Algebras de Nichols sobre grupos no abelianos

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Álgebras de Nichols sobre grupos no abelianos

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Álgebras de Nichols sobre grupos no abelianos

En esta tesis se estudian álgebras de Nichols sobre grupos no abelianos. Un álgebra de Nichols viene dada por un par (X, q), donde X es un rack y q es un 2-cociclo en una teoría de cohomología no abeliana de racks.

En este trabajo definimos racks de tipo D y demostramos que los racks simples y finitos de tipo D dan álgebras de Nichols de dimensión infinita para todo 2-cocyclo q. Muchas de las clases de conjugación de los grupos esporádicos, o de $\mathbb{A}_n$, o de $\mathbb{S}_n$ son racks de tipo D. Con estos resultados y las técnicas que desarrollamos a partir de la clasificación de álgebras de Nichols de tipo diagonal de dimensión finita hecha por Heckenberger demostramos que no existen álgebras de Nichols de dimensión finita sobre G, donde $G = \mathbb{A}_n$ o $G = \mathbb{S}_n$ o G es un grupo simple esporádico distinto de Fi_{22}, $\mathbb{B}$ o $\mathbb{M}$. Como corolario, el método del levante nos da la clasificación de álgebras de Hopf punteadas de dimensión finita sobre estos grupos.

En un apéndice presentamos brevemente un software que desarrollamos para poder realizar cálculos relacionados con racks y álgebras de Nichols. Este software resultó ser una poderosa herramienta para crear, entender y aplicar las técnicas que dan condiciones que garantizan que un par (X, q) (o una clase de conjugación de un grupo finito dado) dé solamente álgebras de Nichols de dimensión finita.

Palabras clave: Álgebras de Hopf punteadas, álgebras de Nichols, racks.

Nichols algebras over non-abelian groups

In this thesis Nichols algebras over non-abelian groups are studied. A Nichols algebra is given by a pair (X,q), where X is a rack and q is a 2-cocycle in a non-abelian rack cohomology theory for racks.

We define the notion of racks of type D and we prove that finite simple racks of type D give infinite-dimensional Nichols algebras for every 2-cocyle q. Most of the conjugacy classes of the sporadic simple groups, or $\mathbb{A}_n$, or $\mathbb{S}_n$ are racks of type D. These results combined with techniques that we developed from Heckenberger's classification of Nichols algebras of diagonal type lead us to to prove that there are no finite-dimensinal Nichols algebras over G, where $G = \mathbb{A}_n$ or G is a sporadic simple group different from Fi_{22}, $\mathbb{B}$ and $\mathbb{M}$. As a corollary, the lifting method allows us to complete the classification of pointed Hopf algebras over these groups.

In an appendix we present a computer package that we developed for computations related with racks and Nichols algebras. This package turns out to be a powerful tool, very useful for creating, understanding and applying the techniques that give conditions that guarantee that a pair (X,q) (or a conjugacy class in a given finite group) gives only infinite-dimensional Nichols algebras.

Keywords: Pointed Hopf algebras, Nichols algebras, racks.

Contents

Introduction

This thesis contributes to the classification of finite-dimensional pointed Hopf algebras over $\mathbb{C}$ (or over any algebraically closed field $\mathbb{K}$ of characteristic 0). There are different possible approaches to this problem. One of them is to fix a finite group G and to address the classification of finite-dimensional pointed Hopf algebras H such that $G(H) \simeq G$. Due to the intrinsic difficulty of this problem, it is natural to consider separately different classes of finite groups. With the Weyl-Heckenberger groupoid [**Hec06**, **Hec09**], a considerable progress in the case when G is abelian was possible (see [**AS10**]). In this work we concentrate on the non-abelian case.

If G denotes a finite group, to classify all complex pointed Hopf algebras H with group of group-likes $G(H) \simeq G$ and $\dim H < \infty$, we need to determine the irreducible Yetter-Drinfeld modules over $\mathbb{C}G$ such that the dimension of the corresponding Nichols algebra is finite. In other words, recalling that irreducible Yetter-Drinfeld modules are parameterized by pairs $(\mathcal{O}, \rho)$ (where $\mathcal{O}$ is a conjugacy class of G, $\sigma \in \mathcal{O}$ fixed, and ρ is an irreducible representation of the centralizer $C_G(\sigma)$), and writing $\mathfrak{B}(\mathcal{O}, \rho)$ for the associated Nichols algebra, we need to know for which pairs $(\mathcal{O}, \rho)$ is $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$.

Our main goal, towards the classification of finite-dimensional pointed Hopf algebras, is to answer the following question.

Question 1. *For any finite group G and for any $V \in {}^G_G\mathcal{YD}$, determine if* $\dim \mathfrak{B}(V) < \infty$.

Since the category ${}^G_G\mathcal{YD}$ is semisimple, the question splits into two cases:

(1) V irreducible,

(2) V completely reducible (a direct sum of at least 2 irreducibles).

The case (1) was addressed in several recent papers for some groups and some conjugacy classes [**AG03**, **AF07b**, **AF07a**, **Fan07b**, **AFZ09**, **FGV07**, **FGV09**, **AZ07**]. The case (2) was considered in [**AHS08**, **HS08**]. Of course, the Nichols algebras of the simple submodules of a completely reducible V such that $\mathfrak{B}(V)$ is finite-dimensional should be finite-dimensional too. But the interaction between the two cases goes also in the other way. To explain this, we need to recall that our first question can be rephrased in terms of racks. Indeed, the Nichols algebra of a Yetter-Drinfeld module depends only on its braiding, which in the case of a group algebra is defined in terms of the conjugation. Question 1 is equivalent to the following one.

Question 2. *For any finite rack X, for any $n \in \mathbb{N}$, and for any non-principal 2-cocycle $\mathbf{q}$ determine if* $\dim \mathfrak{B}(X, \mathbf{q}) < \infty$.

In fact, the consideration of the rack-theoretical question is more economical than the consideration of the group-theoretical one, since different Yetter-Drinfeld modules over different groups may give rise to the same pair $(X, \mathbf{q})$, X a rack and $\mathbf{q}$ a 2-cocycle. This point of view, advocated in [**Gra00, AG03**], is analogous to the similar consideration of braided vector spaces of diagonal type in the classification of finite-dimensional pointed Hopf algebras with abelian group.

The consideration of the rack-theoretical question has another advantage. A basic and useful property of Nichols algebras says: if W is a braided subspace of a braided vector space V, then $\mathfrak{B}(W) \hookrightarrow \mathfrak{B}(V)$. For instance, consider a simple $V = M(\mathcal{O}, \rho) \in {}^G_G\mathcal{YD}$, say $\dim \rho = 1$ for simplicity. If X is a proper subrack of $\mathcal{O}$, then $M(\mathcal{O}, \rho)$ has a braided subspace of the form $W = (\mathbb{C}X, c^q)$, which is clearly not a Yetter-Drinfeld submodule but can be realized as a Yetter-Drinfeld module over smaller groups, that could be reducible if X is decomposable. If we know that $\dim \mathfrak{B}(X, q) = \infty$, say because we have enough information on one of these smaller groups, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ too.

Both questions have the common drawback that there is no structure theorem neither for finite groups nor for finite racks. Therefore, and in order to collect evidence about what groups or what racks might afford finite-dimensional Nichols algebras, it is necessary to attack different classes of groups (or racks). Prominent candidates are the finite simple groups and the finite simple racks. Finite simple racks have been classified in [**AG03, Joy82**] (see Chapter 1, Theorem 1.22). In particular, non-trivial conjugacy classes of finite simple groups, and conjugacy classes of symmetric groups that do not split (seen as orbits for the action of the alternating subgroup) are simple racks. Therefore, it is natural to begin by families of simple groups.

To prove the theorems on Nichols algebras over conjugacy classes of a simple group G, we first establish that $\dim \mathfrak{B}(X, q) = \infty$ for many conjugacy classes X in G and any 2-cocycle q. This relies on a result on Nichols algebras of *reducible* Yetter-Drinfeld modules [**HS08**, Theorem 8.6]. Indeed, let us say (informally) that a rack *collapses* if $\dim \mathfrak{B}(X, q) = \infty$ for any cocycle q (see the precise statement of this notion in Definition 1.28). To translate one of the hypothesis of [**HS08**, Theorem 8.6] to rack-theoretical terms, we introduce the notion of rack of type D. We deduce from [**HS08**, Theorem 8.6] our Theorem 2.33, that says that any rack of type D collapses. It is easy to see that if $\pi : Z \to X$ is an epimorphism of racks and X is of type D, then so is Z. But any indecomposable rack Z has a simple quotient X; this justifies further why we look at simple racks, starting with non-trivial conjugacy classes in simple groups. This is one of the consequences of the study of Nichols algebras of decomposable Yetter-Drinfeld modules in the analogous study of indecomposable ones. We stress that the computation of a second rack-cohomology group is a difficult task. By [**EG03**], it coincides with a first group-cohomology group, but this does not make the problem easier. The point of view taken in this thesis allows to disregard sometimes these considerations about rack-cohomology groups.

Actually, along this work we give a list of conjugacy classes of simple groups which are of type D; hence, if X belongs to this list and $\pi : Z \to X$ is an epimorphism of racks, then the Nichols algebra $\mathfrak{B}(Z, q)$ has infinite dimension for an arbitrary 2-cocycle q.

In this thesis it is proved that several simple groups give only rise to infinite-dimensional Nichols algebras. So, this work does not provide any new examples of finite-dimensional pointed Hopf algebras. In fact, very few examples of finite-dimensional non-trivial pointed Hopf algebras with non-abelian group are known (only few examples are known, see [**Gra**]). At the present moment, it is not clear what is the class of non-abelian finite groups that may afford finite-dimensional pointed Hopf algebras. Therefore, it is important to narrow down as many examples as possible in order to have a feeling of what this class might be.

Definition. *We shall say that a finite group G* **collapses** *if for any finite-dimensional pointed Hopf algebra H, with $G(H) \simeq G$, then $H \simeq \mathbb{C}G$.*

In this work many examples of groups that collapse are presented: alternating and symmetric groups, some finite groups of Lie type, the sporadic simple groups.

Symmetric groups and alternating groups. In the early 90's, Susan Montgomery raised the question of finding a *non-trivial* finite-dimensional complex pointed Hopf algebra H with non-abelian group G (here "non-trivial" means that H is neither a group algebra, nor is cooked out of a pointed Hopf algebra with abelian group by some kind of extension). This question was addressed by Alexander Milinski and Hans-Jürgen Schneider around 1995, who produced two examples, one with $G = \mathbb{S}_3$, another with $G = \mathbb{S}_4$. The main idea of the proof is to check that a quadratic algebra $\mathfrak{B}_m$ built from the conjugacy class of transpositions in $\mathbb{S}_m$ is finite-dimensional. They were able to do it for $m = 3, 4$ using Gröbner bases. These results were published later in [**MS00**]. Independently, Sergei Fomin and Anatol Kirillov considered closely related quadratic algebras $\mathcal{E}_m$, also constructed from the transpositions in $\mathbb{S}_m$, and they determined the dimensions of $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ [**FK99**]. With the introduction of the Lifting Method, see [**AS02b**], it became clear that $\mathfrak{B}_m$ and $\mathcal{E}_m$ should be Nichols algebras. This was indeed checked in [**MS00**] for $m = 3, 4$ and by Matías Graña $m = 5$ [**Gra**].

Recently, there was some progress on pointed Hopf algebras over $\mathbb{S}_m$. The classification of the finite-dimensional Nichols algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$ is concluded in [**AHS08**]. Also, most of the Nichols algebras over the Symmetric group $\mathbb{S}_m$ have infinite dimension, with the exception of a short list of open possibilities [**AZ07, AF07b, AFZ09**].

One of our main results improves drastically the list given in [**AFZ09**, Theorem 1].

Theorem. *Let $n \geq 5$ and let $\sigma \in \mathbb{S}_n$. If the conjugacy class $\sigma^{\mathbb{S}_n}$ gives finite-dimensional Nichols algebras, then σ belongs to one of the following:*

(1) *The conjugacy class of transpositions in $\mathbb{S}_n$;*
(2) *The conjugacy class of $(1\,2)(3\,4\,5)$ in $\mathbb{S}_5$;*
(3) *The conjugacy class of $(1\,2)(3\,4)(5\,6)$ in $\mathbb{S}_6$.*

Notice that the conjugacy class of $(1\ 2)(3\ 4)(5\ 6)$ in $\mathbb{S}_6$ is isomorphic, as a rack, to the conjugacy class of the transpositions in $\mathbb{S}_6$, since any non-inner automorphism of $\mathbb{S}_6$ applies $(1\ 2)$ in $(1\ 2)(3\ 4)(5\ 6)$ (see [**JR82**]). Thus, the case of the conjugacy class of $(1\,2)(3\,4)(5\,6)$ in $\mathbb{S}_6$ is contained in the study of the conjugacy class of transpositions, which is the case studied by Fomin and Kirillov.

Also, we prove the following result about Nichols algebras over the alternating simple groups.

Theorem. *Let $m \geq 5$. Every Nichols algebra over $\mathbb{A}_m$ is infinite-dimensional. Hence, $\mathbb{A}_m$ collapses.*

This result was known for the particular cases $m = 5$ and $m = 7$, see [**AF07a**, **Fan07a**].

Finite Lie groups. We prove that many simple linear groups collapse. This result, of course, is a consequence of the Lifting Method and the non-existence of finite-dimensional Nichols algebras over these simple groups. The first example is the family of groups $\mathsf{PSL}(2, q)$ for $q = 2^m$ for $m > 2$. These groups are studied only with abelian techniques. Other simple Lie groups studied in this thesis are: the exceptional Lie groups $G_2(q)$ for $q = 3, 4, 5$; the Orthogonal groups $O_7(3)$, $O_8^+(2)$, $O_{10}^-(2)$; the Symplectic groups $S_6(2)$ and $S_8(2)$.

Some of these results were proved in [**FGV07**, **FGV09**]. Other results were proved to simplify the proofs of the theorems concerning Nichols algebras over the sporadic simple groups, because some finite Lie groups appear as subgroups or subquotients of the sporadic simple groups.

Sporadic simple groups. With computational techniques we study some of the simple racks coming from the sporadic simple groups. Roughly speaking, we prove that almost every simple rack (or every conjugacy class) of a sporadic simple group is of type D. Therefore, with the help of some abelian techniques based on the Heckenberger's Classification Theorem of Nichols algebras of diagonal type [**Hec06**, **Hec09**], we have the following theorem.

Theorem. *Let G be a sporadic simple group, with the exception of the Fischer group Fi_{22}, the Baby Monster group $\mathbb{B}$, or the Monster group $\mathbb{M}$. Then G has no finite-dimensional Nichols algebras. Hence, G collapses.*

All the results concerning Nichols algebras over the sporadic simple groups were proved in [**AFGV09b**, **AFGV09c**, **AFGV09a**].

Organization

The first chapter is devoted to recall all the basic facts about racks and its cohomologies, Nichols algebras and the Lifting Method. This is mainly based on [**AG03**, **Gra00**, **AG99**, **AS02b**].

Chapter 2 is devoted to developing the techniques that will be used to study Nichols algebras. These techniques, mainly based on the Weyl-Heckenberger groupoid (see [**Hec09**, **Hec06**, **AHS08**, **HS08**]), allow us to translate the problem of studying Nichols algebras (or Yetter-Drinfeld modules or racks and 2-cocycles) to a group-theoretical problem, which seems to be easier to solve. The techniques developed in this chapter include some of the results obtained in [**AF09**].

In the last three chapters we study different families of simple groups. In Chapter 3 we study Nichols algebras over alternating groups and symmetric groups. In Chapter 4 we study Nichols algebras over some linear groups. We study some linear groups such as $\mathsf{PSL}(2, q)$, for q even, some symplectic groups, some orthogonal groups, etc. As said, some of these results are useful for studying Nichols algebras over the sporadic simple groups, because these linear groups appear as subgroups or subquotients of them. Chapter 5 is devoted to Nichols algebras over the sporadic simple groups and some groups of automorphisms of sporadic groups. Most of the results obtained in chapters 4 and 5 strongly depend on computational methods. So, in order to understand the proofs of the main theorems of these chapters we include references to log files and GAP scripts.

This thesis has three appendices. The first appendix is devoted to list all real and quasi-real conjugacy classes in sporadic simple groups. The second one is devoted to present a computer package which we find useful for studying racks and Nichols algebras: The GAP package RiG is a joint work with Matías Graña, and can be used for computations related to racks, such as the computation of subracks, quotients, (co)homology groups, etc. This free software is available at [**GnV08**].

The last appendix is devoted to list all the notations used along this thesis. In this appendix we also include all group theoretic notations used in the ATLAS (The red book: [**CCN+85**], the Web interface: [**WWT+**] and the GAP package: [**WPN+08**]).

Chapter 1

Preliminaries

1.1. Yetter-Drinfeld Modules

In this section we review the basic facts about Yetter-Drinfeld modules over a given finite group. See [**AG99, RT93**] for details. We use the Heyneman-Sweedler notation (see [**Swe69, Mon93**]).

Definition 1.1. *Let H be a Hopf algebra (over $\mathbb{C}$) with bijective antipode. A* **Yetter Drinfeld** *module over H is a left H-module M that also is a left H-comodule with the following compatibility condition:*

$$(hm)_{(-1)} \otimes (hm)_{(0)} = h_{(1)}m_{(-1)}Sh_{(-3)} \otimes h_{(2)}m_{(0)},$$

for all $h \in H$, $m \in M$.

Notice that the category of Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$ is a monoidal category with the usual tensor product of vector spaces (over $\mathbb{C}$), where, for $M, N \in {}^H_H\mathcal{YD}$, $M \otimes N$ has the Yetter-Drinfeld structure given by

$$h(m \otimes n) = h_{(1)}m \otimes h_{(2)}n,$$
$$(m \otimes n)_{(-1)} \otimes (m \otimes n)_{(0)} = m_{(-1)}n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}.$$

It is also a braided category, where the braiding $c : M \otimes N \to N \otimes M$ is given by

$$c(m \otimes n) = m_{(-1)}n \otimes m_{(0)}.$$

Let G be a finite group. We consider the case where $H = \mathbb{C}G$, the group algebra of G. Notice that a Yetter-Drinfeld module over $\mathbb{C}G$ is a G-module M provided with a G-grading $M = \oplus_{g\in G}M_g$ such that $h \cdot M_g = M_{ghg^{-1}}$ for all $g, h \in G$. The category of Yetter-Drinfeld modules over $\mathbb{C}G$ is written ${}^G_G\mathcal{YD}$. The **support** of $M \in {}^G_G\mathcal{YD}$ is $\mathsf{sup}(M) = \{g \in G \mid M_g \neq 0\}$.

For any $g \in G$, denote by g^G the conjugacy class of g in G and by $C_G(g)$ the centralizer of g in G. Let $\rho \in \mathsf{Irr}(C_G(g))$, an irreducible representation of the centralizer of g in G, and let

$$M(g, \rho) = \mathsf{Ind}^G_{C_G(g)}V = \mathbb{C}G \otimes_{\mathbb{C}C_G(g)} V.$$

We have

$$h \cdot (x \otimes v) = hx \otimes v, \quad \delta(x \otimes v) = xgx^{-1} \otimes (x \otimes v),$$

for $x \in G$. Then $M(g, \rho)$ is an object of ${}^G_G\mathcal{YD}$. Notice that

$$\dim M(g, \rho) = [G : C_G(g)] \cdot \deg(\rho).$$

Proposition 1.2. *The objects $M(g, \rho)$ are simple. Moreover, any simple object of ${}^G_G\mathcal{YD}$ is isomorphic to a unique $M(g, \rho)$, where g belongs to the set of representatives of conjugacy classes of G, and $\rho \in \mathsf{Irr}(C_G(g))$.*

Proof. See [**AG99**, Proposition 3.1.2]. □

Corollary 1.3. *If G is abelian, every object of ${}^G_G\mathcal{YD}$ can be decomposed as a direct sum of Yetter-Drinfeld modules of dimension 1.*

Proof. It is easy, since $[G : C_G(g)] = \deg \rho = 1$ for every $g \in G$ and $\rho \in \mathsf{Irr}(C_G(g))$. □

Proposition 1.4. *${}^G_G\mathcal{YD}$ is a braided rigid category.*

Proof. See [**AG99**, Proposition 2.1.1]. □

1.2. Nichols algebras

The definition of a Nichols algebra of a braided vector space (V, c) appears in various different ways (see [**AG99**]).

Definition 1.5. *A* **braided vector space** *is a pair (V, c), where V is a vector space and $c \in \mathsf{Aut}(V \otimes V)$ is a solution of the* **braid equation***:*

$$(c \otimes 1)(1 \otimes c)(c \otimes 1) = (1 \otimes c)(c \otimes 1)(1 \otimes c)$$

Let $n \geq 2$. Recall that the symmetric group $\mathbb{S}_n$ can be presented as the group generated by elementary transpositions $\tau_i = (i\ i+1)$ subject to the relations:

$$\begin{aligned} \tau_i\tau_j &= \tau_j\tau_i \quad \text{if } |i-j| > 1,\\ \tau_i\tau_j\tau_i &= \tau_j\tau_i\tau_j \quad \text{if } |i-j| = 1,\\ \tau_i^2 &= 1. \end{aligned}$$

The **Braid group** $\mathbb{B}_n$ is the group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ subject to the relations:

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{if } |i-j| > 1,\\ \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j \quad \text{if } |i-j| = 1. \end{aligned}$$

Notice that there is a projection $\mathbb{B}_n \to \mathbb{S}_n$ given by $\sigma_i \mapsto \tau_i$. Let $x \in \mathbb{S}_n$. For $y \in \mathbb{S}_n$ we denote by $l(y)$ the length of a minimal word in the alphabet $\tau_1^{\pm1}, \tau_2^{\pm1}, \dots, \tau_{n-1}^{\pm1}$ which represents y.

Lemma 1.6. *There exists a unique section $s : \mathbb{S}_n \to \mathbb{B}_n$ to the projection $\mathbb{B}_n \to \mathbb{S}_n$ such that $s(\tau_i) = \sigma_i$ and $s(\alpha\beta) = s(\alpha)s(\beta)$ if $l(\alpha \cdot \beta) = l(\alpha) + l(\beta)$. It is given by*

$$(\alpha = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_j}) \mapsto (\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_j})$$

if $l(\alpha) = j$.

Proof. See [**CR87**, 62.20]. □

Let $V \in {}^G_G\mathcal{YD}$. Notice that $\mathbb{B}_n$ acts on $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n\text{-times}}$ by

$$\sigma_i \mapsto \underbrace{1 \otimes \cdots 1}_{i-1} \otimes c \otimes \underbrace{1 \otimes \cdots 1}_{n-i-1}$$

For $V \in {}^G_G\mathcal{YD}$, we write by $T^n(V) = V^{\otimes n}$ and by $T(V)$ the object

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

We define the **quantum symmetrizer** $S_n : TV \to TV$ as

$$S_n = \sum_{x \in \mathbb{S}_n} s(x).$$

Then, the **Nichols algebra** of V is

$$\mathfrak{B}(V) = \bigoplus_{n \geq 0} \mathfrak{B}^n(V),$$

where $\mathfrak{B}^0(V) = \mathbb{C}$, $\mathfrak{B}^1(V) = V$ and $\mathfrak{B}^n(V) = T^n(V)/\ker(S_n) \simeq S_n(T^n V)$.

1.3. Racks and Cocycles

In this section we define and review basic properties of racks, quandles and crossed sets. For details see for example [**AG03**].

1.3.1. Racks.

Definition 1.7. *A* ***rack*** *is a pair* $(X, \triangleright)$ *where* X *is a non-empty finite set and* $\triangleright : X \times X \to X$ *is a function, such that*

(1.1) $$x \mapsto i \triangleright x \text{ is bijective for all } i \in X,$$

(1.2) $$i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k) \text{ for all } i, j, k \in X.$$

A ***quandle*** *is a rack* $(X, \triangleright)$ *such that*

(1.3) $$i \triangleright i = i \text{ for all } i \in X.$$

A ***crossed set*** *is a quandle* $(X, \triangleright)$ *such that*

(1.4) $$i \triangleright j = j \Leftrightarrow j \triangleright i = i \text{ for all } i, j \in X.$$

Example 1.8. *If* $X = \{1, 2, \ldots, n\}$ *and* $i \triangleright j = j \pmod n$*, then* $(X, \triangleright)$ *is called a* **trivial (or abelian) rack**.

Example 1.9. *If* A *is an abelian group and* $T \in \mathsf{Aut}(A)$*, then* A *becomes a rack with* $a \triangleright b = (1 - T)a + Tb$*. This rack will be denoted* (A, T) *and called an* **affine rack**.

Definition 1.10. *Let* X *be a rack. A non-empty subset* $Y \subset X$ *is a subrack if* $Y \triangleright Y \subseteq Y$.

Example 1.11. *A finite group* G *is a rack with* $x \triangleright y = xyx^{-1}$*. If a subset* $X \subseteq G$ *is stable under conjugation by* G*, then it is a subrack of* G*. In particular, the support of any* $M \in {}^G_G\mathcal{YD}$ *is a subrack of* G.

In appendix C we present other examples of racks.

Notice that a rack $(X, \triangleright)$ can be presented as a matrix $M = (m_{ij})$, where $m_{ij} = i \triangleright j$.

Definition 1.12. *$f : (X, \triangleright) \to (Y, \triangleright)$ is a **morphism of racks** if*

$$f(i \triangleright j) = f(i) \triangleright f(j) \text{ for all } i, j \in X.$$

Morphisms of quandles (resp. crossed sets) are morphisms of racks between quandles (resp. crossed sets).

Definition 1.13. *Let $(X, \triangleright)$ be a rack and let $\mathbb{S}_X$ be the group of bijective funtions $X \to X$. Notice that there exists a map $\varphi : X \to \mathbb{S}_X$ defined as $i \mapsto i \triangleright \_ : X \to X$.*

(1) *The group of automorphisms of X is*

$$\mathsf{Aut}(X) = \{f \in \mathbb{S}_X \mid f(x \triangleright y) = f(x) \triangleright f(y)\}.$$

(2) *The inner group of X is the subgroup of $\mathbb{S}_X$ generated by $\varphi(X)$. We write $\mathsf{Inn}(X)$ for the inner group of X.*

Remark 1.14. *Notice that $\mathsf{Inn}(X)$ is a normal subgroup of $\mathsf{Aut}(X)$. Also notice that in general these groups are not equal (see Appendix C).*

Definition 1.15. *Let $(X, \triangleright)$ a rack. We say that X is **faithful** if the function $\varphi : X \to \mathbb{S}_X$ is injective.*

Definition 1.16. *Let $(X, \triangleright)$ be a rack. We say that X is **decomposable** if it admits a decomposition, which is a disjoint union $X = Y \sqcup Z$ such that Y and Z are both subracks of X. Also, X is **indecomposable** if it is not decomposable.*

Definition 1.17. *Let $(X, \triangleright)$ be a rack. The **orbit** of $x \in X$ is the orbit of x under the natural action of the inner group. Then,*

$$\mathcal{O}_x = \{y_1 \triangleright (y_2 \triangleright \cdots \triangleright (y_s \triangleright x) \cdots) \mid y_1, y_2, \ldots, y_s \in X\}.$$

Lemma 1.18. *Let $(X, \triangleright)$ be a rack, $Y \neq X$ a non-empty subset and $Z = X - Y$. Then the following are equivalent:*

(1) *$X = Y \sqcup Z$ is a decomposition of X;*
(2) *$Y \triangleright Z \subseteq Z$ and $Z \triangleright Y \subseteq Y$;*
(3) *$X \triangleright Y \subseteq Y$.*

Proof. See [AG03, Lemma 1.14]. □

Lemma 1.19. *Let $(X, \triangleright)$ be a rack. Then the following are equivalent:*

(1) *X is indecomposable;*
(2) *$X = \mathcal{O}_x$ for some $x \in X$;*
(3) *$X = \mathcal{O}_x$ for all $x \in X$.*

Proof. See [AG03, Lemma 1.15]. □

Definition 1.20. *A non-trivial rack X is **simple** if any surjective morphism of racks $\pi : X \to Y$ satisfies $\#Y = 1$ or $\#Y = \#X$.*

Remark 1.21. *Notice that a simple rack is indecomposable, since every decomposable rack has a surjective morphism onto the trivial rack with two elements.*

Theorem 1.22. *Any simple rack is isomorphic to one and only one of the following:*

(1) *$|X| = p$ a prime, $X \simeq \mathbb{F}_p$ a permutation rack, that is $x \triangleright y = y + 1$.*

(2) *$|X| = p^t$, p a prime, $t \in \mathbb{N}$, $X \simeq ({\mathbb{F}_p}^t, T)$ is an affine crossed set where T is the companion matrix of a monic irreducible polynomial of degree t, different from X and $X - 1$.*

(3) *$|X|$ is divisible by at least two different primes, and X is twisted homogeneous. That is, there exist a non-abelian simple group L, a positive integer t and $x \in \mathsf{Aut}(L^t)$, where x acts by $x \cdot (l_1, \dots, l_t) = (\theta(l_t), l_1, \dots, l_{t-1})$ for some $\theta \in \mathsf{Aut}(L)$, such that $X = \mathcal{O}_x(n)$ is an orbit of the action $\rightharpoonup_x$ of $N = L^t$ on itself ($n \neq m^{-1}$ if $t = 1$ and x is inner, $x(p) = mpm^{-1}$). Furthermore, L and t are unique, and x only depends on its conjugacy class in $\mathsf{Out}(L^t)$. Here, the action $\rightharpoonup_x$ is given by $p \rightharpoonup_x n = p\, n\, (x \cdot p^{-1})$.*

Proof. See [**AG03**, Theorems 3.9 and 3.12]. □

1.3.2. Cocycles.

Definition 1.23. *Let $(X, \triangleright)$ be a rack. Let $n \in \mathbb{N}$. A map $q : X \times X \to \mathsf{GL}(n, \mathbb{C})$ is a* **principal 2-cocycle of degree** *n if*

$$q_{x, y \triangleright z} q_{y,z} = q_{x \triangleright y, x \triangleright z} q_{x,z},$$

for all $x, y, z \in X$.

Here is an equivalent formulation: let $V = \mathbb{C}X \otimes \mathbb{C}^n$ and consider the linear isomorphism $c^q : V \otimes V \to V \otimes V$,

$$c^q(e_x v \otimes e_y w) = e_{x \triangleright y} q_{x,y}(w) \otimes e_x v,$$

x, $y \in X$, v ,$w \in \mathbb{C}^n$. Then q is a 2-cocycle if and only if c^q is a solution of the braid equation. If this is the case, then the Nichols algebra of (V, c^q) is denoted $\mathfrak{B}(X, q)$.

Definition 1.24. *Let $(X_i)_{i \in I}$ be a decomposition of a rack X and let $\mathbf{n} = (n_i)_{i \in I}$ be a family of natural numbers. Then a* **non-principal 2-cocycle of degree n**, *associated to the decomposition $(X_i)_{i \in I}$, is a family $\mathbf{q} = (q_i)_{i \in I}$ of maps $q_i : X \times X_i \to \mathsf{GL}(n_i, \mathbb{C})$ such that*

$$q_i(x, y \triangleright z) q_i(y, z) = q_i(x \triangleright y, x \triangleright z) q_i(x, z), \tag{1.5}$$

for all $x, y \in X$, $z \in X_i$, $i \in I$.

Again, this notion is related to braided vector spaces. Given a family $\mathbf{q}$, let $V = \oplus_{i \in I} \mathbb{C}X_i \otimes \mathbb{C}^{n_i}$ and consider the linear isomorphism $c^{\mathbf{q}} : V \otimes V \to V \otimes V$,

$$c^{\mathbf{q}}(e_x v \otimes e_y w) = e_{x \triangleright y} q_i(x, y)(w) \otimes e_x v,$$

$x \in X_j$, $y \in X_i$, $v \in \mathbb{C}^{n_j}$, $w \in \mathbb{C}^{n_i}$. Then $\mathbf{q}$ is a 2-cocycle if and only if $c^{\mathbf{q}}$ is a solution of the braid equation. If this is the case, then the Nichols algebra of $(V, c^{\mathbf{q}})$ is denoted $\mathfrak{B}(X, \mathbf{q})$.

Let X be a rack, $\mathbf{q}$ a non-principal 2-cocycle and V as above. Define a map $g : X \to \mathsf{GL}(V)$ by

$$g_x(e_y w) = e_{x \triangleright y} q_i(x, y)(w), \qquad x \in X, y \in X_i, i \in I. \tag{1.6}$$

Note that $g : X \to \mathsf{GL}(V)$ is a morphism of racks.

The next result shows why Nichols algebras associated to racks and 2-cocycles are important for the classification of pointed Hopf algebras.

Theorem 1.25.

(1) *Let X be a rack, $(X_i)_{i \in I}$ a decomposition of X, $\mathbf{n} \in \mathbb{N}^I$ and $\mathbf{q}$ a 2-cocycle as above. If $G \subset \mathsf{GL}(V)$ is the subgroup generated by $(g_x)_{x \in X}$, then $V \in {}^G_G\mathcal{YD}$. If the image of q_i generates a finite subgroup of $\mathsf{GL}(n_i, \mathbb{C})$ for all $i \in I$, then G is finite.*

(2) *Conversely, if G is a finite group and $V \in {}^G_G\mathcal{YD}$, then there exists a rack X, a decomposition $X = \sqcup_{i \in I} X_i$, $\mathbf{n} \in \mathbb{N}^I$ and non-principal 2-cocycle $\mathbf{q}$ such that V is given as above and the braiding $c \in \mathsf{Aut}(V \otimes V)$ in the category ${}^G_G\mathcal{YD}$ coincides with $c^{\mathbf{q}}$.*

Proof. See [**AG03**, Theorem 4.14]. □

If X is indecomposable, then there is only one possible decomposition and only principal 2-cocycles arise. Conversely, the proof of [**AG03**, Theorem 4.14] shows that if $V \in {}^G_G\mathcal{YD}$ as in part (2) is irreducible, then the cocycle $\mathbf{q}$ is actually principal.

For an easy way of reference, we shall say that a 2-cocycle $\mathbf{q}$ is **finite** if the image of q_i generates a finite subgroup of $\mathsf{GL}(n_i, \mathbb{C})$ for all $i \in I$.

Definition 1.26. *Let X be a finite rack and $\mathbf{q}$ a 2-cocycle. We say that $(X, \mathbf{q})$ is* **faithful** *if the morphism of racks $g : X \to \mathsf{GL}(V)$ defined in (1.6) is injective; if X is clear from the context, we shall also say that $\mathbf{q}$ is faithful.*

Remark 1.27. *If a rack X is faithful, then $(X, \mathbf{q})$ is faithful for any $\mathbf{q}$.*

We present next an important definition concerning infinite-dimensional Nichols algebras over racks. It has deep consequences in the study of Nichols algebras over non-abelian simple groups (see [**AFGV08**]).

Definition 1.28. *We shall say that a finite rack X* collapses *if for any finite faithful cocycle $\mathbf{q}$ (associated to any decomposition of X and of any degree $\mathbf{n}$), $\dim \mathfrak{B}(X, \mathbf{q}) = \infty$.*

Here is a useful reformulation.

Lemma 1.29. *Let X be a finite rack. Assume that for any finite group G and any $M \in {}^G_G\mathcal{YD}$ such that X is isomorphic to a subrack of $\mathsf{sup}(M)$, $\dim \mathfrak{B}(M) = \infty$. Then X collapses. The converse is true if X is faithful.*

Proof. Let $\mathbf{q}$ be a finite faithful cocycle. By Theorem 1.25 (1), the braided vector space $(V, c^{\mathbf{q}})$ arises from a Yetter-Drinfeld module over a finite group Γ; since $\mathbf{q}$ is faithful, X can be identified with $\mathsf{sup}(V)$.

Now assume that X is faithful and collapses. Let G, M as in the hypothesis of Lemma 1.29. The rack Y constructed in Theorem 1.25 (ii) is

$Y = \sqcup_{i\in I}\mathcal{C}_i$, where $M = \oplus_{i\in I} M_i$ is a decomposition in irreducible submodules and $\mathcal{C}_i = \mathsf{sup} M_i$. In general, $\mathsf{sup}(M) \neq Y$, but there is an injective morphism of racks $\mathsf{sup}(M) \hookrightarrow Y$, which induces an injective morphism of racks $X \hookrightarrow Y$. Since X is faithful, the restriction of the cocycle $\mathbf{q}$ on Y to X is faithful. □

1.3.3. Some constructions of racks. We now present a general construction that might be of independent interest. Let X be a rack, with operation $x \triangleright y = \varphi_x(y)$, and let j an integer. Let $X^{[j]}$ be a disjoint copy of X, with a fixed bijection $X \to X^{[j]}$, $x \mapsto x^{[j]}$, $x \in X$. We define a multiplication $\triangleright$ in $X^{[j]}$ by

$$x^{[j]} \triangleright y^{[j]} = (\varphi_x^j(y))^{[j]}, \quad x, y \in X. \tag{1.7}$$

Notice that $X^{[j][k]} \simeq X^{[jk]}$, for $j, k \in \mathbb{Z}$, both non-zero.

Lemma 1.30.

(1) *$X^{[j]}$ is a rack, called the j-th power of X.*

(2) *The disjoint union $X^{[1,j]}$ of X and $X^{[j]}$ with multiplication such that X and $X^{[j]}$ are subracks, and*

$$x \triangleright y^{[j]} = (x \triangleright y)^{[j]}, \quad x^{[j]} \triangleright y = \varphi_x^j(y), \quad x, y \in X, \tag{1.8}$$

is a rack.

$X^{[1,j]}$ is a particular case of an amalgamated sum of racks. The rack $X^{[-1]}$ will be called the inverse rack of X and will be denoted X'; the corresponding bijection is denoted $x \mapsto x'$. Note $X'' \simeq X$. The rack $X^{[1,1]}$ will be denoted $X^{(2)}$ in accordance with [**AF09**].

Proof. We first show (1) for $j = -1$. The self-distributivity of Definition 1.7 holds if and only if $\varphi_x\varphi_y = \varphi_{x\triangleright y}\varphi_x$ for all $x, y \in X$, iff $\varphi^{-1}_{\varphi_x^{-1}(u)}\varphi_x^{-1} = \varphi_x^{-1}\varphi_u^{-1}$ for all $x, u \in X$ (setting $u = x \triangleright y$); this is in turn equivalent to the self-distributivity for X'. We next show (a) for $j \in \mathbb{N}$. We check recursively that $\varphi_x\varphi_y^j = \varphi_{x\triangleright y}^j\varphi_x$, $\varphi_x^j\varphi_y = \varphi_{\varphi_x^j(y)}\varphi_x^j$. Hence $\varphi_x^j\varphi_y^j = \varphi^j_{\varphi_x^j(y)}\varphi_x^j$, and we have self-distributivity for $X^{[j]}$. Combining these two cases, we see that self-distributivity holds for $X^{[j]}$, for any $0 \neq j \in \mathbb{Z}$. The proof of (2) is straightforward. □

Example 1.31. *Let $0 \neq j \in \mathbb{Z}$. Assume that X is a subrack of G such that the map $\eta_j : X \to G$, $x \mapsto x^j$, is inyective. Then the image X^j of η_j is also a subrack, isomorphic to the rack $X^{[j]}$. If $X \cap X^j = \emptyset$, then the disjoint union $X \cup X^j$ is a subrack of G isomorphic to $X^{[1,j]}$.*

Chapter 2

Techniques

Let G be a finite group, let g^G be the conjugacy class of $g \in G$ in G, let $C_G(g)$ be the centralizer of g in G and let (ρ, V) an irreducible representation of $C_G(g)$: $\rho \in \mathsf{Irr}(C_G(g))$. Note that, since ρ is irreducible and $g \in \mathsf{Z}(C_G(g))$, the center of $C_G(g)$, then $\rho(g)$ is a scalar (by Schur lemma). Then

$$\rho(g) = 1 \Longrightarrow \dim \mathfrak{B}(\mathcal{C}, \rho) = \infty. \tag{2.1}$$

Definition 2.1. *Let G be a finite group and $g \in G$. We say that the conjugacy class g^G is of* **type B** *if $\dim \mathfrak{B}(g^G, \rho) = \infty$ for every $\rho \in \mathsf{Irr}(C_G(g))$. A finite group G is of* **type B** *if every Nichols algebra over G is infinite-dimensional.*

Notice that if (V, c) is a braided vector space and $W \subseteq V$ is a subspace such that $c(W \otimes W) = W \otimes W$ (such W is called a braided subspace of V), then $\mathfrak{B}(W) \hookrightarrow \mathfrak{B}(V)$ (see for example [AS02b, Corollary 2.3]). In particular, if $\mathfrak{B}(W)$ is infinite dimensional, so is $\mathfrak{B}(V)$.

2.1. The Subgroup Technique

As we said before, if W is a braided subspace of a braided vector space V, then $\mathfrak{B}(W) \hookrightarrow \mathfrak{B}(V)$. Let G be a group, $M \in {}^G_G\mathcal{YD}$. Here are two ways of getting braided subspaces of M:

(1) If Y is a subrack of $\sup(M)$, then $M_Y = \oplus_{y \in Y} M_y$ is a braided subspace of M.

(2) Let $\sigma \in G$, H a subgroup of G such that $\sigma \in H$. If ρ is a representation of $C_G(\sigma)$, then $M(\sigma^H, \rho|_{C_H(\sigma)})$ is a braided subspace of $M(\sigma^G, \rho)$.

These ways are actually closely related, by the following observation.

Lemma 2.2. *If Y is a subrack of $\sup(M)$ and K is the subgroup of G generated by Y, then M_Y is an object in ${}^K_K\mathcal{YD}$.*

Proof. First notice that, by construction, M_Y is K-graded. Furthermore, if $k \in K$ and $y \in Y$, then $k \cdot M_y = M_{k^{-1}yk} \subseteq M_Y$, since Y is closely under conjugation by K. $\square$

Here are two lemmas that will allow us to study Nichols algebras over abelian groups. These lemmas were proved in [**FGV09**, **AFGV08**].

Lemma 2.3. *Let G be a finite group and $H \subseteq G$ be a subgroup. If $h \in H$ and h^H is of type B then h^G is of type B.*

Proof. Since $M = \mathrm{Ind}_{C_G(g)}^G \rho = \oplus_{s\in h^G} V_s$, where $V_s = \{v \in V \mid \delta(v) = s \otimes v\}$, we have that $M^H = \oplus_{s \in h^H} V_s \subseteq M$ is a Yetter-Drinfeld module over H. □

Lemma 2.4. *Let G be a finite group, H be a subgroup of G, $h \in H$. Let $h_1, h_2 \in h^G \cap H$ and assume that h_1 and h_2 are not conjugate in H. If $\dim \mathfrak{B}(M(h_1^H, \lambda_1) \oplus M(h_2^H, \lambda_2)) = \infty$ for all $\lambda_i \in \mathrm{Irr}(C_H(h_i))$, for $1 \leq i \leq 2$, then h^G is of type B.*

Proof. As before, $M = \oplus_{s\in h^G} V_s$. Let $V = \oplus_{s \in h_1^H \cup h_2^H} V_s \subseteq M$. Then, by Lemma 2.2, $V \in {}^H_H\mathcal{YD}$ and the result follows. □

2.2. Abelian Techniques

Following the method given in [**Gra00**], we try to investigate the question if the Nichols algebra of V is finite-dimensional or not by looking for braided subspaces W of V of diagonal type. Using the existing theory of the Weyl-Heckenberger groupoid (see [**Hec06**, **Hec09**]), it is easy to decide if the Nichols algebra of W is infinite-dimensional. In this case the Nichols algebra of V is also infinite-dimensional.

We begin this section with a useful observation that leads us to look for braided subspaces of diagonal type of infinite-type.

Let G be a finite group and $g \in G$. Let $g^G = \{g_i \mid i \in I\}$ be the conjugacy class of g, with $g_i = x_i g x_i^{-1}$, and let $\rho = (\rho, V) \in \mathrm{Irr}(C_G(g))$ of degree 1. Fix $w \in V$ and define $v_i = x_i \otimes w \in V(g, \rho)$, where $\chi = \rho \in \mathrm{Irr}(C_G(g))$ is a character. Let $T \subseteq I$ be a subset such that $g_i g_j = g_j g_i$ for all $i, j \in T$. Let $V_T \subseteq V(g, \chi)$ be the subspace generated by $\{v_i \mid i \in T\}$. Then the braiding restricted to V_T is of *diagonal type*, given by

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \text{where } q_{ij} = \chi(x_j^{-1} g_i x_j) \in \mathbb{C}. \tag{2.2}$$

Indeed, we have

$$\begin{aligned} c(v_i \otimes v_j) &= g_i v_j \otimes v_i = (x_j x_j^{-1} g_i x_j \otimes w) \otimes v_i \\ &= (x_j \otimes x_j^{-1} g_i x_j w) \otimes v_i = (x_j \otimes \chi(x_j^{-1} g_i x_j) w) \otimes v_i, \end{aligned}$$

since $x_j^{-1} g_i x_j \in C_G(g)$.

We write $\mathbf{q} = \mathbf{q}_T = (\mathbf{q}_{ij})$. If $T \subseteq I$ and $T' \subseteq \mathcal{C}$, $T' = \{g_i \mid i \in T\}$, then we make abuse of notation by calling $\mathbf{q}_{T'} = \mathbf{q}_T$ and $V_{T'} = V_T$.

The following Theorem was proved in [**Hec06**] showing that some hypotheses in [**AS00**, Theorem 1] were unnecessary.

Theorem 2.5. *Let (V, c) a braided vector space of Cartan type. Then $\dim \mathfrak{B}(V) < \infty$ if and only if the Cartan matrix is of finite type.*

Proof. See [**Hec06**, Theorem 4]. □

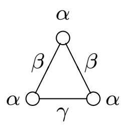
2.2.1. The classification of diagonal braidings. We recall three useful propositions from the Heckernberger's classification of Nichols algebras of diagonal type (see [**Hec09**, **Hec06**]).

Proposition 2.6. *Let W be a two-dimensional braided vector space of diagonal type. Assume that the Dynkin diagram of W is given by* $\overset{\alpha}{\circ}\overset{\beta}{\text{———}}\overset{\alpha}{\circ}$ *and suppose that* $\dim \mathfrak{B}(W) < \infty$. *Then the Dynkin diagram is among the following ones:*

Dynkin diagram	*fixed parameter*
$\overset{\alpha}{\circ}\quad\quad\overset{\alpha}{\circ}$	$\alpha \in \mathbb{C}^{\times}$
$\overset{\alpha}{\circ}\overset{\alpha^{-1}}{\text{———}}\overset{\alpha}{\circ}$	$\alpha \neq 1$
$\overset{-1}{\circ}\overset{\alpha}{\text{———}}\overset{-1}{\circ}$	$\alpha \neq \pm 1$
$\overset{-\alpha^{-2}}{\circ}\overset{-\alpha^{3}}{\text{———}}\overset{-\alpha^{-2}}{\circ}$	$\alpha \in \mathcal{R}_{12}$
$\overset{-\alpha^{-2}}{\circ}\overset{\alpha}{\text{———}}\overset{-\alpha^{-2}}{\circ}$	$\alpha \in \mathcal{R}_{12}$

Proof. By inspection on [**Hec09**, Table 1]. □

Proposition 2.7. *Let W be a three-dimensional connected braided vector space of diagonal type. Assume that the Dynkin diagram of W is given by*



and suppose that $\dim \mathfrak{B}(W) < \infty$. *Then the Dynkin diagram is among the following ones:*

(1) $\alpha = -1$, $\beta = q$, $\gamma = q^{-2}$ *for* $q \notin \{1\} \cup \mathcal{R}_2 \cup \mathcal{R}_3$;
(2) $\alpha = -1$, $\beta = \gamma \in \mathcal{R}_3$;
(3) $\alpha = -1$, $\beta \in \mathcal{R}_3$, $\gamma = 1$.

Proof. By inspection on [**Hec09**, Table 2]. □

Proposition 2.8. *Let W be a finite-dimensional space of diagonal type and assume that the Dynkin diagram of W contains:*

(1) *a cycle of length* ≥ 4; *or*
(2) *a vertex with valency* ≥ 4.

Then $\dim \mathfrak{B}(W) = \infty$.

Proof. Follows from [**Hec09**]. Notice that the first item is [**Hec09**, Lemma 20]. □

2.2.2. Real and quasi-real conjugacy classes. In this subsection we present a useful application of braided vector spaces of Cartan type.

Definition 2.9. *Let G be a finite group and $g \in G$. Classically, $g \in G$ and g^G are called* **real** *if $g^{-1} \in g^G$. If g is not real, but it is conjugate to $g^s \neq g$ for some $s \in \mathbb{N}$, then we say that $g \in G$ and g^G are* **quasi-real** of type s.

Lemma 2.10. *Let G be a finite group, $g \in G$ and $\rho \in \mathsf{Irr}(C_G(g))$. Assume that $\dim \mathfrak{B}(g^G, \rho) < \infty$. If g is real then $\rho(g) = -1$.*

Proof. See [AZ07, Corollary 2.2]. □

The next Lemma is a variation of Lemma 2.10. This result was proved in [FGV07]. A similar result was proved in [AF07a, Lemmas 1.8 and 1.9].

Lemma 2.11. *Let G be a finite group, $g \in G$ and $\rho \in \mathsf{Irr}(C_G(g))$. Assume that $\dim \mathfrak{B}(g^G, \rho) < \infty$. If g is quasi-real of type n then $\rho(g) = -1$ or $\rho(g) \in \mathcal{R}_3$. Moreover, if $g^{n^2} \neq g$ then $\rho(g) = -1$.*

Proof. Let $\alpha = \rho(g)$ and let m be the order of α. We consider first the case $g^{n^2} = g$. After using Proposition 2.6 we obtain that $\alpha^{n+\frac{1}{n}} = 1$ or $\alpha^{n+\frac{1}{n}+1} = 1$. If $m \mid n^2+1$ then, since m divides n^2-1, $\alpha = -1$ ($\alpha = 1$ would imply $\dim \mathfrak{B}(g^G, \rho) = \infty$). If $m \mid n^2+n+1$ then $m \mid n+2$ and then $\alpha \in \mathcal{R}_3$. Now we consider the case $g^{n^2} \neq g$, i.e. g, g^n and g^{n^2} are different and they belong to $C_G(g) \cap g^G$. Then, if $T = \{g, g^n, g^{n^2}\}$ we have

$$\mathbf{q}_T = \begin{pmatrix} \alpha & \alpha^{\frac{1}{n}} & \alpha^{\frac{1}{n^2}} \\ \alpha^n & \alpha & \alpha^{\frac{1}{n}} \\ \alpha^{n^2} & \alpha^n & \alpha \end{pmatrix}.$$

By Propositions 2.7 and 2.6, the only possibilities for $\mathbf{q}_T$ to produce a finite dimensional Nichols algebra are:

- $\alpha = -1$, or
- $\alpha^{\frac{1}{n}+n} = 1$ and $\alpha^{\frac{1}{n^2}+n^2} = 1$ (but then $\alpha = -1$), or
- $\alpha^{\frac{1}{n}+n} = 1$ and $\alpha^{\frac{1}{n^2}+n^2+1} = 1$ (but then $\alpha = 1$), or
- $\alpha^{\frac{1}{n^2}+n^2} = 1$ and $\alpha^{\frac{1}{n}+n+1} = 1$ (but then $\alpha = 1$).

This completes the proof. □

2.2.3. Counting eigenvalues. In this subsection we present a lemma for treating conjugacy classes of involutions. This was proved in [AFGV09c].

Lemma 2.12. *Let G be a finite group, $g \in G$ an involution, and $(\rho, V) \in \mathsf{Irr}(C_G(g))$ of degree n. Assume that there exists an involution x such that $h = xgx^{-1}$ and $gh = hg$. Also assume that there exists a basis*

$$v_1, \dots, v_r, v_{r+1}, \dots, v_n$$

of V such that $\rho(g)v_i = -v_i$ for all i and

$$\rho(h)v_i = \begin{cases} v_i & 1 \leq i \leq r, \\ -v_i & r+1 \leq r \leq n. \end{cases}$$

If $4 \le r \le n-1$ or $1 \le r \le n-4$, then $\dim \mathfrak{B}(g^G, \rho) = \infty$.

Proof. Let $x_1 = 1$ and $x_2 = x$. Define W as the subspace generated by $x_i v_j = x_i \otimes v_j$, where $i = 1, 2$ and $1 \le j \le n$. Then, W is a braided vector space with braiding $c(x_i v_j \otimes x_i v_j) = -x_i v_j \otimes x_i v_j$, and, for $i \neq j$,

$$c(x_1 v_i \otimes x_2 v_j) = x_2 \rho(h) v_j \otimes x_1 v_i,$$
$$c(x_2 v_j \otimes x_1 v_i) = x_1 \rho(h) v_i \otimes x_2 v_j.$$

Therefore, for i, j such that $1 \le i \le r$, $r+1 \le j \le n$, we have

$$c(x_k v_i \otimes x_l v_j) = (-1)^k x_l v_j \otimes x_k v_1.$$

And then the result follows from Lemma 2.8, because the Dynkin diagram has at least one vertex with valency > 3. □

Given a group G and a representation ρ of G, we remark that there exists an interesting formula that allows us to compute the multiplicity of any eigenvalue of $\rho(g)$ for any $g \in G$. With this formula at hand it is not difficult to apply Lemma 2.12.

Remark 2.13. *Let $g \in G$ of order m and (ρ, V) a representation of G of degree n, with character χ. Let ξ be a primitive root of 1 of order m. Let a_i, $i = 0, \dots, m-1$, be the multiplicity of the eigenvalue ξ^i ($a_i = 0$ means that ξ^i is not an eigenvalue). Then,*

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{m-1} \end{pmatrix} = V(\xi)^{-1} \begin{pmatrix} \chi(1) \\ \chi(g) \\ \chi(g^2) \\ \vdots \\ \chi(g^{m-1}) \end{pmatrix},$$

where

$$V(\xi) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & \dots & \xi^{m-1} \\ 1 & \xi^2 & \xi^4 & \dots & \xi^{2(m-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \xi^{m-1} & \xi^{2(m-1)} & \dots & \xi^{(m-1)(m-1)} \end{pmatrix}$$

is the well-known Vandermonde matrix. This is so because the eigenvalues of the matrix $\rho(g^j)$ are just the jth powers of the eigenvalues of $\rho(g)$ with appropriate multiplicities. Thus $\chi(g^j) = \sum a_i (\xi^i)^j$, which is exactly what the matrix equation says.

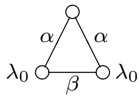
2.2.4. The $\mathbb{A}_4$ techniques. We begin by recalling a simple but very useful result concerning abelian subracks with three elements.

Lemma 2.14. *Let G be a finite group. Assume that $g_0, g_1, g_2 \in G$ are conjugate and commute with each other, that $x_1 x_2$ and $x_2 x_1$ belong to $C_G(g_0)$ (where x_i are such that $g_i = x_i g_0 x_i^{-1}$ for $i = 1, 2$), and that $g_1 g_2 = g_0^m$ for an odd integer m. Then the conjugacy class g_0^G is of type B.*

Proof. Let (V, ρ) be an irreducible representation of $C_G(g_0)$. Since $g_i g_j = g_j g_i$, there exists $0 \neq w \in V$ and $\lambda_i \in \mathbb{C}$ such that $\rho(g_i)(w) = \lambda_i w$ for $i = 0, 1, 2$. For any $0 \leq i, j \leq 2$, we call $\gamma_{ij} = x_j^{-1} g_i x_j$ (here $x_0 = 1$). It is easy to see that $\gamma_{ij} \in C_G(g_0)$ and that

$$\gamma = (\gamma_{ij}) = \begin{pmatrix} g_0 & g_2 & g_1 \\ g_1 & g_0 & g_1^s g_0^{-1} \\ g_2 & g_2^s g_0^{-1} & g_0 \end{pmatrix}.$$

Then, $W = \mathsf{span}_{\mathbb{C}}\{x_0 \otimes w, x_1 \otimes w, x_2 \otimes w\}$ is a braided vector subspace of $M(g_0^G, \rho)$ of abelian type with Dynkin diagram given by



where $\alpha = \lambda_0^s$ and $\beta = \lambda_0^{s^2-2}$. For $\mathfrak{B}(g^G, \rho)$ to be finite dimensional, we should have $\lambda_0 = -1$ (see [**Hec09**, Table 2]) and s should be an even number, which contradicts the hypothesis. ☐

Here are the first two applications of Lemma 2.14.

Lemma 2.15. *The conjugacy class of* $(1\,2)(3\,4)$ *in the Alternating group* $\mathbb{A}_4$ *is of type B.*

Proof. Use Lemma 2.14 with $g_0 = (1\,2)(3\,4)$, $g_1 = (1\,3)(2\,4)$, $g_2 = (1\,4)(2\,3)$, $x_1 = (2\,4\,3)$ and $x_2 = x_1^{-1}$ (we have $g_1 g_2 = g_0$ in this way). ☐

Lemma 2.16. *Let G be a finite group and let $g, h \in G$ with g an involution, h and gh of order 3. Then, the conjugacy class g^G is of type B.*

Proof. The Alternating group $\mathbb{A}_4$ can be presented by generators g and h and relations $g^2 = h^3 = (gh)^3 = 1$. Thus, the subgroup $H = \langle g, h \rangle$ of G is isomorphic to $\mathbb{A}_4$. Indeed, the hypothesis implies that the elements $g, h, h^2, gh, (gh)^2$ of G are all distinct. Then, the result follows from Lemmas 2.3 and 2.15. ☐

There exists a good way of checking if $\mathbb{A}_4$ is a subgroup of a given group G. To this purpose, first recall a very useful result from the character theory of groups.

Proposition 2.17. *Let G be a finite group and take three elements g_1, g_2, g_3 in G Then, if $\xi(g_i^G, g_j^G, g_k^G)$ is the number of times a given element of g_k^G can be expressed as an ordered product of an element of g_i^G and an element of g_j^G, we have*

$$\xi(g_i^G, g_j^G, g_k^G) = \frac{|g_i^G||g_j^G|}{|G|} \sum_{\chi} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)},$$

where χ runs over $\mathsf{Irr}(G)$.

Proof. See [**JL01**, Theorem 30.4]. ☐

Therefore, we have the following application.

Lemma 2.18. *Let G be a finite group, let $\mathcal{C}$ be a conjugacy class of involutions and $\mathcal{A}, \mathcal{B}$ be conjugacy classes of elements of order 3. If $\xi(\mathcal{C}, \mathcal{A}, \mathcal{B}) > 0$ then $\mathcal{C}$ is of type B.*

Proof. By Proposition 2.17, there exists $c \in \mathcal{C}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $ca = b$. Then, Lemma 2.16 applies. □

Another application of Lemma 2.14 is for the group $\mathbb{A}_4 \times C_r$, for r an odd integer.

Lemma 2.19. *Assume $G = \mathbb{A}_4 \times C_r$ with r an odd integer. If $C_r = \langle \tau \rangle$ and $\sigma = (1\,2)(3\,4) \times \tau$, then σ^G is of type B.*

Proof. Use Lemma 2.14 with $g_1 = (1\,2)(3\,4) \times \tau$, $g_2 = (1\,3)(2\,4) \times \tau$, $g_3 = (1\,4)(2\,3) \times \tau$, $x_1 = 1$, $x_2 = (1\,3\,2) \times 1$, $x_3 = x_2^{-1}$ and $h = r + 2$. □

2.3. Non-abelian Techniques

2.3.1. Conjugacy classes of type D. We begin with a powerful result of Heckenberger and Schneider, the proof of which uses the main Theorem of [AHS08].

Theorem 2.20. *Let G be a finite group, $g_i \in G$ and $\rho_i \in \operatorname{Irr}(C_G(g_i))$ for $1 \le i \le 2$. Assume that $M(g_1^G, \rho_1)$ and $M(g_2^G, \rho_2)$ are irreducible objects in ${}^G_G\mathcal{YD}$ such that $\dim \mathfrak{B}(M(g_1^G, \rho_1) \oplus M(g_2^G, \rho_2)) < \infty$. Then, $(rs)^2 = (sr)^2$ for all $r \in g_1^G$, $s \in g_2^G$.*

Proof. See [HS08, Theorem 8.6]. □

Definition 2.21. *Let G be a finite group and $g \in G$. The conjugacy class g^G is of* **type D** *if there exist $r, s \in g^G$ such that $(rs)^2 \neq (sr)^2$ and r and s are not conjugate in the subgroup $H = \langle r, s \rangle$ of G. A finite group G is of* **type D** *if every non-trivial conjugacy class is of type D.*

Proposition 2.22. *Let G be a finite group and $g \in G$. If the conjugacy class g^G is of type D, then it is of type B.*

Proof. Follows from Theorem 2.20 and Lemma 2.4. □

Lemma 2.23. *Let $f : G \to H$ be a group epimorphism and let $g \in G$, $h \in H$ such that $f(g) = h$. If the conjugacy class h^H is of type D, then the conjugacy class g^G is of type D.*

Proof. Let $h \in H$. Since the conjugacy class h^H is of type D, there exist r' and s' in h^H such that $(r's')^2 \neq (s'r')^2$ and r' and s' do not belong to the same conjugacy class in a subgroup S of H. Since f is an epimorphism, there exist r and s in G such that $f(r) = r'$, $f(s) = s'$. Then r and s belong to the conjugacy class of g, $(rs)^2 \neq (sr)^2$, and r and s are not conjugate in $f^{-1}(S)$. Now, the result follows from Proposition 2.22. □

Lemma 2.24. *Let*

$$0 \to K \xrightarrow{i} G \xrightarrow{p} H \to 0$$

be an short exact sequence of groups. Let $g \in G$ of order m. If every conjugacy class in H with representative of order k, for every k such that $k \mid m$ and $\frac{m}{k} \mid \#K$ is of type D, then the class g^G is also of type D.

Proof. If $g \in G$ has order m and $h = p(g)$ has order k, then $k \mid m$. Also, since $g^k \in \ker(p) = i(K)$ and i is a monomorphism, we have that $\frac{m}{k} = \frac{m}{(m,k)} \mid \#K$. Now, the result follows from Lemma 2.23. □

Lemma 2.25. *Let $G = H \times K$ be a direct product of finite groups. Let $h \in H$ and $k \in K$. If h^H is of type D then the conjugacy class of $h \times k$ is of type D.*

Proof. Follows from Lemma 2.23. □

Corollary 2.26. *Let G be any finite group. Assume that the simple factors of G in the Jordan-Hölder decomposition of G are of type D. Then G is of type D.*

2.3.2. The dihedral group and conjugacy class of involutions. Let n be an even number. Recall that the dihedral group of $2n$ elements is given by

$$\mathbb{D}_{2n} = \langle r, s \mid r^n = s^2 = 1,\ srs = r^{-1} \rangle.$$

The involutions of $\mathbb{D}_{2n}$ belong to these three conjugacy classes $\{r^{n/2}\}$, $\{r^{2i}s \mid 0 \leq i \leq \frac{n}{2} - 1\}$ and $\{r^{2i+1}s \mid 0 \leq i \leq \frac{n}{2} - 1\}$.

Lemma 2.27. *Let $n > 4$ be an even number. Let $\sigma_1 = s$ and $\sigma_2 = rs$ be elements of the dihedral group of order $2n$. Then σ_1 and σ_2 belong to different conjugacy classes and $(\sigma_1\sigma_2)^2 \neq (\sigma_2\sigma_1)^2$.* □

Lemma 2.28. *Let $\mathcal{A}$ be a conjugacy class of involutions in a finite group G and let $\mathcal{B}$ be a conjugacy class with representative of order $2m$, with $m > 1$. If $S(\mathcal{A}, \mathcal{A}, \mathcal{B}) > 0$ then the conjugacy class $\mathcal{A}$ is of type D.*

Proof. If x and y are two involutions such that xy has order n then $\langle x, y \rangle \simeq \mathbb{D}_{2n}$. In fact, $r = xy$ and $s = x$. The elements $\sigma_1 = s = x$ and $\sigma_2 = rs = y^{-1}$ of Lemma 2.27, both belong to $\mathcal{A}$, and they are not conjugate in $\mathbb{D}_{2n}$. Then the result follows. □

Example 2.29. *The conjugacy classes of involutions of the Conway group Co_1 are of type D. In fact, $S(2\mathrm{A}, 2\mathrm{A}, 6\mathrm{E}) = 6$, $S(2\mathrm{B}, 2\mathrm{B}, 6\mathrm{A}) = 2592$ and $S(2\mathrm{C}, 2\mathrm{C}, 6\mathrm{A}) = 25920$.*

2.4. Computational techniques

2.4.1. Bases for permutation groups. Let G be a group acting on a set X. A subset B of X is called a *base* for G if the identity is the only element of G which fixes every element in B. In other words,

$$\{g \in G \mid g \cdot b = b, \text{ for all } b \in B\} = 1.$$

Lemma 2.30. *Let G be a group acting on a set X. Let B be a subset of X. The following are equivalent:*

(1) *B is a base for G.*

(2) *For all* $g, h \in G$ *we have:* $g \cdot b = h \cdot b$ *for all* $b \in B$ *implies* $g = h$.

Proof. If B is a base, then $g \cdot b = h \cdot b \Rightarrow (h^{-1}g) \cdot b = b \Rightarrow h^{-1}g = 1 \Rightarrow h = g$. The converse is trivial. □

Let G be a permutation group. With GAP function `BaseOfGroup` we compute a base for G. We use `OnTuples` to encode a permutation and `RepresentativeAction` to decode the information. The following is an example over $\mathbb{A}_4$.

```
gap> gr := AltrenatingGroup(4);;
gap> bg := BaseOfGroup(gr);
[ 1, 2 ]
gap> g := (2,4,3);;
gap> OnTuples(bg, g);
[ 1, 4 ]
gap> RepresentativeAction(gr, bg, last, OnTuples);
(2,4,3)
```

2.4.2. Algorithms for type D. We now explain our algorithms to implement the techniques of the previous sections. The first algorithm is to check if a given conjugacy class in a finite group G is of type D.

Given a finite group G, and $r \in G$, we compute the conjugacy class r^G and use Algorithm 2.1 to decide if r^G is of type D.

Algorithm 2.1: Is the conjugacy class r^G of type D?

for $s \in r^G$ **do**
  if $(rs)^2 \neq (sr)^2$ **then**
    Compute the group $H = \langle r, s \rangle$
    if $r^H \cap s^H = \emptyset$ **then**
      return true `/* the class is of type D */`
    end
  end
end
return false `/* the class is not of type D */`

Notice that in Algorithm 2.1 we need to run over all the conjugacy class r^G to look for the element s such that the hypotheses of Proposition 2.22 are satisfied. This is not always an easy task. To avoid this problem, we have *the random variation* of Algorithm 2.1. The key is to pick randomly an element x (for example, the x is chosen randomly inside the group G or in a smaller subset of G) and then check if r and $s = xrx^{-1}$ satisfy the hypotheses of Proposition 2.22. This naive variation of the Algorithm 2.1 turns out to be very powerful and allows us to study big sporadic groups such as the Janko group J_4 or the Fischer group Fi'_{24}.

Algorithm 2.2: Random variation of Algorithm 2.1

```
forall i : 1 ≤ i ≤ N do                    /* the number of iterations */
    x ∈ G;                                          /* randomly chosen */
    s ← x r x^{-1}
    if (rs)^2 ≠ (sr)^2 then
        Compute the group H = ⟨r, s⟩
        if r^H ∩ s^H = ∅ then
            return true                          /* the class is of type D */
        end
    end
end
return false                                 /* the class is not of type D */
```

In the practice, for large groups, it is more economical to implement the algorithms 2.1 and 2.2 in a recursive way. Let G be a finite group represented faithfully (for example, as a group of permutations or inside a matrix group over a finite field). We compute $g_1^G, \ldots, g_n^G$, the set of conjugacy classes of G. To decide if these conjugacy classes are of type D, we restrict the computations to be done inside a nice subgroup of G.

Assume that the list of all maximal subgroups of G up to conjugation, is known. Say $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$, with increasing order. Also assume that it is possible to restrict our "good" representation of G to every maximal subgroup $\mathcal{M}_i$.

Let $\mathcal{M}$ be a maximal subgroup of G. Let $h \in \mathcal{M}$, and $h^{\mathcal{M}}$ the conjugacy class of h in $\mathcal{M}$. Since $\mathcal{M}$ is a subgroup of G, the element h belongs to a conjugacy class of G, say h^G. So, if the class $h^{\mathcal{M}}$ is of type D then the class h^G is of type D too.

Algorithm 2.3: Type D: Using maximal subgroups

```
g_1^G, ..., g_n^G is the set of conjugacy classes of G
S ← {1, 2, ..., n}
foreach maximal subgroup M do
    Compute h_1^M, ..., h_m^M, the set of conjugacy classes of M
    foreach i : 1 ≤ i ≤ m do
        Identify h_i^M inside a conjugacy class in G: h_i ∈ g_σ(i)^G
        if σ(i) ∈ S then
            if h_i^M is of type D then
                Remove σ(i) from S
                if S = ∅ then
                    return true        /* the group is of type D */
                end
            end
        end
    end
end
foreach s ∈ S do
    if g_s^G is of type D then
        Remove s from S
        if S = ∅ then
            return true                /* the group is of type D */
        end
    end
end
return S                   /* conjugacy classes not of type D */
```

Notice that to implement Algorithm 2.3 we need to have not only a *good representation* for the group G. We need to have a *good representation* of the group G and we need to know how to restrict it to all maximal subgroups. This information appears in the ATLAS for many of the sporadic simple groups. So, using the GAP interface to the ATLAS (see [WPN⁺08]), we could implement Algorithm 2.3 and use it to study Nichols algebras over the sporadic simple groups.

Also notice that the treatment of a maximal subgroup $\mathcal{M}$ can be simplified if it fits into a short exact sequence of groups $0 \to K \to \mathcal{M} \to H \to 0$, where we know the conjugacy classes in H of type D. To apply Lemma 2.23 to a conjugacy class in G, we just need to know that some specific conjugacy classes in H are of type D.

2.5. Rack-theoretical non-abelian techniques

2.5.1. Racks of type D. We now aim to state a rack-theoretical version of Proposition 2.22. Let G be a group, $r, s \in G$. Then

$$(rs)^2 = (sr)^2 \iff r \triangleright (s \triangleright (r \triangleright s)) = s.$$

We next introduce a notion that is central in our considerations.

Definition 2.31. *Let $(X, \triangleright)$ be a rack. We say that X is of* **type D** *if there exists a decomposable subrack $Y = R \sqcup S$ of X such that*

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s, \quad \text{for some } r \in R, s \in S. \tag{2.3}$$

Remark 2.32. *Let G be a finite group and g^G a conjugacy class of G. If the conjugacy class g^G is of type D as in Definition 2.21 then g^G is of type D as a rack (with the conjugation).*

The following theorem is one of the main results of this work. It was proved in [**AFGV08**].

Theorem 2.33. *If X is a finite rack of type D, then X collapses.*

Proof. Let $Y \subseteq X$, $Y = R \sqcup S$ a decomposition as in Definition 2.31. Let G be a finite group, $M \in^G_G \mathcal{YD}$ such that X is isomorphic to a subrack of $\sup M$. We identify X with this subrack, and then we can take M_R and M_S, which are non-trivial objects in ${}^G_G\mathcal{YD}$, K the subgroup of G generated by Y. We may assume that M_R and M_S are irreducible; otherwise, we replace them by irreducible submodules. Now, $\dim \mathfrak{B}(M_R \oplus M_S) = \infty$ by Theorem 2.20, and then $\dim \mathfrak{B}(M) = \infty$. ☐

Remark 2.34. *Theorem 2.33 generalizes* [**AF09**, Corollary 4.12]. *Indeed, if $\mathcal{O}$ is the octahedral rack and $\mathcal{O}^{(2)}$ is a disjoint union of two copies of $\mathcal{O}$ (see Example C.3) then $\mathcal{O}^{(2)}$ is of type D.*

Remark 2.35. *Why is it interesting to consider simple racks of type D? First of all, notice that the following assertions are easy to prove:*

1. *If $Y \subseteq X$ is a subrack of type D, then X is of type D.*
2. *If Z is a finite rack and admits a rack epimorphism $\pi : Z \to X$, where X is of type D, then Z is of type D. For, $\pi^{-1}(Y) = \pi^{-1}(R) \sqcup \pi^{-1}(S)$ is a decomposable subrack of Z satisfying equation (2.3).*

Now, if X is a finite rack and some indecomposable component is of type D, then X is of type D (see [**AG03**, Proposition 1.17]*). And if X is indecomposable, then it admits a projection of racks $\pi : X \to Y$ with Y simple.*

The following Lemma contains some useful observations to detect racks (or conjugacy classes) of type D.

Lemma 2.36.

1. *If X is a rack of type D and Z is a quandle, then $X \times Z$ is of type D.*
2. *Let K be a subgroup of a finite group G and pick $k \in C_G(K)$. We consider the map $\mu_k : K \to G$, $g \mapsto gk$. Then the conjugacy class of any $h \in K$ can be identified with a subrack of the conjugacy class of hk in G. Therefore, if h^K is of type D, then $(hk)^G$ is of type D.*

Proof. The first item is straightforward, since if $r, s \in X$ and $z \in Z$, then $(r,z) \triangleright (s,z) = (r \triangleright s, z)$ because Z is a quandle. The second item follows because the map μ_k is a morphism of racks. $\square$

2.6. The affine technique

In this section we present a rack-theoretical technique that generalizes all the dihedral techniques presented in [**AF09**].

Let (A,T) be a finite affine rack, see Example 1.9. We realize it as a conjugacy class in the following way. Let d be the order of T. Consider the semidirect product $G = A \rtimes \langle T \rangle$, defined by

$$(v, T^h)(w, T^j) = (v + T^h(w), T^{h+j}).$$

The conjugation in G gives

$$(v, T^h) \triangleright (w, T^j) = (T^h(w) + (\mathsf{id} - T^j)(v), T^j). \tag{2.4}$$

Then $\mathcal{Q}^j_{A,T} = \{(w, T^j) : w \in A\}$, $j \in C_d$, is a subrack of G isomorphic to the affine rack (A, T^j). Let $\mathcal{Q}^{[1,j]}_{A,T}$ be the disjoint union $\mathcal{Q}^1_{A,T} \cup \mathcal{Q}^j_{A,T}$, $j \in C_d$; this is a rack with multiplication (2.4). It is called an *affine double rack*. If $j \neq 1$, it can be identified with a subrack of G.

Remark 2.37. *If $(j)_T = \sum_{i=0}^{j-1} T^i$ is an isomorphism, then $\mathcal{Q}^{[j]}_{A,T} \simeq (\mathcal{Q}^1_{A,T})^j$. Indeed, the map $(\mathbb{Q}^1_{A,T})^j \to \mathcal{Q}^j_{A,T}$, $(v,T) \mapsto (v,T)^j = ((j)_T v, T^j)$, is a rack isomorphism. Hence, $\mathbb{Q}^{[1,j]}_{A,T}$ is isomorphic to $(\mathcal{Q}^1_{A,T})^{[1,j]}$, cf. Lemma 1.30.*

Let $A^T = \ker(\mathsf{id} - T)$ be the subgroup of points fixed by T.

Lemma 2.38. *Assume that $A^T = 0$. Then $\mathcal{Q}_{A,T}$ is indecomposable and it does not contain any abelian subrack with more than one element.*

Proof. For, assume that $\mathcal{Q}_{A,T} = R \sqcup S$ is a decomposition, with $(0,T) \in R$. But then $(v,T) \triangleright (0,T) = ((\mathsf{id} - T)(v), T) \in R$ for all $v \in A$. Since $\mathsf{id} - T$ is bijective, $\mathcal{Q}_{A,T} = R$. The second claim follows at once from (2.4). $\square$

Lemma 2.39. *Let $j \in C_d$. If $(\mathsf{id} + T^{j+1})(\mathsf{id} - T) \neq 0$, then $\mathcal{Q}^{[1,j]}_{A,T}$ is of type D.*

Proof. Let $R = Q^1_{A,T}$, $S = Q^j_{A,T}$, $r = (0,T) \in R$. Then $Q^{[1,j]}_{A,T} = R \sqcup S$. Pick $v \notin \ker(\mathsf{id} + T^{j+1})(\mathsf{id} - T)$ and $s = (v, T^j) \in S$. Then

$$r \triangleright (s \triangleright (r \triangleright s)) = ((T - T^{j+1} + T^{j+2})(v), T^j) \neq s,$$

since $(\mathsf{id} - T + T^{j+1} - T^{j+2})(v) \neq 0$. $\square$

We prove an application to affine double racks.

First, we recall a result from [**HS08**].

If G is a finite group and $g, h \in G$, then the conjugacy classes g^G and h^G *commute* if $st = ts$ for all $s \in g^G$ and $t \in h^G$. Let

$$\mathcal{F}(G) = \{\mathcal{O} \text{ conjugacy class of } G : \dim \mathfrak{B}(\mathcal{O}, \rho) < \infty \text{ for some } \rho\}.$$

Theorem 2.40. *Let G be a finite group such that any two conjugacy classes in $\mathcal{F}(G)$ do not commute. Let $U \in {}^G_G\mathcal{YD}$ such that $\dim \mathfrak{B}(U) < \infty$. Then U is irreducible in ${}^G_G\mathcal{YD}$.*

Proof. See [HS08, Theorem 8.2]. □

Lemma 2.41. *Let $G = A \rtimes \langle T\rangle$. Assume that T has order d and that T is irreducible (that is to say that A has no non-trivial T-invariant subgroup). If $U \in {}^G_G\mathcal{YD}$ satisfies $\dim \mathfrak{B}(U) < \infty$, then U is irreducible.*

Proof. By Theorem 2.40, we have to show that any two conjugacy classes in $\mathcal{F}(G)$ do not commute. We claim that

(1) the conjugacy classes of G are either $Q^j_{A,T}$ with $j \neq 0$, or else the orbits of T in A.

(2) $\mathcal{F}(G) \subset \{\mathcal{Q}^j_{A,T} : j \neq 0\}$; hence any two conjugacy classes in $\mathcal{F}(G)$ do not commute.

The first part is elementary, but we sketch the argument. It is evident that the conjugacy class of (v, id) is the orbit of $v = (v, \mathsf{id})$ under T. If $0 < j < d$ then $1 - T^j$ is bijective, since its kernel is a T-invariant subspace, but we are assuming that T is irreducible.

We prove the second claim. Let $0 \neq v \in A$; the centralizer of v is A. Set $\sigma_k = T^k(v)$ and $g_k = (0, T^k)$; thus $g_k \triangleright \sigma_0 = \sigma_k$ and $\sigma_l g_k = g_k \sigma_{l-k}$, $0 \leq \ell, k \leq d-1$. Let $\chi \in \mathsf{Irr}(A)$; then the braiding in $M(\mathcal{O}_v, \chi)$ is given by $c(g_k \otimes g_l) = \chi(\sigma_{k-l}) g_l \otimes g_k$. In other words, this is of diagonal type with matrix $q_{kl} = \chi(T^{k-l}(v))$. Let Δ be the generalized Dynkin diagram associated to (q_{kl}). Now we can identify $A = \mathbb{F}_q$, with $q = p^t$, in such a way that $T(v) = \xi v$, where $\xi \in \mathbb{F}_q^\times$ has order d. Hence $q_{kl} q_{lk} = \chi\big((\xi^{l-k} + \xi^{k-l})v\big)$. Observe that $\xi^l + \xi^{-l} = 0$ implies $\xi^{4l} = 1$. Notice that

$$1 + \xi + \xi^2 + \cdots + \xi^{d-1} = 0. \tag{2.5}$$

We can assume $\chi(v) \neq 1$. We consider different cases.

Suppose that d is odd. If $3 \nmid d$. If there exists l, with $1 \leq l \leq d-1$, such that $q_{0\,l} q_{l\,0} \neq 1$, then $\dim \mathfrak{B}(\mathcal{O}_v, \chi) = \infty$; indeed, Δ contains a cycle of length greater than 3, and the result follows from 2.8. Otherwise, $\chi(v) = 1$ by (2.5), which is a contradiction.

Assume that $3 \mid d$. If there exists l, with $1 \leq l \leq d-1$ and $l \neq \frac{d}{3}$, such that $q_{0l} q_{l0} \neq 1$, then $\dim \mathfrak{B}(\mathcal{O}_v, \chi) = \infty$.

Assume that $q_{0l} q_{l0} = 1$ for all l, with $1 \leq l \leq d-1$ and $l \neq \frac{d}{3}$. If $q_{0\frac{d}{3}} q_{\frac{d}{3}0} = 1$, then $\chi(v) = 1$ by (2.5), a contradiction. On the other hand, if $q_{0\frac{d}{3}} q_{\frac{d}{3}0} \neq 1$, then Δ contains many triangles none of them listed in [Hec09, Table 2].

Now assume that d is even. If there exists l, with $1 \leq l \leq d-1$ and $l \neq \frac{d}{2}, \frac{d}{3}$, $q_{0l} q_{l0} \neq 1$, then $\dim \mathfrak{B}(\mathcal{O}_v, \chi) = \infty$; indeed, Δ contains a cycle of length greater than 3, and the result follows from [Hec09]. Otherwise, we have $q_{0\frac{d}{2}} q_{\frac{d}{2}0} = \chi(v)^{-2}$; hence, 0 is connected to $\frac{d}{2}$ and the sub-diagram spanned by 0 and $\frac{d}{2}$ is of Cartan type $A_1^{(1)}$.

Therefore, $\dim \mathfrak{B}(\mathcal{O}_v, \chi) = \infty$; hence $\mathcal{O}_v \notin \mathcal{F}(G)$. □

If X is a rack that contains a subrack isomorphic to $\mathcal{Q}_{A,T}^{[1,j]}$, for some affine rack satisfying the hypothesis of Lemma 2.39 then X is of type D (therefore it collapses). We now present a way to check this hypothesis.

Proposition 2.42. *Let G be a finite group, $\mathcal{O} \subset G$ a conjugacy class which is quasi-real of type $j \in \mathbb{N}$. Let (A, T) be an affine rack with T of order d, and let $\psi : A \to \mathcal{O}$ be a monomorphism of racks. If $\mathsf{id} - T^j$ is an isomorphism and $\mathsf{id} + T^{j+1} \neq 0$, then $\mathcal{O}$ is of type D.*

Proof. Since $\mathsf{id} - T^j = (\mathsf{id} - T)(j)_T$, both Remark 2.37 and Lemma 2.38 apply. If $Y = \psi(A)$, then $Y \cap Y^j = \emptyset$. If not, pick $y \in Y \cap Y^j$, $y = x^j$ for some $x \in Y$. Then $x = x^j$, because Y does not contain any abelian subrack with more than one element. But this contradicts the definition of a quasi-real conjugacy class. Hence $\mathcal{O}$ contains a subrack isomorphic to $\mathcal{Q}_{A,T}^1 \cup \mathcal{Q}_{A,T}^j$, which is isomorphic to $\mathcal{Q}_{A,T}^{[1,j]}$. Now the statement follows from Lemma 2.42. □

Example 2.43. *Let G be a finite group, $\mathcal{O} \subset G$ a conjugacy class which is quasi-real of type $j \in \mathbb{N}$. Let (A, T) be an affine* simple *rack with $|T| = d$, and let $\psi : A \to \mathcal{O}$ be a monomorphism of racks. If $j \neq \frac{d}{2} - 1$ when p is odd, or if $j \neq d - 1$ when $p = 2$, then $\mathcal{O}$ is of type D.*

Remark 2.44. *If $A = \mathbb{F}_p$, p a prime, and T has order 2, then $\mathcal{Q}_{A,T}^1$ is called a dihedral rack and denoted $\mathbb{D}_p$ in accordance with* [**AF09**, Definition 2.2]*; thus $\mathcal{Q}_{A,T}^{[1,1]}$ is denoted $\mathbb{D}_p^{(2)}$. Therefore, the splitting technique includes (without having to resort to look for cocycles) the case of quasi-real orbits containing a dihedral subrack* [**AF09**, Corollary 2.9].

When $j = \frac{d}{2} - 1$, we still may conclude that the Nichols algebra has infinite dimension, provided that the appropriate hypotheses hold.

Proposition 2.45. *Let G be a finite group, (A, T) an affine simple rack with $|T| = d$, and $\psi : A \rtimes \langle T \rangle \to G$ a monomorphism of groups. Assume that the conjugacy class $\mathcal{O}$ of $\sigma = \psi(0, T)$ is quasi-real of type $j = \frac{d}{2} - 1$. If $\rho \in \mathsf{Irr} C_G(\sigma)$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.*

Proof. This follows from Lemmas 2.4 and 2.41, which can be applied because of Remark 2.37. □

Chapter 3

The alternating and symmetric groups

In this chapter we will be concerned with pointed Hopf algebras over the alternating and symmetric groups. The aim of this chapter is to prove the following theorem.

Theorem 3.1. *Let $n \geq 5$. The only finite-dimensional pointed Hopf algebra with coradical $\mathbb{A}_n$ is the group algebra $\mathbb{C}\mathbb{A}_n$.*

This result is a consequence of the Lifting Method of Andruskiewitsch and Schneider [**AS02b**] and the non-existence of finite-dimensional Nichols algebras over the group $\mathbb{A}_n$.

In a similar fashion, it is shown that the Nichols algebras over the symmetric groups $\mathbb{S}_m$ are all infinite-dimensional, except maybe those related to the transpositions considered in [**FK99**], and the class of type $(2, 3)$ in $\mathbb{S}_5$.

We also show that any simple rack X arising from a symmetric group, with the exception of a small list, collapses, in the sense that the Nichols algebra of $(X, \mathbf{q})$ is infinite dimensional, for $\mathbf{q}$ an arbitrary 2-cocycle.

3.1. Preliminaries

3.1.1. Notations on symmetric groups. Let $\sigma \in \mathbb{S}_n$. Recall that σ is of type

$$(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$$

if the decomposition of σ as product of disjoint cycles contains m_j cycles of length j, for every j, $1 \leq j \leq n$. The odd part of σ is the product of all disjoint cycles of odd length $j > 1$ in the decomposition of σ. Similarly, the even part of σ is the product of all disjoint cycles of even length. We write σ_o (resp. σ_e) for the odd (resp. even) part of σ.

3.1.2. Conjugacy classes. We shall need later to know the conjugacy classes of the alternating and symmetric groups. Recall that for any $\sigma \in \mathbb{S}_n$, the conjugacy class $\sigma^{\mathbb{S}_n}$ consists of all permutations in $\mathbb{S}_n$ which have the same cycle-shape as σ.

Given an even permutation $\sigma \in \mathbb{A}_n$, the conjugacy class

$$\sigma^{\mathbb{A}_n} = \{x\sigma x^{-1} \mid x \in \mathbb{A}_n\}$$

is contained in $\sigma^{\mathbb{S}_n}$ but it might not be equal to $\sigma^{\mathbb{S}_n}$.

The next proposition determines precisely when $\sigma^{\mathbb{A}_n}$ and $\sigma^{\mathbb{S}_n}$ are equal, and what happens when equality fails.

Proposition 3.2. *Let $\sigma \in \mathbb{A}_n$ with $n > 1$.*

(1) *If σ commutes with some odd permutation in $\mathbb{S}_n$, then $\sigma^{\mathbb{S}_n} = \sigma^{\mathbb{A}_n}$.*
(2) *If σ does not commute with any odd permutation in $\mathbb{S}_n$ then $\sigma^{\mathbb{S}_n}$ splits into two conjugacy classes in $\mathbb{A}_n$ of equal size, with representatives σ and $(1\,2)\sigma(1\,2)$.*

Proof. See [JL01, Proposition 12.17]. □

The next proposition is simple but very useful.

Proposition 3.3. *For each element $\sigma \in \mathbb{S}_n$ there is an involution $\tau \in \mathbb{S}_n$ such that $\tau\sigma\tau = \sigma^{-1}$. In particular, every element of $\mathbb{S}_n$ is real.*

Proof. Since every permutation is a product of disjoint cycles, it suffices to prove the result for every j-cycle π. Let

$$\tau = \begin{cases} (1\,j-1)(2\,j-2)\cdots(k-1\,k+1) & \text{if } j = 2k, \\ (1\,j-1)(2\,j-2)\cdots(k\,k+1) & \text{if } j = 2k+1. \end{cases}$$

Then, it is straightforward to check that $\pi^{-1} = \tau\pi\tau^{-1}$. □

Remark 3.4. *This proposition is a particular case of* [Car72, Theorem C], *since the Weyl group of type A_{n-1} for $n \geq 2$ is isomorphic to the symmetric group $\mathbb{S}_n$.*

3.2. Conjugacy classes of type D

Lemma 3.5. *The conjugacy classes of type $(2,4)$, (4^2), $(2^2,3^2)$, $(2^4,3)$, $(1,2^4)$, $(1^3,2^2)$, (2^6), $(1,3^3)$ and (3^3) in the alternating group are of type D.*

Proof. We list in Table 3.1 the permutations σ_1 and x such that σ_1 and $\sigma_2 = x\sigma_1 x^{-1}$ satisfy Proposition 2.22 with the group $H = \langle\sigma_1, \sigma_2\rangle$. In all these cases, it is straightforward to check that these elements fulfill the conditions in Proposition 2.22. It can be done for instance with the help of GAP. □

Of course, classes in $\mathbb{A}_n$ of type D yield classes of type D in $\mathbb{S}_n$. We list in Table 3.2 some classes of type D in symmetric groups which either are not in $\mathbb{A}_n$ or split into two classes not of type D in $\mathbb{A}_n$.

Lemma 3.6. *The conjugacy classes of type $(3,4)$, $(1,4)$, $(2,3^2)$ $(2^3,3)$, $(1,2,3)$, (2^5), (6) and $(1,2^3)$ in the symmetric group are of type D.*

Proof. See the proof of Lemma 3.5 and Table 3.2. □

Table 3.1. Some classes of type D in $\mathbb{A}_n$

Type	σ_1	x
$(2,4)$	$(1\ 2)(3\ 4\ 5\ 6)$	$(1\,2\,3)$
(4^2)	$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$	$(4\,5)(6\,8)$
$(2^2,3^2)$	$(1\ 2)(3\ 4)(5\ 6\ 7)(8\ 9\ 10)$	$(4\,5)(9\,10)$
$(2^4,3)$	$(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10\ 11)$	$(4\,5)(8\,9\,10\,11)$
$(1,2^4)$	$(2\ 3)(4\ 5)(6\ 7)(8\ 9)$	$(1\,2\,3\,4\,6\,7\,8\,5\,9)$
$(1^3,2^2)$	$(4\ 5)(6\ 7)$	$(1\,2\,3\,4)(5\,7)$
(2^6)	$(1\ 2)(3\ 5)(4\ 6)(7\ 9)(8\ 11)(10\ 12)$	$(4\,5)(8\,9)(10\,12\,11)$
$(1,3^2)$	$(2\ 3\ 4)(5\ 6\ 7)$	$(1\,2\,3\,5\,4\,6\,7)$
(3^3)	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$	$(3\,4)(8\,9)$

Table 3.2. Some classes of type D in $\mathbb{S}_n$

Type	σ_1	x
$(3,4)$	$(1\ 2\ 3)(4\ 5\ 6\ 7)$	$(2\,3)(6\,7)$
$(1,4)$	$(2\ 3\ 4\ 5)$	$(1\,2\,4\,3\,5)$
$(2,3^2)$	$(1\ 2)(3\ 4\ 5)(6\ 7\ 8)$	$(2\,3)(7\,8)$
$(2^3,3)$	$(1\ 2)(3\ 4)(5\ 6)(7\ 8\ 9)$	$(2\,3)(6\,7)$
$(1,2,3)$	$(2\ 3)(4\ 5\ 6)$	$(1\,2)(5\,6)$
(2^5)	$(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)$	$(2\,3)(6\,7)(8\,9)$
$(1,2^3)$	$(2\ 3)(4\ 5)(6\ 7)$	$(1\,2\,4\,5\,7)(3\,6)$
(6)	$(1\,2\,3\,4\,5\,6)$	$(3\,5)(4\,6)$

Lemma 3.7. *The class of type (n) in $\mathbb{S}_n$ is of type D if $n > 6$.*

Proof. Let $n \geq 7$. Let $\sigma = (1\,2\,3\,4 \cdots n)$ and take $\tau = (1\ 3)\sigma(1\ 3) = (3\ 2\ 1\ 4 \cdots n)$. Then $\tau\sigma\tau\sigma(1) = 1$. And on the other hand,

$$\sigma\tau\sigma\tau(1) = \begin{cases} 7 & \text{if } n > 7 \\ 4 & \text{if } n = 5 \end{cases}$$

Therefore, $(\sigma\tau)^2 \neq (\tau\sigma)^2$. Let $H = \langle \sigma, \tau \rangle$ be the subgroup of $\mathbb{S}_n$ generated by σ and τ.

If n is odd, then $H \subseteq \mathbb{A}_n$. Since the centralizer of σ in $\mathbb{S}_n$ is included in $\mathbb{A}_n$, and since $\tau = (1\ 3)\sigma(1\ 3)$ with $(1\ 3) \notin \mathbb{A}_n$, then τ and σ belong to different conjugacy classes in $\mathbb{A}_n$. Therefore, they belong to different conjugacy classes in H. Then, the result follows from Proposition 2.22.

If n is even, $H = \langle \sigma, \tau \rangle = \langle \sigma, \tau\sigma^{-1} \rangle = \langle (1\ 2 \cdots n), (1\ 3)(2\ 4) \rangle$. It is easy to see that the elements in H can be written as some products $(\mu_1 \times \mu_2)\sigma^i$, where $\mu_1 \in \mathbb{S}_{\{1,3,5,\cdots,n-1\}}$, $\mu_2 \in \mathbb{S}_{\{2,4,6,\cdots,n\}}$ and the signs $\mathsf{sgn}(\mu_1) = \mathsf{sgn}(\mu_2)$. Now, if $x \in \mathbb{S}_n$ is such that $x\sigma x^{-1} = \tau$, then $x = (1\ 3)\sigma^i$ for some i, which does not belong to H. Hence, σ and τ belong to different conjugacy classes in H. Then, the result follows from Proposition 2.22. □

Lemma 3.8.

(1) *The class of type (n) in $\mathbb{A}_n$ is of type D if n is odd and is not square-free.*

(2) *The class of type* (n, m) *in* $\mathbb{A}_{n+m}$ *is of type D if both* n *and* m *are odd,* $n \geq 3$ *and* $m \geq 5$.

Proof. For the first item, let $n = m^2k$, with $m \geq 3$. Take then $\sigma = (1\ 2 \cdots n)$. For $1 \leq i \leq mk$, let

$$r_i = (i\ (mk + i)\ (2mk + i) \cdots ((m-1)mk + i)),$$

and consider $\tau = r_1 \sigma r_1^{-1}$. Then σ and τ are conjugate in $\mathbb{A}_n$, but they are not conjugate in $H = \langle \sigma, \tau \rangle$. To see this, notice that $\sigma r_1^{-1} \sigma^{-1} = r_2^{-1}$, and then, as in the proof of Lemma 3.7,

$$H = \langle \sigma, \tau\sigma^{-1} \rangle = \langle \sigma, r_1 r_2^{-1} \rangle$$

Then $H \subseteq G := \langle \sigma, r_1, \cdots, r_{mk} \rangle$. Actually, since $\sigma^{mk} = r_1 r_2 \cdots r_{mk}$, G is an extension

$$1 \to \langle r_1, \cdots, r_{mk} \rangle \simeq (C_m)^{mk} \to G \to C_{mk} \to 1.$$

Any element in G can be written uniquely as a product $r_1^{i_1} \cdots r_{mk}^{i_{mk}} \sigma^j$, where $0 \leq j < mk$. We can consider then the homomorphism $\alpha : G \to C_m$, given by $\alpha(r_1^{i_1} \cdots r_{mk}^{i_{mk}} \sigma^j) = \tau^{i_1 + i_2 + \cdots + i_{mk}}$, where τ is a generator of C_m. This homomorphism is well defined, since $\alpha(\sigma^{mk}) = \alpha(r_1 r_2 \cdots r_{mk}) = \tau^{mk} = 1$. On the other hand, the centralizer of σ in $\mathbb{A}_n$ is the subgroup generated by σ. Thus, for σ and τ to be conjugate in H, there should exist an integer j such that $r_1 \sigma^j \in H$. But it is clear that H is in the kernel of α, while $\alpha(r_1 \sigma^j) = \tau$.

To prove the second item we take

$$\sigma_1 = (1\ 2 \cdots n)(n+1\ n+2 \cdots n+m)$$

and $\sigma_2 = x\sigma x^{-1}$, where $x = (1\ 2)(n+1\ n+3)$. Then the subgroup $H = \langle \sigma_1, \sigma_2 \rangle \subseteq \mathbb{A}_n \times \mathbb{A}_m$. Let $\pi : \mathbb{A}_n \times \mathbb{A}_m \to \mathbb{S}_m$ be the projection to the second component, and notice that $\pi(\sigma_1), \pi(\sigma_2)$ are the elements σ, τ in the proof of Lemma 3.7. Then, $(\sigma_1\sigma_2)^2 \neq (\sigma_2\sigma_1)^2$ and they are not conjugate in H, since both statements hold in $\pi(H)$. □

Juxtaposition and classes left. Before proving our main result, we introduce a technique.

Lemma 3.9 (Reduction by juxtaposition)**.** *Let* $m = p + q$, $\mu \in \mathbb{S}_p$, $\tau \in \mathbb{S}_q$ *and* $\sigma = \mu \perp \tau \in \mathbb{S}_m$ *the juxtaposition. If* $\mu^{\mathbb{S}_p}$ *is of type D, then* $\sigma^{\mathbb{S}_m}$ *also is. In the same vein, if* $\mu \in \mathbb{A}_p$ *and its conjugacy class in* $\mathbb{A}_p$ *is of type D, and* $\tau \in \mathbb{A}_q$, *then the conjugacy class of* $\sigma = \mu \perp \tau \in \mathbb{A}_m$ *is of type D.*

This is so because the inclusion $\mathbb{S}_p \times \mathbb{S}_q \hookrightarrow \mathbb{S}_m$ induces an inclusion of racks $\mu^{\mathbb{S}_p} \times \tau^{\mathbb{S}_q} \hookrightarrow \sigma^{\mathbb{S}_m}$.

Observe also that if $\sigma \in \mathbb{A}_n$ and $\sigma^{\mathbb{A}_n}$ is of type D, then so is $\sigma^{\mathbb{S}_n}$. On the other hand, if $\sigma \in \mathbb{A}_n$ is of type $(1^{j_1}, \cdots, n^{j_n})$ and either $j_i > 0$ for some i even or either $j_i > 1$ for some i odd, then $\sigma^{\mathbb{S}_n} = \sigma^{\mathbb{A}_n}$. As an example, if $n \geq 5$ is odd, then the class of type $(1^2, n)$ in $\mathbb{A}_{n+2}$ is of type D. This follows by juxtaposition, since it coincides with the class of type $(1^2, n)$ in $\mathbb{S}_{n+2}$, and the class of type (n) in $\mathbb{S}_n$ is of type D.

As a consequence of the preceding paragraphs, we list the classes which are not of type D. In these cases, we consider all pairs σ_1, σ_2 in them, with the help of GAP, to check that they are not of type D. We list also two families of classes in $\mathbb{A}_m$ for which we do not know the answer.

Corollary 3.10.

- ***Classes not of type D in*** $\mathbb{S}_m$*:* (4)*;* $(3,3)$*;* $(2,2,3)$*;* $(2,3)$*;* $(1^n,3)$*;* (2^4)*;* (2^3)*;* $(1^2,2^2)$*;* $(1,2^2)$*;* (2^2)*;* $(1^n,2)$*.*
- ***Classes not of type D in*** $\mathbb{A}_m$*:* $(3,3)$*;* $(2,2,3)$*;* $(1^n,3)$*;* (2^4)*;* $(1^2,2^2)$*;* $(1,2^2)$*,* (2^2)*.*
- ***Classes not* necessarily *of type D in*** $\mathbb{A}_m$*:* $(1,n)$*;* (n) *(for* $n \geq 5$ *square free).*

We were not able to find out in general whether or not the classes of type (n), $(1,n)$ in $\mathbb{A}_m$, with $n \geq 5$ and square free, are of type D. For instance, the class of type $(1,5)$ is not of type D, while the class of type $(1,7)$ is.

3.3. Main results

3.3.1. The alternating group. Before proving our main result about Nichols algebras over the alternating groups we need to study the conjugacy classes not covered by Corollary 3.10.

Lemma 3.11. *Let* $\sigma \in \mathbb{A}_m$ *be of type* $(1^{n_1}, 2^{n_2}, \sigma_o)$*. Then the conjugacy class* $\sigma^{\mathbb{A}_m}$ *is of type B.*

Proof. If $n_2 = 0$, then the result follows from Lemmas 3.3 and 2.10. Now consider $n_2 = 2k$ an even number. Assume first that $\sigma_o = 1$. For every l, $1 \leq l \leq k$, we define

$$C_l = (4l-3 \quad 4l-2)(4l-1 \quad 4l),$$
$$D_l = (4l-3 \quad 4l-1)(4l-2 \quad 4l),$$
$$\alpha_l = (4l-2 \quad 4l-1)(4l-3 \quad 4l-2) = (4l-1 \quad 4l-2 \quad 4l-3).$$

It is easy to see that the group generated by C_l, D_l and α_l is isomorphic to $\mathbb{A}_4$. Moreover, the group generated by

$$C = C_1 \cdots C_k, \;\; D = D_1 \cdots D_k \text{ and } \alpha = \alpha_1 \cdots \alpha_k$$

is also isomorphic to $\mathbb{A}_4$ and C is an involution, conjugate to σ in $\mathbb{A}_m$. Then, the conjugacy class $\sigma^{\mathbb{A}_m}$ is of type B. Now, if $\sigma_o \neq 1$, as before, we have that σ belongs to a subgroup isomorphic to $\mathbb{A}_4 \times \langle \sigma_o \rangle$. Then, the result follows from Lemma 2.19. □

Lemma 3.12. *The conjugacy classes of type* $(2^2,3)$*,* (2^4)*,* (2^2)*,* $(1,2^2)$ *and* $(1^2,2^2)$ *in the alternating group are of type B.*

Proof. For the conjugacy class of type (2^2) in $\mathbb{A}_4$, use Lemma 2.15. Then, by Lemma 2.3, the conjugacy classes of type $(1,2^2)$ and $(1^2,2^2)$ are also of type B. For the conjugacy class of type $(2^2,3)$ of $\mathbb{A}_7$, notice that the group generated by $(1\,2)(3\,4)(5\,6\,7)$, $(1\,3)(2\,4)(5\,6\,7)$ and $(2\,4\,3)$ is isomorphic to $\mathbb{A}_4 \times C_3$. Then, by Lemmas 2.19 and 2.3, the conjugacy class of type $(2^2,3)$ is of type B. □

Lemma 3.13. *Let $p > 3$ be a prime number. Then, the conjugacy classes of type (p) and $(1, p)$ in the alternating group are of type B.*

Proof. Let σ be an element of type (p). If σ is real, then the result follows from Lemma 2.10. Otherwise, if $\sigma^2 \in \sigma^{\mathbb{A}_p}$ then $\sigma^4 \in \sigma^{\mathbb{A}_p}$ and the result follows from Lemma 2.11, since $\sigma^4 \neq \sigma^2$ and σ has order $p > 3$. If $\sigma^2 \notin \sigma^{\mathbb{A}_p}$ then, by Proposition 3.3, there exists an involution $x \in \mathbb{S}_p$ such that $\sigma^{-1} = x\sigma x$, with x odd. And then, there exists an odd involution y such that $\sigma^2 = y\sigma y^{-1}$. Let $z = xy \in \mathbb{A}_p$. Then, $\sigma^{-2} = z\sigma z^{-1} \in \mathcal{C}$ and $\sigma^4 \in \sigma^{\mathbb{A}_p}$. Therefore, the result follows from Lemma 2.11 since $\sigma^4 = \sigma^{-2}$. The proof for the conjugacy class of type $(1, p)$ in $\mathbb{A}_{p+1}$ is similar. □

Lemma 3.14. *The conjugacy classes of type $(1^n, 3)$ ($n > 2$) and (3^2) in the alternating group are of type B.*

Proof. First we prove that the conjugacy class of type $(1^n, 3)$ is of type B. Let $n > 2$ and σ be of type $(1^n, 3)$. Then, there are at least two fixed points of σ, say $n - 1$ and n. By Proposition 3.3, there exists an involution τ in $\mathbb{S}_{n-2}$ such that $\tau\sigma\tau = \sigma^{-1}$. If $\tau \in \mathbb{A}_{n-2} \subseteq \mathbb{A}_n$, the result follows from Lemma 2.10. Otherwise take the involution $\tau' = \tau(n-1\, n) \in \mathbb{A}_n$ and notice that $\tau'\sigma\tau' = \sigma^{-1}$. Then, the result follows from Lemma 2.10.

For the conjugacy class of type (3^2), let $\sigma = (1\,2\,3)(4\,5\,6)$ and notice that $\sigma^{-1} = x\sigma x$, where $x = (2\,3)(5\,6)$. Therefore, the result follows from Lemma 2.10. □

We have proved the following result.

Theorem 3.15. *Let $n \geq 5$. Every conjugacy class in the alternating group $\mathbb{A}_n$ is of type B.*

This result was known for the particular cases $n = 5$ (see [**AF07a**] or [**FGV07**, because $\mathsf{PSL}(2, 4) \simeq \mathbb{A}_5$]) and $n = 7$ ([**Fan07a**]). Since $\mathbb{A}_3$ is abelian, finite-dimensional Nichols algebras over it are classified and there are 25 of them (see [**AS00**, Theorem 1.3] and [**AS02a**, Theorem 1.8]). Nichols algebras over $\mathbb{A}_4$ are infinite-dimensional except for four pairs corresponding to the classes of (1 2 3) and (1 3 2) and the non-trivial characters of C_3. Actually, these four algebras are connected to each other by an outer automorphism of $\mathbb{A}_4$ or by the Galois group $\mathbb{Q}(\omega)/\mathbb{Q}$ for $\omega \in \mathcal{R}_3$ (the cyclotomic extension by third roots of unity). Therefore, there is only one pair to study.

3.3.2. Proof of Theorem 3.1. Theorem 3.1 now follows from the Lifting Method [**AS02b**] and the Theorem 3.15. □

3.3.3. The symmetric groups. As before, to prove our main result about Nichols algebras over symmetric groups we need to study the conjugacy classes not covered by Corollary 3.10.

Lemma 3.16. *The conjugacy classes of type $(1^n, 3)$ ($n > 2$), (3^2), $(2^2, 3)$, (2^4), (2^2), $(1, 2^2)$ and $(1^2, 2^2)$ in the symmetric group are of type B.*

Proof. Since $\mathbb{A}_m$ is a subgroup of $\mathbb{S}_m$, the result follows from Lemmas 2.3, 3.12, 3.14 and 3.13. □

The classification of finite-dimensional Nichols algebras and finite dimensional pointed Hopf algebras over $\mathbb{S}_3$ was completed in [**AHS08**]. Finite-dimensional Nichols algebras over $\mathbb{S}_4$ were classified in [**AHS08**] and finite-dimensional pointed Hopf algebras over $\mathbb{S}_4$ were classified in [**GG09**].

Nichols algebras over the symmetric groups $\mathbb{S}_n$ for $n \geq 5$ were studied in [**AZ07, AF07b, AFZ09**]. The following result about Nichols algebras over symmetric groups was proved in [**AFGV08**].

Theorem 3.17. *Let $n \geq 5$ and let $\sigma \in \mathbb{S}_n$. If the conjugacy class $\sigma^{\mathbb{S}_n}$ is not of type B, then σ belong to one of the following conjugacy classes:*

(1) *The conjugacy class of transpositions in $\mathbb{S}_n$;*

(2) *The conjugacy class of $(1\,2)(3\,4\,5)$ in $\mathbb{S}_5$;*

(3) *The conjugacy class of $(1\,2)(3\,4)(5\,6)$ in $\mathbb{S}_6$.*

Proof. Let $\sigma \in \mathbb{A}_m$. If $\sigma^{\mathbb{A}_m}$ is of type D, then $\sigma^{\mathbb{S}_m}$ is of type D. Therefore, the result follows from Corollary 3.10 and Lemma 3.16. □

By Proposition 3.3, the conjugacy class of type $(2,3)$ is real. Also, it is easy to see that the centralizer of $\sigma = (1\ 2)(3\ 4\ 5)$ in $\mathbb{S}_5$ is isomorphic to C_6. Therefore, by Lemma 2.10, there is only representacion $\rho \in \mathsf{Irr}(C_6)$ to study: the unique such that $\rho(\sigma) = -1$. The rack and the 2-cocycle can be calculated with GAP and RiG.

The conjugacy class of elements of type (2^3) in $\mathbb{S}_6$ is isomorphic, as a rack, to the conjugacy class of the transpositions in $\mathbb{S}_6$, since any map in the class of the outer automorphism of $\mathbb{S}_6$ applies $(1\ 2)$ in $(1\ 2)(3\ 4)(5\ 6)$ (see [**JR82**]). Thus, the case of the conjugacy class of type (2^3) is contained in the study of the conjugacy class of transpositions.

Let $\sigma = (1\ 2) \in \mathbb{S}_n$, let $\mathcal{C}$ be the conjugacy class of σ in $\mathbb{S}_n$ and let $\rho \in \mathsf{Irr}(C_{\mathbb{S}_n}(\sigma))$ It is easy to see that $C_{\mathbb{S}_n}(\sigma) \simeq C_2 \times \mathbb{S}_{n-2}$. Therefore, if $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$ then $\rho = \mathsf{sgn}\otimes\mathsf{sgn}$ or $\rho = 1\otimes\mathsf{sgn}$, because σ is an involution (here 1 is the trivial representation of C_2 and sgn is the sign representation).

Chapter 4

Some groups of Lie type

The aim of this chapter is to prove that some finite groups of Lie type have no finite-dimensional Nichols algebra. Then, by the lifting method of Andruskiewitsch and Schneider, there will be no finite-dimensional non-trivial pointed Hopf algebra over any of these groups. The groups studied in this chapter are:

- $\mathsf{PSL}(2,q)$ for q even;
- The linear groups $\mathsf{PSL}(5,2)$;
- The Chevalley groups $G_2(q)$ for $q=3,4,5$;
- The symplectic groups $S_6(2)$ and $S_8(2)$;
- The orthogonal groups $O_7(3)$, $O_8^+(2)$ and $O_{10}^-(2)$;
- The automorphism group of the Tits group.

In Section 4.1 we prove that the simple groups $\mathsf{PSL}(2,2^m)$, for $m>1$, are of type B. It is interesting to remark that this result is obtained only with abelian techniques. The other finite groups of Lie type were studied mainly by computational methods.

4.1. The groups $\mathsf{PSL}(2,q)$ for q even

In this section, $q=2^m$ for $m>0$, $E=\mathbb{F}_{q^2}$ will be the quadratic extension of $\mathbb{F}_q$ and $\overline{x}$ will be the Galois conjugate of $x\in E$. Recall that the order of $\mathsf{PSL}(2,q)$ is $(q-1)q(q+1)$. The conjugacy classes of $\mathsf{PSL}(2,q)$ are given in Table 4.1 (see [ZN74]). There are $q+1$ conjugacy classes divided in 4 types:

We first consider the case $q=2$. We have $\mathsf{PSL}(2,2)\simeq\mathbb{S}_3$ and Nichols algebras generated by irreducible Yetter-Drinfeld modules over $\mathbb{S}_3$ were studied in [MS00, AHS08]. They are infinite dimensional except for $\mathfrak{B}(\tau,\mathsf{sgn})$, which is 12-dimensional (here τ is the conjugacy class of transpositions and sgn is the non-trivial character of its centralizer). The classification of finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ was conclued in [AHS08].

We now consider the case $q>2$.

Table 4.1. Conjugacy classes of $\mathsf{PSL}(2,q)$ for q even

Representative	Size	Number
$I = c_1 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	1	1
$c_2 = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	q^2-1	1
$c_3(x) = \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$ $(x \neq 1)$	$q(q+1)$	$\frac{(q-2)}{2}$
$c_4(x) = \begin{pmatrix} & 1 \\ 1 & x+\overline{x} \end{pmatrix}$ $(x \in E \setminus \mathbb{F}_q)$	$(q-1)q$	$\frac{q}{2}$

Proposition 4.1. *Let $q = 2^n$ for $n \geq 2$. The conjugacy classes of c_i in $\mathsf{PSL}(2,q)$ for $i = 1,2,3,4$ are of type B.*

We consider each class separately. The class of c_1 gives the trivial braiding. For the class of c_2, we take, for $a \in \mathbb{F}_q^\times$, $x_a = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$. Then $g_a = x_a c_2 x_a^{-1} = \begin{pmatrix} 1 & a^2 \\ & 1 \end{pmatrix}$. The centralizer of c_2 is the abelian group $\mathbb{F}_q$ embedded in $\mathsf{PSL}(2,q)$ as $\begin{pmatrix} 1 & \mathbb{F}_q \\ & 1 \end{pmatrix}$. If $\chi : \mathbb{F}_q \to \mathbb{C}$ is a character of the centralizer, we get $\mathbf{q}_{ab} = \chi(x_b^{-1} g_a x_b) = \chi(g_{ab^{-1}})$, whence $\mathbf{q}_{ab}\mathbf{q}_{ba} = \chi\begin{pmatrix} 1 & a^2b^{-2} + a^{-2}b^2 \\ & 1 \end{pmatrix}$ (we write $\mathbf{q}_{uv}$ for $\mathbf{q}_{g_u g_v}$). Since $q = 2^n$, χ takes values ± 1. If $\mathbf{q}_{aa} = 1$, then $\dim \mathfrak{B}(\mathcal{C}_2, \chi) = \infty$, so we may assume $\mathbf{q}_{aa} = -1$.

If there exists $a \in \mathbb{F}_q \setminus \{0,1\}$ such that $\chi\begin{pmatrix} 1 & a^2 + a^{-2} \\ & 1 \end{pmatrix} = -1$, then we get

$$\mathbf{q}_{1,a}\mathbf{q}_{a,1} = \mathbf{q}_{a,a^2}\mathbf{q}_{a^2,a} = \cdots = \mathbf{q}_{a^m,1}\mathbf{q}_{1,a^m} = -1,$$

where $a^{m+1} = 1$. This implies that the space V_T contains a cycle of length $m \geq 3$ with edges labelled by -1, whence the conjugacy class of c_i is of type B. Assume then that $\chi\begin{pmatrix} 1 & a^2 + a^{-2} \\ & 1 \end{pmatrix} = 1$ for all $a \neq 0, 1$. But then, for all x in the subgroup generated by the elements of the form $a^2 + a^{-2}$ $(a \neq 0,1)$, we get $\chi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = 1$. Take now any $a \in \mathbb{F}_q \setminus \{0,1\}$, and let r be the order of $a^2 + a^{-2}$. Since r is odd, $1 = (a^2 + a^{-2})^r$ is in the subgroup generated by elements $a^{2i} + a^{-2i}$ for $0 < i \leq r$, but this contradicts the fact that $\mathbf{q}_{aa} = \chi\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = -1$.

For the conjugacy class of c_3, it is easy to see that $c_3(x)$ and $c_3(x^{-1}) = c_3(x)^{-1}$ are conjugate, whence they both have the same odd order. Then by Lemma 2.10, the class of c_3 is of type B.

Finally, for the conjugacy class of c_4, we have that the centralizer of $c_4(x)$ is a subgroup of the cyclic group $E^\times$, hence it is cyclic. Again, it is easy to see that both $c_4(x)$ and $c_4(x)^{-1} = \begin{pmatrix} x+\overline{x} & 1 \\ 1 & \end{pmatrix}$ are conjugate and they have odd order. Then, by Lemma 2.10, the conjugacy class of c_4 is of type B.

4.2. The linear group $\mathsf{PSL}(5,2)$

This group has order 9 999 360. It has 27 conjugacy classes. To study Nichols algebras over this group we use the representation inside $\mathbb{S}_{31}$ given in ATLAS.

Lemma 4.2. *The conjugacy classes of* $\mathsf{PSL}(5,2)$ *with representatives of order* $\neq 31$, *and the conjugacy class* 2A, *of size 465, are all of type D.*

Proof. We use Algorithm 2.1, see the logfile `L5(2).log` for details. □

Theorem 4.3. *The group* $\mathsf{PSL}(5,2)$ *is of type B.*

Proof. The conjugacy classes with representatives of order 31 are quasi-real and Lemma 2.11 applies. For the conjugacy class 2A use Lemma 2.18, since $\xi(2A,3A,3A)=42$. Then the result follows from Lemma 4.2. □

4.3. Some orthogonal groups

In this section we use our computational techniques to obtain some results regarding Nichols algebras of the orthogonal groups that appear as a subgroups or subquotients of the sporadic simple groups.

4.3.1. The group $O_7(3)$. This group has order 4 585 351 680. It has 58 conjugacy classes. For computations we use a representation inside $\mathbb{S}_{351}$ given in ATLAS.

Lemma 4.4. *Every conjugacy class, except the conjugacy class of involutions named* 2A, *is of type D.*

Proof. We use Algorithm 2.2, see the file `O7(3).log` for details. □

Theorem 4.5. *The orthogonal group* $O_7(3)$ *is of type B.*

Proof. By Lemma 4.4 it remains to study the conjugacy class 2A. For this conjugacy class use Lemma 2.12 and Remark 2.13, see the file `O7(3)/2A` for details. □

4.3.2. The group $O_8^+(2)$. This group has order 174 182 400. It has 53 conjugacy classes.

Lemma 4.6. *Every conjugacy class with representative* $\neq 2,3$ *is of type D.*

Proof. We use Algorithm 2.2, see the file `O8+(2)/O8+2(2).log` for details. □

Lemma 4.7. *The conjugacy classes of involutions in* $O_8^+(2)$ *are of type B.*

Proof. For these five conjugacy classes we use Lemma 2.18. See Table 4.2 for details. □

Table 4.2. Involutions in $O_8^+(2)$

Class	Size	
2A	1575	$\xi(2A,3E,3E)=81$
2B	3780	$\xi(2B,3E,3E)=108$
2C	3780	$\xi(2C,3E,3E)=108$
2D	3780	$\xi(2D,3E,3E)=108$
2E	56700	$\xi(2E,3E,3E)=486$

Theorem 4.8. *The group $O_8^+(2)$ is of type B.*

Proof. By Lemmas 4.6 and 4.7, it remains to study the conjugacy classes 3A, 3B, 3C, 3D, 3E. But these classes are real, so Lemma 2.10 applies. □

4.3.3. The group $O_{10}^-(2)$. This group has order 25 015 379 558 400. It has 115 conjugacy classes. For the computations we use the representation inside $\mathbb{S}_{495}$ given in ATLAS.

Lemma 4.9. *Every conjugacy class, except the conjugacy classes named* 2A, 3A, 11A, 11B, 33A, 33B, 33C, 33D *is of type D. Notice that the conjugacy class* 2A *(resp.* 3A*) has 19635 (resp. 47872) elements.*

Proof. We use Algorithm 2.2, see the file `O10-(2)/O10-(2).log` for details. □

Theorem 4.10. *The group $O_{10}^-(2)$ is of type B.*

Proof. By Lemma 4.9 it remains to study the conjugacy classes 2A, 3A, 11A, 11B, 33A, 33B, 33C, 33D. The conjugacy class 2A is of type B, since $\xi(2A,3F,3F)=243$ and Lemma 2.18 applies. For the class 3A use Lemma 2.10, since it is a real conjugacy class. And for the classes 11A, 11B, 33A, 33B, 33C, 33D use Lemma 2.11, since the conjugacy classes 11A, 11B (resp. 33A, 33B, 33C, 33D) are quasi-real of type $j=3$ (resp. $j=4$). □

4.4. Some groups of type G_2

In this section we prove that the groups $G_2(q)$ for $q=3,4,5$ are of type B. Moreover, we will prove that almost all conjugacy classes are of type D.

4.4.1. The exceptional group $G_2(4)$. In this section we prove that the group $G_2(4)$ is of type B. This group has order 251 596 800. It has 32 conjugacy classes. In particular, the conjugacy classes with representatives of order 2 or 3 are the following:

Name	Centralizer size
2A	61440
2B	3840
3A	60480
3B	180

Lemma 4.11. *The conjugacy classes of $G_2(4)$ with representatives of order $\neq 2,3$ are of type D.*

Proof. We use Algorithm 2.3, see the file `G2(4)/G2(4).log` for details. □

Theorem 4.12. *The group $G_2(4)$ is of type B.* □

Proof. By Lemma 4.11, it remains to study the conjugacy classes with representatives of order 2 or 3. For the two conjugacy classes of involutions use Lemma 2.18, because $\xi(2\mathrm{A}, 3\mathrm{B}, 3\mathrm{B}) = 171$ and $\xi(2\mathrm{B}, 3\mathrm{A}, 3\mathrm{A}) = 126$. For the conjugacy classes with representatives of order 3 use Lemma 2.10, because these conjugacy classes are real. □

4.4.2. The exceptional groups $G_2(3)$ and $G_2(5)$. For the orders and number of conjugacy classes of the groups $G_2(3)$ and $G_2(5)$ see Table 4.3. We have the following result.

Theorem 4.13. *The groups $G_2(3)$ and $G_2(5)$ are of type D. Hence, they are of type B.*

Proof. We use Algorithm 2.3, see Table 4.3 for the log files. □

Table 4.3. Some Chevalley groups of type D

Group	Order	Conjugacy classes	Log file
$G_2(3)$	4 245 696	23	`G2(3)/G2(3).log`
$G_2(5)$	5 859 000 000	44	`G2(5)/G2(5).log`

4.5. Some symplectic groups

4.5.1. The symplectic group $S_6(2)$. This group has order 1 451 520. It has 30 conjugacy classes. For the computations we use the representation of $S_6(2)$ inside $\mathbb{S}_{28}$ given in ATLAS.

Lemma 4.14. *Every conjugacy class of $S_6(2)$, with the possible exception of* 2A, 2B, 3A *is of type D.*

Proof. We use Algorithm 2.2, see the file `S6(2)/S6(2).log` for details. □

Notice that the conjugacy class 2A (resp. 2B, 3A) has size 63 (resp. 315, 672).

Theorem 4.15. *The group $S_6(2)$ is of type B.*

Proof. By Lemma 4.14 it remains to study the classes 2A, 2B, 3A. For the conjugacy class 2A use Lemma 2.12 and Remark 2.13, see the file `S6(2)/2A`. For the conjugacy class 2B use Lemma 2.18, since $\xi(2B, 3C, 3C) = 27$. For the conjugacy class 3A use Lemma 2.10, since this conjugacy class is real. □

4.5.2. The symplectic group $S_8(2)$. This group has order 47 377 612 800. It has 81 conjugacy classes. For the computations we use the representation of $S_8(2)$ inside $\mathbb{S}_{120}$ given in ATLAS.

Lemma 4.16. *Every conjugacy class, except* 2A, 2B, 3A *is of type D.*

Proof. We use Algorithm 2.2, see the file `S8(2)/S8(2).log` for details. □

Notice that the conjugacy class 2A (resp. 2B, 3A) has size 5355 (resp. 255, 10880).

Lemma 4.17. *The conjugacy classes* 2A, 2B, 3A *are of type B.*

Proof. For the real conjugacy class 3A use Lemma 2.10. For the conjugacy classes 2A use Lemma 2.12 and Remark 2.13, see the files `S8(2)/2A` for details. For the conjugacy class 2B use Lemma 2.18, since $\xi(2B, 3C, 3C) = 135$. □

4.6. The Tits group

4.6.1. The Tits group. This group, also known as the exceptional group ${}^2F_4(2)'$ has order 17 971 200. It has 22 conjugacy classes and 8 conjugacy classes of maximal subgroups.

Lemma 4.18. *Every conjugacy class of T, with the possible exception of the class* 2A, *of size 1755, is of type D.*

Proof. We use Algorithm 2.3, see the file `T/T.log` for details. □

Theorem 4.19. *The Tits group T is of type B.*

Proof. By Lemma 4.18 it remains to study the conjugacy class 2A. But $\xi(2A, 3A, 3A) = 27$, so Lemma 2.18 applies. □

4.6.2. The automorphism group of the Tits group. Here we study Nichols algebras over the group $\mathsf{Aut}({}^2F_4(2)') \simeq \mathsf{Aut}({}^2F_4(2)) \simeq {}^2F_4(2)$, the automorphism group of the Tits group (see [GL75]). This group has order 35 942 400. It has 29 conjugacy classes.

Lemma 4.20. *Every conjugacy class of ${}^2F_4(2)$, with the exception of the class* 2A, *of size 1755, is of type D.*

Proof. We use Algorithm 2.1. See the file `2F4(2)/2F4(2).log` for details. □

Theorem 4.21. *The group ${}^2F_4(2)$ is of type B.*

Proof. By Lemma 4.20 it remains to study the conjugacy class 2A. For this conjugacy class use Lemma 2.18, because $S(2\mathrm{A}, 3\mathrm{A}, 3\mathrm{A}) = 27$. □

4.7. Some direct products

In this section we study some direct products that appear as subgroups or subquotients of some sporadic simple groups.

Lemma 4.22. *Every non-trivial conjugacy class with representative of order* $\neq 2, 3$ *of* $\mathbb{A}_9 \times \mathbb{S}_3$ *is of type D.*

Proof. Follows from Algorithm 2.1, see the file `A9xS3.log` for details. □

Lemma 4.23. *The conjugacy classes with representatives of order 9 and 18 of* $O_7(3) \times \mathbb{S}_3$ *are of type D.*

Proof. Follows from Lemmas 2.25 and 4.4. □

Lemma 4.24. *Every non-trivial conjugacy class of* $\mathbb{S}_5 \times \mathsf{PSL}(3,2)$ *with representative of order* $\neq 2, 3, 4, 6, 7$ *is of type D.*

Proof. Follows from Algorithm 2.1. See the file `S5xL3(2).log` for the computations. □

Lemma 4.25. *Every conjugacy class in* $\mathbb{A}_6 \times \mathsf{PSU}(3,3)$ *with representative of order 28, 35 is of type D.*

Proof. We use the Algorithm 2.1, see the file `A6xU3(3)/A6xU3(3).log` for details. □

Chapter 5

The sporadic groups

The aim of this chapter is to prove that any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to a sporadic simple group is the group algebra, with the possible exception of the Fischer group Fi_{22}, the Baby Monster $\mathbb{B}$ and the Monster $\mathbb{M}$. This result will be a consequence of the non-existence of finite-dimensional Nichols algebras over the sporadic simple groups studied (and, of course, the Lifting Method of Andruskiewitsch and Schneider).

5.1. Preliminaries

5.1.1. The classification Theorem of finite simple groups. First we recall the classification of finite simple groups (see [Asc00] and the references therein for details).

Classification Theorem. *Every finite simple group is isomorphic to one of the following groups:*

(1) *A cyclic group of prime order;*
(2) *An Alternating group of degree at least* 5*;*
(3) *A group of Lie type;*
(4) *One of* 26 *sporadic groups.*

Of course, in order to understand the Classification Theorem, you need to know what a "group of Lie type" and "sporadic group" mean. A group of Lie type is somehow a finite analog of a Lie group. In addition, there exists twenty-six **sporadic** finite simple groups that are not members of any reasonably defined infinite family of simple groups. The sporadic simple groups are:

- Mathieu groups: M_{11}, M_{12}, M_{22}, M_{23}, M_{24};
- Janko groups: J_1, J_2, J_3, J_4;
- Conway groups: Co_1, Co_2, Co_3;
- Fischer groups: Fi_{22}, Fi_{23}, Fi_{24};

- Higman-Sims group: HS;
- McLaughlin group: McL;
- Held group: He;
- Rudvalis group: Ru;
- Suzuki sporadic group: Suz;
- O'Nan group: ON;
- Harada-Norton group: HN;
- Lyons group: Ly;
- Thompson group: Th;
- Baby Monster group: B;
- Fischer-Griess Monster group: M;

Sometimes the Tits group T is regarded as a sporadic group (because it is not strictly a group of Lie type), in which case there are 27 sporadic groups. Nichols algebras over the Tits group T were studied in Chaper 4, Section 4.6.

5.1.2. The ATLAS of finite groups. The ATLAS of Finite Groups, often simply known as the ATLAS, is a group theory book by John Conway, Robert Curtis, Simon Norton, Richard Parker and Robert Wilson (with computational assistance from J. G. Thackray), published in 1985 (see [**CCN+85**]). It lists basic information about finite simple groups such as presentations, conjugacy classes of maximal subgroups, character tables and power maps on the conjugacy classes.

The ATLAS is being continued in the form of an electronic database: see [**WWT+**]. It currently contains information (including 5215 representations) on about 716 groups. In order to access to the information contained in the ATLAS, we use the `AtlasRep` package (see [**WPN+08**]) for GAP.

5.1.3. Conjugacy classes. With GAP and ATLAS we list real and quasi-real conjugacy classes of the sporadic simple groups studied in this chapter. See Apendix A for details.

5.2. The Mathieu groups

In this section we study Nichols algebras over the Mathieu simple groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}. This study was initiated in [**Fan07b**], where it was proved that the groups M_{22} and M_{24} are of type B. In Table 5.1 we list the order, the number of conjugacy classes and the number of conjugacy classes of maximal subgroups of each Mathieu simple group.

Lemma 5.1. *Let G be any of the Mathieu group M_{11}, M_{12}, M_{22}, M_{23}, M_{24}. Every conjugacy class of G, with the possible exception of the conjugacy classes listed in Table 5.2, is of type D.*

Proof. This was proved by Algorithm 2.3, see the logfiles listed in Table 5.2 for details. □

Table 5.1. Mathieu simple groups

Group	Order	Conjugacy Classes	Maximal subgroups
M_{11}	7920	10	5
M_{12}	95040	15	11
M_{22}	443520	12	8
M_{23}	10200960	17	7
M_{24}	244823040	26	9

Theorem 5.2. *The Mathieu simple groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24} are of type B.*

Proof. By Lemma 5.1, it remains to consider the conjugacy classes 8A, 8B, 11A, 11B of M_{11}, the classes 11A, 11B of M_{12} and M_{22}, and the classes 23A, 23B of M_{23} and M_{24}. For all the conjugacy classes 11A, 11B, 23A, 23B, use Lemma 2.11, because all of these classes are quasi-real (the conjugacy classes with representatives of order 11 are quasi-real with $j = 3$ and the classes with representatives of order 23 are quasi-real with $j = 2$). It remains to prove that the conjugacy classes 8A, 8B of M_{11} are of type B. But this follows from Lemma 2.45, since if $G = \mathbb{F}_3^2 \rtimes \langle T\rangle \simeq (C_3 \times C_3) \rtimes C_8$, there exist a monomorphism of groups ψ (resp. ψ') from G to M_{11} such that $\psi(T)$ (resp. $\psi'(T)$) belongs to the conjugacy class 8A (resp. 8B) of M_{11}. And the conjugacy classes 8A, 8B of M_{11} are quasi-real of type $j = 3$, so the result follows. □

Table 5.2. Proof of Lemma 5.1

Group	Not necessarily of type D	Logfile
M_{11}	8A, 8B, 11A, 11B	`M11/M11.log`
M_{12}	11A, 11B	`M12/M12.log`
M_{22}	11A, 11B	`M22/M22.log`
M_{23}	23A, 23B	`M23/M23.log`
M_{24}	23A, 23B	`M24/M24.log`

As a by-product we prove the following theorem about the Nichols algebras over the automorphism group of the Mathieu groups M_{12} and M_{22}. These groups appear as a maximal subgroups of other sporadic simple groups.

The automorphism group of M_{12} has 21 conjugacy classes and 9 conjugacy classes of maximal subgroups. With the representation inside $\mathbb{S}_{24}$ it is easy to reach the following result.

Theorem 5.3. *The automorphism group* $\mathsf{Aut}(M_{12}) \simeq M_{12} : 2$ *is of type D. Hence, it is of type B.*

Proof. Use Algorithm 2.3, see the file `M12/M12.2.log` for details. □

The automorphism group of M_{22} has 21 conjugacy classes and 7 conjugacy classes of maximal subgroups. With the representation inside $\mathbb{S}_{22}$ it is easy to reach the following result, useful for studying Nichols algebras over the Janko group J_4.

Lemma 5.4. *Every conjugacy class of the group* $\mathsf{Aut}(M_{22}) \simeq M_{22} : 2$*, with the possible exception of the conjugacy class* 2B*, with centralizer of order 2688, is of type D.*

Proof. Use Algorithm 2.3, see the file `M22/M22.2.log` for details. □

5.3. The Higman-Sims group HS

The Higman-Sims simple group has order 44 352 000. It has 24 conjugacy classes and 12 conjugacy classes of maximal subgroups.

Lemma 5.5. *The conjugacy classes of* HS*, with the possible exception of the classes* 11A, 11B*, are of type D.*

Proof. We use Algorithm 2.3, see the file `HS/HS.log` for details. □

Theorem 5.6. *The Higman-Sims group* HS *is of type B.*

Proof. By Lemma 5.5 it remains to consider the conjugacy classes 11A, 11B. But these conjugacy classes are quasi-real of type $j = 3$, so the result follows from Lemma 2.11. □

The following Lemma is useful for studying Nichols algebras over the Conway group Co_1 (see Section 5.11.3). We study Nichols algebras over the group $2.HS.2$. This groups has order 177 408 000 and has 57 conjugacy classes. We have no information about the representations of its maximal subgroups.

Lemma 5.7. *Every conjugacy class of* $2.HS.2$*, with the exception of the conjugacy class* 2A*, of size 1, is of type D.*

Proof. We use Algorithm 2.2. For details, see the file `HS/2.HS.2.log`. □

5.4. The Held group He

The Held group He has order 4 030 387 200. It has 33 conjugacy classes and 11 conjugacy classes of maximal subgroups. The automorphism group of the Held group $\mathsf{Aut}(He) \simeq He : 2$ has 45 conjugacy classes and 12 conjugacy classes of maximal subgroups.

Theorem 5.8. *The groups* He *and* $\mathsf{Aut}(He)$ *are of type D.*

Proof. We use Algorithm 2.3, see the files `He/He.log` and `He/He.2.log` for details. □

5.5. The Suzuki group Suz

5.5.1. The group Suz**.** The Suzuki group Suz has order 448 345 497 600. It has 43 conjugacy classes and 17 conjugacy classes of maximal subgroups.

Lemma 5.9. *Every conjugacy class of* Suz*, with the possible exception of the class* 3A*, of size 4576, is of type D.*

Proof. We use Algorithm 2.1, see the file `Suz/Suz.log` for details. □

Theorem 5.10. *The Suzuki group Suz is of type B.*

Proof. By Lemma 5.9 it remains to study the conjugacy class 3A. But this class is real, so the result follows from Lemma 2.10. □

5.5.2. The automorphism group $\mathsf{Aut}(Suz)$. The automorphism group $\mathsf{Aut}(Suz) \simeq Suz : 2$ has 68 conjugacy classes and 16 conjugacy classes of maximal subgroups. For the computations we use a representation inside $\mathbb{S}_{1782}$.

Lemma 5.11. *Every conjugacy class, with the possible exception of* 3A, *of size 45760, is of type D.*

Proof. We use Algorithm 2.3, see the file `Suz/Suz.2.log` for details. □

Theorem 5.12. *The automorphism group $\mathsf{Aut}(Suz)$ is of type B.*

Proof. By Lemma 5.11 it remains to study the conjugacy class 3A. But this class is real, so Lemma 2.10 applies. □

The following Lemma is useful for studying Nichols algebras over the Conway group Co_1 (see Section 5.11.3). We study some conjugacy classes of the covering group named $3.Suz.2$. This groups has 106 conjugacy classes. We have no information about the representations of its maximal subgroups.

Lemma 5.13. *The conjugacy classes of the group $3.Suz.2$ with representatives of order 7, 8, 10, 13, 15, 21, 30, 33, 39, 42 are of type D.*

Proof. We use Algorithm 2.2, see the file `Suz/3.Suz.2.log` for details. □

5.6. The O'Nan group ON

This group has order 460 815 505 920. It has 30 conjugacy classes and 13 conjugacy classes of maximal subgroups.

Lemma 5.14. *All the conjugacy classes of ON, with the possible exception of the classes* 31A, 31B *are of type D.*

Proof. We use Algorithm 2.3, see the file `ON/ON.log` for details. □

Theorem 5.15. *The O'Nan group ON is of type B.*

Proof. The result follows from Lemmas 5.14 and 2.11, because the conjugacy classes 31A, 31B are quasi-real with $j = 2$. □

5.7. The MacLaughlin group McL

This groups has order 898 128 000. It has 24 conjugacy classes and 12 conjugacy classes of maximal subgroups.

Lemma 5.16. *Every conjugacy class of McL, with the possible exception of* 11A, 11B *is of type D.*

Proof. We use Algorithm 2.3, see the file `McL/McL.log` for details. □

Theorem 5.17. *The MacLaughlin group McL is of type B.*

Proof. The result follows from Lemmas 5.16 and 2.10, because the conjugacy classes 11A, 11B are real. □

5.8. The Rudvalis group Ru

This group has order 145 926 144 000. It has 36 conjugacy classes and 15 conjugacy classes of maximal subgroups.

Lemma 5.18. *Every conjugacy class of Ru, with the possible exception of the classes* 29A, 29B *is of type D.*

Proof. We use Algorithm 2.3, see the file `Ru/Ru.log` for details. □

Theorem 5.19. *The Rudvalis group Ru is of type B.*

Proof. The result follows from Lemmas 5.18 and 2.10, because the conjugacy classes 29A, 29B are real. □

5.9. The Fischer group Fi_{22}

This group has order 64 561 751 654 400. It has 65 conjugacy classes and 14 conjugacy classes of maximal subgroups.

Lemma 5.20. *Every conjugacy class of the Fischer group, with the possible exception of* 2A, 22A, 22B *is of type D.*

Proof. We use Algorithm 2.3, see the file `Fi22/Fi22.log` for details. □

Lemma 5.21. *The conjugacy class* 2A*, of size* 3510*, of the Fischer group Fi_{22} is of type B.*

Proof. Follows from Theorem 4.5, since $O_7(3)$ is a maximal subgroup of Fi_{22}. For the fusion of conjugacy classes $O_7(3) \to Fi_{22}$ see the file `Fi22/2A`. □

Remark 5.22. *We cannot prove that the Fischer group F_{22} is of type B. In fact, we cannot prove that the classes* 22A, 22B *are of type B.*

5.10. The Janko groups

5.10.1. The groups J_1, J_2, J_3. In this subsection we study Nichols algebras over the first three Janko groups: J_1, J_2, J_3. In Table 5.3 we list the order, the number of conjugacy classes and the number of conjugacy classes of maximal subgroups of each of these Janko groups.

Table 5.3. The first three Janko groups

Group	Order	Conjugacy Classes	Maximal subgroups
J_1	175560	15	7
J_2	604800	21	9
J_3	50232960	21	9

Lemma 5.23. *Let G be any of the Janko group J_1, J_2, J_3. Every conjugacy class of G, with the possible exception of the conjugacy classes listed in Table 5.4, is of type D.*

Proof. This was proved by the Algorithm 2.3, see the logfiles listed in Table 5.4 for details. □

Table 5.4. Proof of Lemma 5.23

Group	Not necessarily of type D	Logfile
J_1	15A, 15B, 19A, 19B, 19C	`J1/J1.log`
J_2	2A, 3A	`J2/J2.log`
J_3	5A, 5B, 19A, 19B	`J3/J3.log`

Theorem 5.24. *The Janko groups J_1, J_2, J_3 are of type B.*

Proof. By Lemma 5.23, it remains to consider the conjugacy classes 15A, 15B, 19A, 19B, 19C of J_1, the classes 2A, 3A of J_2 and the classes 5A, 5B, 19A, 19B of J_3. For the Janko group J_1, use Lemma 2.10, because all the conjugacy classes of J_1 are real. For the conjugacy class 2A of J_2 notice that $\xi(2A, 3B, 3B) = 18$ and apply Lemma 2.18. Also, by Lemma 2.10, the conjugacy class 3A of J_2 is also of type B. For the conjugacy classes 5A, 5B of J_3 apply Lemma 2.10, since these classes are real. For the conjugacy classes 19A, 19B of J_3 use Lemma 2.11, since these classes are quasi-real with $j = 4$. □

5.10.2. The Janko group J_4. The Janko group J_4 has order

$$86\,775\,571\,046\,077\,562\,880.$$

It has 62 conjugacy classes. This group is too big to use a script as in the previous cases. Indeed, the previous scripts fail because the best representation of the group is inside $\mathsf{GL}(112, 2)$ and this is not good for computations. However, some of the maximal subgroups have good enough representations and we work with them on a case by case basis. The list of (representatives of conjugacy classes of) maximal subgroups is:

$2^{11} : M_{24}$	$M_{22}.2$	$U_3(3)$
$2^{1+11}.3.M_{22} : 2$	$29 : 28 = F_{812}$	$43 : 14 = F_{602}$
$2^{10} : L_5(2)$	$11_+^{1+2} : (5 \times 2\mathbb{S}_4)$	$37 : 12 = F_{444}$
$2^{3+12}.(\mathbb{S}_5 \times L_3(2))$	$L_2(32).5$	
$U_3(11).2$	$L_2(32).2$	

See the file `J4/fusions` for the fusion of conjugacy classes. We split the proof into several Lemmas.

Lemma 5.25. *The conjugacy classes* 3A, 5A, 6A, 6B, 6C, 7A, 7B, 10A, 10B, 12A, 12B, 12C, 14A, 14B, 14C, 14D, 15A, 16A, 20A, 20B, 21A, 21B, 24A, 24B, 28A, 28B, 30A *of J_4 are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_1 \simeq 2^{11} : M_{24}$. Consider the short exact sequence $0 \to 2^{11} \to 2^{11} : M_{24} \to M_{24} \to 0$. By Lemma 5.1, every conjugacy class of M_{24} with representative of order $\neq 23$ is of type D. By Lemma 2.24, every conjugacy class in $2^{11} : M_{24}$ with representative of order $\neq 1, 2, 4, 8, 16, 23$ is of type D. Hence the conjugacy classes 3A, 5A, 6A, 6B, 6C, 7A, 7B, 10A, 10B, 12A, 12B, 12C, 14A, 14B, 14C, 14D, 15A, 20A, 20B, 21A, 21B, 22B, 24A, 24B, 30A of J_4 are of type D. Also, this maximal subgroup has a primitive permutation representation on 2^{11} points. We use the GAP's function `PrimitiveGroup` and Algorithm 2.2 to see that the conjugacy class 16A of J_4 is of type D (see the file `J4/step1.log`). □

Lemma 5.26. *The conjugacy classes* 35A, 35B, 42A, 42B *of* J_4 *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_4 \simeq 2^{3+12}.(\mathbb{S}_5 \times L_3(2))$. Consider the short exact sequence $0 \to 2^{3+12} \to 2^{3+12}.(\mathbb{S}_5 \times L_3(2)) \to L_3(2) \to 0$. Every conjugacy class of $\mathbb{S}_5 \times L_3(2)$ with representative of order $\neq 2, 3, 4, 6, 7$ is of type D (see Lemma 4.24). Then, by Lemma 2.23, the result follows. □

Lemma 5.27. *The conjugacy classes* 4B, 8C *of* J_4 *are of type D.*

Proof. The result follows from Lemma 5.4, because $\mathsf{Aut}(M_{22}) \simeq M_{22} : 2$ is a maximal subgroup of J_4. □

Lemma 5.28. *The conjugacy classes*

(1) 8A, 8B, 11A, 11B;

(2) 4A, 22A, 22B, 40A, 40B, 44A, 66A, 66B;

(3) 31A, 31B, 31C, 33A, 33B;

(4) 2A, 2B, 23A

of J_4 *are of type D.*

Proof. For all of these conjugacy classes we use Algorithms 2.1, 2.2 in some maximal subgroup of J_4. See Table 5.5 for details. □

Table 5.5. Proof of Lemma 5.28

	Maximal subgroup	Logfile
1	$U_3(11).2$	`J4/step3.log`
2	$11_+^{1+2} : (5 \times 2\mathbb{S}_4)$	`J4/step5.log`
3	$L_2(32).5$	`J4/step6.log`
4	$L_2(23).2$	`J4/step7.log`

Lemma 5.29. *The conjugacy class* 4C *of* J_4 *is of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_{13} \simeq 37 : 12$. By the fusion of conjugacy classes, the classes 4a and 4b of $\mathcal{M}_{13}$ go to the conjugacy class 4C of J_4. Then we find r in 4a, s in 4b such that $(rs)^2 \neq (sr)^2$ (see `J4/step8.log`) and the result follows from Proposition 2.22. Notice that the conjugacy classes written in lowercase letters are not named as in ATLAS. □

Theorem 5.30. *The Janko group J_4 is of type B.*

Proof. It remains to study the conjugacy classes 29A, 37A, 37B, 37C, 43A, 43B, 43C. These conjugacy classes are real, so Lemma 2.10 applies. □

5.11. The Conway groups

5.11.1. The group Co_3. The Conway group Co_3 has order 495 766 656 000. It has 42 conjugacy classes and 14 conjugacy classes of maximal subgroups.

Lemma 5.31. *Every conjugacy class of Co_3, with the possible exception of the classes* 23A, 23B *is of type D.*

Proof. We use the Algorithm 2.3. See the file `Co3/Co3.log` for details. □

Theorem 5.32. *The Conway group Co_3 is of type B.*

Proof. By Lemma 5.31 it remains to study the conjugacy classes 23A, 23B. But these classes are quasi-real with $j = 2$, hence the result follows from Lemma 2.11. □

5.11.2. The group Co_2. The Conway group Co_2 has order

$$42\,305\,421\,312\,000.$$

It has 60 conjugacy classes and 11 conjugacy classes of maximal subgroups.

Lemma 5.33. *Every conjugacy class of Co_2, with the possible exception of the classes* 2A, 23A, 23B *is of type D.*

Proof. We use the Algorithm 2.3. See the file `Co2/Co2.log` for details. □

Theorem 5.34. *The Conway group Co_2 is of type B.*

Proof. By Lemma 5.33 it remains to study the conjugacy classes 2A, 23A, 23B. For the conjugacy class 2A, use Lemma 2.12 (see the file `Co2/2A`). For the conjugacy classes 23A, 23B, use Lemma 2.11, since both classes are quasi-real with $j = 2$. □

5.11.3. The group Co_1. This group has order

$$4\,157\,776\,806\,543\,360\,000.$$

It has 101 conjugacy classes, all real except 23A, 23B, 39A, 39B (quasi-real with $j = 2$ and $g^4 \neq g$). The list of (representatives of conjugacy classes of) maximal subgroups is:

Co_2	$2^{4+12}.(\mathbb{S}_3 \times 3\mathbb{S}_6)$	$(\mathbb{A}_7 \times L_2(7)) : 2$
$3.Suz.2$	$3^2.U_4(3).\mathbb{D}_8$	$(\mathbb{D}_{10} \times (\mathbb{A}_5 \times \mathbb{A}_5).2).2$
$2^{11}.M_{24}$	$3^6 : 2M_{12}$	$5^{1+2} : \mathsf{GL}(2,5)$
Co_3	$(\mathbb{A}_5 \times J_2) : 2$	$5^3 : (4 \times \mathbb{A}_5).2$
$2_+^{1+8}.O_8^+(2)$	$3^{1+4}.2U_4(2).2$	$7^2 : (3 \times 2\mathbb{A}_4)$
$U_6(2) : \mathbb{S}_3$	$(\mathbb{A}_6 \times U_3(3)) : 2)$	$5^2 : 2\mathbb{A}_5$
$(\mathbb{A}_4 \times G_2(4)) : 2$	$3^{3+4} : 2(\mathbb{S}_4 \times \mathbb{S}_4)$	
$2^{2+12} : (\mathbb{A}_8 \times \mathbb{S}_3)$	$\mathbb{A}_9 \times \mathbb{S}_3$	

See the file `Co1/fusions` for the fusion of conjugacy classes. We split the proof into several Lemmas.

Lemma 5.35. *The conjugacy classes with representatives of order 2, 18, 24, 36, 40, 60 are of type D. Also, the conjugacy class* 20A *is of type D.*

Proof. For the conjugacy classes of involutions use Example 2.29. For the conjugacy classes 20A, 36A, 40A, 60A use Lemma 4.6 and Lemma 2.24. For all the other conjugacy classes, we use the Algorithm 2.2 in the maximal subgroup $2_+^{1+8}.O_8^+(2)$ to see that these conjugacy classes are of type D (see the file `Co1/step7.log` for details). □

Lemma 5.36. *The conjugacy classes* 3B, 3C, 4A, 4B, 4C, 4D, 4F, 5B, 5C, 6C, 6D, 6E, 6F, 6G, 6I, 7B, 8B, 8C, 8D, 8E, 9B, 9C, 10D, 10E, 10F, 11A, 14B, 15D, 15E, 16A, 16B, 20B, 20C, 21C, 22A, 28A, 30D, 30E *of Co_1 are of type D.*

Proof. The result follows from Lemmas 5.33 and 5.31, because Co_2 and Co_3 are both maximal subgroups of Co_1. □

Lemma 5.37. *The conjugacy classes* 4E, 5A, 6A, 6B, 6H, 9A, 9B, 10A, 10B, 15A, 15C, 30A, 30C *of Co_1 are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_{16} \simeq \mathbb{A}_9 \times \mathbb{S}_3$. By Lemma 4.22 every conjugacy class of $\mathbb{A}_9 \times \mathbb{S}_3$ with representative of order $\neq 2, 3$ is of type D. Then the result follows. □

Lemma 5.38. *The conjugacy classes* 7A, 8A, 8F, 10C, 13A, 15B, 21A, 21B, 30B, 33A, 39A, 39B, 42A *of Co_1 are of type D.*

Proof. The result follows from Lemma 5.13, because the group $3.Suz.2$ is a maximal subgroup of Co_1. □

Lemma 5.39. *The conjugacy classes of Co_1 with representatives of order 12 are of type D.*

Proof. For the conjugacy classes 12L, 12M we use the maximal subgroup $\mathcal{M}_{16} \simeq \mathbb{A}_9 \times \mathbb{S}_3$, because in this subgroup every conjugacy class with representative of order 12 is of type D (see Lemma 4.22). And for remaining conjugacy classes we use Algorithm 2.2 in the maximal subgroup $2^{1+8}.O_8^+(2)$ (see the file `Co1/step7.log` for details). □

Lemma 5.40. *The conjugacy classes*

(1) 14A, 26A;

(2) 28B, 35A

of Co_1 are of type D.

Proof. For the conjugacy classes 14A, 26A, use the subgroup $\mathbb{A}_4 \times G_2(4)$ of the maximal subgroup $(\mathbb{A}_4 \times G_2(4)) : 2$ of Co_1. By Lemma 4.11, every conjugacy class of $G_2(4)$ with representative of order 7, 13 is of type D. Then, since every non-trivial element of $\mathbb{A}_4$ has order 2 or 3, the result follows from Lemma 2.25. Notice that this maximal subgroup has two conjugacy classes with representative of order 14 and these classes go to the class 14A of Co_1.

For the conjugacy classes 28B, 35A use the subgroup $\mathbb{A}_6 \times \mathsf{PSU}(3,3)$ of the maximal subgroup $(\mathbb{A}_6 \times \mathsf{PSU}(3,3)) : 2$. Then, the result follows from Lemma 4.25. Notice that this maximal subgroup has only one conjugacy class with representative of order 28 and this class goes to the class 28B of Co_1. □

Lemma 5.41. *The conjugacy classes* 3D *of* Co_1 *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_3 \simeq 2^{11} : M_{24}$. Consider the short exact sequence $0 \to 2^{11} \to 2^{11} : M_{24} \to M_{24} \to 0$. By Lemma 5.1, every conjugacy class of M_{24} with representative of order $\neq 23$ is of type D. Then, the result follows from Lemma 2.24. □

Theorem 5.42. *The Conway group* Co_1 *is of type B.*

Proof. It remains to study the classes 3A, 23A, 23B. For the *real* conjugacy class 3A use Lemma 2.10. The conjugacy classes 23A, 23B are quasi-real with $j = 2$, so Lemma 2.11 applies. □

5.12. The Lyons group Ly

The Lyons group Ly has order 51 765 179 004 000 000. It has 53 conjugacy classes. The list of (representatives of conjugacy classes of) maximal subgroups is:

$G_2(5)$	$2.\mathbb{A}_{11}$	$3^{2+4} : 2.\mathbb{A}_5.\mathbb{D}_8$
$3.McL : 2$	$5^{1+4} : 4.\mathbb{S}_6$	$67 : 22 = F_{1474}$
$5^3.L_3(5)$	$3^5 : (2 \times M_{11})$	$37 : 18$

See the file `Ly/fusions` for the fusion of conjugacy classes. We split the proof into several Lemmas.

Lemma 5.43. *The conjugacy classes* 9A, 14A, 18A, 28A, 40A, 40B, 42A, 42B *of* Ly *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_4 \simeq 2.\mathbb{A}_{11}$. Consider the short exact sequence $0 \to 2 \to 2.\mathbb{A}_{11} \to \mathbb{A}_{11} \to 0$. By Corollary 3.10, every conjugacy class of $\mathbb{A}_{11}$ with representative of order $\neq 3, 11$ is of type D. Therefore, the result follows from Lemma 2.24. □

Lemma 5.44. *The conjugacy classes* 2A, 3A, 3B, 4A, 5A, 5B, 6A, 6B, 6C, 7A, 8A, 8B, 10A, 10B, 12A, 12B, 15A, 15B, 15C, 20A, 21A, 21B, 24A 24B, 24C, 25A, 30A, 30B, 31A, 31B, 31C, 31D, 31E *of* Ly *are of type D.*

Proof. The result follows from Thereom 4.13, because $G_2(5)$ is a maximal subgroup. □

Lemma 5.45. *The conjugacy classes* 11A, 11B, 22A, 22B *of* Ly *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_6 \simeq 3^5 : (2 \times M_{11})$. We construct a permutation representation of $\mathcal{M}_6$ and apply Algorithm 2.1 to see that the conjugacy classes of $\mathcal{M}_6$ with representatives of order 11, 22 are of type D (see the file `Ly/step3.log`). Therefore the conjugacy classes 11A, 11B, 22A, 22B of Ly are of type D. □

Theorem 5.46. *The group Ly is of type B.*

Proof. It remains to consider the conjugacy classes 33A, 33B, 37A, 37B, 67A, 67B, 67C. The conjugacy classes 37A, 37B, 67A, 67B, 67C are real, so Lemma 2.10 applies. The conjugacy classes 33A, 33B are quasi-real with $j = 4$ and $g^{16} \neq g$, so Lemma 2.11 applies. Therefore, the result follows. □

5.13. The Fischer group Fi_{23}

This group has order

$$4\,089\,470\,473\,293\,004\,800.$$

It has 98 conjugacy classes. The list of (representatives of conjugacy classes of) maximal subgroups is:

$2.Fi_{22}$	$2^{11}.M_{23}$	$2^{6+8}:(\mathbb{A}_7 \times \mathbb{S}_3)$
$O_8^+(3).3.2$	$3^{1+8}.2^{1+6}.3^{1+2}.2\mathbb{S}_4$	$\mathbb{S}_4 \times S_6(2)$
$2^2.U_6(2)$	$[3^{10}].(L_3(3) \times 2)$	$S_4(4).4$
$S_8(2)$	$\mathbb{A}_{12}.2$	$L_2(23)$
$\mathbb{S}_3 \times O_7(3)$	$(2^2 \times 2^{1+8}).(3 \times U_4(2)).2$	

See the file `Fi23/fusions` for the fusion of conjugacy classes. We split the proof into several Lemmas.

Lemma 5.47. *Every conjugacy class of Fi_{23} with representative of order 3, 5, 6, 7, 8, 10, 11, 13, 14, 16, 20, 21, 26, 42 is of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_1 \simeq 2.Fi_{22}$. Consider the short exact sequence $0 \to 2 \to 2.Fi_{22} \to Fi_{22} \to 0$. By Lemma 5.20, every conjugacy class of Fi_{22} with representative of order $\neq 2, 22$ is of type D. Then, by Lemma 2.24, every conjugacy class in $2.Fi_{22}$ with representative of order $\neq 2, 22$ is of type D. Hence, the result follows. □

Lemma 5.48. *The conjugacy classes* 2B, 2C, 4A, 4B, 4C, 4D, 15A, 15B, 17A *of Fi_{23} are of type D. Also, the conjugacy class* 2A *is of type B.*

Proof. Follows from Lemma 4.16 and the fusion of conjugacy classes, because $S_8(2)$ is a maximal subgroup of Fi_{23}. □

Lemma 5.49. *Every conjugacy class of Fi_{23} with representative of order 12, 39, 60 is of type D. Also, the conjugacy classes* 9B, 9C, 9D, 9E, 18A, 18B, 18C, 18E, 18F, 18H, 39A, 39B *are of type D.*

Proof. Follows from Lemma 4.23 and the fusion of conjugacy classes, because $O_7(3) \times \mathbb{S}_3$ is a maximal subgroup of Fi_{23}. □

Lemma 5.50. *The conjugacy classes*

(1) 22A, 22B, 22C;
(2) 27A;
(3) 9A, 18D, 18G, 24A, 24B, 24C, 36A;
(4) 36B.

of Fi_{23} are of type D.

Proof. We use Algorithms 2.1 or 2.2 in a suitable maximal subgroup and then the result follows from the fusion of conjugacy classes. See Table 5.6 for the details about the maximal subgroups used and the logfiles. □

Table 5.6. Proof of Lemma 5.50

	Maximal subgroup	Logfile
1	$2^2.U_6(2).2$	`Fi23/step4.log`
2	$O_8^+(3):\mathbb{S}_3$	`Fi23/step5.log`
3	$(2^2\times 2^{1+8}).(3\times U_4(2)).2$	`Fi23/step6.log`
4	$\mathbb{S}_4\times S_6(2)$	`Fi23/step7.log`

Lemma 5.51. *The conjugacy classes* 28A, 30A, 30B, 30C, 35A *of* Fi_{23} *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_9\simeq\mathbb{S}_{12}$. By Corollary 3.10, every conjugacy class with representative of order 28, 30, 35 is of type D. Hence, the result follows. □

Theorem 5.52. *The Fischer group* Fi_{23} *is of type B.*

Proof. It remains to study the classes 23A, 23B. And these conjugacy classes are quasi-real of type $j=2$, so the result follows Lemma 2.11. □

5.14. The Thompson group Th

This group has order

$$90\,745\,943\,887\,872\,000$$

It has 48 conjugacy classes. The list of (representatives of conjugacy classes of) maximal subgroups is:

${}^3D_4(2).3$	$3^2.3^3.3^2.3^2:2\mathbb{S}_4$	$\mathsf{PSL}(3,3)$
$2^5.\mathsf{PSL}(5,2)$	$3^5:2\mathbb{S}_6$	$\mathbb{A}_6.2_3$
$2^{1+8}.\mathbb{A}_9$	$5^{1+2}:4\mathbb{S}_4$	$31:15=F_{465}$
$U_3(8).6$	$5^2:4\mathbb{S}_5$	$\mathbb{A}_5.2$
$(3\times G_2(3)):2$	$7^2:(3\times 2\mathbb{S}_4)$	
$3.3^2.3.(3\times 3^2).3^2:2\mathbb{S}_4$	$\mathsf{PSL}(2,19).2$	

See the file `Th/fusions` for the fusion of conjugacy classes. We split the proof into several Lemmas.

Lemma 5.53. *The conjugacy classes* 5A, 7A, 10A, 14A, 15A, 15B, 18A, 18B, 20A, 28A, 30A, 30B, 36A, 36B, 36C *of* Th *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_3\simeq 2^{1+8}.\mathbb{A}_9$. Consider the short exact sequence $0\to 2^{1+8}\to 2^{1+8}.\mathbb{A}_9\to\mathbb{A}_9\to 0$. By Corollary 3.10, every conjugacy class of $\mathbb{A}_9$ not of type (3) is of type D. Then, by Lemma 2.24, the result follows. □

Lemma 5.54. *The conjugacy classes* 3A, 3C, 12D, 21A, 24A, 24B *of* Th *are of type D*

Proof. We use the maximal subgroup $\mathcal{M}_2 \simeq 2^5.\mathsf{PSL}(5,2)$. Consider the exact sequence $0 \to 2^5 \to 2^5.\mathsf{PSL}(5,2) \to \mathsf{PSL}(5,2) \to 0$. In $\mathsf{PSL}(5,2)$ every conjugacy class with representative of order $\neq 31$ is of type D (see Lemma 4.2). Therefore, the result follows from Lemma 2.24. □

Lemma 5.55. *The conjugacy classes*

(1) 2A, 3B, 19A;

(2) 4B, 6A, 6B, 6C, 8B, 9A, 9B, 9C, 24C, 24D, 27A, 27B, 27C;

(3) 4A, 8A, 12A, 12B, 12C

of Th are of type D.

Proof. For all of these conjugacy classes we use the Algorithm 2.1 in some maximal subgroup of Th. Notice that instead of using the matrix representation of the maximal subgroups inside $\mathsf{GL}(248,2)$, with GAP we construct permutation representations for the maximal subgroups listed in Table 5.7. See the logfiles for details. □

Table 5.7. Proof of Lemma 5.55

	Maximal subgroup	Logfile
1	$\mathsf{PSL}(2,19).2$	`Th/step3.log`
2	$3^2.3^3.3^2.3^2 : 2\mathbb{S}_4$	`Th/step6.log`
3	$3.3^2.3.(3 \times 3^2).3^2 : 2\mathbb{S}_4$	`Th/step5.log`

Lemma 5.56. *The conjugacy classes* 13A, 39A, 39B *of Th are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_5 \simeq (3 \times G_2(3)) : 2$. In $G_2(3)$ every conjugacy class with representative of order 13 is of type D (see Theorem 4.13). Therefore, the conjugacy class 13A of Th is of type D. Also, by Proposition 2.25 the conjugacy classes 39A, 39B of Th are of type D (for the fusion of conjugacy classes $3 \times G_2(3) \to (3 \times G_2(3)) : 2$ see the file `Th/step4.g`). □

Lemma 5.57. *The conjugacy classes* 31A, 31B *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_2 \simeq 2^5.\mathsf{PSL}(5,2)$. In this group there are six conjugacy classes with representatives of order 31. Notice that three of these classes go to the class named 31A of Th and the other three go to 31B. Let $\mathcal{A}$ and $\mathcal{B}$ be two conjugacy classes of $\mathcal{M}_2$ that go to the class 31A (resp. 31B) of Th. Then, it is possible to find $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $(ab)^2 \neq (ba)^2$ and the result follows. For details, see the file `Th/step2.log`. □

Theorem 5.58. *The Thompson group Th is of type D.* □

5.15. The Harada-Norton group HN

This group has order

$$273\,030\,912\,000\,000.$$

It has 54 conjugacy classes. The list of (representatives of conjugacy classes of) maximal subgroups is:

$\mathbb{A}_{12}$	$5^{1+4}:2^{1+4}.5.4$	$M_{12}.2$
$2.HS.2$	$2^6.U_4(2)$	$M_{12}.2$
$U_3(8).3_1$	$(\mathbb{A}_6\times\mathbb{A}_6).\mathbb{D}_8$	$3^4:2(\mathbb{A}_4\times\mathbb{A}_4).4$
$2^{1+8}.(\mathbb{A}_5\times\mathbb{A}_5).2$	$2^3.2^2.2^6.(3\times\mathsf{PSL}(3,2))$	$3^{1+4}:4\mathbb{A}_5$
$(\mathbb{D}_{10}\times U_3(5)).2$	$5^2.5.5^2.4\mathbb{A}_5$	

Lemma 5.59. *The conjugacy classes* 2A, 2B, 5A, 5E, 6A, 6B, 6C, 7A, 9A, 11A, 15A, 20C, 21A, 30A, 35A, 35B *of HN are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_1\simeq\mathbb{A}_{12}$. By Corollary 3.10, in this maximal subgroup every conjugacy class with representative of order $\neq 3,11$ is of type D. Therefore, the result follows for every conjugacy class with representative of order $\neq 11$. It remains to prove that the conjugacy class 11A of HN is of type D. For that purpose, let $r=(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)$ and $s=(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 11\ 10)$ be elements in $\mathbb{A}_{12}$. It is easy to see that $(rs)^2\neq(sr)^2$ and that r and s belong to different conjugacy classes in the group $\langle r,s\rangle\simeq\mathbb{A}_{11}$. Then, by Proposition 2.22 and the fusion of conjugacy classes, the conjugacy class 11A of HN is of type D. □

Lemma 5.60. *The conjugacy classes*

(1) 3A, 3B, 4A, 4B, 4C, 12C;

(2) 5B, 8A, 8B, 10A, 10B, 10C, 10F, 10G, 10H, 12A, 12B, 14A, 20A, 20B, 22A, 40A, 40B

of HN are of type D.

Proof. The first item follows from Theorem 5.3, because $\mathsf{Aut}(M_{12})\simeq M_{12}.2$ is a maximal subgroup. The second follows from Lemma 5.7, because $2.HS.2$ is also a maximal subgroup of HN. □

Lemma 5.61. *The conjugacy classes*

(1) 5C, 5D, 10D, 10E, 15B, 15C, 20D, 20E, 30B, 30C;

(2) 25A, 25B;

(3) 19A, 19B

of HN are of type D.

Proof. We use the Algorithm 2.2 in some maximal subgroup of HN. We have used the representation of HN inside $\mathsf{GL}(760,2)$ given in ATLAS for the first two items. For the last item we used a representation of the maximal subgroup $U_3(8).3_1$ inside $\mathsf{GL}(133,5)$. See Table 5.8 for details. □

Theorem 5.62. *The Harada-Norton group HN is of type D, hence it is of type B.* □

Table 5.8. Proof of Lemma 5.61

	Maximal subgroup	Logfile
1	$3^{1+4}:4\mathbb{A}_5$	`HN/step4.log`
2	$5^{2+1+2}.4.\mathbb{A}_5$	`HN/step6.log`
3	$U_3(8).3_1$	`HN/step5.log`

5.16. The Fischer group Fi'_{24}

This group has order

$$1\,255\,205\,709\,190\,661\,721\,292\,800.$$

It has 108 conjugacy classes. The list of (representatives of conjugacy classes of) maximal subgroups is (see [LW91]):

Fi_{23}	$3^3.[3^{10}].\mathsf{GL}(3,3)$	$\mathbb{A}_6\times\mathsf{PSL}(2,8):3$
$2.Fi_{22}.2$	$3^2.3^4.3^8.(\mathbb{A}_5\times 2\mathbb{A}_4).2$	$7:6\times\mathbb{A}_7$
$(3\times O_8^+(3):3):2$	$(\mathbb{A}_4\times O_8^+(2).3).2$	$U_3(3).2$
$O_{10}^-(2)$	$He:2$	$U_3(3).2$
$3^7.O_7(3)$	$He:2$	$\mathsf{PSL}(2,13).2$
$3^{1+10}:U_5(2):2$	$2^{3+12}.(\mathsf{PSL}(3,2)\times\mathbb{A}_6)$	$\mathsf{PSL}(2,13).2$
$2^{11}.M_{24}$	$2^{6+8}.(\mathbb{S}_3\times\mathbb{A}_8)$	$29:14=F_{406}$
$2^2.U_6(2).3.2$	$(3^2:2\times G_2(3)).2$	
$2^{1+12}.3_1.U_4(3).2_2$	$(\mathbb{A}_5\times\mathbb{A}_9):2$	

Lemma 5.63. *The conjugacy classes* 3A, 3B, 3C, 3D, 4A, 4B, 4C, 5A, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 7A, 8A, 8B, 9A, 9B, 9C, 9E, 9F, 10A, 10B, 11A, 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12K, 12L, 12M, 13A, 14A, 15A, 15C, 16A, 17A, 18A, 18B, 18C, 18D, 18E, 18F, 20A, 21A, 22A, 24A, 24B, 24E, 26A, 27A, 28A, 30A, 30B, 35A, 36C, 36D, 39A, 39B, 42A, 60A *of* Fi'_{24} *are of type D.*

Proof. We use the maximal subgroup $\mathcal{M}_1\simeq Fi_{23}$. By Section 5.13, we know that every conjugacy class of Fi_{23} with representative of order $\neq 2,23$ is of type D (see Section 5.13). Hence, the result follows. ☐

Lemma 5.64. *The conjugacy classes* 2A, 2B, 3E, 6K, 7B, 12I, 12J, 14B, 21B, 21C, 21D, 24C, 24D, 42B, 42C *of* Fi'_{24} *are of type D.*

Proof. The result follows from Theorem 5.8 and the fusion of conjugacy classes, because Fi'_{24} has two conjugacy classes of maximal subgroups isomorphic to $\mathsf{Aut}(He)\simeq He.2$. ☐

Lemma 5.65. *The conjugacy classes* 8C, 15B, 18G, 18H, 20B *of* Fi'_{24} *are of type D.*

Proof. The result follows from Lemma 4.9, because $O_{10}^-(2)$ is a maximal subgroup. ☐

Lemma 5.66. *The conjugacy classes* 24F, 24G, 36A, 36B, 45A, 45B *of* Fi'_{24} *are of type D.*

Proof. The maximal subgroup $\mathcal{M}_5 \simeq 3^7.O_7(3)$. We consider the short exact sequence $0 \to 3^7 \to 3^7.O_7(3) \to O_7(3) \to 0$. By Theorem 4.4, every conjugacy class of $O_7(3)$ with representative of order $\neq 2$ is of type D. Therefore, by Lemma 2.24, the result follows. □

Lemma 5.67. *The conjugacy class* 9D *is of type D.*

Proof. We use GAP to construct a representation of the maximal subgroup $\mathcal{M}_{19} \simeq \mathbb{A}_6 \times \mathsf{PSL}(2,8) : 3$ and use Algorithm 2.1 to prove that every conjugacy class of $\mathcal{M}_{19}$ with representative of order 9 is of type D (see the file `+F3+/step5.log` for details). □

Theorem 5.68. *The Fischer group Fi_{24}' is of type B.*

Proof. It remains to study the conjugacy classes 23A, 23B, 27A, 27B, 27C, 29A, 29B, 33A, 33B, 39C, 39D. For the conjugacy classes 23A, 23B use Lemma 2.11, because these classes are quasi-real with $j = 2$ and $g^4 \neq g$. For the remaining conjugacy classes use Lemma 2.10 because all of them are real. □

5.17. Conclusions

In this chapter we studied Nichols algebras almost every sporadic simple group. The Baby Monster and the Monster were studied in [**AFGV09c**, **AFGV09a**]. In Table 5.9 we review the sporadic groups stutied mostly with Algorithm 2.3. In Table 5.10 we list the conjugacy classes not necessarily of type D.

Table 5.9. The sporadic groups studied with Algorithm 2.3

G	**Log file**	G	**Log file**
M_{11}	`M11/M11.log`	Ru	`Ru/Ru.log`
M_{12}	`M12/M12.log`	HS	`HS/HS.log`
M_{22}	`M22/M22.log`	He	`He/He.log`
M_{23}	`M23/M23.log`	McL	`McL/McL.log`
M_{24}	`M24/M24.log`	Co_3	`Co3/Co3.log`
J_1	`J1/J1.log`	Co_2	`Co2/Co2.log`
J_2	`J2/J2.log`	$O'N$	`ON/ON.log`
J_3	`J3/J3.log`	Fi_{22}	`Fi22/Fi22.log`
Suz	`Suz/Suz.log`	T	`T/T.log`

Table 5.10. Conjugacy classes not known of type D.

G	**Conjugacy classes not necessarily of type D**	**Reference**
M_{11}	8A, 8B, 11A, 11B	§5.2
M_{12}	11A, 11B	§5.2
M_{22}	11A, 11B	§5.2
M_{23}	23A, 23B	§5.2
M_{24}	23A, 23B	§5.2
J_1	15A, 15B, 19A, 19B, 19C	§5.10
J_2	2A, 3A	§5.10
J_3	5A, 5B, 19A, 19B	§5.10
Suz	3A	§5.5
Ru	29A, 29B	§5.8
HS	11A, 11B	§5.3
He	all collapse	§5.4
McL	11A, 11B	§5.7
Co_3	23A, 23B	§5.11
Co_2	2A, 23A, 23B	§5.11
ON	31A, 31B	§5.6
Fi_{22}	2A, 22A, 22B	§5.9
Co_1	3A, 23A, 23B	§5.11.3
Fi_{23}	2A, 23A, 23B	§5.13
HN	all collapse	§5.15
Th	all collapse	§5.14
Ly	33A, 33B, 37A, 37B, 67A, 67B, 67C	§5.12
J_4	29A, 37A, 37B, 37C, 43A, 43B, 43C	§5.10.2
Fi'_{24}	23A, 23B, 27B, 27C, 29A, 29B, 33A, 33B, 39C, 39D	§5.16

Conjugacy classes in sporadic groups

In this apendix we list all real and quasi-real conjugacy classes of the sporadic simple groups studied in Chapter 5. The information about real conjugacy classes of a given group G is easy to obtain from the character table of G.

The kth powers of the elements of a given conjugacy class form another conjugacy class. This information is stored in the ATLAS. With this information at hand it is easy to determine the quasi-real conjugacy classes of a given group.

```
gap> QuasiRealClasses := function( ct )
>  local nc, oc, a, b, p, c, j, rc;
>
>  nc := NrConjugacyClasses(ct);
>  oc := OrdersClassRepresentatives(ct);
>  rc := RealClasses(ct);
>
>  a := [];
>  b := [];
>
>  for c in [1..nc] do
>    if not c in rc then
>      for j in [2..oc[c]-2] do
>        p := PowerMap(ct, j);
>        if p[c] = c then
>          if j-1 mod oc[c] <> 0 then
>            if not c in b then
>              Add(a, [c, j]);
>              Add(b, c);
>            fi;
>          fi;
>        fi;
```

```
>       od;
>     fi;
>   od;
>   return a;
>end;
```

The Tits group. In this group the conjugacy classes 16A, 16B, 16C, 16D are quasi-real of type $j = 9$. The conjugacy classes 8A, 8B are quasi-real of type $j = 5$. The remaining conjugacy classes are real.

The Mathieu groups. In any of the Mathieu simple groups, every conjugacy class is real or quasi-real. See Table A.1 for the details concerning not real but quasi-real conjugacy classes.

Table A.1. Mathieu groups: Quasi-real classes

	Classes	Type
M_{11}	8A, 8B, 11A, 11B	$j = 3$
M_{12}	11A, 11B	$j = 3$
M_{22}	7A, 7B	$j = 2$
	11A, 11B	$j = 3$
M_{23}	7A, 7B, 15A, 15B, 23A, 23B	$j = 2$
	11A, 11B	$j = 3$
	14A, 14B	$j = 9$
M_{24}	7A, 7B, 15A, 15B, 21A, 21B, 23A, 23B	$j = 2$
	14A, 14B	$j = 9$

The Conway groups. In the Conway groups Co_1, Co_2 and Co_3 every conjugacy class is real or quasi-real. Not real but quasi-real conjugacy classes are listed in Table A.2.

Table A.2. Conway groups: Quasi-real classes

	Classes	Type
Co_1	23A, 23B, 39A, 39B	$j = 2$
Co_2	15B, 15C, 23A, 23B	$j = 2$
	14B, 14C	$j = 9$
	30B, 30C	$j = 17$
Co_3	23A, 23B	$j = 2$
	11A, 11B, 20A, 20B, 22A, 22B	$j = 3$

The Janko groups. In the Janko groups J_1 and J_2 every conjugacy class is real. In the Janko group J_3 the conjugacy classes 19A, 19B are quasi-real of type $j = 4$ and the remaining conjugacy classes are real. In the group J_4 every conjugacy class is real, with the exceptions of these classes, all of them quasi-real:

(1) 7A, 7B, 21A, 21B, 35A, 35B (of type $j = 2$);

(2) 14A, 14B, 14C, 14D, 28A, 28B (of type $j = 9$);

(3) 42A, 42B (of type $j = 11$).

The Fischer groups. In the Fischer groups Fi_{22}, Fi_{23} and Fi'_{24} every conjugacy class is real or quasi-real. Not real but quasi-real conjugacy classes are listed in Table A.3.

Table A.3. Fischer groups: Quasi-real classes

	Classes	Type
Fi_{22}	11A, 11B, 16A, 16B, 22A, 22B	$j = 3$
	18A, 18B	$j = 7$
Fi_{23}	16A, 16B, 22B, 22C	$j = 3$
	23A, 23B	$j = 2$
Fi'_{24}	23A, 23B	$j = 2$
	18G, 18H	$j = 7$

The Highman-Sims group. In this group the conjugacy classes 11A, 11B, 20A, 20B are quasi-real of type $j = 3$. The remaining conjugacy classes are real.

The Lyons group. In this group the conjugacy classes 11A, 11B, 22A, 22B are quasi-real of type $j = 3$. The conjugacy classes 33A, 33B are quasi-real of type $j = 4$. The remaining conjugacy classes are real.

The Harada-Norton group. In this group every conjugacy class is real, with the exceptions of these classes, all of them quasi-real:

(1) 19A, 19B (of type $j = 4$);
(2) 35A, 35B (of type $j = 3$);
(3) 40A, 40B (of type $j = 7$).

The Held-group. In this group every conjugacy class is real, with the exceptions of these classes, all of them quasi-real:

(1) 7A, 7B, 7D, 7E, 21C, 21D (of type $j = 2$);
(2) 14A, 14B, 14C, 14D, 28A, 28B (of type $j = 9$).

The MacLaughlin group. In this group every conjugacy class is real, with the exceptions of these classes, all of them quasi-real:

(1) 7A, 7B, 15A, 15B (of type $j = 2$);
(2) 11A, 11B (of type $j = 3$);
(3) 9A, 9B (of type $j = 4$);
(4) 14A, 14B (of type $j = 9$);
(5) 30A, 30B (of type $j = 17$).

The O'Nan group. In this group every conjugacy class is real, with the exceptions of these classes, all of them quasi-real:

(1) 31A, 31B (of type $j = 2$);
(2) 20A, 20B (of type $j = 3$).

The Rudvalis group *Ru***.** In this group the conjugacy classes 16A, 16B are quasi-real of type $j = 5$. The remaining conjugacy classes are real.

The Suzuki group *Suz***.** In this groups the conjugacy classes 6B, 6C, with centralizers of size 1296 are neither real nor quasi-real. The classes 9A, 9B are quasi-real of type $j = 4$, and the classes 18A, 18B are quasi-real of type $j = 7$. The remaining conjugacy classes are real.

The Thompson group. In this group every conjugacy class is real, with the exceptions of these classes, all of them quasi-real:

(1) 15A, 15B, 31A, 31B, 39A, 39B (of type $j = 2$);
(2) 27B, 27C (of type $j = 4$);
(3) 24C, 24D (of type $j = 5$);
(4) 12A, 12B, 24A, 24B, 36B, 36C (of type $j = 7$);
(5) 30A, 30B (of type $j = 17$).

Appendix B

Notations

B.1. Notations for group structures

We use the ATLAS notations (see for example [**CCN+85**, page xx]). There are various ways to combine groups or abreviate some groups structures:

- $A \times B$ is the direct product of A and B;
- A^m denotes the direct product of m groups isomorphic to A;
- p^m, for p prime, denotes the elementary abelian group of order p^m;
- $A.B$ or AB denotes any group having a normal subgroup isomorphic to A for which the corresponding quotient is isomorphic to B;
- $A : B$ indicates the case $A.B$ which is a semi-direct product;
- $[m]$, for $m \in \mathbb{N}$, denotes an abitrary group of order m;
- m denotes the cyclic group C_m;
- p^{n+m} indicates a case of $p^n.p^m$.
- p^{1+2n} or p_+^{1+2n} or p_-^{1+2n} is used for the particular case of an extraspecial group.

Product of three or more groups are left-associated. This, $A.B.C$ means $(A.B).C$, and implies the existence of a normal subgroup isomorphic to A.

B.2. Notations for conjugacy classes

We also use the ATLAS notations for conjugacy classes. The conjugacy classes that contain elements of order n are named nA, nB, nC..., and remark that the alphabet used here is potentially infinite). In some cases we will note the classes with lowercase letters na, nb, nc..., in order to remark that this ordering for the conjugacy classes might be not equal to the one given in the ATLAS.

A computer package for racks

The GAP package RiG provides a free library of functions for computations related to racks. It is a joint work with Matías Graña. It can be downloaded from [**GnV08**]. This package can be used to:

(1) Compute racks and 2-cocycles associated to a given group;
(2) Compute subracks and quotients;
(3) Compute rack and quandle (co)homology groups;
(4) Compute group theoretic structures related to racks.
(5) Compute some relations of $\mathfrak{B}(X, q)$.

To load the package:

```
gap> LoadPackage("rig");
true
```

We present some examples of racks.

Example C.1 (Trivial rack)**.** *Let $n \in \mathbb{N}$. The trivial rack of order n is given by $X = \{1, \dots, n\}$ and the action $i \triangleright j = j$. The RiG function used to define this rack is* `TrivialRack`*. Here is an example with $n = 3$.*

```
gap> TrivialRack(3);
[ [  1,  2,  3 ],
  [  1,  2,  3 ],
  [  1,  2,  3 ] ]
```

Example C.2 (Dihedral rack)**.** *Let $n \geq 2$. A dihedral rack (of order n) is given by $X = \{1, \dots, n\}$ and $i \triangleright j = 2i - j \pmod{n}$. The RiG function used to define this rack is* `DihedralRack`*. Here is an example with $n = 3$:*

```
gap> DihedralRack(3);
[ [  1,  3,  2 ],
  [  3,  2,  1 ],
  [  2,  1,  3 ] ]
```

Example C.3 (Octahedral rack). $(X,\triangleright)$ *is the octahedral rack if* X *is the conjugacy class of transpositions in* $\mathbb{S}_4$ *and* $\triangleright$ *is given by the conjugation. With the GAP function* `RackFromConjugacyClass` *we construct this rack. Here is the output:*

```
gap> gr := SymmetricGroup(4);;
gap> RackFromAConjugacyClass(gr, (1,2));
[ [  1,  3,  2,  4,  6,  5 ],
  [  3,  2,  1,  5,  4,  6 ],
  [  2,  1,  3,  6,  5,  4 ],
  [  1,  5,  6,  4,  2,  3 ],
  [  6,  4,  3,  2,  5,  1 ],
  [  5,  2,  4,  3,  1,  6 ] ]
```

With RiG function `IsomorphismRacks` it is possible to study isomorphisms of racks. Here is a simple example:

Example C.4. *Let* R *(resp.* S*) be the conjugacy class (as a rack) of* $(1\ 2\ 3)$ *(resp.* $(1\ 3\ 2)$*) in* $\mathbb{A}_4$*. These two racks are isomorphic:*

```
gap> gr := AlternatingGroup(4);;
gap> r := RackFromAConjugacyClass(gr, (1,2,3));;
gap> s := RackFromAConjugacyClass(gr, (1,3,2));;
gap> IsomorphismRacks(r,s);
(3,4)
```

This means that the elements of R *are* $(1\ 2\ 3)$, $(1\ 4\ 2)$, $(1\ 3\ 4)$, $(2\ 4\ 3)$*. The elements of* S *are* $(1\ 3\ 2)$, $(1\ 2\ 4)$, $(1\ 4\ 3)$, $(2\ 3\ 4)$*. And the isomorphism between* R *and* S *is given by*

$$(1\ 2\ 3) \mapsto (1\ 3\ 2)$$
$$(1\ 4\ 2) \mapsto (1\ 2\ 4)$$
$$(1\ 3\ 4) \mapsto (2\ 3\ 4)$$
$$(2\ 4\ 3) \mapsto (1\ 4\ 3)$$

Here is an interesting example, mentioned in Chapter 3.

Example C.5. *The conjugacy class of elements of type* (2^3) *in* $\mathbb{S}_6$ *is isomorphic, as a rack, to the conjugacy class of the transpositions in* $\mathbb{S}_6$*, since any map in the class of the outer automorphism of* $\mathbb{S}_6$ *applies* $(1\ 2)$ *in* $(1\ 2)(3\ 4)(5\ 6)$ *(see* [**JR82**]*).*

```
gap> c := (1,2);;
gap> d := (1,2)(3,4)(5,6);;
gap> r := RackFromAConjugacyClass(SymmetricGroup(6), c);;
gap> s := RackFromAConjugacyClass(SymmetricGroup(6), d);;
gap> IsomorphismRacks(r,s);
(2,5,13,15)(3,9,8,6,14,4,11,7,10)
```

In the following example we compute the inner group and the automorphism group of a decomposable rack.

Example C.6. *It is not true in general that* $\mathsf{Inn}(X) = \mathsf{Aut}(X)$*.*

```
gap> r := DihedralRack(4);
[ [  1,  4,  3,  2 ],
  [  3,  2,  1,  4 ],
  [  1,  4,  3,  2 ],
  [  3,  2,  1,  4 ] ]
gap> StructureDescription(InnerGroup(r));
"C2 x C2"
gap> StructureDescription(AutomorphismGroup(r));
"D8"
```

With RiG functions `RackHomology` and `RackCohomology` it is possible to compute racks (co)homology groups. For the definitions of this (co)homology groups see for example [**AG03**].

Example C.7. *Let X be a dihedral rack of order 3. We have:*

(1) $H^2_R(X,\mathbb{Z}) \simeq H^3_R(X,\mathbb{Z}) \simeq \mathbb{Z}$;
(2) $H^4_R(X,\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/3$;
(3) $H^5_R(X,\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$;
(4) $H^6_R(X,\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

Here is the output:

```
gap> RackCohomology(DihedralRack(3),2);
[ 1, [  ] ]
gap> RackCohomology(DihedralRack(3),3);
[ 1, [  ] ]
gap> RackCohomology(DihedralRack(3),4);
[ 1, [ 3 ] ]
gap> RackCohomology(DihedralRack(3),5);
[ 1, [ 3, 3 ] ]
gap> RackCohomology(DihedralRack(3),6);
[ 1, [ 3, 3, 3, 3 ] ]
```

Example C.8. *Let X be the octahedral rack. Then,*

(1) $H_2^R(X,\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$;
(2) $H_3^R(X,\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6$.

Here is the output:

```
gap> RackHomology(r, 2);
[ 1, [ 2 ] ]
gap> RackHomology(r, 3);
[ 1, [ 2, 6 ] ]
```

This package can also be used to compute Quandle (co)homology groups. To this purpose we have the function `QuandleHomology`. For the definition of this (co)homology theory see for example [**CJK+03**].

Example C.9. *Let X be a dihedral rack of order 5. Then, we have these quandle homology groups:*

(1) $H_2^Q(X,\mathbb{Z}) \simeq 1$;

(2) $H_3^Q(X,\mathbb{Z}) \simeq H_4^Q(X,\mathbb{Z}) \simeq \mathbb{Z}/5$.

Here is the output:

```
gap> QuandleHomology(DihedralRack(5), 2);
[ 0, [  ] ]
gap> QuandleHomology(DihedralRack(5), 3);
[ 0, [ 5 ] ]
gap> QuandleHomology(DihedralRack(5), 4);
[ 0, [ 5 ] ]
```

As we said, RiG can also be used to compute low degree relations of the Nichols algebra $\mathfrak{B}(X,q)$. To this purpose we use GBNP, a non-commutative GAP Gröbner basis package, written by Arjeh M. Cohen (see [**CK10**] for details).

Example C.10. *Let ω be a cubic root of 1. Let X be the conjugacy class (as a rack) of* (1 2 3) *in $\mathbb{A}_4$. And let* **q** *be the constact 2-cocycle equal to ω. It is not know if the Nichols algebra $\mathfrak{B}(X,q)$ is finite dimensional. With RiG function* `Dimension` *it is possible to compute the first dimensions:*

```
gap> for n in [0..7] do
> Print(Dimension(r, q, n), "\n");
> od;
1
4
16
52
172
544
5312
16412
```

So, the first dimensions are:

$$1, 4, 16, 52, 172, 544, 1712, 5312, 16412, \ldots$$

(Notice that for this example a lot of memory was needed.) With RiG we compute the degree three relations and with GBNP function `PrintNPList` *we get a nice presentation of these relations. The code*

```
gap> LoadPackage("gbnp");;
gap> PrintNPList(Relations4GAP(r, q, 3));;
```

gives us these twelve relations:

$$a^3 = b^3 = c^3 = d^3 = 0,$$
$$a^2b + abc + aca + bc^2 + cac + c^2d + cda + da^2 = 0,$$
$$a^2d + aba + adb + bab + b^2c + bca + ca^2 + db^2 = 0,$$
$$acb - \omega adc - \omega bcd + bda + cad - \omega cba - \omega dab + dbc = 0,$$
$$abd - \omega^2 adc + bac - \omega^2 bcd - \omega^2 cba + cdb - \omega^2 dab + dca = 0,$$
$$ab^2 + b^2d + bcb + bdc + cab + cbc + c^2a + dc^2 = 0,$$
$$ac^2 + bd^2 + cbd + c^2b + cdc + dac + dcd + d^2a = 0,$$
$$a^2c + acd + ada + ba^2 + cd^2 + dad + dba + d^2b = 0,$$
$$ad^2 + bad + b^2a + bdb + cb^2 + dbd + dcb + d^2c = 0.$$

Also, it is possible to find the relations of degree 6 (there are twenty of them). These relations are omitted owing to length.

Example C.11. *Let ω be a cubic root of 1. We consider the Nichols algebra given by the abelian rack $X = \{a, b, c\}$ with the 2-cocycle given by*

$$\mathbf{q} = \begin{pmatrix} -1 & \omega & \omega \\ 1 & -1 & \omega \\ 1 & 1 & -1 \end{pmatrix}$$

By the work of Heckenberger, we know that this Nichols algebra has dimension 432. With RiG *(and the Gröbner package* GBNP*) we obtain these relations:*

$$a^2 = b^2 = c^2 = 0,$$
$$abc + \omega^2 acb + \omega^2 bac + \omega bca + \omega cab + cba = 0,$$
$$cbcbcb - bcbcbc = bababa - ababab = cacaca - acacac = 0$$

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