Within Q we can express that 'permutations are not observable', without necessarily introducing symmetry postulates, and in particular to derive 'in a natural way' the quantum statistics (see [48], [31, Chap. 7]).

3.2 Quantity in Quantum Mechanics

As is well known, performing a single measurement in a quantum system does not allow to attribute the result of this measurement to a property which the system possesses before the measurement is performed without giving rise to serious problems [58]. What is the relationship between this fact and the quantity of particles in a quantum system? Take for example an electromagnetic field (with a single frequency for simplicity) in the following state:

$$| \psi \rangle = \alpha \, | \, 1 \rangle + \beta \, | \, 2 \rangle \tag{3.2.1}$$

where $| \, 1 \rangle$ and $| \, 2 \rangle$ are eigenvectors of the particle number operator with eigenvalues 1 and 2 respectively, and α and β are complex numbers which satisfy $| \, \alpha \, |^2 + | \, \beta \, |^2 = 1$. If a measurement of the number of particles of the system is made, one or two particles will be detected, with probabilities $| \, \alpha \, |^2$ and $| \, \beta \, |^2$ respectively. And any other possibility is excluded. Suppose that in a single measurement two particles are detected. What allows us to conclude that the system had two particles before the measurement was performed? The assertion that the number of particles is varying in time because particles are being constantly created and destroyed is also problematic, because it assumes that at each instant the number of particles is well defined. Only in case that it is known with certainty that the system is in an eigenstate of the particle number operator we can say that the system has a well defined cardinal. There would be no problem too if it is known with certainty that the system is prepared in an statistical mixture. In this case, the corresponding density operator would be:

$$\rho_m = | \, \alpha \, |^2 | \, 1 \rangle \langle 1 \, | + | \, \beta \, |^2 | \, 2 \rangle \langle 2 \, | \tag{3.2.2}$$

where the subindex "m" stands for statistical mixture. But the density operator corresponding to (3.2.1) is:

$$\rho = (\alpha \mid 1\rangle + \beta \mid 2\rangle)(\alpha^* \langle 1 \mid + \beta^* \langle 2 \mid) \qquad (3.2.3)$$

which is the same as:

$$\rho = \mid \alpha \mid^2 \mid 1\rangle\langle 1 \mid + \mid \beta \mid^2 \mid 2\rangle\langle 2 \mid + \alpha\beta^* \mid 1\rangle\langle 2 \mid + \alpha^*\beta \mid 2\rangle\langle 1 \mid \qquad (3.2.4)$$

The presence of interference terms in the last equation implies that difficulties will appear in stating that, after a single measurement, the system has the quantity of particles obtained as the result of the measurement. In this case, the incapability of knowing the particle number would not come from our ignorance about the system, but from the fact that in this state, the particle number is not even well defined.

Taking into account these considerations, it is worth asking: is it possible to represent a system prepared in the state (3.2.4) in the frame of Q? Which place would correspond to a system like (3.2.4) in that theory? If such system could be represented as a qset, then it should have an associated quasi-cardinal, for every qset has it. But this does not seem to be proper, considering what we have discussed in this section. It follows that it does not appear reasonable to assign a quasi-cardinal to every qset if Q has to include all bosonic and fermionic systems (in all their possible many particle states). Therefore, a system in the state (3.2.4) cannot be included in Q as a qset. Yet, it would be interesting to study the possibility of including systems in those states in the formalism. A possible way out is to reformulate Q in such a way that the quasi-cardinal is not to be taken as a primitive concept, but as a derived one, turning into a property that some qsets have and some others do not (in analogy with the property "being a prime number" of the integers). Those qsets for which the property of having a quasi-cardinal is not satisfied, would be suitable to represent quantum systems with particle number not defined. This property would also fit well with the position that asserts that particle interpretation is not adequate in, for example, quantum electrodynamics. With such a modification of Q, a field (in any state) could always be represented as a qset, avoiding the necessity of regarding the field as a collection of classical "things". On the contrary, the field would be described by a qset which has a defined quasi-cardinal only in special cases, but not in general.

In the following section we discuss the concept of particle number as the result of a process (the measurement process). We will discuss its relationship

with the idea of individuality suggested by experiments, and relate it with the possibility of developing quasi-cardinal as a derived concept.

3.3   Particle number as the result of a process

In the last section, we suggested that the development of a *Q*-like theory in which quasi-cardinal is a derived concept, could be useful if one aims to represent quantum systems with particle number not defined as qsets. In this section, we discuss the experimental relation of the concept of "particle number" and find new arguments for the development of quasi-cardinal as a derived concept. We start posing the question: In which sense do we talk about quantum systems composed, for example, of a single photon? We certainly know about the existence of the electromagnetic field, and that this field obeys the rules of quantum theory. How do we decide if the field is in a single photon state or not? What do we mean when we use the words "single photon"? These questions find an answer in our laboratory experience, i.e., making measurements on the system. The measurement process (which in the case of photons could be described by the theory of interaction of the electromagnetic field with matter) allows us to construct an idea of individuality which in time allows us to speak about the photon as a particle. In a similar way, and always mediated by a measuring process, we talk about the other particles (electrons, protons, etc.). But these corpuscular features of quantum systems differ notably from the classical ones, and though experiments suggest an idea of individuality, it is well established that this does not enable us to consider particles as individuals, at least not in an equal sense to classical individuality. Elementary particles cannot be considered as individuals, as E. Schrödinger pointed out in the early days of quantum mechanics [76, 77]. In spite of these difficulties, we continue speaking about photons, electrons, etc., using a jargon which has a lot of points in common with classical physics, source of conceptual confusion.

Let us consider an example to illustrate how particle number arises as a result of the measuring process. A photoelectric detector consists in its fundamental aspects of an atom that can be ionized due to the interaction with the electromagnetic field. The signal (a current originated by the ionized atom) must be amplified in order to be detected. The amplified signal is a (*macroscopical*) current, and we say that the intensity of this current is

proportional to the quantity of "absorbed photons" in the volume of the detector (in practice, composed of many atoms). In the limit of single photon states, we would observe a single current pulse each time a photon is detected. Thus, we see that once the detection mechanisms are considered, it is possible to assign to some quantum systems an associated number, which represents the "particle number". It is important to point out again that the so called "particle number" only appears, in general, after the measurement process is performed. And we have already mentioned that the measurement process almost always implies the modification of the original state, and that the result of the measurement cannot be attributed in general to a property pertaining to the system before the measure is performed. In particular, it is not true that a particle number can be always assigned in a consistent manner, as we saw in the last section. Thus, counting the quantity of elements in quantum mechanics (here understood as measuring particle number) is qualitatively different from counting the quantity of elements of a classical system. In particular, in quantum mechanics the system is usually destroyed or modified when counted, alike the classical case, where the counting process can be made in principle without disturbing the system.

Nevertheless, we know that there exist in nature systems for which it is possible to assign a cardinal in a consistent manner (a well defined number of particles, as for example the electrons of a Litium atom, or single photon states). But they cannot simply be considered as aggregates of individuals as if they were distinguishable. This is to say that there should exist the possibility of counting without distinguishing. If this were not the case, physicists would have never talked about something like "number of indistinguishable particles". Here, the word "counting" is taken in the sense of assigning in a consistent manner a "number of elements" to a system which is not so simple as an individuals aggregate. For example, we could count how many electrons has an Helium atom imagining the following process (perhaps not the best, but possible in principle). Put the atom in a cloud chamber and use radiation to ionize it. Then we would observe the tracks of both, an ion and an electron. It is obvious that the electron track represents a system of particle number equal to one and, of course, we cannot ask about the identity of the electron (because this question has no meaning in the standard interpretation of quantum mechanics), but the counting process does not depend on this query. The only thing that cares is that we are sure that the track is due to a single electron state, and for that purpose, the identity of the electron does not matter. If we ionize the atom again, we will see the track of a new

ion (of charge $2e$), and a new electron track. Which electron is responsible of the second electron track? This query is ill defined, but we still do not care. Now, the counting process has finished, for we cannot extract more electrons. The process finished in two steps, and so we say that an Helium atom has two electrons, and we know that, as the wave function of the electrons is an eigenstate of the particle number operator, no problem of consistence will arise in any other experiment if we make this assertion. In [18] Dalla Chiara and Toraldo di Francia had already noted that we know experimentally that the Helium atom has two electrons, because we can ionize it and extract two separate electrons. They were looking for experimental and theoretical grounds for developing a Quaset theory (for a comparison between Q and Quaset theory see [19]).

From the example of the ionized Helium atom, we find that the process of counting the elements of a given "collection" extracting them one by one can be applied to some quantum systems without giving rise to serious contradictions. Then, we should be able to count the Urelemente of some quasisets too. A radical difference between counting the electrons of an atom and counting the elements in a collection of classical objects in the way shown above is that, in the classical case we can ask about the identity of the extracted element at each step while, in the case of the atom, this cannot be done. But this fact, does not alter the essence of the counting process and we will exploit this fact. In the following section, we will translate this idea to the language of Q. As we have already mentioned, Q describes collections of truly indistinguishable objects as quasisets and the quasicardinal is introduced as a primitive concept. The latter is justified arguing that indistinguishability prohibits well ordering, and for that reason the possibility of counting *à la ZF*. We agree that quasisets cannot be counted in the same form as in ZF, but our point is that it should be interesting to search for other ways of counting, motivated by physical examples.

The possibility of modifying the axiomatic of Q in order to include quantum systems with undefined quasi-cardinal, was explored in [21]. The basic idea was to define a quasi-cardinal as a derived concept. For that proposal, a notion of "descendent chains" was proposed. The basic idea behind this concept is that it is possible to count the elements of a collection of electrons extracting them one by one, but never identifying them in the process. Using the notion of descendant chains, it is possible to define finite qsets for which quasi-cardinal is a derived concept. In particular, a notion of "individual qset" is developed, and it is used as a basis for developing the notion of

quantity of elements of a certain kind of given qsets. But it turns out that within that particular axiomatic variant (as in Q), there are no qsets with undefined quasi-cardinal, and so, it is not possible to represent systems in states like 3.2.4. It is an open problem to find an axiomatic variant with the characteristics mentioned above.

Yet there is another interesting possibility, which is to use the non-classical part of Q (without modifying its axioms) in order to represent sates such as 3.2.4. This possibility will be considered in the following, and as we shall see, it has interesting consequences.

3.4 The Q-space

In the standard formulation of quantum mechanics, pure states of quantum systems are represented by normalized to unit vectors in a Hilbert space. In the case of identical particles, the vectors representing their states are symmetrized or antisymmetrized, as mentioned above. In this section, we will use Q to construct a vector space, which we will call Q-*space*, in which the states are defined without labeling particles for they are represented by m-atoms. The structure of this space will result analogous to that of the Fock-space.

3.4.1 Motivation

Let us analyze with a deeper detail how quantum mechanics deals with a system of two indistinguishable particles, just to introduce some notation and to motivate our construction. Recall that the usual construction of a vector space –and of the whole formalism of quantum mechanics– makes use of set theory, which presupposes the individuality and distinguishability of the elements of any set. First the Hilbert space $\mathcal{H} = \mathcal{H}_1 \bigotimes \mathcal{H}_2$ is constructed up from the one particle spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We use Dirac notation for simplicity. Let $\{|\alpha\rangle\}$ be a basis set of $\mathcal{H}_i$. Then, $\{|\alpha\rangle \otimes |\beta\rangle\}$ is a basis for $\mathcal{H}$. α and β run over all possible values of the corresponding physical magnitudes and it is understood that the first ket corresponds to the particle labeled "1" and the second to the one labeled "2".

The scalar product of any two basis vectors is given by:

$$(\langle\alpha|\otimes\langle\beta|)(|\alpha'\rangle\otimes|\beta'\rangle) = \langle\alpha|\alpha'\rangle\langle\beta|\beta'\rangle \tag{3.4.1}$$

and, in general, the scalar product between two product vectors $|\psi\rangle\otimes|\varphi\rangle$ and $|\psi'\rangle\otimes|\varphi'\rangle$ of the product space is given by:

$$(\langle\psi|\otimes\langle\varphi|)(|\psi'\rangle\otimes|\varphi'\rangle) = \langle\psi|\psi'\rangle\langle\varphi|\varphi'\rangle. \tag{3.4.2}$$

It is worth noting that when α and β are different, $|\alpha\rangle\otimes|\beta\rangle$ will be not the same vector as $|\beta\rangle\otimes|\alpha\rangle$. Thus, in general, if $|\psi'\rangle$ and $|\varphi'\rangle$ are linear combinations of basis vectors:

$$(\langle\psi|\otimes\langle\varphi|)(|\psi'\rangle\otimes|\varphi'\rangle) \neq (\langle\psi|\otimes\langle\varphi|)(|\varphi'\rangle\otimes|\psi'\rangle) \tag{3.4.3}$$

for the general case.

We aim to develop a procedure that takes into account indistinguishability from the start, so we recall in which steps artificial labeling has occurred. First of all, one assigns vector states to each particle in their corresponding Hilbert spaces and "names" in some way these spaces to perform the tensor product. Informally, we say that we make the product of the Hilbert space of particle "1" and the Hilbert space of particle "2", and does the same thing for the resulting states. At this step particles are assumed to be distinguishable. In order to recreate indistinguishability we apply the symmetrization postulate. Then, when defining the scalar product, the differentiation of state spaces is maintained when taking brackets, the bra of particle "1" with the ket of particle "1" and the same for particle "2". Thus, there are two steps to be avoided: the use of the tensor product and this differentiation in the scalar product, which expresses itself in equation 3.4.3.

To introduce the formal construction which will be developed in the next sections, consider the possibility of a definition of a scalar product resembling the following: Let $\{|\alpha\rangle\otimes|\beta\rangle\}$ and $\{|\alpha'\rangle\otimes|\beta'\rangle\}$ be two basis vectors of the state space of the two particle system, then we could define a different scalar product "$\circ$" such that

$$(\langle\alpha|\otimes\langle\beta|)\circ(|\alpha'\rangle\otimes|\beta'\rangle) = \delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\alpha\beta'}\delta_{\beta\alpha'}. \tag{3.4.4}$$

For any two vectors $|\psi\rangle\otimes|\varphi\rangle$ and $|\psi'\rangle\otimes|\varphi'\rangle$ that are linear combinations of basis vectors, one should obtain:

$$(\langle\psi|\otimes\langle\varphi|)\circ(|\psi'\rangle\otimes|\varphi'\rangle) = \langle\psi|\psi'\rangle\langle\varphi|\varphi'\rangle + \langle\psi|\varphi'\rangle\langle\varphi|\psi'\rangle. \tag{3.4.5}$$

It is easy to verify that this product satisfies:

$$(\langle\psi|\otimes\langle\varphi|)\circ(|\psi\rangle\otimes|\varphi\rangle) = |\psi|^2|\varphi|^2 + |\langle\psi|\varphi\rangle|^2 \geq 0, \qquad (3.4.6)$$

and also:

$$(\langle\psi'|\otimes\langle\varphi'|)\circ(|\psi\rangle\otimes|\varphi\rangle) = (((\langle\psi|\otimes\langle\varphi|)\circ(|\psi'\rangle\otimes|\varphi'\rangle)))^*. \qquad (3.4.7)$$

where * stands for complex conjugation. Another possibility to be considered is a "•" such that

$$(\langle\psi|\otimes\langle\varphi|)\bullet(|\psi'\rangle\otimes|\varphi'\rangle) = \langle\psi|\psi'\rangle\langle\varphi|\varphi'\rangle - \langle\psi|\varphi'\rangle\langle\varphi|\psi'\rangle. \qquad (3.4.8)$$

This "product" clearly depends on the order of the terms, and it is defined up to a minus sign. But recall that in quantum mechanics we are interested in squared probability amplitudes and its square does not depend on the order. Furthermore, the "•" has the following interesting property:

$$(\langle\psi|\otimes\langle\psi|)\bullet(|\psi\rangle\otimes|\psi\rangle) = \langle\psi|\psi\rangle\langle\psi|\psi\rangle - \langle\psi|\psi\rangle\langle\psi|\psi\rangle = 0, \qquad (3.4.9)$$

and this will turn of great importance, because if we interpret $|\psi\rangle\otimes|\psi\rangle$ as a vector with two fermions in the same state, then it will be a vector of null norm, and thus, null probability, and also its scalar product with any other vector is zero:

$$(\langle\varphi|\otimes\langle\phi|)\bullet(|\psi\rangle\otimes|\psi\rangle) = \langle\phi|\psi\rangle\langle\varphi|\psi\rangle - \langle\phi|\psi\rangle\langle\varphi|\psi\rangle = 0 \qquad (3.4.10)$$

Moreover, using Cauchy-Schwartz inequality we have that

$$(\langle\psi|\otimes\langle\varphi|)\bullet(|\psi\rangle\otimes|\varphi\rangle) = |\psi|^2|\varphi|^2 - |\langle\psi|\varphi\rangle|^2 \geq 0, \qquad (3.4.11)$$

or

$$(\langle\psi|\otimes\langle\varphi|)\bullet(|\psi\rangle\otimes|\varphi\rangle) = -|\psi|^2|\varphi|^2 + |\langle\psi|\varphi\rangle|^2 \leq 0. \qquad (3.4.12)$$

These two possibilities come from the ambiguity in the sign when we define "•". This ambiguity will be solved later.

3.4.2   Construction of the Q-space

In the following we apply the guiding ideas discussed above to define a product in a Q-*space* constructed using the non-classical part of Q.

Quasi-functions

Let us consider a Q-set $\epsilon =_E \{\epsilon_i\}_{i \in I}$, where I is an arbitrary (denumerable) collection of indexes (remember that a Q-set is a qset from the copy of ZFU in Q). From now own, by a "set" we mean a Q-set, and "=" stands for "$=_E$", except if explicitly mentioned. We also recall that all the usual mathematical concepts mentioned below can be obtained in the "classical" part of Q. We take the elements ϵ_i to represent the eigenvalues of a physical magnitude of interest, and so, they are real numbers. To fix the ideas, they may be the energy eigenvalues of the Hamiltonian H of a (one particle) system, $H|\varphi_i\rangle = \epsilon_i|\varphi_i\rangle$, with $|\varphi_i\rangle$ being the corresponding eigenstates. The construction we present is, of course, independent of this particular choice. Consider then the quasi-functions f, $f : \epsilon \longrightarrow \mathcal{F}_p$, where $\mathcal{F}_p$ is the quasi-set formed of all finite and pure quasi-sets. f is the quasi-set formed of ordered pairs $\langle \epsilon_i; x \rangle$ with $\epsilon_i \in \epsilon$ and $x \in \mathcal{F}_p$. Let us choose these quasi-functions in such a way that whenever $\langle \epsilon_{i_k}; x \rangle$ and $\langle \epsilon_{i_{k'}}; y \rangle$ belong to f and $k \neq k'$, then $x \cap y = \emptyset$. Let us further assume that the sum of the quasi-cardinals of the quasi-sets which appear in the image of each of these quasi-functions is finite, and then, $qc(x) = 0$ for almost every x in the image of f, except for a finite number of elements of ϵ. Let us call $\mathcal{F}$ the quasi-set formed of these quasi-functions. If $\langle \epsilon_i; x \rangle$ is a pair of $f \in \mathcal{F}$, we will interpret that the energy level ϵ_i has occupation number $qc(x)$. These quasi-functions will be represented by symbols such as $f_{\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_m}}$ (or by the same symbol with permuted indexes). This indicates that the levels $\epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_m}$ are occupied. It will be taken as convention that if the symbol ϵ_{i_k} appears j-times, then the level ϵ_{i_k} has occupation number j. For example, the symbol $f_{\epsilon_1\epsilon_1\epsilon_1\epsilon_2\epsilon_3}$ means that the level ϵ_1 has occupation number 3 while the levels ϵ_2 and ϵ_3 have occupation numbers 1. We could also choose the expression $f_{3\epsilon_1\epsilon_2\epsilon_3}$, but it will not be necessary. The levels that do not appear have occupation number zero.

These quasi-functions will be used to construct quantum states. It is worth to say that, because of the utilization of pure qsets with indistinguishable elements, there is no reference to particle indexation. The only reference is to the occupation numbers, because permutations make no sense here, as it should be. Let us consider, for example, the quasi-function $f_{\epsilon_1\epsilon_1\epsilon_1\epsilon_2\epsilon_3}$. As we have said above, we interpret this as a state in which the level 1 has occupation number three, the levels 2 and 3 only one, and the others zero. Thus, a permutation of particles makes no difference because the quasi-function $f_{\epsilon_1\epsilon_1\epsilon_1\epsilon_2\epsilon_3}$ is a collection of ordered pairs. These pairs are $\langle \epsilon_1; x \rangle$, $\langle \epsilon_2; y \rangle$,

$\langle \epsilon_3; z \rangle$ and $\langle \epsilon_n; \emptyset \rangle$ (for $n > 3$), where x, y and z are pure and disjoint quasi-sets which satisfy $qc(x) = 3$ and $qc(y) = 1 = qc(z)$. Thus, permutation of two particles is formally represented by the procedure that takes an element of, say, x and interchanges it with an element of y (or z). But it is a theorem of Q that permutation of m-atoms gives place to indistinguishable quasi-sets (unobservability of permutations). By definition, we have $\langle \epsilon_1; x \rangle = [[\epsilon_1]; [\epsilon_1; x]]$. Also by definition, $[\epsilon_1; x]$ is the collection of all the indistinguishable from either ϵ_1 or x (taken from some previously given qset). For this reason, if we replace x by x', with $x \equiv x'$ we will obtain $[\epsilon_1; x] = [\epsilon_1; x']$. Thus, we obtain $\langle \epsilon_1; x \rangle = \langle \epsilon_1; x' \rangle$ and the ordered pairs of the 'permuted' quasi-function will be the same and, consequently, the new quasi-function is again $f_{\epsilon_1 \epsilon_1 \epsilon_1 \epsilon_2 \epsilon_3}$. We thus see that the permutation of indistinguishable elements does not produce changes in the quasi-functions and, then, in any vector space constructed using them, the permutation operation will be reduced to identity.

It is important to point out that the order of the indexes in a quasi-function $f_{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}}$ has no meaning at all because up to now, there is no need to define any particular order in ϵ, the domain of the quasi-functions of $\mathcal{F}$. Nevertheless, we may define an order in the following way. For each quasi-function $f \in \mathcal{F}$, let $\{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_m}\}$ be the quasi-set formed by the elements of ϵ such that $\langle \epsilon_{i_k}, X \rangle \in f$ and $qc(X) \neq 0$ ($k = 1 \ldots m$). We call $supp(f)$ this quasi-set (the *support* of f). Then consider the pair $\langle o, f \rangle$, where o is a bijective quasi-function:

$$o : \{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_m}\} \longrightarrow \{1, 2, \ldots, m\}. \tag{3.4.13}$$

Each of these quasi-functions o define an order on $supp(f)$. For each $f \in \mathcal{F}$, if $qc(supp(f)) = m$, then, there are $m!$ orderings. Then, let $\mathcal{OF}$ be the quasi-set formed by all the pairs $\langle o, f \rangle$, where $f \in \mathcal{F}$ and o is a a particular ordering in $supp(f)$. Thus, $\mathcal{OF}$ is the quasi-set formed by all the quasi-functions of $\mathcal{F}$ with ordered support. For this reason, if we now say that $f_{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}} \in \mathcal{OF}$, we will be speaking of a quasifunction $f \in \mathcal{F}$ and of an special ordering of $\{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}\}$. For the sake of simplicity, we will use the same notation as before. But now the order of the indexes *is meaningful*. It is also important to remark, that the order on the indexes must not be understood as a labeling of particles, for it easy to check that, as above, the permutation of particles does not give place to a new element of $\mathcal{OF}$. This is so because a permutation of particles operating on a pair $\langle o, f \rangle \in \mathcal{OF}$ will not change f, and so, will not alter the ordering. We will use the elements

of $\mathcal{OF}$ later, when we deal with fermions.

Vector space structure

A linear space structure is required to adequately represent quantum states. To equip $\mathcal{F}$ and $\mathcal{OF}$ with such a structure, we need to define two operations "$\star$" and "$+$", a product by scalars and an addition of their elements, respectively. We will construct a vector space starting from the quasi-functions of the quasi-sets $\mathcal{F}$ (or equivalently $\mathcal{OF}$) defined above. Call C the collection of quasi-functions which assign to every $f \in \mathcal{F}$ (or $f \in \mathcal{OF}$) a complex number. That is, a quasi-function $c \in C$ is a collection of ordered pairs $\langle f; \lambda \rangle$, where $f \in \mathcal{F}$ (or $f \in \mathcal{OF}$) and λ a complex number. Let C_0 be the subset of C such that, if $c \in C_0$, then $c(f) = 0$ for almost every $f \in \mathcal{OF}$ (i.e., $c(f) = 0$ for every $f \in \mathcal{OF}$ except for a finite number of quasi-functions). We can define in C_0 a sum and a product by scalars in the same way as it is usually done with functions as follows.

Definition 7. *Let α, β and $\gamma \in \mathcal{C}$, and c, c_1 and c_2 be quasi-functions of C_0, then*

$$(\gamma * c)(f) := \gamma(c(f)) \tag{3.4.14}$$

$$(c_1 + c_2)(f) := c_1(f) + c_2(f) \tag{3.4.15}$$

The quasi-function $c_0 \in C_0$ such that $c_0(f) = 0$, for any $f \in F$, acts as the null element of the sum, for

$$(c_0 + c)(f) = c_0(f) + c(f) = 0 + c(f) = c(f), \forall f. \tag{3.4.16}$$

With the sum and the multiplication by scalars defined above we have that $(C_0, +, *)$ is a complex vector space. Each one of the quasi-functions of C_0 should be interpreted in the following way. If $c \in C_0$ (and $c \neq c_0$), let f_1, f_2, f_3,..., f_n be the only functions of C_0 which satisfy $c(f_i) \neq 0$ ($i = 1, \ldots, n$). These quasi-functions exist because, as we have said above,

the quasi-functions of C_0 are zero except for a finite number of quasi-functions of $\mathcal{F}$. If λ_i are complex numbers which satisfy that $c(f_i) = \lambda_i$ $(i = 1, \ldots, n)$, we will make the association

$$c \approx (\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n). \tag{3.4.17}$$

The symbol $\approx$ must be understood in the sense that we use this notation to represent the quasi-function c. The idea is that the quasi-function c represents the pure state which is a linear combination of the states represented by the quasi-functions f_i according to the interpretation given above. As a particular case of this notation, we have that if $c_j \in C_0$ are the quasi-functions such that $c_j(f_i) = \delta_{ij}$ (δ_{ij} is the Kronecker symbol), then $c_j \approx f_j$ and in a similar way $\lambda * c_j \approx \lambda f_j$. In this space, the vectors c_j are the "natural" basis vectors, while the others are linear combinations of them.

Scalar products

With the aid of a vector space structure, we can express quantum superpositions. In order to calculate probabilities and mean values, we need to introduce the notion of scalar product. In the following, we will introduce two different products for bosons and fermions separately, following the ideas of Section 3.4.1. Let us do it first for bosons.

Definition 8. *Let δ_{ij} be the Kronecker symbol and $f_{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}}$ and $f_{\epsilon_{i'_1} \epsilon_{i'_2} \ldots \epsilon_{i'_m}}$*

two basis vectors which belong to $\mathcal{F}$, then

$$f_{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}} \circ f_{\epsilon_{i'_1} \epsilon_{i'_2} \ldots \epsilon_{i'_m}} := \delta_{nm} \sum_p \delta_{i_1 p i'_1} \delta_{i_2 p i'_2} \ldots \delta_{i_n p i'_n} \tag{3.4.18}$$

The sum is extended over all the permutations of the index set $i' = (i'_1, i'_2, \ldots, i'_n)$

and for each permutation p, $pi' = (pi'_1, pi'_2, \ldots, pi'_n)$.

This product can be easily extended over linear combinations:

$$\left(\sum_k \alpha_k f_k \right) \circ \left(\sum_k \alpha'_k f'_k \right) := \sum_{kj} \alpha_k^* \alpha'_j (f_k \circ f'_j) \tag{3.4.19}$$

On the other hand, we can consider another "$\bullet$" product as follows, which will be adequate for fermions:

Definition 9. *Let δ_{ij} be the Kronecker symbol and $f_{\epsilon_{i_1}\epsilon_{i_2}...\epsilon_{i_n}}$ and $f_{\epsilon_{i'_1}\epsilon_{i'_2}...\epsilon_{i'_m}}$ two basis vectors which belong to $\mathcal{OF}$, then*

$$f_{\epsilon_{i_1}\epsilon_{i_2}...\epsilon_{i_n}} \bullet f_{\epsilon_{i'_1}\epsilon_{i'_2}...\epsilon_{i'_m}} := \delta_{nm} \sum_p \sigma_p \delta_{i_1 p i'_1} \delta_{i_2 p i'_2} \ldots \delta_{i_n p i'_n} \qquad (3.4.20)$$

where:

$$\sigma_p = \begin{cases} 1 & \text{if } p \text{ is even,} \\ -1 & \text{if } p \text{ is odd.} \end{cases}$$

The result of this product is an antisymmetric sum of the indexes which appear in the quasi-functions. In order that the product is well defined, the quasi-functions must belong to $\mathcal{OF}$. Once this product is defined over the basis functions, we can extend it to linear combinations, in a similar way as in (3.4.19). If the occupation number of a product is more or equal than two, then the vector has null norm, as in equations 3.4.9 and 3.4.10. As it is a vector of null norm, the product of this vector with any other vector of the space would yield zero, and thus the probability of observing a system in a state like this vanishes. This means that we can add to any physical state an arbitrary linear combination of null norm vectors for they do not contribute to the scalar product, which is the meaningful quantity.

We have defined two products, "∘" and "•", which are adequate for bosons and fermions, respectively. We will return to this point in the following section.

We point out that to formulate quantum mechanics in such a way that no reference to particle individuality is made, we need to avoid labeling in state vectors as much as in operators representing observable quantities. As said above, in the Fock-space formalism, the observables can be written in terms of creation and annihilation operators, avoiding particle labeling. In the following section we will introduce creation and annihilation operators in order to express observable quantities, without making appeal to particle labeling in the operators themselves. We will review Fock-space formalism also.

3.5 The construction of quantum mechanics using Q-spaces

In this section we will first briefly review for completeness the formulation of quantum mechanics using the Fock-space formalism. After that, we will rewrite that formulation using the language of the Q-space developed above.

3.5.1 Fock-space formalism

As is well known, the standard formulation of quantum mechanics and the Fock-space formulation are deeply connected. Equivalence with wave mechanics is studied (for example) in [73]. It is important to remark that all the mathematics of this section is contained in the classical part of Q. Here we briefly recall some basic notions of the standard formalism to fix notation. We call T_1 the kinetic energy of a single particle and V_1 the external potential acting on it. For n particles we have:

$$T_n := \sum_{i=1}^{n} T_1(r_i) \qquad (3.5.1)$$

and the same for the external potential. We represent by

$$V_n := \sum_{i>j=1}^{n} V_2(\mathbf{r}_i, \mathbf{r}_j) \qquad (3.5.2)$$

the pairwise interaction potential. Thus, the total hamiltonian operator is given by

$$H_n = \sum_{i=1}^{n} [T_1(\mathbf{r}_i) + V_1(\mathbf{r}_i) + \sum_{i>j=1}^{n} V_2(\mathbf{r}_i, \mathbf{r}_j)] \qquad (3.5.3)$$

The n-particles wave function is written as

$$\Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) \qquad (3.5.4)$$

The standard Fock-space is built up from the one particle Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space and define:

$$
\begin{aligned}
\mathcal{H}^0 &= \mathcal{C} \\
\mathcal{H}^1 &= \mathcal{H} \\
\mathcal{H}^2 &= \mathcal{H} \otimes \mathcal{H} \\
&\vdots \\
\mathcal{H}^n &= \mathcal{H} \otimes \cdots \otimes \mathcal{H}
\end{aligned}
\tag{3.5.5}
$$

If no symmetry condition is required for the states, the Fock-space is constructed as the direct sum of n particles Hilbert spaces:

$$
\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n
\tag{3.5.6}
$$

When dealing with bosons or fermions, the standard procedure to obtain the physical space state is as follows. Given a vector $v = v_1 \otimes \cdots \otimes v_n \in \mathcal{H}^n$, define:

$$
\sigma^n(v) = \left(\frac{1}{n!}\right) \sum_P P(v_1 \otimes \cdots \otimes v_n)
\tag{3.5.7}
$$

and:

$$
\tau^n(v) = \left(\frac{1}{n!}\right) \sum_P s^p P(v_1 \otimes \cdots \otimes v_n)
\tag{3.5.8}
$$

where:

$$
s^p = \left\{
\begin{array}{ll}
1 & \text{if p is even,} \\
-1 & \text{if p is odd.}
\end{array}
\right.
$$

It is important to realize that in this construction, particles are labeled and then symmetry conditions are imposed by state symmetrization. Thus, indistinguishability is not taken from the beginning. Calling

$$
\mathcal{H}_\sigma^n = \{\sigma^n(v) : v \in \mathcal{H}^n\}
\tag{3.5.9}
$$

and:

$$
\mathcal{H}_\tau^n = \{\tau^n(v) : v \in \mathcal{H}^n\}
\tag{3.5.10}
$$

we have the Fock-space

$$
\mathcal{F}_\sigma = \bigoplus_{n=0}^{\infty} \mathcal{H}_\sigma^n
\tag{3.5.11}
$$

for bosons and

$$\mathcal{F}_\tau = \bigoplus_{n=0}^{\infty} \mathcal{H}_\tau^n \tag{3.5.12}$$

for fermions. Once each Fock-space is constructed, the usual procedure runs as follows. Let $\psi(\mathbf{r})$ and its hermitian conjugate $\psi(\mathbf{r})^\dagger$ be operators acting on the Fock-space and satisfying:

$$[\psi(\mathbf{r}), \psi(\mathbf{r}')]_\mp = 0$$
$$[\psi(\mathbf{r})^\dagger, \psi(\mathbf{r}')^\dagger]_\mp = 0$$
$$[\psi(\mathbf{r}), \psi(\mathbf{r}')^\dagger]_\mp = \delta_{\mathbf{r}-\mathbf{r}'} \tag{3.5.13}$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function. For any operators A and B, the brackets are defined by:

$$[A, B]_\mp := AB \mp BA \tag{3.5.14}$$

Corresponding to the n particle wave function $\Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ of the standard formulation, the n particle state in the Fock-space is defined:

$$|\psi_n\rangle := (n!)^{-\frac{1}{2}} \int d^3 r_1 \cdots \int d^3 r_n \psi(\mathbf{r}_1)^\dagger \cdots \psi(\mathbf{r}_n)^\dagger |0\rangle \Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) \tag{3.5.15}$$

which can be shown to be an eigenvector (with eigenvalue n) of the particle number operator:

$$N := \int d^3 r \psi(\mathbf{r})^\dagger \psi(\mathbf{r}) \tag{3.5.16}$$

The connection between the two representations is given by:

$$\Psi_n(\mathbf{r}_1, \cdots, \mathbf{r}_n) = (n!)^{-\frac{1}{2}} \langle 0|\psi(\mathbf{r}_1) \cdots \psi(\mathbf{r}_n)|\Psi_n\rangle \tag{3.5.17}$$

In general, an arbitrary vector of the Fock-space:

$$|\Psi\rangle = \sum_{n=0}^{\infty} |\Psi_n\rangle \tag{3.5.18}$$

will not be an eigenstate of the particle number operator (and so, it has undefined particle number).

Corresponding to the kinetic energy operator of standard wave mechanics, an operator in the Fock-space is defined as:

$$T := \int d^3 r \psi^\dagger(\mathbf{r}) T_1(r) \psi(\mathbf{r}) \tag{3.5.19}$$

and it is easy to see that:

$$T^\dagger = T \tag{3.5.20}$$

It can also be shown that:

$$T|\Psi_n\rangle = (n!)^{-\frac{1}{2}} \int d^3r_1 \cdots \int d^3r_n \Psi^\dagger(\mathbf{r}_1) \cdots \Psi^\dagger(\mathbf{r}_n)|0\rangle \sum_{i=1}^{n} T_1(\mathbf{r}_i)\Psi_n(\mathbf{r}_1, \cdots, \mathbf{r}_n) \tag{3.5.21}$$

Analogously, if there is a pairwise interaction potential $V_2(\mathbf{r}, \mathbf{r}')$, the operator:

$$V := \frac{1}{2} \int d^3r \int d^3r' \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')V_2(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \tag{3.5.22}$$

is defined on the Fock-space. Its action on $|\Psi_n\rangle$ is given by:

$$V|\Psi_n\rangle = (n!)^{-\frac{1}{2}} \int d^3r_1 \cdots \int d^3r_1 [V, \psi^\dagger(\mathbf{r}_n) \cdots \psi^\dagger(\mathbf{r}_1)]|0\rangle \Psi_n(\mathbf{r}_1 \ldots \mathbf{r}_n) \tag{3.5.23}$$

and it follows that:

$$
\begin{aligned}
V|\Psi_n\rangle &= (n!)^{-\frac{1}{2}} \int d^3r_1 \cdots \int d^3r_1 \psi^\dagger(\mathbf{r}_n) \cdots \psi^\dagger(\mathbf{r}_1)|0\rangle \\
&\quad \times \sum_{i=1}^{n}\sum_{j=1}^{i-1} V_2(\mathbf{r}_i, \mathbf{r}_j)\Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_n)
\end{aligned} \tag{3.5.24}
$$

It can be shown that that the following equations holds:

$$T_n\Psi_n(\mathbf{r}_1, \cdots, \mathbf{r}_n) = (n!)^{-\frac{1}{2}}\langle 0|\Psi(\mathbf{r}_1) \cdots \Psi(\mathbf{r}_n)T|\Psi_n\rangle \tag{3.5.25}$$

$$V_n\Psi_n(\mathbf{r}_1, \cdots, \mathbf{r}_n) = (n!)^{-\frac{1}{2}}\langle 0|\Psi(\mathbf{r}_1) \cdots \Psi(\mathbf{r}_n)V|\Psi_n\rangle \tag{3.5.26}$$

where:

$$T_n := \sum_{i=1}^{n} T_1(\mathbf{r}_i) \tag{3.5.27}$$

$$V_n := \sum_{i>j=1}^{n} V_2(\mathbf{r}_i, \mathbf{r}_j) \tag{3.5.28}$$

The equivalence with wave mechanics can now be established as follows. If $\Psi_n(\mathbf{r}_1, \cdots, \mathbf{r}_n)$ satisfies the n particle Schrödinger wave equation with Hamiltonian (3.5.3), it follows that in the Fock-space formulation $|\Psi_n\rangle$ must satisfy the Fock-space Schrödinger equation:

$$[i\hbar(\frac{\partial}{\partial t}) - H]|\Psi_n\rangle = 0 \tag{3.5.29}$$

with $H := T + V$: given by:

$$
\begin{aligned}
H &= \int d^3r \psi^\dagger(\mathbf{r})[T_1 + V_1(\mathbf{r})]\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3r \int d^3r' \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')V_2(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \qquad (3.5.30)
\end{aligned}
$$

It is important to remark that the n particle Schrödinger wave equation is not completely equivalent to its analogue in the Fock-space formalism. Only solutions of the Fock-space equation which are eigenvectors of the particle number operator with particle number n can be solutions of the corresponding n particle Schrödinger wave equation. On the other hand, not all the solutions of the n particle Schrödinger wave equation can be solutions of the Fock equation, only those which are adequately symmetrized do. So, both conditions, defined particle number and symmetrization, must hold in order that both formalisms be equivalent.

3.5.2   Creation and annihilation operators

The standard manner to handle with the equations in Fock-space is to write physical magnitudes in terms of creation and annihilation operators. To do so, one makes the following expansion:

$$
\psi(\mathbf{r}) = \sum_k a_k u_k(\mathbf{r}) \qquad (3.5.31)
$$

using basis functions $\{u_k(\mathbf{r})\}$. The coefficients of the expansion are the annihilation operators:

$$
a_k := \int d^3r\, u_k^*(\mathbf{r})\psi(\mathbf{r}) \qquad (3.5.32)
$$

A similar expansion stands for the creation operator: $a_k^\dagger$. In quantum field theory, it is commonly assumed that the action of the operator $a_k^\dagger$ describes the "creation of a particle" with wave function $u_k(\mathbf{r})$. In a similar way, is interpreted that the action of a_k describes the "annihilation of a particle". It can be shown, that these operators satisfy the commutation relations:

$$[a_k, a_l]_{\mp} = 0$$
$$[a_k^\dagger, a_l^\dagger]_{\mp} = 0$$
$$[a_k, a_l^\dagger]_{\mp} = \delta_{kl} \tag{3.5.33}$$

and we can define the particle number operator associated with level k:

$$N_k := a_k^\dagger a_k \tag{3.5.34}$$

We can cast these equations in a more familiar form, using the "$[,]$" symbol for bosonic commutation relations and the "$\{,\}$" symbol for fermionic (anti)commutation relations. Then for bosons we have:

$$[a_\alpha; a_\beta^\dagger] = a_\alpha a_\beta^\dagger - a_\beta^\dagger a_\alpha = \delta_{\alpha\beta} I \tag{3.5.35}$$

$$[a_\alpha^\dagger; a_\beta^\dagger] = 0 \tag{3.5.36}$$

$$[a_\alpha; a_\beta] = 0 \tag{3.5.37}$$

and for fermions, (with $C_\alpha^\dagger$ and C_α playing the role of fermionic creation and annihilation operators respectively) we have:

$$\{C_\alpha; C_\beta^\dagger\} = C_\alpha C_\beta^\dagger + C_\beta^\dagger C_\alpha = \delta_{\alpha\beta} I \tag{3.5.38}$$

$$\{C_\alpha^\dagger; C_\beta^\dagger\} = 0 \tag{3.5.39}$$

$$\{C_\alpha; C_\beta\} = 0 \tag{3.5.40}$$

Substitution of (3.5.31) in (3.5.30) yields:

$$H = \sum_{kl} a_k^\dagger T_{kl} a_l + \frac{1}{2} \sum_{klpq} a_k^\dagger a_l^\dagger V_{klpq} a_p a_q \tag{3.5.41}$$

where the matrix elements T_{kl} and V_{klpq} are given by:

$$T_{kl} = \int d^3 r\, u_k^*(\mathbf{r})[(-\tfrac{\hbar^2 \nabla^2}{2m}) + V_1(\mathbf{r})]u_l(\mathbf{r})$$
$$V_{klpq} = \int d^3 r \int d^3 r'\, u_k^*(\mathbf{r})u_l^*(\mathbf{r}')V_2(\mathbf{r}, \mathbf{r}')u_p(\mathbf{r}')u_q(\mathbf{r}) \tag{3.5.42}$$

and similar expressions can be found for more general obserables.

3.5.3 Using the Q-space

We have constructed two spaces whose vectors make no reference to particle indexation and, besides, particles are not labeled in any step of the formal construction. This is possible because these spaces are constructed using the non classical part of Q, which may refer to intrinsically indistinguishable particles. Vectors in these spaces are only distinguished by the occupation number in each (energy) level. With these tools and using the language of Q, the formalism of quantum mechanics may be completely rewritten giving a straightforward answer to the problem of giving a formulation of quantum mechanics in which intrinsical indistinguishability is taken into account from the beginning, without artificially introducing extra postulates.

Let us first show that the Q-space is useful to provide a states space analogous to Fock-space. With this aim, we make the following association in order to turn the notation similar to that of standard quantum mechanics. For each quasi-function $f_{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}}$ of the qsets $\mathcal{F}$ or $\mathcal{OF}$ constructed above, we will write:

$$\alpha f_{\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}} := \alpha |\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}) \tag{3.5.43}$$

with the obvious corresponding generalization for linear combinations.

Let us recall again that in $|\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}) \in \mathcal{F}$, the order of the indexes has no meaning. But in $|\epsilon_{i_1} \epsilon_{i_2} \ldots \epsilon_{i_n}) \in \mathcal{OF}$, the order makes sense.

As we have already pointed out, in order to avoid particle labeling in the expressions for observables, no reference to particle indexation should appear in their corresponding operators. For that reason we will only use creation and annihilation operators. To do so, we construct creation and annihilation operators acting on Q-spaces. We will first develop the construction for bosons and later fermions. We will use creation and annihilation operators and instead of postulating commutation relations, we will deduce them from their definitions and the properties of the vectors of the Q-spaces (following an analogous procedure as that exposed, for example, in [11, Chap. 17]).

3.5.4 Bosonic states

For bosons, the procedure is similar to the procedure of the standard approach for, as we have remarked earlier, a scalar product naturally arises from the symmetric product (3.4.18). This implies that, once normalized

to unity, the vectors $|\alpha\beta\gamma\ldots)$ constructed using Q, are equivalent to the symmetrized vectors $|\alpha\beta\gamma\ldots\rangle$ for bosonic states. This is so, because permutations do not alter the vector in none of the spaces.

Suppose then that vectors $|\alpha\beta\gamma\ldots)$ are normalized to unity. If ζ represents an arbitrary collection of indexes, we want define the operators "$a_\alpha^\dagger$" in such a way that they satisfy:

$$a_\alpha^\dagger|\zeta) = K|\alpha\zeta) \tag{3.5.44}$$

and the proportionality constant K be such that

$$a_\alpha^\dagger a_\alpha|\zeta) = n_\alpha|\zeta) \tag{3.5.45}$$

with n_α being the occupation number of level α in $|\zeta)$. Then, it follows that:

$$((\zeta|a_\alpha^\dagger)(a_\alpha|\zeta)) = n_\alpha \tag{3.5.46}$$

Making the definition

Definition 10.

$$a_\alpha|\ldots n_\alpha\ldots) = \sqrt{n_\alpha}\,|\ldots n_\alpha - 1\ldots) \tag{3.5.47}$$

we obtain

$$a_\alpha a_\alpha^\dagger|\ldots n_\alpha\ldots) = K\sqrt{n_\alpha + 1}\,|\ldots n_\alpha\ldots) \tag{3.5.48}$$

If we apply $a_\alpha^\dagger$ once again:

$$a_\alpha^\dagger a_\alpha a_\alpha^\dagger|\ldots n_\alpha\ldots) = K^2\sqrt{n_\alpha + 1}\,|\ldots n_\alpha + 1\ldots) \tag{3.5.49}$$

and using (3.5.45):

$$(a_\alpha^\dagger a_\alpha)a_\alpha^\dagger|\ldots n_\alpha\ldots) = (n_\alpha + 1)K|\ldots n_\alpha + 1\ldots) \tag{3.5.50}$$

so $K = \sqrt{n_\alpha + 1}$. Then we define

Definition 11.

$$a_\alpha^\dagger|\ldots n_\alpha\ldots) = \sqrt{n_\alpha + 1}\,|\ldots n_\alpha + 1\ldots) \tag{3.5.51}$$

Once this is established, let us obtain the commutation relations. By a straightforward computation, we see that:

$$(a_\alpha a_\beta^\dagger - a_\beta^\dagger a_\alpha)|\psi) = \delta_{\alpha\beta}|\psi) \tag{3.5.52}$$

which is the same as:

$$[a_\alpha; a_\beta^\dagger] = \delta_{\alpha\beta}I \tag{3.5.53}$$

In an analogous way we can show that:

$$[a_\alpha; a_\beta] = [a_\alpha^\dagger; a_\beta^\dagger] = 0 \tag{3.5.54}$$

This shows that the (bosonic) commutation relations that are obtained in Q-space are the same ones as in the standard Fock-space.

3.5.5 Fermionic states

For the fermionic case, we will use C_0 equipped with the antisymmetric product given by equation (3.4.20). We define the creator operator $C_\alpha^\dagger$ as follows:

Definition 12. *Let ζ represent a collection of indexes with non null occupation number, then*

$$C_\alpha^\dagger|\zeta) = |\alpha\zeta) \tag{3.5.55}$$

Note that if α was already in the collection ζ, then $|\alpha\zeta)$ is a vector with null norm (as in 3.4.9 and 3.4.10). To have null norm implies that $(\psi|\alpha\zeta) = 0$ for all $|\psi)$. Then, if a given vector has null norm, its scalar product with any other vector in the space is zero. It follows that in the case that a system were eventually in a state of null norm, the probability of observing it would be zero. In the same way, if a linear combination of null norm vectors were added to the vector representing the state of a system, this addition would not give place to observable results. It follows then that null norm vectors do not represent real physical states, and the same holds for linear combinations of them. Moreover, adding a vector of null norm to any other one does not produce observable affects, because the terms of null norm do not contribute to the mean values or to the probabilities. In order to express this situation, we define the following relation:

Definition 13. *Two vectors $|\varphi\rangle$ and $|\psi\rangle$ are similar (and we will write $|\varphi\rangle \cong |\psi\rangle$))) if the difference between them is a linear combination of null norm vectors.*

Let us now compute the effect of applying C_α to the vectors of $\mathcal{OF}$. Using Definition (12) we find that:

$$(\zeta|C_\alpha = (\alpha\zeta| \tag{3.5.56}$$

Then, for any vector $|\psi\rangle$:

$$(\zeta|C_\alpha|\psi) = (\alpha\zeta|\psi) = 0 \tag{3.5.57}$$

for $\alpha \in \zeta$ or $(\psi|\alpha\zeta) = 0$. Then, if we choose $|\psi\rangle = |0\rangle$ it follows that:

$$(\zeta|C_\alpha|0) = (\alpha\zeta|0) = 0 \tag{3.5.58}$$

and thus we obtain that $C_\alpha|0)$ is orthogonal to any vector which contains α and to any vector which does not contain α. Then, it is orthogonal to any vector, and for that reason, it has to be a linear combination of null norm vectors. Then, we do not loose anything if we establish $C_\alpha|0) = \vec{0}$. In an analogous way we can assert that:

$$C_\alpha|(\sim \alpha)\cdots) = \vec{0} \tag{3.5.59}$$

where $(\sim \alpha)$ means that α has occupation number zero, and the dots mean that the other levels have arbitrary occupation numbers. Using (13) we can also write:

$$C_\alpha|0) \cong \vec{0} \tag{3.5.60}$$

and

$$C_\alpha|(\sim \alpha)\ldots) \cong \vec{0} \tag{3.5.61}$$

In what follows we will use $\cong$ if necessary, but the same results are obtained if we replace $\cong$ by the equality symbol. Making $|\psi\rangle = |\alpha\rangle$ in (3.5.57) it follows that:

$$(\zeta|C_\alpha|\alpha) = (\alpha\zeta|\alpha) = 0 \tag{3.5.62}$$

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in any case except for $|\zeta) = |0)$. In that case, $(0|C_\alpha|\alpha) = 1$. Then, it follows that $C_\alpha|\alpha) \cong |0)$. In an analogous way we obtain:

$$C_\alpha|\alpha\zeta) \cong |(\sim \alpha)\zeta) \tag{3.5.63}$$

if α does not belongs to ζ. But in the case that α belongs to ζ, (i.e., the occupation number is greater than 1) we have that $|\alpha\zeta)$ has null norm, and so:

$$(\alpha\zeta|C_\alpha^\dagger|\psi) = (\alpha\zeta|\alpha\psi) = 0, \forall|\psi) \tag{3.5.64}$$

From this equation it follows that:

$$(\psi|C_\alpha|\alpha\zeta) = 0, \forall|\psi) \tag{3.5.65}$$

and so, $C_\alpha|\alpha\zeta)$ has null norm too.

Now, let us find the anti-commutation relations obeyed by the fermionic creation and annihilation operators. To do so, let us first study the relationship between $|\alpha\beta) \in \mathcal{OF}$ and $|\beta\alpha) \in \mathcal{OF}$. With this aim, consider the vector $|\alpha\beta) + |\beta\alpha)$ and perform the product of this sum with another arbitrary vector. It suffices to study what happens with basis vectors. The product yields trivially zero for any vector different from $|\alpha\beta)$ or $|\beta\alpha)$. Making the product with $|\alpha\beta)$ we obtain:

$$(\alpha\beta|[|\alpha\beta) + |\beta\alpha)] = (\alpha\beta|\alpha\beta) + (\alpha\beta|\beta\alpha) =$$
$$\delta_{\alpha\alpha}\delta_{\beta\beta} - \delta_{\alpha\beta}\delta_{\beta\alpha} + \delta_{\alpha\beta}\delta_{\alpha\alpha} - \delta_{\alpha\alpha}\delta_{\beta\beta} = 1 - 0 + 0 - 1 = 0 \tag{3.5.66}$$

The same conclusion holds if we multiply it by $|\beta\alpha)$. Then, it follows that $|\alpha\beta) + |\beta\alpha)$ is a linear combination of null norm vectors (which we will denote by $|nnlc)$) and, thus:

$$|\alpha\beta) = -|\beta\alpha) + |nnlc) \tag{3.5.67}$$

We do not care about which is the particular null norm linear combination, because it has no observable effects. Now, we can compute

$$C_\alpha^\dagger C_\beta^\dagger|\psi) = |\alpha\beta\psi) = -|\beta\alpha\psi) + |nnlc) = -C_\beta^\dagger C_\alpha^\dagger|\psi) + |nnlc) \tag{3.5.68}$$

and thus

$$\{C_\alpha^\dagger; C_\beta^\dagger\}|\psi) = |nnlc) \tag{3.5.69}$$

Then, we do not loose generality if we set

$$\{C_\alpha^\dagger; C_\beta^\dagger\}|\psi) = 0 \tag{3.5.70}$$

In an analogous way, we conclude that

$$\{C_\alpha; C_\beta\}|\psi) = 0 \tag{3.5.71}$$

Now let us calculate the commutation relation between C_α and $C_\beta^\dagger$. Suppose first that $\alpha \neq \beta$. If $\alpha \notin \psi$ or $\beta \in \psi$ then it is clear that

$$\{C_\alpha; C_\beta^\dagger\}|\psi) \approx \overrightarrow{0} \tag{3.5.72}$$

If $\alpha \in \psi$ and $\beta \notin \psi$, suppose (without loss of generality), that α is the first symbol in the list of ψ. Then,

$$\{C_\alpha; C_\beta^\dagger\}|\psi) = C_\alpha|\beta\psi) + C_\beta^\dagger|\psi(\sim \alpha)) \cong$$
$$\cong -|\beta\psi(\sim \alpha)) + |\beta\psi(\sim \alpha)) = \overrightarrow{0} \tag{3.5.73}$$

If $\alpha = \beta$, and $\alpha \in \psi$, then

$$\{C_\alpha; C_\alpha^\dagger\}|\psi) = C_\alpha|\alpha\psi) + C_\alpha^\dagger|\psi(\sim \alpha)) \cong$$
$$\cong \overrightarrow{0} + |\psi) = |\psi) \tag{3.5.74}$$

If $\alpha = \beta$, and $\alpha \notin \psi$, then

$$\{C_\alpha; C_\alpha^\dagger\}|\psi) = C_\alpha|\alpha\psi) + C_\alpha^\dagger|\psi(\sim \alpha)) \cong$$
$$\cong |\psi) + \overrightarrow{0} = |\psi) \tag{3.5.75}$$

So, in any case, we recover the relation

$$\{C_\alpha; C_\alpha^\dagger\}|\psi) \cong \delta_{\alpha\beta}|\psi) \tag{3.5.76}$$

and then, we can set

$$\{C_\alpha; C_\alpha^\dagger\} = \delta_{\alpha\beta}. \tag{3.5.77}$$

Thus, we have shown that the same commutation relations hold, as the standard formalism hold in Q-space. This means that both formulations are equivalent, for all the interesting information is contained in the commutation relations. In the following section, we show an example of this new formulation.

3.5.6 Observables

To avoid particle labeling in the expressions for observables, in Fock-space formalism they are written in terms of creation and annihilation operators. This is also the case in the Q-space. For example, we have shown that operators T acting over a single particle states are of the form:

$$T = \sum_{\alpha\beta} t_{\alpha\beta} a_\alpha^\dagger a_\beta = \sum_k (\alpha|k) t_k (k|\beta) a_\alpha^\dagger a_\beta = \sum_k \sum_j (\alpha|k)(k|T(1)|j)(j|\beta) a_\alpha^\dagger a_\beta$$

$$(3.5.78)$$

Interaction operators act over spaces of a greater number of particles. The expression of an interaction operator V between two particles is:

$$V = \frac{1}{2} \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta V_{\alpha\beta,\gamma\delta} a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\delta = \frac{1}{4} \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta (\kappa\lambda|V|\mu\nu) a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\delta$$

$$(3.5.79)$$

3.6 Correlation in a two-particle state

In this section we show an application of the use of the formalism in Q-space to illustrate how the usual results of the standard QM formalism are obtained. To do so, let us consider a two spin 1/2 quanta regarding only to spin degrees of freedom. Let $S_i = (\hbar/2)\sigma_i$ be the spin operator, σ_i the Pauli matrices. We use eq. (3.5.78) to write σ_z:

$$\sigma_z = \sum_{\alpha\beta} (\sigma_z)_{\alpha\beta} \, C_\alpha^\dagger C_\beta = C_+^\dagger C_+ - C_-^\dagger C_-. \qquad (3.6.1)$$

To obtain the spin operator in an arbitrary direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we propose σ_n in the form:

$$\sigma_n = \cos\theta \, C_+^\dagger C_+ + e^{-i\phi} \sin\theta C_+^\dagger C_- + e^{i\phi} \sin\theta C_-^\dagger C_+ - \cos\theta \, C_-^\dagger C_- \quad (3.6.2)$$

In fact, this operator rotates the basis vectors as usual. Thus, the mean value of σ_n in the one particle state 'up' in direction $\hat{z}$ results:

$$(+|\sigma_n|+) = \cos\theta(+|+) + e^{i\phi}\sin\theta(+|-) = \cos\theta \qquad (3.6.3)$$

Now we consider the a pair of indistinguishable fermions, one with spin 'up' and the other with spin 'down'. In the Q-space its state is $|+-)$. It has not to be confused with the standard $|+-\rangle$ of the tensor product Hilbert space, which is not an antisymmetric state. We first show that in the same spatial direction, say $\hat{z}$, the spin components are in perfect anticorrelation. As usual, the correlation is evaluated as the mean value of an operator that represents the measurement of σ_z for both components over the state. Differently from the standard formulation, where this operator is obtained in the labeled tensor product space, here it is obtained from eq. (3.5.79) for the fermionic case:

$$
\begin{aligned}
\sigma_{\mathbf{zz}} &= \frac{1}{2}\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\delta}(\sigma_{zz})_{\alpha\beta,\gamma\delta}C_{\alpha}^{\dagger}C_{\beta}^{\dagger}C_{\delta}C_{\gamma} \\
&= \frac{1}{2}[C_{+}^{\dagger}C_{+}^{\dagger}C_{-}C_{+} - C_{+}^{\dagger}C_{-}^{\dagger}C_{+}C_{-} + C_{-}^{\dagger}C_{-}^{\dagger}C_{-}C_{-} - C_{-}^{\dagger}C_{+}^{\dagger}C_{+}C_{-}]
\end{aligned} \qquad (3.6.4)
$$

When applied to a state $|+-)$ it yields

$$
\begin{aligned}
\sigma_{\mathbf{zz}}|+-) &= \frac{1}{2}[C_{+}^{\dagger}C_{+}^{\dagger}C_{+}C_{+}|+-) - C_{+}^{\dagger}C_{-}^{\dagger}C_{-}C_{+}|+-) \\
&\quad + C_{-}^{\dagger}C_{-}^{\dagger}C_{-}C_{-}|+-) - C_{-}^{\dagger}C_{+}^{\dagger}C_{+}C_{-}]|+-) \\
&= \frac{1}{2}[-|+-) + |-+)] = -|+-)
\end{aligned} \qquad (3.6.5)
$$

Thus, the mean value results

$$(+-|\sigma_{\mathbf{zz}}|+-) = -(+-|+-) = -1 \qquad (3.6.6)$$

which is the usual result.

To obtain the correlation between components in two arbitrary directions, say $\hat{z}$ and $\hat{n}$, we have to follow an analogous procedure. First we write the operator σ_{zn} that acts over the state space of the two particles without distinguishing them:

$$\sigma_{\mathbf{zn}} = \frac{1}{2}[\cos\theta C_+^\dagger C_+^\dagger C_+ C_+ + e^{-i\phi}\sin\theta C_+^\dagger C_+^\dagger C_- C_+$$
$$+ e^{i\phi}\sin\theta C_+^\dagger C_-^\dagger C_+ C_+ - \cos\theta C_+^\dagger C_-^\dagger C_- C_+$$
$$+ \cos\theta C_-^\dagger C_-^\dagger C_- C_- - e^{-i\phi}\sin\theta C_-^\dagger C_-^\dagger C_+ C_-$$
$$- e^{-i\phi}\sin\theta C_-^\dagger C_+^\dagger C_- C_- - \cos\theta C_-^\dagger C_+^\dagger C_+ C_-] \qquad (3.6.7)$$

Applied to the state $|+-)$ it yields:

$$\sigma_{\mathbf{zn}}|+-) = \frac{1}{2}[e^{-i\phi}\sin\theta C_+^\dagger C_+^\dagger|0) - \cos\theta C_+^\dagger C_-^\dagger|0)$$
$$+ e^{-i\phi}\sin\theta C_-^\dagger C_-^\dagger|0) + \cos\theta C_-^\dagger C_+^\dagger|0)]$$
$$= \frac{1}{2}[-\cos\theta|+-) + \cos\theta|-+)] = \frac{1}{2}[-\cos\theta|+-) - \cos\theta|+-)]$$
$$= -\cos\theta|+-)$$

$$(3.6.8)$$

Thus, the mean value which gives the correlation results

$$(+-|\sigma_{\mathbf{zn}}|+-) = -\cos\theta \qquad (3.6.9)$$

as it must be. It is important to remark that the state $|+-)$ takes into account indistinguishability and antisymmetry without 'tricks', just because it is constructed in the Q-space.

3.7   The Lattice of Convex Subsets and The Symmetrization Operators $P^\pm$

In this section we apply the tools developed in Section 2.6 to the indistinguishable particles case. The application of lattice theoretical techniques for indistinguishable particles is not widely studied. Even so, there are some interesting discussions as the one given in [35].

A lot of effort has been devoted to the characterization of entanglement [66] and there are new relatively recent efforts to characterize entanglement

of indistinguishable particles [65, 32, 74, 41, 9, 81]. The distinction between entanglement and exchange correlations has been a matter of recent debate (see for example [65] and [41]). We cannot concentrate in all this discussion here, but we only apply some tools of Chapter 2 to the identical particle case, hoping that further development on the logic-algebraic approach can contribute to these questions.

3.7.1 The Symmetrization Operators $P^{\pm}$

Given two indistinguishable particles, the state of the system in which one particle is in state $|\varphi\rangle$ and the other in state $|\phi\rangle$, with $|\phi\rangle \neq |\varphi\rangle$, is given by the symmetrization of $|\varphi\rangle \otimes |\phi\rangle$ (as we saw in section 3.5.1). Let $P^{\pm}$ denote the operators which yield a symmetrized and normalized state. If P_{12} is the permutation operator between the two particles, it is easy to see that in this case:

$$P^+ = (1/\sqrt{2})(1 + P_{12}) \tag{3.7.1}$$

$$P^- = (1/\sqrt{2})(1 - P_{12}) \tag{3.7.2}$$

yield the correct states when applied to $|\varphi\rangle \otimes |\phi\rangle$. The "admissible" pure states of the compound system of two identical particles represented by the Hilbert space $\mathcal{H}$ will be the rays of the subspaces of $\mathcal{H} \otimes \mathcal{H}$

$$\mathcal{H}^+ := (1/\sqrt{2})(1 + P_{12})(\mathcal{H} \otimes \mathcal{H}) \tag{3.7.3}$$

for bosons and

$$\mathcal{H}^- = (1/\sqrt{2})(1 - P_{12})(\mathcal{H} \otimes \mathcal{H}) \tag{3.7.4}$$

for fermions. If we form the density matrix $\rho^{\pm}$ which corresponds to the state $\frac{1}{\sqrt{2}}(|\varphi\rangle \otimes |\phi\rangle \pm |\phi\rangle \otimes |\varphi\rangle)$, (with $|\varphi\rangle \neq |\phi\rangle$), then. it is easy to verify that

$$\rho^{\pm} = P^{\pm}\rho(P^{\pm})^{\dagger} = [(1/\sqrt{2})(1 \pm P_{12})]\,\rho\,[((1/\sqrt{2})(1 \pm P_{12}))^{\dagger}] \tag{3.7.5}$$

where ρ is the density matrix corresponding to the non-symmetrized state $|\varphi\rangle \otimes |\phi\rangle$.

As the pure states of the compound system will be given by the rays of $\mathcal{H}^{\pm}$, we can form the convex set of states of the symmetrized subspaces. In doing that, we must consider all hermitian matrixes $\mathcal{A}^{\pm}$ acting on $\mathcal{H}^{\pm}$ which satisfy the symmetrization condition. Then, we can define:

$$\mathcal{C}^{\pm} := \{\rho^{\pm} \in \mathcal{A}^{\pm} \mid \mathrm{tr}(\rho^{\pm}) = 1, \ \rho^{\pm} \geq 0\} \tag{3.7.6}$$

Suppose that $\rho^{\pm} \in \mathcal{C}^{\pm}$. Then, $\rho^{\pm}$ is hermitian, and so, we can always find a diagonal decomposition such that:

$$\rho^{\pm} = \sum_{i=1}^{N} p_i P_i \tag{3.7.7}$$

The P_i are projection operators over $\mathcal{H}\pm$. Suppose for simplicity that $\rho^{\pm}$ is non-degenerate. Then, $P_i = |\psi_i\rangle\langle\psi_i|$, for some $|\psi_i\rangle \in \mathcal{H}^{\pm}$. But then $|\psi_i\rangle = P^{\pm}|\varphi\rangle_i$, with $|\varphi_i\rangle$ in $\mathcal{H} \otimes \mathcal{H}$. And so, we can write:

$$\rho^{\pm} = P^{\pm}[\sum_{i=1}^{N} p_i |\varphi_i\rangle\langle\varphi_i|](P^{\pm})^{\dagger} = P^{\pm}\rho(P^{\pm})^{\dagger} \tag{3.7.8}$$

where $\rho \in \mathcal{H} \otimes \mathcal{H}$. It is easy to see that the same relation holds if $\rho^{\pm}$ is degenerate. After this discussion, we can state the following:

Proposition 26. $\mathcal{C}^{\pm} = P^{\pm}\mathcal{C}(P^{\pm})^{\dagger}$

Now we are ready to construct a lattice for indistinguishable particles.

3.7.2   The Indistinguishable Particles Lattice $\mathcal{L}_{\mathcal{C}^{\pm}}$

Proceeding in an analogous way as in 2.6, we define:

$$\mathcal{L}_{\mathcal{C}^{\pm}} := \{C \subseteq \mathcal{C}^{\pm} \mid C \text{ is a convex subset of } \mathcal{C}^{\pm}\} \tag{3.7.9}$$

In order to give $\mathcal{L}_{\mathcal{C}\pm}$ a lattice structure similar to that of $\mathcal{L}_{\mathcal{C}}$ we introduce the following operations:

Definition 14. *For all $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}^{\pm}}$*

$\wedge^{\pm} : C_1 \wedge^{\pm} C_2 := C_1 \cap C_2$

$\vee^{\pm} : C_1 \vee^{\pm} C_2 := conv(C_1, C_2)$. *It is again a convex set, and it is included*

    in $\mathcal{C}^{\pm}$

$\neg^{\pm} : \neg^{\pm}C := C^{\perp} \cap \mathcal{C}^{\pm}$

$\longrightarrow^{\pm} : C_1 \longrightarrow^{\pm} C_2 := C_1 \subseteq C_2$

    With the operations of definition 14, it is apparent that $(\mathcal{L}_{\mathcal{C}^{\pm}}; \longrightarrow^{\pm}; \wedge^{\pm}; \vee^{\pm})$ is a complete bounded lattice, setting $\emptyset = \mathbf{0}$ and $\mathcal{C}^{\pm} = \mathbf{1}$. All of this is granted by the following proposition:

Proposition 27. $(\mathcal{L}_{\mathcal{C}^{\pm}}; \longrightarrow^{\pm}; \wedge^{\pm}; \vee^{\pm})$ *satisfies*

(a) $C_1 \wedge^{\pm} C_1 = C_1$

(b) $C_1 \wedge^{\pm} C_2 = C_2 \wedge^{\pm} C_1$

(c) $C_1 \vee^{\pm} C_2 = C_2 \vee^{\pm} C_1$

(d) $C_1 \wedge^{\pm} (C_2 \wedge^{\pm} C_3) = (C_1 \wedge^{\pm} C_2) \wedge^{\pm} C_3$

(e) $C_1 \vee^{\pm} (C_2 \vee^{\pm} C_3) = (C_1 \vee^{\pm} C_2) \vee^{\pm} C_3$

(f) $C_1 \wedge^{\pm} (C_1 \vee^{\pm} C_2) = C_1$

(g) $C_1 \vee^{\pm} (C_1 \wedge^{\pm} C_2) = C_1$

    As any $\rho^{\pm} \in \mathcal{C}^{\pm}$ satisfies that $\rho^{\pm} = P^{\pm}\rho(P^{\pm})^{\dagger}$ for some $\rho \in \mathcal{H}$, it is easy to show that

Proposition 28. *If* $C^{\pm} \in \mathcal{L}_{\mathcal{C}^{\pm}}$ *then* $C^{\pm} = P^{\pm}C(P^{\pm})^{\dagger}$, *for some* $C \in \mathcal{L}_{\mathcal{C}}$.

Thus, in this section we have shown how to apply the methods of Section 2.6 to systems of indistinguishable particles. It is important to remark that the orthodox approach of quantum logic does not pay too much attention to systems of indistinguishable particles. As far as we know, an analysis such as the one presented here is completely new in the lattice theoretical approach to quantum mechanics.

The connection of the approach of this section and the approach of the Q-space is obvious, because when we restrict to the two particle case (or in general, for fixed particle number), the Q-space approach is completely equivalent to that of standard quantum mechanics. So the same lattice can be constructed using the Q-space.

Chapter 4

Conclusions

In chapter 2 we have shown that it is possible to construct a lattice theoretical framework which incorporates mixed states as atoms. This is done in order to overcome a problem of the standard QL formalism posed in Section 2.3, namely that the conjunction of all actual properties of the system does not yield its actual state when compound systems are considered. We showed that this is directly linked with the fact that QL treats mixtures as measures over the projection lattice, in an analogous way as classical statistical distributions are measures over the phase space. But alike classical mixtures, mixed states in quantum mechanics are of a very different nature, and its interpretation gives rise to well known difficulties. While each state (pure or mixed) induces a measure in the lattice of projections, this has nothing to do with the identification of these measures with classical mixtures. Indeed, any pure state is not dispersion free also and so induces a measure over $\mathcal{L}_{v\mathcal{N}}$ that has a radical different nature than that of classical measures. This was at the origin of the problems posed in Section 2.3. Our construction is a quite natural extension of the von Neumann lattice, and its properties and characteristics are consistent with the constraints imposed by quantum mechanics. More precisely, in the standard quantum logical approach, when the whole system is in a pure entangled state there are no elements available in the lattices of the subsystems to represent the states of the subsystems as elements of the lattice. This is expressed in the absence of projection functions which map the states of the lattice of the whole system to the the states of the lattices of the subsystems which satisfy in turn, to be compatible with the physical description. Alike the standard approach, the projections defined in

the frame of the enlarged structures satisfy this condition. They are also the canonical ones in the sense that they are constructed using partial traces, in accordance with the quantum formalism. This was shown in Sections 2.5.2 and in 2.7.2.

Traditionally, the difference between classical and quantum lattices is said to be that the classical lattice is a Boolean lattice while von Neumann lattice is an orthomodular one. We claim that this is not the only difference, the other one –although not independent– being their behavior with respect to the coupling of two or more systems. The necessity of the enlargement of the von Neumann lattice in order to preserve the condition that the meet of actual properties defines the state of the system may be seen as an algebraic expression of the existence of entanglement. The approach presented here shows, in an algebraic fashion, the radical difference between quantum mechanics and classical mechanics when two systems interact. If the systems are classical, no non-trivial enlargement of the lattice is needed even in the presence of interactions. It is enough in order to describe all relevant physics about the subsystems. But the existence of entanglement in quantum mechanics forces an enlargement of the state space of pure states to the convex set $\mathcal{C}$ to deal with the states of subsystems and thus the enlargement of $\mathcal{L}_{vN}$. Two possible candidates to fulfill this task, namely the lattices $\mathcal{L}$ and $\mathcal{L}_{\mathcal{C}}$, has been presented in this work and the relations among $\mathcal{L}$ and $\mathcal{L}_i$ (and also for $\mathcal{L}_{\mathcal{C}}$ and $\mathcal{L}_{\mathcal{C}i}$) have been analyzed. We think that paying more attention to this kind of approaches would shed new light on the algebraic properties of quantum non-separability.

The structures presented in this work do not have the problems of the orthodox approach. This is so because they satisfy the conditions listed in section 2.3. In our approach, mixed states are in the same status than pure states and induce measures over $\mathcal{L}_{v\mathcal{N}}$ as well as pure states, but measures and states are not identified. This situation is very different than that of CM, in which the measures induced by pure states are trivial.

Our approach presents itself as a natural logic-algebraic language for the study of topics which involve compound quantum systems, such as quantum information processing and decoherence, which concentrate on the study of $\mathcal{C}$ instead of the lattice of projections. They also capture the physics behind the fact that we can mix states according to the "mixing principle" of section 2.2.3.

On the other hand, as we shown in section 2.7.3, our approach sheds new light into algebraic properties of quantum entanglement via the study of

the natural arrows defined between the lattice of the system and its subsystems. The study of these arrows reveals itself as adequate for the of algebraic characterization of entanglement, as shown in section 2.7.3, providing an entanglement criteria.

In chapter 3 have shown that it is possible to construct the quantum mechanical formalism for indistinguishable particles without labeling them in any step, and in particular, avoiding the LTPHSF. So, it gives an alternative answer to the problems posed in [72], and also contributes to an answer to Manin's problem posed in [52]. To do so, we have built a vector space with inner product, the Q-space, using the non-classical part of Q, a generalization of ZFU, to deal with indistinguishable elements. Vectors in Q-space refer only to occupation numbers and permutations operators act as the identity operator, reflecting in the formalism the fact of unobservability of permutations, already expressed in terms of the formalism of Q.

We have also argued that it is useful to represent operators (which are intended to represent observable quantities) as combinations of creator and annihilation operators, in order to avoid particle indexation in the expression of observable quantities. We have shown that creation and annihilation operators which act on Q-space can be constructed. We have proved that they obey the usual commutation and anticommutation relations for bosons and fermions respectively, and this means that our construction is equivalent to that of the Fock-space formulation of quantum mechanics. Thus, using the results reviewed in section 3.5, this implies that we can recover the n-particles wave equation using Q-space in the same way as in the standard theory. Though both formulations are equivalent 'for all practical purposes', when subjected to careful analysis, the conceptual difference turns very important. And so, we give an alternative approach in which we show explicitly that a formulation of QM which incorporates indistinguishability avoiding the LTPHSF is possible. Thus, our construction gives a different turn to the questions posed in [59, 60, 75]. In section 3.6 we gave an example of how our construction can be applied to the singlet state.

Finally, in section 3.7 we applied some of the tools developed in section 2.6 to the indistinguishable particle case. This construction opens the door for new developments on the lattice theoretical approach of the indistinguishable particles case, which is not widely studied in the literature.

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