Una inversa a derecha para el operador divergencia en dominios con cúspides

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Resumen

En esta tesis estudiamos la existencia de soluciones del problema de la divergencia en dominios con cúspides exteriores. Es sabido que los resultados clásicos en espacios de Sobolev standard, los cuales son una herramienta básica para el análisis variacional de las ecuaciones de Stokes, no valen para este tipo de dominios. Una clase importante de dominios con cúspides exteriores es la de los Hölder-α, con $0 < \alpha < 1$.

Primero, probamos que si Ω es un dominio Hölder-α plano simplemente conexo existen soluciones de $\operatorname{div} \mathbf{u} = f$ en un espacio de Sobolev con peso apropiado. Los pesos considerados son potencias de la distancia al borde de dominio.

Luego, para una clase particular de dominios Hölder-α acotados $\Omega \subset \mathbb{R}^n$, con cúspides exteriores de dimensión entera $m \leq n-2$, mostramos la existencia de soluciones en espacios de Sobolev con peso de la ecuación de divergencia. Los pesos considerados en este caso son potencias de la distancia a la cúspide. Este resultado es más fuerte que el que involucra la distancia a $\partial\Omega$. También, obtenemos una versión de la desigualdad de Korn con peso para esta clase de dominios y pesos. Las potencias en los pesos de los resultados obtenidos en este trabajo resultan óptimas.

Como una aplicación de los resultados previos, probamos la existencia y unicidad de soluciones variacionales de las ecuaciones de Stokes en espacios de Sobolev con peso apropiados. Como consecuencia, obtenemos la existencia de una solución $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L^r(\Omega)$, con $r < 2$ dependiendo de la potencia de la cúspide, donde $\mathbf{u}$ denota la velocidad y p la presión.

Por otro lado, damos condiciones suficientes para que una potencia de la distancia a un compacto esté en la clase de Muckenhoupt A_p. Este resultado es auxiliar en este trabajo aunque nos parece que tiene interés en sí mismo.

Finalmente, definimos nuevos contraejemplos para el problema de la divergencia y la desigualdad de Korn en dominios cuspidales, donde las cúspides no son necesariamente de tipo potencia.

Palabras clave: Operador Divergencia, dominios con cúspides exteriores, Ecuaciones de Stokes, Desigualdad de Korn, Espacios de Sobolev con peso.

A right inverse of the divergence operator on domains with cusps

Abstract

This thesis deals with solutions of the divergence equation on domains with external cusps. It is known that the classic results in standard Sobolev spaces, which are basic in the variational analysis of the Stokes equations, are not valid for this kind of domains. An important class of domains which could present external cusps is the Hölder-α, with $0 < \alpha < 1$.

First, we prove that if Ω is a planar simply connected Hölder-α domain there exist solutions of $\operatorname{div} \mathbf{u} = f$ in appropriate weighted Sobolev spaces. The weights considered are powers of the distance to the boundary.

Then, for particular bounded Hölder-α domains $\Omega \subset \mathbb{R}^n$ which have cusps of integer dimension $m \leq n-2$, we show existence of solutions of the divergence equation in weighted Sobolev spaces. The weights used in this case are powers of the distance to the cusp. It provides a result stronger that the one with the distance to $\partial\Omega$. Also, we obtain weighted Korn type inequalities for this class of domains and weights. Moreover, we show that the powers of the distance in the results obtained in this thesis are optimal.

As an application of the previous divergence results, we prove the well posedness of the Stokes equations in appropriate weighted Sobolev spaces. In consequence, we obtain the existence of a solution $(\mathbf{u}, p) \in H^1_0(\Omega)^n \times L^r(\Omega)$ for some $r < 2$ depending on the power of the cusp, where $\mathbf{u}$ is the velocity and p the pressure.

On the other hand, we give sufficient conditions in order to determinate when a power of the distance to a compact set belongs to the Muckenhoupt class A_p. In this thesis this is an auxiliary result, however, we consider it interesting in itself.

Finally, we define new counterexamples for the divergence problem and Korn inequality on domains with external cusps arbitrarily narrow.

Key words : Divergences Operator, domains with external cusps, Stokes equations, Korn inequality, weighted Sobolev spaces.

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Contents

Introducción

El problema de la divergencia en dominios regulares

Sea $\Omega \subset \mathbb{R}^n$ un dominio acotado con ciertas condiciones de regularidad, por ejemplo tener borde suave. Dada $f \in L^p(\Omega)$ de integral cero, con $1 < p < \infty$, es sabido que existe una solución $\mathbf{u} \in W^{1,p}(\Omega)^n$ con traza nula del problema

$$\operatorname{div} \mathbf{u} = f \tag{Ec. 1}$$

que satisface

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}, \tag{Ec. 2}$$

donde la constante C depende sólo de Ω y p. En otras palabras, existe una inversa a derecha continua de la divergencia considerada como un operador del espacio de Sobolev $W_0^{1,p}(\Omega)^n$ en $L_0^p(\Omega)$, donde $L_0^p(\Omega)$ denota el espacio de funciones en $L^p(\Omega)$ de integral cero y $W_0^{1,p}(\Omega)^n$ la clausura de $C_0^\infty(\Omega)^n$ en $W^{1,p}(\Omega)^n$.

Este resultado tiene varias aplicaciones. Por ejemplo, en el caso particular $p = 2$, es una herramienta básica para el análisis variacional de las ecuaciones de Stokes, las cuales modelan el desplazamiento de un fluido viscoso incompresible en Ω . Precisamente, si existe una solución de (Ec. 1) verificando la condición de continuidad (Ec. 2) obtenemos una única solución variacional $(\mathbf{u}, p)$ en el espacio de Hilbert $H_0^1(\Omega)^n \times L_0^2(\Omega)$ del siguiente sistema de ecuaciones

$$\begin{cases} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{en } \Omega \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{en } \Omega \\ \mathbf{u} &= 0 \quad \text{en } \partial\Omega, \end{cases} \tag{Ec. 3}$$

para toda $f \in H^{-1}(\Omega)^n$, donde $H^{-1}(\Omega)^n$ denota el dual del espacio de Sobolev $H_0^1(\Omega)^n$. Además, vale la siguiente estimación a priori

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)^n},$$

donde la constante C depende sólo del dominio Ω. El análisis variacional de este sistema de ecuaciones, y por lo tanto la existencia de soluciones de (Ec. 1), es fundamental para el desarrollo de las aproximaciones numéricas por elementos finitos de sus soluciones.

Por otro lado, existen distintos resultados equivalentes a la existencia de una inversa para la divergencia en dominios regulares. Uno de los más conocidos es la desigualdad de Korn, un resultado clave para el estudio de las ecuaciones de elasticidad lineal. Se encuentran en la bibliografía distintas versiones de esta desigualdad una de ellas es la siguiente conocida como la desigualdad clásica de Korn,

$$\|D\mathbf{v}\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(\mathbf{v})\|_{L^p(\Omega)^{n\times n}},$$

para todo campo $\mathbf{v} \in W^{1,p}(\Omega)^n$ bajo ciertas condiciones que impidan que $\varepsilon(\mathbf{v}) = 0$ mientras que $D\mathbf{v} \neq 0$, donde $D\mathbf{v}$ denota la matriz diferencial de $\mathbf{v}$ y $\varepsilon(\mathbf{v})$ su parte simétrica, es decir

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i}\right).$$

Las dos condiciones consideradas por Korn fueron $\mathbf{v} = 0$ en $\partial\Omega$ (usualmente llamada de primer caso) y $\int_\Omega \operatorname{rot}\mathbf{v} = 0$ (segundo caso). Mostraremos luego otros resultados equivalentes a la existencia de soluciones para el problema de la divergencia.

Debido a la variedad de aplicaciones y resultados equivalentes que posee, este problema ha sido ampliamente estudiado y han sido desarrollados diversos métodos para probar la existencia de una solución $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ de (Ec. 1) verificando (Ec. 2) para diferentes tipos de dominios. Nos gustaría mencionar a continuación algunos de estos trabajos. Por ejemplo, si Ω es un dominio de borde suave en $\mathbb{R}^2$ o un polígono convexo se puede encontrar una solución vía el problema de Neumann para el operador de Laplace (ver [ASV, BA, BB, BS, L]). En efecto, es sabido que para este tipo de dominios existe una solución $\mathbf{v} \in H^2(\Omega)$ de

$$\begin{cases} -\Delta\mathbf{v} &= f \qquad \text{en } \Omega \\ \partial\mathbf{v}/\partial\eta &= 0 \qquad \text{en } \partial\Omega, \end{cases}$$

donde η denota la normal al borde de Ω. Así, $\tilde{\mathbf{u}} = \nabla\mathbf{v}$ es una solución de la divergencia en $H^1(\Omega)^n$ verificando (Ec. 2), propiedad que se desprende de las estimaciones a priori de $\mathbf{v}$.

Por otro lado, en [B, DM[1]] se define una solución explícita para el problema de la divergencia válida en dominios estrellados respecto de una bola (en la página 5 recordamos la definición de este tipo de dominios). Para demostrar la acotación (Ec. 2) se utiliza la teoría de operadores integrales singulares de Calderón-Zygmund. Posteriormente, en [ADM] los autores generalizan este último resultado a una clase más grande de dominios, los John domains (ver página 5).

Recientemente, en [DRS] se probó que toda función de integral cero en un John domain se puede descomponer como una suma numerable de funciones de integral cero soportadas en cubos. Así, resolviendo (Ec. 1) en cada uno de los cubos se obtiene una demostración alternativa a la propuesta previamente en [ADM]. Utilizando también una descomposición de funciones de integral cero aunque en este caso para dominios arbitrarios se prueba en [DMRT] la solubilidad de (Ec. 1) con un condición similar a (Ec. 2) en espacios de Sobolev con peso.

Por otro lado, se han publicado algunos trabajos donde se muestran distintos dominios para los cuales la existencia de una solución al problema de la divergencia en las condiciones planteadas previamente o alguno de sus resultados equivalentes no se satisface. Se sabe que las cúspides exteriores, no así las interiores, son en general un impedimento para la validez de estos resultados aunque aún no se conoce una clasificación de los dominios con esta propiedad más allá de la publicada en [ADM] para dominios planos simplemente conexos. El primero de estos trabajos se debe a Friedrichs (ver [F]) y muestra que cierta desigualdad para funciones analíticas de variable compleja, desigualdad que sigue fácilmente de la existencia de **u** con las propiedades (Ec. 1) y (Ec. 2), no vale para ciertos dominios planos con una cúspide exterior de orden cuadrático (ver más abajo "algunos resultados equivalentes en dominios regulares"). Otros dominios planos donde no se verifica la desigualdad de Korn se pueden encontrar en [GG, D]. Para dominios en $\mathbb{R}^3$ podemos mencionar [W], donde también se estudian contraejemplos para la desigualdad de Korn.

Existen otros puntos de interés en relación al problema de la divergencia en un dominio Ω además de la existencia de soluciones en $W_0^{1,p}(\Omega)^n$. Por ejemplo, para aquellos dominios donde se conoce la existencia de soluciones, se intenta estimar la constante de continuidad en (Ec. 2) dependiendo de la geometría del dominio. Este resultado es relevante en la estimación del error en el análisis numérico de las ecuaciones de Stokes (ver [KuOp]). Otro punto de interés, central en este trabajo, es determinar si existen soluciones con una condición de continuidad más débil que (Ec. 2) en aquellos dominios donde las soluciones standard no existen, por ejemplo dominios Hölder-α. Finalmente, como hemos mencionado existen varios resultados equivalentes a la solubilidad del problema de la divergencia. Estas equivalencias valen en general para dominios planos simplemente conexos en $\mathbb{R}^2$. Así, un problema de interés es el estudio de estas equivalencias en dominios arbitrarios.

Algunos resultados equivalentes en dominios regulares

En esta sección recordamos algunos resultados equivalentes a la existencia de soluciones de la ecuación (Ec. 1) verificando (Ec. 2).

El primer caso que vamos a mencionar es la desigualdad de **Korn**. Como comentamos previamente existen distintas versiones de este resultado, una de ellas es la siguiente la cual utilizaremos frecuentemente en este trabajo. Dada una bola B contenida compactamente en Ω existe una constante C dependiendo únicamente de Ω, B y p tal que para todo campo $\mathbf{v}$ en $W^{1,p}(\Omega)^n$ se verifica

$$\|D\mathbf{v}\|_{L^p(\Omega)^{n\times n}} \;\leq\; C\Big\{\|\varepsilon(\mathbf{v})\|_{L^p(\Omega)^{n\times n}} + \|\mathbf{v}\|_{L^p(B)^n}\Big\}.$$

Es sabido que la desigualdad clásica de Korn, definida en la página 2, se puede obtener a partir de esta utilizando un argumento de compacidad.

Otro resultado equivalente al problema de la divergencia para un dominio regular plano Ω fue obtenido por **Friedrichs** en [F]. En este trabajo el autor muestra que si Ω, en el caso $p=2$, es cierto dominio plano simplemente conexo y $\mathbf{w}(z)=f(x,y)+ig(x,y)$ es una función analítica con $z=x+iy$ y f y g dos funciones reales tal que $\int_\Omega f=0$ entonces

$$\|f\|_{L^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}$$

donde C depende sólo de Ω. Para más detalles sobre las equivalencias entre el problema de la divergencia y las desigualdades de Korn y Friedrichs se puede ver [HP].

El resultado conocido como **Lema de Lions** es otro resultado equivalente a la existencia de soluciones para el problema de la divergencia en el caso $p=2$, el cual afirma que

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{H^{-1}(\Omega)^n} + \|f\|_{H^{-1}(\Omega)}\right),$$

para toda $f\in L^2(\Omega)$ donde C depende únicamente de Ω y $H^{-1}(\Omega)$ denota el dual del espacio de Sobolev $H_0^1(\Omega)$.

En el caso particular de funciones de integral cero es posible concluir la siguiente desigualdad

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{H^{-1}(\Omega)^n}.$$

Observemos que este resultado es más fuerte que la desigualdad de Poincaré donde se utiliza la norma de $L^2(\Omega)^n$ en lugar de $H^{-1}(\Omega)^n$ en el lado derecho. Dado que la demostración es muy sencilla veamos como obtener este resultado a partir de la existencia de soluciones del problema de la divergencia. Dada $f\in L^2(\Omega)$ de integral cero sea $\mathbf{u}\in H_0^1(\Omega)^n$ una solución de (Ec. 1) verificando (Ec. 2). Así, dividiendo por $\|f\|_{L^2(\Omega)}$ la siguiente desigualdad demostramos lo que estabamos buscando

$$\|f\|^2_{L^2(\Omega)} \;=\; \int_\Omega f\,\mathrm{div}\,\mathbf{u} \leq \|\nabla f\|_{H^{-1}(\Omega)^n}\|\mathbf{u}\|_{H_0^1(\Omega)^n} \leq C\|\nabla f\|_{H^{-1}(\Omega)^n}\|f\|_{L^2(\Omega)}.$$

Una reescritura del lema de Lions para funciones de integral cero, muy utilizada en análisis numérico para el estudio variacional de las ecuaciones de Stokes, es la siguiente

$$\inf_{0\neq q\in L_0^2(\Omega)} \sup_{0\neq\mathbf{u}\in H_0^1(\Omega)^n} \frac{\int_\Omega q\,\mathrm{div}\,\mathbf{u}}{\|q\|_{L_0^2(\Omega)}\|\mathbf{u}\|_{H_0^1(\Omega)^n}} \geq C, \tag{Ec. 4}$$

donde C es una constante positiva. Específicamente, la validez de (Ec. 4) implica la existencia de una única solución $(\mathbf{u}, p)$ en $H_0^1(\Omega)^n \times L_0^2(\Omega)$ del sistema

$$\begin{cases} \int_\Omega D\mathbf{u} : D\mathbf{v} - \int_\Omega p \operatorname{div} \mathbf{v} & = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in H_0^1(\Omega)^n \\ \int_\Omega q \operatorname{div} \mathbf{u} & = 0 \qquad \forall q \in L_0^2(\Omega), \end{cases}$$

donde $\mathbf{f} \in H^{-1}(\Omega)^n$ y el producto entre dos matrices $A = (a_{ij})$ y $B = (b_{ij})$ en $\mathbb{R}^{n\times n}$ está definido por $A : B = \sum_{i,j=1}^n a_{ij}b_{ij}$. Además, se obtiene la siguiente estimación a priori

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)^n}.$$

La desigualdad (Ec. 4) es conocida como la condición **inf-sup**.

Definición de algunos dominios de interés

Mencionemos las definiciones y algunas características de los distintos tipos de dominios considerados en este trabajo.

Un conjunto $\mathcal{C} \subset \mathbb{R}^n$ es un cono si existen $r_1, r_2 \in \mathbb{R}_{>0}$ tal que, en algún sistema de coordenadas ortogonal $(x_1, \ldots, x_n)$,

$$\mathcal{C} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, 0 < x_n < r_1 \text{ y } x_n^{-1}x' \in B_{r_2}\}, \tag{Ec. 5}$$

donde B_{r_2} es la bola de radio r_2 centrada en el origen de $\mathbb{R}^{n-1}$. Así, dado un dominio acotado $\Omega \subset \mathbb{R}^n$ decimos que es **Lipschitz** si para todo $x_0 \in \partial\Omega$ existe un cono $\mathcal{C} \subset \mathbb{R}^n$ y un entorno U de x_0 tal que $x + \mathcal{C} \subset \Omega$ para todo $x \in U \cap \bar{\Omega}$.

Decimos que $\Omega \subset \mathbb{R}^n$ es **estrellado respecto de una bola** $B \subset \mathbb{R}^n$ si para todo $x \in \Omega$ e $y \in B$ el segmento que los tiene por extremos está contenido en Ω. Esta clase de dominios contiene estrictamente a los conjuntos convexos y a su vez está contenida estrictamente en los dominios Lipschitz. Por otro lado, se sabe que los dominios Lipschitz son unión finita de estrellados respecto de una bola (ver [G]) y que si existe solución de la divergencia para una colección finita de abiertos entonces existe para la unión de ellos (ver [B]). Así, los resultados que se obtengan en relación al problema de la divergencia sobre estrellados se pueden extender a Lipschitz.

Otra clase de dominios muy importante es la de **John domains**. Esta clase es una generalización de los dominios Lipschitz donde es posible "torcer" el cono $\mathcal{C}$ de forma tal de incluir $x + \mathcal{C}$ en Ω. Es decir, decimos que un dominio acotado Ω es un John domain con un punto distinguido $x_0 \in \Omega$ si existe una constante positiva C tal que para todo $x \in \Omega$ existe una curva rectificable parametrizada por longitud de arco $\sigma : [0, l] \to \Omega$ tal que $\sigma(0) = x$ y $\sigma(l) = x_0$ verificando

$$\operatorname{dist}(\sigma(t), \partial\Omega) \geq Ct. \tag{Ec. 6}$$

Los dominios **Hölder-α**, con $0 < \alpha \leq 1$, pueden ser definidos del mismo modo que los Lipschitz reemplazando los conos en (Ec. 5) por α-cúspides. Un conjunto $\mathcal{C} \subset \mathbb{R}^n$ es una α-cúspide si existen $r_1, r_2 \in \mathbb{R}_{>0}$ tal que, en algún sistema de coordenadas ortogonal $(x_1, \ldots, x_n)$,

$$\mathcal{C} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, 0 < x_n < r_1 \text{ and } x_n^{-\gamma} x' \in B_{r_2}\},$$

donde $\gamma = 1/\alpha$ y B_{r_2} es la bola de radio r_2 centrada en el origen de $\mathbb{R}^{n-1}$. Observemos que en el caso particular en que $\alpha = 1$ obtenemos precisamente la definición de Lipschitz. Por otro lado, si $0 < \alpha < 1$ los dominios Hölder-α pueden presentar cúspides exteriores de tipo potencia. Es más, todos los contraejemplos conocidos, previos a este trabajo, para la no existencia de soluciones de (Ec. 1) con (Ec. 2) pertenecen a esta clase.

Una generalización de los John domain es la clase de **s-John domains**, con $s \geq 1$. La definición de un s-John domain se obtiene reemplazando en la definición de John domain la condición (Ec. 6) por

$$\text{dist}(\sigma(t), \partial\Omega) \geq Ct^s.$$

Objetivos principales de la tesis

El objetivo central de este trabajo es probar existencia de soluciones en espacios de Sobolev con peso de la ecuación $\text{div}\,\mathbf{u} = f$ para dominios donde las soluciones standard (sin peso) no existen. Antes de continuar, definamos los espacios de Sobolev con peso con los que vamos a trabajar. Dada $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ una función localmente integrable, a la cual llamaremos un peso, definimos los espacios de Sobolev con peso $L^p(\Omega, \omega)$ y $W^{1,p}(\Omega, \omega)$ formados por funciones localmente integrables con las siguientes normas acotados

$$\|\varphi\|^p_{L^p(\Omega,\omega)} = \int_\Omega |\varphi(x)|^p\, \omega(x)\mathrm{d}x$$

y

$$\|\varphi\|^p_{W^{1,p}(\Omega,\omega)} = \int_\Omega |\varphi(x)|^p\, \omega(x)\mathrm{d}x \,+\, \sum_{i=1}^n \int_\Omega |\partial\varphi(x)/\partial x_i|^p\, \omega(x)\mathrm{d}x,$$

respectivamente. Definimos también el subespacio $W^{1,p}_0(\Omega, \omega)$ como la clausura de $C^\infty_0(\Omega)$ en $W^{1,p}(\Omega, \omega)$.

Dado $\Omega \subset \mathbb{R}^n$ un dominio acotado con cúspides exteriores buscamos determinar los pesos apropiados que garanticen la existencia de una inversa para el operador divergencia. Es decir, queremos encontrar pesos ω_1 y ω_2 tal que para toda $f \in L^p(\Omega, \omega_2)$ de integral cero exista una solución $\mathbf{u} \in W^{1,p}_0(\Omega, \omega_1)^n$ de

$$\text{div}\,\mathbf{u} = f$$

tal que

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,\omega_1)^n} \leq C\|f\|_{L^p(\Omega,\omega_2)}, \tag{Ec. 7}$$

para una constante C que dependa únicamente de Ω, ω_1, ω_2 y p.

Dado que las cúspides son un impedimento para la existencia de soluciones standard es razonable utilizar pesos que involucren la distancia al borde del dominio. Por otro lado, en el caso particular de dominios Hölder-α, donde las cúspides son de tipo potencia, es apropiado considerar pesos del tipo potencia de la distancia al borde.

Por otro lado, para dominios con una sola singularidad puede resultar interesante considerar pesos que involucren la distancia a la singularidad.

Otro objetivo central de este trabajo es determinar si es posible utilizar las soluciones con peso de la divergencia para obtener resultados de existencia y unicidad de soluciones para las ecuaciones de Stokes en dominios con cúspides. Resultados que hasta el momento no se conocían en este tipo de dominios.

Si bien los dos objetivos principales de la tesis ya fueron mencionados existen otros problemas considerados en este trabajo. Uno de ellos es generalizar las equivalencia mencionadas previamente a espacios de Sobolev con peso. Otro es encontrar nuevos dominios para los cuales no exista una solución de (Ec. 1) verificando (Ec. 7), por ejemplo considerar dominios con cúspides que no sean de tipo potencia.

Estructura de la tesis y resultados obtenidos

Capítulo 1

En el primer capítulo estudiamos el problema de la divergencia en dominios Hölder-α planos simplemente conexos, donde α es un número real en $(0, 1)$. Para ello, utilizamos los resultados de Korn y Poincaré con peso para este tipo de dominios publicados en [ADL].

En la sección 1.1 adaptamos la versión de Korn de [ADL] a una forma generalizada del segundo caso que resulta más conveniente para nuestro propósito. Si bien vamos a utilizar esta versión de Korn en el caso plano la escribimos en $\mathbb{R}^n$ dado que no representa una dificultad adicional y nos parece que tiene interés en sí misma.

En la sección 1.2 mostramos el resultado principal del capítulo, la existencia de una solución $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_2) \in W^{1,p}(\Omega, \omega_1)^2$ de (Ec. 1) con una condición de borde dada por la siguiente propiedad

$$\int_\Omega \left(\frac{\partial \mathrm{u}_i}{\partial x_2}, -\frac{\partial \mathrm{u}_i}{\partial x_1}\right) \cdot \nabla\phi = 0 \tag{Ec. 8}$$

para ciertas funciones de prueba $\phi \in C^\infty(\Omega)$ apropiadas, con $i = 1, 2$. La condición (Ec. 8) implica que $\mathbf{u}$ es constante en $\partial\Omega$ en un sentido distribucional. Además, la condición de continuidad que se obtiene, más débil que (Ec. 2), es

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,\omega_1)^2} \leq C\|f\|_{L^p(\Omega,\omega_2)},$$

donde ω_1 es una potencia positiva de la distancia al borde y ω_2 una potencia negativa.

En la sección 1.3 mostramos que en cierta clase de dominios planos Hölder-α toda función que verifica (Ec. 8) está en la clausura de $C_0^\infty(\Omega)^2$, restando una constante apropiada. Además, mostramos que los pesos utilizados en la sección anterior resultan óptimos en este tipo de dominios.

En la sección 1.4 utilizamos la inversa a derecha de la divergencia con la noción de continuidad (Ec. 7) para $\omega_1 = 1$ y $p = 2$ en los dominios particulares de la sección 1.3 para probar existencia y unicidad de soluciones de las ecuaciones de Stokes en espacios de Hilbert apropiados.

Por último, en la sección 1.5 mostramos que para dominios planos simplemente conexos (no es necesaria ninguna condición de regularidad sobre el borde) la desigualdad de Korn con peso resulta equivalente a la existencia de soluciones en espacios de Sobolev con peso de $\operatorname{div}\mathbf{u} = f$ con la condición de traza constante considerada en la sección 1.2. De este modo, se puede ver que la equivalencia entre estos dos resultados sobre dominios planos se generaliza del caso Lipschitz, tratado en [KuOp], a dominios arbitrarios y espacios de Sobolev con peso.

Capítulo 2

En este capítulo mostramos la existencia de una inversa a derecha continua para el operador de divergencia en dominios estrellados respecto de una bola y espacios de Sobolev con peso, donde los pesos utilizados coinciden en los espacios de salida y llegada del operador.

En la sección 2.1 mostramos que si $F \subset \mathbb{R}^n$ es un conjunto compacto contenido en un m-regular set K, es decir, un conjunto K tal que la medida de Hausdorff $m-$dimensional de $B(x,r) \cap K$ es equivalente a r^m para todo $x \in K$ y r suficientemente chico, entonces

$$d_F^\mu \in A_p \qquad \text{si} \qquad -(n-m) < \mu < (n-m)(p-1),$$

donde d_F denota la distancia a F. Recordamos la definición de la clase de Muckenhoupt A_p en (0.5).

En la sección 2.2 y con la idea de hacer una tesis autocontenida, recordamos la muy conocida formulación de Bogovskii. En la cual se puede leer que en dominios estrellados respecto de una bola existe una inversa del operador divergencia la cual puede escribirse vía operadores integrales singulares (ver [B, DM[1], G]). Una demostración alternativa para John-domains se puede encontrar en el reciente trabajo [ADM]. La mayoría de los resultados desarrollados en los dos primeros capítulos fueron incluidos en [DLg[1]].

Capítulo 3

En este capítulo mostramos que para ciertos dominios Hölder-α en $\mathbb{R}^n$ con una cúspide de dimensión natural $m \leq n-2$ es posible encontrar un resultado más fuerte que el exhibido en el capítulo 1 considerando pesos del tipo potencias de la distancia a la singularidad.

En la sección 3.1 mostramos el resultado principal del capítulo, una solución para la divergencia involucrando la distancia a la cúspide.

En la sección 3.2 y usando el resultado obtenido en la sección previa, obtenemos una versión con peso de la desigualdad de Korn. En ambas secciones los pesos obtenidos resultan óptimos lo que será demostrado en el capítulo 4 .

En la sección 3.3 mostramos la segunda aplicación de los resultados obtenidos en la sección 3.1. Mostramos existencia y unicidad de soluciones variacionales para las ecuaciones de Stokes en dominios con cúspides en espacios de Sobolev con peso, donde los pesos son potencias de la distancia a la singularidad.

Los temas tratados en este capítulo fueron incluidos en [DLg[2]].

Capítulo 4

En este capítulo construimos dominios cuspidales, con cúspides no necesariamente del tipo potencia, para los cuales no existen soluciones de (Ec. 1) verificando (Ec. 2) o falla alguno de los resultados equivalentes.

En la sección 4.1 hacemos una recopilación de distintos contraejemplos que fueron apareciendo comenzando por el primero de ellos enunciado en 1937 por Friedrichs.

En la sección 4.2 construimos una serie de contraejemplos con cúspides arbitrarias (hasta el momento sólo se habían mostrado cúspides de tipo potencia) y condiciones necesarias sobre los pesos cuando está involucrada la distancia a la singularidad.

En la sección 4.3 mostramos el exponente necesario para obtener soluciones de la divergencia cuando está involucrada la distancia al borde del dominio.

Los contraejemplos exhibidos en este último capítulo serán incluidos en el trabajo [ADLg].

Introduction

The divergence problem on regular domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with some regularity conditions, for example smooth boundary. Given $f \in L^p(\Omega)$ with vanishing mean value, with $1 < p < \infty$, it is known that there exists a solution $\mathbf{u} \in W^{1,p}(\Omega)^n$, with trace zero, of the equation

$$\operatorname{div} \mathbf{u} = f \tag{Eq. 1}$$

satisfying

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}, \tag{Eq. 2}$$

where the constant C depends only on Ω and p. In others words, there exists a continuous right inverse for the divergence as an operator from the Sobolev space $W_0^{1,p}(\Omega)^n$ to $L_0^p(\Omega)$, where $L_0^p(\Omega)$ denotes the space of function in $L^p(\Omega)$ with vanishing mean value and $W_0^{1,p}(\Omega)^n$ the closure of $C_0^\infty(\Omega)^n$ in $W^{1,p}(\Omega)^n$.

This result has several applications. For example, in the particular case $p = 2$, it is fundamental for the variational analysis of the Stokes equations, which modeling the displacement of a viscous incompressible fluid contained in Ω . Precisely, if there exists a solution of (Eq. 1) satisfying the condition (Eq. 2) we obtain a unique variational solution $(\mathbf{u}, p)$ in the Hilbert space $H_0^1(\Omega)^n \times L_0^2(\Omega)$ of the following system of equations,

$$\begin{cases} -\Delta \mathbf{u} + \nabla p & = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} & = 0 & \text{in } \Omega \\ \mathbf{u} & = 0 & \text{in } \partial\Omega, \end{cases} \tag{Eq. 3}$$

for all $f \in H^{-1}(\Omega)^n$, where $H^{-1}(\Omega)^n$ denotes the dual of the Sobolev space $H_0^1(\Omega)^n$. Also, it holds the following a priori estimation

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)^n},$$

where the constant C depends only on Ω. The variational analysis of the Stokes equations, and in consequence the existence of solutions for the divergence problem, is fundamental for the development of the finite element numerical approximations of these solutions.

On the other hand, there are several results equivalent to the existence of solutions for the divergence problem on regular domains. One of the most known is the inequality of Korn which is basic in the analysis of the linear elasticity equations. It can be found on the literature different versions of this result, one of them is the so called classic Korn inequality which states that

$$\|D\mathbf{v}\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(\mathbf{v})\|_{L^p(\Omega)^{n\times n}},$$

for all $\mathbf{v} \in W^{1,p}(\Omega)^n$ satisfying some particular conditions which prevents that $\varepsilon(\mathbf{v}) = 0$ while $D\mathbf{v} \neq 0$, where $\varepsilon(\mathbf{v})$ denotes the symmetric part of the differential matrix of $\mathbf{v}$, namely,

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i}\right).$$

The conditions considered by Korn were $\mathbf{v} = 0$ in $\partial\Omega$ (usually called first case) and $\int_\Omega \operatorname{rot} \mathbf{v} = 0$ (second case). We will show later another results equivalent to the existence of solutions for the divergence problem.

According to the variety of applications and equivalent results, this problem has been hardly studied and diverse methods have been developed to prove the existence of a solution $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ of (Eq. 1) verifying (Eq. 2) on different kind of domains. Let us mention some articles where this problem was analyzed. For example, if the domain $\Omega \subset \mathbb{R}^2$ has smooth boundary or if it is a convex polygon then existence of a solution can be proved via the Neumann problem for the Laplace operator (see [ASV, BA, BB, BS, L]). Indeed, it is well known that for this kind of domains there exists $\mathbf{v} \in H^2(\Omega)$ satisfying

$$\begin{cases} -\Delta \mathbf{v} &= f \qquad \text{in } \Omega \\ \partial \mathbf{v}/\partial \eta &= 0 \qquad \text{in } \partial\Omega, \end{cases}$$

where η denotes the external normal to the boundary of Ω. Thus, $\tilde{\mathbf{u}} = \nabla \mathbf{v}$ is a solution of the divergence problem in $H^1(\Omega)$ satisfying (Eq. 2) as a consequence of the a priori estimation of $\mathbf{v}$.

In [B, DM[1]] it was defined an explicit solution for the divergence problem on domains which are star-shaped with respect to a ball (in page 15 we recall the definition of this type of domains). In order to prove the condition (Eq. 2) the authors use the theory of singular integral operator introduced by Calderón and Zygmund. This approach was extended in [ADM] to John domains (see page 15).

Recently, in [DRS] it was proved that a function with vanishing mean value in a John domain can be decomposed as a countable sum of function with vanishing mean value supported in cubes. Thus, solving (Eq. 1) in cubes the authors show an alternative proof to the one previously published in [ADM]. Similarly, in [DMRT] it was shown the solvability of (Eq. 1) with a condition similar to (Eq. 2) in weighted Sobolev spaces using a decomposition as in [DRS] but for arbitrary domains.

On the other hand, some counterexamples have been given in order to show that for some domains there is not a solution of the divergence equation verifying (Eq. 2), or some equivalent result fails. However, there is not a complete characterization of the domains where these properties fail or not. It has only been done for simply connected planar domains in [ADM].

The first counterexample was given by Friedrichs (see [F]). Indeed, Friedrichs introduced a class of planar domains with a quadratic external cusp where a related inequality for analytic functions in complex variable does not hold (see below "some equivalent results on regular domains" for details). This inequality can be deduced from the existence of $\mathbf{u}$ satisfying (Eq. 1) and (Eq. 2). Another plane domain where the Korn inequality fails was published in [GG, D]. For domains in $\mathbb{R}^3$ we can cite [W].

Apart from the existence of solutions of the divergence in $W_0^{1,p}(\Omega)^n$, there exist other interest related problems. For example, for domains where solutions exist it is of interest to estimate the optimal constant in (Eq. 2) depending on the shape of the domain. This result is relevant in applications in numerical analysis of the Stokes equations (see [KuOp]).

For domains where solutions of (Eq. 1) and (Eq. 2) do not exist, for example in Hölder-α domains, it is of interest to see if there are solutions of the divergence satisfying a condition weaker than (Eq. 2). This is the main problem of this work.

Finally, as we have mentioned, there exist several results equivalent to the solvability of the divergence problem in standard Sobolev spaces. These equivalences hold in general for simply connected Lipschitz domains in $\mathbb{R}^2$. Thus, a problem of interest is the analysis of these equivalences on arbitrary domains for weighted Sobolev spaces.

Some equivalent results on regular domains

In this section we recall some results equivalent to the existence of a solution of (Eq. 1) verifying (Eq. 2).

The first one is the **Korn** inequality. Even if we have mentioned previously this result, we will introduce a new version which we will use frequently in this thesis. Given B a ball compactly contained in Ω there exists a constant C depending only on Ω, B and p such that

$$\|D\mathbf{v}\|_{L^p(\Omega)^{n\times n}} \leq C\Big\{\|\varepsilon(\mathbf{v})\|_{L^p(\Omega)^{n\times n}} + \|\mathbf{v}\|_{L^p(B)^n}\Big\},$$

for all $\mathbf{v}$ in $W^{1,p}(\Omega)^n$. It is well known that the classic Korn inequality mentioned at the beginning of this introduction can be derived from this one by using compactness arguments.

Another equivalent result was introduced by **Friedrichs** in [F]. This result state that if Ω is a simply connected planar domain with some additional property and $\mathbf{w}(z) =$

$f(x,y)+ig(x,y)$ is an analytic function with $z=x+iy$ and f and g two real functions such that $\int_\Omega f=0$ then

$$\|f\|_{L^2(\Omega)} \leq C\|g\|_{L^2(\Omega)},$$

where C is depending only on Ω. For more details about the equivalence between the divergence problem and the inequalities Korn and Friedrichs see [HP].

The result known as **Lions lemma** is also equivalent to the existence of solutions of the divergence problem, with $p=2$. This result asserts that

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{H^{-1}(\Omega)^n} + \|f\|_{H^{-1}(\Omega)}\right),$$

for all $f \in L^2(\Omega)$ where C depends only on Ω and $H^{-1}(\Omega)$ denotes the dual of the Sobolev space $H_0^1(\Omega)$. In the particular case of functions with vanishing mean value, it is possible to conclude the following inequality

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{H^{-1}(\Omega)^n}.$$

Observe that this result is stronger than the usual Poincaré inequality where the norm $H^{-1}(\Omega)^n$ is replaced by the one in $L^2(\Omega)^n$ in the right side. As the proof is very short, let us show that the existence of solutions of the divergence problem implies this simplified version of Lions. Given $f \in L^2(\Omega)$ integrating zero, let $\mathbf{u} \in H_0^1(\Omega)^n$ a solution of (Eq. 1) verifying (Eq. 2). Thus, dividing by $\|f\|_{L^2(\Omega)}$ we prove what we had claimed

$$\|f\|^2_{L^2(\Omega)} = \int_\Omega f \operatorname{div}\mathbf{u} \leq \|\nabla f\|_{H^{-1}(\Omega)^n}\|\mathbf{u}\|_{H_0^1(\Omega)^n} \leq C\|\nabla f\|_{H^{-1}(\Omega)^n}\|f\|_{L^2(\Omega)}.$$

A rewriting of Lions lemma for functions with vanishing mean value, strongly used in the analysis on the Stokes equation, is the following

$$\inf_{0\neq q\in L_0^2(\Omega)} \sup_{0\neq \mathbf{u}\in H_0^1(\Omega)^n} \frac{\int_\Omega q \operatorname{div}\mathbf{u}}{\|q\|_{L_0^2(\Omega)}\|\mathbf{u}\|_{H_0^1(\Omega)^n}} \geq C, \qquad \text{(Eq. 4)}$$

where C denotes a positive constant. In fact, the validity of (Eq. 4) implies the existence of a unique solution $(\mathbf{u},p)$ in $H_0^1(\Omega)^n \times L_0^2(\Omega)$ of the system of equations

$$\begin{cases} \int_\Omega D\mathbf{u} : D\mathbf{v} - \int_\Omega p \operatorname{div}\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v} & \forall \mathbf{v} \in H_0^1(\Omega)^n \\ \int_\Omega q \operatorname{div}\mathbf{u} = 0 & \forall q \in L_0^2(\Omega), \end{cases}$$

where $\mathbf{f} \in H^{-1}(\Omega)^n$, $D\mathbf{v}$ denotes the matrix of partial derivatives of $\mathbf{v}$ and the product between two matrices $A=(a_{ij})$ and $B=(b_{ij})$ in $\mathbb{R}^{n\times n}$ is defined by $A:B=\sum_{i,j=1}^n a_{ij}b_{ij}$. Furthermore, it holds the following a priori estimate

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)^n}.$$

The inequality (Eq. 4) is usually called **inf-sup** condition.

Definition of some relevant domains

Let us mention the definition and some important properties of the different domains considered in this thesis.

We say that a set $\mathcal{C} \subset \mathbb{R}^n$ is a cone if there exist $r_1, r_2 \in \mathbb{R}_{>0}$ such that, in some orthogonal coordinate system $(x_1, \ldots, x_n)$,

$$\mathcal{C} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, 0 < x_n < r_1 \text{ and } x_n^{-1} x' \in B_{r_2}\}, \tag{Eq. 5}$$

where B_{r_2} is the ball with center in the origin of $\mathbb{R}^{n-1}$ and radius r_2. Thus, given a bounded domain $\Omega \subset \mathbb{R}^n$ we say that it is **Lipschitz** if for all $x_0 \in \partial\Omega$ there exists a cone $\mathcal{C} \subset \mathbb{R}^n$ and a neighborhood U of x_0 such that $x + \mathcal{C} \subset \Omega$ for all $x \in U \cap \bar{\Omega}$.

A domain $\Omega \subset \mathbb{R}^n$ is a **star-shaped domain with respect to a ball** $B \subset \mathbb{R}^n$ if for all $x \in \Omega$ and $y \in B$ the segment joining x and y is included in Ω. This class contains the convex domains and is included in the Lipschitz class. On the other hand, it is known that any Lipschitz domain can be written as a finite union of domains which are star-shaped with respect to a Ball (see [G]), and that if there exists a solution of the divergence problem for each domain in a finite sequence of domains then there exists for the union of them (see [B]). Thus, the solvability of the divergence equation for star-shaped domains can be generalized to Lipschitz.

The class of **John domains** is a generalization of the Lipschitz one. In this case, it is possible "to twist" the cone $\mathcal{C}$ in order to include $x + \mathcal{C}$ in Ω. In fact, we say that a bounded domain Ω is a John domain with respect to $x_0 \in \Omega$ if for all $x \in \Omega$ there exists a rectifiable curve $\sigma : [0, l] \to \Omega$ parameterized by arc length such that $\sigma(0) = x$ and $\sigma(l) = x_0$ satisfying

$$\operatorname{dist}(\sigma(t), \partial\Omega) \geq Ct, \tag{Eq. 6}$$

where C depends only on Ω.

Given $0 < \alpha \leq 1$, it is possible to define the class of **Hölder-α** domains as the Lipschitz one replacing the cones in (Eq. 5) by α-cusps. A set $\mathcal{C} \subset \mathbb{R}^n$ is an α-cusp if there exists $r_1, r_2 \in \mathbb{R}_{>0}$ such that, in some orthogonal coordinate system $(x_1, \ldots, x_n)$,

$$\mathcal{C} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, 0 < x_n < r_1 \text{ and } x_n^{-\gamma} x' \in B_{r_2}\},$$

where $\gamma = 1/\alpha$ and B_{r_2} is the ball with radius r_2 centered at the origin of $\mathbb{R}^{n-1}$. Observe that in the particular case $\alpha = 1$ we obtain the Lipschitz domains. In addition, the solvability of the divergence in standard Sobolev spaces fails in general on Hölder-α domains if $0 < \alpha < 1$.

A generalization of John domains containing cuspidal domains is the class of **s-John**, with $s \geq 1$. We get the definition of s-John domains replacing the condition (Eq. 6) by

$$\operatorname{dist}(\sigma(t), \partial\Omega) \geq Ct^s.$$

Observe that a Hölder-α domain is in particular an s-John domain with $s = 1/\alpha$.

Goals of the thesis

The aim of this work is to prove the existence of solutions in weighted Sobolev spaces of the equation $\operatorname{div} \mathbf{u} = f$ on domains where the standard solutions (without weights) do not exist. Let us define first the spaces that we will use. Given $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ a function locally integrable, namely a weight, we define the weighted Sobolev spaces $L^p(\Omega, \omega)$ and $W^{1,p}(\Omega, \omega)$ as the space of functions locally integrable satisfying

$$\|\varphi\|^p_{L^p(\Omega,\omega)} = \int_\Omega |\varphi(x)|^p \, \omega(x) \mathrm{d}x$$

and

$$\|\varphi\|^p_{W^{1,p}(\Omega,\omega)} = \int_\Omega |\varphi(x)|^p \, \omega(x) \mathrm{d}x \, + \sum_{i=1}^{n} \int_\Omega |\partial\varphi(x)/\partial x_i|^p \, \omega(x) \mathrm{d}x,$$

respectively. We denote by $W^{1,p}_0(\Omega, \omega)$ the closure $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega, \omega)$.

Thus, given $\Omega \subset \mathbb{R}^n$ a bounded domain with external cusps we want to find weights ω_1 and ω_2 such that for all $f \in L^p(\Omega, \omega_2)$, integrating zero, there exists a solution $\mathbf{u} \in W^{1,p}_0(\Omega, \omega_1)^n$ of

$$\operatorname{div} \mathbf{u} = f$$

verifying

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,\omega_1)^n} \leq C\|f\|_{L^p(\Omega,\omega_2)}, \tag{Eq. 7}$$

where C denotes a positive constant depending only on Ω, p, ω_1 and ω_2. It is known that the solvability in standard Sobolev spaces of the divergence fails on some domains with external cusps. Thus, it may be appropriate to consider weights involving the distance to the boundary or to the cusp. In particular, when the cusps are quadratic or a general power we will consider powers of the distance.

Another goal of this thesis is to apply the results obtained in weighted spaces for cuspidal domains to get existence and uniqueness of solution of the Stokes equations on this kind of domains using weighted norms.

In this thesis we also analyzed other problems. First we generalize the equivalences with the divergence problem to the case of weighted spaces and irregular domains. Second, we introduce new counterexamples for the solvability of (Eq. 1) and (Eq. 2). In particular, we show that the result fails for very general external cusps.

Outline of the thesis and obtained results

Chapter 1

In the first chapter we study the divergence problem in planar simply connected Hölder-α domains, where α is a real number in $(0,1)$. To prove existence of solutions in Sobolev weighted spaces of this problem, we use the weighted Korn and Poincaré results published in [ADL].

In section 1.1 we adapt the Korn inequality published in [ADL] to obtain a new version more convenient for our propose. Although we will need this Korn result in the planar case, we will give the proof in $\mathbb{R}^n$ because it has the same difficulty and, furthermore, we find it of interest in itself.

In section 1.2 we show the main result of the chapter, the existence of a solution $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_2) \in W^{1,p}(\Omega, \omega_1)^2$ of (Eq. 1) with a boundary condition given by the following property

$$\int_\Omega \left(\frac{\partial \mathrm{u}_i}{\partial x_2}, -\frac{\partial \mathrm{u}_i}{\partial x_1} \right) \cdot \nabla \phi = 0 \qquad \text{(Eq. 8)}$$

for appropriate test function $\phi \in C^\infty(\Omega)$ with $i = 1, 2$. Condition (Eq. 8) implies that $\mathbf{u}$ is constant in a distributional sense in $\partial\Omega$. Furthermore, we obtain the following continuity condition, weaker than (Eq. 2),

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,\omega_1)^2} \leq C\|f\|_{L^p(\Omega,\omega_2)},$$

where ω_1 is a positive power of the distance to the boundary and ω_2 a negative one.

In section 1.3 we prove that every function satisfying (Eq. 8) is in the closure of $C_0^\infty(\Omega)^2$, adding an appropriate constant, for certain class of planar Hölder-α domains. In addition, we demonstrate that the weights used in the previous section can not be improved for this type of domains

In sección 1.4 we use the solvability of the divergence problem developed previously with the condition (Eq. 7) for $\omega_1 = 1$ and $p = 2$ to prove existence and uniqueness of solutions of the Stokes equations in appropriate Hilbert spaces on the domains introduced in section 1.3.

Finally, in section 1.5 we show that for simply connected planar domains (it is not necessary any regularity condition on the boundary) the weighted Korn inequality is equivalent to the existence of solutions in weighted Sobolev spaces of $\operatorname{div} \mathbf{u} = f$ with the boundary condition considered in section 1.2. Thus, the equivalence between these results for planar domains can be generalized from Lipschitz, studied in [KuOp], to arbitrary domains and weighted Sobolev spaces.

Chapter 2

In this chapter we deal with the existence of a continuous right inverse for the divergence operator in star-shaped domains with respect to a ball and weighted Sobolev spaces, when the weights used in the continuity condition are the same in both sides.

In section 2.1 we show that if $F \subset \mathbb{R}^n$ is a compact set included in a m-regular set K, namely, K is a set such that the Hausdorff measure $m-$dimensional of $B(x,r) \cap K$ is equivalent to r^m for all $x \in K$ and r small, then

$$d_F^\mu \in A_p \qquad \text{if} \qquad -(n-m) < \mu < (n-m)(p-1),$$

where d_F denotes the distance to F. The definition of the Muckenhoupt class A_p is recalled in (0.5).

In section 2.2, in order to make this thesis self-contained, we recall the well known Bogovskii formula. It asserts that, for domains which are star-shaped with respect to a ball, there exists a right inverse for the divergence operator given by an explicit integral operator (see [B, DM[1], G]). Analogously, in [DRS] the authors show an explicit solution of $\operatorname{div} \mathbf{u} = f$ for John domains.

The main results developed in the first two chapters were included in [DLg[1]].

Chapter 3

In this chapter we show that for certain Hölder-α domains in $\mathbb{R}^n$ with a cusp of natural dimension $m \leq n-2$, it is possible to find a result stronger than the one proven in chapter 1 considering as weights powers of the distance to the singularity.

In section 3.1 we give the main result of the chapter, the existence of a weighted solution for the divergence where the weights involved are powers of the distance to the cusp.

In section 3.2, and using the result obtained in the previous section, we obtain a weighted version of the Korn inequality. In both sections, the weights obtained are optimal, this will be proved in chapter 4.

In section 3.3 we show another application of the results obtained in section 3.1. We prove existence and uniqueness of variational solutions for the Stokes equations in cuspidal domains in weighted Sobolev spaces where the weights are powers of the distance to the singularity.

The results developed in this chapter were included in [DLg[2]].

Chapter 4

In this chapter we construct counterexamples for very general cuspidal domains (not necessarily of power type) for the existence of solution of (Eq. 1) satisfying (Eq. 2), or some equivalent result.

In section 4.1 we recall different counterexamples which have been published over the years, starting with the first one given by Friedrichs in 1937.

In section 4.2 we construct a class of counterexamples for general cusps. Also, we show that the results obtained in Chapter 3 are optimal in the sense that the powers of the distance to the cusp involved in the estimates cannot be improved.

In section 4.3 we show the optimality of the results of Chapter 1.

The counterexamples given in this last chapter were included in [ADLg].

Preliminaries and Notations

In this thesis we deal with the solvability of the divergence problem in weighted Sobolev spaces for bounded domains. Let us introduce the definitions and notations that we will use.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < \infty$. We will say that $\mathbf{(div)_p}$ is solvable in Ω if there exists a solution $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ of the equation

$$\operatorname{div} \mathbf{u} = f, \tag{0.1}$$

for $f \in L^p(\Omega)$ integrating zero, such that

$$\|\mathbf{u}\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{0.2}$$

where the constant C depends only on Ω and p.

Now, let us introduce the weighted Sobolev spaces that we will utilize later.

We say that a function ω on $\mathbb{R}^n$ is a **weight** if it is a locally integrable and takes values in $(0, \infty)$ almost everywhere. Therefore, weights are permitted to be zero or infinite only on a set of Lebesgue measure zero.

Thus, given a domain $\Omega \subset \mathbb{R}^n$ we define for $1 < p < \infty$ the weighted Lebesgue space $L^p(\Omega, \omega)$ as the space of locally integrable functions $\varphi : \Omega \to \mathbb{R}$ equipped the following norm

$$\|\varphi\|^p_{L^p(\Omega,\omega)} = \int_\Omega |\varphi(x)|^p \, \omega(x)\mathrm{d}x.$$

Analogously, given weights $\omega_1, \omega_2 : \mathbb{R}^n \to [0, \infty]$ we define the following weighted Sobolev space

$$W^{1,p}(\Omega, \omega_1, \omega_2) = \left\{ \varphi \in L^p(\Omega, \omega_1) \, : \, \varphi \text{ is locally integrable and } \frac{\partial \varphi}{\partial x_i} \in L^p(\Omega, \omega_2), \, \forall i \right\},$$

where the partial derivative $\frac{\partial \varphi}{\partial x_i}$ is in the sense of distributions and the norm is

$$\|\varphi\|^p_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \int_\Omega |\varphi(x)|^p \, \omega_1(x)\mathrm{d}x + \sum_{i=1}^n \int_\Omega |\partial\varphi(x)/\partial x_i|^p \, \omega_2(x)\mathrm{d}x. \tag{0.3}$$

In the case that $\omega_1 = \omega_2 = \omega$ we will write $W^{1,p}(\Omega,\omega)$ instead of $W^{1,p}(\Omega,\omega,\omega)$ to simplify notation.

Since no confusion is possible we will use the same notations for the norms of vector or tensor fields.

We will essentially work with two different classes of weights, the first one is composed by powers of the distance to a subset M included in the border of the domain. And, we denote

$$\omega(x) = d_M^{\beta}(x) = (\mathrm{dist}(x,M))^{\beta},$$

where β is a real number. In the particular case in which M is equal to the border of the domain we write $d(x)$ instead of $d_{\partial\Omega}(x)$. In addition, for the real number β we introduce the followings notation

$$L^p(\Omega,\beta) = L^p(\Omega, d_{\partial\Omega}^{\beta}) \qquad \text{and} \qquad W^{1,p}(\Omega,\beta) = W^{1,p}(\Omega, d_{\partial\Omega}^{\beta}). \tag{0.4}$$

The second class of weights is the Muckenhoupt class A_p, for $1 < p < \infty$. Recall that a weight ω is said to be an A_p weight, if it satisfies that

$$\sup_{B\subset\mathbb{R}^n} \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty, \tag{0.5}$$

where the supremum is taken over all the balls $B \subset \mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of B (see for details [Du, S^3]).

It is known that, if ω, ω_1 and ω_2 are powers of the distance to a subset in $\partial\Omega$ or weights in the Muckenhoupt class it happens that the spaces $W^{1,p}(\Omega,\omega_1,\omega_2)$ and $L^p(U,\omega)$ are Banach spaces (see [Ku] or [GU], respectively).

Now, using that weights are locally integrable in Ω it follows that $C_0^{\infty}(\Omega)$ is included in $W^{1,p}(\Omega,\omega_1,\omega_2)$. Thus, this enables us to introduce the subspace of functions with trace zero $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as the closure of $C_0^{\infty}(\Omega)$ in the norm (0.3).

On the other hand, whenever $L^p(\Omega,\omega) \subset L^1(\Omega)$ we will call $L_0^p(\Omega,\omega)$ the subspace of $L^p(\Omega,\omega)$ formed by functions of vanishing mean value. In the follows remark, we show that if ω belongs to the A_p class or $\omega = d_M^{\gamma}$, with $M \subset \partial\Omega$ and $\gamma \le 0$, then the inclusion in $L^1(\Omega)$ is satisfied.

From now on q will denote $\frac{p}{p-1}$, the dual exponent of p.

Remark 0.1. *If Ω is a bounded domain and $\omega = d_M^{\gamma}$, with $M \subset \Omega$ and $\gamma \le 0$ then, $L^p(\Omega,\omega) \subset L^1(\Omega)$. Effectively, as Ω is bounded there exists a positive constant C such that $d_M \le C$ on Ω. Thus,*

$$\int_\Omega |\varphi| = \int_\Omega |\varphi|\omega^{1/p}\omega^{-1/p} \le \left(\int_\Omega |\varphi|^p\omega\right)^{1/p}\left(\int_\Omega \omega^{-q/p}\right)^{1/q} \le C^{-\gamma/p}|\Omega|^{1/q}\left(\int_\Omega |\varphi|^p\omega\right)^{1/p}.$$

Remark 0.2. *If Ω is a bounded domain and $\omega \in A_p$ then, $L^p(\Omega,\omega) \subset L^1(\Omega)$. Indeed, let B a ball containing U. We have,*

$$\begin{aligned}\int_\Omega |f| &= \int_\Omega |f|\omega^{1/p}\omega^{-1/p} \leq \left(\int_\Omega |f|^p\omega\right)^{1/p}\left(\int_\Omega \omega^{-q/p}\right)^{1/q} \\ &\leq |B|^{(p-1)/p}\|f\|_{L^p(\Omega,\omega)}\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\right)^{(p-1)/p}.\end{aligned}$$

In view of the previous remarks the space $L^p_0(\Omega,\omega)$ is well defined for the mentioned weights.

In order to analyze the solvability of the divergence problem in domains when $(\mathrm{div})_p$ is not solvable, we replace the condition (0.2) for another one more general involving weighted norms. Thus, we say that $\mathbf{(div)_{p,w}}$ is solvable in Ω for weights w_1 and w_2 if there exists a solution $\mathbf{u} \in W^{1,p}_0(\Omega,\omega_1)^n$ of the equation

$$\operatorname{div}\mathbf{u} = f,$$

with $f \in L^p_0(\Omega,\omega_2)$, such that

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,\omega_1)} \leq C\|f\|_{L^p(\Omega,\omega_2)}, \tag{0.6}$$

where C depends only on Ω, w_1, w_2 and p.

The following definition is fundamental in order to determinate when a power of the distance to a compact set belongs to the Muckenhoupt class.

Definition 0.3. *For $0 \leq m \leq n$, a compact set $F \subset \mathbb{R}^n$ is an* **m-regular set**, *if there exists a positive constant C such that*

$$C^{-1}r^m < \mathcal{H}^m(B(x,r)\cap F) < Cr^m,$$

for every $x \in F$ and $0 < r \leq diam\, F$, where $\mathcal{H}^m$ is the m-dimensional Hausdorff measure and $B(x,r)$ is the ball with radius r and center x. The restriction $0 < r \leq diam\, F$ is eliminated if F is a set with only one point.

Examples of these sets could be a smooth curve in the plane with $m = 1$ or the well known Von Koch snowflake with $m = \ln(4)/\ln(3)$.

This kind of regular sets is also known on the literature as m-dimensional Ahlfors-regular.

Finally, let us recall the Whitney decompositions of an open set. If F is a compact non-empty subset of $\mathbb{R}^n$, then $\mathbb{R}^n \setminus F$ can be represented as a union of closed dyadic cubes with pairwise disjoint interior Q^k_j satisfying

$$\mathbb{R}^n \setminus F = \bigcup_{k\in\mathbb{Z}}\bigcup_{j=1}^{N_k} Q^k_j, \tag{0.7}$$

where the edge length of Q_j^k is 2^{-k}. The previous decomposition is called a *Whitney decomposition of* $\mathbb{R}^n \setminus F$ and the collection $\{Q_j^k : j = 1, ..., N_k\}$ is called the k^{th} generation of Whitney cubes. Furthermore, the Whitney cubes satisfy

$$\ell_k \leq d(Q_j^k, F) \leq 4\ell_k, \tag{0.8}$$

where $d(Q_j^k, F)$ denotes the distance of the cube to F and ℓ_k the diameter of Q_j^k (see for example [S^2]).

Observe that this decomposition can be used to represent a bounded domain considering F to be the border of the domain.

Chapter 1

Weighted solutions of the divergence on planar domains

It is well known that the solvability of $(\mathrm{div})_p$ is equivalent to the Korn inequality on regular domains. However, it is unknown if this relationship can be extended to weighted Sobolev spaces on irregular domains. In this chapter, we give a positive answer about the existence of this relationship. In particular, we use the weighted Korn inequality on Hölder-α domains formulated in [ADL] to get the solvability of $(\mathrm{div})_{p,w}$ in this kind of domains for appropriate weights with a weaker boundary condition. As we have not been able to generalize appropriately that boundary condition to an arbitrary dimension we will consider in this chapter planar domains, although, as it can be seen some results hold in a more general context.

1.1 Korn type Inequalities on Hölder-α domains

The classic Korn inequality states that for a vector field $\mathbf{u} = (\mathrm{u}_1, \cdots, \mathrm{u}_n)$ defined in Ω with some condition it follows that

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega)}, \tag{1.1}$$

where $D\mathbf{u}$ denotes the jacobian matrix, namely, $(D\mathbf{u})_{ij} = \frac{\partial \mathrm{u}_i}{\partial x_j}$ and $\varepsilon(\mathbf{u})$ its symmetric part that is $\varepsilon(\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right)$. The condition imposed in the field has to remove the fields where the right side in the inequality vanishes while the left one does not. The two conditions considered by Korn were $\mathbf{u}(x) = 0$ in $\partial\Omega$ (called the first case) and $\int_\Omega \mathrm{rot}\,\mathbf{u} = 0$ (the second case), where $\mathrm{rot}\,\mathbf{u} = -\frac{\partial \mathrm{u}_1}{\partial x_2} + \frac{\partial \mathrm{u}_2}{\partial x_1}$. It is important to observe that the Korn inequality in the first case holds on arbitrary domains while in the second does not. It may fail if the domain has a cusp. On the other hand, there exists another inequality, equivalent to (1.1) in both cases for Lipschitz domains, which states that

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leq C \left\{ \|\mathbf{u}\|_{L^p(\Omega)} + \|\varepsilon(\mathbf{u})\|_{L^p(\Omega)} \right\}, \tag{1.2}$$

for all fields $\mathbf{u} \in W^{1,p}(\Omega)^n$.

As we mentioned before, the goal of this chapter is to use a known weighted Korn inequality on Hölder-α domains introduced in [ADL] to obtain existence of solutions for an appropriate divergence problem.

During this section, we will show that the weighted inequality in the mentioned article implies a weighted Korn inequality which is more appropriate for our purpose. With this goal, we will prove first that the result in Theorem 3.1 [ADL] admits a statement slightly stronger. Therefore, we include the proof for the sake of completeness although the arguments, as we said, are essentially those given in that reference. In particular we will make use of the following improved Poincaré inequality proved in Theorem 2.1 [ADL]. If Ω is a Hölder-α domain, $0 < \alpha \leq 1$, $B \subset \Omega$ a ball and $\phi \in C_0^\infty(B)$ is such that $\int_B \phi = 1$ then, for $\alpha \leq \beta \leq 1$ and f such that $\int_B f\phi = 0$ there exists a constant C depending only on Ω, B and ϕ such that,

$$\|f\|_{L^p(\Omega, p(1-\beta))} \leq C \, \|\nabla f\|_{L^p(\Omega, p(1+\alpha-\beta))}. \tag{1.3}$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a Hölder-α domain, $B \subset \Omega$ a ball and $1 < p < \infty$. Then, for $\alpha \leq \beta \leq 1$ the following inequality holds,*

$$\|D\mathbf{u}\|_{L^p(\Omega, p(1-\beta))} \leq C \Big\{ \|\varepsilon(\mathbf{u})\|_{L^p(\Omega, p(\alpha-\beta))} + \|\mathbf{u}\|_{L^p(B)} \Big\},$$

where the constant C depends only on Ω, B and p.

Proof. Following [KO], we can show that there exists $\mathbf{v} \in W^{1,p}(\Omega)^n$ such that

$$\Delta \mathbf{v} = \Delta \mathbf{u} \qquad \text{in } \Omega \tag{1.4}$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq C \, \|\varepsilon(\mathbf{u})\|_{L^p(\Omega)}. \tag{1.5}$$

Now, let $\phi \in C_0^\infty(B)$ be such that $\int_B \phi \, dx = 1$. For $i = 1, ..., n$ define the linear functions

$$L_i(x) := \left(\int_B \nabla(\mathrm{u}_i - \mathrm{v}_i)\phi \right) \cdot x$$

and $\mathbf{L}(x)$ as the vector with components $L_i(x)$.

Then,

$$D\mathbf{L} = \int_B D(\mathbf{u} - \mathbf{v})\phi$$

and, integrating by parts and applying the Hölder inequality we obtain

$$|D\mathbf{L}| \leq \|\mathbf{u}-\mathbf{v}\|_{L^p(B)}\|\nabla\phi\|_{L^q(B)}.$$

Therefore, it follows from (1.5) that there exists a constant C depending only on Ω, p and ϕ such that

$$\|D\mathbf{L}\|_{L^p(\Omega)} \leq C\Big\{\|\mathbf{u}\|_{L^p(B)} + \|\varepsilon(\mathbf{u})\|_{L^p(\Omega)}\Big\}. \tag{1.6}$$

Let us now introduce

$$\mathbf{w} := \mathbf{u}-\mathbf{v}-\mathbf{L}.$$

Then, in view of the bounds (1.5) and (1.6), it only remains to estimate $\mathbf{w}$. But, from (1.4) and the fact that $\mathbf{L}$ is linear we know that

$$\Delta\mathbf{w} = 0$$

and consequently,

$$\Delta\varepsilon_{ij}(\mathbf{w}) = 0.$$

But, if f is a harmonic function in Ω, the following estimate holds

$$\|\nabla f\|_{L^p(\Omega,p-\mu)} \leq C\|f\|_{L^p(\Omega,-\mu)},$$

for all $\mu \in \mathbb{R}$. Indeed, this estimate was proved in [De] (see also Lema 3.1 in [ADL] or [KO] for a different proof in the case $p = 2$ and $\mu = 0$).

Therefore, taking $\mu = p(\beta-\alpha)$ we obtain

$$\|\nabla\varepsilon_{ij}(\mathbf{w})\|_{L^p(\Omega,p(1+\alpha-\beta))} \leq C\|\varepsilon_{ij}(\mathbf{w})\|_{L^p(\Omega,p(\alpha-\beta))}$$

and using the well known identity

$$\frac{\partial^2 \mathrm{w}_i}{\partial x_j\partial x_k} = \frac{\partial\varepsilon_{ik}(\mathbf{w})}{\partial x_j} + \frac{\partial\varepsilon_{ij}(\mathbf{w})}{\partial x_k} - \frac{\partial\varepsilon_{jk}(\mathbf{w})}{\partial x_i}$$

we conclude that

$$\left\|\frac{\partial^2 \mathrm{w}_i}{\partial x_j\partial x_k}\right\|_{L^p(\Omega,p(1+\alpha-\beta))} \leq C\|\varepsilon(\mathbf{w})\|_{L^p(\Omega,p(\alpha-\beta))}, \tag{1.7}$$

for any i, j and k.

Since $\int \frac{\partial \mathrm{w}_i}{\partial x_j}\phi = 0$ (indeed, we have defined $\mathbf{L}$ in order to have this property), it follows from the improved Poincaré inequality (1.3) that

$$\left\|\frac{\partial \mathrm{w}_i}{\partial x_j}\right\|_{L^p(\Omega,p(1-\beta))} \leq C\left\|\nabla\frac{\partial \mathrm{w}_i}{\partial x_j}\right\|_{L^p(\Omega,p(1+\alpha-\beta))}.$$

Therefore, using (1.7), we obtain

$$\|D\mathbf{w}\|_{L^p(\Omega,p(1-\beta))} \leq C\|\varepsilon(\mathbf{w})\|_{L^p(\Omega,p(\alpha-\beta))} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,p(\alpha-\beta))},$$

concluding the proof. □

In the following corollary we give a weighted Korn inequality on Hölder-α domains which can be seen as a generalization of the so-called second case of Korn inequality. To state this inequality we need to introduce the space of infinitesimal rigid motions, namely,

$$\mathcal{N} = \{\mathbf{v} \in W^{1,p}(\Omega)^n \,:\, \varepsilon(\mathbf{v}) = 0\}.$$

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a Hölder-α domain and $1 < p < \infty$. Then, for $\alpha \leq \beta \leq 1$ the following inequality holds,*

$$\inf_{\mathbf{v}\in\mathcal{N}} \|\mathbf{u}-\mathbf{v}\|_{W^{1,p}(\Omega,p(1-\beta))} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,p(\alpha-\beta))}. \tag{1.8}$$

Proof. Take B and ϕ as in the previous theorem with $\overline{B} \subset \Omega$. Define $\overline{x}_i = \int_B x_i\phi(x)\,dx$ and $\mathbf{v} \in W^{1,p}(\Omega)^n$ defined by

$$\mathrm{v}_i(x) = a_i + \sum_{j=1}^{n} b_{ij}(x_j - \overline{x}_j)$$

with

$$a_i = \int_B \mathrm{u}_i\phi \qquad \text{and} \qquad b_{ij} = \frac{1}{2|B|}\int_B \left(\frac{\partial \mathrm{u}_i}{\partial x_j} - \frac{\partial \mathrm{u}_j}{\partial x_i}\right).$$

It is easy to check that $\mathbf{v} \in \mathcal{N}$. Now, since $\int_B(\mathbf{u}-\mathbf{v})\phi = 0$, it follows from (1.3) (actually we are using only a weaker standard Poincaré inequality with weights) and Theorem 1.1 that

$$\|\mathbf{u}-\mathbf{v}\|_{W^{1,p}(\Omega,p(1-\beta))} \leq C\Big\{\|\varepsilon(\mathbf{u}-\mathbf{v})\|_{L^p(\Omega,p(\alpha-\beta))} + \|\mathbf{u}-\mathbf{v}\|_{L^p(B)}\Big\}$$

and using now the Poincaré inequality in B we have

$$\|\mathbf{u}-\mathbf{v}\|_{W^{1,p}(\Omega,p(1-\beta))} \leq C\Big\{\|\varepsilon(\mathbf{u}-\mathbf{v})\|_{L^p(\Omega,p(\alpha-\beta))} + \|D(\mathbf{u}-\mathbf{v})\|_{L^p(B)}\Big\}. \tag{1.9}$$

But,

$$\int_B \left(\frac{\partial(\mathrm{u}-\mathrm{v})_i}{\partial x_j} - \frac{\partial(\mathrm{u}-\mathrm{v})_j}{\partial x_i}\right) = 0$$

and therefore, the so-called second case of Korn inequality applied in B gives

$$\|D(\mathbf{u}-\mathbf{v})\|_{L^p(B)} \leq C\|\varepsilon(\mathbf{u}-\mathbf{v})\|_{L^p(B)}.$$

Using this inequality in (1.9) and that $\varepsilon(\mathbf{v}) = 0$ we obtain

$$\|\mathbf{u} - \mathbf{v}\|_{W^{1,p}(\Omega,p(1-\beta))} \leq C\Big\{\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,p(\alpha-\beta))} + \|\varepsilon(\mathbf{u})\|_{L^p(B)}\Big\},$$

which implies (1.8) because $\overline{B} \subset \Omega$. □

Remark 1.3. *It is possible to prove the above corollary directly, i.e., without using the Korn inequality in the ball B, by using a standard compactness argument. Indeed, assuming that (1.8) does not hold and using that $W^{1,p}(\Omega, p(1-\beta))$ is compactly embedded in $L^p(\Omega, \gamma)$ for any $\gamma > p(1-\beta-\alpha)/\alpha$ (see Theorem 19.11 in Ref. [KuOp]) and Theorem 1.1 one obtains a contradiction.*

1.2 Solutions of the Divergence on Hölder-α Domains

As we mentioned in the introduction, it is known that $(\mathrm{div})_p$ is not solvable on general Hölder-α domains. Hence, in this section we analyze the solvability of $(\mathrm{div})_{p,w}$ for this class of domains where the weights considered, ω_1 and ω_2, are powers of the distance to the boundary. In fact, we obtain a positive answer for this problem imposing a new boundary condition which will be defined bellow.

Let us introduce some notations and assumptions. During this section Ω will denote a planar simply connected Hölder-α domains. For a scalar function ψ we write $\mathbf{curl}\,\psi = (\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1})$ and, for a vector field $\Psi = (\psi_1, \psi_2)$, $\mathrm{Curl}\,\Psi$ denotes the matrix which has $\mathbf{curl}\,\psi_i$ as it rows. Furthermore, if $\sigma \in L^p(\Omega)^{2\times 2}$, $\mathbf{Div}\,\sigma$ denotes the vector field with components obtained by taking the divergence of the rows of σ.

Let us explain this distributional boundary condition for solutions of the divergence problem in standard Sobolev spaces. Thus, to solve the problem $(\mathrm{div})_p$ it is enough to find a solution $\mathbf{u}$ of $\mathrm{div}\,\mathbf{u} = f$ such that the restriction to $\partial\Omega$ of both components of $\mathbf{u}$ are constant (whenever the domain is such that this restriction makes sense). Of course, we should replace the estimate (0.2) by

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Afterwards, (0.2) would follow by applying the Poincaré inequality to the solution obtained by adding an appropriate constant vector field to $\mathbf{u}$ in order to obtain the vanishing boundary condition.

Now, assume that Ω is a Lipschitz domain. Then, if $\psi \in W^{1,p}(\Omega)$ satisfies

$$\int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall \phi \in W^{1,q}(\Omega) \tag{1.10}$$

it follows by integration by parts that

$$\int_{\partial\Omega} \frac{\partial\psi}{\partial t}\phi = 0 \qquad \forall\phi \in W^{1,q}(\Omega), \tag{1.11}$$

where $\frac{\partial\psi}{\partial t}$ indicates the tangential derivative of ψ. Therefore $\frac{\partial\psi}{\partial t} = 0$ and then the restriction of ψ to $\partial\Omega$ is constant.

For a general domain Ω the tangential derivative on the boundary might not even be defined and therefore (1.11) would not make sense. However, condition (1.10) is well defined in any domain and this is the condition that we will use. Therefore we introduce the space

$$W^{1,p}_{const}(\Omega) \subset W^{1,p}(\Omega)$$

defined by

$$W^{1,p}_{const}(\Omega) = \left\{ \psi \in W^{1,p}(\Omega) \,:\, \int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall\phi \in W^{1,q}(\Omega) \right\} \tag{1.12}$$

and more generally, for any $\gamma \in \mathbb{R}$,

$$W^{1,p}_{const}(\Omega,\gamma) = \left\{ \psi \in W^{1,p}(\Omega,\gamma) \,:\, \int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall\phi \in W^{1,q}(\Omega,(1-q)\gamma) \right\}.$$

We do not know if this distributional boundary condition is equivalent to the standard one on arbitrary planar domains and weights. However, this relationship is satisfied on Lipschitz domains, as we observed before, and for some cuspidal domains, as we will see in section 1.3.

Thus, throughout this section we will analyze the solvability of the divergence problem $\operatorname{div}\mathbf{u} = f$ in the weighted Sobolev spaces $W^{1,p}_{const}(\Omega,\gamma_1)^2$ with the condition

$$\|D\mathbf{u}\|_{L^p(\Omega,\gamma_1)} \leq C\|f\|_{L^p(\Omega,\gamma_2)}, \tag{1.13}$$

where γ_1 and γ_2 are real numbers and denote the powers in the weights introduced in (0.4).

1.2.1 The weight in the left side

In order to simplify the computation and the conditions over the domain we will consider first $\gamma_2 = 0$ in (1.13).

For $1 < p < \infty$ and $\gamma \in \mathbb{R}$, $L^p_{sym}(\Omega,\gamma)^{2\times 2}$ denotes the subspace of symmetric tensors in $L^p(\Omega,\gamma)^{2\times 2}$. The proof of the following lemma uses ideas introduced in [GK] with different goals.

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a Hölder-α domain and $\mathbf{u} \in W^{1,p}(\Omega, p(\beta - 1))^2$, with $\alpha \leq \beta \leq 1$, such that $\int_\Omega \operatorname{div} \mathbf{u} = 0$. Then, there exists $\sigma \in L^p_{sym}(\Omega, p(\beta - \alpha))^{2\times 2}$ satisfying*

$$\int_\Omega \sigma : D\,\mathbf{w} \;=\; \int_\Omega \operatorname{Curl} \mathbf{u} : D\mathbf{w}, \qquad \forall \mathbf{w} \in W^{1,q}(\Omega, q(\alpha - \beta))^2$$

and

$$\|\sigma\|_{L^p(\Omega, p(\beta-\alpha))} \leq C \|\operatorname{Curl} \mathbf{u}\|_{L^p(\Omega, p(\beta-1))}.$$

Proof. Let $H \subset L^q_{sym}(\Omega, q(\alpha - \beta))^{2\times 2}$ the subspace defined as

$$H = \{\tau \in L^q_{sym}(\Omega, q(\alpha - \beta))^{2\times 2} \,:\, \tau = \varepsilon(\mathbf{w}) \text{ with } \mathbf{w} \in W^{1,q}(\Omega, q(\alpha - \beta))^2\}.$$

Let us see that the application

$$T \,:\, \varepsilon(\mathbf{w}) \mapsto \int_\Omega \operatorname{Curl} \mathbf{u} : D\mathbf{w} \tag{1.14}$$

defines a continuous linear functional on H.

First of all observe that T is well defined. Indeed, it is enough to check that the expression on the right of (1.14) vanishes whenever $\varepsilon(\mathbf{w}) = 0$. But, it is known that in that case $\mathbf{w}(x, y) = (a - cy, b + cx)$ and therefore

$$\int_\Omega \operatorname{Curl} \mathbf{u} : D\mathbf{w} = c \int_\Omega \operatorname{div} \mathbf{u} = 0.$$

Now, we want to show that T is continuous on H. Using again that $\int_\Omega \operatorname{Curl} \mathbf{u} : D\mathbf{v} = 0$ if $\varepsilon(\mathbf{v}) = 0$ and applying Corollary 1.2 we have, for $\tau = \varepsilon(\mathbf{w}) \in H$,

$$\begin{aligned}
|T(\tau)| &= \left| \int_\Omega \operatorname{Curl} \mathbf{u} : D\mathbf{w} \right| \\
&\leq \|\operatorname{Curl} \mathbf{u}\|_{L^p(\Omega, p(\beta-1))} \inf_{\mathbf{v} \in \mathcal{N}} \|D(\mathbf{w} - \mathbf{v})\|_{L^q(\Omega, q(1-\beta))} \\
&\leq C \|\operatorname{Curl} \mathbf{u}\|_{L^p(\Omega, p(\beta-1))} \|\varepsilon(\mathbf{w})\|_{L^q(\Omega, q(\alpha-\beta))} \\
&= C \|\operatorname{Curl} \mathbf{u}\|_{L^p(\Omega, p(\beta-1))} \|\tau\|_{L^q(\Omega, q(\alpha-\beta))}.
\end{aligned}$$

By the Hahn-Banach theorem the functional T can be extended to $\mathrm{L}^q_{sym}(\Omega, q(\alpha - \beta))^{2\times 2}$ and therefore, by the Riesz representation theorem, there exists $\sigma \in L^p_{sym}(\Omega, p(\beta - \alpha))^{2\times 2}$ such that

$$T(\tau) \;=\; \int_\Omega \sigma : \tau \qquad \forall \tau \in L^q_{sym}(\Omega, q(\alpha - \beta))^{2\times 2}$$

and

$$\|\sigma\|_{L^p(\Omega,p(\beta-\alpha))} \leq C\|\mathrm{Curl}\,\mathbf{u}\|_{L^p(\Omega,p(\beta-1))},$$

where C depends on the constant in Corollary 1.2. In particular,

$$\int_\Omega \sigma : \varepsilon(\mathbf{w}) \;=\; \int_\Omega \mathrm{Curl}\,\mathbf{u} : D\mathbf{w}, \tag{1.15}$$

for every $\mathbf{w} \in W^{1,q}(\Omega, q(\alpha-\beta))^2$. Then, we conclude the proof observing that, since σ is symmetric, we can replace $\varepsilon(\mathbf{w})$ in (1.15) by $D\mathbf{w}$. □

It is a very well known result that a divergence free vector field is a rotational of a scalar function ϕ. Indeed, for smooth vector fields the proof is usually given at elementary courses on calculus in several variables. On the other hand, if the vector field is only in $L^p(\Omega)^2$ but $\partial\Omega$ is Lipstchiz, it is not difficult to see that the vector field can be extended to a divergence free vector field defined in $\mathbb{R}^2$ and then, the existence of ϕ can be proved by using the Fourier transform. However, we need to use the existence of ϕ in the case where the domain and the vector field are both non-smooth. We have not been able to find a proof of this result in the literature and so we include the following lemma.

Lemma 1.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\omega : \Omega \to \mathbb{R}_{>0}$ a weight such that ω^{-1} is locally bounded. Given a vector field $\mathbf{v} \in L^p(\Omega,\omega)^2$ such that $\mathrm{div}\,\mathbf{v} = 0$, there exists $\phi \in W^{1,p}_{loc}(\Omega)$ such that $\mathbf{curl}\,\phi = \mathbf{v}$.*

Proof. Take $\psi \in C_0^\infty(B_1)$ satisfying $\int \psi = 1$, where B_1 is the unit ball centered at the origin. For $k \geq 1$, define $\psi_k(x) = k^2\psi(kx)$ and, extending $\mathbf{v}$ by zero to $\mathbb{R}^2$, $\mathbf{v}_k = \psi_k * \mathbf{v}$.

Let Ω_n be a sequence of Lipschitz simply connected open subsets of Ω such that

$$\overline{\Omega}_n \subset \Big\{x \in \Omega :\, d(x) > 1/n\Big\} \qquad \text{and} \qquad \Omega_n \nearrow \Omega.$$

We will prove in Lemma A.2 that this sequence exists. Using that the distance between Ω_n and $\partial\Omega$ is greater than $1/n$ and $\mathrm{supp}\,\psi_k \subset B(0,\frac{1}{k})$, it is not difficult to see that $\mathrm{div}\,\mathbf{v}_k = 0$ in Ω_n for every $k \geq n$.

Then, since $\mathbf{v}_n \in C_0^\infty(\mathbb{R}^2)^2$, there exists $\phi_n \in C_0^\infty(\Omega_n)$ such that $\mathbf{curl}\,\phi_n = \mathbf{v}_n$. Moreover, adding a constant we can take ϕ_n such that $\int_{\Omega_1} \phi_n = 0$.

Now, by the Poincaré inequality on the Lipschitz domain Ω_n we have that, for each n, there exists a constant C depending only on Ω_n such that

$$\|\phi_k - \phi_{k'}\|_{L^p(\Omega_n)} \leq C\|\mathbf{curl}\,(\phi_k - \phi_{k'})\|_{L^p(\Omega_n)} = C\|\mathbf{v}_k - \mathbf{v}_{k'}\|_{L^p(\Omega_n)} \to 0,$$

for $k, k' \to \infty$.

Then, there exists $\phi \in W^{1,p}_{loc}(\Omega)$ such that $\phi_k|_{\Omega_n} \to \phi$ in $W^{1,p}(\Omega_n)$ and so $\mathbf{curl}\,\phi = \mathbf{v}$ in $\Omega_n, \forall n$ and consequently in Ω. □

We can now state and prove our results about solutions of the divergence on Hölder-α domains satisfying (1.13) in the particular case in which $\gamma_2 = 0$ to avoid technical complications.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected Hölder-α domain, $0 < \alpha \leq 1$. Given $f \in L^p_0(\Omega)$, $1 < p < \infty$, there exists $\mathbf{u} \in W^{1,p}_{const}(\Omega, p(1-\alpha))^2$ such that*

$$\operatorname{div} \mathbf{u} = f$$

and

$$\|D\mathbf{u}\|_{L^p(\Omega,p(1-\alpha))} \leq C\|f\|_{L^p(\Omega)}. \tag{1.16}$$

Proof. Take $\mathbf{v} \in W^{1,p}(\Omega)^2$ such that

$$\operatorname{div} \mathbf{v} = f \tag{1.17}$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{1.18}$$

The existence of such a $\mathbf{v}$ is well known, for example, since no boundary condition on $\mathbf{v}$ is required, we can extend f by zero and take the solution of problem (0.1) and (0.2) in a ball containing Ω.

To prove the theorem it is enough to show that there exists $\mathbf{w} \in W^{1,p}(\Omega, p(1-\alpha))^2$ satisfying $\operatorname{div} \mathbf{w} = 0$ and such that

$$\mathbf{v} - \mathbf{w} \in W^{1,p}_{const}(\Omega, p(1-\alpha))^2$$

and

$$\|D\mathbf{w}\|_{L^p(\Omega,p(1-\alpha))} \leq C\|f\|_{L^p(\Omega)}. \tag{1.19}$$

Indeed, in view of (1.17), $\mathbf{u} := \mathbf{v} - \mathbf{w}$ will be the desired solution.

But, since $\operatorname{div} \mathbf{v}$ has vanishing mean value, we know from Lemma 1.4 that there exists $\sigma \in L^p_{sym}(\Omega, p(1-\alpha))^{2\times 2}$ satisfying

$$\|\sigma\|_{L^p(\Omega,p(1-\alpha))} \leq C\, \|\operatorname{Curl} \mathbf{v}\|_{L^p(\Omega)} \tag{1.20}$$

and

$$\int_\Omega \sigma : D\mathbf{r} = \int_\Omega \operatorname{Curl} \mathbf{v} : D\mathbf{r} \quad , \quad \forall \mathbf{r} \in W^{1,q}(\Omega, q(\alpha-1))^2.$$

Then,

$$\int_\Omega \mathbf{Div}\ \sigma \cdot \mathbf{r} = -\int_\Omega \sigma : D\mathbf{r} = -\int_\Omega \operatorname{Curl} \mathbf{v} : D\mathbf{r} = \int_\Omega \mathbf{Div}\ \operatorname{Curl} \mathbf{v} \cdot \mathbf{r} = 0,$$

for every $\mathbf{r} \in C^\infty_0(\Omega)^2$ and therefore $\mathbf{Div}\, \sigma = 0$.

Now, from Lemma 1.5 we can assert that there exists $\mathbf{w} \in W^{1,p}_{loc}(\Omega)^2$ such that $\text{Curl}\,\mathbf{w} = \sigma$. Thus, as Ω is a Hölder-α domain using Theorem 2.1 of [ADL] with $\beta = \alpha$ we have

$$\begin{aligned}\|\mathbf{w}\|_{L^p(\Omega,p(1-\alpha))} &\leq C\|D\mathbf{w}\|_{L^p(\Omega,p(1-\alpha))}\\ &= C\|\text{Curl}\,\mathbf{w}\|_{L^p(\Omega,p(1-\alpha))} \leq C\|\sigma\|_{L^p(\Omega,p(1-\alpha))}. \end{aligned} \tag{1.21}$$

We have to check that $\text{div}\,\mathbf{w} = 0$, but since σ is a symmetric tensor we have

$$\text{div}\,\mathbf{w} = \frac{\partial \mathbf{w}_1}{\partial x_1} + \frac{\partial \mathbf{w}_2}{\partial x_2} = -\sigma_{12} + \sigma_{21} = 0.$$

To conclude the proof observe that in view of (1.18), (1.20) and (1.21) we have (1.19). □

1.2.2 The weights in both sides

As we mentioned before, it is natural to ask whether part or all the weights in the estimate (1.16) can be moved to the right hand side. We will give a positive answer to this question. The proof of this more general result is similar to that of Theorem 1.6 but it requires some non-trivial preliminary results. In particular, we will need an extra hypothesis on the domain.

We are going to use that some singular integral operators are continuous in weighted L^p-norms, $1 < p < \infty$, for weights in the Muckenhoupt class A_p. Recall that a weight $\omega : \mathbb{R}^n \to \mathbb{R}$ belongs to the A_p class if and only if the Hardy-Littlewood maximal is continuous in $L^p(\mathbb{R}^n, \omega)$. This well known result can be seen for example in the book [S[3]].

In what follows we consider the distance to $\partial\Omega$, $d(x)$, defined for every $x \in \mathbb{R}^n$ and not only for $x \in \Omega$. We will give sufficient conditions on $\partial\Omega$ and on the exponent μ such that d^μ belongs to the A_p class.

Lemma 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $\partial\Omega$ is included in a* **m-regular set***, with $n-1 \leq m < n$. If*

$$-(n-m) < \mu < (n-m)(p-1)\,,$$

then d^μ belongs to the class A_p, with $1 < p < \infty$.

Proof. This result will be proved in Chapter 2 in the more general situation of the distance to a compact set F contained in $\mathbb{R}^n$ since this result can be of interest in other situations and its proof does not require any extra effort. □

As a consequence we have the following result on weighted estimates for solutions of the divergence problem.

Lemma 1.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that its boundary is contained in a* **m-regular set***, with $1 \le m < 2$. Given $f \in L^p(\Omega, \gamma)$, with $-(2-m) < \gamma < (2-m)(p-1)$ and $1 < p < \infty$, there exists $\mathbf{v} \in W^{1,p}(\Omega, \gamma)^2$ such that*

$$\operatorname{div} \mathbf{v} = f$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega,\gamma)} \le C\|f\|_{L^p(\Omega,\gamma)}.$$

Proof. Extend f by zero to $\mathbb{R}^2$. Then, it is well known that

$$\phi(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^n} \log|x-y|\, f(y)\, dy$$

is a solution of $\Delta\phi = f$. Moreover, it follows from the theory of singular integral operators (see for example [S³]) that, if $w \in A_p$,

$$\int_{\mathbb{R}^2} \left|\frac{\partial^2\phi(x)}{\partial x_i \partial x_j}\right|^p w(x)\, dx \le \int_{\mathbb{R}^2} |f(x)|^p\, w(x)\, dx.$$

Now, using Lemma 1.7 it follows that $d^\mu \in A_p$ and therefore $\mathbf{v} := \nabla\phi$ is the desired solution. $\square$

Now, we can give our more general result on solutions of the divergence.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected Hölder-α domain, $0 < \alpha \le 1$, such that its boundary is contained in a* **m-regular set***, with $1 \le m < 2$.*

Given $f \in L^p_0(\Omega, p(\beta-1))$, with $1 < p < \infty$, $\alpha \le \beta \le 1$ and $-(2-m) < p(\beta-1)$, there exists $\mathbf{u} \in W^{1,p}_{const}(\Omega, p(\beta-\alpha))^2$ such that

$$\operatorname{div} \mathbf{u} = f$$

and

$$\|D\mathbf{u}\|_{L^p(\Omega,p(\beta-\alpha))} \le C\|f\|_{L^p(\Omega,p(\beta-1))}. \tag{1.22}$$

Proof. Since $-(2-m) < p(\beta-1)$, it follows from Lemma 1.8 that there exists $\mathbf{v} \in W^{1,p}(\Omega, p(\beta-1))^2$ such that

$$\operatorname{div} \mathbf{v} = f \tag{1.23}$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega,p(\beta-\alpha))} \le C\|\mathbf{v}\|_{W^{1,p}(\Omega,p(\beta-1))} \le C\|f\|_{L^p(\Omega,p(\beta-1))}. \tag{1.24}$$

The rest of the proof follows as that of Theorem 1.6. Indeed, we have to show that there exists $\mathbf{w} \in W^{1,p}(\Omega, p(\beta-\alpha))^2$ satisfying $\operatorname{div}\mathbf{w} = 0$ and such that

$$\mathbf{v} - \mathbf{w} \in W^{1,p}_{const}(\Omega, p(\beta-\alpha))^2$$

and

$$\|D\mathbf{w}\|_{L^p(\Omega,p(\beta-\alpha))} \leq C\|f\|_{L^p(\Omega,p(\beta-1))}.$$

The reader can easily check that the existence of $\mathbf{w}$ follows by using Lemma 1.4 as in Theorem 1.6. □

1.3 Some particular Hölder-α domains with an external cusp

In this section we consider the particular case of the Hölder-α domain defined as

$$\Omega = \Big\{(x,y) \in \mathbb{R}^2 \,:\, 0 < x < 1\,,\, 0 < |y| < x^{1/\alpha}\Big\}, \tag{1.25}$$

with $0 < \alpha \leq 1$.

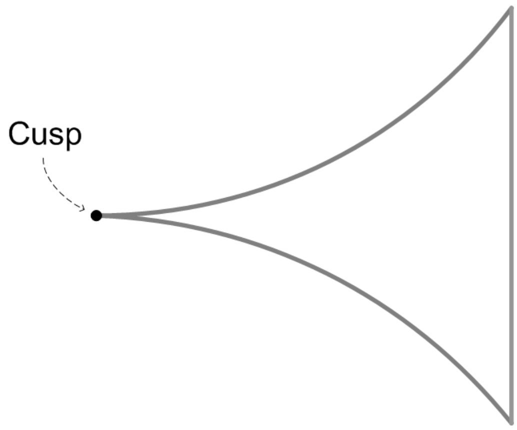

Figure 1.1: Zero dimensional external cusp in the plane.

We are going to show that in this case the weaker boundary condition imposed in Theorem 1.9 is equivalent to the standard one, i.e., that the solution of the divergence obtained in that theorem can be modified, by adding a constant vector field, to obtain a solution which vanishes on the boundary in the classic sense.

We will consider the particular case $\beta = \alpha$ and $m = 1$ of our general Theorem 1.9. Extension of the arguments to other cases might be possible but it is not straightforward.

Theorem 1.10. *Let $\Omega \subset \mathbb{R}^2$ be the domain defined in (1.25) and $1 < p < \infty$. If $1 - 1/p < \alpha \leq 1$ then, given $f \in L^p_0(\Omega, p(\alpha-1))$ there exists $\mathbf{u} \in W^{1,p}_0(\Omega)^2$ such that*

$$\operatorname{div}\mathbf{u} = f \tag{1.26}$$

and

$$\|\mathbf{u}\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega,p(\alpha-1))}, \tag{1.27}$$

with a constant depending only on Ω, p *and* α.

Proof. It is easy to see that Ω satisfies the hypotheses of Theorem 1.9. Therefore, it follows from that theorem that there exists $\mathbf{u} \in W^{1,p}_{const}(\Omega)^2$ which verifies (1.26).

We are going to prove that, for any $\psi \in W^{1,p}_{const}(\Omega)$, there exists a constant $\psi_0 \in \mathbb{R}$ such that

$$\psi - \psi_0 \in W_0^{1,p}(\Omega) := \overline{C_0^\infty(\Omega)}.$$

Consequently, $\mathbf{u}$ can be modified by adding a constant to each of its components to obtain the desired solution. Indeed, the estimate (1.27) will follow from (1.22) by the Poincaré inequality.

Given $\psi \in W^{1,p}_{const}(\Omega)$, let us show first that ψ is constant on $\partial\Omega$. From the definition of $W^{1,p}_{const}(\Omega)$ we have that

$$\int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall \phi \in W^{1,q}(\Omega).$$

Now, let (x_0, y_0) be a point in $\partial\Omega$ different from the origin and B an open ball centered at (x_0, y_0) such that $0 \notin B$. Taking $\phi \in C_0^\infty(B)$ we have

$$0 = \int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = -\int_{B\cap\partial\Omega} \psi \frac{\partial \phi}{\partial t} \qquad \forall \phi \in C^\infty(B)$$

where $\frac{\partial\phi}{\partial t}$ indicates the tangential derivative of ϕ. Consequently $\frac{\partial\psi}{\partial t} = 0$ in the distributional sense on $B \cap \partial\Omega$ and then, since $\partial\Omega - (0,0)$ is a connected set, we conclude that there exists a constant ψ_0 such that $\psi = \psi_0$ on $\partial\Omega$. To simplify notation we assume in what follows that $\psi_0 = 0$ and so, we have to see that $\psi \in W_0^{1,p}(\Omega)$.

Now, let $\zeta \in C^\infty(\mathbb{R}_+)$ be such that

$$\zeta \equiv 1 \text{ in } [0,1] \qquad \zeta \equiv 0 \text{ in } \mathbb{R}_+ \setminus (0,2) \qquad 0 \leq \zeta \leq 1.$$

We decompose ψ as

$$\psi(x,y) = \zeta(3x)\psi(x,y) + (1 - \zeta(3x))\,\psi(x,y) =: \psi_1 + \psi_2.$$

It is easy to see that $\psi_2 \in W_0^{1,p}(\Omega_2)$ where Ω_2 is the Lipschitz domain

$$\Omega_2 := \Omega \cap \Big\{ x > \frac{1}{3} \Big\}.$$

Thus, we can suppose that $\psi = \psi_1$. Let now $\phi_n \in C^\infty(\Omega)$ be a sequence satisfying $\phi_n \to \psi$ in $W^{1,p}(\Omega)$ and let $\gamma := 1/\alpha$.

It is easy to check that, for $y \in (0,1)$,

$$|\phi_n(x, x^\gamma - y)| \leq |\phi_n(x, x^\gamma)| + \int_0^y \left| \frac{\partial \phi_n}{\partial y}(x, x^\gamma - t) \right| dt.$$

Therefore, integrating and using the Hölder inequality we have

$$\int_{y^\alpha}^1 |\phi_n(x, x^\gamma - y)|^p \, dx \leq C \left(\int_{y^\alpha}^1 |\phi_n(x, x^\gamma)|^p \, dx + y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial \phi_n}{\partial y}(x, x^\gamma - t) \right|^p dt \, dx \right).$$

Thus, using the continuity of the trace in the Lipschitz domain $\Omega \cap \{x > y^\alpha\}$ we have

$$\begin{aligned}
\int_{y^\alpha}^1 |\psi(x, x^\gamma - y)|^p \, dx &= \lim_{n\to\infty} \int_{y^\alpha}^1 |\phi_n(x, x^\gamma - y)|^p \, dx \\
&\leq C \lim_{n\to\infty} \left(\int_{y^\alpha}^1 |\phi_n(x, x^\gamma)|^p \, dx \, + \, y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial \phi_n}{\partial y}(x, x^\gamma - t) \right|^p dt \, dx \right) \\
&= C \, y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial \psi}{\partial y}(x, x^\gamma - t) \right|^p dt \, dx. \qquad (1.28)
\end{aligned}$$

Now we will show that the sequence ψ_m defined by

$$\psi_m(x, y) := \psi(x, y) \, (1 - \zeta_m(x^\gamma - |y|))$$

where $\zeta_m(t) := \zeta(mt)$, converges to ψ in $W^{1,p}(\Omega)$. Moreover, it is easy to see that $\operatorname{supp} \psi_m \subset \Omega$.

By symmetry we can assume that $\Omega = \Omega \cap \{y > 0\}$. Using the dominated convergence theorem we obtain

$$\lim_{m\to\infty} \|\psi - \psi_m\|^p_{L^p(\Omega)} = \lim_{m\to\infty} \int_\Omega |\psi(x, y) \zeta_m(x^\gamma - y)|^p = 0.$$

On the other hand,

$$\frac{\partial \psi_m}{\partial x}(x, y) = \frac{\partial \psi}{\partial x}(x, y) \Big(1 - \zeta_m(x^\gamma - y) \Big) - m \, \psi(x, y) \, \zeta' \gamma x^{\gamma - 1}$$

and then,

$$\begin{aligned}
\int_\Omega \left| \frac{\partial \psi}{\partial x} - \frac{\partial \psi_m}{\partial x} \right|^p \quad &\leq \quad \int_\Omega \left| \frac{\partial \psi}{\partial x}(x, y) \, \zeta_m(x^\gamma - y) \right|^p + C m^p \int_\Omega \left| \psi(x, y) \chi_{\{y > \psi(x) - 2/m\}} \right|^p \\
&=: \quad I \, + \, II.
\end{aligned}$$

Thus, using again dominated convergence, it is easy to check that $I \to 0$. So, it only remains to analyze II.

Now, by the change of variables defined by $(x,y) \longmapsto (x, x^\gamma - y)$ and using (1.28) it follows that

$$\begin{aligned}
II &= C\, m^p \int_0^{2/m} \int_{y^\alpha}^1 |\psi(x, x^\gamma - y)|^p \, dx\, dy \\
&\leq C\, m^p \int_0^{2/m} y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial \psi}{\partial y}(x, x^\gamma - t) \right|^p dt\, dx\, dy \\
&\leq C\, m^p \int_0^{2/m} y^{p-1} \int_0^{2/m} \int_{t^\alpha}^1 \left| \frac{\partial \psi}{\partial y}(x, x^\gamma - t) \right|^p dx\, dt\, dy \\
&\leq C\, m^p \left(\frac{2}{m} \right)^p \int_0^{2/m} \int_{t^\alpha}^1 \left| \frac{\partial \psi}{\partial y}(x, x^\gamma - t) \right|^p dx\, dt \\
&\leq C \int_\Omega \left| \frac{\partial \psi}{\partial y}(x,y) \chi_{\{y > \psi(x) - 2/m\}} \right|^p \longrightarrow 0.
\end{aligned}$$

An analogous argument can be applied to prove that $\frac{\partial \psi_m}{\partial y} \to \frac{\partial \psi}{\partial y}$ in $L^p(\Omega)$.

Consequently, we conclude the proof by observing that ψ_m belongs to $W_0^{1,p}(\Omega)$.

□

We will show in Chapter 4 that the estimate (1.27) is optimal in the sense that it is not possible to improve the power of the distance in the right hand side.

1.4 An Application to the Stokes Equations

The goal of this section is to explain the motivation of the main results exhibited in section 1.3, namely, the existence of right inverses of the divergence in weighted Sobolev spaces.

As we will show the Stokes system of equations is not well posed in the usual Sobolev spaces for domains with external cusps. In view of this fact we introduce a generalization of the classic analysis for this kind of domains. We will use the usual notation for Sobolev spaces.

The Stokes equations are given by

$$\begin{cases} -\Delta \mathbf{u} + \nabla p & = \mathbf{f} \qquad \text{in } \Omega \\ \operatorname{div} \mathbf{u} & = 0 \qquad \text{in } \Omega \\ \mathbf{u} & = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.29}$$

For a bounded domain Ω which is Lipschitz (or more generally a John domain [ADM]) it is known that, if $\mathbf{f} \in H^{-1}(\Omega)^n$, then there exists a unique solution

$$(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L_0^2(\Omega).$$

Moreover, the following a priori estimate holds

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)},$$

where the constant C depends only on the domain Ω.

Our general existence and uniqueness results on domains with cusps follow from the classic theory but replacing the usual Sobolev spaces by appropriate weighted spaces.

The classic analysis of the Stokes equations is based on the abstract theory for saddle point problems given by Brezzi in [Br] (see also the book [BDF]).

Indeed, the weak formulation of (1.29) can be written as

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) & = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in V \\ b(\mathbf{u}, q) & = 0 \qquad \forall q \in Q, \end{cases} \tag{1.30}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega D\mathbf{u} : D\mathbf{v}$$

and

$$b(\mathbf{v}, p) = \int_\Omega p \operatorname{div} \mathbf{v},$$

where, for $\mathbf{v} \in H^1(\Omega)^n$, $D\mathbf{v}$ is its differential matrix and, given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{n\times n}$, $A : B = \sum_{i,j=1}^n a_{ij} b_{ij}$.

The abstract theory gives existence and uniqueness for (1.30) when a and b are continuous bilinear forms, a is coercive on the kernel of the operator $B : V \to Q'$ associated with b, and b satisfies the inf-sup condition

$$\inf_{0\neq q\in Q} \sup_{0\neq \mathbf{v}\in V} \frac{b(\mathbf{v}, q)}{\|q\|_Q \|\mathbf{v}\|_V} > 0.$$

In the case of the Stokes problem, if we choose the spaces $V = H_0^1(\Omega)^n$ and $Q = L_0^2(\Omega)$, continuity of the bilinear forms and coercivity of a follow immediately by Schwarz and Poincaré inequalities. Therefore, the problem reduces to prove the inf-sup condition for b which reads

$$\inf_{0\neq q\in L_0^2(\Omega)} \sup_{0\neq \mathbf{v}\in H_0^1(\Omega)^n} \frac{\int_\Omega q \operatorname{div} \mathbf{v}}{\|q\|_{L_0^2(\Omega)}\|\mathbf{v}\|_{H_0^1(\Omega)}} > 0. \tag{1.31}$$

But, it is well known that this condition is equivalent to the existence of solutions of $\operatorname{div} \mathbf{u} = f$, for any $f \in L_0^2(\Omega)$, with $\mathbf{u} \in H_0^1(\Omega)^n$ satisfying $\|\mathbf{u}\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$, which fails on domains with external cusps. So, the condition (1.31) does not hold, neither.

For domains such that (1.31) is not valid, the idea is to replace this condition by a weaker one. With this goal we will work with weighted norms.

We will use the following facts for $\omega \in L^1(\Omega)$ which are easy to see. First, $L^2(\Omega, \omega^{-1}) \subset L^1(\Omega)$ and therefore $L_0^2(\Omega, \omega^{-1})$ is well defined, and second, the integral $\int_\Omega q\,\omega$ is well defined for $q \in L^2(\Omega, \omega)$ and therefore we can define the space

$$L^2_{\omega,0}(\Omega, \omega) = \left\{ q \in L^2(\Omega, \omega) \,:\, \int_\Omega q\,\omega = 0 \right\}.$$

We have the following generalization of the classic result which will be useful on cuspidal domains.

Theorem 1.11. *Let $\omega \in L^1(\Omega)$ be a positive weight. Assume that for any $f \in L_0^2(\Omega, \omega^{-1})$ there exists $\mathbf{u} \in H_0^1(\Omega)^n$ such that $\operatorname{div} \mathbf{u} = f$ and*

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C_1 \|f\|_{L^2(\Omega,\omega^{-1})},$$

with a constant C_1 depending only on Ω and ω. Then, for any $\mathbf{f} \in H^{-1}(\Omega)^n$, there exists a unique $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L^2_{\omega,0}(\Omega, \omega)$ solution of the Stokes problem (1.29). Moreover,

$$\|\mathbf{u}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega,\omega)} \leq C_2 \|\mathbf{f}\|_{H^{-1}(\Omega)},$$

where C_2 depends only on C_1 and Ω.

Proof. We apply the general abstract theory for saddle point problems with appropriate spaces.

For the pressure we introduce the space $Q = L^2_{\omega,0}(\Omega, \omega)$ with the norm $\|q\|_Q = \|q\|_{L^2(\Omega,\omega)}$.

Since we are modifying the pressure space, we have to enlarge the H^1-norm of the velocity space in order to preserve continuity of the bilinear form b. Then, we define

$$V = \left\{ \mathbf{v} \in H_0^1(\Omega)^n \,:\, \operatorname{div} \mathbf{v} \in L^2(\Omega, \omega^{-1}) \right\}$$

with the norm given by

$$\|\mathbf{v}\|_V^2 = \|\mathbf{v}\|_{H^1(\Omega)}^2 + \|\operatorname{div}\mathbf{v}\|_{L^2(\Omega,\omega^{-1})}^2.$$

Since $\|\mathbf{v}\|_{H^1(\Omega)} \le \|\mathbf{v}\|_V$ the continuity of a in $V \times V$ follows immediately by Schwarz inequality. Also, from the definitions of the spaces it is easy to see that b is continuous on $V \times Q$.

On the other hand, coercivity of a, in the norm of V, on the kernel of the operator B follows from Poincaré inequality because this kernel consists of divergence free vector fields.

Therefore, to apply the general theory it only rests to prove the inf-sup condition

$$\inf_{0\ne q\in Q}\sup_{0\ne\mathbf{v}\in V}\frac{\int_\Omega q\operatorname{div}\mathbf{v}}{\|q\|_Q\|\mathbf{v}\|_V} > 0. \tag{1.32}$$

But this follows in a standard way. Indeed, given $q \in Q$ it follows from our hypothesis that there exists $\mathbf{u} \in H_0^1(\Omega)^n$ such that $\operatorname{div}\mathbf{u} = q\,\omega$ and

$$\|\mathbf{u}\|_{H^1(\Omega)} \le C_1\|q\,\omega\|_{L^2(\Omega,\omega^{-1})} = C_1\|q\|_Q.$$

Moreover, since $\|\operatorname{div}\mathbf{u}\|_{L^2(\Omega,\omega^{-1})} = \|q\|_Q$ we have

$$\|\mathbf{u}\|_V \le C\|q\|_Q,$$

with C depending only on C_1.

Then,

$$\|q\|_Q = \frac{\int_\Omega q\,q\,\omega}{\|q\|_Q} \le C\frac{\int_\Omega q\operatorname{div}\mathbf{u}}{\|\mathbf{u}\|_V}$$

and therefore (1.32) holds. □

As an immediately corollary of this theorem in the particular case where ω is $d^{2(1-\alpha)}$ with d the distance to the boundary we obtain existence and uniqueness for the Stokes equations in the domain (1.25).

Corollary 1.12. *Let Ω be the domain defined in (1.25) with $1/2 < \alpha \le 1$. Then, if $\mathbf{f} \in H^{-1}(\Omega)^2$, there exists a unique weak solution $(\mathbf{v}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega, 2(1-\alpha))$ of the Stokes equations (1.29). Moreover, there exists a constant C depending only on α such that*

$$\|\mathbf{v}\|_{H_0^1(\Omega)} + \|p\|_{L^2(\Omega,2(1-\alpha))} \le C\|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{1.33}$$

Proof. This result can be easily proved combining Theorem 1.11 and Theorem 1.10 with $\omega = d^{2(1-\alpha)}$. □

We can eliminate the weight in (1.33) estimating the pressure with a standard L^r-norm for a r smaller than 2.

Corollary 1.13. *Let Ω be the domain defined in (1.25) with $1/2 < \alpha \leq 1$ and $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega, 2(1-\alpha))$ be the solution of the Stokes equations (1.29). If $\mathbf{f} \in H^{-1}(\Omega)^2$ and $1 \leq r < 2/(3-2\alpha)$ then $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L^r(\Omega)$. Moreover, there exists a constant C depending only on α such that*

$$\|\mathbf{u}\|_{H_0^1(\Omega)} + \|p\|_{L^r(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

Proof. We only have to prove that $p \in L^r(\Omega)$ and that

$$\|p\|_{L^r(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{1.34}$$

Observe that $\int_\Omega d^\beta < +\infty$ for any $\beta > -1$. Indeed, this follows easily by using that $d(x,y) \simeq x^{1/\alpha} - |y|$, this equivalence will be analyzed in Chapter 4. Then, applying the Hölder inequality with exponent $2/r$, we have

$$\|p\|_{L^r(\Omega)}^r = \int_\Omega |p|^r d^{(1-\alpha)r} d^{(\alpha-1)r} \leq \|p\|_{L^2(\Omega,2(1-\alpha))}^r \left(\int_\Omega d^{\frac{2(\alpha-1)r}{2-r}}\right)^{\frac{2-r}{2}}$$

but the integral in the right hand side is finite because $(2(\alpha-1)r)/(2-r) > -1$. So $\|p\|_{L^r(\Omega)} \leq C\|p\|_{L^2(\Omega,2(1-\alpha))}$ and therefore, (1.34) follows immediately from (1.33). $\square$

1.5 An equivalence between Korn and divergence with general weights

It is very well known that the Korn inequality and the divergence problem are equivalent on regular domains. For example, Horgan and Payne showed in [HP] that these results are equivalent, and also they are equivalent to Friedrichs inequality, on Lipschitz simply connected planar domains for $p = 2$. However, it is not known if the equivalence holds in weighted Sobolev spaces on irregular domains. Extending the techniques used previously on Hölder-α domains and some ideas in [BS], we will extend the equivalence to weighted Sobolev spaces on arbitrary simple connected planar domains.

Let $\Omega \subset \mathbb{R}^2$ be a bounded planar domain without any assumption on the regularity of its boundary and $\omega : \Omega \to \mathbb{R}$ a bounded weight satisfying that

$$\frac{1}{C} \leq \omega(x) \leq C \tag{1.35}$$

for all $x \in K$ and for all compact $K \subset \Omega$, where C is a positive constant depending only on K. Powers of the distance to a subset of $\partial\Omega$ are examples of these weights.

Let us introduce the Banach quotient space

$$V(\Omega, p, \omega) = \left\{ \mathbf{v} \in W^{1,p}_{loc}(\Omega)^2 \,:\, D\mathbf{v} \in L^p(\Omega, \omega)^{2\times 2} \text{ and } \operatorname{div} \mathbf{v} \in L^p(\Omega) \right\} / \left\{ Constants \right\},$$

with the norm

$$\|\mathbf{v}\|_V^p := \|D\mathbf{v}\|^p_{L^p(\Omega,\omega)} + \|\operatorname{div} \mathbf{v}\|^p_{L^p(\Omega)}.$$

Analogously, we introduce a space which will be like a dual space for $V(\Omega, p, \omega)$

$$W(\Omega, p, \omega) = \left\{ \mathbf{w} \in W^{1,q}_{loc}(\Omega)^2 \,:\, D\mathbf{w} \in L^q(\Omega)^{2\times 2} \text{ and } \varepsilon(\mathbf{w}) \in L^q(\Omega, \omega^{-q/p})^{2\times 2} \right\} / \left\{ Constants \right\},$$

with the norm

$$\|\mathbf{w}\|_W^q := \|D\mathbf{w}\|^q_{L^q(\Omega)} + \|\varepsilon(\mathbf{w})\|^q_{L^q(\Omega,\omega^{-q/p})}.$$

We are going to generalize the boundary condition introduced in (1.12) for arbitrary weights. Thus, let us define the following subspace:

$$V_{const}(\Omega, p, \omega) = \left\{ \mathbf{v} \in V(\Omega, p, \omega) \,:\, \int_\Omega D\mathbf{w} : \operatorname{Curl} \mathbf{v} = 0, \text{ for all } \mathbf{w} \in W(\Omega, p, \omega) \right\}.$$

Observe that the integral in the previous definition is well defined. In fact, it is easy to check that the product coordinate by coordinate between two matrixes, denoted with two points, is zero if one of them is symmetric and the other one is antisymmetric. So,

$$\int_\Omega \operatorname{Curl} \mathbf{v} : D\mathbf{w} = \int_\Omega (\operatorname{Curl} \mathbf{v})_s : (D\mathbf{w})_s + \int_\Omega (\operatorname{Curl} \mathbf{v})_a : (D\mathbf{w})_a,$$

where the indexes s and a denote the symmetric and antisymmetric part of the matrix, respectively. Thus,

$$\int_\Omega \operatorname{Curl} \mathbf{v} : D\mathbf{w} = \int_\Omega (\operatorname{Curl} \mathbf{v})_s : \varepsilon(\mathbf{w}) + \int_\Omega \operatorname{div} \mathbf{v}. \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : (D\mathbf{w})_a}_{\operatorname{rot}(\mathbf{w})}.$$

Therefore, using Schwarz inequality we obtain:

$$\begin{aligned} \left| \int_\Omega \operatorname{Curl} \mathbf{v} : D\mathbf{w} \right| &\leq \|(\operatorname{Curl} \mathbf{v})_s\|_{L^p(\Omega,\omega)} \|\varepsilon(\mathbf{w})\|_{L^q(\Omega,\omega^{-q/p})} + \|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} \|\operatorname{rot}(\mathbf{w})\|_{L^q(\Omega)} \\ &\leq C\|\mathbf{v}\|_V \|\mathbf{w}\|_W. \end{aligned}$$

When no confusion can arise we will write $V = V(\Omega, p, \omega)$, $W = W(\Omega, p, \omega)$ and $V_{const} = V_{const}(\Omega, p, \omega)$. Observe that in the previous section we defined another weighted space V lightly different from this one, although, both spaces are introduced in order to obtain a well definition of the divergence operator and its inverse.

Let us formulate a new weighted version of the divergence problem and the Korn inequality on arbitrary bounded planar domains. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\omega : \Omega \to (0, \infty)$ a bounded weight satisfying (1.35), and $1 < p < \infty$.

We say that (Ω, p, ω) satisfies the divergence property (1.36) if, given $f \in L^p(\Omega)$ with vanishing mean value, there exists a field $\mathbf{v} \in V_{const}$ such that

$$\operatorname{div} \mathbf{v} = f \qquad \text{and} \qquad \|\mathbf{v}\|_V \leq C\|f\|_{L^p(\Omega)}, \tag{1.36}$$

where the constant C depends only on Ω.

On the other hand, we say that (Ω, p, ω) satisfies the Korn property (1.37) if there exists a constant C depending only on Ω such that

$$\inf_{\mathbf{z}\in\mathcal{N}} \|\mathbf{w} - \mathbf{z}\|_W \leq C\|\varepsilon(\mathbf{w})\|_{L^q(\Omega,\omega^{-q/p})}, \tag{1.37}$$

for all $\mathbf{w} \in W$, where $\mathcal{N} = \{\mathbf{z} \in W \,:\, \varepsilon(\mathbf{z}) = 0\}$.

Observe that these new properties (1.36) and (1.37) are equivalent to the standard ones on Lipschitz domains where $\omega \equiv 1$.

1.5.1 Divergence implies Korn

In the next proposition we will prove that the divergence property (1.36) implies the Korn property (1.37) when $p = 2$. We generalize the ideas used in [BS] to show some version of Korn inequality in standard Sobolev spaces.

Proposition 1.14. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\omega : \Omega \to (0, \infty)$ a bounded weight such that $(\Omega, 2, \omega)$ satisfies the divergence property (1.36), then $(\Omega, 2, \omega)$ satisfies the Korn property (1.37).*

Proof. Let $\mathbf{w} \in W$ and let us suppose that $\operatorname{rot}(\mathbf{w}) := -\frac{\partial \mathrm{w}_1}{\partial x_2} + \frac{\partial \mathrm{w}_2}{\partial x_1}$ integrates zero. Thus, from (1.36) there exists $\mathbf{v} \in V$ such that $\operatorname{div} \mathbf{v} = \operatorname{rot}(\mathbf{w})$ and

$$\|\mathbf{v}\|_V \leq C\|\operatorname{rot}(\mathbf{w})\|_{L^2(\Omega)}.$$

Now, by a simple computation we can observe that

$$\varepsilon(\mathbf{w}) \;=\; D\mathbf{w} - \frac{1}{2}\operatorname{rot}(\mathbf{w}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, using that $\operatorname{div}\mathbf{v} = \operatorname{rot}(\mathbf{w})$ we can see

$$\begin{aligned}
\varepsilon(\mathbf{w}) : (D\mathbf{w} - \operatorname{Curl}\mathbf{v}) &= \left(D\mathbf{w} - \frac{1}{2}\operatorname{rot}(\mathbf{w})\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) : (D\mathbf{w} - \operatorname{Curl}\mathbf{v}) \\
&= D\mathbf{w} : D\mathbf{w} - D\mathbf{w} : \operatorname{Curl}\mathbf{v} - \frac{1}{2}\operatorname{rot}(\mathbf{w})\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} : (D\mathbf{w} - \operatorname{Curl}\mathbf{v}) \\
&= D\mathbf{w} : D\mathbf{w} - D\mathbf{w} : \operatorname{Curl}\mathbf{v} - \frac{1}{2}\operatorname{rot}(\mathbf{w})\,(\operatorname{rot}(\mathbf{w}) - \operatorname{div}(\mathbf{v})) \\
&= D\mathbf{w} : D\mathbf{w} - D\mathbf{w} : \operatorname{Curl}\mathbf{v}.
\end{aligned}$$

For that reason, using that $\mathbf{v} \in V_{const}$ and ω is bounded it follows that

$$\begin{aligned}
\|D\mathbf{w}\|^2_{L^2(\Omega)} &= \int_\Omega \varepsilon(\mathbf{w}) : (D\mathbf{w} - \operatorname{Curl}\mathbf{v}) + \int_\Omega D\mathbf{w} : \operatorname{Curl}\mathbf{v} \\
&\le \|\varepsilon(\mathbf{w})\|_{L^2(\Omega,\omega^{-1})}\left(\|D\mathbf{w}\|_{L^2(\Omega,\omega)} + \|\operatorname{Curl}\mathbf{v}\|_{L^2(\Omega,\omega)}\right) \\
&\le C\|\varepsilon(\mathbf{w})\|_{L^2(\Omega,\omega^{-1})}\left(\|D\mathbf{w}\|_{L^2(\Omega)} + \|\mathbf{v}\|_V\right) \\
&\le C\|\varepsilon(\mathbf{w})\|_{L^2(\Omega,\omega^{-1})}\|D\mathbf{w}\|_{L^2(\Omega)}.
\end{aligned}$$

Now, dividing by $\|D\mathbf{w}\|_{L^2(\Omega)}$ it is easy to conclude that

$$\|\mathbf{w}\|_W \le C\|\varepsilon(\mathbf{w})\|_{L^2(\Omega,\omega^{-1})}$$

for all $\mathbf{w} \in W$ such that $\operatorname{rot}(\mathbf{w})$ has a vanishing mean value.

Finally, any $\mathbf{z} \in \mathcal{N}$ can be written by $\mathbf{z} = (ay + b, -ax + c)$, with $a, b, c \in \mathbb{R}$. Thus, given an arbitrary $\mathbf{w} \in W$ there exists $\mathbf{z} \in N$ such that $\int \operatorname{rot}(\mathbf{w} - \mathbf{z}) = 0$. In consequence,

$$\|\mathbf{w} - \mathbf{z}\|_W \le C\|\varepsilon(\mathbf{w} - \mathbf{z})\|_{L^2(\Omega,\omega^{-1})} = C\|\varepsilon(\mathbf{w})\|_{L^2(\Omega,\omega^{-1})},$$

concluding the proof. □

1.5.2 Korn implies divergence

Now, we will prove that (1.37) implies (1.36) for $1 < p < \infty$. In particular, we can conclude that both results are equivalent for $p = 2$.

Proposition 1.15. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain, $\omega : \Omega \to (0, \infty)$ a bounded weight with the property (1.35) and $1 < p < \infty$ such that (Ω, p, ω) satisfies the Korn property (1.37), then (Ω, p, ω) satisfies the divergence property (1.36).*

Proof. Take $\mathbf{u} \in W^{1,p}(\Omega)^2$ such that

$$\operatorname{div} \mathbf{u} = f$$

and

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Thus, using that ω is bounded it follows that $\mathbf{u} \in V(\Omega, p, \omega)$ and

$$\|\mathbf{u}\|_V \leq C\|f\|_{L^p(\Omega)}.$$

So, it is enough to show that there exists $\mathbf{v} \in V$ with $\operatorname{div} \mathbf{v} = 0$ satisfying

$$\mathbf{u} - \mathbf{v} \in V_{const}(\Omega, p, \omega)$$

and

$$\|\mathbf{v}\|_V \leq C\|f\|_{L^p(\Omega)}.$$

Let us introduce the application

$$T(\tau) := \int_\Omega \operatorname{Curl} \mathbf{u} : D\mathbf{w}$$

for all $\tau \in L^q_{sym}(\Omega, \omega^{-q/p})^{2\times 2}$ which can be written by $\tau = \varepsilon(\mathbf{w})$ with $\mathbf{w} \in W(\Omega, p, \omega)$.

As $\operatorname{div} \mathbf{u}$ has vanishing mean it follows that T is well defined. Furthermore, applying the Korn property (1.37) we obtain the continuity of T in $L^q_{sym}(\Omega, \omega^{-q/p})^{2\times 2}$ as we can see:

$$\begin{aligned}
|T(\tau)| &= \left| \int_\Omega \operatorname{Curl} \mathbf{u} : D\mathbf{w} \right| \\
&\leq \|\operatorname{Curl} \mathbf{u}\|_{L^p(\Omega)} \inf_{\mathbf{v} \in \mathcal{N}} \|D(\mathbf{w} - \mathbf{v})\|_{L^q(\Omega)} \\
&\leq C\|\operatorname{Curl} \mathbf{u}\|_{L^p(\Omega)} \|\varepsilon(\mathbf{w})\|_{L^q(\Omega, \omega^{-q/p})} \\
&= C\|\operatorname{Curl} \mathbf{u}\|_{L^p(\Omega)} \|\tau\|_{L^q(\Omega, \omega^{-q/p}).}
\end{aligned}$$

By the Hahn-Banach theorem the functional T can be extended to $L^q_{sym}(\Omega, \omega^{-q/p})^{2\times 2}$ and therefore, by the Riesz representation theorem, there exists $\sigma \in L^p_{sym}(\Omega, \omega)^{2\times 2}$ such that

$$T(\tau) = \int_\Omega \sigma : \tau \qquad \forall \tau \in L^q_{sym}(\Omega, \omega^{-q/p})^{2\times 2}$$

and

$$\|\sigma\|_{L^p(\Omega,\omega)} \le C\|\mathrm{Curl}\,\mathbf{u}\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)},$$

where C depends on the constant in Korn property (1.37) and on the bound for the weight ω. In particular,

$$\int_\Omega \sigma : \varepsilon(\mathbf{w}) \;=\; \int_\Omega \mathrm{Curl}\,\mathbf{u} : D\mathbf{w}, \tag{1.38}$$

for every $\mathbf{w} \in W$. Then, since σ is symmetric, we can replace $\varepsilon(\mathbf{w})$ in (1.38) by $D\mathbf{w}$.

Now, using that ω satisfies the condition (1.35) it follows that $\sigma \in L^1_{loc}(\Omega)$. Thus, as it makes sense to calculate the divergence, let us show that $\mathbf{Div}\,\sigma$ is zero:

$$\int_\Omega \mathbf{Div}\;\sigma \cdot \mathbf{r} = -\int_\Omega \sigma : D\mathbf{r} = -\int_\Omega \mathrm{Curl}\,\mathbf{v} : D\mathbf{r} = \int_\Omega \mathbf{Div}\,\mathrm{Curl}\,\mathbf{v} \cdot \mathbf{r} = 0,$$

for every $\mathbf{r} \in C_0^\infty(\Omega)^2$ and therefore $\mathbf{Div}\,\sigma = 0$.

Now, from Lemma 1.5 we know that there exists $\mathbf{v} \in W^{1,p}_{loc}(\Omega)^2$ such that

$$\mathrm{Curl}\,\mathbf{v} = \sigma.$$

So, we obtain

$$\|D\mathbf{v}\|_{L^p(\Omega,\omega)} = \|\mathrm{Curl}\,\mathbf{v}\|_{L^p(\Omega,\omega)} = \|\sigma\|_{L^p(\Omega,\omega)}.$$

We have to check that $\mathrm{div}\,\mathbf{v} = 0$, but since σ is a symmetric tensor we have

$$\mathrm{div}\,\mathbf{v} = \frac{\partial \mathrm{v}_1}{\partial x_1} + \frac{\partial \mathrm{v}_2}{\partial x_2} = -\sigma_{12} + \sigma_{21} = 0.$$

Thus, it follows that $\mathbf{v} \in V$ with

$$\|\mathbf{v}\|_V = \|D\mathbf{v}\|_{L^p(\Omega,\omega)} = \|\sigma\|_{L^p(\Omega,\omega)} \le C\|f\|_{L^p(\Omega)}$$

and, from (1.38) it happens that $\mathbf{u} - \mathbf{v} \in V_{const}(\Omega, p, \omega)$, concluding the proof. □

Chapter 2

Weighted solutions of the divergence on regular domains

It has been mentioned several times in this thesis that if $\Omega \subset \mathbb{R}^n$ is a star-shaped domain with respect to a ball then there exists a right inverse of the divergence operator continuous from $L^p_0(\Omega)$ to $W^{1,p}_0(\Omega)^n$. In addition, the continuity can be showed via the singular integral theory of Calderón-Zygmund. Thus, we use this property to prove that this operator can be extended continuously to weighted Sobolev spaces where the weight is an appropriate power of the distance to a compact set.

In first section, we will give sufficient condition over $\mu \in \mathbb{R}$ and a compact set $F \subset \mathbb{R}^n$ to imply that $\omega = d_F^\mu$ belongs to the the Muckenhoupt class A_p, where d_F denotes the distance to F. And, in the second one, we will apply this result to obtain the solvability of $(\mathrm{div})_{p,w}$ on star-shaped domain with respect to a ball.

2.1 Powers of the distance and A_p classes

As we have commented, we deal in this section with weights $\omega : \mathbb{R}^n \to \mathbb{R}$ for which the Hardy-Littlewood operator M is bounded from $L^p(\mathbb{R}^n, \omega)$ into itself, with $1 < p < \infty$, that is

$$\int_{\mathbb{R}^n} |Mf(x)|^p \, \omega(x) \mathrm{d}x \leq C \int_{\mathbb{R}^n} |f(x)|^p \, \omega(x) \mathrm{d}x, \tag{2.1}$$

for all $f \in L^p(\mathbb{R}^n, \omega)$. Here $Mf(x)$ denotes the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y,$$

where the supreme is taken over all the cubes Q containing x. Muckenhoupt showed in 1972 that the class of functions for which the inequality (2.1) holds is defined by the condition (0.5) mentioned in the preliminaries that we remind here

$$\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}<\infty,$$

where the supreme is taken over all the balls $B\subset\mathbb{R}^n$.

Example 2.1. *One of the most known examples of weights in the A_p class $\omega:\mathbb{R}^n\to\mathbb{R}$ is $\omega(x)=|x|^\mu$, with $-n<\mu<n(p-1)$, which considers powers of the distance to the compact set $F=\{0\}$. Furthermore, it was shown that the condition over μ is optimal.*

Let us announce the main result of this section which generalizes the result proved in [DST] for smooth domains.

Theorem 2.2. *Let $F\subset\mathbb{R}^n$ be a compact set included in a m-regular set K. If*

$$-(n-m)<\mu<(n-m)(p-1)\,,$$

then d_F^μ belongs to the class A_p.

In order to prove the theorem we introduce first the following lemma.

Lemma 2.3. *Let $F\subset\mathbb{R}^n$ be a compact set included in an m-regular set K. Given $x_0\in F$ and $0<R<diam\,F/3$, there exists a constant C depending only on K such that*

$$N_k\left(B(x_0,R)\right)\;\le\;C\,R^m\,2^{km},$$

where $N_k\left(B(x_0,R)\right)$ denotes the number of Whitney cubes of F^c in the k^{th} generation contained in $B(x_0,R)$.

Proof. We can assume that $2^{-k}\le R$, if not $N_k\left(B(x_0,R\right)=0$. The number of Whitney cubes of F^c in the k^{th} generation contained in the ball $B(x_0,R)$ can be estimated in terms of the number of balls of radius 2^{-k} and center contained in F necessary to cover $F\cap B(x_0,2R)$. Indeed, suppose there exist balls $B(x_i,2^{-k})$ with $x_i\in F$, $1\le i\le N$, such that

$$F\cap B(x_0,2R)\subseteq\bigcup_{i=1}^N B(x_i,2^{-k}) \tag{2.2}$$

and let Q^k be a Whitney cube in the k^{th} generation contained in $B(x_0,R)$. Then, it is easy to check that

$$d(Q^k,F)=d(Q^k,F\cap B(x_0,2R)).$$

Thus, if $y_Q\in F$ is a point satisfying $d(Q^k,F)=d(Q^k,y_Q)$, there exists some i, $1\le i\le N$, such that $y_Q\in B(x_i,2^{-k})$. So, using that Q^k is a Whitney cube in the k^{th} generation it follows that

$$Q^k\subset B(x_i,6\ell_k).$$

But, $B(x_i, 6\ell_k)$ cannot contain more than a finite number $c(n)$ of Whitney cubes Q^k. Then, by (2.2) it follows that

$$N_k\left(B(x_0, R)\right) \leq c(n)N.$$

Therefore, to complete the proof we have to show that there exists N balls satisfying (2.2) with $N \leq C\, R^m\, 2^{km}$.

Let $r = 2^{-(k+1)}$. For $K_0 := K \cap B(x_0, 2R)$ we define the numbers

$$H_m(K_0, r) := \min\left\{ N r^m \;:\; K_0 \subseteq \bigcup_{i=1}^{N} B(x_i, r), \text{with } x_i \in K_0 \right\}$$

and

$$P(K_0, r) := \max\left\{ N \;:\; \text{there exists disjoint balls } B(x_i, r),\; i = 1, \ldots, N,\; x_i \in K_0 \right\}.$$

Then, using that K is an m-regular set we have

$$\begin{aligned} H_m(K_0, r) &\leq P\left(K_0, \frac{r}{2}\right) r^m \;=\; 2^m\, P\left(K_0, \frac{r}{2}\right)\left(\frac{r}{2}\right)^m \\ &< 2^m\, C \sum_{i=1}^{P(K_0, r/2)} \mathcal{H}^m\left(B\left(x_i, \frac{r}{2}\right) \cap K\right) \\ &= 2^m\, C \sum_{i=1}^{P(K_0, r/2)} \mathcal{H}^m\left(B\left(x_i, \frac{r}{2}\right) \cap K \cap B(x_0, 3R)\right) \\ &\leq 2^m\, C\, \mathcal{H}^m\left(K \cap B(x_0, 3R)\right) \;<\; C^2 6^m\, R^m. \end{aligned}$$

Thus, using the definition of $H_m(K_0, r)$ we obtain

$$K \cap B(x_0, 2R) \subseteq \bigcup_{i=1}^{N} B(x_i, 2^{-(k+1)}) \qquad \text{and} \qquad N \;\leq\; C\, R^m\, 2^{(k+1)m}. \tag{2.3}$$

Now, F is contained in K and therefore it is possible to cover $F \cap B(x_0, 2R)$ with $\bigcup_{i=1}^{N} B(x_i, 2^{-(k+1)})$. Then, if $B(x_i, 2^{-(k+1)})$ intersects F, for $x_i' \in F \cap B(x_i, 2^{-(k+1)})$ we have that

$$B(x_i, 2^{-(k+1)}) \subseteq B(x_i', 2^{-k}).$$

Thus, it is easy to see that the balls $B(x_i', 2^{-k})$ satisfy (2.2), concluding the proof. $\square$

Proof of the Theorem 2.2. Let B be a ball in $\mathbb{R}^n$, r_B its radius and $d(B)$ the distance of B to F.

If $r_B \leq d(B)$, given x in B we have $d(B) \leq d(x) \leq 3d(B)$. Then,

$$\begin{aligned} & \left(\frac{1}{|B|}\int_B d^{\mu}\right)\left(\frac{1}{|B|}\int_B d^{-\frac{\mu}{p-1}}\right)^{p-1} \\ \leq \;& C\left(\frac{1}{|B|}\int_B d(B)^{\mu}\right)\left(\frac{1}{|B|}\int_B d(B)^{-\frac{\mu}{p-1}}\right)^{p-1} \leq C. \end{aligned}$$

On the other hand, if $r_B \geq d(B)$, there exists $x_0 \in \partial\Omega$ such that $B \subseteq B(x_0, 3r_B)$. Then, without loss of generality, we can assume that B is centered at a point in F.

We consider two cases:

(a) If $r_B < diam\, F/6$, from the Whitney decomposition of F we have

$$\begin{aligned} & \left(\frac{1}{|B|}\int_B d^{\mu}\right)\left(\frac{1}{|B|}\int_B d^{-\frac{\mu}{p-1}}\right)^{p-1} \\ \leq \;& C r_B^{-np}\left(\sum_{Q^k}\int_{Q^k} d^{\mu}\right)\left(\sum_{Q^k}\int_{Q^k} d^{-\frac{\mu}{p-1}}\right)^{p-1} =: \mathrm{I}, \end{aligned}$$

where the sum is taken over all Whitney cubes Q^k intersecting B. But it is enough to consider the Whitney cubes contained in the ball $2B$.

Observe that if Q^k is contained in $2B$ then the diagonal of Q^k is smaller than the diagonal of $2B$. So, $2^{-k} \leq \frac{4}{\sqrt{n}} r_B$. Thus, if we call k_0 the minimum k such that there exists Q^k contained in B, it satisfies that $2^{-k_0} \leq C r_B$.

Now, using that $d(x) \simeq d(Q^k) \simeq 2^{-k}$ for every $x \in Q^k$ and Lemma 2.3 we obtain

$$\begin{aligned} \mathrm{I} \;&\leq\; C r_B^{-np}\left(\sum_{Q_k} 2^{-k\mu}2^{-kn}\right)\left(\sum_{Q_k} 2^{\frac{\mu k}{p-1}}2^{-kn}\right)^{p-1} \\ &\leq\; C r_B^{-np}\left(\sum_{k=k_0}^{\infty} N_k(B(x_0,2r_B))2^{-k\mu}2^{-kn}\right)\left(\sum_{k=k_0}^{\infty} N_k(B(x_0,2r_B))2^{\frac{\mu k}{p-1}}2^{-kn}\right)^{p-1} \\ &\leq\; C r_B^{-np}\left(\sum_{k=k_0}^{\infty} r_B^m 2^{-k(\mu+n-m)}\right)\left(\sum_{k=k_0}^{\infty} r_B^m 2^{-k\left(n-m-\frac{\mu}{p-1}\right)}\right)^{p-1} = \mathrm{II}. \end{aligned}$$

Then, since $-(n-m) < \mu < (p-1)(n-m)$, we obtain

$$\begin{aligned} \text{II} &\le C\, r_B^{-p(n-m)} \left(2^{-k_0(\mu+n-m)}\right) \left(2^{-k_0\left(n-m-\frac{\mu}{p-1}\right)}\right)^{p-1} \\ &\le C\, r_B^{-p(n-m)} \left(2^{-k_0}\right)^{p(n-m)} \le C. \end{aligned}$$

(b) If $r_B \ge diam\, F/6$, let x_F be a point in F independent of B. Then, since $x_0 \in F$ and $r_B > diam\, F/6$ we can assume that B is the ball with radius $r_B > 3diam\, F$ and center x_F. On the other hand, if B_1 denotes the ball of radius $2\, diamK$ centered at x_F we can see that d^μ and $d^{-\frac{\mu}{p-1}}$ are locally integrable. Then,

$$\int_{B_1} d^\mu \le C \qquad \text{y} \qquad \int_{B_1} d^{-\frac{\mu}{p-1}} \le C.$$

But, it is easy to see that $d(x) \simeq d(x, x_F)$ for all $x \in B \setminus B_1$. Therefore,

$$\begin{aligned} \int_B d^\mu &= \int_{B_1} d^\mu + \int_{B\setminus B_1} d^\mu \le C \left(1 + \int_{B\setminus B_1} |x - x_F|^\mu\right) \\ &\le C \left(1 + \int_{2\, diam\, F}^{r_B} \rho^\mu \rho^{n-1}\right) \le C r_B^{\mu+n}. \end{aligned}$$

Analogously we can show that

$$\int_B d^{-\frac{\mu}{p-1}} \le C r_B^{-\frac{\mu}{p-1}+n}$$

and therefore,

$$\left(\frac{1}{|B|}\int_B d^\mu\right)\left(\frac{1}{|B|}\int_B d^{-\frac{\mu}{p-1}}\right)^{p-1} \le C \frac{1}{|B|^p} r_B^{\mu+n} r_B^{-\mu+n(p-1)} \le C$$

and the lemma is proved. □

Remark 2.4. *A result very similar to Theorem 2.2 was obtained independently in [HaPi], considering that F is a m-regular set instead of a subset of one of them.*

Remark 2.5. *Theorem 2.2 is true if F satisfy the more general condition:*

$$F \subset \bigcup_{j=1}^{j_0} K_i,$$

where K_j is a m_j -regular set, with $m_j \le m$.

Proof. We only give the main ideas of the proof. To observe that the regularity condition in K is only used in the Lemma 2.3 to prove the property (2.2). It is more, as we can see in the proof of Lemma 2.3, it is enough to show the condition (2.3) where $K := \bigcup_{j=1}^{j_0} K_j$. In this case K is a union of m_j -regular set and not a m-regular set.

If $B(x_0, 2R) \cap K_j \neq \emptyset$, given y_j in the intersection, it follows that

$$K \cap B(x_0, 2R) \subseteq \bigcup_j K_j \cap B(y_j, 4R).$$

Thus, if $2R < diam\, K_j/3$, from (2.3) we can assert that

$$K_j \cap B(y_j, 4R) \subseteq \bigcup_{i=1}^{N_j} B(x_{ij}, 2^{-k}) \qquad \text{and} \qquad N_j \,\leq\, C\, R^{m_j}\, 2^{km_j} \,\leq\, C\, R^m\, 2^{km}.$$

Therefore, summing over all K_j we obtain that

$$K \cap B(x_0, 2R) \subseteq \bigcup_{j=1}^{j_0} K_j \cap B(y_j, 4R) \subseteq \bigcup_{j=1}^{j_0} \bigcup_{i=1}^{N_j} B(x_{ij}, 2^{-k})$$

if

$$4R \leq min\{diam\, K_j/3 \,:\, K_j \text{ is not a singleton }\}.$$

As we can see in its proof (part **2)b)**), it is sufficient to show the Theorem 2.2. □

2.2 The weighted divergence operator on star shaped domains

In this section we recall the well known explicit right inverse of the divergence operator introduced by Bogovskii on star-shaped domains with respect to a ball (see for more details [B, DM[1], G]). Also, using some properties published in [N], we can conclude that this operator is continuous in A_p weighted Sobolev spaces.

A similar result on John domains was given in [DRS].

Let us present the Bogovskii's formulation. Let $U \subset \mathbb{R}^n$ be a bounded star-shaped domain with respect to a ball $B \subset U$ and $\varphi \in C_0^\infty(B)$ such that $\int \varphi = 1$. Thus, given $f \in L^1(U)$ integrating zero, with $1 < p < \infty$, the solution introduced by Bogovskii is

$$\mathbf{u}(x) = \int_U \mathbf{G}(x, y)\, f(y)\, dy \qquad x \in \mathbb{R}^n \tag{2.4}$$

where $\mathbf{G}(x,y) = (G_1, \dots, G_n)$ is given by

$$\mathbf{G}(x,y) = \int_0^1 \frac{(x-y)}{s^{n+1}} \varphi\left(y + \frac{x-y}{s}\right) ds.$$

Since the proof is short let us recall why this function satisfies $\operatorname{div} \mathbf{u} = f$ in U. It is enough to prove that

$$-\int_U \mathbf{u}(y) \cdot \nabla\phi(y)\, \mathrm{d}y = \int_U f(y)\phi(y)\, \mathrm{d}y,$$

for all $\phi \in C_0^\infty(\mathrm{U})$. So, if $\bar{\phi}$ denotes $\int_U \phi\,\varphi$ and $y \in U$ it follows that

$$\begin{aligned}
\phi(y) - \bar{\phi} &= \int_U \left(\phi(y) - \phi(z)\right) \varphi(z)\, \mathrm{d}z \\
&= \int_U \int_0^1 -\frac{\mathrm{d}}{\mathrm{d}s}\phi\left(y + s(z-y)\right) \varphi(z)\, \mathrm{d}s\, \mathrm{d}z \\
&= \int_U \int_0^1 -(z-y) \cdot \nabla\phi\left(y + s(z-y)\right) \varphi(z)\, \mathrm{d}s\, \mathrm{d}z.
\end{aligned}$$

Making the change of variable $x = y + s(z-y)$, it will be justified later by (2.5), we obtain

$$\phi(y) - \bar{\phi} = \int_U \int_0^1 \frac{-1}{s^{n+1}} (x-y) \cdot \nabla\phi(x)\, \varphi\left(y + \frac{x-y}{s}\right) \mathrm{d}s\, \mathrm{d}x.$$

Then, as f integrates zero, it is easy to see interchanging the order of integration that

$$\begin{aligned}
\int_U f(y)\phi(y)\, \mathrm{d}y &= \int_U f(y) \left(\phi(y) - \bar{\phi}\right)\, \mathrm{d}y \\
&= -\int_U \int_U \int_0^1 f(y) \frac{1}{s^{n+1}} (x-y) \cdot \nabla\phi(x)\, \varphi\left(y + \frac{x-y}{s}\right) \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y \\
&= -\int_U \mathbf{u}(x) \cdot \phi(y)\, \mathrm{d}y.
\end{aligned}$$

It can be found in Lemma 2.1 from [DM[1]] that if $y \in U$ it follows that

$$|\mathbf{G}(x,y)| \leq \frac{C}{|x-y|^{n-1}}. \tag{2.5}$$

This property will be used to prove the continuity of Bogovskii's operator but, in particular, it warranties that the change of variable and the interchange of the order of integration realized before can be done.

The condition imposed over the domain U is utilized to obtain a solution $\mathbf{u}$ which vanishes on U^c. Indeed, given $x \notin U$ we will show that $G(x,y) = 0$ for all $y \in U$. Suppose that $y + (x-y)s^{-1}$ belongs to $Supp\,(\varphi) \subset B$ then

$$x = (1-s)y + s\left(y + \frac{x-y}{s}\right) \in U.$$

Thus, $\varphi(y + (x-y)s^{-1}) = 0$ for all $s \in (0,1)$ and $y \in U$. So, $G(x,y) = 0$ for all $y \in U$ and, in consequence $\mathbf{u}(x) = 0$. The density of $C_0^\infty(U)^n$ can be obtained by density using Theorem 2.6.

Before to demonstrate the goal of this section, the continuity of Bogovskii's operator in A_p weighted Sobolev spaces, we recall the following result published in [S^2, page 221].

Let T a bounded operator from $L^2(\mathbb{R}^n)$ to itself of the form

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|y-x|>\varepsilon} K(x,y) f(y)\, dy$$

where the kernel K satisfies

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \tag{2.6}$$

and the so called Hörmander conditions, namely,

$$|K(x,y) - K(x',y)| \leq C\frac{|x-x'|}{|x-y|^{n+1}} \qquad \text{if } |x-y| \geq 2|x-x'|, \tag{2.7}$$

and

$$|K(x,y) - K(x,y')| \leq C\frac{|y-y'|}{|x-y|^{n+1}} \qquad \text{if } |x-y| \geq 2|y-y'|. \tag{2.8}$$

Then, T is continuous from $L^p(\mathbb{R}^n, \omega)$ to itself for all $1 < p < \infty$ and ω a weight in the class A_p.

Theorem 2.6. *Given $\omega \in A_p$ and $1 < p < \infty$ the Bogovskii's formulation defined in (2.4) satisfies that*

$$\|\mathbf{u}\|_{W^{1,p}(\mathbb{R}^n,\omega)} \leq C\|f\|_{L^p(U,\omega)}. \tag{2.9}$$

Proof. It has been proved in Remark 0.2 that $L^p(U,\omega)$ is included in $L^1(U)$. Thus, the solution $\mathbf{u}$ is well defined.

In what follows the letter C denotes a generic constant which may depend on n, p, φ, ω, and the diameter of U, that we will call d, but it is independent of f and $\mathbf{u}$.

Let us first see that $\mathbf{u} \in L^p(\mathbb{R}^n,\omega)^n$. Using (2.5) we have

$$\begin{aligned}
|\mathbf{u}(x)| &\leq C \int_{\mathrm{U}} \frac{1}{|x-y|^{n-1}}\,|f(y)|\,dy \,\leq\, C \int_{B(x,d)} \frac{1}{|x-y|^{n-1}}\,|f(y)|\,dy \\
&\leq C \sum_{k=0}^{\infty} \int_{\frac{d}{2^{k+1}}<|y-x|<\frac{d}{2^k}} \frac{1}{|x-y|^{n-1}}\,|f(y)|\,dy \\
&\leq C \sum_{k=0}^{\infty} \int_{\frac{d}{2^{k+1}}<|y-x|<\frac{d}{2^k}} \left(\frac{2^{k+1}}{d}\right)^{n-1} |f(y)|\,dy \\
&\leq C \sum_{k=0}^{\infty} 2^{-k}\, \frac{1}{|B(x,\frac{d}{2^k})|} \int_{B(x,\frac{d}{2^k})} |f(y)|\,dy \,\leq\, C\, Mf(x),
\end{aligned}$$

where Mf denotes the Hardy-Littlewood maximal function of f. Since $\omega \in A_p$ the maximal operator is bounded in $L^p(\mathbb{R}^n,\omega)$ (see for example [Du, S[3]]) and therefore

$$\|\mathbf{u}\|_{L^p(\mathbb{R}^n,\omega)} \leq C\|f\|_{L^p(U,\omega)}. \tag{2.10}$$

Now, to see that the derivative of the components u_j of $\mathbf{u}$ with respect to x_i is also in $L^p(U,\omega)$ we use the decomposition published in Lemma 2.4 from [DM[1]],

$$\frac{\partial \mathrm{u}_j}{\partial x_i}(x) = \varphi_{ij}(x)f(x) + T^*_{ij}f(x) \qquad \text{for } x \in U$$

where φ_{ij} is a function bounded by a constant depending only on φ and

$$T^*_{ij}f(x) = \lim_{\varepsilon\to 0} \int_{|y-x|>\varepsilon} \chi_U(y)\, \frac{\partial G_j}{\partial x_i}(x,y) f(y)\,dy.$$

So, it is enough to prove that T^*_{ij} is bounded in $L^p(U,\omega)$.

In [DM[1]], the authors show the continuity of T^*_{ij} in $L^p(U)$ extending this operator to functions in $L^p(\mathbb{R}^n)$ and using the Calderón-Zygmund singular integral operator theory developed in [CZ]. However, we have not been able to find in the literature that a general operator in the form considered in [CZ] is continuous in $L^p(\mathbb{R}^n,\omega)$ for $\omega \in A_p$. Thus, we will extend the operator T^*_{ij} to $L^p(\mathbb{R}^n)$ in a way slightly different from the used one in [DM[1]] to warrant the continuity in weighted spaces using the theory introduced in [S[2]].

Let T be the singular integral operator defined by

$$Tg(x) = \lim_{\varepsilon\to 0} \int_{|y-x|>\varepsilon} \underbrace{\psi(y)\, \frac{\partial G_j}{\partial x_i}(x,y)}_{K(x,y)}\, g(y)\,dy,$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfies that $\psi(y) = 1$ for any $y \in U$ and it is supported in the ball B^* with radius d and the same center of B that we will suppose it is zero.

By a simple computation we obtain

$$\frac{\partial G_j}{\partial x_i}(x,y) = \int_0^1 \frac{\delta_{ij}}{s^{n+1}}\varphi\left(y+\frac{x-y}{s}\right) + \frac{x_j-y_j}{s^{n+2}}\frac{\partial \varphi}{\partial x_i}\left(y+\frac{x-y}{s}\right)ds,$$

where δ_{ij} denotes the Kronecker symbol.

Now, given $x \in \mathbb{R}^n$ and $y \in B^*$ we can observe that if $y + \frac{x-y}{s}$ belongs to the support of φ then

$$\frac{|x-y|}{s} \leq |y| + \left|y+\frac{x-y}{s}\right| \leq 2d.$$

Thus, we can conclude that

$$K(x,y) = \psi(y)\int_{\min\{1,\frac{|x-y|}{2d}\}}^1 \frac{\delta_{ij}}{s^{n+1}}\varphi\left(y+\frac{x-y}{s}\right) + \frac{x_j-y_j}{s^{n+2}}\frac{\partial \varphi}{\partial x_i}\left(y+\frac{x-y}{s}\right)ds. \quad (2.11)$$

It can be see that T is continuous from $L^p(\mathbb{R}^n)$ to itself repeating the proof introduced in [DM[1]]. Thus, to prove the continuity in $L^p(\mathbb{R}^n,\omega)$ to itself it is enough to prove that conditions (2.6), (2.7) and (2.8) hold.

From (2.11) it follows (2.6). The conditions of Hörmander can be proved in a similar way and, as (2.7) was proved in [N] let us show (2.8).

Given $x, y, y' \in \mathbb{R}^n$ such that $|x-y| \geq 2|y-y'|$ it follows that

$$\frac{|x-y'|}{2d} \geq \frac{|x-y|}{4d}.$$

Thus,

$$K(x,y) - K(x,y') = (i) + (ii) + (iii) + (iv),$$

where

$$\begin{aligned}
(i) &= (\psi(y)-\psi(y'))\int_{\min\{1,\frac{|x-y|}{4d}\}}^1 \frac{\delta_{ij}}{s^{n+1}}\varphi\left(y+\frac{x-y}{s}\right) + \frac{x_j-y_j}{s^{n+2}}\frac{\partial \varphi}{\partial x_i}\left(y+\frac{x-y}{s}\right)ds \\
(ii) &= \psi(y')\int_{\min\{1,\frac{|x-y|}{4d}\}}^1 \frac{\delta_{ij}}{s^{n+1}}\left[\varphi\left(y+\frac{x-y}{s}\right) - \varphi\left(y'+\frac{x-y'}{s}\right)\right]ds \\
(iii) &= \psi(y')\int_{\min\{1,\frac{|x-y|}{4d}\}}^1 \frac{1}{s^{n+1}}\left[\frac{y'_j-y_j}{s}\frac{\partial \varphi}{\partial x_i}\left(y+\frac{x-y}{s}\right)\right]ds \\
(iv) &= \psi(y')\int_{\min\{1,\frac{|x-y|}{4d}\}}^1 \frac{1}{s^{n+1}}\left[\frac{x_j-y'_j}{s}\left(\frac{\partial \varphi}{\partial x_i}\left(y+\frac{x-y}{s}\right) - \frac{\partial \varphi}{\partial x_i}\left(y'+\frac{x-y'}{s}\right)\right)\right]ds.
\end{aligned}$$

It is easy to observe that the expressions between brackets in (ii), (iii) and (iv) are bounded by $C\frac{|y-y'|}{|x-y|}$. Thus, we conclude (2.8) by a straight forward computation. $\square$

Remark 2.7. *If the weight ω in the previous theorem is a power of the distance to the origin (which is one of the case of interest in our applications to Stokes) it is not necessary to use the Hörmander conditions. Indeed, in this case (2.9) can be proved using the results in ([S^1]).*

Chapter 3

The divergence operator on domains with m−dimensional cusp in $\mathbb{R}^n$

In chapter 1 we proved the solvability of $(\text{div})_{p,w}$ on planar Hölder-α domains, where the weights are appropriate powers of the distance to the border. Now, for some domains where the set of singularities is a singleton or an interval, as the one introduced in (1.25), it can be interested to consider the distance to the cusp instead of the distance to the border. A simple example of this domains in $\mathbb{R}^2$ can be defined leaning a circle over a line, where the cusp is the contacted point.

In this chapter, we will show that there exists a weighted right inverse for the divergence for a particular class of Hölder-α domains in $\mathbb{R}^n$, where the weights are powers of the distance to the cusp. Furthermore, we will adapt this approach in order to obtain a positive answer for the existence of weighted solutions for the Stokes equations and the validity of the weighted Korn inequality.

3.1 Weighted solutions of the divergence on domains with a cusp

We consider the following class of domains in $\mathbb{R}^n$. Given integer numbers $k \geq 1$ and $m \geq 0$ we define

$$\Omega = \left\{(x, y, z) \in I \times \mathbb{R}^k \times I^m \,:\, |y| < x^\gamma\right\} \subset \mathbb{R}^n, \tag{3.1}$$

where $n = m + k + 1$, I is the interval $(0, 1)$ and $\gamma \geq 1$.

For $\gamma = 1$, Ω is a convex domain while, for $\gamma > 1$, Ω has an external cusp. The set of singularities of the boundary, which has dimension m, will be called M. Namely,

$$M = \{\mathbf{0}\} \times [0,1]^m \subset \mathbb{R}^{k+1} \times \mathbb{R}^m. \tag{3.2}$$

We show the graphic of the two different kind of domains Ω in $\mathbb{R}^3$. The dimension of the cusp in the first one is 0 while in the second one is 1.

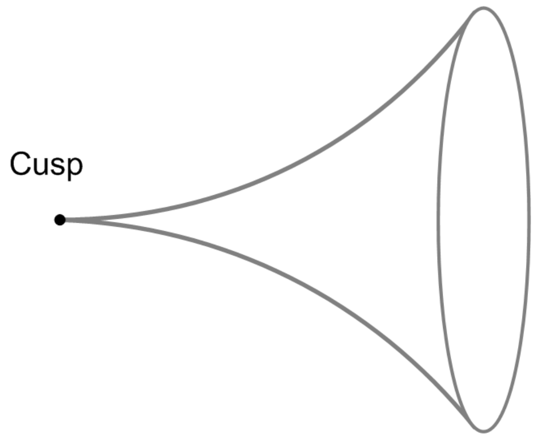

Figure 3.1: Zero dimensional external cusp in $\mathbb{R}^3$

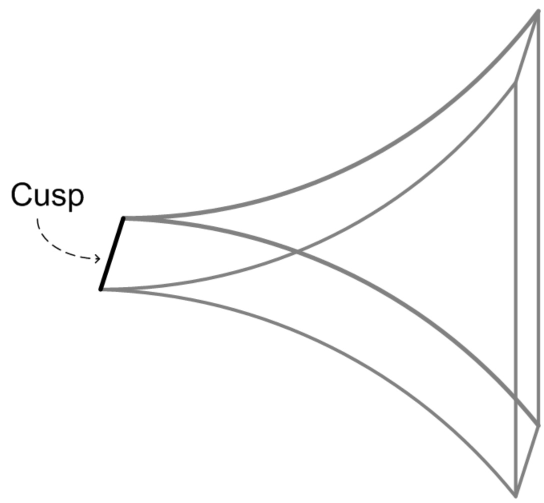

Figure 3.2: One dimensional external cusp in $\mathbb{R}^3$

We will work with weighted Sobolev spaces where the weights are powers of the distance to M that will be called d_M. Precisely, we will use the spaces $L^p(\Omega, d_M^{p\beta})$ and $W^{1,p}(\Omega, d_M^{p\beta_1}, d_M^{p\beta_2})$ where β, β_1 and β_2 are real numbers. For $\beta_1 = \beta_2 = \beta$ we will write $W^{1,p}(\Omega, d_M^{p\beta})$ instead of $W^{1,p}(\Omega, d_M^{p\beta}, d_M^{p\beta})$. It is well known that these spaces are Banach spaces (see [Ku], Theorem 3.6. for details).

Considering d_M defined everywhere in $\mathbb{R}^n$, from Theorem 2.2 it can be easily deduced the following result that we state as a lemma for the sake of clarity.

Lemma 3.1. *If $-(n-m) < \mu < (n-m)(p-1)$ then, $d_M^{\mu} \in A_p$.*

In what follows we will use several times that, for $(x, y, z) \in \Omega$, $d_M(x, y, z) \simeq x$, where the symbol $\simeq$ denotes equivalence up to multiplicative constants. Indeed, it is easy to see that $x \le d_M(x, y, z) = |(x, y)| \le (\sqrt{2})x$.

In the proof of the main result of this section we will use the Hardy type inequality given in the next lemma.

Lemma 3.2. *Let Ω be the domain defined in (3.1) and $1 < p < \infty$. Given $\kappa \in \mathbb{R}$, if $v \in W_0^{1,p}(\Omega, d_M^{p\kappa})$ then, $v/x \in L^p(\Omega, d_M^{p\kappa})$ and there exists constant C, depending only on p and κ, such that*

$$\left\| \frac{v}{x} \right\|_{L^p(\Omega, d_M^{p\kappa})} \le C \left\| \frac{\partial v}{\partial x} \right\|_{L^p(\Omega, d_M^{p\kappa})}. \tag{3.3}$$

Consequently, $W_0^{1,p}(\Omega, d_M^{p\kappa})$ is continuously imbedded in $W_0^{1,p}(\Omega, d_M^{p(\kappa-1)}, d_M^{p\kappa})$.

Proof. By density it is enough to prove (3.3) for $v \in C_0^{\infty}(\Omega)$. Writing $x^{p\kappa-p} = \frac{\partial(x^{p\kappa-p}y_1)}{\partial y_1}$ and integrating by parts we have

$$\begin{aligned} \int_\Omega |v(x, y, z)|^p x^{p\kappa-p} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_\Omega |v(x, y, z)|^p \, \frac{\partial(x^{p\kappa-p}y_1)}{\partial y_1} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= -\int_\Omega \frac{\partial(|v(x, y, z)|^p)}{\partial y_1} \, x^{p\kappa-p} y_1 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \end{aligned}$$

Then, since $|y_1| \le x$, we have

$$\int_\Omega |v(x, y, z)|^p x^{p\kappa-p} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le p \int_\Omega |v(x, y, z)|^{p-1} \left| \frac{\partial v(x, y, z)}{\partial y_1} \right| x^{p\kappa-p+1} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and so, writing now $x^{p\kappa-p+1} = x^{\frac{p\kappa-p}{q}} x^{\kappa}$ and applying the Hölder inequality, we obtain

$$\begin{aligned} &\int_\Omega |v(x, y, z)|^p x^{p\kappa-p} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\quad \le p \left(\int_\Omega |v(x, y, z)|^p x^{p\kappa-p} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right)^{(p-1)/p} \left(\int_\Omega \left| \frac{\partial v(x, y, z)}{\partial y_1} \right|^p x^{p\kappa} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right)^{1/p} \end{aligned}$$

and therefore,

$$\left(\int_\Omega |v(x, y, z)|^p x^{p\kappa-p} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right)^{1/p} \le p \left(\int_\Omega \left| \frac{\partial v(x, y, z)}{\partial y_1} \right|^p x^{p\kappa} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right)^{1/p}.$$

To conclude the proof we use that $d_M(x, y, z) \simeq x$ and then, that $W_0^{1,p}(\Omega, d_M^{p\kappa})$ is continuously imbedded in $W_0^{1,p}(\Omega, d_M^{p(\kappa-1)}, d_M^{p\kappa})$ follows from (3.3). □

We can now prove the main result of this chapter.

Theorem 3.3. *Let Ω be the domain defined in (3.1) for a fixed $\gamma > 1$, M defined as in (3.2), and $1 < p < \infty$. If $\beta \in \left(\frac{-\gamma(n-m)}{p} - \frac{\gamma-1}{q}, \frac{\gamma(n-m)}{q} - \frac{\gamma-1}{q}\right)$ and $\eta \in \mathbb{R}$ is such that $\eta \geq \beta + \gamma - 1$ then, given $f \in L^p_0(\Omega, d_M^{p\beta})$, there exists $\mathbf{u} \in W^{1,p}_0(\Omega, d_M^{p(\eta-1)}, d_M^{p\eta})^n$ satisfying*

$$\text{div }\mathbf{u} = f \tag{3.4}$$

and

$$\|\mathbf{u}\|_{W^{1,p}(\Omega, d_M^{p(\eta-1)}, d_M^{p\eta})} \leq C\|f\|_{L^p(\Omega, d_M^{p\beta})} \tag{3.5}$$

with a constant C depending only on γ, β, η, p and n.

Proof. It is enough to prove the result for the case $\eta = \beta + \gamma - 1$. Therefore we are going to consider this case.

Define

$$\hat{\Omega} = \left\{(\hat{x}, \hat{y}, \hat{z}) \in I \times \mathbb{R}^k \times I^m \;:\; |\hat{y}| < \hat{x}\right\} \subset \mathbb{R}^n \tag{3.6}$$

and let $F : \hat{\Omega} \to \Omega$ be the one-to-one application given by

$$F(\hat{x}, \hat{y}, \hat{z}) = (\hat{x}^\alpha, \hat{y}, \hat{z}) = (x, y, z),$$

where $\alpha = 1/\gamma$.

By this change of variables we associate functions defined in Ω with functions defined in $\hat{\Omega}$ in the following way,

$$h(x, y, z) = \hat{h}(\hat{x}, \hat{y}, \hat{z}).$$

Now, for $f \in L^p_0(\Omega, d_M^{p\beta})$, we define $\hat{g} : \hat{\Omega} \to \Omega$ by

$$\hat{g}(\hat{x}, \hat{y}, \hat{z}) := \alpha \hat{x}^{\alpha-1} \hat{f}(\hat{x}, \hat{y}, \hat{z}).$$

We want to apply Theorem 2.6 for $\hat{g}$ on the convex domain $\hat{\Omega}$ and then obtain the desired solution of (3.4) by using the so called Piola transform for vector fields.

In the rest of the proof we will use several times that, for $(x, y, z) \in \Omega$, $d_M(x, y, z) \simeq x$, $\det DF(\hat{x}, \hat{y}, \hat{z}) = \alpha\hat{x}^{\alpha-1}$ and $\det DF^{-1}(x, y, z) = \gamma x^{\gamma-1}$.

First let us see that, for $\hat{\beta} = \alpha\left(\beta + (\gamma - 1)/q\right)$, we have

$$\hat{g} \in L^p_0(\hat{\Omega}, d_M^{p\hat{\beta}}) \qquad \text{and} \qquad \|\hat{g}\|_{L^p(\hat{\Omega}, d_M^{p\hat{\beta}})} \simeq \|f\|_{L^p(\Omega, d_M^{p\beta})}. \tag{3.7}$$

Indeed, we have

$$\begin{aligned}\|\hat{g}\|^p_{L^p(\hat{\Omega},d_M^{p\hat{\beta}})} &\simeq \int_{\hat{\Omega}} |\hat{g}|^p \hat{x}^{p\hat{\beta}} = \alpha^p \int_{\hat{\Omega}} |\hat{f}|^p \hat{x}^{p(\alpha-1)} \hat{x}^{\alpha p(\beta+(\gamma-1)/q)} \\ &= \alpha^p \int_{\Omega} |f|^p x^{p\beta+1-\gamma} \gamma x^{\gamma-1} \simeq \|f\|^p_{L^p(\Omega,d_M^{p\beta})}\end{aligned}$$

and

$$\int_{\hat{\Omega}} \hat{g} = \alpha \int_{\hat{\Omega}} \hat{f} \hat{x}^{\alpha-1} = \alpha \int_{\Omega} f x^{1-\gamma} \gamma x^{\gamma-1} = \int_{\Omega} f = 0.$$

Thus, (3.7) holds.

Observe that, from Lemma 3.1 and our hypothesis on β, we have $d_M^{p\hat{\beta}} \in A_p$. In particular, it follows from Remark 0.2 that $\hat{g} \in L^1(\hat{\Omega})$ and therefore the mean value of f in Ω is well defined.

Now, from Theorem 2.6 we know that there exists $\hat{\mathbf{v}} \in W_0^{1,p}(\hat{\Omega}, d_M^{p\hat{\beta}})^n$ such that

$$\operatorname{div} \hat{\mathbf{v}} = \hat{g} \tag{3.8}$$

and

$$\|\hat{\mathbf{v}}\|_{W^{1,p}(\hat{\Omega},d_M^{p\hat{\beta}})} \leq C \|\hat{g}\|_{L^p(\hat{\Omega},d_M^{p\hat{\beta}})}. \tag{3.9}$$

Now, we define $\mathbf{u}$ as the Piola transform of $\hat{\mathbf{v}}$, namely,

$$\mathbf{u}(x,y,z) = \frac{1}{\det DF} DF(\hat{x},\hat{y},\hat{z}) \hat{\mathbf{v}}(\hat{x},\hat{y},\hat{z})$$

or equivalently, if $\hat{\mathbf{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_n)$,

$$\mathbf{u}(x,y,z) = \gamma x^{\gamma-1} \left(\alpha x^{1-\gamma} \hat{\mathrm{v}}_1(x^\gamma,y,z), \hat{\mathrm{v}}_2(x^\gamma,y,z), \ldots, \hat{\mathrm{v}}_n(x^\gamma,y,z)\right).$$

Then, using (3.8), it is easy to see that

$$\operatorname{div} \mathbf{u} = f.$$

To prove (3.5) we first show that

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,d_M^{p\eta})} \leq C \|f\|_{L^p(\Omega,d_M^{p\beta})}. \tag{3.10}$$

In view of the equivalence of norms given in (3.7) and the estimate (3.9), to prove (3.10) it is enough to see that

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,d_M^{p\eta})} \leq C \|\hat{\mathbf{v}}\|_{W^{1,p}(\hat{\Omega},d_M^{p\hat{\beta}})}. \tag{3.11}$$

But, we have

$$\|\mathrm{u}_1\|^p_{L^p(\Omega,d_M^{p\eta})} \simeq \int_\Omega |\mathrm{u}_1|^p x^{p\eta} = \alpha \int_{\hat\Omega} |\hat{\mathrm{v}}_1|^p \hat{x}^{\alpha p\eta}\hat{x}^{\alpha-1} \simeq \|\hat{\mathrm{v}}_1\|^p_{L^p(\Omega,d_M^{p\hat\beta})}, \tag{3.12}$$

where in the last step we have used $\alpha p\eta + \alpha - 1 = p\hat\beta$. In an analogous way we can show that, for $j = 2, \dots, n$,

$$\|\mathrm{u}_j\|_{L^p(\Omega,d_M^{p\eta})} \le C\|\hat{\mathrm{v}}_j\|_{L^p(\Omega,d_M^{p\hat\beta})}.$$

Then, it only remains to bound the derivatives of the components of $\mathbf{u}$. That

$$\left\|\frac{\partial \mathrm{u}_1}{\partial y_1}\right\|^p_{L^p(\Omega,d_M^{p\eta})} \simeq \left\|\frac{\partial \hat{\mathrm{v}}_1}{\partial \hat{y}_1}\right\|^p_{L^p(\hat\Omega,d_M^{p\hat\beta})}$$

follows exactly as (3.12). Let us now estimate $\frac{\partial \mathrm{u}_2}{\partial x}$. Using

$$\left|\frac{\partial \mathrm{u}_2}{\partial x}\right| = \gamma^2 \left|\frac{\gamma-1}{\gamma}\frac{\hat{\mathrm{v}}_2(x^\gamma,y,z)}{x^\gamma} + \frac{\partial \hat{\mathrm{v}}_2(x^\gamma,y,z)}{\partial \hat{x}}\right| x^{2(\gamma-1)}$$

and Lemma 3.2 for $\hat\Omega$ we have

$$\begin{aligned}\left\|\frac{\partial \mathrm{u}_2}{\partial x}\right\|^p_{L^p(\Omega,d_M^{p\eta})} &\simeq \int_\Omega \left|\frac{\partial \mathrm{u}_2}{\partial x}\right|^p x^{p\eta} \le C\int_{\hat\Omega}\left(\left|\frac{\hat{\mathrm{v}}_2}{\hat{x}}\right|^p + \left|\frac{\partial \hat{\mathrm{v}}_2}{\partial \hat{x}}\right|^p\right)\hat{x}^{2p(1-\alpha)}\hat{x}^{\alpha p\eta+\alpha-1} \\ &\le C\int_{\hat\Omega}\left(\left|\frac{\hat{\mathrm{v}}_2}{\hat{x}}\right|^p + \left|\frac{\partial \hat{\mathrm{v}}_2}{\partial \hat{x}}\right|^p\right)\hat{x}^{p\hat\beta} \le C\left(\left\|\frac{\partial \hat{\mathrm{v}}_2}{\partial \hat{y}_1}\right\|^p_{L^p(\hat\Omega,d_M^{p\hat\beta})} + \left\|\frac{\partial \hat{\mathrm{v}}_2}{\partial \hat{x}}\right\|^p_{L^p(\hat\Omega,d_M^{p\hat\beta})}\right),\end{aligned}$$

where we have used again $\alpha p\eta + \alpha - 1 = p\hat\beta$ and that $2p(1-\alpha) > 0$.

All the other derivatives of the components of $\mathbf{u}$ can be bounded in an analogous way and therefore (3.11) holds.

Now, since

$$\mathbf{u}|_{\partial\Omega} = \frac{1}{\det DF} DF\hat{\mathbf{v}}|_{\partial\hat\Omega},$$

it is easy to check that $\mathbf{u}$ belongs to the closure of $C_0^\infty(\Omega)^n$,i. e., $\mathbf{u} \in W_0^{1,p}(\Omega, d_M^{p\eta})^n$ and by Lemma 3.2 $\mathbf{u} \in W_0^{1,p}(\Omega, d_M^{p(\eta-1)}, d_M^{p\eta})^n$ as we wanted to show. □

Remark 3.4. *The hypothesis that* $\beta < \frac{\gamma(n-m)}{q} - \frac{\gamma-1}{q}$ *is necessary in order to have the condition* $\int_\Omega f = 0$ *is well defined for* $f \in L^p(\Omega, d_M^{p\beta})$*. Indeed, if* $\beta \ge \frac{\gamma(n-m)}{q} - \frac{\gamma-1}{q}$*, it is easy to check that* $f(x,y,z) = (1-\log x)^{-1}x^{\gamma-1-\gamma(n-m)}$ *belongs to* $L^p(\Omega, d_M^{p\beta}) \setminus L^1(\Omega)$*.*

Remark 3.5. *It can be shown that the condition* $\eta \ge \beta + \gamma - 1$ *assumed in the theorem is also necessary. Indeed, as a particular case of Theorem 4.3 it could be proved that if* $\eta - \beta < \gamma - 1$ *there exists* $f \in L_0^p(\Omega, d_M^{p\beta})$ *such that a solution* $\mathbf{u}$ *of (3.4) satisfying (3.5) does not exist.*

3.2 Application to the Stokes equations

In this section we show how the results obtained in the previous section can be applied to prove the well posedness of the Stokes equations

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 \qquad \text{in } \Omega \\ \mathbf{u} &= 0 \qquad \text{on } \partial\Omega \end{cases}$$

in appropriate weighted Sobolev spaces on cuspidal domains.

Indeed, combining the variational analysis given in Section 1.4 with the results in Section 3.1 we obtain the following theorem.

Theorem 3.6. *Given $\gamma \geq 1$, let Ω be the domain defined in (3.1). If $\mathbf{f} \in H^{-1}(\Omega)^n$ then, there exists a unique $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L^2(\Omega, d_M^{2(\gamma-1)})$, with p satisfying $\int_\Omega p \, d_M^{2(\gamma-1)} = 0$, weak solution of the Stokes equations. Moreover,*

$$\|\mathbf{u}\|_{H_0^1(\Omega)} + \|p\|_{L^2(\Omega, d_M^{2(\gamma-1)})} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)} \tag{3.13}$$

with a constant C depending only on γ and n.

Proof. Consider the particular case $\eta = 0$, $\beta = 1-\gamma$ and $p = 2$ in Theorem 3.3. It is easy to check that in this case β satisfies the hypothesis of that theorem for any values of n and m (recall that $m \leq n-2$), i. e,

$$\beta = 1 - \gamma \in \left(\frac{-\gamma(n-m)}{2} - \frac{\gamma - 1}{2}, \frac{\gamma(n-m)}{2} - \frac{\gamma - 1}{2} \right).$$

Then, given $f \in L_0^2(\Omega, d_M^{2(1-\gamma)})$ there exists $\mathbf{u} \in H_0^1(\Omega)^n$ satisfying

$$\operatorname{div} \mathbf{u} = f$$

and

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega, d_M^{2(1-\gamma)})}$$

with a constant C depending only on γ and n. Therefore, the result follows immediately from Theorem 1.11. $\square$

In the next corollary we show the well posedness of the Stokes equations in standard spaces.

Corollary 3.7. *Given $\gamma \geq 1$, let Ω be the domain defined in (3.1) and $\mathbf{f} \in H^{-1}(\Omega)^n$. If r_0 is defined by*

$$r_0 = 2 - \frac{4(\gamma-1)}{\gamma(k+2)-1}.$$

Then, $r_0 > 0$, and for $0 < r < r_0$, there exists $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L^r(\Omega)$, with p satisfying $\int_\Omega p\, d_M^{2(\gamma-1)} = 0$, weak solution of the Stokes equations (1.29). Moreover, there exists a constant C depending only on n, γ and r such that

$$\|\mathbf{u}\|_{H_0^1(\Omega)} + \|p\|_{L^r(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

In particular, if $k \geq 2$, or $k = 1$ and $\gamma < 3$, $p \in L^1(\Omega)$.

Proof. Since $\gamma > 1$ and $k \geq 1$ it follows that $r_0 > 0$. Now, given a positive $r < r_0$ it is enough to see that, if $(\mathbf{u}, p)$ is the solution given by Theorem 3.6, then $p \in L^r(\Omega)$ and

$$\|p\|_{L^r(\Omega)} \leq C\|p\|_{L^2(\Omega, d_M^{2(\gamma-1)})}. \tag{3.14}$$

It is easy to see that $\int_\Omega d_M^s < +\infty$ for any $s > -\gamma k - 1$. Then, applying the Hölder inequality with $2/r$ and its dual exponent we have

$$\|p\|_{L^r(\Omega)}^r = \int_\Omega |p|^r d_M^{(\gamma-1)r} d_M^{(1-\gamma)r} \leq \|p\|_{L^2(\Omega, d_M^{2(\gamma-1)})}^r \left(\int_\Omega d_M^{\frac{2(1-\gamma)r}{2-r}}\right)^{\frac{2-r}{2}}.$$

Since $r < r_0$, we have $(2(1-\gamma)r)/(2-r) > -\gamma k - 1$, and so the integral on the right hand side is finite. Therefore, (3.14) is proved. Finally, if $k \geq 2$, or $k = 1$ and $\gamma < 3$, it is easy to check that $r_0 > 1$ and therefore $p \in L^1(\Omega)$. □

Let us show the results of the above theorem and corollary in the particular cases $n = 2$ and $n = 3$. We will use here the usual notation $x = (x_1, x_2) \in \mathbb{R}^2$ or $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

For $n = 2$ we have $m = 0$ and, for $\gamma \geq 1$, the domain is

$$\Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, |x_2| < x_1^\gamma\right\}.$$

In this case $M = \{(0,0)\}$ and therefore $d_M(x) = |x|$. Then, for $\mathbf{f} \in H^{-1}(\Omega)^2$, there exists a unique

$$(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L^2(\Omega, |x|^{2(\gamma-1)}), \tag{3.15}$$

with p satisfying $\int_\Omega p\, |x|^{2(\gamma-1)} = 0$, weak solution of the Stokes equations . Moreover,

$$\|\mathbf{u}\|_{H_0^1(\Omega)} + \|p\|_{L^2(\Omega, |x|^{2(\gamma-1)})} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)} \tag{3.16}$$

and, for $r < 2 - \frac{4(\gamma-1)}{3\gamma-1}$,

$$\|p\|_{L^r(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)} \tag{3.17}$$

with a constant C depending only on γ and r.

For $n = 3$ we have the two possible cases $m = 0$ or $m = 1$. In the first case the domain has a cuspidal point and is given by

$$\Omega = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, 0 < x_1 < 1 \, , \, \sqrt{x_2^2 + x_3^2} < x_1^\gamma \right\}.$$

In this case we obtain exactly the same estimates (3.16) and (3.17) with obvious changes of dimension. The only difference is that now $r < 2 - \frac{4(\gamma-1)}{4\gamma-1}$. Observe that in particular, in this case $p \in L^1(\Omega)$.

Finally, when $m = 1$, the domain has a cuspidal edge and is given by

$$\Omega = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, 0 < x_1 < 1 \, , \, 0 < x_3 < 1 \, , \, |x_2| < x_1^\gamma \right\}$$

and, defining $\bar{x} = (x_1, x_2)$, we have $d_M(x) = |\bar{x}|$ and the a priori estimates

$$\|\mathbf{u}\|_{H_0^1(\Omega)} + \|p\|_{L^2(\Omega, |\bar{x}|^{2(\gamma-1)})} \leq C \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

and, for $r < 2 - \frac{4(\gamma-1)}{3\gamma-1}$,

$$\|p\|_{L^r(\Omega)} \leq C \|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

Finally, it is natural to ask if the a priori estimate on the unique solution $(\mathbf{u}, p)$ in Theorem 3.6 can be improved on this kind of domains. Thus, to end this section we will show that the weight involved in the condition (3.13) is optimal. In particular, it follows that there is not a standard Sobolev solution in general for the Stokes equations on these domains.

Let Ω be the domain defined in (3.1) for the particular case $n = 2$. Thus, if $p(x_1, x_2) = x_1^s$, with $s < 0$, let us find conditions on s for ∇p to belong to $H^{-1}(\Omega)^2$. We only have to show that $\frac{\partial p}{\partial x_1} \in H^{-1}(\Omega)$. Now, from the following equality

$$\frac{\partial p}{\partial x_1} = \frac{\partial (s x_1^{s-1} x_2)}{\partial x_2}$$

we can see that it is enough to prove that $s x_1^{s-1} x_2 \in L^2(\Omega)$. Thus, by an elementary computation we can assert that $\nabla p \in H^{-1}(\Omega)^2$ if $s > 1 - \gamma - \frac{1+\gamma}{2} = s_0$.

On the other hand, it is easy to check that $p \in L^2(\Omega, |x|^{2(\gamma-1)})$ if and only if $s > s_0$. Thus, if $s > s_0$ it follows that $(\mathbf{0}, p)$ is the unique solution in (3.15) with $\mathbf{f}(x) = \nabla p$, up to an additive constant in the pressure. Concluding that the power in (3.16) can not be improved.

These kind of examples were introduced by G. Acosta to study the Korn inequality on cuspidal domains.

3.3 Weighted Korn type inequalities

Important and well-known consequences of the existence of a right inverse of the divergence operator in Sobolev spaces are the different cases of Korn inequalities. It is also known that the classic first and second cases (in the terminology introduced by Korn) can be derived from the following inequality,

$$\|D\mathbf{v}\|_{L^p(\Omega)} \leq C\left\{\|\varepsilon(\mathbf{v})\|_{L^p(\Omega)} + \|\mathbf{v}\|_{L^p(\Omega)}\right\}, \tag{3.18}$$

where, as we have mentioned, $\varepsilon(\mathbf{v})$ denotes the symmetric part of the differential matrix $D\mathbf{v}$.

For the cuspidal domains that we are considering this inequality is not valid (counterexamples will be included in Chapter 4). In view of our results on solutions of the divergence it is natural to look for Korn type inequalities in weighted Sobolev spaces. For general Hölder-α domains, inequalities of this kind were obtained in [ADL] using weights which are powers of the distance to the boundary. Here we are interested in stronger results for the particular class of Hölder-α domains defined in (3.1). We are going to prove estimates in norms involving the distance to the cusp.

It is not straightforward to generalize the classic arguments to derive Korn inequalities from the existence of right inverses of the divergence to the weighted case. We do not know how to do it if we work with weighted norms in both sides of the inequality (3.18). Therefore, we are going to prove a result for a general weight and afterwards, we will obtain more general inequalities for the case of weights which are powers of the distance to the cusp, using an argument introduced in [BK]. In fact, in order to obtain Korn inequalities we prove first a generalization of Lions lemma for weighted Sobolev spaces.

Let us mention that in what follows we state and prove several inequalities assuming that the left hand side is finite. Afterwards, by density arguments, one can conclude that these inequalities are valid whenever the right hand side is finite. This is a usual procedure.

Given $1 < p < \infty$, a domain $U \subset \mathbb{R}^n$, and a weight ω, we denote with $W^{-1,q}(U,\omega^{1-q})$ the dual space of $W_0^{1,p}(U,\omega)$. Observe that $W^{-1,p}(U,\omega) = W_0^{1,q}(U,\omega^{1-q})'$.

Lemma 3.8. *Given a weight ω, a bounded domain $U \subset \mathbb{R}^n$, and $1 < p < \infty$, assume that for any $g \in L^q_0(U)$ there exists $\mathbf{u} \in W_0^{1,q}(U,\omega^{1-q})^n$ such that* $\operatorname{div}\mathbf{u} = g$ *and*

$$\|\mathbf{u}\|_{W^{1,q}(U,\omega^{1-q})} \leq C\|g\|_{L^q(U)},$$

with a constant C depending only on U, p, and ω. Fix an open ball $B \subset U$. Then, for any $f \in L^p(U)$,

$$\|f\|_{L^p(U)} \leq \ C\left\{\|f\|_{W^{-1,p}(B)} + \|\nabla f\|_{W^{-1,p}(U,\omega)}\right\},$$

where the constant C depends only on U, B, p, and ω.

Proof. Take $f \in L^p(U)$. If $\bar{f}$ denotes the mean value of f over U we have, for $g \in L^q(U)$,

$$\int_U (f-\bar{f})g = \int_U (f-\bar{f})(g-\bar{g}).$$

But, from our hypothesis, there exists a solution $\mathbf{u} \in W_0^{1,q}(U,\omega^{1-q})^n$ of $\operatorname{div}\mathbf{u} = g-\bar{g}$ satisfying

$$\|\mathbf{u}\|_{W^{1,q}(U,\omega^{1-q})} \leq C\|g-\bar{g}\|_{L^q(U)}.$$

Thus,

$$\begin{aligned} \int_U (f-\bar{f})g = \int_U (f-\bar{f})\operatorname{div}\mathbf{u} &\leq \|\nabla f\|_{W^{-1,p}(U,\omega)}\|\mathbf{u}\|_{W^{1,q}(U,\omega^{1-q})} \\ &\leq C\|\nabla f\|_{W^{-1,p}(U,\omega)}\|g-\bar{g}\|_{L^q(U)}. \end{aligned}$$

Therefore, by duality,

$$\|f-\bar{f}\|_{L^p(U)} \leq C\|\nabla f\|_{W^{-1,p}(U,\omega)}. \tag{3.19}$$

Now, we decompose f as

$$f = (f-f_\varphi) + f_\varphi,$$

where $f_\varphi := \int_B f\varphi$ with $\varphi \in C_0^\infty(B)$ such that $\int_B \varphi = 1$. Thus,

$$f - f_\varphi = f - \bar{f} + \int_B \left(\bar{f} - f\right)\varphi,$$

and so, using (3.19),

$$\|f-f_\varphi\|_{L^p(U)} \leq \left(1 + \|\varphi\|_{L^q(B)}\right)\|f-\bar{f}\|_{L^p(U)} \leq C\|\nabla f\|_{W^{-1,p}(U,\omega)}.$$

Therefore, to conclude the proof we have to estimate $\|f_\varphi\|_{L^p(U)}$. But,

$$\|f_\varphi\|_{L^p(U)} \leq |U|^{1/p}\left|\int_B f\varphi\right| \leq |U|^{1/p}\|f\|_{W^{-1,p}(B)}\|\varphi\|_{W_0^{1,q}(B)}.$$

□

Using the inequality obtained in previous lemma we can generalize a classic argument to prove a Korn type inequality obtaining the following result.

Lemma 3.9. *Given ω, U, B, and p as in previous Lemma, assume that for any $f \in L^p(U)$,*

$$\|f\|_{L^p(U)} \leq C\left\{\|f\|_{W^{-1,p}(B)} + \|\nabla f\|_{W^{-1,p}(U,\omega)}\right\},$$

where the constant C depends only on U, B, p, and ω. Then, for any $\mathbf{v} \in W^{1,p}(U)^n$,

$$\|D\mathbf{v}\|_{L^p(U)} \leq C\left\{\|\varepsilon(\mathbf{v})\|_{L^p(U,\omega)} + \|\mathbf{v}\|_{L^p(B)}\right\},$$

where the constant C depends only on U, B, p, and ω.

Proof. It is known that, for any $g \in L^p(B)$,

$$\left\| \frac{\partial g}{\partial x_j} \right\|_{W^{-1,p}(B)} \leq \|g\|_{L^p(B)}. \tag{3.20}$$

Analogously, for any $g \in L^p(U,\omega)$, we have

$$\left\| \frac{\partial g}{\partial x_j} \right\|_{W^{-1,p}(U,\omega)} = \sup_{0 \neq \phi \in W_0^{1,q}(U,\omega^{1-q})} \frac{\left| \int_U g \frac{\partial \phi}{\partial x_j} \right|}{\|\phi\|_{W^{1,q}(U,\omega^{1-q})}} \leq \|g\|_{L^p(U,\omega)}. \tag{3.21}$$

On the other hand, using the property assumed, we have

$$\left\| \frac{\partial \mathrm{v}_i}{\partial x_j} \right\|_{L^p(U)} \leq C \left\{ \left\| \frac{\partial \mathrm{v}_i}{\partial x_j} \right\|_{W^{-1,p}(B)} + \left\| \nabla \frac{\partial \mathrm{v}_i}{\partial x_j} \right\|_{W^{-1,p}(U,\omega)^n} \right\}.$$

Using now the well known identity

$$\frac{\partial^2 \mathrm{v}_i}{\partial x_j \partial x_k} = \frac{\partial \varepsilon_{ik}(\mathbf{v})}{\partial x_j} + \frac{\partial \varepsilon_{ij}(\mathbf{v})}{\partial x_k} - \frac{\partial \varepsilon_{jk}(\mathbf{v})}{\partial x_i}$$

in the last term on the right hand side, and the inequalities (3.20) and (3.21), we conclude the proof. □

An immediate consequence of Lemmas 3.8 and 3.9, and Theorem 3.3 is the following.

Corollary 3.10. *Given $\gamma \geq 1$, let Ω be the domain defined in (3.1), M defined in (3.2), $1 < p < \infty$, and $B \subset \Omega$ an open ball. Then, there exists a constant C, which depends only on Ω, B, and p, such that for all $\mathbf{u} \in W^{1,p}(\Omega)^n$,*

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leq C \left\{ \|\varepsilon(\mathbf{u})\|_{L^p(\Omega, d_M^{p(1-\gamma)})} + \|\mathbf{u}\|_{L^p(B)} \right\}.$$

Proof. According to Theorem 3.3, for any $g \in L^q_0(\Omega)$ there exists $\mathbf{u} \in W_0^{1,q}(\Omega, d_M^{q(\gamma-1)})^n$ such that $\operatorname{div} \mathbf{u} = g$ and

$$\|\mathbf{u}\|_{W^{1,q}(\Omega, d_M^{q(\gamma-1)})} \leq C \|g\|_{L^q(\Omega)},$$

with a constant C depending only on γ and p. Therefore, Lemmas 3.8 and 3.9 applies for $\omega = d_M^{p(1-\gamma)}$. □

We conclude this chapter proving a more general Korn type inequalities for the cuspidal domains defined in (3.1). To obtain these inequalities we use an argument introduced in [BK].

Theorem 3.11. *Given $\gamma \geq 1$, let Ω be the domain defined in (3.1), M defined in (3.2), $1 < p < \infty$, $B \subset \Omega$ an open ball, and $\beta \geq 0$. Then, there exists a constant C, which depends only on Ω, B, p, and β, such that for all $\mathbf{u} \in W^{1,p}(\Omega, d_M^{p\beta})^n$*

$$\|D\mathbf{u}\|_{L^p(\Omega, d_M^{p\beta})} \leq C \left\{ \|\varepsilon(\mathbf{u})\|_{L^p(\Omega, d_M^{p(\beta+1-\gamma)})} + \|\mathbf{u}\|_{L^p(B)} \right\}.$$

Proof. To simplify the notation we will assume that $m = 0$ in the definition of Ω. The other cases can be treated analogously.

Let $n' \in \mathbb{N}_0$ and $0 < s \leq \gamma$ be such that $sn' = p\beta$. As in [BK] we introduce

$$\Omega^{n',s} = \{(x, y, z') \in \mathbb{R}^{n+n'} \, : \, (x, y) \in \Omega, \; z' \in \mathbb{R}^{n'} \text{ with } |z'| < x^s\}. \tag{3.22}$$

Suppose that the hypothesis in Lemma 3.8 on solutions of the divergence is verified for $U = \Omega^{n',s}$ and $\omega = x^{p(1-\gamma)}$. Then, if $B' \subset \Omega^{n',s}$ is a ball with the same radius and center than B, from Lemma 3.9 we have

$$\|D\mathbf{v}\|_{L^p(\Omega^{n',s})} \leq C \left\{ \|\varepsilon(\mathbf{v})\|_{L^p(\Omega^{n',s}, x^{p(1-\gamma)})} + \|\mathbf{v}\|_{L^p(B')} \right\}, \tag{3.23}$$

for all $\mathbf{v} \in W^{1,p}(\Omega^{n',s})^{n+n'}$.

Now, given $\mathbf{u}$ in $W^{1,p}(\Omega, d_M^{p\beta})^n$ we define

$$\mathbf{v}(x, y, z') = (\mathbf{u}(x, y), \underbrace{0, \ldots, 0}_{n'}).$$

Then, using that for $(x, y) \in \Omega$, $d_M(x, y) \simeq x$, it is easy to check that (3.23) is equivalent to

$$\|D\mathbf{u}\|_{L^p(\Omega, d_M^{p\beta})} \leq C \left\{ \|\varepsilon(\mathbf{u})\|_{L^p(\Omega, d_M^{p(\beta+1-\gamma)})} + \|\mathbf{u}\|_{L^p(B)} \right\}.$$

Hence, to finish the proof we have to verify the hypothesis of Theorem 3.9 for the domain $\Omega^{n',s}$ with the weight $\omega = x^{p(1-\gamma)}$. Since in this case $\omega^{1-q} = x^{q(\gamma-1)}$, we have to show that, for any $g \in L^q_0(\Omega^{n',s})$, there exists $\mathbf{w} \in W^{1,q}_0(\Omega^{n',s}, x^{q(\gamma-1)})^n$ such that $\operatorname{div} \mathbf{w} = g$ and

$$\|\mathbf{w}\|_{W^{1,q}(\Omega^{n',s}, x^{q(\gamma-1)})} \leq C \|g\|_{L^q(\Omega^{n',s})}.$$

But this can be proved exactly as Theorem 3.3, using now the convex domain

$$\hat{\Omega}^{n',s} := \{(\hat{x}, \hat{y}, \hat{z}') \in \mathbb{R}^{n+n'} \, : \, (\hat{x}, \hat{y}) \in \hat{\Omega}, \; z' \in \mathbb{R}^{n'} \text{ with } |z'| < x^{\alpha s}\},$$

with $\hat{\Omega}$ defined as in (3.6), and the one-to-one map $F : \hat{\Omega}^{n',s} \to \Omega^{n',s}$ defined by

$$F(\hat{x}, \hat{y}, \hat{z}') := (\hat{x}^\alpha, \hat{y}, \hat{z}').$$

□

Chapter 4

Counterexamples and optimal weights

It is well known that the solvability of the divergence problem $(\text{div})_p$ and some related results may fail if the domain has an external cusp, however it is rare to find examples of that in the literature. In particular, for the 2-dimensional case we can only refer to [F] and [GG] and, for the 3-dimensional case, [W] and [ADL].

The goal of this chapter is to present new counterexamples for a class of domains with external cusps, and moreover to show that the weights considered in Theorems 1.10, 3.3 and 3.11 are optimal in the sense that the powers of the distances can not be improved. We will begin this chapter recalling some old counterexamples.

4.1 Some studied counterexamples

The first example where the solvability of $(\text{div})_p$ fails can be deduced from a related result given by Kurt Friedrichs in 1937. In fact, Friedrichs proved in [F] that if $\mathbf{w}(z) = f(x,y) + ig(x,y)$ is analytic, with f and g real functions such that f has vanishing mean value and $z = x + iy$, then there exists a positive constant Γ such that

$$\|f\|_{L^2(\Omega)} \le \Gamma \|g\|_{L^2(\Omega)} \tag{4.1}$$

under some assumption on the shape of the planar domain Ω. Specifically, he assumed that the boundary $\partial\Omega$ can be defined by some particular curve with continuous tangent except at a finite number of corners and he proved that the inequality (4.1) holds if and only if none of these corners is an external cusp.

Indeed, he defined, using polar coordinates (r, θ),

$$\Omega = \{(r,\theta) \,:\, 0 < r < R,\, \theta_1(r) < \theta < \theta_2(r)\}, \tag{4.2}$$

with

$$\theta_1(r) = -kr + O(r^2) \qquad \text{and} \qquad \theta_2(r) = kr + O(r^2),$$

where k is a positive constant. In figure 4.1, we show the domain Ω for particular θ_1 and θ_2.

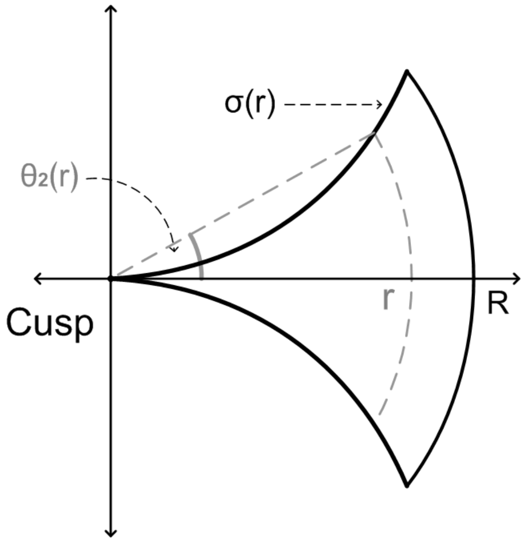

Figure 4.1: Cuspidal domains considered by Friedrichs.

Furthermore, for $\alpha > 0$ he introduced the complex function $\mathbf{w}_\alpha(z) = (2\alpha)^{1/2} z^{\alpha-3/2}$ which satisfies

$$\frac{\int_\Omega \mathrm{Re}\,\mathbf{w}_\alpha^2}{\int_\Omega |\mathbf{w}_\alpha|^2} \xrightarrow[\alpha\to 0^+]{} \frac{2k}{2k} = 1$$

and $\int_\Omega \mathbf{w}_\alpha \to 0$. This can be checked by an elementary computation.

Now, if $\mathbf{w} = f + ig$ is an arbitrary complex function not identically zero we can observe the following elementary equivalence

$$\begin{aligned}
2\int_\Omega f^2 \quad &\leq \quad 2\Gamma \int_\Omega g^2 \\
\underbrace{\Gamma\int_\Omega f^2}_{(\mathbf{1})} + \int_\Omega f^2 - \Gamma\int_\Omega g^2 - \underbrace{\int_\Omega g^2}_{(\mathbf{2})} \quad &\leq \quad \underbrace{\Gamma\int_\Omega f^2}_{(\mathbf{1})} - \int_\Omega f^2 + \Gamma\int_\Omega g^2 - \underbrace{\int_\Omega g^2}_{(\mathbf{2})} \\
(\Gamma+1)\int_\Omega \left(f^2 - g^2\right) \quad &\leq \quad (\Gamma-1)\int_\Omega \left(f^2 + g^2\right) \\
\frac{\int_\Omega \mathrm{Re}\,\mathbf{w}^2}{\int_\Omega |\mathbf{w}|^2} \quad &\leq \quad \frac{\Gamma-1}{\Gamma+1} < 1.
\end{aligned}$$

Thus, using the functions $\mathbf{w}_\alpha$ introduced by Friedrichs it is possible to assert that the inequality (4.1) does not hold if Ω is the domain defined in (4.2).

Now, let us relate Friedrichs's result with the solvability of $(\mathrm{div})_2$. Thus, let $\mathbf{w}(z) = f(x,y) + ig(x,y)$ be an analytic function such that the real part f has a vanishing mean value and suppose that there exists a solution for the divergence problem $\mathrm{div}\,\mathbf{u} = f$. Then, from the Cauchy-Riemann equation it follows that

$$\begin{aligned} \int_\Omega f^2 &= \int_\Omega f \mathrm{div}\,\mathbf{u} = -\int_\Omega \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).\mathbf{u} \\ &= -\int_\Omega \left(\frac{\partial g}{\partial y}, -\frac{\partial g}{\partial x}\right).\mathbf{u} = \int_\Omega g\left(\frac{\partial \mathrm{u}_1}{\partial y} - \frac{\partial \mathrm{u}_2}{\partial x}\right) \le C\left(\int_\Omega g^2\right)^{1/2}\left(\int_\Omega f^2\right)^{1/2}. \end{aligned}$$

Hence, the solvability of $(\mathrm{div})_2$ implies Friedrichs. In fact, both results are equivalent and the other implication can be found in [HP]. Then, we can conclude that there are no solutions for the divergence problem if Ω is the planar domain defined in (4.2). Furthermore, subtracting an appropriate constant the real part of the functions $\mathbf{w}_\alpha$ are a counterexample for the divergence problem in this domain.

Observe that the domains considered by Friedrichs present a quadratic cusp as the Hölder-α domains defined in (3.1) for $\alpha = 1/2$. In this case, by a quadratic cusp we mean that the two curves which define the cusp, denoted by $\sigma = (\sigma_1, \sigma_2)$, satisfy

$$\lim_{r\to 0^+} \frac{\sigma_2}{\sigma_1^2} = \lim_{r\to 0^+} \frac{r\sin(\theta_i(r))}{r^2\cos(\theta_i(r))} = l > 0,$$

for $i = 1, 2$. In the follows sections we will consider cuspidal domains with other orders.

More recently, other two counterexamples have been given in papers [W] and [GG]. In [W], N. Weck presents a counterexample for the Korn inequality in a 3-dimensional cuspidal domain. In [GG], G. Geymonat and G. Gilardi define a planar counterexample for the Korn inequality and the Lions lemma.

Another important counterexample for the Korn inequality has been defined by G. Acosta. For its simplicity, this result can be adapted to the divergence problem in domains with an arbitrary dimension and also for the weighted case (see [ADLg] for details). In fact, in section 4.3 we will adapt Acosta's counterexample for the weighted case.

Let us recall this counterexample here. For $\gamma > 1$, he considered

$$\Omega := \left\{(x,y) \in \mathbb{R}^2 \,:\, 0 < x < 1,\ |y| < x^\gamma\right\}$$

and the field $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2) = ((s-1)yx^{-s}, x^{1-s})$, with $s \in \mathbb{R}$ to be chosen below.

Now, it follows that

$$D\mathbf{v} = \begin{pmatrix} -s(s-1)yx^{-s-1} & (s-1)x^{-s} \\ (1-s)x^{-s} & 0 \end{pmatrix}$$

and

$$\varepsilon(\mathbf{v}) = \begin{pmatrix} -s(s-1)yx^{-s-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

By a straightforward computation we obtain that

$$\|\varepsilon(\mathbf{v})\|^p_{L^p(\Omega)} \leq C \int_0^1 x^{\gamma(p+1)-p(1+s)} dx \qquad \text{and} \qquad \|\mathbf{v}\|^p_{L^p(\Omega)} \leq C \int_0^1 x^{p-ps+\gamma} dx.$$

Thus, if $s < \min\left\{\frac{(\gamma+1)}{p} + (\gamma-1), \frac{(\gamma+1)}{p} + 1\right\}$ it follows that $\gamma(p+1) - p(1+s) > -1$ and $p - ps + \gamma > -1$, hence, we have

$$\|\varepsilon(\mathbf{v})\|_{L^p(\Omega)} < \infty \qquad \text{and} \qquad \|\mathbf{v}\|_{L^p(\Omega)} < \infty. \tag{4.3}$$

However,

$$\left\|\frac{\partial \mathrm{v}_1}{\partial y}\right\|^p_{L^p(\Omega)} = C \int_0^1 x^{-sp+\gamma} dx.$$

So, $\|\frac{\partial \mathrm{v}_1}{\partial y}\|_{L^p(\Omega)}$ is finite if and only if $s < \frac{\gamma+1}{p}$. Finally, for any $\gamma > 1$ it is always possible to take s such that,

$$\frac{\gamma+1}{p} \leq s < \min\left\{\frac{(\gamma+1)}{p} + (\gamma-1), \frac{(\gamma+1)}{p} + 1\right\}.$$

In consequence, from (4.3) the Korn inequality

$$\|\nabla \mathbf{w}\|_{L^p(\Omega)} \leq C \left\{\|\varepsilon(\mathbf{w})\|_{L^p(\Omega)} + \|\mathbf{w}\|_{L^p(\Omega)}\right\}$$

fails on Ω.

Although the goal of this section is to analyze different explicit counterexamples for the divergence problem and related results on bad domains, it is appropriate to mention the classification published in [ADM] where the authors showed that, for planar simply connected domains, there exists a right inverse for the divergence operator if and only if the domain is John.

We have been recalling different bad domains in which there is no solution for $(\text{div})_p$ or where some related results do not hold. However, a more general problem could be find necessary conditions on the weights ω in order to obtain weighted solutions for the divergence problem $(\text{div})_{p,w}$ in cuspidal domains. In particular, if the weight is a power of the distance to a closed set, it could be interesting to determine the optimal power. In [ADL], the authors generalized the counterexample exhibited by N. Weck in order to obtain necessary and sufficient conditions on the weight which warrants some weighted Korn inequality.

4.2 Domains with a general external cusp in $\mathbb{R}^n$

In this section, we will introduce some domains in $\mathbb{R}^n$ with an external cusp arbitrarily narrow and we will give necessary conditions on the weights in order to obtain a weighted right inverse for the divergence. These domains are a generalization of the ones previously considered in (3.1), which have a m Hausdorff dimensional cusp, where m is a natural number no bigger than $n-2$.

Let us first recall a result published by Dobrowolski in [Do] which is of interest for our propose. In that article the author studies the constant of the inf-sup condition on some particular domains. In fact, he proves that, if $\Omega \subset \mathbb{R}^n$ is a bounded domain which contains a cylinder $U_a = (0,a) \times U$, with $U \subset \mathbb{R}^{n-1}$ Lipschitz, such that $(0,a) \times \partial U \subset \partial\Omega$, then the constant L_Ω in the inf-sup condition must be smaller than $\frac{C(U)}{a}$, where $C(U)$ depends only on U. Indeed, the author shows that if $p_0 \in L^2_0(\Omega)$ is the function

$$p_0(x_1,\ldots,x_n) = \begin{cases} \sin(\frac{2\pi}{a}x_1) & \text{in } U_a \\ 0 & \text{in } \Omega \backslash U_a \end{cases}$$

then

$$L_\Omega \leq \sup_{\mathbf{v}\in H^1_0(\Omega)^n} \frac{\int_\Omega p_0 \operatorname{div} \mathbf{v}}{\|p_0\|_{L^2(\Omega)} \, \|D\mathbf{v}\|_{L^2(\Omega)}} \leq \frac{C(U)}{a}.$$

Now, we will show that these examples can be used to obtain conditions on the constant of the divergence problem. As a first approach we will consider the problem without weight and leave the more general case to be analyzed in subsection 4.2.1.

It has been mentioned that the solvability of the divergence problem implies inf-sup condition. In addition, if C_Ω is the constant involved on $(\text{div})_2$ then $L_\Omega \geq \frac{1}{C_\Omega}$. So, we can conclude that $C_\Omega \geq C(U)a$. Thus, if Ω contains a cylinder U_a with a arbitrarily large it is possible to assert that there does not exist a solution of $(\text{div})_2$ on Ω. However, this argument does not work for bounded domains. So, let us analyze these examples on bounded domains paying attention on the constant $C(U)$.

Let $\Omega \subset \mathbb{R}^n$ and $p_0 \in L^2_0(\Omega)$ be defined as before with the extra assumption that the diameter of U is smaller than b. In fact, in order to simplify the calculus we will suppose that U is included in a ball in $\mathbb{R}^{n-1}$ centered at the origin with radius b. So, suppose that there exists $\mathbf{u} \in H^1_0(\Omega)^n$ such that $\operatorname{div} \mathbf{u} = p_0$ and

$$\|\mathbf{u}\|_{H^1_0(\Omega)} \leq C_\Omega \|p_0\|_{L^2(\Omega)}. \tag{4.4}$$

In consequence, we obtain

$$\begin{aligned}
\|p_0\|^2_{L^2(U_a)} &= \int_{U_a} p_0 \operatorname{div} \mathbf{u} = -\int_{U_a} \tfrac{2\pi}{a}\cos(\tfrac{2\pi}{a}x_1)\mathrm{u}_1 + \int_{\partial U_a} \overbrace{\sin(\tfrac{2\pi}{a}x_1)\mathbf{u}}^{=0}.\nu \\
&= -\tfrac{2\pi}{a}\int_{U_a} \frac{\partial[x_2\cos(\frac{2\pi}{a}x_1)]}{\partial x_2}\mathrm{u}_1 \\
&= \tfrac{2\pi}{a}\int_{U_a} x_2\,\cos(\tfrac{2\pi}{a}x_1)\frac{\partial \mathrm{u}_1}{\partial x_2} - \tfrac{2\pi}{a}\int_{\partial U_a} x_2\,\cos(\tfrac{2\pi}{a}x_1)\,\overbrace{\mathrm{u}_1\nu_2}^{=0} \\
&\le \tfrac{2\pi}{a}\|x_2\,\cos(\tfrac{2\pi}{a}x_1)\|_{L^2(U_a)}\|\nabla\mathbf{u}\|_{L^2(U_a)} \le C_\Omega 2\pi\frac{b}{a}\|\cos(\tfrac{2\pi}{a}x_1)\|_{L^2(U_a)}\|p_0\|_{L^2(U_a)}.
\end{aligned}$$

Then,

$$\|\sin(\tfrac{2\pi}{a}x_1)\|_{L^2(U_a)} \le C_\Omega 2\pi\frac{b}{a}\|\cos(\tfrac{2\pi}{a}x_1)\|_{L^2(U_a)}.$$

Now, using that $\cos^2(t) - \sin^2(t) = \cos(2t)$ it is easy to observe that

$$\|\sin(\tfrac{2\pi}{a}x_1)\|^2_{L^2(U_a)} = \|\cos(\tfrac{2\pi}{a}x_1)\|^2_{L^2(U_a)},$$

concluding that $C_\Omega \ge \dfrac{a}{2\pi b}$.

On this way, if Ω contains a cylinder U_a with $\frac{a}{b}$ arbitrarily large the solvability of $(\mathrm{div})_2$ fails.

Let us define an explicit domain of the type of rooms and corridor containing this kind of cylinders. Given a convergent series $\sum_{k=1}^\infty b_k$, with positive terms b_k, we define Ω as

$$\Omega = (-1,0)\times(0,2S_1)\cup\bigcup_{n=1}^\infty R_n,$$

where $R_n = [0,1)\times(2S_n - b_n, 2S_n)$ and $S_n = \sum_{k=n}^\infty b_k$.

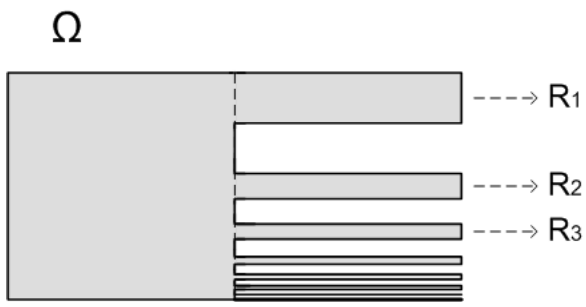

Figure 4.2: A bad domain without cusps

The rectangle R_n is a particular case of U_a with $a = 1$ and $b = b_n$. Then, we can assert that $C_\Omega \geq \dfrac{a}{2\pi b_n}$. But, since $\sum b_k$ is convergent it follows that

$$C_\Omega \geq \lim_{n\to\infty} \frac{a}{2\pi b_n} = \infty,$$

concluding the insolvability of $(\text{div})_2$.

Note that this result can be obtained using the characterization published in [ADM] where the authors showed that for simply connected planar domains the existence of a continuous right inverse for the divergence holds if and only if the domain is John. However, this technique can be generalized to different domains in which that characterization does not exist.

4.2.1 Counterexamples for general cusps

Let $\varphi : [0,1] \to \mathbb{R}$ be a C^1 function such that $\varphi(0) = 0$, its derivatives φ_x is strictly increasing and $\varphi_x(0) = 0$. Two examples are:

i) $\varphi(x) = x^\gamma$, with $\gamma > 1$.

ii) $\varphi(x) = e^{-1/x^2}$ in $(0,1]$ and $\varphi(0) = 0$.

Associated with φ we introduce the cuspidal domain $\Omega_\varphi \subset \mathbb{R}^n$ given by:

$$\Omega_\varphi = \left\{(x,y,z) \in I \times \mathbb{R}^k \times I^{n-k-1} \,:\, |y| < \varphi(x)\right\} \subset \mathbb{R}^n, \tag{4.5}$$

where I is the interval $(0,1)$ and $k \geq 1$.

Observe that, in the first example, the function φ defines the γ-John domain studied in chapter 3. And, in the second one, φ defines a new domain which is not a s-John domain for any s.

Before we build the counterexamples we have to prove some technical lemmas. Hence, let $(x_m)_{m\geq m_0} \subset (0,1]$ be a sequence such that

$$\varphi'(x_m) = 2^{-m}. \tag{4.6}$$

From now on we will denote $x_m - x_{m+1}$ as r_m and, to simplify the notation, we will assume that $m_0 = 1$ and $x_1 = 1$.

In the next lemma we will show two properties of $\varphi(x)$ which will be used to analyze the narrowness of Ω_φ when $x_{m+1} < x < x_m$.

Lemma 4.1. *If $(x_m)_{m\geq 1}$ is the sequence defined in (4.6), φ satisfies:*

$$\frac{1}{4}2^{-m} \leq \frac{\varphi(x)}{r_m} \qquad \text{if } (x_{m+1}+x_m)/2 \leq x \leq x_m \tag{4.7}$$

and

$$\frac{\varphi(x)}{r_{m_j}} \leq 2\,2^{-m_j} \qquad \text{if } x_{m_j+1} \leq x \leq x_{m_j}, \tag{4.8}$$

where $(x_{m_j})_j$ is a subsequence of $(x_m)_m$.

Proof. Given $x \in (x_{m+1}, x_m]$ and using elementary tools, we can observe that:

$$\begin{aligned} \varphi(x) &= \varphi(x) - \varphi(x_{m+1}) + \varphi(x_{m+1}) \geq \varphi(x) - \varphi(x_{m+1}) \\ &= \varphi'(\xi_x)(x - x_{m+1}) \geq 2^{-(m+1)}(x - x_{m+1}), \end{aligned}$$

where ξ_x belongs to (x_{m+1}, x). So, if $x \in [\frac{x_{m+1}+x_m}{2}, x_m]$ it follows that $(x - x_{m+1}) \geq \frac{r_m}{2}$ and we can conclude (4.7).

On the other hand, using that $\varphi(0) = 0$ and an inductive argument, we can assert that

$$\begin{aligned} \varphi(x_m) &\leq \varphi(x_{m+1}) + 2^{-m}(x_m - x_{m+1}) \\ &\leq \varphi(x_{m+2}) + 2^{-(m+1)}(x_{m+1} - x_{m+2}) + 2^{-m}(x_m - x_{m+1}) \\ &\vdots \\ &\leq \sum_{i=m}^{\infty} 2^{-i}\,(x_i - x_{i+1}) = \sum_{i=m}^{\infty} 2^{-i} r_i. \end{aligned}$$

Now, we choose a subsequence $(r_{m_j})_j$ of $(r_m)_m$ such that $\frac{r_i}{r_{m_j}} \leq 1$ for all $i \geq m_j$. For example, we can choose r_{m_1} the maximum of r_i over all i and r_{m_j} the maximum of r_i over all $i > m_{j-1}$. Then, it follows that

$$\frac{\varphi(x_{m_j})}{r_{m_j}} \leq \sum_{i=m_j}^{\infty} 2^{-i} = 2^{-m_j+1}.$$

So, using that φ is increasing, we obtain the second property (4.8). □

In the next theorem we will prove some necessary condition for weighted Korn type inequalities in Ω_φ. Although generalizations for non power type φ of the results given in previous chapters have not been proved, we believe that powers of φ_x are the natural weights to be considered. We use in the construction of the following theorem some ideas from [Do].

Theorem 4.2. *Let $\Omega_\varphi \subset \mathbb{R}^n$ be the domain defined in (4.5), $\beta_1,\ \beta_2 \in \mathbb{R}$, $1 < p < \infty$ and B a ball compactly contained in Ω_φ. If there exists a positive constant C such that*

$$\|D\mathbf{v}\|_{L^p(\Omega_\varphi,\varphi_x^{p\beta_1})} \le C\left\{\|\varepsilon(\mathbf{v})\|_{L^p(\Omega_\varphi,\varphi_x^{p\beta_2})} + \|\mathbf{v}\|_{L^p(B)}\right\}, \tag{4.9}$$

for all $\mathbf{v} \in W^{1,p}(\Omega_\varphi, \varphi_x^{p\beta_1})^n$, then $\beta_1 \ge \beta_2 + 1$.

Proof. Let $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, \overbrace{0, \cdots, 0}^{n-k-1})$ in $W^{1,p}(\Omega_\varphi, \varphi_x^{p\beta_1})^n$ defined as:

$$\mathrm{v}_1(x,y,z) = \chi(x)\,\sin\left(\frac{2\pi}{r_m}(x - x_{m+1})\right)\frac{2\pi}{r_m}(y_1 + \cdots + y_k)$$

and

$$\mathrm{v}_i(x,y,z) = \chi(x)\,\left(\cos\left(\frac{2\pi}{r_m}(x - x_{m+1})\right) - 1\right)$$

for $2 \le i \le k+1$, where $\chi(x)$ is the characteristic function of the interval $[x_{m+1}, x_m]$.

Now, it is easy to check that $\varepsilon(\mathbf{v})_{i,j}$ vanishes if (i,j) is different from $(1,1)$. So, as B is compactly contained in Ω_φ we can assert from (4.9) that

$$\|(D\mathbf{v})_{2,1}\|^p_{L^p(\Omega_\varphi,\varphi_x^{p\beta_1})} \le C\|\varepsilon(\mathbf{v})_{1,1}\|^p_{L^p(\Omega_\varphi,\varphi_x^{p\beta_2})}, \tag{4.10}$$

for m sufficiently large.

Now, using that the weight in the left hand side of (4.10) is equivalent to $2^{-m\,p\beta_1}$, if x belongs to $[x_{m+1}, x_m]$, and property (4.7) we obtain

$$\begin{aligned}
\|(D\mathbf{v})_{2,1}\|^p_{L^p(\Omega_\varphi,\varphi_x^{p\beta_1})} &\simeq 2^{-mp\beta_1}\int_{\Omega_\varphi}\left|\sin\left(\frac{2\pi}{r_m}(x - x_{m+1})\right)\frac{2\pi}{r_m}\right|^p\chi(x) \\
&\simeq \frac{2^{-mp\beta_1}}{r_m^{p-k-1}}\int_{x_{m+1}}^{x_m}\left|\sin\left(\frac{2\pi}{r_m}(x - x_{m+1})\right)\right|^p\left(\frac{\varphi(x)}{r_m}\right)^k\frac{2\pi}{r_m}\mathrm{d}x \\
&\ge \frac{2^{-mp\beta_1}}{r_m^{p-k-1}}\int_{(x_{m+1}+x_m)/2}^{x_m}\left|\sin\left(\frac{2\pi}{r_m}(x - x_{m+1})\right)\right|^p\left(\frac{\varphi(x)}{r_m}\right)^k\frac{2\pi}{r_m}\mathrm{d}x \\
&\ge \frac{2^{-m(p\beta_1+k)}}{4^k\,r_m^{p-k-1}}\int_\pi^{2\pi}|\sin(t)|^p\,\mathrm{d}t \qquad (4.11) \\
&\simeq \frac{2^{-m(p\beta_1+k)}}{r_m^{p-k-1}}.
\end{aligned}$$

Analogously, if $m = m_j$ for some j we can conclude from property (4.8) that

$$\begin{aligned}
\|\varepsilon(\mathbf{v})_{1,1}\|^p_{L^p(\Omega_\varphi,\varphi_x^{p\beta_2})} &\simeq 2^{-mp\beta_2}\int_{\Omega_\varphi}\left|\cos\left(\frac{2\pi}{r_m}(x-x_{m+1})\right)\left(\frac{2\pi}{r_m}\right)^2(y_1+\cdots+y_k)\right|^p\chi(x)\\
&\simeq 2^{-mp\beta_2}\int_{x_{m+1}}^{x_m}\int_0^{\varphi(x)}\left|\cos\left(\frac{2\pi}{r_m}(x-x_{m+1})\right)\right|^p\left(\frac{2\pi}{r_m}\right)^{2p}\rho^{p+k-1}\,\mathrm{d}\rho\,\mathrm{d}x\\
&\simeq \frac{2^{-mp\beta_2}}{r_m^{p-k-1}}\int_{x_{m+1}}^{x_m}\left|\cos\left(\frac{2\pi}{r_m}(x-x_{m+1})\right)\right|^p\left(\frac{\varphi(x)}{r_m}\right)^{p+k}\frac{2\pi}{r_m}\mathrm{d}x\\
&\leq \frac{2^{-m(p\beta_2+p+k)}}{r_m^{p-k-1}}2^{p+k}\int_0^{2\pi}|\cos(t)|^p\,\mathrm{d}t \qquad (4.12)\\
&\simeq \frac{2^{-m(p\beta_2+p+k)}}{r_m^{p-k-1}}.
\end{aligned}$$

Finally, from (4.10), (4.11) and (4.12) it follows that there exists a positive constant C which does not depend on m such that

$$\frac{2^{-m(p\beta_1+k)}}{r_m^{p-k-1}} \leq C\frac{2^{-m(p\beta_2+p+k)}}{r_m^{p-k-1}},$$

for all $m = m_j$. Thus, dividing the inequality for an appropriate factor we obtain that $2^{-m_j p(\beta_1-\beta_2-1)} \leq C$ for all $j \geq 1$. So, we can assert that $\beta_1 \geq \beta_2 + 1$. □

Finally, we show an optimality result on solutions of the divergence in Ω_φ. As in the previous theorem we will consider weights which are powers of φ_x.

Theorem 4.3. *Let $\Omega_\varphi \subset \mathbb{R}^n$ be the domain defined in (4.5), $\beta_1,\ \beta_2 \in \mathbb{R}$ and $1 < p < \infty$. If for any $f \in L^p_0(\Omega_\varphi, \varphi_x^{p\beta_2})$ there exists $\mathbf{v} \in W^{1,p}_0(\Omega_\varphi, \varphi_x^{p\beta_1})^n$ such that $div\,\mathbf{v} = f$ and*

$$\|\mathbf{v}\|_{W_0^{1,p}(\Omega_\varphi,\varphi_x^{p\beta_1})^n} \leq C_\varphi\|f\|_{L^p(\Omega_\varphi,\varphi_x^{p\beta_2})}, \qquad (4.13)$$

where φ_x denotes the derivative of φ and C depends only on Ω_φ, β_1, β_2 and p, then $\beta_1 \geq \beta_2 + 1$.

Proof. To prove the result, we will build a function with zero integral for which there is no solution for the divergence problem satisfying (4.13) if $\beta_1 \ngeq \beta_2 + 1$. But, there are two technical complications to build such example, one of them is the weight and the other one is to impose the vanishing mean value condition. For this reason, we will see that the divergence problem in the cuspidal domain Ω_φ can be transformed in a sequence of problems for symmetric domains with constant weight. In relation to the mean value condition, we will work with odd functions which integrate zero in symmetric domains by an appropriate translation.

The proof will be divided into three parts. In the first one, we will define a subset $U_m \subset \Omega_\varphi$ in which the weight can be considered constant. In the second part, we will define a symmetric domain V_m in order to easily impose the vanishing mean value. And, in the last one, we will show the theorem.

Part 1. Given $m \in \mathbb{N}$, we introduce the domain $U_m \subset \Omega_\varphi$ by

$$U_m = \{(x, y, z) \in \Omega_\varphi \,:\, x_{m+1} < x < x_m\}.$$

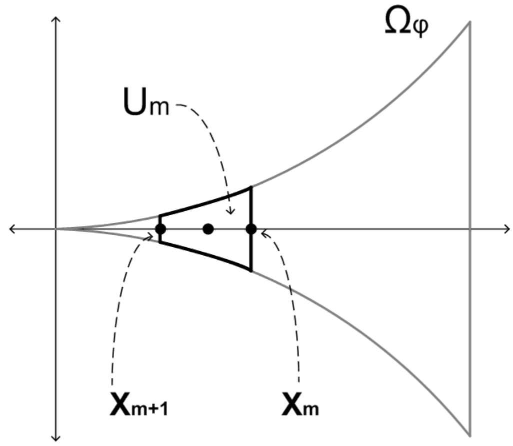

Figure 4.3: A subset where the weight is constant.

Now, given $f \in L^p_0(U_m)$ we can see, extending by zero, that there are solutions for the divergence in $W^{1,p}_0(\Omega_\varphi, \varphi_x^{p\beta_1})$. Moreover, as $2^{-(m+1)} \leq \varphi'(x) \leq 2^{-m}$ if $x_{m+1} \leq x \leq x_m$, it is easy to observe that there exists $\mathbf{u} \in W^{1,p}(U_m)$ satisfying

$$\begin{cases} \operatorname{div} \mathbf{u} = f & \text{in } U_m \\ \|\mathbf{u}\|_{W^{1,p}(U_m)} \leq C_m \|f\|_{L^p(U_m)} & \text{in } U_m \\ \mathbf{u} = 0 & \text{in } \partial U_m \text{ if } x \neq x_m,\, x_{m+1}, \end{cases} \tag{4.14}$$

where $C_m = C_\varphi\, 2^{|\beta_1|+|\beta_2|}\, 2^{m(\beta_1-\beta_2)}$. To simplify the notation we will write $C_m \simeq 2^{m(\beta_1-\beta_2)}$.

Part 2. In this part, we will define a symmetric domain V_m satisfying a condition similar to (4.14) in order to simplify the computations in the next part.

Let U'_m be the domain defined by

$$U'_m = \{(2a_m - x, y, z) \,:\, (x, y, z) \in U_m\},$$

where $a_m = (x_{m+1} + x_m)/2$. It is immediate to realize that U'_m is obtained from U_m by an isometric application. In this way, using Lemma 4.4 we can assert that U'_m satisfies the condition (4.14) with the same constant C_m.

Now, let V_m be the Lipschitz domain defined by

$$V_m = U_m \cup U'_m,$$

which is symmetric with respect to $x = a_m$.

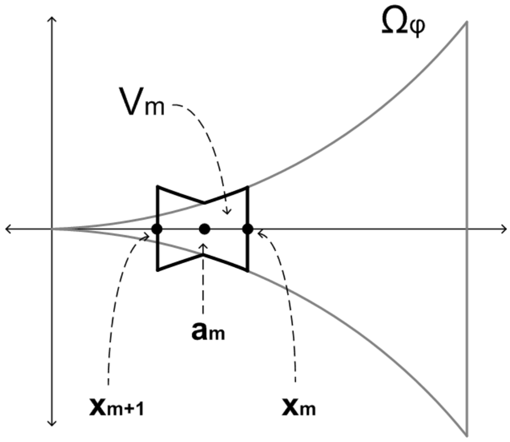

Figure 4.4: A symmetric domain.

We will use some ideas introduced in [B] in order to show that V_m admits a solution for the divergence satisfying the conditions (4.14) too. Indeed, given $g \in L^p_0(V_m)$ suppose that it could be decomposed as a sum of functions g_1, $g_2 \in L^p_0(V_m)$ with g_1 and g_2 supported in U_m and U'_m, respectively, and satisfying that

$$\|g_i\|_{L^p(V_m)} \leq C\|g_i\|_{L^p(V_m)}, \tag{4.15}$$

where C is not depending on m. Hence, the field $\mathbf{u} = \mathbf{v} + \mathbf{v}'$ is the solution required, where $\mathbf{v} \in W^{1,p}(U_m)^n$ is a solution for the divergence problem satisfying (4.14) for g_1 and $\mathbf{v}' \in W^{1,p}(U'_m)^n$ for g_2.

Therefore, it remains to prove that g can be decomposed as a sum of function $g_1 \in L^p_0(U_m)$ and $g_2 \in L^p_0(U'_m)$ verifying (4.15). Thus, we define $g_2 = g - g_1$ and

$$g_1(x,y,z) = \begin{cases} g(x,y,z) - \frac{\chi(x,y,z)}{|U_m \cap U'_m|}\int_{U_m} g & \text{in } U_m \\ \\ 0 & \text{in } V_m \setminus U_m, \end{cases}$$

where χ is the characteristic function of $U_m \cap U'_m$. Then, it follows that g_1 and g_2 have vanishing mean value and they satisfy

$$\begin{aligned} \|g_1\|_{L^p(U_m)} &\leq \|g\|_{L^p(V_m)} + \frac{1}{|U_m \cap U'_m|^{1/q}}\int_{U_m} |g| \\ &\leq \|g\|_{L^p(V_m)}\left(1 + \frac{|U_m|^{1/q}}{|U_m \cap U'_m|^{1/q}}\right) \end{aligned}$$

and

$$\|g_2\|_{L^p(U'_m)} \;\leq\; \|g\|_{L^p(V_m)} + \|g_1\|_{L^p(V_m)} \leq \|g\|_{L^p(V_m)} \left(2 + \frac{|U_m|^{1/q}}{|U_m \cap U'_m|^{1/q}}\right).$$

Now, using Lemma 4.1 it is easy to check that if $m = m_j$, for some j, it follows that

$$1 \leq \frac{|U_m|}{|U_m \cap U'_m|} \leq C,$$

where the constant C does not depend on m, concluding part 2.

Part 3. Now, we are ready to prove the theorem. Indeed, let $f \in L^q(V_m)$ be defined by $f(x,y,z) = \sin(\frac{2\pi}{r_m}(x - a_m))$, where $g \in L^p_0(V_m)$. Thus, integrating by parts and using a solution $\mathbf{u}$ for $\operatorname{div}\mathbf{u} = g$ satisfying (4.14) we can see that

$$\begin{aligned}
\int_{V_m} f\,g \;&=\; \int_{V_m} f \operatorname{div}\mathbf{u} = -\int_{V_m} \frac{2\pi}{r_m}\cos(\frac{2\pi}{r_m}(x-a_m))\mathrm{u}_1 + \int_{\partial V_m} \overbrace{\sin(\frac{2\pi}{r_m}(x-a_m))\mathbf{u}\,.\nu}^{=0} \\
&=\; -\frac{2\pi}{r_m}\int_{V_m} \frac{\partial y_1}{\partial y_1}\cos(\frac{2\pi}{r_m}(x-a_m))\mathrm{u}_1 \\
&=\; \frac{2\pi}{r_m}\int_{V_m} y_1\,\cos(\frac{2\pi}{r_m}(x-a_m))\frac{\partial \mathrm{u}_1}{\partial y_1} - \frac{2\pi}{r_m}\int_{\partial V_m} y_1\,\cos(\frac{2\pi}{r_m}(x-a_m))\overbrace{\mathrm{u}_1\nu_2}^{=0} \\
&\leq\; \frac{2\pi}{r_m}\|y_1\,\cos(\frac{2\pi}{r_m})(x-a_m))\|_{L^q(V_m)}\|D\mathbf{u}\|_{L^p(V_m)} \\
&\leq\; C\frac{2^{m(\beta_1-\beta_2)}}{r_m}\|y_1\,\cos(\frac{2\pi}{r_m}(x-a_m))\|_{L^q(V_m)}\|g\|_{L^p(V_m)}.
\end{aligned}$$

So, as f is odd with respect to $x = a_m$ and V_m is symmetric with respect to the same parameter we can assert that $\int_{V_m} f = 0$. Thus,

$$\|\sin(\tfrac{2\pi}{r_m}(x-a_m))\|_{L^q(V_m)} = \sup_{0\neq g\in L^p_0(V_m)} \frac{\int_{V_m} f\,g}{\|g\|_{L^p(V_m)}} \leq C\frac{2^{m(\beta_1-\beta_2)}}{r_m}\|y_1\,\cos(\tfrac{2\pi}{r_m}(x-a_m))\|_{L^q(V_m)}.$$

Finally, computing the norms we obtain that for all $m = m_j$ it follows

$$r_m^{(k+1)/q}2^{-mk/q} \leq C\frac{2^{m(\beta_1-\beta_2)}}{r_m}r_m^{(q+k+1)/q}2^{-m(k+q)/q}.$$

So, dividing for an appropriate factor we obtain

$$2^{-m(\beta_1-\beta_2-1)} \leq C.$$

Thus $\beta_1 \geq \beta_2 + 1$.

□

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $T : W_0^{1,p}(\Omega)^n \to L_0^p(\Omega)$ a continuous right inverse for the divergence operator. Thus, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one affine function, the domain $\tilde{\Omega} := F(\Omega)$ admits a divergence inverse $\tilde{T} : W_0^{1,p}(\tilde{\Omega})^n \to L_0^p(\tilde{\Omega})$. Furthermore, if F is an isometry then T and $\tilde{T}$ have the same norm.*

Proof. Let us denote F as $F(x) = Ax + b$. Thus, given $f \in L_0^p(\tilde{\Omega})$ we define, using the Piola transform,

$$\tilde{T}(f)(\tilde{x}) = A\,T(g)(F^{-1}(\tilde{x})),$$

where $g = f \circ F$. To simplify the notation let us write $\mathbf{v} = T(g)$ and $\mathbf{u} = \tilde{T}(f)$. Thus, by the chain rule it follows that the differential matrix

$$D\mathbf{u}(\tilde{x}) = A\,D\mathbf{v}(F^{-1}(\tilde{x}))\,A^{-1}.$$

Hence, using that the trace is invariant under conjugation we can assert that

$$\operatorname{div}\mathbf{u}(\tilde{x}) = \operatorname{div}\mathbf{v}(F^{-1}(\tilde{x})) = g(F^{-1}(\tilde{x})) = f(\tilde{x}).$$

On the other hand, it may be concluded that

$$\|\mathbf{u}\|_{L^p(\tilde{\Omega})} \leq C\|f\|_{L^p(\tilde{\Omega})},$$

where C is a constant depending on the norm of T and the matrixes A and A^{-1}. A possible C could be the norm of T multiplied by $\max\{\|A\|_\infty, \|A^{-1}\|_\infty\}$. But, in the particular case that F is an isometry it is easy to observe that T and $\tilde{T}$ have the same norm. □

Remark 4.5. *It was proved in Theorem 3.9 chapter 3 that the divergence problem in the form of Theorem 4.3 with $\beta_2 = 0$ implies the Korn inequality in the form of Theorem 4.2 with $\beta_1 = 0$. Thus, from Theorem 4.2 we can prove Theorem 4.3 when $\beta_2 = 0$.*

4.3 Domains with an external cusp and the distance to the boundary

In this section, we will give conditions on the weights in order to obtain an inverse for the divergence in weighted Sobolev spaces where the weights are a power of the distance to the boundary of the Hölder-α domain defined by

$$\Omega = \Big\{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} \,:\, 0 < x < 1\,,\, 0 < |y| < x^\gamma\Big\} \tag{4.16}$$

where $\gamma = \frac{1}{\alpha}$, with $0 < \alpha \leq 1$. Observe that this is a trivial generalization of the domain introduced in (1.25) to arbitrary n dimension.

In fact,we will consider the distance to a subset of $\partial\Omega$ instead of the whole boundary, specifically, the distance to $\Gamma = \partial\Omega \cap \{x \neq 1\}$. But, it can be seen that it is equivalent to consider any subset of $\partial\Omega$ containing a neighborhood of the cusp.

Theorem 4.6. *Let Ω be as in (4.16). If div : $W_0^{1,p}(\Omega, d_\Gamma^{p\eta})^n \to L_0^p(\Omega, d_\Gamma^{p\beta})$ admits a continuous right inverse for some $\eta, \beta < \frac{1}{q}$ then, $\eta - \beta \geq 1 - \alpha$.*

Proof. For simplicity of notation we will write $L^p(\Omega, p\beta)$ instead of $L^p(\Omega, d_\Gamma^{p\beta})$ and, analogously, we will extend the notation to $W^{1,p}(\Omega, d_\Gamma^{p\beta})$.

For $s < \frac{n-1-\beta q+\alpha}{\alpha q}$ define $f_s(x,y) = x^{-\frac{s}{p-1}} d_\Gamma(x,y)^{-q\beta}$. Then, we have

$$\|f_s\|^p_{L^p(\Omega,p\beta)} = \int_\Omega x^{-sq} d_\Gamma(x,y)^{-\beta pq+\beta p}\, dxdy = \int_\Omega x^{-sq} d_\Gamma(x,y)^{-\beta q}\, dxdy.$$

Therefore, using that $d_\Gamma(x,y) \simeq x^\gamma - |y|$ (see Lemma A.1 for details) and changing to polar coordinates on y, we obtain

$$\|f_s\|^p_{L^p(\Omega,p\beta)} \simeq \int_0^1 \int_0^{x^\gamma} x^{-sq}(x^\gamma - \rho)^{-\beta q} \rho^{n-2}\, d\rho dx.$$

However, integrating by parts $n-2$ times in ρ,

$$\int_0^1 \int_0^{x^\gamma} x^{-sq}(x^\gamma - \rho)^{-\beta q} \rho^{n-2}\, d\rho dx \simeq \int_0^1 x^{-sq} \int_0^{x^\gamma} (x^\gamma - \rho)^{n-2-\beta q}\, d\rho dx$$

$$\simeq \int_0^1 x^{-sq} x^{\gamma(n-1-\beta q)}\, dx = \frac{1}{q(\frac{n-1-\beta q+\alpha}{\alpha q} - s)},$$

where we have used $s < \frac{n-1-\beta q+\alpha}{\alpha q}$. Therefore,

$$\|f_s\|^p_{L^p(\Omega,p\beta)} \simeq \frac{1}{A-s} \tag{4.17}$$

where $A := \frac{n-1-\beta q+\alpha}{\alpha q}$ and the constants in the equivalence are independent of s.

Now, let B be a ball such that $\overline{B} \subset \Omega$ and $\omega \in C_0^\infty(B)$ such that $\int_B \omega = 1$. From our hypothesis we know that, if $c_s = \int_\Omega f_s$, there exists $\mathbf{v}_s \in W_0^{1,p}(\Omega, p\eta)^n$ such that

$$\operatorname{div} \mathbf{v}_s = f_s - c_s\omega \qquad \text{and} \qquad \|\mathbf{v}_s\|_{W_0^{1,p}(\Omega,p\eta)} \leq C\|f_s - c_s\omega\|_{L^p(\Omega,p\beta)}.$$

But, since $\beta < \frac{1}{q}$,

$$|c_s| = \|f_s\|_{L^1(\Omega)} \leq \|1\|_{L^q(\Omega,-q\beta)} \|f_s\|_{L^p(\Omega,p\beta)} \leq C\|f_s\|_{L^p(\Omega,p\beta)} \tag{4.18}$$

and so,

$$\|\mathbf{v}_s\|_{W_1^p(\Omega)} \leq C\|f_s\|_{L^p(\Omega,p\beta)} \tag{4.19}$$

where we have used that $\|\omega\|_{L^p(\Omega,p\beta)} \leq C$ because the support of ω is contained in B. Then,

$$\begin{aligned} \|f_s\|^p_{L^p(\Omega,p\beta)} &= \int_\Omega f_s^{p-1}\,(f_s - c_s\omega)\, d_\Gamma^{p\beta} + \int_\Omega f_s^{p-1}\, c_s\omega\, d_\Gamma^{p\beta} \\ &= \int_\Omega f_s^{p-1} \operatorname{div}\mathbf{v}_s\, d_\Gamma^{p\beta} + \int_\Omega f_s^{p-1}\, c_s\omega\, d_\Gamma^{p\beta} \\ &= \int_\Omega x^{-s}\operatorname{div}\mathbf{v}_s + \int_\Omega f_s^{p-1}\, c_s\omega\, d_\Gamma^{p\beta}. \end{aligned}$$

Then, from (4.18), it follows that

$$\begin{aligned} \int_\Omega f_s^{p-1}\, c_s\omega\, d_\Gamma^{p\beta} &\leq |c_s| \int x^{-s} d_\Gamma^{-q(p-1)\beta} d_\Gamma^{p\beta} w \\ &\leq \left(\int x^{-s} w\right) |c_s| \leq C \left(\int x^{-s} w\right) \|f_s\|_{L^p(\Omega,p\beta)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_\Omega x^{-s}\operatorname{div}\mathbf{v}_s &= s\int_\Omega x^{-s-1}\,\mathrm{v}_{s,1} = s\int_\Omega \frac{\partial(y_1\, x^{-s-1})}{\partial y_1}\mathrm{v}_{s,1} \\ &= -s\int_\Omega y_1\, x^{-s-1}\frac{\partial \mathrm{v}_{s,1}}{\partial y_1} \leq s\|y_1\, x^{-s-1}\|_{L^q(\Omega,-q\eta)}\, \|\mathbf{v}_s\|_{W_0^{1,p}(\Omega,p\eta)} \\ &\leq Cs\|y_1\, x^{-s-1}\|_{L^q(\Omega,-q\eta)}\, \|f_s\|_{L^p(\Omega,p\beta)} \end{aligned}$$

where for the last inequality we have used (4.19).

Therefore,

$$\|f_s\|^{p-1}_{L^p(\Omega,p\beta)} \leq C\left\{ s\|y_1\, x^{-s-1}\|_{L^q(\Omega)} + \left(\int x^{-s} w\right)\right\} \tag{4.20}$$

But, an elementary computation shows that

$$\|y_1\, x^{-s-1}\|^q_{L^q(\Omega,-q\eta)} \simeq \frac{1}{B-s} \tag{4.21}$$

where $B := \frac{n-1-\eta q+(1-\alpha)q+\alpha}{\alpha q}$ and with the constants in the equivalence independent of s.

Thus, using again that the support of ω is at a positive distance from the boundary, together with (4.17), (4.20) and (4.21) we conclude that there exists a constant independent of s such that

$$\frac{1}{A-s} \leq C\frac{1}{B-s}.$$

Therefore, $B \leq A$ and it follows immediately that $\eta - \beta \geq 1-\alpha$. $\square$

Appendix A

It is reasonable that the distance from (x, y) to $\Gamma = \partial\Omega \cap \{x \neq 1\}$ is equivalent to $x^\gamma - |y|$, where Ω is defined in (4.16), but we decided to prove this technical result in order to demonstrate Theorem 4.6 as complete as possible.

Lemma A.1. *If Ω is the domain defined in (4.16) and $\Gamma = \partial\Omega \cap \{x \neq 1\}$ then*

$$\mathrm{dist}_\Gamma(x, y) \simeq x^\gamma - |y|.$$

Proof. Given (x, y) in Ω it is easy to see that $x^\gamma - |y|$ is the distance from (x, y) to a point in Γ. In fact,

$$\begin{aligned} \mathrm{dist}_\Gamma(x, y) &\leq \mathrm{dist}\left((x, y); (x, \tfrac{x^\gamma}{|y|}y)\right) \\ &= \left|\tfrac{x^\gamma}{|y|}y - y\right| = \left|(x^\gamma - |y|)\tfrac{1}{|y|}y\right| = x^\gamma - |y|. \end{aligned}$$

So, it is enough to show the other inequality. We will suppose that $n = 2$ and leave the general case to be analyzed later. Let $(x', y') = (x, \mathrm{sg}(y)x^\gamma)$ and (x'', y'') the point in M where the distance to (x, y) is realized (the case $y = 0$, where the distance is realized in two points, is analyzed similarly).

Thus, let L be the secant line which joins (x', y') with (x'', y'') and let (x''', y''') be the point in L which realizes the distance to (x, y). Observe that we have to show that

$$\mathrm{dist}_\Gamma\left((x, y), (x', y')\right) \leq C\,\mathrm{dist}_\Gamma\left((x, y), (x'', y'')\right).$$

But, using that (x'', y'') belongs to L and (x''', y''') realizes the distance between (x, y) and L it is enough to prove that

$$\mathrm{dist}_\Gamma\left((x, y), (x', y')\right) \leq C\,\mathrm{dist}_\Gamma\left((x, y), (x''', y''')\right).$$

Now, if we define the triangle with vertices (x, y), (x', y') and (x''', y''') we can observe that it has an right angle in (x''', y'''). So, it is sufficient to prove that the side opposite

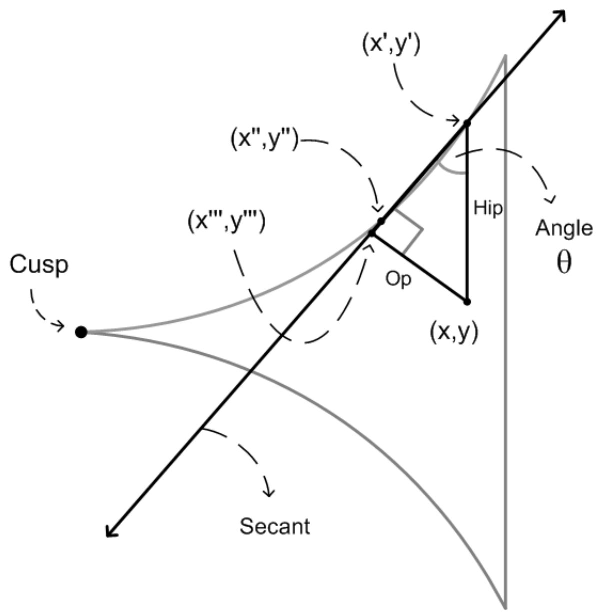

Figure 1.1: Distance to the boundary.

to θ **Op** and the hypotenuse **Hip** in the picture satisfy **Hip**$\leq C$**Op**. But, using that the slope of the secant line is bounded by a constant independent of (x, y) we can assert that there exists $\theta_0 > 0$ such that $\theta \geq \theta_0$. Then,

$$\begin{aligned}\frac{\mathrm{dist}_\Gamma(x,y)}{x^\gamma - |y|} &= \frac{\mathrm{dist}_\Gamma(x,y)\left((x,y),(x'',y'')\right)}{\mathrm{dist}_\Gamma(x,y)\left((x,y),(x',y')\right)} \geq \frac{\mathrm{dist}_\Gamma(x,y)\left((x,y),(x''',y''')\right)}{\mathrm{dist}_\Gamma(x,y)\left((x,y),(x',y')\right)} \\ &= \frac{\mathbf{Op}}{\mathbf{Hip}} = \sin(\theta) \geq \sin(\theta_0),\end{aligned}$$

concluding the case $n = 2$.

Now, suppose $n \geq 3$ is an arbitrary natural number and define (x'', y'') and (x''', y''') as before where (x', y') is equal to $(x, \frac{x^\gamma}{|y|}y)$ in this general case. Using that Ω is a revolution domain we will prove that (x, y), (x', y'), (x'', y''), (x''', y''') and x-axis are contained in a plane and thus it can be reduced to the case $n = 2$. We will prove that (x, y) is included in the plane generated by (x'', y'') and x-axis, the rest follows easily. Now, it is a straightforward computation to see that if $z \in \mathbb{R}^{n-1}$ is orthogonal to y'' then $(0, z)$ is a vector tangent to $\partial\Omega$ at the point (x'', y''). Thus,

$$(0, z).\big((x, y) - (x'', y'')\big) = 0.$$

In consequence, $y.z = 0$ for all $z \in \mathbb{R}^{n-1}$ orthogonal to y''. Hence, y is a multiple of y'' and (x, y), (x'', y'') and x-axis are included in a plane, concluding the proof. □

Later, we will prove some technical result.

Lemma A.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then, there exists a sequence Ω_n of Lipschitz simply connected open subsets of Ω such that*

$$\overline{\Omega}_n \subset \left\{ x \in \Omega : \, d(x) > 2^{-n} \right\} \qquad \text{and} \qquad \Omega_n \nearrow \Omega.$$

Proof. First, we define a particular Whitney decomposition of Ω. Let Q be a dyadic cube in $\mathbb{R}^2$ in the n^{th} generation with $n \in \mathbb{Z}$. Namely,

$$Q := \left(j_1 2^{-n}, (j_1+1)2^{-n} \right) \times \left(j_2 2^{-n}, (j_2+1)2^{-n} \right),$$

where $j_1, j_2 \in \mathbb{Z}$. Given a dyadic cube Q, we will denote by Q^* the unique dyadic cube in the $(n\text{-}1)^{th}$ generation containing Q.

Thus, we say that Q belongs to the Whitney decomposition W if it satisfies that

$$3Q \subseteq \Omega \qquad \text{and} \qquad 3Q^* \nsubseteq \Omega, \tag{A.1}$$

where $3Q$ is the cube (non dyadic) with the same center of Q and side 3 times bigger than the side of Q.

Now, we are ready to prove the lemma. Let Q_0 in W with the biggest side. To simplify the notation we will suppose that the biggest side is 1. Thus, if U_n denotes the closure of the sum of all Whitney cubes with side bigger than or equal to 2^{-n}, we will define by Ω_n the connected component of ${U_n}^\circ$ which contains Q_0. We consider the closure in the definition of U_n to obtain that two cubes sharing a side belong to the same connected component.

Finally, we will show that Ω_n is simply connected, the others properties follow immediately. Let σ be a simple closed curve included in Ω_n and V_σ the open set limited by σ. So, it is enough to prove $V_\sigma \subseteq \Omega_n$. But, as $V_\sigma \subseteq \Omega$ and V_σ is surrounded by Whitney cubes with side bigger than 2^{-n} we can conclude that any dyadic cube Q in the n^{th} generation intersecting V_σ satisfies that $3Q \subseteq \Omega$. Thus, it is included in Ω_n. $\square$

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