# Agujeros negros y solitones gravitatorios en teorías de gravedad con términos superiores en la curvatura

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2009

# UNIVERSIDAD DE BUENOS AIRES


**Facultad de Ciencias Exactas y Naturales**

**Departamento de Física**


*Agujeros Negros y Solitones Gravitatorios en Teorías de Gravedad con Términos Superiores en la Curvatura*


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Trabajo de Tesis para optar por el título de Doctor de la Universidad de Buenos Aires en el **área de Ciencias Físicas**.

Octubre de 2009

**Resumen**

La teoría de gravedad de Lovelock es la extensión natural de la teoría de Einstein a dimensiones mayores que cuatro. Esta es una teoría de gran importancia en física teórica ya que incluye a la Relatividad general y a las llamadas Gravedades de Chern-Simons como ejemplos particulares. Además, la teoría de Lovelock emerge como una corrección de segundo orden en la parte gravitacional de la acción efectiva de bajas energías de ciertas teorías de cuerdas.

En esta tesis, estudiamos soluciones de agujero negro y solitones gravitatorios en la teoría de Lovelock. Comezamos nuestro estudio discutiendo las motivaciones para considerar esta teoría de gravedad en particular. Luego, revemos cómo, en el régimen de distancias cortas, la física de los agujeros negros resulta modificada con respecto a lo que aprendimos de la mano de la Relatividad General. Discutimos en detalle los aspectos geométricos y termodinámicos. Luego, atacamos el problema de incluir términos de borde, y usamos éstos para construir soluciones de vacío que pueden ser pensadas como solitones gravitatorios de la teoría de Lovelock en cinco dimensiones. Analizamos detalladamente la (in)estabilidad y la estructura global de las nuevas soluciones. De la gran familia de nuevas soluciones que encontramos, prestamos particular atención a las soluciones de vacío tipo *wormhole* y tipo *vacuum-shells*, esféricamente simétricas. La existencia de este tipo de soluciones nos lleva a mostrar que los teoremas tipo Birkhoff en la teoría de Lovelock resultan ser válidos sólo localmente, a diferencia de lo que ocurre en la teoría de Einstein. También estudiamos la naturaleza de la singularidad en la teoría de Lovelock. En esta teoría, existen soluciones de masa positiva que, aún así, exhiben singularidades de curvatura desnudas. Probamos que, cuando estas singularidades son analizadas con el método de *quantum probes*, estos espacios singulares pueden ser considerados como regulares en el contexto cuántico.

Esta tesis está basada en los resultados que la autora ha publicado en las referencias [81, 80, 82]. El contenido también fue presentado por la autora en dos seminarios dictados en el Martin A. Fisher Physics Department de Brandeis University.



PALABRAS CLAVE: Agujeros Negros, Gravedad Cuántica, Dimensiones adicionales

**Abstract**

Lovelock theory of gravity is the natural extension of Einstein theory to higher dimensions. This is a theory of great importance in theoretical physics because it includes General Relativity and the so called Chern-Simons gravities as particular examples. Besides, Lovelock theory arises as next-to-leading corrections in the gravitational part of the low energy effective action of certain string theories.

In this thesis we study black hole solutions and gravitational solitons in Lovelock theory. We begin by discussing the motivation for considering this particular theory of gravity. Then, we review how, in the short-distance regime, black hole physics gets modified with respect to what we know from General Relativity. Geometrical and thermodynamical aspects are discussed in detail. Then, we address the problem of including boundary terms, and use them to construct vacuum solutions that can be thought of as gravitational solitons of five-dimensional Lovelock theory. We carefully analyse the (in)stability and global structure of the new solutions. Among the large family of new exact solutions we found, particular attention is focused on vacuum spherically symmetric wormholes and vacuum-shells. The existence of such solutions leads us to show that Birkhoff-like theorems only hold locally in Lovelock theory, in contrast to Einstein theory. We also study the nature of the singularity. Solutions of positive mass exhibiting naked curvature singularity exist, and we prove that when testing with quantum probes these singular spaces can be regarded as regular within a quantum mechanical context.

This thesis is based on the results the author has published in references [81, 80, 82]. The material was also presented by the author in two lectures delivered in the Martin A. Fisher Physics Department of Brandeis University.

KEY WORDS: Black Holes, Quantum Gravity, Higher-Dimensions

# Contents

# Chapter 1

# Introduction

## 1 Why higher-curvature corrections?

It is a common belief that General Relativity, despite its fabulous success in describing our Universe at middle and large scale, has to be corrected at short distance. In particular, the apparent tension between Einstein's theory and quantum field theory supports the idea that General Relativity is merely an effective model that would be replaced in the UV regime by a different theory, and such a *new* theory would ultimately permit to make sense of what we call Quantum Gravity. The natural scale at which one expects such short distance corrections to manifestly appear is the Planck scale $l_P$, determined by the Newton's coupling constant $G = l_P^2/16\pi$.

At present, the most successful candidate to represent a quantum theory of gravity is String Theory (or its mother theory, M-theory). In fact, one of the predictions of string theory is the existence of a massless particle of spin 2 whose dynamics at classical level is governed by Einstein equation

$$R_{\mu\nu} = 0. \tag{1.1}$$

In addition, string theory also predicts next-to-leading corrections to (1.1), which would be relevant at distances comparable with the typical length scale of the theory $l_s = \sqrt{\alpha'} \lesssim l_P$. These short-distance corrections are typically described by supplementing Einstein-Hilbert action by adding higher-curvature terms, correcting General Relativity in the UV regime. As a result, the spin 2 interaction turns out to be renormalizable, and this raises the hope to finally have access to a consistent theory of quantum gravity.

In addition to higher-curvature corrections to Einstein theory, string theory makes other strong predictions about nature. Probably, the most important ones are: the existence of supersymmetry, and the existence of extra dimensions. In fact, one of the requirements for superstring theory to be consistent is the space-time to have $9+1$ dimensions. Besides, we learn from our daily experience that six of these extra dimensions have to be hidden somehow.

This digression convinces us that studying higher-curvature modification of General Relativity in higher dimensions seems to be crucial to address the problem of quantum gravity. This is precisely the subject we will study in this Thesis. More precisely, in this Thesis we will investigate how the string inspired higher-curvature corrections to Einstein-Hilbert action modify the black hole physics in the UV regime. This turns out to be a very important question since the black holes are known to be a fruitful arena to explore gravitational phenomena.

To investigate black hole physics in higher-curvature gravity theories, the first question we have to answer is whether such theories actually induce short-distance modifications to the black hole geometry or not. Actually, despite one naively expects that the inclusion of higher-curvature terms in the gravitational action yields modifications to General Relativity, it is not necessary the case that such modifications manifestly appear in the static spherically symmetric sector of the space of solutions. In fact, as we will see below, Schwarzschild geometry usually resists to be modified. In turn, first it is important to identify which are the theories of gravity that yield modifications to the spherically symmetric Schwarzschild solution.

## 1.1 Schwarzschild metric as a persistent solution

To warm up, let us start by considering a very simple example of higher-curvature term. Consider the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda + \alpha R^2 \right) \tag{1.2}$$

which corresponds to Einstein-Hilbert action in four dimensions augmented with the square of the curvature scalar. This action is a particular case of the so-called $f(R)$-gravity theories, which are defined by adding to the Einstein-Hilbert Lagrangian a function of the Ricci scalar $f(R)$. It is well known that $f(R)$-gravity theories are equivalent (after field redefinition that involves a conformal transformation) to General Relativity coupled to a scalar field $\phi$, provided a suitable self-interaction potential $V(\phi)$ that depends on the function $f$ (see [149] and references therein). In this sense, these theories are not different from particular models of quintessence. Here, we are interested in less simple models; however, let us consider (1.2) as the starting point of our discussion.

A remarkable point is that the theory defined by action (1.2) admits (Anti-) de Sitter-Schwarzschild metric as its static spherically symmetric solution.

Theory (1.2) is not the only theory of gravity that admits Schwarzschild metric as a persistent solution. Actually, this is a common feature of theories with higher-curvature terms. A second example is given by Einstein gravity coupled to conformally invariant gravity; namely

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\bigg(R - 2\Lambda + \alpha\ C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\bigg), \qquad (1.3)$$

where $\alpha$ is a coupling constant and $C_{\alpha\beta\mu\nu}$ is the Weyl tensor, whose quadratic contraction reads

$$C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = \frac{1}{3}R^2 - 2R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \qquad (1.4)$$

The equations of motion associated to this action read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha W_{\mu\nu} = 0, \qquad (1.5)$$

where $W_{\mu\nu}$ is the Bach tensor,

$$\begin{aligned} W_{\mu\nu} \;=\; & \Box R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}\Box R - \frac{1}{3}\nabla_\mu\nabla_\nu R + 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} \\ & -\frac{1}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - \frac{2}{3}RR_{\mu\nu} + \frac{1}{6}g_{\mu\nu}R^2, \end{aligned} \qquad (1.6)$$

It is easy to show that, when $\Lambda = 0$, Scwarzschild metric solves equations (1.5) as well. This follows from the fact that Bach tensor (1.6) vanishes if Ricci tensor vanishes, and thus all solutions to General Relativity are also solutions to (1.5).

Another example of a modified theory that admits Schwarzschild metric as a solution is the Jackiw-Pi theory [102]. This theory has recently attracted much attention due to its phenomenological predictions[1]. It is defined by the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - 2\Lambda + \frac{\theta}{4} {}^*R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right), \tag{1.7}$$

where the function $\theta$ is a Lagrange multiplier that couples to the Pontryagin density ${}^*R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$, constructed via the dual curvature tensor

$${}^*R^{\alpha}{}_{\beta}{}^{\mu\nu} = \frac{1}{2} \varepsilon^{\rho\sigma\mu\nu} R^{\alpha}{}_{\beta\rho\sigma},$$

where $\varepsilon_{\rho\sigma\mu\nu}$ is the volume 4-form. The inclusion of the non-dynamical field $\theta$ comes from the fact that the Pontryagin form ${}^*R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$ is a total derivative. Action (1.7) is often called Chern-Simons modified gravity; however, this has to be distinguished from the Chern-Simons gravitational theories we will discuss in this Thesis.

The equations of motion derived from the Jackiw-Pi action (1.7) take the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + C_{\mu\nu} = 0, \tag{1.8}$$

where,

$$C^{\mu\nu} = \nabla_\alpha \left( \nabla_\beta \theta \, {}^*R^{\alpha\mu\beta\nu} \right) + \nabla_\alpha \left( \nabla_\beta \theta \, {}^*R^{\alpha\nu\beta\mu} \right). \tag{1.9}$$

In addition, we have the constraint

$${}^*R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} = 0 \tag{1.10}$$

which implies the conservation of the field equations

$$\nabla^\mu C_{\mu\nu} = \frac{1}{8} {}^*R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} \nabla_\nu \theta = 0, \tag{1.11}$$

It is not hard to see that equations (1.8) and (1.10) are solved by Schwarzschild metric. This is because the Pontryagin density ${}^*R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma}$ of Schwarzschild metric vanishes. In contrast, Kerr metric has not vanishing Pontryagin form, and thus it is not a solution of Jackiw-Pi theory. In fact, the rotating solution of this theory has not yet been found, and this represents an interesting open problem as the Jackiw-Pi theory is considered as a phenomenologically viable correction to Einstein theory.

Summarizing, there are several models that, while representing short distance corrections to General Relativity, still admit the Schwarzschild metric

[1] For instance, this theory predicts polarization and birefingence in gravitational waves.

as an exact solution. In particular, this implies that such models can not be the solution to problems like the issue of singularity. On the other hand, there are other examples which, still being integrable, do yield deviations from General Relativity solutions even in the static spherically symmetric sector. In this Thesis we will be concerned with one of such models. We will study a very special case of higher-curvature corrections to Einstein gravity in higher dimensions, and we will see that substantial modifications to Schwarzschild solution are found at short distances. The model we will study is the five-dimensional Lovelock theory, which arises in the low energy effective action of M-theory.

## 1.2 Higher-curvature terms in higher dimensions

Before introducing the Lovelock theory in five-dimensions, let us begin by considering a much more general example. Consider the action

$$S[g_{\mu\nu}] = \int d^D x\sqrt{-g}\left(R - 2\Lambda + \alpha R^2 + \beta R_{\alpha\beta}R^{\alpha\beta} + \gamma R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right) \quad (1.12)$$

where the constants $\alpha$, $\beta$, and $\gamma$ are the coupling constants for each quadratic term. The field equations obtained by varying the action (1.12) with respect to the metric read

$$\begin{aligned} 0 \;=\; & G_{\mu\nu} + \Lambda g_{\mu\nu} + (\beta + 4\gamma)\,\Box R_{\mu\nu} + \frac{1}{2}(4\alpha + \beta)\,g_{\mu\nu}\Box R + \qquad (1.13)\\ & -(2\alpha + \beta + 2\gamma)\,\nabla_\mu\nabla_\nu R + 2\gamma R_{\mu\gamma\alpha\beta}R_\nu^{\ \gamma\alpha\beta} + 2(\beta + 2\gamma)\,R_{\mu\alpha\nu\beta}R^{\alpha\beta} + \\ & -4\gamma R_{\mu\alpha}R_\nu^{\ \alpha} + 2\alpha R R_{\mu\nu} - \frac{1}{2}\left(\alpha R^2 + \beta R_{\alpha\beta}R^{\alpha\beta} + \gamma R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) g_{\mu\nu} \end{aligned}$$

Action (1.12) is the most general quadratic action one can write down in $D$-dimensions. For $D \leq 4$, the Gauss-Bonnet theorem permits to fix $\gamma = 0$ without loss of generality. In $D > 4$, however, three quadratic invariants are needed to describe the most general Lagrangian of this type.

For generic values of the coupling constants $\alpha$, $\beta$ and $\gamma$, the equations of motion (1.13) are fourth-order differential equations for the metric (e.g. there are terms like $\nabla_\mu\nabla_\nu R$, and $\Box R_{\mu\nu}$, etc.). Nevertheless, a remarkable property of (1.12) is that there exists one particular choice of the coupling constants $\alpha$, $\beta$ and $\gamma$ that results in the cancellation of the higher order terms, yielding second order differential equations. It is not hard to see that this choice is $\alpha = \gamma = -\beta/4$, which only gives a non-trivial modification to Einstein theory for $D > 4$. It is worth emphasizing that this choice of coupling constants is *unique* (up to a free parameter $\alpha$), and this feature is a consequence of a more general result known as "the Lovelock theorem", which we will be discuss in Chapter 2.

In this Thesis we will be mainly concerned with the theory defined by action (1.12) with $\alpha = \gamma = -\beta/4$. That is, the five-dimensional Lovelock theory of gravity. Besides the *uniqueness* of the choice $\alpha = \gamma = -\beta/4$, what already makes this model interesting in its own right, let us say that this is exactly the effective Lagrangian that appears in the low energy action of some string theories (or M-theory).

# 2 Higher-curvature terms from M-theory

Now, let us discuss how the five-dimensional Lovelock theory arises in the low energy limit of M-theory (and, consequently, of string theory) when the theory is compactified from $11D$ (resp. $10D$) to $5D$.

## 2.1 The M-theory effective action

M-theory is supposed to be a generalization of string theory; a theory in eleven dimensions that, in certain regime, would flow to string theory [166]. Presumably, the basic degrees of freedom of this theory are extended objects (often called M-branes).

This Mother-theory, if it exists, is yet to be found; nevertheless, we do know what it has to look like in the low energy limit: it has to look like eleven-dimensional supergravity augmented with higher-curvature terms. That is, the bosonic sector of the M-theory effective action is given by the graviton $g_{\mu\nu}$ (i.e. the metric) and the 3-form gauge field $A_{\mu\nu\rho}$ (with field strength $F_{\mu\nu\rho\sigma} = \frac{1}{4}\partial_{[\mu}A_{\nu\rho\sigma]}$). Including the pure gravitational fourth order corrections

$\mathcal{O}(R^4)$, this effective action takes the form[2] [157]

$$
\begin{aligned}
S_{\rm M} \;=\; & \frac{1}{(2\pi)^5 l_P^9}\left[\int d^{11}x\sqrt{g}R \;-\frac{1}{48}\int d^{11}x\sqrt{g}F_{\mu_1\mu_2\mu_3\mu_4}F^{\mu_1\mu_2\mu_3\mu_4}+\right. \\
& -\frac{1}{36(4!)^2}\int d^{11}x\varepsilon_{\mu_1\mu_2..\mu_{11}}A^{\mu_1\mu_2\mu_3}F^{\mu_4\mu_5\mu_6\mu_7}F^{\mu_8\mu_9\mu_{10}\mu_{11}}+ \\
& +\frac{l_P^6}{27}\left(\frac{3}{2^{13}}\int d^{11}x\sqrt{g}t^{\mu_1\mu_2\dots\mu_8}t_{\nu_1\nu_2\dots\nu_8}R^{\nu_1\nu_2}{}_{\mu_1\mu_2}R^{\nu_3\nu_4}{}_{\mu_3\mu_4}R^{\nu_5\nu_6}{}_{\mu_5\mu_6}R^{\nu_7\nu_8}{}_{\mu_7\mu_8}\right. \\
& \left.\left.-\frac{1}{2^{16}}\int d^{11}x\sqrt{g}\varepsilon^{\mu_1\mu_2\dots\mu_8\alpha\beta\gamma}\varepsilon_{\nu_1\nu_2\dots\nu_8\alpha\beta\gamma}R^{\nu_1\nu_2}{}_{\mu_1\mu_2}R^{\nu_3\nu_4}{}_{\mu_3\mu_4}R^{\nu_5\nu_6}{}_{\mu_5\mu_6}R^{\nu_7\nu_8}{}_{\mu_7\mu_8}\right)\right]+\;...
\end{aligned}
\tag{1.14}
$$

where the ellipses stand for the fermionic content and higher-order contributions. These higher order contributions include terms like $\mathcal{O}(F^4)$ and also couplings of the form $\mathcal{O}(A\ R^4)$; we will not consider these terms here.

The tensor $t^{\mu_1\cdots\mu_8}$ in the third line of (1.14) is defined in terms of the way it acts on antisymmetric tensors of second rank, namely

$$
t^{\mu_1\mu_2\dots\mu_8}B_{\mu_1\mu_2}B_{\mu_3\mu_4}B_{\mu_5\mu_6}B_{\mu_7\mu_8}=24\text{tr}(B^4)-6\text{tr}(B^2)^2,
$$

where $\text{tr}(B^n)$ refers to the trace of $B^n$.

The term in the fourth line in (1.14) is actually one of the terms that appear in the Lagrangian of Lovelock theory (see Section 2, where this term is expressed in an alternative way). However, the term in the third line, which is of the same order, does not correspond to a term in the Lovelock theory[3].

## 2.2 Calabi-Yau compactifications and Gauss-Bonnet

Now, let us analyze what happens when the M-theory effective action we discussed above (including the higher-curvature terms $\mathcal{R}^4$) is compactified to five dimensions. Let us assume we reduce from $11D$ to $5D$ by compactifying six of the eleven dimensions in compact Calabi-Yau ($\text{CY}_6$) threefold. It is known that, in that case, the effective action of the five-dimensional theory takes the form

[2]The eleven-dimensional Newton constant is given by the Planck scale $G_{(11D)}=2\pi^4 l_P^9$.

[3]Actually, while second-order terms of heterotic string theory expressed in a particular frame agree with the second-order term of the Lovelock theory, the fourth-order terms of Type IIA and IIB string theories (and M-theory) do not agree with the fourth-order term of the Lovelock theory.

$$S_{\text{eff}} = \int d^5x\sqrt{-g}\left(R - 2\Lambda + \frac{1}{16}c^I_{(2)}V_I(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu})\right), \quad (1.15)$$

where we used units such that $G_{(5D)} = 1/16\pi$, and where the coupling $c^I_{(2)}V_I$ is a quantity that depends on the "details" of the internal $\text{CY}_6$ manifold (more precisely, $c^I_{(2)}$ are the components of the second Chern-class of the $6D$ Calabi-Yau space, while $V^I$ are the so-called scalar components of the vector multiplet, which are proportional to the Kähler moduli of the Calabi-Yau).

We observe that action (1.15), at least in the approximation where the coupling $c^I_{(2)}V_I = 16\alpha$ can be considered constant, corresponds to a particular case of (1.12); namely the case $D = 5$, $\alpha = \gamma = -\beta/4$. And this is exactly the (only) case for which the equations of motion (1.13) turn out to be of second order. This is precisely the theory we will study in this Thesis: the most general quadratic theory of gravity with equations of motion of second order, which, on the other hand, is the one that arises as Calabi-Yau compactifications of M-theory. It is worth mentioning that this quadratic action also appears in the low energy effective action of heterotic string theory.

## 2.3 AdS/CFT correspondence and holography

Because action (1.15) also appears in the effective action of the heterotic string, it is usually called "string inspired higher-curvature corrections". In turn, it represents a nice model to explore the effects of next-to-lading contributions of string theory to gravitational physics. In particular, this five-dimensional (Lovelock) model of gravity was recently considered in the context of AdS/CFT holographic correspondence: One of the applications of the Lovelock theory to AdS/CFT that has attracted attention recently was that of showing that the so-called Kovtun-Starinets-Son (KSS) bound may violated in a theory that contains higher-curvature corrections. The KSS bound is a conjecture that states that: the ratio between the shear viscosity $\eta$ to the entropy $s$ of all the materials obey the relation

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \quad (1.16)$$

In Refs. [26] it was observed that when action (1.12) with $\alpha = \gamma = -\beta/4$ and $\Lambda = -l^{-2} < 0$ is considered in asymptotically locally $\text{AdS}_5$ space, then the conformal field theory (CFT) that would be dual to such a theory of gravity would satisfy

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - \frac{4\alpha}{3l^2}\right) \tag{1.17}$$

what then would violate (1.16) for $\alpha > 0$. Therefore, the KSS bound would be violated for all the CFTs with a Einstein-Gauss-Bonnet gravity duals with positive $\alpha$, and this is precisely the sign of $\alpha$ predicted by string theory.

The consideration of five-dimensional Lovelock theory as a working example to study the effects of including higher-curvature terms in AdS/CFT has been one of the most active lines of research in the last year. This represents one of the main motivations to study this theory in detail herein.

# 3 Overview

## 3.1 Motivation

In summary, we can mention two main reasons to study Lovelock gravity in five dimensions:

- First, this theory is interesting in its own right as it is the natural generalization of Einstein gravity to higher dimensions. It contains General Relativity and Chern-Simons gravity as particular cases, and can be thought of as a *unique* theory under certain natural assumptions.
- Secondly, as discussed above, the five-dimensional Lovelock theory emerges in a low energy effective action of M-theory. Because of it, this model of gravity has recently been considered as a prototype to explore the effects of higher-curvature terms in the context of AdS/CFT correspondence.
- The aim of this Thesis is to investigate the implications of including higher-curvature corrections to Einstein equations, studying the UV effects in black hole physics. Surprising phenomena, absent in General relativity, are discovered.

## 3.2 Overview

This Thesis is organized as follows: In Chapter 2, we will introduce Lovelock theory of gravity. In Chapter 3, the black hole solutions of this theory will be studied. In particular, we will analyze in detail the Boulware-Deser static spherically symmetric black hole solution. The geometrical aspects, causal structure, and thermodynamical properties, will be discussed in detail. In Chapter 4, we will consider the boundary terms and use such contributions to construct special cases of gravitational solitons of the theory. We will show how vacuum solutions with non-trivial topological properties arise. Wormhole-like objects and vacuum-shells will be explicitly constructed by geometric surgery. In Chapter 5 we will describe the dynamical configurations of these gravitational solitons and study their (in)stability. In Chapter 6, we will study naked singularities in Lovelock theory using quantum probes. We will show in what sense the Lovelock spaces can be regarded as regular spaces in a quantum mechanical context.

# Chapter 2

# The Lovelock Theory

As we mentioned in the introduction, Lovelock theory is the most general metric theory of gravity yielding conserved second order equations of motion in arbitrary number of dimensions $D$. In turn, it is the natural generalization of Einstein's general relativity (GR) to higher dimensions [115, 116]. In three and four dimensions Lovelock theory coincides with Einstein theory [113], but in higher dimensions both theories are actually different. In fact, for $D > 4$ Einstein gravity can be thought of as a particular case of Lovelock gravity since the Einstein-Hilbert term is one of several terms that constitute the Lovelock action. Besides, Lovelock theory also admits other quoted models as particular cases; for instance, this is the case of the so called Chern-Simons gravity theories, which in a sense are actual gauge theories of gravity.

On the other hand, Lovelock theory resembles also string inspired models of gravity as its action contains, among others, the quadratic Gauss-Bonnet term, which is the dimensionally extended version of the four-dimensional Euler density. This quadratic term is present in the low energy effective action of heterotic string theory [34, 36, 89], and it also appears in six-dimensional Calabi-Yau compactifications of M-theory; see [91] and references therein. In [169] Zwiebach earlier discussed the quadratic Gauss-Bonnet term within the context of string theory, with particular attention on its property of being free of ghost about the Minkowski space. Besides, the theory is known to be free of ghosts about other exact backgrounds [24]. For a nice and concise review on stringy corrections to gravity actions [35, 134, 133] see the introduction of [130] and references therein. For interesting recent discussions on higher order curvature terms see [91, 90, 18, 108, 151] and related works.

The Lovelock theory represents a very interesting scenario to study how the physics of gravity results corrected at short distance due to the presence of higher order curvature terms in the action. In this work we will be concerned with the black hole solutions of this theory, and we will discuss how short distance corrections to black hole physics substantially change the qualitative features we know from our experience with black holes in GR. So, let us introduce the Lovelock theory.

The Lagrangian of the theory is given as a sum of dimensionally extended Euler densities, and it can be written as follows[1] [115, 116]

$$\mathcal{L} = \sqrt{-g}\, \sum_{n=0}^{t} \alpha_n\, \mathcal{R}^n, \qquad \mathcal{R}^n = \frac{1}{2^n} \delta^{\mu_1\nu_1\dots\mu_n\nu_n}_{\alpha_1\beta_1\dots\alpha_n\beta_n} \prod_{r=1}^{n} R^{\alpha_r\beta_r}{}_{\mu_r\nu_r} \tag{2.1}$$

where the generalized Kronecker $\delta$-function is defined as the antisymmetric product

$$\delta^{\mu_1\nu_1\dots\mu_n\nu_n}_{\alpha_1\beta_1\dots\alpha_n\beta_n} = \frac{1}{n!}\delta^{\mu_1}_{[\alpha_1}\delta^{\nu_1}_{\beta_1}\dots\delta^{\mu_n}_{\alpha_n}\delta^{\nu_n}_{\beta_n]}. \tag{2.2}$$

Each term $\mathcal{R}^n$ in (2.1) corresponds to the dimensional extension of the Euler density in $2n$ dimensions[2], so that these only contribute to the equations of motion for $n < D/2$. Consequently, without lack of generality, $t$ in (2.1) can be taken to be $D = 2t$ for even dimensions and $D = 2t + 1$ for odd dimensions[3].

The coupling constants $\alpha_n$ in (2.1) have dimensions of $[\text{length}]^{2n-D}$, although it is convenient to normalize the Lagrangian density in units of the Planck scale $\alpha_1 = (16\pi G)^{-1} = l_P^{2-D}$. Expanding the product in (2.1) the Lagrangian takes the familiar form

$$\mathcal{L} = \sqrt{-g}\, (\alpha_0 + \alpha_1 R + \alpha_2 \left(R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}\right) + \alpha_3\mathcal{O}(R^3)), \tag{2.3}$$

where we see that coupling $\alpha_0$ corresponds to the cosmological constant $\Lambda$, while $\alpha_n$ with $n \geq 2$ are coupling constants of additional terms that represent ultraviolet corrections to Einstein theory, involving higher order contractions of the Riemann tensor $R^{\alpha\beta}{}_{\mu\nu}$. In particular, the second order term $\mathcal{R}^2 =$

[1]Here we are ignoring the boundary terms. We will consider these terms in section 6.

[2]The $2n$-dimensional Euler density $\chi$ is given by $\chi(\mathrm{M}) = \frac{(-)^{n+1}\Gamma(2n+1)}{2^{2+n}\pi^n\Gamma(n+1)}\int_{\mathrm{M}} d^{2n}x\sqrt{-g}\ \mathcal{R}^n$, where, again, we are not considering the boundary terms.

[3]See [51] for a related discussion on gravitational dynamics and Lovelock theory.

$R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}$ is precisely the Gauss-Bonnet term discussed above. The cubic term still has a moderate form [131], namely

$$\begin{aligned}\mathcal{R}^3 = R^3 + 3RR^{\mu\nu\alpha\beta}R_{\alpha\beta\mu\nu} - 12RR^{\mu\nu}R_{\mu\nu} + 24R^{\mu\nu\alpha\beta}R_{\alpha\mu}R_{\beta\nu} + 16R^{\mu\nu}R_{\nu\alpha}R^{\alpha}_{\mu}+ \\ + 24R^{\mu\nu\alpha\beta}R_{\alpha\beta\nu\rho}R^{\rho}_{\mu} + 8R^{\mu\nu}{}_{\alpha\rho}R^{\alpha\beta}{}_{\nu\sigma}R^{\rho\sigma}{}_{\mu\beta} + 2R_{\alpha\beta\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\mu\nu}.\end{aligned} \tag{2.4}$$

The fourth order term $\mathcal{R}^4$ has a more abstruse expression. It can be shown to coincide with that of the last line in (1.14).

Even though the way of writing Lovelock action in its tensorial form (2.3)-(2.4) may result clear to introduce the theory, it is not the most efficient way for most of the calculations one usually deal with. A more convenient way of working out these expressions is to resort to the so-called first-order formalism, which turns out to be useful both for formal purposes and for practical ones. Nevertheless, it is important to point out that the first-order formalism is not necessarily equivalent to the second-order formalism, so it should not be regarded merely as a different nomenclature. In the first-order formalism, both the vielbein $e^a_\mu$ and the spin connection $\omega^{ab}_\mu$ are considered as independent degrees of freedom, and the torsion acquires in general propagating degrees of freedom [156]. It is only in the torsion-free sector where both formulations are equivalent; notice that the vanishing torsion condition is always allowed by the equations of motion; see [167], see also [66]. We will make use of the first-order formalism in section 4, as it is almost unavoidable in the discussion of Chern-Simons theory. However, with the intention to make the exposition as friendly as possible, we will avoid abstruse technology in the rest of this work. In any case, since we could not afford to give all the definitions necessary to introduce the subject, we will assume the reader is familiarized with basic notions of the theory of gravity and with the standard nomenclature.

# Chapter 3

# Black holes in Lovelock theory

## 1 Spherically symmetric solutions

Let us first consider the theory in five dimensions. Since in $D < 7$ the $\mathcal{R}^3$ and higher order terms do not contribute to the equations of motion, the five-dimensional Lovelock theory basically corresponds to Einstein gravity coupled to the dimensional extension of the four dimensional Euler density, i.e. the theory that is usually referred as Einstein-Gauss-Bonnet theory (EGB), defined by the following Lagrangian

$$\mathcal{L} = \sqrt{-g}\, \left(\alpha_0 + \alpha_1 R + \alpha_2 \left(R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}\right)\right), \qquad (3.1)$$

The spherically symmetric static solution of EGB theory was obtained by Boulware and Deser in Ref. [24]. The metric takes the simple form [70]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_3^2 \qquad (3.2)$$

where $d\Omega_3^2$ is the metric of a unitary 3-sphere, and where the metric function $f(r)$ is given by

$$f(r) = 1 + \frac{r^2}{4\alpha} + \xi \frac{r^2}{4\alpha}\sqrt{1 + \frac{16\alpha M}{r^4} + \frac{4\alpha\Lambda}{3}}, \qquad (3.3)$$

with $\xi^2 = 1$. Here we used the standard convention $\alpha_0/\alpha_1 = -2\Lambda$, $\alpha_2/\alpha_1 = \alpha$, and, besides, we have set the Newton constant to a specific value for short. From (3.3) we notice that there exist two different branches of solutions to the spherically symmetric ansatz (3.2), namely $f_+(r)$ in the case $\xi = +1$ and $f_-(r)$ in the case $\xi = -1$, and this reflects the fact that the equations of motion give a differential equation quadratic in the metric function $f(r)$. As usual, the parameter $M$ arises here as an integration constant, and it corresponds to the mass of the solution[1], up to the factor we absorbed[2] in $M$.

It is worth mentioning that (3.2)-(3.3) is the most general spherically symmetric solution to EGB theory, provided the fact that the metric is smooth everywhere and that the parameters $\Lambda$ and $\alpha$ are generic enough. In turn, a Birkhoff theorem holds for this model [67, 168, 43, 73]. It is important to emphasize that for very particular choices of the set of parameters $\alpha_n$, degeneracy in the space of solutions can appear, and in those special cases the Birkhoff's theorem can be circumvented; see [168] for a very interesting discussion. To our knowledge, the most complete analysis of the EGB analogue of Birkhoff's theorem was performed in [120], where the Nariai-type solutions [40] where also discussed.

As it was already mentioned, the Boulware-Deser metric presents two different branches, corresponding to $\xi = -1$ and $\xi = +1$. Interestingly, these two branches have substantially different behaviors, and only one of them tends to the GR solution in the small $\alpha$ limit. In fact, in the limit $\alpha \to 0$ the branch $\xi = -1$ looks like

$$f_-^2(r) \simeq 1 - \frac{2M}{r^2} - \frac{\Lambda}{6}r^2, \tag{3.4}$$

where we see it approaches the five-dimensional (Anti)-de Sitter-Schwarzschild-Tangherlini solution [152]. On the other hand, in the $\alpha \to 0$ limit the solution corresponding to the branch $\xi = +1$ behaves like

$$f_+^2(r) \simeq 1 + \frac{2M}{r^2} + \frac{\Lambda}{6}r^2 + \frac{1}{2\alpha}r^2, \tag{3.5}$$

and we see it acquires a large effective cosmological constant term $\sim r^2/2\alpha$. In particular, this implies that microscopic (A)dS space-time is a solution of

[1]For the discussion on the computation of charges in this theory see the list of references [5, 128, 127, 7, 8, 126, 59, 112, 120, 71]; see also [142, 141, 72, 64, 28].

[2]More precisely, in the definition of $M$ we absorbed a factor $\frac{8\pi G}{(D-2)\Omega_{D-2}}$ where $\Omega_n = \frac{(n+1)\pi^{(n+1)/2}}{\Gamma((n+3)/2)}$ is the surface of the $n$-sphere, and where $G$ is the Newton constant, given by $G \sim \alpha_1^{-1}$, which has been fixed to a specific values such that $\alpha_1 = 1$.

the theory even for $\Lambda = 0$. This feature was expressed by Boulware and Deser [24] by saying that EGB theory has its own cosmological constant problem, with $\Lambda_{\text{eff}} \sim -1/\alpha$. In a sense, the branch $\xi = +1$ is commonly believed to be a false vacuum of the theory, and it is known to present ghost instabilities [24]; see also [25].

The branch $\xi = -1$, on the other hand, is well-behaved, and it represents short distance corrections to GR black holes (3.4). While at short distances the black hole solutions of both theories are substantially different due to the effects of the Gauss-Bonnet term, in the large distance regime $r^2 >> \alpha$, and in the case of non-vanishing cosmological constant, the Lovelock black hole (3.2) with $\xi = -1$ behaves like a GR black hole whose parameters $M$ and $\Lambda$ get corrected by finite-$\alpha$ subleading contributions $\mathcal{O}(\alpha\Lambda)$ [2]. Namely

$$f_{-}(r) = 1 - \frac{2m_d}{\pi r^2} - \frac{\Lambda_d}{6} r^2 + \mathcal{O}(\alpha r^{-6}) \tag{3.6}$$

where the *dressed* parameters $m_d$ and $\Lambda_d$ are given by

$$\Lambda_d = \Lambda \quad 1 + \sum_{n=2}^{\infty} c_n \; x^{n-1} \Bigg) = 1 - \sqrt{1+x}, \quad m_d = m \quad 1 + \sum_{n=2}^{\infty} n \; c_n \; (-x)^{n-1} \Bigg),$$

with

$$c_n = \frac{(2n-3)!!}{2^{n-1} n!} \; , \quad x := \frac{4}{3}\Lambda\alpha.$$

It is important to emphasize the difference existing between (3.4) and (3.6): While the first corresponds to the actual limit $\alpha \to 0$, the second represents the large $r^2/\alpha$ regime which takes into account finite-$\alpha$ contributions. For instance, the finite-$\alpha$ corrections to the mass are found by simply collecting the coefficients of the Newtonian term $\sim r^2$. The parameter $x$ controls the *dressing* of the whole set of black hole parameters. The above power expansion converges for values such that $x < 1$. On the other hand, for $x > 1$ we find a different expansion, leading to the following *dressed* parameters in the large $r$ regime

$$m_d = \frac{m}{\sqrt{|x|}} \quad 1 + \sum_{n=2}^{\infty} n \; c_n \; (-x)^{1-n} \Bigg)$$

Thus, we note that the Newtonian term $\sim m_d r^{-2}$ vanishes in the limit $|\Lambda\alpha| \to \infty$. The particular case $x = 1$ is discussed below. Moreover, it is possible to see that, if one considers the case $\alpha\Lambda > 0$, the effective cosmological constant in the large $x$ limit turns out to be

$$\Lambda_d = \sqrt{\frac{3\Lambda}{\alpha}} - \frac{3}{2\alpha} + \mathcal{O}(1/\sqrt{|x|}) \ .$$

One of the relevant differences existing between the black hole solutions in Einstein theory and in Einstein-Gauss-Bonnet theory is the fact that, in the latter, the metric does not diverge at the origin of Schwarzschild coordinates, $r = 0$, though its curvature is still singular. From (4.9), we easily observe

$$f_{\pm}(r = 0) = 1 \pm \sqrt{\frac{M}{\alpha}}.$$

In particular, this implies that the metric presents a angular deficit around the origin, and, also, that massive objects with no even horizon exist; thus, these correspond to naked singularities.

So in the large $r$ limit, the next-to-leading $r$-dependent contribution to (3.4) goes like $\mathcal{O}(\alpha r^{-6})$. The damping of this additional term, which in $D$ dimensions goes like $\mathcal{O}(\alpha r^{4-2D})$, is actually strong, and, for distance large enough, it is negligible even in comparison with semiclassical corrections to the metric due to field theory backreaction, which typically go like $\mathcal{O}(\hbar r^{5-2D})$.

For $\Lambda = 0$, the $\xi = +1$ branch depends on $\alpha$ asymptotically, while the asymptotically flat branch $\xi = -1$ does not. Also, the sign of the Schwarzschild type of term depends on the branch: the two branches view the energy $M$ differently, i.e. the exotic metric of the Boulware-Deser solution does not reduce to Einstein solution in the "infrared" limit.

If $\alpha > 0$, the solution corresponding to $\xi = -1$ in (3.3) may represent a black hole solution whose horizon, in the case $\Lambda = 0$, is located at $r_+ = \sqrt{2(M - \alpha)}$. On the other hand, as long as $\alpha > 0$ and $M > 0$, the branch $\xi = +1$ has no horizon but presents a naked singularity at $r = 0$.

The sign $\xi$ is in some sense a charge which determines how a certain energy $M$ enters a metric and thus if the field will be attractive or repulsive. As noted in [25], the graviton is a ghost on the asymptotic $\xi = +1$ branch, because the linear Einstein tensor appears to have the opposite overall sign (that is, this metric is classically unstable). This wrong sign is reflected in the inverted sign of the Schwarzschild term.

Another interesting feature of the presence of the Gauss-Bonnet term is that, for the particular choice of the parameters $\alpha\Lambda = -\frac{3}{4}$, the solution takes the form

$$f_{\pm}(r) = \frac{r^2}{4\alpha} - \mathcal{M} \tag{3.7}$$

where we have considered $\Lambda < 0$ and $\alpha > 0$, and where $\mathcal{M} + 1 = \sqrt{\frac{M}{\alpha}}$. This solution resembles the Bañados-Teitelboim-Zanelli black hole [14, 12]. Actually, the solution (3.7) shares several properties with the three-dimensional black hole geometry, as it is the case of its thermodynamics properties. Parameter $\mathcal{M}$ in Eq. (3.7) plays the role of the mass $M$ in the BTZ solution. For instance, just like $AdS_3$ space-time is obtained as a particular case of the BTZ geometry by setting the negative mass $M = -(8G)^{-1}$, the five-dimensional Anti-de Sitter space corresponds to setting $\mathcal{M} = -1$ in Eq. (3.7). Moreover, notice that in the large $\mathcal{M}^{-1}$ limit the solution becomes the metric to which $AdS_5$ tends in the near boundary limit. Similarly, the massless BTZ corresponds to the boundary of $AdS_3$. Besides, as it was already mentioned, a conical singularity is found in the range $0 < M < \alpha$ (corresponding to $-1 < \mathcal{M} < 0$), and this completes the parallelism with the three-dimensional black hole. We will study this case with further detail in section 4.

All these features are essentially due to the nature of the Gauss-Bonnet term, and also hold in higher dimensions. In fact, it is straightforward to generalize solution (3.2) to the case of EGB gravity in $D > 5$ dimensions, and the metric is seen to adopt a very similar form [24]. Actually, it is given by simply replacing the element of the 3-sphere in (3.2) by the element of the unitary $(D-2)$-sphere $d\Omega^2_{D-2}$, and by replacing the piece $16\alpha/r^4$ in (3.3) by $16\alpha/r^{D-1}$.

In spite of the non-polynomial form of (3.3), the horizon structure of Boulware-Deser solution is quite simple, and in $D$ dimensions the horizon location is given by the roots of the polynomial

$$\frac{\Lambda}{6} r^{D-1} - r^{D-3} - 2\alpha r^{D-5} + 2M = 0, \tag{3.8}$$

where $\Lambda$ has been appropriately rescaled by a $D$-dimensional constant factor.

The five-dimensional case is actually a remarkable example since, among other special features, it allows to have massive solutions with naked singularities. Actually, the metric (3.2) turns out to be finite at the origin, namely $f^2_{(r=0)} = 1 + \xi\sqrt{M/\alpha}$, nevertheless, the curvature still diverges at the origin, although not in a dramatic way. We will return to this point in chapter 5 where we will discuss naked singularities. We mentioned above that if $D = 5$ and $\Lambda = 0$ the black hole horizon is located at $r_+^2 = 2(M - \alpha)$, and this implies a lower bound for the spherical solution not to develop a naked singularity, namely $M > \alpha$. That is, for $0 < M < \alpha$ we do find naked singularities even for the well-behaved branch $\xi = -1$ with positive $M$. For the model with a second order term $\mathcal{R}^2$ this only occurs in $D = 5$. In seven dimensions, for instance, the Boulware-Deser solution with $\Lambda = 0$ develops horizons at $r_+^2 = \alpha\sqrt{1 + 2M/\alpha^2} - \alpha$ and then the horizon always exists provided $\alpha > 0$, $M > 0$. Naked singularities in $D = 2n + 1$ dimensions usually arise when a term of order $\mathcal{R}^n$ is present in the action. So, for the EGB theory this only occurs for $D = 5$.

It could be important to mention that the analysis of the dynamical stability of EGB black holes is also special for $D = 5$. The stability analysis under tensor mode perturbations has been explored recently, and it has been shown that the EGB theory exhibits some differences with respect to Einstein theory; at least, it seems to be the case for sufficiently small values of the mass in five and six dimensions [74] where instabilities arise; see also Refs. [75, 21, 85]. In this sense, the cases $D = 5$ and $D = 6$ are special ones. See Ref. [96] for an interesting recent discussion.

Now, let us be reminded of the fact that in $D > 6$ dimensions the Lovelock action (2.1) presents also additional terms of higher order $n > 2$, so that in $D \geq 7$ the Boulware-Deser black hole geometry (3.2)-(3.3) only corresponds to a very special example of Lovelock black hole.

Spherically symmetric solutions in higher dimensions containing arbitrary higher order terms $\mathcal{R}^n$ in (2.1) can be implicitly found by solving a polynomial equation of degree $n$ whose solutions give the metric function $f(r)$; this was originally noticed by Wheeler in [160, 161]. Moreover, several explicit examples containing arbitrary amount of terms $\mathcal{R}, \mathcal{R}^2, ...\ \mathcal{R}^{n-1}, \mathcal{R}^n$ are also known. These correspond to particular choices of the couplings $\alpha_n$ in (2.1). One of these explicitly solvable cases corresponds to the Chern-Simons theory, which exists in odd dimensions. We will briefly discuss this special case in section 4. A remarkable fact is that in the case a term $\mathcal{R}^n$ of the Lovelock expansion (2.1) is considered in the action, then the spherically symmetric solution may still take a very simple expression, and, depending on the coupling constants $\alpha_n$, it may merely correspond to replacing the square root in (3.3) by a power $1/n$; see [63, 58, 32, 49] for explicit examples.

On the other hand, it is quite remarkable that electrically charged black hole solutions in Lovelock theory also present a very simple form. The solutions charged under both Maxwell and Born-Infeld electrodynamics have been known for long time [164, 163], and these solutions were reconsidered recently [2]. In general, the metric function of a charged solution takes the form (3.3) but replacing the mass parameter $M$ by a mass function $M(r)$ that depends on the radial coordinate $r$. Function $M(r)$ depends on the particular electromagnetic Lagrangian one considers. In the case of Maxwell theory, and in five dimensions, this function is given by the energy contribution $M(r) \sim \int_{\varepsilon}^{r} dr\; Q^2/r^3 \sim -Q^2/r^2 + M_0$, where $Q$ represents the electric charge of the black hole, and where the UV cut-off in the integral is absorbed in the definition of the additive constant $M_0$. More precisely, for charged black holes in Einstein-Gauss-Bonnet-Maxwell theory we have $M(r) - M_0 = -Q^2/6r^2$, as it was originally noticed by Wiltshire [164].

In the next section we will consider a generalization of the black hole solutions reviewed here. We will discuss extended black objects in EGB theory.

# 2 Topological black holes

One of the interesting aspects of Lovelock theory is that it admits another class of black objects, whose horizons are not necessarily positive curvature hypersurfaces. These solutions are usually called topological black holes, and their metric are obtained by replacing the $(D-2)$-sphere $d\Omega_{D-2}^2$ in (3.2) by a base manifold $d\Sigma_{D-2}^2$ of constant (but not necessarily positive) curvature, provided a suitable shifting in the metric function $f(r)$. Namely, these solutions read

$$ds^2 = -K^2(r)dt^2 + K^{-2}(r)dr^2 + r^2 d\Sigma_{D-2}^2 \tag{3.9}$$

where the metric function is now given by $K^2(r) = f(r) + k - 1$, being $k$ the sign of the curvature of the horizon hypersurface whose line element is $r_+^2 d\Sigma_{D-2}^2$. For $k = 1$ the Boulware-Deser solution (3.2)-(3.3) is recovered. In general, the base manifold $d\Sigma_{D-2}^2$ here may be given by a more general constant curvature space: For instance, it can be given by the product of hyperbolic spaces $d\Sigma_{D-2}^2 = dH_{D-2}^2$ for the case of negative curvature $k = -1$, or merely by a flat space piece $d\Sigma_{D-2}^2 = dx_i dx^i$. In turn, solutions (3.9) correspond to black brane type geometries. Such black objects represent fibrations over constant curvature $(D-2)$-dimensional hypersurfaces, implying that the event horizon, in the cases it exists, is not necessarily a compact simply connected manifold.

Consider for example the five-dimensional EGB theory with negative cosmological constant $\Lambda < 0$, and its black brane solution of the form

$$ds^2 = -K^2_{(k=0)}(r)dt^2 + K^{-2}_{(k=0)}(r)dr^2 + r^2 dx^i dx_i \tag{3.10}$$

with

$$K^2_{(k=0)}(r) = \frac{r^2}{4\alpha} - \sqrt{\frac{r^4}{16\alpha^2}\left(1 - 4|\Lambda|\alpha/3\right) + \frac{M}{\alpha}}, \tag{3.11}$$

where $x^i = x^1, x^2, x^3$. These objects (brane-like configurations and topological black holes) have attracted some attention recently due to their curious properties, and, more recently, these were considered in applications inspired in string theory; see for instance [26, 27].

In [76], an exhaustive classification of static topological black hole solutions of five-dimensional Lovelock theory was presented. The authors considered an ansatz such that spacelike sections are given by warped product of the radial coordinate $r$ and an arbitrary base manifold $d\Sigma^2_{D-2}$, and they showed that, for values of the coupling constant $\alpha_2$ generic enough, the base manifold must be necessarily of constant curvature, and then the solutions of the theory reduce to the topological extension of the Boulware-Deser metric of the form (3.9). In addition, they showed that for the special case where the coupling $\alpha_2$ is appropriately tuned in terms of the cosmological constant $\alpha_0$, then the base manifold could admit a wider class of geometries, and such enhancement of the freedom in choosing $d\Sigma^2_{D-2}$ allows to construct very curious solutions with non-trivial topology. We will return to this point in section 4.

The existence of black holes with generic horizon structure was also analyzed in [76], where selection criteria for the base manifold $d\Sigma^2_{D-2}$ were discussed, and the authors concluded that sensible physical models strongly restrict most of the examples of exotic black holes with non-constant curvature horizons. Moreover, the different horizon structures were also studied in [9, 155] together with its relation to the asymptotic behavior of the corresponding solutions; see also [54, 55, 62, 56, 140]. Recently, the electrically charged topological black hole solutions were also analyzed, both for the case of the second order Lovelock theory in [154, 54] and for the case of the third order[3] Lovelock theory in [56].

[3]Recently, references [95, 61, 60, 93, 92, 4] discussed other classes of solutions. We will

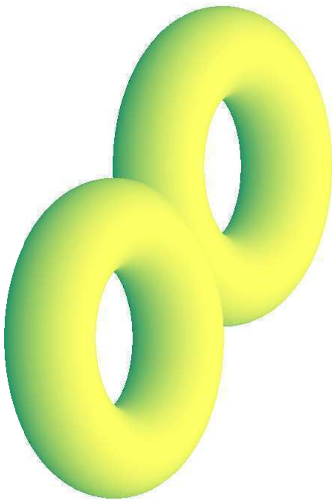

Figure 3.1: Horizons with non-trivial topology are admitted.

One of the most interesting aspects of these objects with non-trivial horizon geometries is that they enable to construct a very simple class of Kaluza-Klein black holes with interesting properties from the four-dimensional viewpoint. For instance, such a solution was recently studied by Maeda and Dadhich in Ref. [118]. These Kaluza-Klein black holes are given by a product $\mathrm{M}_4 \times \mathrm{H}_{D-4}$ between a four-dimensional manifold $\mathrm{M}_4$ and a $(D-4)$-dimensional hyperbolic space $\mathrm{H}_{D-4}$. It turns out that the four-dimensional piece of the geometry asymptotically approaches the charged black hole in locally $\mathrm{AdS}_4$ space. In turn, the Gauss-Bonnet term acts by emulating the Reissner-Nordström term for large $r$, while it changes the geometry at short distances [125, 52, 119]. In addition to these solutions, other exotic Kaluza-Klein Lovelock black hole solutions with arbitrary order terms of the form $\mathcal{R}^n$ and for a specific values of the coefficients $\alpha_n$ were studied in [84]. These black holes are different from those studied in [118], and are obtained by considering black $p$-brane geometries of the form $\mathrm{M}_{D-p} \times \mathrm{T}^p$ in the Lovelock theory with $\alpha_i = \delta_{i,n}$ and $2n = D - p$. These solutions exist for $D - p$ even, and, in addition, the horizon structure also depends on $n$. Analogous toric compactifications of the form $\mathrm{M}_{D-p} \times \mathrm{T}^p$ were studied in [107], and warped brane-like configurations were also discussed in both [84] and [107].

It was shown in [84] that, in spite of the difference between Lovelock theory and Einstein theory, the qualitative features of thermodynamic stability of brane-like configurations in both theories are considerable similar, although the higher order terms $\mathcal{R}^n$ can be seen to contribute. For example, the thermodynamical analogue of Gregory-Laflamme transition between black hole and black string configurations was discussed in [84]. Extended string-like objects in Lovelock theory and their thermodynamics were also discussed in [111, 146, 28]. We discuss black hole thermodynamics in the next section.

# 3 Black hole thermodynamics

The purpose of this section is to describe the general aspects of black hole thermodynamics in Lovelock theory. In fact, one of the most interesting features of the Lovelock theory regards the thermodynamics of its black hole solutions. This is because it is in the analysis of the black hole thermodynam-

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not comment on these solutions here.

ics where the substantial differences between Lovelock theory and Einstein theory manifest themselves.

Pioneer works where the Lovelock black hole thermodynamics was discussed in detail are references [135, 162]; see also [103, 105, 104]. In [106], Jacobson and Myers derived a close expression for the entropy of these solutions in $D$ dimensions, and they showed that the entropy of these black holes does not satisfy the area law, but contains additional terms that are given by a sum of intrinsic curvature invariants integrated over the horizon.

The thermodynamics of charged solutions was originally studied by Wiltshire in Refs. [164, 163], while the thermodynamics of topological black holes was studied more recently, in Refs. [33, 9, 29]. The study of charged topological black holes in presence of cosmological constant was addressed in [50], where the most general solution of this type in EGB theory was obtained. References [45, 136, 137] also analyze topological black holes and their thermodynamics; see also [62, 30].

The aim of this section is to discuss the more relevant thermodynamical features of Lovelock solutions. To do this, we will consider again the five-dimensional case (3.2)-(3.3). Actually, besides it represents a simple instructive example, the five-dimensional case is also special in what concerns thermodynamical properties. It is the best example to see that substantial differences between Lovelock gravity and Einstein gravity exist.

It is easy to verify that the Hawking temperature associated to the solution in $D = 5$ with $\Lambda = 0$ is given by

$$T = \frac{\hbar}{2\pi} \frac{r_+}{4\alpha + r_+^2}. \tag{3.12}$$

Then, we see that, as expected, (3.12) behaves like the Hawking temperature of a GR solution if the black hole is large enough, $r_+ >> \alpha$, going like $T \simeq \hbar/8\pi r_+ - \mathcal{O}(\alpha/r_+^3)$. On the other hand, temperature tends to zero for small values of $r_+$, going like $T \simeq \hbar r_+/8\pi\alpha + \mathcal{O}(r_+^3/\alpha^2)$. This implies that the specific heat changes its sign at length scales of order $r_+ \sim \sqrt{\alpha}$, and a direct consequence of this phenomenon is that five-dimensional Lovelock black holes turn out to be thermodynamically stable, as they yield eternal remnants. This can be easily verified by considering the rate of thermal radiation which goes like $\partial_t M \sim -T^5 r_+^3$, behaving like $dt \sim -dr_+/r_+^7$ at short distances.

Nevertheless, it is worth pointing out that for dimension $D > 5$ the functional form of the temperature is substantially different from the case

$D = 5$, as it includes an additional term which is actually proportional to $(D-5)$. The general formula reads

$$T = \frac{\hbar}{4\pi} \frac{(D-3)r_+^2 + 2\alpha(D-5)}{4\alpha r_+ + r_+^3}. \tag{3.13}$$

which implies that, in $D > 5$, the short distance limit is given by $T \simeq (D-5)\hbar/8\pi r_+$, and the specific heat is then negative. This is the reason why the thermodynamic behavior of higher dimensional Einstein-Gauss-Bonnet black holes turns out to be more similar to that in Einstein theory if $D \neq 5$. In general, eternal black holes arise in $D = 2n+1$ dimensions if an $n^{\text{th}}$-order term $\mathcal{R}^n$ is present in the action.

So, let us return to our instructive example of five dimensions. The entropy associated to (3.12) is given by

$$S = \frac{\mathcal{A}}{4G\hbar} + \mathcal{O}(\alpha r_+) \sim r_+^3 + 12\alpha r_+, \tag{3.14}$$

from what we observe that black holes of Lovelock theory do not in general obey the Bekenstein-Hawking area law. Actually, some particular solutions, corresponding to topological black holes with flat horizon geometry $d\Sigma_3^2 = dx_i dx^i$, do obey the area law [57, 30], but it is not the case for spherically symmetric static solutions. A very interesting discussion on the area law[4] is that of Ref. [140], where a version of the area law for symmetric dynamical black holes defined by a future outer trapping horizon was derived. There, the authors discussed the differences between the branches of solutions with GR limit and those without it, and argue how for the latter one still can define a concept of increasing dynamical entropy.

Notice that the second term in the right hand side of (3.14) implies that if $\alpha < 0$ then the entropy turns out to be negative for sufficiently small black holes[5]. This was discussed in [47], where it was argued there that an additive ambiguity in the definition of the entropy could be a solution for the negative entropy contributions; see also the related discussion in [9]. In any case, the theory for negative values of the coupling constant $\alpha$ is somehow pathological in several respects. It not only gives negative contributions to the entropy,

[4] In [147] other corrections to area law were studied. The authors thank S. Shankaranarayanan for pointing out this references to them.

[5] Refs. [138, 114] discuss related features. The authors thank S. Odintsov for pointing out these references to them.

but also ghost instabilities and strange causal structure arise if $\alpha < 0$. We will not consider the negative values of $\alpha$ here.

Because of the current interest in black hole thermodynamics of higher order theories, we consider convenient to mention that the entropy function formalism, recently proposed by A. Sen [150] within the context of the attractor mechanism, works nicely for the case of Lovelock black holes. In particular, this was recently studied in [129] for the case of EGB black holes, and it was explicitly shown that (3.14) is recovered by analyzing the near horizon geometry. A rather general analysis was presented in Ref. [41]. Very interesting discussions are those of Refs. [11, 10].

The thermodynamic properties of topological black holes are also very interesting; see for instance [31, 30]. As we already mentioned, it can be shown that those black objects whose horizons are of zero curvature do obey the area law for the entropy density. For instance, consider the black brane geometry (3.10), which is solution of the theory with negative cosmological constant, $\Lambda < 0$. It is straightforward to check that the Hawking temperature of this solution is given by

$$T = \frac{\hbar}{6\pi}|\Lambda| r_+, \tag{3.15}$$

and that the area formula for the entropy density does hold in this special case. Remarkably, identical expression for the temperature is obtained in the particular case of the Chern-Simons theories of gravity, which we discuss in the next section.

# 4 Chern-Simons Black Holes

Now, let us move on, and analyze a very particular case of Lovelock theory which exist in odd dimensions. This is the so-called Chern-Simons gravity (CS), and can be thought of as a higher-dimensional generalization of the Chern-Simons description of three-dimensional Einstein gravity [165]. Basically, these theories are those particular cases of Lovelock Lagrangian (2.1) that admit a formulation in terms of a Chern-Simons action. As we will discuss, these models are given by a very precise choice of the set of coefficients $\alpha_n$.

To discuss CS gravity theories[6] it is convenient to resort to the first-order

[6]It is worth pointing out that the CS theories we are referring to herein are different

formalism which, in spite of its advantage, it is paradoxically avoided in physics discussions. So, let us first review some basic notions: Consider the vielbein $e^a_\mu$, which defines the metric as $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, where we are using the standard notation such that the greek indices $\mu, \nu, ...$ correspond to the space-time while the latin indices $a, b, ...$ are reserved for the tangent space. Now, consider the 1-form associated to the vielbein, defined by $e^a = e^a_\mu dx^\mu$, and the corresponding 1-form associated to the spin connection $\omega^{ab}_\mu$, defined by $\omega^{ab} = \omega^{ab}_\mu dx^\mu$. These quantities enable us to define the so-called curvature 2-form, which is given by

$$R^{ab} = d\omega^{ab} + \omega^a_{\ c} \wedge \omega^{cb} = R^{ab}_{\ \ \ \mu\nu}\ dx^\mu \wedge dx^\nu \equiv \frac{1}{2} R^{ab}_{\ \ \ \mu\nu}\ (dx^\mu dx^\nu - dx^\nu dx^\mu),$$

and is related to the Riemann tensor by $R^\alpha_{\ \beta\mu\nu} = \eta_{bc} e^\alpha_a e^c_\beta R^{ab}_{\ \ \ \mu\nu}$. The torsion-free condition is then given by

$$T^a = de^a + \omega^a_b \wedge e^b = 0.$$

In this language, local Lorentz invariance of the theory is expressed in terms of the covariant derivative

$$\delta_\lambda \omega^a_b = d\lambda^a_b + \omega^a_c \wedge \lambda^c_b - \omega^c_b \wedge \lambda^a_c, \qquad \delta_\lambda e^a = -\lambda^a_b e^b, \tag{3.16}$$

where $\lambda^a_b$ represent the parameters of the transformation.

The remarkable fact is that, for particular cases of the action (2.1), if the coupling constants are chosen appropriately, the theory exhibits an additional local symmetry. For instance, if we consider the case $\Lambda = 0$, such additional symmetry turns out to be given by the invariance of the Lagrangian density under the gauge transformation

$$\delta_\lambda e^a = d\lambda^a + \omega^a_b \wedge \lambda^b, \qquad \delta_\lambda \omega^a_b = 0. \tag{3.17}$$

That is, the CS theory possesses a local symmetry under gauge transformation $\delta_\lambda e^a_\mu = \partial^a_\mu \lambda + \omega^a_{b\mu} \lambda^b$, with $\lambda^a$ being a parameter. This is actually an off-shell local gauge symmetry of the theory (2.1) that arises for special choices of the coupling constants $\alpha_n$, as far as the boundary conditions are also chosen in the appropriate way. Besides, it can be easily verified that transformation (3.17), once considered together with (3.16), satisfies

to those discussed in Refs. [69, 68].

the Poincaré algebra $ISO(2,1)$, and this is why these theories are usually referred as Poincaré-Chern-Simons gravitational theories [17]; see also [167] for an excellent introduction to Chern-Simons gravity.

So, let us specify which are the theories that possess the gauge symmetry like[7] (3.16)-(3.17), namely the CS theories. To do this, first it is convenient to rewrite the Lovelock Lagrangian. In the first-order formalism, the Lovelock action corresponding to (2.1) in $D = 2t + 1$ dimensions can be written as

$$S = \int \varepsilon_{a_1 b_1 a_2 b_2 \ldots a_t b_t c} \bigwedge\nolimits_{n=1}^{t} \left( R^{a_n b_n} + l_n^{-2} e^{a_n} \wedge e^{b_n} \right) \wedge e^c \tag{3.18}$$

where $l_n^{-2}$ correspond to $t$ independent coefficients that are a rearrangement of the coefficients $\alpha_n$. In (3.18), the convention is such that the $t^{\text{th}}$ coupling $\alpha_{n=t}$ has been set to 1 (or, alternatively speaking, it has been absorbed in the definition of the curvature $R^{ab}$), so that in this notation we have $|\Lambda| \sim \prod_{n=1}^{t} l_n^{-2}$, and $G^{-1} \sim \sum_{m=1}^{t} \prod_{n \neq m} l_n^{-2}$.

It is worth noticing that, in order to represent the most general form of (2.1), the coefficients $l_n^{-2}$ in (3.18) should be allowed to take complex values. In fact, Lovelock action (2.1) with real coefficients $\alpha_n$ can correspond to (3.18) with imaginary $l_n^{-2}$. An example is given by the five-dimensional theory whose action reads $S = \int \varepsilon_{abcdf} \left( R^{ab} + i\beta^2\ e^a \wedge e^b \right) \wedge \left( R^{cd} - i\beta^2\ e^c \wedge e^d \right) \wedge e^f$, which leads to the particular form of (2.1) where no Einstein-Hilbert contribution is present, but only the cosmological constant and the Gauss-Bonnet term appear, with $\alpha/\Lambda \sim \beta^{-4}$ for a real $\beta$.

The CS gravity theories, however, are given by real values of $l_n^{-2}$. More precisely, CS theory correspond to the special case where all the coupling $l_n^2$ in (3.18) are equal, namely $l_1^2 = l_2^2 = ... = l_t^2 \equiv l^2$. In terms of the Lagrangian density (2.1) this corresponds to taking the coupling constants $\alpha_n$ to be $\alpha_n = (-1)^{n+1} l^{2n-D} m!/((D-2n)(m-n)!n!)$ for $n > 0$, while $\alpha_0$ is given by the cosmological constant $\Lambda = -\alpha_0/2\alpha_1$. It is important to mention that (3.18) corresponds to the case of negative cosmological constant, which yields the CS theory with the AdS$_D$ group (i.e. the group $SO(D-1,2)$) as the one that generates the gauge symmetry. The case of positive $\Lambda$ is simply obtained by changing $l^2 \to -l^2$, while the Poincaré invariant theory is obtained through the Inonu-Winger contraction of (A)dS group; see [167]

[7]Notice that, as mentioned, (3.17) is the transofrmation that corresponds to the case $\Lambda = 0$. The analogous tranformation for the case $l^2 \neq 0$ takes a slightly different form, see [167].

for details. An example of Poincaré invariant CS is given by the Lagrangian containing only the quadratic Gauss-Bonnet $\sqrt{-g}\mathcal{R}^2$ term in five dimensions, without the Einstein-Hilbert term and with $\Lambda = 0$.

As it is well known, an example of the CS gravity theory is given by three-dimensional Einstein theory, whose action[8],

$$S = \int d^3x \ \mathcal{L} = \int d^3x \sqrt{-g} \left( R - 2\Lambda \right), \tag{3.19}$$

admits to be formulated as a CS theory. To see this, and then extend the construction to higher dimensional cases, let us first point out that (5.1) can be written as follows,

$$S = \int_{\mathrm{M}_3} \varepsilon_{abc}(R^{ab} \wedge e^c - l^{-2} e^a \wedge e^b \wedge e^c), \tag{3.20}$$

with $\Lambda \sim l^{-2}$.

It turns out that (5.1)-(3.20) admits to be formulated as a CS theory [165] for the groups $SO(2,2)$, $SO(3,1)$ and $ISO(2,1)$, depending on whether the cosmological constant $\Lambda$ is negative, positive or zero, respectively. To make contact with the usual form of the CS action, let us introduce a $(D+1)$-dimensional 1-form $A^{ab}$ whose indices run over $a, b = 0, 1, 2, ..., 2t+1$ (recall $D = 2t+1$), and its strength field $F^{ab} = dA^{ab} + A^a_{\ c} \wedge A^{cb}$, which are given by

$$A^{ab} = \begin{pmatrix} \omega^{ab} & e^a/l \\ -e^b/l & 0 \end{pmatrix}, \quad F^{ab} = \begin{pmatrix} R^{ab} - l^{-2} e^a \wedge e^b & l^{-1} \left( de^a + \omega^a_{\ c} \wedge e^c \right) \\ -l^{-1} \left( de^b + \omega^b_{\ c} \wedge e^c \right) & 0 \end{pmatrix}.$$

That is, $A^{ab} = \omega^{ab}$ for $a, b = 0, 1, 2, ..., 2t$, while $A^{aD} = -A^{Da} = e^a/l$ for $a = 0, 1, 2...2t$. Analogously, $F^{ab} = R^{ab} - l^{-2} e^a \wedge e^b$ for $a, b = 0, 1, ...2t$, while $F^{aD} = -F^{Da} = T^a/l$ for $a = 0, 1, 2, ...2t$.

Then, making use of these definitions, (5.1)-(3.20) can be alternatively expressed in its Chern-Simons form

$$S = \int_{\mathrm{M}_3} \mathrm{Tr} \ (A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{3.21}$$

[8]For simplicity here we have fixed the Newton constant according to $16\pi G = 1$.

where the trace is over the indices $a, b$ that run from 0 to 3 (corresponding to $D = 3$, i.e. $t = 1$). Local symmetry under (3.16) and (3.17) is then gathered by gauge symmetry of (3.21).

The next example we could consider is the five-dimensional one, which corresponds to the Lovelock theory (3.1) for the particular case $\alpha_0\alpha_2 = 3/2$ (i.e. $\alpha\Lambda = -3/4$). Then, the action reads

$$S = \int d^5x\,\mathcal{L} = \int d^5x\sqrt{-g}(R+\frac{2}{l^2}-\frac{3l^2}{4}(R+R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R^{\mu\nu}R_{\mu\nu})) \quad (3.22)$$

where $\Lambda = -l^{-2}$ and $\alpha_2 = 3/2\alpha_0 = -3/4\Lambda = 3l^2/4$. This can be also written as

$$S = \int_{\mathrm{M}_5} \varepsilon_{abcdf}\ (R^{ab} \wedge R^{cd} + \frac{2}{3l^2}R^{ab} \wedge e^c \wedge e^d + \frac{1}{5l^4}e^a \wedge e^b \wedge e^c \wedge e^d) \wedge e^f \quad (3.23)$$

and, again, it admits to be written in its Chern-Simons form

$$S = \frac{1}{\kappa^2}\int_{\mathrm{M}_5} \mathrm{Tr}\ (A \wedge (dA)^{\wedge 2} + \frac{3}{2}\,(A)^{\wedge 3} \wedge dA + \frac{3}{5}\,(A)^{\wedge 5}) \quad (3.24)$$

Actually, this structure goes on as $D$ increases, and it expands a whole family of theories which, still being particular cases of Lovelock theory (2.1), represent odd-dimensional field theories with local off-shell symmetry under the (A)dS (or Poincaré) group.

Now, once we have introduced the theories, let us analyze their black hole solutions. Going back to solution (3.2), and considering again the five-dimensional case as an example, we observe that replacing the Chern-Simons condition[9] $\alpha\Lambda = -3/4$ in the metric function (3.3) leads to a rather different geometry, given by

$$f(r) = \frac{r^2}{4\alpha} - \mathcal{M} \qquad \text{with} \qquad \mathcal{M} + 1 = -\xi\sqrt{M/\alpha}. \quad (3.25)$$

This solution still may represent a black hole, provided $\mathcal{M} > 0$, with the horizon located at $r_+ = 2\sqrt{\mathcal{M}/\alpha}$. However, this is a black hole of a different sort. In particular, it does not present a limit where GR is recovered,

[9]It is helthy to consider the case $\alpha > 0$ and $\Lambda < 0$.

and this can be understood in terms of the condition $\alpha = -3/4\Lambda$ in the following way: While the cosmological constant $\Lambda$ introduces an infrared cut-off (the length scale $1/\sqrt{|\Lambda|}$) where the cosmological term dominates over the Einstein-Hilbert term, the Gauss-Bonnet term introduces an ultraviolet cut-off (the length scale $\sqrt{\alpha}$) where the quadratic terms dominate. Therefore, the condition $\alpha = -3/4\Lambda$ basically states that in Chern-Simons theory both length scales are of the same order, and consequently there is no range where the Einstein-Hilbert term is the leading one. This explains why there is no range where (3.25) approaches Schwarzschild-Tangherlini solution. This asphyxia of the Einstein-Hilbert term is a typical feature of Chern-Simons theories for $D > 3$, where a unique free parameter $l^2$ appears in the action.

The Hawking temperature associated to black hole solution (3.25) is given by

$$T = \frac{\hbar}{8\alpha\pi} r_+ = \frac{\hbar}{6\pi} |\Lambda| r_+, \tag{3.26}$$

which in turn agrees with (3.15), although now it corresponds to a spherically symmetric solution. As it is well known [14, 14] in $D = 3$ formula (3.26) agrees with the area law.

Certainly, solution (3.25) is reminiscent of the Bañados-Teitelboim-Zanelli three-dimensional black hole (BTZ), which, after all, also corresponds to a CS black hole. In fact, this is not a coincidence, and regarding this, let us make a historical remark: It turns out that, even though one could imagine that CS black holes (3.25) were discovered as higher-dimensional extensions of the BTZ, the story was precisely the opposite: In 1992, Bañados, Teitelboim and Zanelli discovered the BTZ as a particular case of a family of Lovelock black holes they were studying at that time [13, 16, 15].

The analogy between the BTZ black hole and those solutions for higher-dimensional CS theories was discussed in detail in [2]. In particular, it was emphasized there that five-dimensional solution (3.25) shares several properties with its three-dimensional analogue. For instance, it is the case of their thermodynamics properties, which, after all, are actually encoded in the function $f(r)$. This is also why all CS black holes have infinite lifetime.

Notice that the parameter $\mathcal{M}$ in Eq. (3.25) plays the role that the mass $M$ plays in the BTZ solution. Also, as in the three-dimensional case, the Anti-de Sitter space is obtained for a particular value of this parameter, namely $\mathcal{M} = -1$, and a naked singularity is developed for the range $-1 < \mathcal{M} < 0$.

In [49] the CS black holes and their dimensional extensions were exhaustively studied, together with their topological and charged extensions. There, a very interesting class of black holes was found by considering the particular choice of coefficients that leads to the $(2t+1)$-dimensional CS theory, but dimensionally extending the action from $D = 2t+1$ to $D \geq 2t+1$. The metrics of such solutions are given by replacing the constant $\mathcal{M}$ in (3.25) by the quantity $1 - \mathcal{M}r^{(2t+1-D)/t}$. A further generalization of the solutions of [49] would be given by adding a volume term to the gravitational action, which in turn corresponds to shifting the coupling $\alpha_0 \to \alpha_0 + \delta\Lambda$ but keeping the rest of $\alpha_{n>0}$ tuned as they are in the $(2t+1)$-dimensional CS theory, given in terms of the length scale $l^2$. The solution for this case is given by replacing the constant $\mathcal{M}$ in (3.25) by a term $1 - \left(r^{2t} + \lambda r^{2t} + \mathcal{M}r^{2t+1-D}\right)^{1/t}/l^2$, where $\lambda + 1 \sim \delta\Lambda/\alpha_0$. These black holes do have a GR limit since now the cosmological length scale can be pushed away by choosing $\delta\Lambda$ appropriately.

It is also important to mention that black hole solution (3.25) is also a solution of the CS theory with torsion [38, 39, 37].

The solutions of Chern-Simons theory are very special ones, and this is due to the fact that for that specific choice of the coupling constants $\alpha_n$ the equations of motion of Lovelock theory somehow degenerate. In particular, it is remarkable that the obstruction imposed by Birkhoff-like theorems does not hold for CS theories.

# 5 Towards Rotating Black Holes

The problem of finding a rotating solution in Einstein-Gauss-Bonnet gravity, which would generalize the Kerr's spinning black hole of GR, is a very interesting and still unsolved problem. Some recent claims in the literature [3] turned out to be incorrect, and this gave rise to some activity in this direction.

Recently, it was proven in [6] that the Kerr-Schild ansatz doesn't work in Lovelock theory (except for very special cases as Einstein theory and Chern-Simons theory) which manifestly shows how difficult this classical problem can be.

Nevertheless, some advances in this area were recently achieved: In [6] an exact analytic rotating solution was found for Chern-Simons gravity in five dimensions. This Einstein-Gauss-Bonnet solution, however, does not present a horizon, and thus it does not represent a black hole. Nevertheless, the numerical analysis carried out in [28] suggests that the rotating solutions actually exist. Besides, approximate analytic solutions at first order in the angular momentum parameter were found in [109]. Other solutions are known which represent rotating flat branes. However, these are, indeed, a trivial extension of the topological black holes with $k = 0$.

Despite these results, the problem of finding an exact analytic rotating black hole solution in Lovelock theory still remains an open problem.

# Chapter 4

# Gravitational solitons in Lovelock Theory

## 1 Why do we expect wormholes?

The next class of solutions we would like to discuss is a class of vacuum solutions of Lovelock theory which represents wormhole geometries that connect two disconnected asymptotic regions of the space-time. Recently, several examples of such solutions were found [81, 22, 79, 121, 145, 48], describing vacuum wormholes with different asymptotic behaviors, and in different number of dimensions. So, the first question we might ask is: why do wormholes exist in Lovelock theory?

The main reason why vacuum wormholes exist in a theory like (3.1) is actually simple, and it can be heuristically explained as follows: Consider the equations of motion corresponding to Lagrangian (3.1), which can be always written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu} = 0 \tag{4.1}$$

where the higher order terms act as an effective stress tensor that here we denoted $T_{\mu\nu}$. In the case of EGB theory it reads

$$\frac{1}{\alpha} T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left( R_{\rho\sigma\alpha\beta} R^{\rho\sigma\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) - 2 R R_{\mu\nu} + 4 R_{\mu\rho} R^{\rho}_{\nu} + R_{\alpha\beta} R^{\alpha\beta}{}_{\mu\nu} - 2 R_{\mu\alpha\beta\rho} R_{\nu}{}^{\alpha\beta\rho},$$

where, as usual, $\alpha = \alpha_2/\alpha_1$, $2\Lambda = -\alpha_0/\alpha_1$. The key point is that this effective stress tensor $T_{\mu\nu}$, thought of as a kind of matter contribution, can be shown to violate the energy conditions for $\alpha$ large enough. Actually, this does not represent an actual problem from the conceptual point of view since this "matter" is actually made of pure gravity. However, a consequence of this violation of the energy conditions is that Eqs. (4.1) allow the existence of vacuum wormhole solutions at scales of order $\sqrt{\alpha}$, unlike the case of GR, where such solutions of this sort require the consideration of exotic matter.

Furthermore, there is a second reason for such curious solutions to exist in Lovelock theory. As mentioned above, when the coefficients $\alpha_n$ in (3.1) correspond to the CS theory, the space of solutions experiments an unusual enhancement, which translates into a large degeneracy of the metric of spaces with enough symmetry. Roughly speaking, for such particular cases, Lovelock theory is somehow degenerated enough to admit metric with very special properties, and wormholes are some of them [82].

Nevertheless, here we will focus our attention on wormhole solutions that exist in five-dimensional EGB theory without requiring the coefficients $\Lambda$ and $\alpha$ to be those that correspond to CS theory. Therefore, the existence of such solutions, regarded as an anomaly, is ultimately attributed to the issue of the energy conditions mentioned above.

The particular configurations we will consider are the so-called thin-shell wormholes, which correspond to connecting two regions of the space through a codimension-one hypersurface that plays the role of the wormhole throat. For such a geometry to be constructed, we have to make use of the junction conditions of the EGB theory [132, 81]. In particular, we will consider the configuration of two Boulware-Deser spaces connected through a hypersurface on which the induced stress-tensor vanishes. Such geometries are not possible in GR, where wormholes require the energy conditions to be violated on the thin-shell. However, in Lovelock theory, and because of the higher order terms, spherically symmetric vacuum wormholes with positive mass can be constructed, as shown by Gravanis and Willison in [88]. Let us review the procedure here.

Let $\Sigma$ be a four-dimensional timelike orientable hypersurface of codimension one, whose normal vector is denoted by $n^\mu$. Suppose $\Sigma$ separates two regions of the space, which we call $\mathcal{M}_I$ and $\mathcal{M}_{II}$. Then, junction conditions read

$$\langle K_{ij} - K h_{ij} \rangle_\Sigma + 2\alpha \left\langle 3J_{ij} - J h_{ij} + 2P_{iklj} K^{kl} \right\rangle_\Sigma = 8\pi S_{ij} \qquad (4.2)$$

where $\langle X \rangle_\Sigma$ denotes the jump of the quantity $X$ across the hypersurface $\Sigma$, which means $\langle X \rangle_\Sigma = X_{|_{II}} \pm X_{|_I}$, where the sign $\pm$ depends on the relative orientation of the regions. Above, tensor $S_{ij}$ represents the induced stress-tensor on the hypersurface $\Sigma$, in complete analogy with the Israel junction conditions in Einstein theory. In fact, we see that the first two terms in (4.2) actually correspond to the Israel junction conditions constructed with the extrinsic curvature $K_i^j$ and its trace $K$. In addition, the junction conditions corresponding to the EGB theory contains contributions cubic in the extrinsic curvature[1],

$$J_{ij} = \frac{1}{3}(K_{kl}K^{kl}K_{ij} + 2KK_{ik}K_j^k - K^2K_{ij} - 2K_{ik}K^{kl}K_{lj}), \qquad (4.3)$$

and also contributions that involve the Riemann curvature tensor of the hypersurface

$$P_{ijkl} = R_{ijkl} + R_{jk}h_{il} - R_{jl}h_{ik} + R_{il}h_{jk} - R_{ik}h_{jl} + \frac{1}{2}Rh_{ik}h_{jl} - \frac{1}{2}Rh_{il}h_{jk}. \quad (4.4)$$

The notation used here is such that latin indices $i, j, k, l$ refer to coordinates on the four-dimensional hypersurface that separate the two five-dimensional regions of the space. The induced metric is denoted by $h_{ij}$. In this work we sudy the most general case, including the case of space-like junctures, which corresponds to a cosmological-type geometries that experiment a change of behavior at a given time characterized by the hypersurface $\Sigma$. It was pointed out by H. Maeda that this kind of space-like junction conditions could be used to construct regular black hole solutions by means of geometric surgery procedure inside the black hole horizon.

# 2 Towards wormholes in Lovelock theory

As we discussed in the previous sections, the presence of the Gauss-Bonnet term introduces some exotic features not found in General Relativity. One such feature is related to the problem of causality; this was treated in Ref. [153] in the Hamiltonian formalism (see also Ref. [46] for an alternative treatment of the Cauchy problem). Because of the non-linearity of the theory, the canonical momenta are not linear in the extrinsic curvature; and there exist quite generically points in the phase space where the Hamiltonian turns out to be multiple-valued. In such a situation, there is a breakdown in the deterministic evolution of the metric from the initial data. This can also be

[1] See [42] for a recent review. See also [124] where boundary terms in odd-dimensions are discussed.

seen explicitly using the junction conditions [53, 87]. In fact, it can be shown that there exist vacuum solutions where the extrinsic curvature can jump spontaneously at some spacelike hypersurface in a way that is not predicted by the initial data[2]. This breakdown in predictability is induced by the presence of terms in the junction conditions which, unlike the Israel conditions valid for Einstein's theory, contain non-linear contributions coming from the Gauss-Bonnet term.

On the other hand, the timelike version of such a jump in the extrinsic curvature is also of great interest. This is realized by the existence of a kind of gravitational solitons in the theory, which resemble a kink solution. These solitons correspond to spacetimes that contain timelike hypersurfaces where the metric is $C^0$ continuous but where the extrinsic curvature jumps. Although the Riemann curvature tensor contains delta-function singularities on the hypersurface, these spacetimes can still be vacuum solutions because of a nontrivial cancelation coming from additional terms in the junction conditions. Some explicit examples have appeared in the literature [123, 98, 110, 94], and a spherically symmetric realization of such solutions were studied in detail in Ref. [88] for the case of pure Gauss-Bonnet gravitational theory. Here, the systematical analysis made in Ref. [88] will be extended to the more phenomenologically important case where Einstein-Hilbert term and cosmological constant are included in the gravitational action. We will show that vacuum shell solutions are indeed found in Einstein-Gauss-Bonnet theory described by the following action :

$$\mathcal{S} = \int \sqrt{-g}\,\left(\alpha_0 + \alpha_1 R + \alpha_2 \left(R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}\right)\right. \tag{4.5}$$

More precisely, we will consider the junction conditions for spherical thin shells in Einstein-Gauss-Bonnet theory, and discuss the case where the induced stress tensor on the shell vanishes. Then, we will show that geometries associated with two spherically symmetric spaces with different masses (and different effective cosmological constant) can be joined without resorting to the introduction of matter fields as a source. Depending on the orientation of the two spaces to be joined, different global structures may arise. For instance, for one choice of orientation the resulting geometries corresponds

[2] The junction condition in vacuum gives precisely that the jump in the canonical momenta is zero. The existence of solutions with non-zero jump in the extrinsic curvature at a spacelike shell is therefore equivalent to the problem of a multiple-valued Hamiltonian.

to vacuum wormholes in five-dimensions. These wormholes are gravitational solitons that connect two regions with different masses and/or effective cosmological constants. The regions are asymptotically either flat, Anti de Sitter (AdS) or de Sitter (dS) depending on the sign of the effective cosmological constant. Besides, other choices in the orientation are possible and lead to vacuum shells that separate two different spherically symmetric region of the space. All the cases we will study in detail are such that the singular hypersurfaces where the jump in extrinsic curvature is located correspond to a sphere. We will call them"vacuum shells".

# 3 The setup

First, we will present some introductory material and notation and conventions and then we will show how these junction conditions permit to join two spherically symmetric spaces without resorting to the introduction of matter source.

With respect to the style of presentation, we have chosen to organize our results in a series of remarks, propositions and theorems in order to highlight key facts, but descriptions such as 'theorem' should not be taken in the most strict mathematical sense.

## 3.1 The bulk metric

Let us consider the Einstein-Gauss-Bonnet theory. The field equations associated with the action (3.1) coupled to some matter action take the form

$$G_B^A + \Lambda\delta_B^A + \alpha H_B^A = \kappa^2 T_B^A \,, \tag{4.6}$$

where $T_B^A$ is the stress tensor, $G_B^A \equiv -\frac{1}{4}\,\delta_{BEF}^{ACD}\,\mathcal{R}_{AB}^{EF} = \mathcal{R}^A{}_B - \frac{1}{2}\delta_B^A\,\mathcal{R}$ is the Einstein tensor and

$$H_B^A \equiv -\frac{1}{8}\,\delta_{BD_1\ldots D_4}^{AC_1\ldots C_4}\,\mathcal{R}^{D_1D_2}{}_{C_1C_2}\mathcal{R}^{D_3D_4}{}_{C_3C_4}\,,$$

and where the antisymmetrized Kronecker delta is defined as $\delta_{B_1\ldots B_p}^{A_1\ldots A_p} \equiv p!\delta_{[B_1}^{A_1}\cdots\delta_{B_p]}^{A_p}$.

We are mainly interested in the static spherically symmetric solution (without matter) to Einstein-Gauss-Bonnet theory in five dimensions. In this case, of space-times fibered over (constant radius) 3-spheres, the solutions correspond to the analogues of the Schwarzschild geometry, and its form was found by D. Boulware and S. Deser in Ref. [24]. More generally, the solutions that correspond to fiber bundles over 3-surfaces of constant negative (or vanishing) curvature were subsequently studied in Ref. [29] (and also [49] in a special class of Lovelock theories in arbitrary dimension). Let us discuss these solutions here. First, let us write the ansatz for the metric as follows

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega_k^2\,, \tag{4.7}$$

where $d\Omega_k^2$ is the metric of the constant curvature three-manifold (of normalized curvature $k = +1$, $-1$ or $0$). From $T_0^0 = 0$ (the other field equations are equivalent to it) one obtains

$$f'\big\{r^2 + 4\alpha\,(k - f)\big\} = -2r^3\frac{\Lambda}{3} + 2r\,(k - f)\,. \tag{4.8}$$

This is integrated for $(k - f)$ to give

$$f(r) = k + \frac{r^2}{4\alpha}\left(1 + \xi\sqrt{1 + \frac{4\Lambda\alpha}{3} + \frac{16M\alpha}{r^4}}\right) \tag{4.9}$$

where $\xi^2 = 1$. The case $\xi = +1$ corresponds to the "exotic branch" of the Boulware-Deser metrics which for $\Lambda = 0$ and $M = 0$ gives a "microscopic" anti-de Sitter or de Sitter metric, with $f(r) = 1 + r^2/2\alpha$. It is usually argued that this exotic branch turns out to be an unstable vacuum of the theory, containing ghost excitations [24, 169]. Unlike the case $\xi = -1$, this branch does not have a well defined $\alpha \to 0$ limit. As in the case of Schwarzschild solution, $M$ here is a constant of integration, and is also associated with the mass of the solution. In fact, when there is an asymptotic region at the infinity of the coordinate $r$, i.e. $1 + \frac{4\alpha\Lambda}{3} \geq 0$, the total energy w.r.t. each constant curvature background is calculated to be

$$\text{mass} = \xi^2\, M\, \frac{6\pi^2}{\kappa^2} = M\, \frac{6\pi^2}{\kappa^2}\,, \tag{4.10}$$

so that, in general, we will call $M$ the mass parameter or simply the mass of the metric[3].

[3]It should be kept in mind that the masses $M$ in each branch $\xi$, by being the total energy w.r.t. the $M = 0$ spacetime in *that branch*, can not be directly compared.

The general features of the black holes (4.7)-(4.9), such as horizons structure, singularities, etc, were studied systematically in Ref. [155]; for further details see the Appendix. Unlike General Relativity, the Einstein-Gauss-Bonnet theory admits massive solutions with no horizon but with a naked singularity at the origin. From (4.9) we see that this always happens for the exotic branch $\xi = +1$, and might also happen for the branch $\xi = -1$, provided $M < \alpha$. A related feature occurs for electrically charged solutions [163, 164]. Among other interesting properties, it can be seen that charged black holes in Einstein-Gauss-Bonnet theory have a single horizon if the mass reaches a certain critical value. Another substantial difference between the Schwarzschild solution and the Boulware-Deser solution concerns thermodynamics. Unlike black holes in General Relativity, the Einstein-Gauss-Bonnet black holes turn out to be eternal. The thermal evaporation process leads to eternal remnants due to a change of the sign in the specific heat for sufficiently small black holes. This and the other unusual phenomena discussed above are ultimately due to the ultraviolet corrections introduced by the Gauss-Bonnet term.

The discussion about a spherically symmetric solution of a given theory of gravity immediately raises the obvious question about its uniqueness. Regarding this, there is a subtlety that deserves to be pointed out. The uniqueness of the Boulware-Deser solution, discussed previously in Refs. [160, 161, 43], is only valid under certain assumptions. This was formalized in a theorem proven by R. Zegers [168], and which also holds for generic Lovelock theory in any dimension. Let us state the result as applies for Einstein-Gauss-Bonnet theory in five dimensions:

**Theorem 1** (Ref.[168])**.** *Any solution with spherical (or planar or hyperbolic) symmetry in the second-order Einstein-Gauss-Bonnet theory of gravity has to be locally static and given by the Boulware-Deser solution provided two key conditions are satisfied: i) The coefficients of the Lovelock expansion are generic enough, which means that the exceptional combination* $\alpha\Lambda = -3/4$ *is excluded; ii) the solution is* $C^2$ *smooth.*

Condition *i)* is certainly a necessary assumption. Indeed, the non-uniqueness in the case of $\alpha\Lambda = -3/4$, corresponding to the (A)dS-invariant Chern-Simons theory, is a well-known result and was explicitly shown in Refs. [43, 78]. In this work, we will see that condition *ii)* is also necessary. In fact, the vacuum shell solutions we will present are $C^0$ spacetimes which are only piecewise of the Boulware-Deser form.

In order to analyze $C^0$ solutions, we will need to use the junction conditions in the theory, which will now be discussed.

## 3.2 Junction Conditions

The next ingredient in our discussion is the junction conditions in Einstein-Gauss-Bonnet theory. These are the analogues of the Israel conditions [101] in General Relativity, and were worked out in Refs. [53, 87]. In particular, the junction conditions will be employed to join two different spherically symmetric spaces.

We will organize the discussion as follows: First, we will discuss the timelike junction condition; namely, the case where the surgery is performed on a timelike hypersurface. We will call this case the "timelike shell". After studying this we will briefly discuss its spacelike analogue.

### Timelike shell

Let $\Sigma$ be a timelike hypersurface separating two bulk regions of spacetime, region $\mathcal{V}_L$ and region $\mathcal{V}_R$ ("left" and "right"). Conveniently, we introduce the coordinates $(t_L, r_L)$ and $(t_R, r_R)$ and the metrics

$$ds_L^2 = -f_L\, dt_L^2 + \frac{dr_L^2}{f_L} + r_L^2 d\Omega^2\,, \tag{4.11}$$

$$ds_R^2 = -f_R\, dt_R^2 + \frac{dr_R^2}{f_R} + r_R^2 d\Omega^2\,, \tag{4.12}$$

in the respective regions. We shall be interested in the case where the bulk regions are empty of matter so $f_L(r_L)$ and $f_R(r_R)$ are the Boulware-Deser metric functions given by equation (4.9). In general, the mass parameter $M_R$ will be different from $M_L$. Moreover, we will also consider the possibility of having $\xi_R$ different from $\xi_L$, so that the two different branches of the Boulware-Deser solution can be considered to the two spaces to be joined.

It is convenient to parameterize the shell's motion in the $r-t$ plane using the proper time $\tau$ on $\Sigma$. In region $\mathcal{V}_L$ we have $r_L = a(\tau)$, $t_L = T_L(\tau)$ and in region $\mathcal{V}_R$ we have $r_R = a(\tau)$, $t_R = T_R(\tau)$. The induced metric on $\Sigma$ induced from region $\mathcal{V}_L$ is the same as that induced from region $\mathcal{V}_R$, and is given by

$$d\hat{s}^2 = -d\tau^2 + a(\tau)^2 d\Omega^2\,. \tag{4.13}$$

This guarantees the existence of a coordinate system where the metric is continuous ($C^0$).

Here, $d\Omega^2$ will be chosen to be the line element of a 3-manifold with (intrinsic) curvature $k = \pm 1, 0$ i.e. it is a unit sphere, a hyperboloid or flat space respectively. Although our interest will be mainly focused on the spherical shell similar features to those we will discuss hold also for the cases $k = 0$ and $k = -1$. The hypersurface $\Sigma$ is the shell's world-volume, i.e. the four-dimensional history of the shell in spacetime. The intrinsic geometry is well defined on $\Sigma$ and given by (4.13). However, since the metric is only $C^0$ and not necessarily differentiable, the geometry of the embedding of $\Sigma$ into $\mathcal{V}_L$ is independent of the embedding of $\Sigma$ into $\mathcal{V}_R$. The geometric information about the embedding is quantified by the extrinsic curvature as well as the orientation of $\Sigma$ with respect to each bulk region.

To be precise, let us consider the following conventions for a timelike shell outside of any event horizon:

- The hypersurface $\Sigma$ has a single unit normal vector $\boldsymbol{n}$ which points from left to right.
- The orientation factor $\eta$ of each bulk region is defined as follows: $\eta = +1$ if the radial coordinate $r$ points from left to right, while $\eta = -1$ if the radial coordinate $r$ points from right to left.

This is depicted in Fig. 4.1, where the wormhole-like geometry on the left corresponds to the orientation $\eta_L \eta_R < 0$, while the "standard shell" on the right corresponds to the case $\eta_L \eta_R > 0$. Notice also that the wormhole depicted in such figure is not the only possibility for among those of orientation $\eta_L \eta_R < 0$. While this geometry roughly speaking corresponds to join two "exterior regions" of a spherical solution, it is also feasible to construct a space by joining two "interior regions", instead. This corresponds to the case $\eta_L \eta_R < 0$ as well.

**Definition 2.** *The orientation defined by $\eta_L \eta_R > 0$ will be called the standard orientation. A shell with standard orientation will be called a standard shell. The orientation defined by $\eta_L \eta_R < 0$ will be called the wormhole orientation. Such a shell shall be referred to as a wormhole.* [*This makes actual sense when $\eta_R = +1$. When $\eta_R = -1$ the latter case represents a closed universe, containing singularities.*]

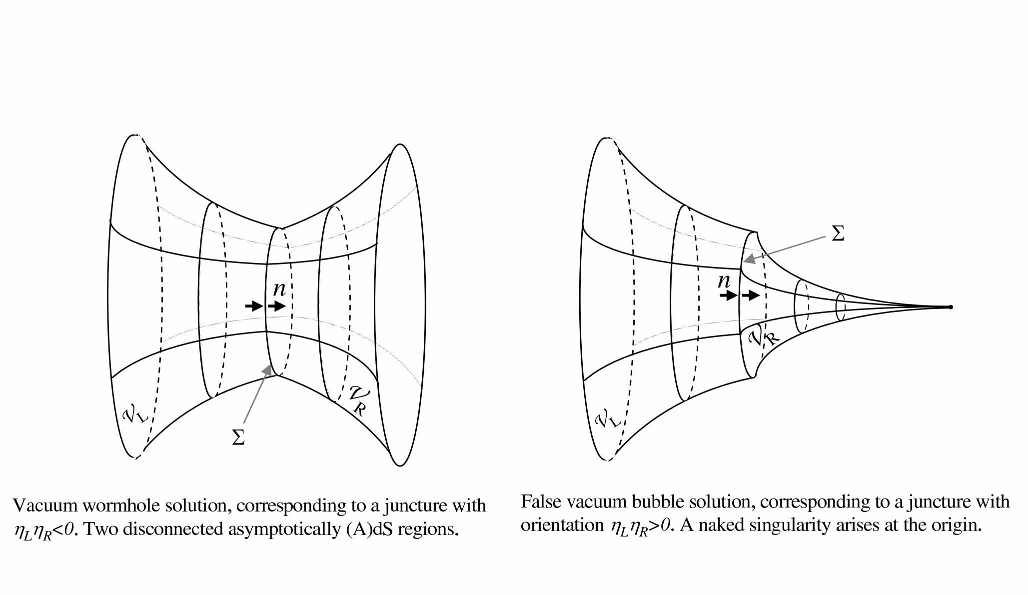



Figure 4.1: The figure on the left corresponds to a wormhole-like solution, defining the orientation $\eta_L\eta_R < 0$. The throat connects two different asymptotically de-Sitter spaces. The figure on the right corresponds to a vacuum shell, connecting two Boulware-Deser solutions of different branches. In this second case the orientation is such that $\eta_L\eta_R > 0$.

The components of the normal vector with respect to the basis $\boldsymbol{e}_A := (\partial_{t_L}, \partial_{r_L}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_\mathcal{L}$ and the basis $\boldsymbol{e}_{A'} := (\partial_{t_R}, \partial_{r_R}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_\mathcal{R}$ are respectively given by

$$n^A = \eta_L \left( \frac{\dot{a}}{f_L}, \sqrt{f_L + \dot{a}^2}, 0, 0, 0 \right) , \qquad n^{A'} = \eta_R \left( \frac{\dot{a}}{f_R}, \sqrt{f_R + \dot{a}^2}, 0, 0, 0 \right) .$$

where dot denotes differentiation with respect to $\tau$. This formula for the normal vector extends the definition of the orientation factors to the situation where the shell is inside the horizon when $r$ is a timelike coordinate.

We can introduce the basis $\boldsymbol{e}_a = (\partial_\tau, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ intrinsic to $\Sigma$. The extrinsic curvature is then defined as $K_{ab} := \boldsymbol{e}_a \cdot \nabla_{\boldsymbol{e}_b} \boldsymbol{n} = -\boldsymbol{n} \cdot \nabla_{\boldsymbol{e}_b} \boldsymbol{e}_a$. In terms of a coordinate basis we have $e^A_a = \frac{\partial X^A}{\partial \zeta^a}$ and the extrinsic curvature takes the explicit form

$$K_{ab} = -n_A \left( \frac{\partial^2 X^A}{\partial \zeta^a \partial \zeta^b} + \Gamma^A_{BC} \frac{\partial X^B}{\partial \zeta^a} \frac{\partial X^C}{\partial \zeta^b} \right) ,$$

and in our case the components read

$$K^\tau_\tau = \eta \frac{\ddot{a} + \frac{1}{2} f'}{\sqrt{\dot{a}^2 + f}} , \qquad K^\theta_\theta = K^\chi_\chi = K^\varphi_\varphi = \frac{\eta}{a} \sqrt{\dot{a}^2 + f} .$$

We denote the extrinsic curvature with respect to the embedding into $\mathcal{V}_L$ and $\mathcal{V}_R$ by $(K_L)_{ab}$ and $(K_R)_{ab}$ respectively. At a singular shell $(K_L)_{ab} \neq (K_R)_{ab}$, i.e. the extrinsic curvature jumps from one side to the other. This is a covariant way of expressing the fact that the metric is not $C^1$ (i.e. there does not exist any coordinate system where the metric is $C^1$). In General Relativity this amounts to saying that (non-null) vacuum shells do not exist since Israel conditions cannot be satisfied without the introduction of a induced stress tensor on the spherical shell. Things are different in the case of the gravity theory defined by action (4.5). This is because the Gauss-Bonnet term induces additional terms in the junction conditions, which supplements the Israel equation. In section 3.3 we will show how both contributions can be combined to yield vacuum spherically symmetric thin shells. First we briefly discuss spacelike shells.

### Spacelike shell

Solutions of a different sort are those constructed by joining two spaces through a spacelike juncture. Let us suppose now that $\Sigma$ is now a spacelike hypersurface. The motion of the shell in the $r-t$ plane is parameterized by $(t,r) = (T(\tau), a(\tau))$, where it is necessary to remember that $\tau$ is now a spacelike coordinate on $\Sigma$. The induced metric on $\Sigma$ is then given by

$$d\hat{s}^2 = +d\tau^2 + a(\tau)^2 d\Omega^2 . \qquad (4.14)$$

The components of the normal vector with respect to the basis $\boldsymbol{e}_A := (\partial_{t_L}, \partial_{r_L}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_L$ and the basis $\boldsymbol{e}_{A'} := (\partial_{t_R}, \partial_{r_R}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_R$ are respectively:

$$n^A = \eta_L \left( \frac{\dot{a}}{f_L}, \sqrt{\dot{a}^2 - f_L}, 0, 0, 0 \right) , \qquad n^{A'} = \eta_R \left( \frac{\dot{a}}{f_R}, \sqrt{\dot{a}^2 - f_R}, 0, 0, 0 \right) .$$

This defines the orientation factors in the case of a spacelike shell. The components of the extrinsic curvature are:

$$K^\tau_\tau = \eta \frac{\ddot{a} - \frac{1}{2} f'}{\sqrt{\dot{a}^2 - f}}\,, \qquad K^\theta_\theta = K^\phi_\phi = K^\chi_\chi = \frac{\eta}{a}\sqrt{\dot{a}^2 - f}\,.$$

## 3.3 The junction condition for a vacuum shell

The Einstein-Gauss-Bonnet field equations are well-defined distributionally at $\Sigma$ due to the property of quasi-linearity in second derivatives (see e.g. Refs [65, 88] ). Thus, one can define a distributional stress tensor $T_{AB} = \delta(\Sigma)\, e^b_A e^b_B S_{ab}$, where $S_{ab}$ is the intrinsic stress tensor induced on the shell and $\delta(\Sigma)$ denotes a Dirac delta function with support on the shell world-volume $\Sigma$.

Integrating the field equations from left to right in an infinitesimally thin region across $\Sigma$ one obtains the junction condition. This relates the discontinuous change of spacetime geometry across $\Sigma$ with the stress tensor $S^b_a$. For the Einstein-Gauss-Bonnet theory the general formulas can be found in the Refs. [53, 87, 88].

$$(\mathfrak{Q}_R)^b_a - (\mathfrak{Q}_L)^b_a = -\kappa^2 S^b_a\,, \tag{4.15}$$

where the symmetric tensor $\mathfrak{Q}^a{}_b$ is given by

$$\mathfrak{Q}^a{}_b = \mp\delta^{ac}_{bd} K^d_c + \alpha\, \delta^{acde}_{bfgh}\Big( \mp K^f_c R^{gh}{}_{de} + \frac{2}{3} K^f_c K^g_d K^h_e \Big)\,. \tag{4.16}$$

Above, the sign $\mp$ depends on the signature of the junction hypersurface: it is minus for the timelike case and plus for the spacelike case. In this expression, lower case Roman letters from the beginning of the alphabet $a$, $b$ etc. represent four-dimensional tensor indices on the tangent space of the world-volume of the shell. The symbol $K^a_b$ refers to the extrinsic curvature, while the symbol $R^{ab}{}_{cd}$ appearing here corresponds to the four-dimensional intrinsic curvature (see the appendix for details).

Once applied to the spherically symmetric (or $k = -1, 0$) case the tensor $\mathfrak{Q}^b_a$ turns out to be diagonal with components

$$\mathfrak{Q}^\tau_\tau = -3\sigma\, a^{-3}\Big(\eta\, a^2 \sqrt{\dot{a}^2 + f} + 4\alpha\,\eta \sqrt{\dot{a}^2 + f}\, \big(k + \frac{2}{3}\sigma\dot{a}^2 - \frac{1}{3} f\big)\Big)\,, \tag{4.17}$$

$$\mathfrak{Q}^\theta_\theta = \mathfrak{Q}^\chi_\chi = \mathfrak{Q}^\varphi_\varphi\,. \tag{4.18}$$

The precise form of $\mathfrak{Q}_\theta^\theta$ will not be needed but is given in the appendix for completeness. The above formula was written in a way that is valid for both timelike and spacelike shells, where we have defined

$$\sigma = +1 \quad (\text{timelike shell})\,, \qquad \sigma = -1 \quad (\text{spacelike shell})\,.$$

Also, let us be reminded of the fact that $\eta_L$ and $\eta_R$ (with $\eta^2 = 1$) are the orientation factors in each region, which are independent one from each other. Above, the subscripts $L$, $R$ signify the quantity evaluated on $\Sigma$ induced by regions $\mathcal{V}_L$ and $\mathcal{V}_R$ respectively (e.g. $\mathfrak{Q}_L$ is a function of $\eta_L$ and $f_L(a)$)[4].

One may verify that the following equation is satisfied

$$\frac{d}{d\tau}\left(a^3 \mathfrak{Q}_\tau^\tau\right) = \dot{a}\, 3a^2 \mathfrak{Q}_\theta^\theta \,, \tag{4.19}$$

which expresses the conservation of $S_a^b$. The reason why one obtains exact conservation, i.e. no energy flow to the bulk, is that the normal-tangential components of the energy tensor in the bulk is the same in both sides of the junction hypersurface [53, 88].

The main point here is that, unlike the Israel conditions in Einstein gravity, non-trivial solutions to (4.15) are possible even when $S_a^b = 0$. That is, the extrinsic curvature can be discontinuous across $\Sigma$ with no matter on the shell to serve as a source. The discontinuity is then self-supported gravitationally and this is due to non-trivial cancelations between the terms of the junction conditions. Similar configurations are impossible in Einstein gravity. From now on we consider the vacuum case

$$S_a^b = 0 \,. \tag{4.20}$$

In the next section we will treat the static shell in detail. An exhaustive study of the space of solutions describing both static and dynamical shells is left to sections 1 through 2.3. Let us now first briefly introduce the basic features of the general solution for a dynamical shell.

Equation (4.19) tells us that when $\dot{a} \neq 0$, the components of the junction condition are not independent; namely

$$(\mathfrak{Q}_R)_\tau^\tau - (\mathfrak{Q}_L)_\tau^\tau = 0 \quad \Rightarrow \quad (\mathfrak{Q}_R)_\theta^\theta - (\mathfrak{Q}_L)_\theta^\theta = 0 \,.$$

[4] From now on we shall be concerned with $f(a)$, i.e. the metric function evaluated at the shell. In an abuse of notation we shall just write $f$ instead of $f(a)$.

Therefore, for time-dependent solutions it suffices to impose only the first condition. This can be factorized as follows,

$$\Big(\eta_R\sqrt{\dot a^2+\sigma f_R}-\eta_L\sqrt{\dot a^2+\sigma f_L}\Big)\times$$
$$\times\left\{a^2+4\alpha(k+\sigma\dot a^2)-\sigma\frac{4\alpha}{3}\Big(f_R+f_L+2\sigma\dot a^2+\eta_R\eta_L\sqrt{\sigma f_R+\dot a^2}\sqrt{\sigma f_L+\dot a^2}\Big)\right\}=0\,. \tag{4.21}$$

Equation (4.21) contains all the information about the spherically symmetric junctions in empty space, which we generically call "vacuum shells". The solutions to this equation includes both wormhole-like and bubble-like geometries, depending on whether the orientation is $\eta_L\eta_R<0$ or $\eta_L\eta_R>0$ respectively. Certainly, there exist several cases to be explored. First of all, there are the parameters $k$, $M$ and $\xi$, which characterize each of the two Boulware-Deser metrics to be joined. On the other hand, there are two possible orientation for each one of both spaces, and this is given by the sign of the respective $\eta$. The convention for the orientations is such that $\eta=+1$ if the $\partial_r$ is parallel to the normal vector $\boldsymbol{n}$, and $\eta=-1$ in the opposite case. Furthermore, there is the sign of $\sigma$, what tells us whether the signature of the junction hypersurface is timelike ($\sigma=+1$) or spacelike ($\sigma=-1$). So, this permits a very interesting catalogue of geometries which we survey in section 5 and further explore in subsequent sections.

The vanishing of the first factor in (4.21) would imply that the metric is smooth across $\Sigma$. Rejecting this as the trivial solution, we demand that the second factor vanishes. From the second factor, squaring appropriately, we obtain

$$\dot a^2=\sigma\,\frac{\Big(f_R+f_L-3(k+a^2/4\alpha)\Big)^2-f_Rf_L}{3\Big(f_R+f_L-2(k+a^2/4\alpha)\Big)}=:-V(a)\,, \tag{4.22}$$

along with two inequalities discussed below. The system, because of the symmetry, has reduced to an essentially one-dimensional problem, given by the ordinary differential equation (4.22). It is seemingly equivalent to the problem of a particle moving in a potential[5] $V(a)$. Nevertheless, it is worth pointing out that, unlike the equation for a single particle, here we find that

[5]Notice that the effective potential for the spacelike shell is simply minus the potential for the timelike shell.

the energy $h$ is unavoidably fixed to zero instead of arising as an constant of motion. An important difference arises in the case where there is a minimum of $V(a)$ precisely at $V = 0$. The constraint $h = 0$, provided the fact that the minimum of $V(a)$ is precisely at zero energy, would lead to the conclusion that the shell can not move but it would be stacked at the bottom of the potential. Actually, this is the case if no external system acts as a perturbation. One such perturbation can be thought of as being an incoming particle which, after perturbing the shell, scatters back to infinity spending an energy $\delta h$ through the process. This would provide energy for the vacuum shell to move. One can also think about a slight change in the parameters of the solution yielding a shifting $V(a) \to V(a) - \delta h$, see [158]. We will discuss the dynamical case in detail in the next Chapter.

Now, let us notice that since we have squared the junction condition, we must substitute (4.22) back into (4.21) to check the consistency. When doing so, the solutions of equation (4.22) are solutions of the junction condition if and only if the following restrictions are obeyed

$$-\eta_R \eta_L \, (2f_R + f_L - 3(k + a^2/4\alpha)) \, (2f_L + f_R - 3(k + a^2/4\alpha)) \geq 0 \; ; \quad (4.23)$$

and

$$(f_R + f_L - 2(k + a^2/4\alpha)) > 0 \qquad \text{timelike shell} \qquad (4.24)$$
$$(f_R + f_L - 2(k + a^2/4\alpha)) < 0 \qquad \text{spacelike shell}\,. \qquad (4.25)$$

Furthermore, we also have an inequality which is not an extra condition but rather follows as a consequence of equation (4.22). The fact that $\dot{a}^2$ is positive in (4.22) implies that

$$\Big(f_R + f_L - 3(k + a^2/4\alpha)\Big)^2 - f_R f_L \geq 0 \; .$$

for both timelike and spacelike. This inequality provides further information about the space of solutions of (4.22).

**Proposition 3.** *For a dynamical vacuum shell with a timelike world-volume $\Sigma$, the scale factor of the metric (4.13) on $\Sigma$ is governed by (4.22), under the inequalities (4.23) and (4.24).*
*On the other hand, for a dynamical vacuum shell with a spacelike world-volume $\Sigma$, the scale factor of the metric (4.14) on $\Sigma$ is governed by (4.22), under the inequalities (4.23) and (4.25).*

Now, let us begin by studying the inequalities to give idea of what kinds of solutions exist. With this in mind, let us translate the restrictive inequalities (4.23-4.25) into simpler terms. The metric function evaluated on the hypersurface is

$$f_L(a) = k + \frac{a^2}{4\alpha}\Big(1 + \xi_L Y_L(a)\Big)\,, \qquad Y_L(a) \equiv \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{16\alpha M_L}{a^4}}\,, \tag{4.26}$$

and similarly for $f_R$. Recall that $\xi_L$ and $\xi_R$ are independent of each other, with $\xi = +1$ being the exotic branch of the Boulware-Deser solution. It is convenient to write the inequalities in terms of the square roots $Y_L(a)$ and $Y_R(a)$; namely

$$-\eta_R\eta_L\ (2\xi_R Y_R + \xi_L Y_L)\ (2\xi_L Y_L + \xi_R Y_R) \geq 0\ ; \tag{4.27}$$

and

$$\alpha(\xi_R Y_R + \xi_L Y_L) > 0 \qquad \text{timelike shell} \tag{4.28}$$

$$\alpha(\xi_R Y_R + \xi_L Y_L) < 0 \qquad \text{spacelike shell}\,. \tag{4.29}$$

These inequalities contain relevant information about the global structure of the solutions. Let us summarize this information in the following table

| **Timelike shells** ($\sigma = +1$) | Product of orientation factors ($\eta_L\eta_R$) | Product of branch signs ($\xi_L\xi_R$) | Inequalities imposed on solutions |
|---|---|---|---|
| Standard orientation | +1 | +1 | No solution |
| | +1 | -1 | $\frac{1}{2}Y_L \leq Y_R(a) \leq 2Y_L(a)$ ; $\xi_R(M_R - M_L) > 0$ |
| "Wormhole" orientation | -1 | +1 | $\alpha\xi_R > 0$ ; $Y_L, Y_R > 0$ |
| | -1 | -1 | $Y_R \geq 2Y_L$ or $Y_R \leq \frac{1}{2}Y_L$ ; $\xi_R(M_R - M_L) > 0$ |

| **Spacelike shells** ($\sigma = -1$) | Product of orientation factors ($\eta_L \eta_R$) | Product of branch signs ($\xi_L \xi_R$) | Inequalities imposed on solutions |
|---|---|---|---|
| Standard orientation | +1 | +1 | No solution |
| | +1 | -1 | $\frac{1}{2} Y_L \leq Y_R(a) \leq 2Y_L(a)$ ; $\xi_R(M_R - M_L) > 0$ |
| "Wormhole" orientation | -1 | +1 | $\alpha \xi_R < 0$ ; $Y_L, Y_R > 0$ |
| | -1 | -1 | $Y_R \geq 2Y_L$ or $Y_R \leq \frac{1}{2} Y_L$ ; $\xi_R(M_R - M_L) < 0$ |

From the conditions obtained here we conclude the following:

**Remark 4.** *Vacuum shells with the standard orientation always involve the gluing of a plus branch ($\xi = +1$) metric with a minus branch ($\xi = -1$) metric.*

Now the plus branch has a different effective cosmological constant to the minus branch. In this sense, standard shells are a kind of false vacuum bubble. This is discussed further in section 5.1.

**Remark 5.** *Vacuum shells which involve the gluing of two minus branch ($\xi = -1$) metrics exist only when the Gauss-Bonnet coupling constant $\alpha$ satisfies $\alpha < 0$. They always have the wormhole orientation.*

In the analysis above it has been explicitly assumed that $\dot{a} \neq 0$. Nevertheless, the case $\dot{a} = 0$ is also of considerable interest. This describes static shells in the timelike case, and also an analogous situation for the spacelike case which we call instantaneous shells. In the next section, the case of constant $a$ shells is considered in detail. It can be checked that, as expected, all the information about the constant $a$ solutions can be obtained from the dynamical case by imposing both $V(a_0) = 0$ and $V'(a_0) = 0$. Thus, proposition 3 gives the general solution of all the vacuum shells, including the static ones.

Closing the general discussion of the dynamical vacuum we note the following. The potential $V(a)$ in (4.22) and the restrictive inequalities (4.23) and (4.24), (4.25) are symmetric in the exchange

$$\xi_L, M_L \leftrightarrow \xi_R, M_R \ . \tag{4.30}$$

That is, the same kinds of motion are possible for the two situations obtained if we swap the values of the parameters $\xi, M$ in $\mathcal{V}_L$ and $\mathcal{V}_R$. In the constant $a$ case, governed by $V(a_0) = 0 = V'(a_0)$, the symmetry means that the solution is left unchanged under the swapping.

# 4 Static vacuum shells

Now, let us discuss the solutions at constant $a$. That is, the static and instantaneous solutions, depending on whether the juncture corresponds to the timelike or spacelike case respectively.

The bulk metric in each of the two region is assumed to be of the Boulware-Deser form (4.9) with ($k = \pm 1, 0$) and considering $a = a_0$ fixed. Although the main focus will be on the spherically symmetric case $k = +1$, the analysis can be straightforwardly extended to the cases $k = -1$ and $k = 0$. Then, there are two possibilities to be distinguished; namely,

- Static shell: For the timelike case the shell is located at fixed radius $r_L = r_R = a_0$. The proper time on the shell's world-volume is $\tau = t_L\sqrt{f_L(a)} = t_R\sqrt{f_R(a)}$ so that the induced metric on $\Sigma$ turns out to be $d\hat{s}^2 = -d\tau^2 + a_0^2 d\Omega^2$. Then, the extrinsic curvature components are $K^\tau_\tau = \eta\frac{f'}{2\sqrt{f}}$, $K^\theta_\theta = K^\chi_\chi = K^\varphi_\varphi = \frac{\eta\sqrt{f}}{a}$ and the intrinsic curvature components are $R^{\theta\varphi}{}_{\theta\varphi} = k/a_0^2$, etc.

- Instantaneous shell: In the spacelike case there is an exotic kind of shell, which exists when $f$ is negative. The metric function is negative inside of an event horizon or outside of a cosmological horizon, where $r$ actually plays the role of a timelike coordinate. Matching two metrics at time $r_\pm = a_0$ therefore describes an instantaneous transition from one smooth metric to another. We can introduce $\tau = t_L\sqrt{-f_L(a)} = t_R\sqrt{-f_R(a)}$ which is a spacelike intrinsic coordinate on the shell, so that the induced metric on $\Sigma$ is $ds^2 = +d\tau^2 + a_0^2 d\Omega^2$. The extrinsic curvature components are $K^\tau_\tau = -\eta\frac{f'}{2\sqrt{-f}}$, $K^\theta_\theta = K^\chi_\chi = K^\varphi_\varphi = \frac{\eta\sqrt{-f}}{a}$.

It is worth noticing that both the static and instantaneous shells can be analyzed together, provided the presence of $\sigma$ in the equations. Recall that the sign of $\sigma$ carries the information about the signature of the junction hypersurface. Then, by considering the quantities introduced above, and by substituting this in the junction conditions with $S^a_b = 0$, we get

$$S^\tau_\tau = 0 \;\Rightarrow\; \left(\eta_R\sqrt{f_R} - \eta_L\sqrt{f_L}\right)\left(a_0^2 + \frac{4\alpha}{3}\{3k - f_R - f_L - \sigma\eta_L\eta_R\sqrt{f_L f_R}\}\right) = 0 \; . \tag{4.31}$$

$$S^\theta_\theta = 0 \;\Rightarrow\; \left(\frac{\eta_R}{\sqrt{f_R}} - \frac{\eta_L}{\sqrt{f_L}}\right)\left(k - \frac{\Lambda a_0^2}{3} - \sigma\eta_L\eta_R\sqrt{f_L f_R}\right) = 0 \; , \tag{4.32}$$

where $\sigma = +1$ is the static shell and $\sigma = -1$ is the instantaneous shell. The l.h.s. of (4.8) conveniently appears in the $\theta - \theta$ component of the junction condition and we have used it to eliminate the derivative of $f$ from the formula. This is why $\Lambda$ appears explicitly in equation (4.31).

In both equations (4.32) and (4.31), the first factor vanishes if and only if the metric is smooth. Again, rejecting this as the trivial solution, we demand that the second factor vanishes in both equations. So, we have

**Proposition 6.** *A static vacuum shell is described by*

$$f_L + f_R = 2k + \frac{3a_0^2}{4\alpha} + \frac{\Lambda a_0^2}{3} \; , \tag{4.33}$$

$$\eta_L\eta_R\sqrt{f_L f_R} = k - \frac{\Lambda a_0^2}{3} \; , \tag{4.34}$$

*under the condition* $f_L, f_R > 0$. *On the other hand, an instantaneous vacuum shell is described by*

$$f_L + f_R = 2k + \frac{3a_0^2}{4\alpha} + \frac{\Lambda a_0^2}{3} \; , \tag{4.35}$$

$$-\eta_L\eta_R\sqrt{f_L f_R} = k - \frac{\Lambda a_0^2}{3} \; , \tag{4.36}$$

*under the condition* $f_L, f_R < 0$.

We have included for completeness the instantaneous shells. Now, let us consider some examples of the static case with more attention. As mentioned, a more complete analysis of the space of solutions will be given in sections 1 and 2.

## 4.1 The moduli space of solutions

Now, to continue the study of the different solutions we find it convenient to introduce some notation. For the rest of this section it is convenient to define the dimensionless parameters

$$x \equiv \frac{4\alpha\Lambda}{3}\,, \qquad y \equiv \frac{\Lambda}{3}a_0^2\,, \qquad \bar{M} \equiv \frac{M}{\alpha}\,. \tag{4.37}$$

By $x$ and $y$ we measure the Gauss-Bonnet coupling and the vacuum shell radius respectively in units of $\Lambda$. The parameter $y$ is useful for our purposes but it is meaningful only when $\Lambda \neq 0$. In terms of these parameters, the Boulware-Deser solution evaluated at $r = a_0$ has the form

$$f_{L,R}(a_0) \equiv 1 + \frac{y}{x} \quad k + \xi_{L,R}\sqrt{1 + x + \frac{x^2}{y^2}\bar{M}_{L,R}}\ \Bigg)\,. \tag{4.38}$$

The general solution will be derived in the following way: We will solve the junction conditions for $\bar{M}_L$ and $\bar{M}_R$ in terms of $(x, y)$. The range of admissible values of $(x, y)$ turns out to be restricted by inequalities coming from demanding the metric to be real-valued. So there is a continuous space of solutions.

**Definition 7.** *The range of values of $(x, y)$ for which solutions exist will be called the moduli space.*

The parameters $x$ and $y$ are coordinates of this moduli space. The complete description of the moduli space will be given in more appropriate parameters introduced in section 2.3. For the moment, let us consider $x$, $y$ and $\bar{M}$.

Since the moduli space is two dimensional, it can be plotted. So by obtaining a formula for the masses and by plotting the moduli space, we obtain all the solutions. Let us now do this explicitly for the case of non-vanishing cosmological constant.

## 4.2 Static spherical shells with $\Lambda \neq 0$

Consider static spherically symmetric shells with $\Lambda \neq 0$. For definiteness, let us focus on the case of timelike shells with $k = 1$. From Proposition 6 we have the following pair of equations

$$f_L + f_R = \frac{y}{x}(3+x) + 2 \ , \tag{4.39}$$

$$\sqrt{f_L f_R} = \eta_L \eta_R \, (1-y) \ , \tag{4.40}$$

where $f_L, f_R > 0$. We can see immediately from (4.40) that solutions with the wormhole orientation, i.e. $\eta_L \eta_R = -1$, only exist for $y \equiv \Lambda a_0^2/3 > 1$.

**Remark 8.** *Static vacuum shell wormholes exist only when $\Lambda > 0$.*

Solving the equations above we see that $f_L$ and $f_R$ obey the same quadratic equation where one $f$ has the $+$ root of the solution and the other has the $-$ root. So we define a solution $f_{(+)}$ which corresponds to the $+$ root of the solution and an $f_{(-)}$ which corresponds to the $-$ root. So there are two solutions to the problem:

$$f_L = f_{(-)} \ \ , \ \ f_R = f_{(+)} \qquad \text{or,} \qquad f_L = f_{(+)} \ \ , \ \ f_R = f_{(-)} \ . \tag{4.41}$$

Substituting the explicit expression (4.38) for $f_{L,R}(a_0)$ we have: In the first case of (4.41), $M_L = M_{(-)}$, $\xi_L = \xi_{(-)}$ and $M_R = M_{(+)}$, $\xi_R = \xi_{(+)}$, and in the second case $+ \leftrightarrow -$, for constants $\xi_{(\pm)}$ and $M_{(\pm)}$ satisfying

$$1 + x - \sqrt{3}\sqrt{x(1+x)\Big(\frac{4}{y} + \frac{3}{x} - 1\Big)} = 2\xi_{(-)}\sqrt{1 + x + \frac{x^2 \bar{M}_{(-)}}{y^2}} \ , \tag{4.42}$$

$$1 + x + \sqrt{3}\sqrt{x(1+x)\Big(\frac{4}{y} + \frac{3}{x} - 1\Big)} = 2\xi_{(+)}\sqrt{1 + x + \frac{x^2 \bar{M}_{(+)}}{y^2}} \ . \tag{4.43}$$

For a solution to exist, the square root in the l.h.s. of must be real, so that we demand

$$x(1+x)\Big(\frac{4}{y} + \frac{3}{x} - 1\Big) \geq 0 \qquad \text{(Existence of solutions)} \, . \tag{4.44}$$

Since we have squared the equations we must substitute back to check the consistency. So we get the following inequalities:[6]

$$\frac{y}{x}(3+x) + 2 > 0 \qquad \text{(Timelike shells)} \, ; \tag{4.45}$$

[6] Note that the timelike condition (4.45), when combined with the reality condition (4.44) can be equivalently stated

$$xy(1+x) > 0 \qquad \text{(Timelike shell)} \, .$$

This is useful for plotting the graphs.

$$y < 1 \quad \text{(Standard orientation)}\,, \qquad y > 1 \qquad \text{(Wormhole orientation)}\,. \tag{4.46}$$

The above inequalities are plotted in figures 8.2 and 8.3. Also we find the regions of the moduli space corresponding to the allowed branch signs $(\xi_{(-)}, \xi_{(+)})$.

| $\xi_{(-)}$ | $\xi_{(+)}$ | Inequality |
|---|---|---|
| $+1$ | | $1 + x > 0 \bigcap x\left(\frac{3}{y} + \frac{2}{x} - 1\right) < 0$ |
| $-1$ | | $1 + x < 0 \bigcup x\left(\frac{3}{y} + \frac{2}{x} - 1\right) > 0$ |
| | $+1$ | $1 + x > 0 \bigcup x\left(\frac{3}{y} + \frac{2}{x} - 1\right) < 0$ |
| | $-1$ | $1 + x < 0 \bigcap x\left(\frac{3}{y} + \frac{2}{x} - 1\right) > 0$ |

(Branches).

The standard shells are always $(-, +)$. The regions $(+, +)$, $(-, +)$ and $(-, -)$ for the wormholes are shown in figure 8.4.

**Remark 9.** *Provided $\Lambda > 0$ and assuming the existence of two asymptotic regions we find $\alpha > 0$. Consequently, at least one of the two spherically symmetric spaces connected through the throat turns out to be asymptotically Anti-de Sitter.*

The next step is computing the masses. We can solve (4.42) and (4.43) to give the parameter $M$ in each region, namely

$$\bar{M}_{(-)} = \frac{y(1+x)}{2x^2}\left\{6x + 3y - xy - y\sqrt{3}\sqrt{x(1+x)\Big(\frac{4}{y} + \frac{3}{x} - 1\Big)}\right\}, \tag{4.47}$$

$$\bar{M}_{(+)} = \frac{y(1+x)}{2x^2}\left\{6x + 3y - xy + y\sqrt{3}\sqrt{x(1+x)\Big(\frac{4}{y} + \frac{3}{x} - 1\Big)}\right\}. \tag{4.48}$$

As mentioned above, relations (4.41), the left-metric can be either a metric with parameters $(\xi_{(-)}, M_{(-)})$ or a metric with $(\xi_{(+)}, M_{(+)})$, and the other way around for the right-metric. For wormholes the two solutions (4.41) correspond to the same spacetime looked at from the opposite way around. In the case of standard shells, they correspond to swapping the mass and branch sign of the interior with those of the exterior region.

The metrics with parameters $(\xi_{(-)}, M_{(-)})$ and $(\xi_{(+)}, M_{(+)})$ as determined by the solutions we found above have different properties. We will call these metrics minus- and plus-metrics respectively.

Also we note the following useful expression: we can eliminate $y$ to get an implicit equation for the masses and $x$. The solution lies on sections of the curves

$$1 + \frac{9}{4}\frac{x(x+1)}{3-x}\frac{1}{\bar{M}_{(+)} + \bar{M}_{(-)}} \quad 1 + (-1)^p \sqrt{1 + \frac{4}{9}\frac{(3-x)}{x(x+1)}(\bar{M}_{(+)} + \bar{M}_{(-)})}\Bigg) = (-1)^q \sqrt{1 + \frac{(3-x)}{(1+x)}\frac{(\bar{M}_{(+)} - \bar{M}_{(-)})^2}{(\bar{M}_{(+)} + \bar{M}_{(-)})^2}} \tag{4.49}$$

where the signs $(-1)^p$ and $(-1)^q$ are to be determined by consistency.

## 4.3 Instantaneous shells

Before concluding this section, let us briefly comment on spacelike junction conditions with $\dot{a} = 0$. For instance, consider the case $\Lambda \neq 0$. From Proposition 6 we have the following pair of equations:

$$f_L + f_R = \frac{y}{x}(3+x) + 2 \, , \tag{4.50}$$

$$\sqrt{f_L f_R} = -\eta_L \eta_R \,(1-y) \, , \tag{4.51}$$

The solution is exactly the same as the above except that the inequalities (4.45) and (4.46) are reversed. That means

$$\frac{y}{x}(3+x) + 2 < 0 \qquad \text{(Spacelike shells)}\,; \tag{4.52}$$

$$y > 1 \quad \text{(Standard orientation)}\,, \qquad y < 1 \qquad \text{(Wormhole orientation)}\,. \tag{4.53}$$

The inequality (4.44) and mass formulae are the same. The moduli space of these solutions is plotted in figure 8.5. They exist for $\alpha < 0$.

## 4.4 Static spherical shells with $\Lambda = 0$

First, we can consider the case of static spherically symmetric shells with $\Lambda = 0$. This is an interesting special case. The analysis simplifies considerably and, besides, there are some qualitative differences between this and the case $\Lambda \neq 0$. In this case, the equations reduce to

$$f_L + f_R = 2 + \frac{3a_0^2}{4\alpha} \; , \tag{4.54}$$

$$\eta_L \eta_R \sqrt{f_L f_R} = 1 \; , \tag{4.55}$$

We see from the second equation that $\eta_L\eta_R$ must be $+1$, i.e. static wormholes do not exist for $\Lambda = 0$. Then, the solution is either $M_L = M_{(-)}, M_R = M_{(+)}$ or $M_L = M_{(+)}, M_R = M_{(-)}$ where

$$\frac{M_{(\pm)}}{\alpha} = \frac{1}{2}\frac{a_0^2}{4\alpha}\left(6 + \frac{3a_0^2}{4\alpha} \pm \sqrt{12\frac{a_0^2}{4\alpha} + 9\left(\frac{a_0^2}{4\alpha}\right)^2}\right) . \tag{4.56}$$

The consistency of the solution requires

$$\alpha > 0 \, , \qquad (\xi_{(-)}, \xi_{(+)}) = (-1, +1) \, , \tag{4.57}$$

so that $M_{(-)}$ and $M_{(+)}$ correspond to minus branch and exotic plus branch metrics respectively. There are solutions for all positive values of $M_{(-)}$ (the plus branch mass parameter is also positive but in that case the bulk spacetime asymptotically takes the form of a negative mass AdS-Schwarzschild solution). When the throat radius is small compared to the scale set by the Gauss-Bonnet coupling constant, $a_0^2 << \alpha$, the masses are also small compared to $\alpha$, namely $M_{(-)}/\alpha \sim M_{(+)}/\alpha \sim 3a_0^2/4\alpha$. On the other hand, for large radius $a_0^2 >> \alpha$, the masses are large, $M_{(-)}/\alpha \sim a_0^2/2\alpha$, $M_{(+)}/\alpha \sim 3a_0^4/16\alpha^2$. Figure 8.1 shows a plot of the masses as a function of $\alpha$ and also an implicit plot of $M_{(+)}$ as a function of $M_{(-)}$.

# 5 Surveying static vacuum solutions

In the previous section we have shown the existence of static vacuum shells in the spherically symmetric case and found some basic qualitative features, as well as a formula for the mass parameters in each region. A more exhaustive treatment of the static shells will be left for section 2. Before going any further let us summarize the catalogue of vacuum solutions that arise through the geometric surgery we described above. The first cases of interest are those corresponding to the standard orientation $\eta_L\eta_R > 0$, what we have called “standard shells”.

## 5.1 Standard shells and false vacuum bubbles

The vacuum shells with the standard orientation are always $(\xi_{(-)}, \xi_{(+)}) = (-1, +1)$ branch. So region $\mathcal{V}_L$ has a different effective cosmological constant to region $\mathcal{V}_R$, as can be seen from the expansion of the metric for large $r$. For example, when the bare cosmological constant $\Lambda = 0$ we have on one side of the shell the effective cosmological constant $\Lambda_d^{(+)} = -3/2\alpha$ and on the other $\Lambda_d^{(-)} = 0$. In the region with $\Lambda_d^{(+)}$ the graviton is expected to have ghost instability. In this sense the shell is like the false vacuum bubbles[7] studied in Refs.[19, 122, 23, 1, 44, 148, 44], but for a false vacuum which is of purely gravitational origin.

Being allowed to cut out a region of the space (here described by a given Boulware-Deser metric) and replace such region with a different piece of geometry (such as a Boulware-Deser metric with different parameters $M$ and $\xi$) might lead to curious implications. For instance, let us consider the following construction: Suppose we have a "well behaved" minus branch ($\xi_L = -1$) solution with positive mass $M_L$; where by "well behaved" we mean a solution in which the singularity is hidden behind an event horizon and for which we get a suitable GR limit for small $\alpha$. Now, let us cut out the black hole at some radius $r = a(\tau) > r_H$ and then replace it with the interior of a plus branch ($\xi_R = +1$) solution, i.e. a naked singularity. By doing this we would be constructing a vacuum solution whose geometry, from the point of view of an external observer, would coincide with that of a black hole but, instead, would not possess a horizon. A particle in free fall would not find a horizon but rather a naked singularity as soon as it passes through the $C^0$ junction hypersurface located at $r = a > r_H$. The solutions with $\Lambda = 0$ and $\alpha > 0$ are a clear example of this. As can be seen from figure 8.1 there are solutions for all positive $M_{(-)}$; so that we can indeed cut out the event horizon and replace it with a naked singularity!

Also, for $\Lambda \neq 0$ "false vacuum bubble" solutions gluing a positive mass Boulware-Deser branch with a naked singularity do exist. This is seen by looking at the moduli space described in figure 8.3. One might expect that such cosmic-censorship-spoiling shells be unstable and in section 1.1 we will confirm that they are unstable with respect to small perturbations.

[7]Strictly speaking, this label of false vacuum bubble would be correct if the minus branch metric were lower total energy with respect to the plus branch metric and if the classical transition were impossible.

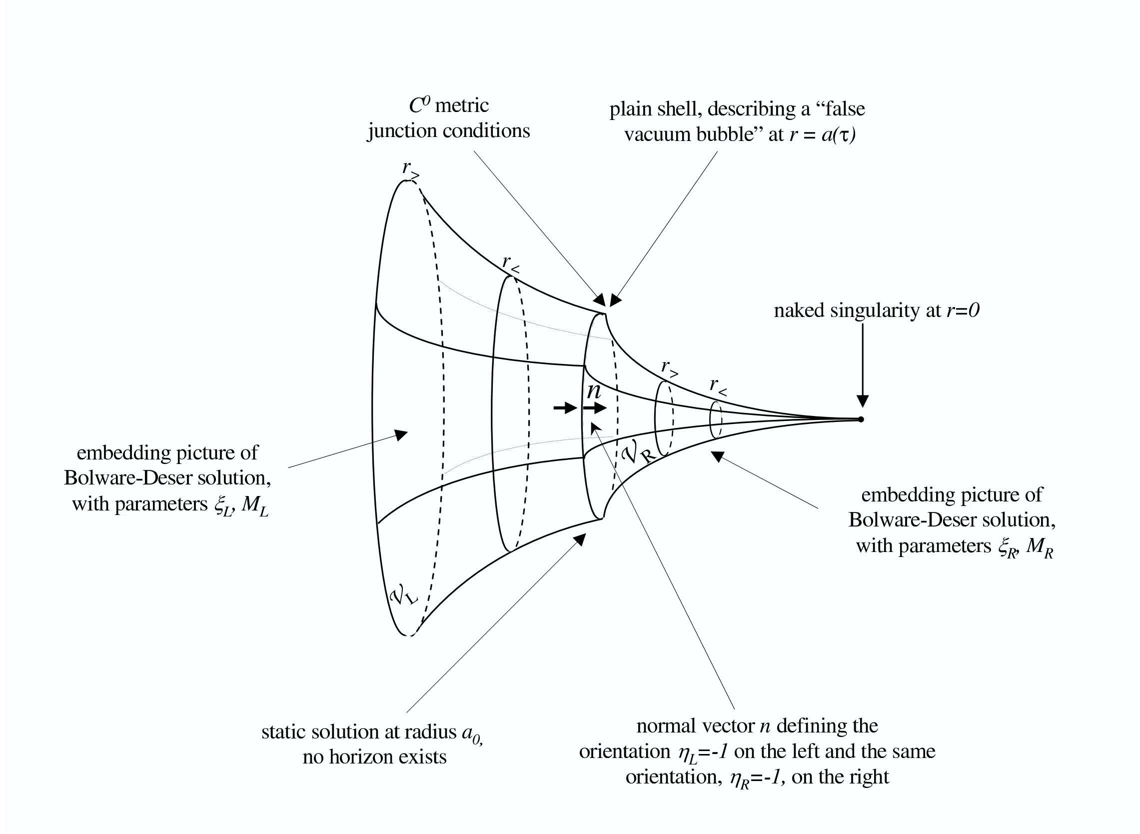



Figure 4.2: A spherically symmetric spacetime with metric of the class $C^0$. A vacuum shell with the standard orientation always connects two regions with different branch signs $\xi$ (and generically with different mass parameters $M$). Each region has a different effective cosmological constant.

## 5.2 Vacuum wormhole-like geometries

So far, we have discussed different kinds of geometries constructed by a cut and paste procedure of two spaces that were initially provided with the Boulware-Deser metric on them. The strategy was to make use of the junction conditions holding in Einstein-Gauss-Bonnet theory and, in particular, we have shown that solutions with non-trivial topology, which have no analogues in Einstein gravity, do arise through this method. A remarkable example is the existence of vacuum wormhole-like geometries, corresponding to the case $\eta_L \eta_R < 0$. These "wormholes" can be thought of as belonging to two different classes: The first class describes actual wormholes, presenting two different asymptotic regions which are connected through a throat located at radius $r_L = r_R = a$; the radius of the throat being larger than the radius where the event horizons (or naked singularities) would be. The two asymptotic regions are $r_L \to \infty$ and $r_R \to \infty$ as measured by the radial coordinate in the respective sides of the junction. This type of geometry is an example of a vacuum spherically symmetric wormhole solution in Lovelock theory and its existence is a remarkable fact on its own. On the other hand, a second class of wormhole-like geometry with no asymptotic regions also exists. This second class is obtained also by considering the orientation $\eta_L \eta_R < 0$, this time cutting away the exterior region of both geometries and gluing the two interior regions together. We shall discuss this later; first let us discuss the static wormhole solutions with two asymptotic regions (actual wormholes).

### Geometries presenting two asymptotic regions

Let us begin by emphasizing that such static wormhole solutions only exist if at least one of the two bulk regions corresponds to $\xi = +1$. That is, at least one of the two Boulware-Deser metrics has to correspond to what we have called the exotic branch. This could have deep implications in what regards semiclassical stability [24]. It is also remarkable that for these static wormholes to exist it is necessary that $\Lambda > 0$. Furthermore, the existence of two asymptotic regions demands $\alpha > 0$ (for values $\alpha < 0$ there are only solutions with"closed universe" geometry to be discussed below). Moreover, since the static wormholes only exist if at least one of the branches corresponds to $\xi = +1$, then at least one of the regions connected through the throat possesses a negative effective cosmological constant.

Another interesting feature concerns the stability under radial perturbations. This is seen in Fig 8.6. In particular, it can be shown that stable static wormholes only exist for the case $\xi_L = \xi_R = +1$; namely, the case where both Boulware-Deser metrics correspond to the exotic branch. Nevertheless, no stable wormholes exist for the case $M_L = M_R$, and thus, concisely, the static symmetric wormholes are unstable under perturbations that preserve the spherical symmetry.

An interesting possibility is that of having wormhole solutions whose Boulware-Deser metrics would correspond to negative mass parameters. For instance, one can construct a static wormhole with one side being of the "good branch" $\xi_L = -1$ and having a negative mass $M_L < 0$. In that case, from the point of view of a naive external observer, the vacuum solution would seem to correspond to a naked singularity. However, now we know

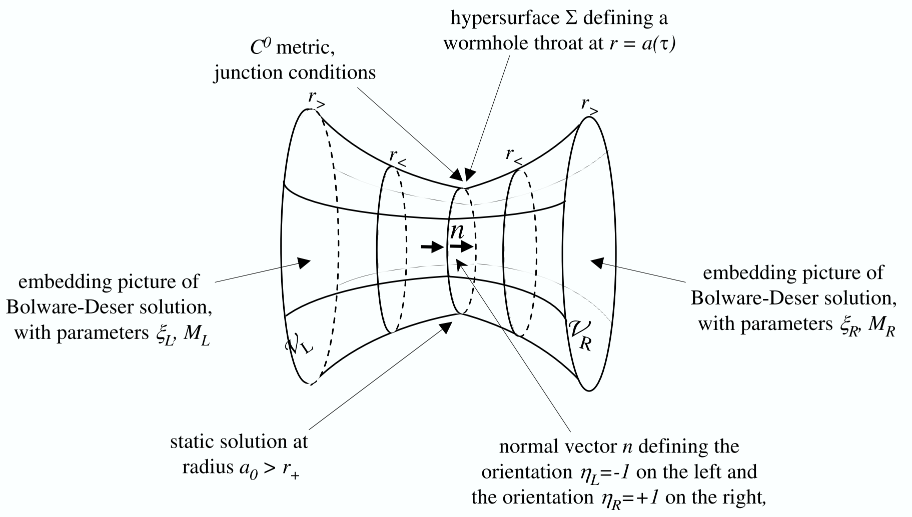



Figure 4.3: Static vacuum wormhole solution. The junction conditions are satisfied on the timelike hypersurface $\Sigma$ for the case of the orientation $\eta_L\eta_R < 0$. This solution presents two disconnected asymptotically de-Sitter regions.

that the inclusion of non-trivial junctures makes it possible to replace such a singularity by an exterior region on the other side of a non-smooth wormhole throat. This has a deep implication in what concerns the "cosmic censorship principle" since for the appropriate values of the coupling constants, and unlike what usually happens in pure gravitational theories, the spherically symmetric vacuum solutions presenting naked singularity cannot be unambiguously classified (and consequently systematically excluded) in terms of the mass parameter.

Another particular case that deserves to be mentioned as a special one is that of having a massless solution in one of the sides of the wormhole geometry. For instance, such a construction is achieved if the massless side corresponds to the exotic branch $\xi = +1$ and the massive side to the branch $\xi = -1$. In these cases, the wormhole throat turns out to be a kind of puncture of the (A)dS spacetime, let us call it a "hole in the vacuum". Since (A)dS is homogeneously isotropic, a spherically symmetric matching can be done anywhere: remarkably, several of these "holes" could be located at different places in the spacetime and each "hole" would not influence the others. We shall discuss this kind of geometry in more detail in section 3.2.

The massless side may then correspond to a microscopic de-Sitter geometry and, presumably, its cosmological horizon, yielding thermal radiation, could be seen from the massive sides. This is an intriguing possibility that deserves to be further explored.

**Closed universe type geometries**

Now, let us comment on the second class of wormholes; namely those with no asymptotic regions. As mentioned, these geometries are constructed by gluing the interior of the throat of both regions, instead of the exterior. One can perform the matching by keeping the region that is inside the throat but still outside the horizons. Consequently, one gets a geometry that resembles a "static closed universe" with horizons. This exotic geometry has no asymptotic regions at all, and, because of this, this second type of geometry does not represent what one would usually call a wormhole. Nevertheless, we shall abuse the notation and call "wormhole" any timelike junction with the orientation $\eta_L\eta_R < 0$.

Static solutions of this kind without naked singularities (i.e. with horizon) exist for negative values of the coupling $\alpha$ and $x \equiv 4\alpha\Lambda/3 < -1$. In this range of the coupling constants the Boulware-Deser metric develops a branch singularity at fixed radius $r_c^4 = \frac{16M\alpha}{|x|-1}$, where the curvature diverges. This branch singularity represents the maximum three-sphere radius: the metric becomes non-real for $r > r_c$. In addition there is a curvature singularity at $r = 0$. In this region of the space of parameters we would say that the Boulware-Deser geometry is somehow pathological. However, if junction conditions are appropriately applied, then a well-behaved $C^0$ vacuum geometry can be constructed by simply taking a pair of such pathological spaces, cutting out the naked singularities and joining them together. To see that this is possible, it is sufficient to consider the symmetrical case. It can be checked from equations (4.47) and (4.48) that two bulk regions with equal masses $M_L/\alpha = M_R/\alpha = \frac{4(1+x)}{(x-3)}$ can be matched at a throat radius $a = \frac{16|\alpha|}{3-x}$. Consulting figure 8.4 (these solutions are located on the upper bounding curve of the left part of the moduli space) we see that wormhole solutions exist when the bulk regions have branch signs $(\xi_L, \xi_R) = (-1, -1)$. The bulk metric has a horizon $r_H$, which separates $r = 0$ (a timelike naked singularity) from $r = r_c$, which is a spacelike singularity. The static shell is located at $a < r_H$. So by cutting out the regions $r < a$ and joining with the wormhole orientation the naked singularities can be removed. The causal diagram of

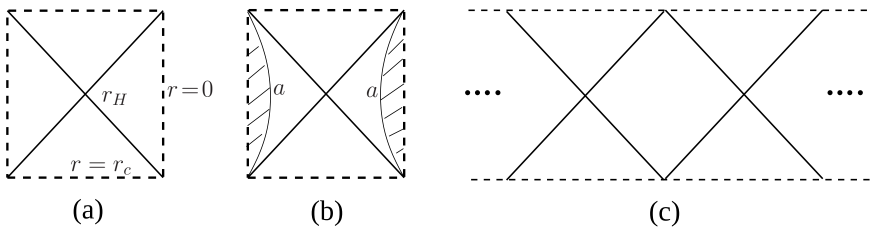



Figure 4.4: (a) The causal diagram of the smooth spherically symmetric solution for $\alpha < 0$, $1 + \frac{4\alpha\Lambda}{3} < 0$. (b) The vacuum shell is introduced at radius $r = a$ cutting out the $r = 0$ singularity. (c) Causal diagram of the resulting spacetime (a $C^0$, spherically symmetric vacuum solution).

the original pathological spacetimes and the extended causal diagram of the $C^0$ closed universe, which results from the matching, with horizons are shown in figure 4.4.

# Chapter 5

# Dynamical Solitons of Lovelock Theory

## 1 Dynamical vacuum shells

In general, vacuum shells will be dynamical objects. We discuss the dynamics here and also discuss the issue of radial stability of the static solutions.

### 1.1 General solution

Let us briefly recapitulate upon the equation (4.22), which governs the dynamics of the shells. We can treat both the timelike and spacelike together since, as we noticed in section 3, the analysis is completely analogous. A dynamical vacuum shell is governed by a differential equation of the form

$$\dot{a}^2 + V(a) = 0\,; \tag{5.1}$$

see (4.22) above. It is useful to express $V(a)$ in terms of the non-negative quantity $Y = \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{16M\alpha}{a^4}}$, and the effective potential then reads

$$V(a) = \sigma\left(k + \frac{a^2}{4\alpha}\right) - \frac{\sigma a^2}{4\alpha}\left(\frac{3(\xi_R Y_R + \xi_L Y_L)^2 + (\xi_R Y_R - \xi_L Y_L)^2}{12(\xi_R Y_R + \xi_L Y_L)}\right)\,. \tag{5.2}$$

In addition to the differential equation, the solution must obey the inequalities (4.27)-(4.28). It is convenient to rewrite them as follows

$$-\eta_R\eta_L\Big(9(\xi_R Y_R + \xi_L Y_L)^2 - (\xi_R Y_R - \xi_L Y_L)^2\Big) \geq 0\,, \tag{5.3}$$

$$\sigma\alpha(\xi_R Y_R + \xi_L Y_L) > 0\,. \tag{5.4}$$

Note that the effective potential (5.2), $V(a) = \frac{\sigma a^2}{4\alpha} + \sigma k + \Delta V(a)$, consists of a quadratic piece, a constant determined by the three-dimensional curvature $k$ of the shell, and another piece which, by inequality (5.4) obeys $\Delta V < 0$.

To analyze the motion of a shell we shall need to know the derivatives of the potential. This is worked out in appendix 3. Differentiating the potential we get the following expression for the acceleration of a moving shell,

$$\ddot{a} = -\sigma\,\frac{a}{4\alpha}\Big[1 - \frac{1 + 4\alpha\Lambda/3}{\xi_R Y_R + \xi_L Y_L}\Big]\,. \tag{5.5}$$

Considering the sign of this acceleration and making use of inequality (5.4) we can make some general observations:

**Remark 10.** *For a timelike shell ($\sigma = +1$):*
*When $1 + \frac{4\alpha\Lambda}{3} \geq 0$ and $\alpha < 0$ a vacuum shell always experiences a repulsive force away from $r = 0$;*
*When $1 + \frac{4\alpha\Lambda}{3} \leq 0$ and $\alpha > 0$ a vacuum shell always experiences an attractive force towards $r = 0$.*

In the situations not covered by Remark 10 the potential may have an extremum. From (5.5) we deduce that there is an extremum at $r = a_e$ iff

$$\xi_R Y_R(a_e) + \xi_L Y_L(a_e) = 1 + \frac{4\alpha\Lambda}{3}\,. \tag{5.6}$$

Recalling inequality (5.4), we conclude that an extremum can exist only if

$$\sigma\alpha\left(1 + \frac{4\alpha\Lambda}{3}\right) > 0\,. \tag{5.7}$$

The extremum will be a minimum or maximum depending on the sign of the second derivative of the potential evaluated there,

$$V''(a_e) = \frac{\sigma}{\alpha}\left(\frac{1 + 4\alpha\Lambda/3}{\xi_R\xi_L Y_R(a_e) Y_L(a_e)} - 1\right)\,. \tag{5.8}$$

There is a general result for vacuum shells separating different branch metrics. From (5.7) we see that $V''(a_e)$ in (5.8) must be negative for $1 + \frac{4\alpha\Lambda}{3} \geq 0$.

**Proposition 11.** *In the range*[1] $1+\frac{4\alpha\Lambda}{3}\geq 0$ *for the product of Gauss-Bonnet coupling and cosmological constant: Let* $\Sigma$ *be a vacuum shell such that* $\xi_L\xi_R = -1$*. Then the potential never has a minimum. If* $\Sigma$ *is a timelike shell it will either be in an (unstable) static state, or, if it is moving, will continue to expand or collapse, it can not be bound.*

We have already remarked in section 3 that any shell with standard orientation *must* match two bulk metrics of opposite branch sign ($\xi_L\xi_R = -1$), except in the trivial case of a smooth matching. So we obtain the following general result about instability of standard shells:

**Corollary 12.** *Let* $1+\frac{4\alpha\Lambda}{3}\geq 0$*. A timelike shell with standard orientation is either in an (unstable) static motion, or, if it is moving, will continue either to expand or collapse.*

We have already seen in section 4 that static shells with standard orientation are always in a state of unstable equilibrium in the (physical) regime $1+\frac{4\alpha\Lambda}{3}\geq 0$. The proposition above strengthens this result to include dynamical shells. A dynamical shell with standard orientation can not be oscillatory. It must either disappear into a singularity or fly out towards spatial infinity.

There is not such a strong result for shells with the wormhole orientation. Indeed in section 4 we found stable static wormholes for $1+\frac{4\alpha\Lambda}{3}\geq 0$ which matched two bulk metrics of the plus branch. We can however derive a strong result about instability concerning bulk metrics of the minus branch. When $\xi_L = \xi_R = -1$ we see from (5.6) that for $1+\frac{4\alpha\Lambda}{3}\geq 0$ an extremum is not possible. Furthermore, combining the results of Remarks 5 and 10 we see that the shell is always expanding:

**Proposition 13.** *Let* $1+\frac{4\alpha\Lambda}{3}\geq 0$ *and let* $\Sigma$ *be a timelike vacuum shell with wormhole orientation, and* $\mathcal{V}_L$ *and* $\mathcal{V}_R$ *be minus branch bulk metrics* $(\xi_L,\xi_R) = (-1,-1)$*. Then the shell always experiences a repulsive force away from* $r = 0$*.*

So in summary, we have found some general results for the range of parameters $1+\frac{4\alpha\Lambda}{3}\geq 0$. This range is important because it includes the case $|\alpha\Lambda| << 1$ and therefore applies when the Gauss-Bonnet term is a small correction. Combining these results, we conclude that, in this range of parameters, all timelike vacuum shells involving the minus branch are unstable.

[1] In the case of the wormhole orientation, by using the inequality (5.3), the result can be extended to apply to the range $1+\frac{4\alpha\Lambda}{3} > -\frac{1}{2}$.

The only vacuum shell solutions which can be static or oscillatory are wormholes which match two regions of the exotic plus branch.

## 1.2 Comment on the stability of static shells

Dynamical equation (5.1) resembles the equation for a particle moving under the influence of an effective potential (5.2). Nevertheless, as pointed out in section 3, this is not strictly the case due to the presence of the vanishing energy constraint. This is important for the case when the extremum of the potential is at $a = a_0$ with $V(a_0) = 0$, i.e. when static solutions exist. When $V''(a_0) < 0$ we conclude that the shell is unstable with respect to the radial component of a perturbation- a slight shift $a \to a_0 + \delta a$ will cause the shell to accelerate away from the (unstable) equilibrium radius. When $V''(a_0) > 0$ we conclude that the shell is stable with respect to radial perturbations. There is however a slight subtlety: as mentioned previously the energy is unavoidably fixed instead of arising as a constant of motion. So for a fixed potential, there is no real solution for $a$ when $V > 0$. We can consider spherically symmetric solutions which are close-by in the space of the solutions, i.e. with slightly different parameters $M_{L,R}$ and $w$ such that the value of the potential at $a_e$ is slightly negative: let us say $V(a_e) = 0^-$. This means that such a solution oscillates between two radii around $a_e$ at which the potential vanishes. This is certainly a stable solution though not static, a 'bounded excursion' [158]. Now if we let $a_e$ coincide with the $a_0$ of the original static solution, this means that for slightly different parameters than those for which $a_0$ is a static solution, there exists an oscillating solution around $a_0$. Therefore a static solution $a_0$ which is a minimum of the potential gives information about when infinitesimal bounded excursions can happen. More generally, the dynamics of the perturbed shell can be thought of as corresponding to a perturbation of the above equation $V(a) \to V(a) - \delta h$, provided energy $\delta h$ from an external excitation. The stable regions of the moduli space of static solutions are plotted in figures 8.6 and 8.7. The graph will take an elegant form in terms of the change of variables to be introduced in section 2 (see fig 5.4).

In the rest of this section we present some illustrative examples of dynamical vacuum shells, first in symmetrical wormhole solutions and then in the context of Chern-Simons gravity.

## 1.3 Symmetric dynamical wormholes

Now let us consider the case where the masses in each bulk region are the same, being $M_L = M_R = M$. The inequalities (5.3) and (5.4) are equivalent to:

**Remark 14.** *If $\Sigma$ is a vacuum shell joining two bulk regions with the same mass $M_R = M_L$ then:*
*i) The bulk solutions must have the same branch sign $\xi_L = \xi_R = sign(\sigma\alpha)$;*
*ii) The shell must have wormhole orientation.*

So the spacetime is completely left-right mirror-symmetric. The equation of motion reads

$$\sigma \dot{a}^2 + \frac{a^2}{4\alpha}\left[1 - \xi \frac{Y(a)}{2}\right] + k = 0 \ . \tag{5.9}$$

The general solution is rather complicated. Next we proceed to consider a simple case where both masses vanish.

The case where $M_{L,R} = 0$ is an interesting special case of the symmetric wormholes, which exists for $1 + \frac{4\alpha\Lambda}{3} > 0$. The equation of motion reduces to

$$\sigma \dot{a}^2 + \frac{a^2}{4\alpha}\left(1 - \xi \frac{\sqrt{1 + \frac{4\alpha\Lambda}{3}}}{2}\right) + k = 0 \ . \tag{5.10}$$

**Remark 15.** *Consider a timelike shell ($\sigma = +1$), that is sign($\alpha$) $= \xi$.*
*Bounded motions: $\xi = +1$, $\frac{4\alpha\Lambda}{3} < 3$, $k = +1$ ; $\xi = +1$, $\frac{4\alpha\Lambda}{3} = 3$, $k = 0$.*
*Unbounded motions: $\xi = +1$, $\frac{4\alpha\Lambda}{3} = 3$, $k = -1$ ; $\xi = +1$, $\frac{4\alpha\Lambda}{3} > 3$, any $k$ ; $\xi = -1$, $\frac{4\alpha\Lambda}{3} > -1$, any $k$.*
*The same bounded or unbounded configurations exist in the spacelike case provided one replaces $k$ with $-k$, for the opposite sign of $\alpha$.*

The hyperbolic shell, $k = -1$, admits a stationary vacuum wormhole solution: for sign($\alpha$) $= \xi = +1$ and $4\alpha\Lambda/3 = 3$ we have that $\ddot{a} = 0$ and $\dot{a}^2 = 1$.

When $\Lambda = 0$ and $\xi = -1$ the two bulk regions have flat Minkowski metrics, spherical timelike vacuum wormholes with an expanding throat of minimum radius squared $8|\alpha|/3$ are possible when spacetime is Minkowski on both sides. On the other hand, we may cut a Minkowski spacetime along the spacelike hypersurface $t^2 - r^2 = 8|\alpha|/3$ (up to shifts of $t$) and match it with a similar region of another Minkowski spacetime.

## 1.4 Chern-Simons dynamical vacuum junctions

When $1 + \frac{4\alpha\Lambda}{3} = 0$ some very special things happen. For this choice of coupling constants the Einstein-Gauss-Bonnet theory (in first order formalism) is equivalent to a Chern-Simons theory for the deSitter ($\alpha < 0$) or Anti de Sitter ($\alpha > 0$) group[2]. In this case the metric function takes the very simple form

$$f(r) = 1 + \frac{r^2}{4\alpha} - \mu \ , \tag{5.11}$$

where $\mu$ is a constant and the mass is proportional to $\mu^2 - 1$[49]. When $\mu > 0$ the bulk solution is a black hole. When $\mu < 0$ the bulk metric would have a naked singularity at the origin.

The dynamics of vacuum shells takes a very simple form. The quantity $a^2\xi Y/4\alpha = -\mu$ for each bulk region is a constant and therefore the non-harmonic part of the potential $\Delta V$ is a constant. The equation of motion takes the form

$$\dot{a}^2 + \frac{\sigma}{4\alpha}a^2 = \mathcal{E} \quad , \qquad \mathcal{E} = -\sigma\left(k + \frac{3(\mu_R+\mu_L)^2 + (\mu_R-\mu_L)^2}{12(\mu_R+\mu_L)}\right) \ . \tag{5.12}$$

The potential is like that of a harmonic oscillator potential (or an upside-down harmonic potential if $\sigma\alpha$ is negative), although it should be remembered that the origin $r = 0$ of the bulk spacetimes is singular so the shell can not really oscillate. The solution is constrained according to the two inequalities (5.3) and (5.4), which now read

$$-\eta_L\eta_R(9(\mu_R+\mu_L)^2 - (\mu_R-\mu_L)^2) \geq 0 \ , \tag{5.13}$$

$$-\sigma\,(\mu_R+\mu_L) > 0 \ . \tag{5.14}$$

The last inequality tells that $\mathcal{E} > -\sigma k$. These inequalities are generally consistent with $\mathcal{E} > 0$ so that solutions do indeed exist.

For instance, consider the timelike shells in this theory. Note that, from inequality (5.14), at least one out of $\mu_R$ or $\mu_L$ must be negative. So it is not possible to match two black hole spacetimes. From inequality (5.13) we see that shells with the standard orientation must obey $\mu_R\mu_L < 0$.

**Remark 16.** *For the Chern-Simons combination* $1 + \frac{4\alpha\Lambda}{3} = 0$*, timelike vacuum shells always represent either:*

[2]The case of Poincaré Chern-Simons theory was discussed in Ref. [88].

*i) a matching between a bulk region of a black hole spacetime with bulk region of a naked singularity spacetime; or*
*ii) a matching, with wormhole orientation, between two bulk regions of naked singularity spacetimes.*

Now, let us analyze the de-Sitter invariant Chern-Simons gravity, which corresponds to $\alpha\Lambda = -3/4$ with $\alpha < 0$. In this case, the potential is like an inverted harmonic oscillator centered at the origin. There are solutions for $\mathcal{E}$ positive, negative and zero.

Let us just focus on the case $\mathcal{E} > 0$. The trajectory of a timelike shell is then given by

$$a(\tau) = 2\sqrt{|\alpha\mathcal{E}|}\ \sinh\Big(\pm\frac{\tau}{2\sqrt{|\alpha|}} + \text{const.}\Big)\ , \tag{5.15}$$

which is a shell either emerging from the past white hole or falling into the future black hole, depending on the sign $\pm$ in the argument.

For $\mathcal{E} < 0$ the hyperbolic sine is replaced by the cosine. $\mathcal{E} = 0$ gives an increasing and a decreasing exponential.

On the other hand, for $\mathcal{E} > 0$ and $k = 1$, one could consider Euclideanization of the problem. Presumably, this could be relevant in describing the decay of the exotic negative $\mu$ spacetime. Define an angle $\chi$ by

$$\chi = \frac{\tau_E}{2\sqrt{|\alpha|}} \tag{5.16}$$

up to a constant, where $\tau_E$ is the Euclidean proper time of the shell. The metric on the Euclidean world sheet of the shell reads

$$ds_\Sigma^2 = 4|\alpha|\big(d\chi^2 + \mathcal{E}\sin^2\chi d\Omega^2\big)\ . \tag{5.17}$$

When $\mathcal{E} < 1$ we have a deficit solid angle, and, when $\mathcal{E} > 1$, an excess. In both cases the space has a curvature singularity at the poles $\chi = 0$ and $\chi = \pi$. Therefore, the smoothness of the Euclidean shell requires $\mathcal{E} = 1$. This metric is spherically symmetric in the five dimensional sense in this case, whence it describes a 4-sphere. The 4-sphere separates a ball of Euclidean black-hole solution with mass parameter $\mu_R$ from another solution with $\mu_L$, obeying the relation

$$\mu_L^2 + \mu_R^2 + \mu_L\mu_R + 6(\mu_L + \mu_R) = 0\,. \tag{5.18}$$

It is interesting to note that the size of the Euclidean world sheet depends essentially only on $\alpha$ and not on the $\mu_{L,R}$; the latter change its shape, which is fixed to spherical by the above relation.

The curve (5.18) is an ellipse. It is symmetrical around the line $\mu_L = \mu_R$ and tangential with the line $\mu_L + \mu_R = 0$ at $\mu_L = \mu_R = 0$. It exists completely in the region $\mu_L + \mu_R \leq 0$. In view of the inequality (5.14) all points of the curve are included except $\mu_L = \mu_R = 0$. Therefore the 4-sphere Euclidean world sheet does exists for certain values of the parameters. Whether this interesting configuration is a mere curiosity or it is related to semiclassical transitions between the $\mu_L$ and $\mu_R$ spacetime is an open question.

We can also consider the Anti-de Sitter invariant Chern-Simons theory, corresponding to $\alpha > 0$. In this case, the effective potential $V$ turns out to be a quadratic potential centered at the singularity at the origin. The analysis is similar to that of the dS case except that there are solutions only with $\mathcal{E} > 0$.

On the other hand, we can also think about the spacelike shells for this case of Chern-Simons couplings $\Lambda\alpha = -3/4$. These shells represent a sudden classical transition from a spacetime with some mass parameter $\mu_{\rm in}$ to another with a different mass parameter $\mu_{\rm out}$. Such transitions occur for quite general values of $\mu_{\rm in}$, and this is a concise manifestation of the extreme degeneracy of the field equations of the Chern-Simons theories.

# 2 Constant solutions revisited

In this section we will perform an exhaustive analysis of the space of constant solutions, what we have called the moduli space.

## 2.1 Static and instantaneous spherical vacuum shells

To begin, it will be convenient to introduce new dimensionless parameters, defined as follows

$$u \equiv \sqrt{3}\sqrt{x(x+1)\Big(\frac{4}{y}+\frac{3}{x}-1\Big)} \quad , \quad w \equiv x+1 \; . \tag{5.19}$$

On should think of $u$ and $w$ as functions of $\alpha$, $\Lambda$ and $a_0^2$, via the definitions (4.37). Here,

$$u \geq 0 \; .$$

The inverse transformation is given by

$$y = \frac{12w(w-1)}{u^2+3(w^2-4w)} \quad , \quad x = w-1 \; . \tag{5.20}$$

Each point on the $(w,u)$ plane such that $u^2+3(w^2-4w) \neq 0$ uniquely determines the values of the basic dimensionless ratios $x$ and $y$ and therefore the solution[3]. The line $w=0$ is peculiar as it implies that either $a_0=0$, or $\alpha=\infty$ and $\Lambda=0$. The latter is the case of pure Gauss-Bonnet gravity. We will see that for $|\alpha|<\infty$ the line $w=0$ is excluded from the moduli space as it corresponds to smooth geometries.

**Definition 17.** *We will call the allowed domain on the $(u,w)$ plane as the $(u,w)$ parameter space representing the moduli space of the vacuum shell. Similarly the allowed domain on the plane of $x$ and $y$ is the $(x,y)$ parameter space for the vacuum shell for non-zero $\Lambda$. The various possible pairs of parameters that uniquely represent all possible points of the moduli space can be thought of its coordinates.*

In terms of these new variables we have that the vacuum shells are described by equations

$$f_L+f_R = \frac{2(u^2+9w^2)}{u^2+3(w^2-4w)} \, , \qquad \sqrt{f_L f_R} = \sigma\eta_R\eta_L \, \frac{u^2-9w^2}{u^2+3(w^2-4w)} \, , \tag{5.22}$$

with $f_{L,R}>0$ for the timelike vacuum shell (which corresponds to $\sigma=+1$), and $f_{L,R}<0$ for the spacelike vacuum shell ($\sigma=-1$). After squaring the equations above, we obtain

$$f_{(\pm)} = \frac{(3w\pm u)^2}{u^2+3(w^2-4w)} \; . \tag{5.23}$$

which turns out to be always real. The solution for $f_{L,R}$ is given by (4.41) as discussed there. Then from (5.22) we first have

[3] It is useful to remember that the radius $a_0$ of the vacuum shell is given in terms of these variables by

$$a_0^2 = 4\alpha \cdot \frac{12w}{u^2+3(w^2-4w)} \; . \tag{5.21}$$

**Proposition 18.** *Let the total moduli space be described in the* $(w, u)$ *parameter space. Then it necessarily is a subset of the upper half plane* $u \geq 0$ *from which the points on the curves* $\pm u = 3w$ *and* $u^2 + 3(w^2 - 4w) = 0$ *are excluded. The four disconnected regions are divided according to the type of the matching by combinations of the following. Timelike:* $u^2 + 3(w^2 - 4w) > 0$*. Spacelike:* $u^2+3(w^2-4w) < 0$*. Same orientation i.e.* $\eta_L \eta_R > 0 : u^2-9w^2 > 0$*. Opposite orientation, i.e.* $\eta_L \eta_R < 0 : u^2 - 9w^2 < 0$*.*

It is good to remember

**Remark 19.** *The points* $(0, 0)$*,* $(1, 3)$ *and* $(0, 4)$ *in the* $(w, u)$ *plain do not belong to the moduli space. The point* $(1, 3)$ *corresponds to the line* $x = 0$ *and* $y \neq 0$*.*

We have already used the fact that $f_{L,R}(r)$ are the Boulware-Deser metric functions. In order to completely solve our problem we must substitute for $f_{L,R}$ using the Boulware-Deser expression evaluated at $r = a_0$ given in equation (4.38),

$$f_L = f_L(a_0) \quad , \quad f_R = f_R(a_0) \ . \tag{5.24}$$

Recall (4.41). Similarly to equations (4.42) and (4.43) we have that, within the space of Proposition 18, (5.24) amount to

$$w\,(w \pm u) = 2w\,\xi_{(\pm)} \sqrt{w + \frac{\left(u^2 + 3(w^2 - 4w)\right)^2}{144w^2} \bar{M}_{(\pm)}} \ . \tag{5.25}$$

The solution $w = 0$ is possible only if $\bar{M}_{L,R} = 0$. Then for $|\alpha| < \infty$ we have that $M_{L,R} = 0$ and the bulk metrics are simply $f_{L,R}(r) = 1 + r^2/(4\alpha)$. We have

**Remark 20.** *The line* $w = 0$*, which lies in the "cone"* $u^2 - 9w^2 > 0$ *and entirely within the timelike standard shell region, is excluded from the moduli space as it merely corresponds to smooth geometries.*

Therefore we work with non-zero $w$. Squaring the previous relation we find the mass parameters of $f_{(\pm)}(r)$ which are consistent with the vacuum shell solution; namely

$$\bar{M}_{(\pm)} = \frac{36w^2((w \pm u)^2 - 4w)}{(u^2 + 3(w^2 - 4w))^2} \ . \tag{5.26}$$

Substituting back into (5.25) we have the condition

$$\xi_{(\pm)} |w \pm u| = w \pm u \ . \tag{5.27}$$

The sign of $\xi_{(+)}$ is completely determined over the moduli space if $u + w \neq 0$ by $\xi_{(+)}(w + u) > 0$. Similarly, the sign of $\xi_{(-)}$ is determined if $w - u \neq 0$ by $\xi_{(-)}(w - u) > 0$. Now for $w + u = 0$ we find that $\xi_{(-)}|w| = w = -u < 0$. Similarly for $w - u = 0$ we find that $\xi_{(+)} > 0$, and this happens for $w > 0$.

We see that the signs $\xi_{\pm}$ are specified for each point on the moduli space, i.e. a solution of the vacuum shell. We will say that this is a solution of the vacuum shell of *type* $(\xi_{(-)}, \xi_{(+)})$. The exception is along the *partition curve* $u^2 - w^2 = 0$ where one of the signs is undetermined. We can summarize

**Proposition 21.** *The moduli space consists of the regions of the parameter space* $(w, u)$ *given in Proposition 18 such that: i) the line* $w = 0$ *is excluded, ii) according to the branch signs* $(\xi_{(-)}, \xi_{(+)})$ *of the bulk regions the parameter space is divided as follows:* $(+, +)$ *for* $u < w$; $(-, +)$ *for* $-u < w < u$, $(-, -)$ *for* $w < -u$.

*The points along the partition curve* $u^2 - w^2 = 0$ *satisfy: if* $w > 0$ *then* $\xi_{(+)} > 0$ *and* $\xi_{(-)}$ *arbitrary, if* $w < 0$ *then* $\xi_{(-)} < 0$ *and* $\xi_{(+)}$ *arbitrary. The mass parameters* $M_{(\pm)}$ *are well defined and given over the moduli space by formula (5.26).*

Propositions 18 and 21 categorize the allowed spherically symmetric vacuum shell solution at constant $r$ in terms of spacelike/timelike and branch signs. This is plotted in figure 5.1. In what follows we further categorize the solutions according to other physical properties. The entire information we will get is given in the Fig.5.2.

Note also the following: Formula (5.26) says that we can define a function

$$\bar{M}_*(w, u) := \frac{36w^2((w + u)^2 - 4w)}{(u^2 + 3(w^2 - 4w))^2} \ , \tag{5.28}$$

defined on the whole of the $(w, u)$ plain (minus the curve $u^2 + 3(w^2 - 4w) = 0$) and not only on the upper half. Then for $u > 0$, $\bar{M}_{(+)} = \bar{M}_*(w, u)$ and $\bar{M}_{(-)} = \bar{M}_*(w, -u)$. More generally, recalling also equations (5.23), and (5.27), one may extend also $f(a_0)$ and $\xi$, regarded as functions of $w$ and $u$, over the whole of the $(w, u)$ plane.

**Lemma 22.** *Let $X$ denote any of the quantities $\bar{M}$, $f$, $\xi$, or combinations of them. One may define a function $X_*(w,u)$ such that $X_{(+)} = X_*(w,u)$ for $u \geq 0$. Then, $X_{(-)} = X_*(w,-u)$. At $u = 0$ we have $X_{(+)} = X_{(-)}$ i.e. $X_R = X_L$.*

*The parameter space can be extended over the whole of the $(u,w)$ plane. The mirror transformation $u \to -u$ has the effect of sending $(+) \leftrightarrow (-)$. So one may specify one type of quantities $\xi_{(+)}$, $\bar{M}_{(+)}$ on the whole plane and mirror image the results to obtain the values of $\xi_{(-)}$ and $\bar{M}_{(-)}$.*

Now, we will also return to discuss in a more detailed manner the two most basic distinct types of constructions here (recall Definition 2): matching with the same orientation, i.e. standard shell solutions, and matching with opposite orientation, which we call collectively wormholes. Though the following definition has been already in use in our work, it is useful to formalize the following

**Definition 23.** *A plus-metric, corresponding to the metric function $f_{(+)}(r)$, is one whose mass parameters is given by $\bar{M}(w,u) = M_{(+)}$ and branch by $(w+u)/|w+u| = \xi_{(+)}$ over the moduli space. A minus-metric, corresponding to the metric function $f_{(-)}(r)$), is one whose mass parameters is given by $\bar{M}(w,-u) = \bar{M}_{(-)}$ and branch by $(w-u)/|w-u| = \xi_{(-)}$ over the moduli space.*

Now, let us make a remark on the sign of $\alpha$. As $a_0^2 > 0$ it can be determined by the sign of $y/x$ and is given by

$$\mathrm{sign}(\alpha) = \mathrm{sign}\big(w\,(u^2 + 3(w^2 - 4w))\big)\ . \tag{5.29}$$

Therefore we have

**Remark 24.** $\alpha > 0$ *only for timelike vacuum shells and in the region $w > 0$ (for standard or wormhole orientation). Inside the ellipse of the spacelike vacuum shells (see fig. 5.1), or for $w < 0$, we have*: $\alpha < 0$.

From the definition of $w$, the sign of the cosmological constant $\Lambda$ is determined according to

$$\mathrm{sign}(\Lambda) = \mathrm{sign}(\alpha)\,\mathrm{sign}(w-1). \tag{5.30}$$

When $\Lambda = 0$ i.e. $w = 1 > 0$, the sign of $\alpha$ depends on the whether the shell is time- or space-like as we mentioned just above.

## 2.2 The masses over the moduli space

Important physical properties of the solutions have to do with what values the masses $M_{(\pm)}$ take, w.r.t. sign, magnitude and relative magnitude, over the moduli space. Let us comment on it below.

### Equal mass solutions

A question with a very simple answer is where on the moduli space we could have $M_{(+)} = M_{(-)}$. We have seen that this happens at $u = 0$. Explicitly, from (5.26) we have

$$\bar{M}_{(+)} - \bar{M}_{(-)} = \frac{144uw^3}{\big(u^2 + 3(w^2 - 4w)\big)^2} \ . \tag{5.31}$$

**Proposition 25.** *$M_{(+)} = M_{(-)}$ only at the boundary $u = 0$. Therefore such solutions exist only for wormholes.*

From Proposition 21 we have that at the points where $M_{(+)} = M_{(-)}$ we have also that $\xi_{(+)} = \xi_{(-)}$.

**Lemma 26.** *Symmetric configuration are such $M_L = M_R$ and $\xi_L = \xi_R$. They exist only at the boundary $u = 0$ of the moduli space and they can be either time- or space-like shell wormholes.*

The equal mass $\bar{M} = \bar{M}_{(\pm)}$ of the symmetric case reads

$$\bar{M} = \frac{4w}{w - 4} \ . \tag{5.32}$$

So, symmetric configurations exist for all $w \neq 4$ and $\bar{M}$ can take all real values except 4.

### Zero mass solutions

The masses $\bar{M}_{(\pm)}$ change sign crossing the curves where they vanish, and of course these curves are where $M_{(\pm)}$ vanish too. From the formula (5.26) and Proposition 21 we have that $\bar{M}_{(\pm)} = 0$ along the curves

$$(u \pm w)^2 = 4w \ , \tag{5.33}$$

respectively. They exist only for $w > 0$. The masses cannot vanish for $w < 0$.

Using Lemma 22 we may look only at one of the curves e.g. $(u-w)^2 = 4w$, which also reads $u = \pm 2\sqrt{w} + w$, over the whole plane. The other mass is

$$\bar{M}_0^{(\pm)} = \frac{9w\sqrt{w}}{(w-3)\sqrt{w} \pm 2} \ . \tag{5.34}$$

The curve $(u-w)^2 - 4w = 0$ goes to negative values of $u$ for $0 < w < 4$. On the $u > 0$ side appears disconnected emerging into two pieces ar $w = 0$ and $w = 4$, fig. 5.2. Therefore

**Proposition 27.** *$\bar{M}_{(-)} = 0$ for $u = \pm 2\sqrt{w} + w > 0$ where $\bar{M}_{(+)} = \bar{M}_0^{(\pm)}$ respectively. $\bar{M}_{(+)} = 0$ for $u = 2\sqrt{w} - w > 0$ where $\bar{M}_{(-)} = \bar{M}_0^{(-)}$.*

Independently of whether the mass that vanishes is an $M_{(+)}$ or an $M_{(-)}$ note also the following

**Remark 28.** *When the zero mass is of branch $\xi$ the massive side has mass $\bar{M}_0^{(-\xi)}$ and the matching happens according to $u = |w - 2\xi\sqrt{w}| > 0$. The branch of the massive side depends, as always, on which side of the line $u = w$ we are.*

### Sign of the mass parameters

Now, let us discuss the positivity of the mass parameters.

The signs of $\bar{M}_{(\pm)}$ behave quite simply. From formula (5.26) we have:

**Proposition 29.** *$\bar{M}_{(\pm)} < 0$ at the convex region defined by the curves (5.33) i.e. where $(u \pm w)^2 - 4w < 0$ respectively. They have an overlap for $0 \leq u < 2\sqrt{w} - w$, inside the spacelike shell wormhole region. $\bar{M}_{(+)} < 0$ only in this overlap.*

**Remark 30.** *The entire curve $u - w = 0$ exists within the region where $\bar{M}_{(-)} < 0$. This is also seen by the fact that the r.h.s. of (5.25) vanishes there. $\bar{M}_{(+)} > 0$ along $u - w = 0$.*

The above mean that $\bar{M}_{(-)} < 0$ in a very large part of the moduli space for $w > 0$. Therefore the metrics $f_{(-)}$ will have inner branch singularities, discussed in appendix 1.

The signs of $M_{(\pm)} = \alpha\bar{M}_{(\pm)}$ themselves are depicted in the figure 5.3 using also formula (5.29).

### Mass as a function of the radius of the shell

We see from (5.26) that given a mass $M_{(\pm)}$, for given $\alpha$ and $\Lambda$, the radius $a_0$ of the shell where the matching takes place is determined by a fourth order polynomial of $u$, namely

$$(u^2 + 3(w^2 - 4w))^2 \, \bar{M}_* - 36w^2 \, (u^2 + 2wu + w^2 - 4w) = 0 \; ; \qquad (5.35)$$

given a $u$ we can obtain $a_0^2$ by (5.21). As discussed above, Lemma 22, $\bar{M}_*$ is an $\bar{M}_{(+)}$ when $u$ is non-negative and an $\bar{M}_{(-)}$ when $u$ is non-positive.

The equation above does not seem to be very enlightening. However, we can combine it with some pieces of information we have: First we know that $u$ takes values on the entire real line. Secondly, there is at least one real solution $u$, since $\bar{M}_*$ is *defined* by (5.26) to correspond to some real $u$. Besides, $\bar{M}_*$ takes all real values itself as one may verify.

So, one may ask the following: For a given $\bar{M}_*$, and a given $w$, how many *different* real solutions $u$ exist and what is their sign? Now, the l.h.s. of (5.35) is an even order polynomial. Then we know that there must be at least a second $u$ producing the same $M_*$. What we a priori do not know is whether the second $u$ is of the same sign or of the opposite.

There is one case where the second solution coincides with the first, and therefore has the same sign. This is when the root $u$ is also an extremum of the polynomial. It is easy to verify when this happens. We simply differentiate the polynomial w.r.t. $u$ and use (5.26) to find the following answer

$$(u + 3w)(u^2 - w^2 + 4w) = 0 \; . \qquad (5.36)$$

The points on the orientation curve $u+3w = 0$ are not included in the moduli space. Therefore we have that there is a single $u$ when $\bar{M}_*$ and $w$ are such that $u^2 - w^2 + 4w = 0$. We will see below that this is the *stability curve* i.e. the curve which separates the radially stable from the unstable solutions on the moduli space as we will see below (see also figure 5.4).

A related fact is given in the following

**Remark 31.** *For fixed $\alpha$ and $\Lambda$ we can think of the masses as functions of the radius of the shell $a_0$: $M_{(\pm)} = M_{(\pm)}(a_0)$. The function $M_{(+)}(a_0)$ has a global minimum and the function $M_{(-)}(a_0)$ has a global maximum for radii $a_0$ given by $u^2 - w^2 + 4w = 0$.*

Thus, there is simpler question one may ask: Given *pair* of masses $M_{(+)}$ and $M_{(-)}$, when can the matching happen at more than one shell radii $a_0$?

The answer is that this can never happen:

**Proposition 32.** *For any $w$, any $u$ such that $(w,u)$ belongs to the moduli space gives a pair of mass parameters $\bar{M}_{(+)}$ and $\bar{M}_{(-)}$. Then, this is the unique $u$ that gives these mass parameters.*

Let us prove this proposition. For a given $w$, let $u_0$ be such that the corresponding point $(u_0, w)$ belongs to the moduli space. We have

$$\bar{M}_{(+)} = 36w^2 \frac{(u_0+w)^2-4w}{(u_0^2+3(w^2-4w))^2} \quad , \quad \bar{M}_{(-)} = 36w^2 \frac{(u_0-w)^2-4w}{(u_0^2+3(w^2-4w))^2}. \tag{5.37}$$

Of course $u_0 \geq 0$.

There are two special cases to deal with before proceeding. First, consider $\bar{M}_{(+)} = \bar{M}_{(-)}$. We know that this is possible if and only if $u_0 = 0$. So for non-unique solutions we may restrict ourselves to $u_0 > 0$. The second case is when $\bar{M}_{(+)} + \bar{M}_{(-)} = 0$. This happens in the moduli space along the circle: $u_0^2 + w^2 - 4w = 0$. Clearly there is a unique positive $u_0$ solving this equation.

Therefore it is adequate to consider $u_0 > 0$ and masses such that $\bar{M}_{(+)} \pm \bar{M}_{(-)} \neq 0$. The proof is by contradiction. Let us suppose that $u_0$ is not unique in the sense that there exists some $u_1 > 0$ in the moduli space such that $u_1 \neq u_0$ and which gives the same masses

$$\bar{M}_{(+)} = 36w^2 \frac{(u_1+w)^2-4w}{(u_1^2+3(w^2-4w))^2} \quad , \quad \bar{M}_{(-)} = 36w^2 \frac{(u_1-w)^2-4w}{(u_1^2+3(w^2-4w))^2} . \tag{5.38}$$

With a little rearranging subtracting the respective equations we have

$$(u_1-u_0)\Big\{(u_1+u_0)\big(u_1^2+u_0^2+6(w^2-4w)\big)\bar{M}_{(+)} - 36w^2\,(u_1+u_0+2w)\Big\} = 0 \ ,$$
$$(u_1-u_0)\Big\{(u_1+u_0)\big(u_1^2+u_0^2+6(w^2-4w)\big)\bar{M}_{(-)} - 36w^2\,(u_1+u_0-2w)\Big\} = 0 \ .$$

$u_1 \neq u_0$ so the quantities in the big brackets vanish. Adding and subtracting them we obtain the equations

$$u_1^2+u_0^2+6(w^2-4w) = \frac{72w^2}{\bar{M}_{(+)}+\bar{M}_{(-)}} \quad , \quad u_1+u_0 = 2w\,\frac{\bar{M}_{(+)}+\bar{M}_{(-)}}{\bar{M}_{(+)}-\bar{M}_{(+)}} \ . \tag{5.39}$$

Via (5.37) these equations express $u_1$ in terms of $u_0$ and $w$. The second of these tells us that

$$u_1 = -\frac{3w^2}{u_0} < 0 \ . \tag{5.40}$$

So we conclude that $u_1$ is negative, contradicting the assumption. This completes our proof.

Therefore, remembering that $u$ is single valued in terms of the shell radius $a_0$, the junction conditions define a single-valued function $a_0 = a_0(M_{(-)}, M_{(+)})$, in fact one-to-one on the space of the allowed values of $M_{(\pm)}$. As we know from section 3.3 $a_0$ is a symmetric function of $M_L$ and $M_R$ and it is given by $a_0 = a_0(M_{(-)}, M_{(+)})$, via the correspondence implied in (4.41). Thus given the bulk metrics, the $a$ =constant vacuum shell is *unique*. So we see that a weakened version of uniqueness of solutions does survive. Note that for shells with standard orientation there are exactly two inequivalent configurations corresponding to the same shell radius, depending on whether $M_{(+)}$ is the mass of the inner or the outer region.

## 2.3 The spectrum of curves

We notice that, throughout the computations, the quantity

$$W \equiv w^2 - 4w = w(w-4) \ , \tag{5.41}$$

appears often. Now we comment on how it turns out to be convenient to extract information on the moduli space. First, notice that $W$ clearly vanishes at $w = 0$ and $w = 4$. We also encounter the curves

$$\begin{aligned} u^2 &= \pm W, \\ u^2 &= \pm 3W, \\ u &= \pm w, \\ u &= \pm 3w, \\ u &= 0 \ ; \end{aligned} \tag{5.42}$$

which in detail they correspond to

$$\begin{aligned} &u^2 = 3(w^2-4w): \quad \bar{M}_{(+)} + \bar{M}_{(-)} = 2\bar{M} \text{ curve}, \\ &u^2 = -3(w^2-4w): \quad \text{causality curve}, \\ &u^2 = (w^2-4w): \quad \text{stability curve}, \\ &u^2 = -(w^2-4w): \quad \bar{M}_{(+)} + \bar{M}_{(-)} = 0 \text{ curve}, \\ &u = \pm 3w: \quad \text{orientation curve}, \\ &u = \pm w: \quad \text{branch curve}, \\ &u = 0: \quad \text{boundary curve (where } \bar{M}_{(\pm)} = \bar{M}) \ . \end{aligned} \tag{5.43}$$

We also found the curve where $\bar{M}_{(\pm)} = 0$ to be

$$(u-w)^2 - 4w = 0 \ , \quad \text{i.e.} \quad u = u^{(\pm)} = \pm 2\sqrt{w} + w : \quad \text{zero minus-mass curve} \ ,$$

$$(u+w)^2 - 4w = 0 \ , \quad \text{i.e.} \quad u = -u^{(-)} = 2\sqrt{w} - w : \quad \text{zero plus-mass curve} \ ,$$

respectively. And notice that in terms of $W$ this simply reads

$$u^{(+)}u^{(-)} = W \ . \tag{5.44}$$

The first four curves in our list, which involve $W$, are conic sections with symmetry axes the lines $w = 2$ and $u = 0$. The orientation and branch curves on the other hand have symmetry axes the lines $w = 0$ and $u = 0$. The conic sections and especially the causality curve which is an ellipse, $u^2 + 3(w-2)^2 = 12$, break the symmetry between positive and negative values of $w$. The image of the causality curve around $w = 0$ would be centered at $w = -2$ i.e. $x = -3$.

The above analysis manifestly shows that the quantity $W \equiv w^2 - 4w$ captures much important information about the moduli space.

Actually, the parameterization of the space of solutions in terms of variables $(u, w)$ had shown to present advantages in order to classify the whole set of solutions. To emphasize this, and for completeness, let us also express the regions of radial stability over the moduli space in terms of these variables. Such regions are known to be characterized by the second derivative of the effective potential, which in terms of $u$ and $w$ is seen to be

$$V''(a_0) = -\frac{1}{a_0^2}\,\frac{w\left(u^2 - w^2 + 4w\right)}{(u^2 - w^2)\big(u^2 + 3(w^2 - 4w)\big)} \ . \tag{5.45}$$

The regions where $V''(a_0) > 0$ are shown in figure 5.4. $V''(a_0) = 0$ along the curve $u^2 - w^2 + 4w = 0$ which we have already called the stability curve, for reasons that become clear now. According to remark 31 this is where the mass $\bar{M}_{(\pm)}(a_0)$ have extrema.

# 3 Nontrivial features of $C^0$ metrics

## 3.1 Topology

In Einstein gravity in four dimensions there is a variety of smooth, everywhere non-singular vacuum configurations in general characterized by some nontrivial topological property, e.g. Euler number. Any topological property they may have is an intrinsic feature of the smooth solution.

In five dimensions and in Einstein-Gauss-Bonnet gravity similar configurations may exist as well. The equations of motions though are such that one can manufacture, by cut and paste along the world-volume of vacuum shells, similar kind of solutions with the difference that they are not smooth, i.e. not $C^1$. In this case there is no intrinsic property in the vacuum solution, we are simply building objects which are much simpler locally. For that reason one may call these objects non-topological, though they certainly have non-trivial topological features. An analogy for this would be the difference existing between an object with an exactly given smooth metric which has the topology, e.g. of the sphere, and tetrahedra built out of flat pieces.

This digression leads us to recognize a great difference with respect to four dimensions. Unlike in four dimensions, in five-dimensional Lovelock theory, spacetimes which are defined by some simple property locally, for example being vacuum and spherically symmetric, are by no means well defined globally, if smoothness is given up. For each such metric, which may itself have non-trivial topological features, one can construct infinitely many other spacetimes by cut and paste which locally are given by the same simple property. That is, the theory allows for many different topologies where one would expect it to allow only for different coordinates.

A general analysis of the objects obtained by geometric surgery along vacuum shells is an interesting problem and contains much of the actual physics of five-dimensional Lovelock gravity (that is, Einstein plus the Gauss-Bonnet term). In this work we mainly focus on the direct implications of their existence illustrated by appropriate examples. A systematic analysis is left for future work. Below we analyze how a constant curvature vacuum is modified by wormholes (and related configurations). It turns out that, the smaller such constructions with wormholes are with respect to the scale set by $\alpha$, the more complicated the topology can be.

## 3.2 Holes in the vacuum

An interesting special case of a wormhole is when on one side we have pure vacuum, as mentioned already in section 5. Starting from a constant curvature background, by introducing the vacuum wormhole we cut a hole in the constant curvature manifold, replacing it with an "outgoing" spacetime region of mass parameter $M$. Of course the topology of the vacuum is not the same anymore; there are now non-contractible 3-spheres. Nevertheless, it turns out that these configurations are everywhere non-singular in the following sense: *the only singularities that exist in spacetime are integrable*[4].

Besides, there is a second kind of feasible construction: A natural possibility for this construction is that of being a standard shell instead of a wormhole. That is, we may consider to match two metrics with the same orientation, with one having mass parameter equal to zero. In this case the topology is the same as before, up to the inhomogeneities introduced by the open balls which belong to a different spacetime. These configurations contain singularities since there is a singularity at the origin of each ball. This is the case where the inner spacetime is the massive one. A third conceivable type of configurations is when the inner spacetime is the one with zero mass. (This is actually not of "holes in the vacuum" kind exactly but it is obviously related).

The analysis gets simplified and clarified if we express everything in terms of the constant curvature. When the mass parameter is zero the metric is defined as

$$f(r) = 1 - Kr^2 \ . \tag{5.46}$$

This means that for the metric of branch $\xi$ we have:

$$4\alpha K = -(1 + \xi\sqrt{w}) \ . \tag{5.47}$$

These configurations exist for $w > 0$.

We consult Remark 28 and also Proposition 27. The relevant points on the moduli space are on the curve $u = |w - 2\xi\sqrt{w}| > 0$. According to Proposition 18 the points on this curve such that $u > 3w$ correspond to standard shell configurations. In detail, standard shells are the configurations corresponding to: $u = 2\sqrt{w} - w$ for $w \in (0, 1/4)$, and $u = 2\sqrt{w} + w$ for $w \in (0, 1)$.

$u = |w - 2\xi\sqrt{w}|$ is a continuous curve. The points with $w = 0$ and $w = 4$ do not belong in the moduli space. The same for the points with $w = 1/4$ and $w = 1$. So in all, from (5.47) we have that $4\alpha K \neq -3, -3/2, -1, 0$.

Now from Remark 28 and Proposition 27 one finds that the mass in all cases is

$$M = 9\alpha \, \frac{(4\alpha K + 1)^3}{(4\alpha K)^2 (4\alpha K + 3)} \ . \tag{5.48}$$

Also

$$\Lambda = 6K + 12\alpha K^3.$$

[4]The curvature and Lovelock tensor are singular at the shell but only in the sense of delta functions. Local integrals of these quantities are finite and the physical laws defined by the field equations do not break down there. In this sense the solutions are not singular.

We have that this construction is possible when $w > 0$. Therefore, from Remark 24, we have that the sign of $\alpha$ depends solely on the causal character of the vacuum shell. Namely, it is $\alpha > 0$ when the shell is timelike, and $\alpha < 0$ when the shell is spacelike. We have the following

**Remark 33.** *All standard orientation shell configurations with zero mass in one of the bulk regions are spacelike.*

**Remark 34.** *The variable $4\alpha K$ takes values on the entire real line with the exception of the points $-3, -3/2, -1, 0$. With these exceptions in mind we have:*

*Spacelike shells i.e. $\alpha < 0$: $4\alpha K \in (-3, 0)$. In the interval $(-3/2, 0)$ exist all the standard shell configurations.*

*Timelike shells i.e. $\alpha > 0$: $4\alpha K \in (-\infty, -3) \cup (0, \infty)$.*

The mass $M$ has poles at the boundary of the spacelike shell region. One may note that thought of as a function of $\alpha$ both poles are of first order.

From the formula $u = |w - 2\xi\sqrt{w}| > 0$ we find for the radius of the shell in all cases is

$$a_0^2 = K^{-1}\left(1 + \frac{4\alpha K}{3}\right)^{-1} . \tag{5.49}$$

The vacuum of constant curvature $K$ is a locally homogeneous spacetime and in particular is locally spatially homogeneous. Having placed one vacuum shell around some arbitrarily chosen origin, we have seen that outside of the shell the homogeneity is everywhere maintained. As long as it does not cross the first, we may place a second vacuum shell and in fact an arbitrary number of them modifying the manifold in a way depicted in Fig. 5.5.

Let $K > 0$ and $\alpha > 0$. It is useful to rewrite this as

$$\alpha K = \frac{3}{8}\left\{\sqrt{\frac{16}{3}\frac{\alpha}{a_0^2} + 1} - 1\right\} . \tag{5.50}$$

It is clear from both the last formula that in units of $\alpha$ the radius of the shell is an increasing function of the radius of universe $1/\sqrt{K}$. When the shell is microscopic, i.e. small in units of $\alpha$, we have that $Ka_0^2 \ll 1$. When the shell is macroscopic we have that $Ka_0^2 \simeq 1$. A microscopic universe could fit roughly

$$(Ka_0^2)^{-2} = \frac{1}{4}\left.\sqrt{\frac{16}{3}\frac{\alpha}{a_0^2} + 1} + 1\right)^2 \tag{5.51}$$

vacuum shells of radius $a_0$. So the more microscopic the universe the more complicated its topology can be.

## 3.3 Black hole spectrum and degeneracy

Reversing in a sense our viewpoint from the previous section, we may think of the matching of a massive metric with one of constant curvature along a vacuum shell, as a way to eliminate the singularity at the origin, or better to replace it with an integrable singularity along the vacuum shell.

Consider for example configurations along the curve $u = 2\sqrt{w} + w$ and $\alpha > 0$ i.e. $w > 1$ (and the vacuum shell is static). Then $K > 0$, that is $M > 0$ and $\Lambda > 0$, and the massive branch is an exotic branch ($\xi = +1$). This metric alone has a naked singularity at the origin. By constructing the vacuum wormhole we have managed to replace a region around the origin with a region of a de Sitter spacetime which contains the horizon. That is, the spacetime which asymptotically looks like an exotic branch, massive, Boulware-Deser spacetime is actually everywhere non-singular and has horizons. We might reasonably expect that thermal effects of the horizons will be felt in this would-be singular spacetime.

The mass parameter $M$ in the massive region is determined by the curvature $K$ of the de Sitter region. So then, the de Sitter space mimics a particle, or some fairly localized mass, as viewed from sufficiently far away. The entropy $S$ related to the existence of the de Sitter horizon depends on $K$ and therefore on $M$. We expect $\partial S/\partial K > 0$. We know that $\partial M/\partial K < 0$. Therefore that entropy will decrease with $M$. This is not surprising since a positive mass in the exotic branch behaves effectively like a negative gravitational mass.

The previous example shows that the spectrum of black holes in Einstein gravity modified by the Gauss-Bonnet term is not the same when $C^0$ metrics are allowed, compared to the smooth Boulware-Deser metrics. A space which by an asymptotic observer who thinks in terms of smooth metrics would not be recognized as a black hole might actually be one. Conversely, a spacetime which asymptotically would be a recognized as a Boulware-Deser black hole, could actually be a spacetime with a naked singularity, or a black hole different to the one expected.

Consider, for example, the case $\Lambda = 0$ and $\alpha < 0$. From the analysis in appendix 1 we see that this spacetime is a black hole for $M > |\alpha|$ (and $\xi = -1$). The horizon hides an inner branch singularity. Our analysis in section 2 shows that we can cut this spacetime and match it with wormhole orientation along a spacelike shell, i.e. within the horizon, with another spacetime of exotic branch metric. In that spacetime $r$ is everywhere a timelike variable. Thus although outside the horizon spacetime looks like a specific Boulware-Deser black hole spacetime it can actually be a different one. The two different states have the same energy as measured at spatial infinity and horizons with the same properties: as black holes they must be degenerate. Whether the usual entropy calculations take into account the effects of this degeneracy in the number of states is not clear to us. It is amusing to think that the modifications to the usual Bekenstein-Hawking formula in the presence of the Gauss-Bonnet term, see e.g. [29], are essentially due to such degeneracies.

### 3.4 Other types of shells

The analysis has focused on the spherically symmetric case ($k = 1$). This can readily be extended to the case of $k = -1$ (where the bulk metrics are taken to be topological black holes, with horizons some compactified hyperbolic space, or the corresponding naked singularity spacetime). Similar features are expected to occur (wormholes and shells of standard orientation exist, typically involving the exotic plus branch.) Also the case of $k = 0$ for toroidal black holes or naked singularity spacetimes, can be investigated.

We have seen that spacelike shells exist, representing a sudden transition from one solution to another. These present problems in terms of the predictability of the field equations. It would be useful to know whether the shells are generic or if they only occur for a certain range of the coupling constants and mass parameters.

The Euclidean version of the $C^0$ wormholes may be important for estimating the transition rate between the (unstable) plus branch and the (stable) minus branch solutions.

These are left for future work.

### 3.5 On uniqueness and staticity of solutions

In this work we construct and analyze solutions of Einstein-Gauss-Bonnet gravity whose metric is class $C^0$, piecewise analytic in the coordinates. The solutions are made by joining together two spherically symmetric pieces.

Since the shell itself admits $SO(4)$ isometry group, the resulting global spacetime is spherically symmetric. To put things into this context and discuss the special implications of low differentiability we start by reviewing the existing relevant theorems in Einstein and Lovelock gravity.

We start with a uniqueness and staticity theorem, applying to Lovelock gravity in general, which imposes the stronger conditions on differentiability.

**Theorem 35** (Ref. [168]). *For generic values of the couplings (including the cosmological constant), class $C^2$ solutions of the Lovelock gravity field equations with spherical, planar or hyperbolic symmetry are isometric to the corresponding static solutions.*

*In particular, in Einstein-Gauss-Bonnet gravity in five dimensions $C^2$ solutions with spherical symmetry are isometric to the Boulware-Deser solutions when $\Lambda \neq -3/4\alpha$.*

When we let the metric become merely continuous at hypersurfaces, we have seen already that one can construct many different time-independent solutions of the vacuum field equations: for example, when $\Lambda = 0$ with $\alpha > 0$, one can construct multiple concentric vacuum discontinuities separating Boulware-Deser solutions. So uniqueness of black hole solutions does not hold for $C^0$ metrics in Lovelock gravity. In fact neither does staticity. We return to discuss this below, after we revisit the corresponding theorems in Einstein gravity.

**Theorem 36** (Ref. [143][20]). *A differentiability class $C^0$ and spherically symmetric vacuum solution of Einstein gravity is: i) static, ii) equivalent to the Schwarzschild solution.*

That a spherically symmetric vacuum solution is static can be shown by finding a timelike Killing vector, which also happens to be hypersurface orthogonal, even when the solution is given in forms that don't look very much like the usual Schwarzschild metric and which assume lower differentiability [144], see [20].

**Theorem 37** (Ref. [143]). *A $C^0$ solution of the Einstein gravity field equations is well defined as the limit of a sequence of (at least) $C^2$ solutions. The metric is assumed to become $C^0$ only at smooth hypersurfaces.*

Fields of low differentiability, e.g. with a discontinuous first derivative, can be understood as solutions of field equations in the weak sense, as limits of sequences of smoother fields. The fact that this limit is well defined makes the junction conditions of Israel well defined (the above work appeared earlier than Israel's famous work). Now based on the junction conditions one may conclude: any hypersurfaces where the metric is not smooth must be a null hypersurfaces (we may call them shock waves). Then one may show that there are no spherically symmetric shock waves in Einstein gravity, see e.g. [20].

The result regarding limits of smooth metrics holds in Einstein-Gauss-Bonnet and in fact in Lovelock gravity in general (see the appendix of Ref. [88]).

**Theorem 38.** *A $C^0$ solution of Lovelock gravity field equations is well defined as the limit of a sequence of (at least) $C^2$ solutions. The metric is assumed to become $C^0$ only at smooth hypersurfaces and their intersections.*

So considerations related to uniqueness or non-uniqueness similar to the above are meaningful in Lovelock gravity as well. In this paper we have demonstrated:

**Theorem 39.** *There exist spherically symmetric $C^0$ solutions of Einstein-Gauss-Bonnet gravity in five dimensions which are not given by the Boulware-Deser metric, but rather they are piecewise of the Boulware-Deser form. There exist solutions which are not static.*

In section 2 we found that for any value of the couplings $\alpha$ and $\Lambda$ such that[5] $\Lambda > -3/4\alpha$, there exist static (time-independent) vacuum shells: spherically symmetric $C^0$ vacuum metrics are not unique for a wide range of couplings $\alpha$ and $\Lambda$ in Einstein-Gauss-Bonnet gravity in five dimensions. One can in fact construct arbitrarily complicated spherically symmetric configurations by having an infinity of concentric discontinuities. The exotic branch ($\xi = +1$) is typically involved. Though the radius of a static vacuum shell is uniquely fixed by the metrics in the bulk, $C^0$ static metrics are to a high degree non-unique as one does not a priori know how many vacuum shells there may be in spacetime.

[5] According to that section, $y(w-1)w = yx(x+1) > 0$ for timelike i.e. static shells and $yx(x+1) < 0$ for spacelike i.e. instantaneous shells. Via the simple relations of $x$ and $y$ to the couplings these read for non-zero $\Lambda$: $3/4\alpha + \Lambda > 0$ and $3/4\alpha + \Lambda < 0$ respectively. As we saw in the end of 2.1 they actually hold for $\Lambda = 0$ as well.

Now recall section 1.1. The time-dependent solutions, i.e. non-static ones, exist *always*: For any non-zero value of $\alpha$ and any value of $\Lambda$ there exists[6] a time-dependent vacuum shell solution $a(\tau)$. The shell radius function $a(\tau)$ and the orientation signs $\eta_L$ and $\eta_R$, completely define the world-volume of the shell intrinsically as well as its embedding in spacetime (section 3.2). That is, they define a $C^0$ metric in spacetime. Therefore a non-static $C^0$ metric which respects everywhere spherical symmetry can always be constructed in Einstein-Gauss-Bonnet gravity with cosmological constant (which can be also zero).

Looking back at the propositions and remarks of the previous sections one may come up with a conjecture.

**Conjecture 40.** *Consider a subspace of the space of $C^0$ solutions of Einstein-Gauss-Bonnet gravity such that, i) they are piecewise smooth, such that all smooth regions have a well-defined Einstein gravity limit ($\alpha \to 0$), ii) they do not contain naked singularities. Then, vacuum wormhole or standard shell solutions are not possible and uniqueness and staticity theorems essentially hold.*

Theorem 39 shows that uniqueness does not apply to $C^0$ metrics. How is this to be interpreted? One could simply reject non-smooth metrics as unphysical. However, according to Theorem 38 these $C^0$ solutions are well defined as the limit of a family of smooth geometries. As such, they approximate arbitrarily closely to some smooth solution of the theory. Now suppose $g^{(n)}_{\mu\nu}$ is a family of smooth metrics which converge to a spherically symmetric vacuum shell solution as $n \to \infty$. For finite $n$, $g^{(n)}_{\mu\nu}$ can not be a spherically symmetric vacuum solution, because the uniqueness theorem holds for smooth metrics. So it must either deviate slightly from spherical symmetry or have some small amount of matter as source. Assuming that suitable $g^{(n)}_{\mu\nu}$ can be constructed which obey the energy conditions, our results can be taken as evidence for the generic existence of such exotic features as smooth wormholes in this theory.

[6]In fact, it exists for a wide range of the bulk metric masses $M_L$ and $M_R$, possibly for all values of the masses for which the metrics are real. What is more important, for given values of the couplings $\alpha$ and $\Lambda$, for any given Boulware-Deser metric one can construct a time-dependent vacuum shell for some other Boulware-Deser metric on the other side.

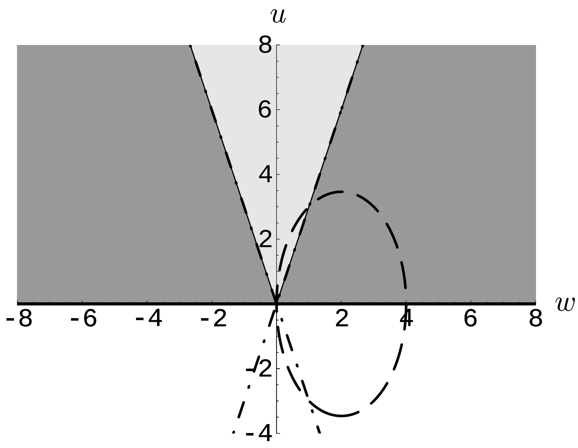



Figure 5.1: The space of *constant radius* solutions, which we have called the moduli space, is depicted here. The dimensionless variables $w$ and $u$ are defined at the beginning of section 2.1. The ellipse divides solutions into spacelike (inside) and timelike (outside); The diagonal lines divide solutions into standard orientation (light grey) which have well-defined inner and outer region of the shell, and wormhole orientation (dark grey), where both regions can be thought of as exterior or interior depending on whether a non-compact or compact region is maintained. Solutions exist for $u \geq 0$.
The line $w = 0$, $u > 0$ for finite $\alpha$ does not actually belong to the moduli space as being trivial: the junction condition require the metric across the shell must be continuous in this case. In terms of the couplings $\alpha$ and $\Lambda$, $w$ is given simply by $w \equiv 1 + \frac{4\alpha\Lambda}{3}$. The combination of the couplings $w = 0$ corresponds to the case where Einstein-Gauss-Bonnet gravity can be written as a Chern-Simons theory with (A)dS gauge group. It for this combination that the smooth $C^2$ metrics fail to be unique [43, 168]. Note that the pure Gauss-Bonnet case, which formally corresponds to $w = 0$, $\Lambda = 0$ in the limit that $\alpha \to \infty$ but $M\alpha$ is finite, does have nontrivial solutions, which were considered separately in Ref. [88].

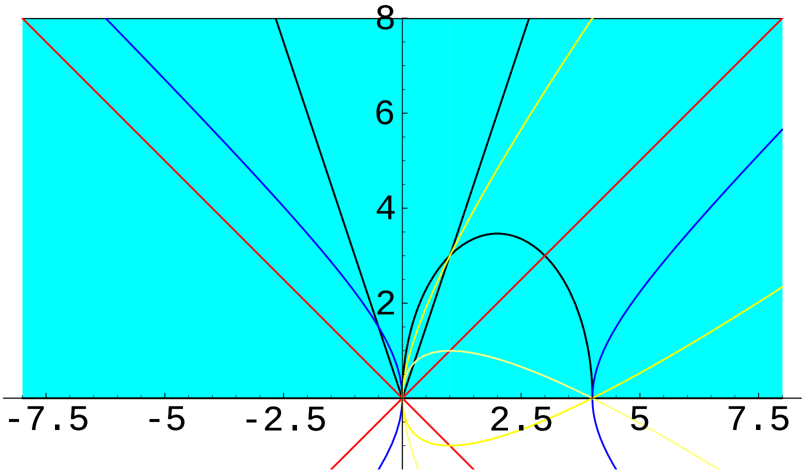


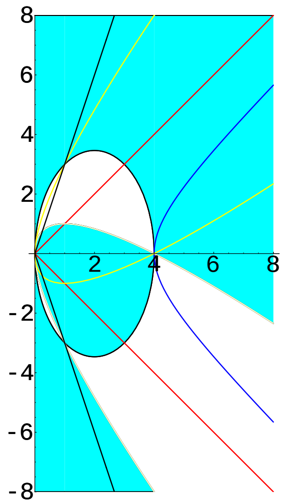

Figure 5.2: The moduli space showing the various curves listed in section 2.3. The basic division according to the time- or space-likeness of the world-volume of the shell and the orientation of the matching were described in the figure 5.1 and are shown in black lines here: the points along them they do not belong to the moduli space.

The diagonal (red) lines divide the space according to the branches of the bulk metrics which we classify by the pair of signs $(\xi_{(-)}, \xi_{(+)})$ explained in section 4.2 and 2.1: in the region on the left the branch signs are $(+,+)$ i.e. both the bulk metrics on each side of the shell belong to the "exotic" Boulware-Deser branch which does not have a well-defined limit $\alpha \to 0$; the region in between the diagonal lines is $(-,+)$; in the region on the left the branches are $(-,-)$ i.e. both metrics belong to the branch with a well-defined $\alpha \to 0$ limit (however the vacuum juncture requires that these solutions only exist for $w \equiv 1 + \frac{4\alpha\Lambda}{3} < 0$, i.e. they have no asymptotics: certain curvature singularities appear at finite radius [78]).

The hyperbola that exists on the outside of and touches the elliptic region of spacelike solutions only at the border of the ellipse at the points $w = 0$ and $w = 4$ (blue line), is what we have called the stability curve: crossing this curve the second derivative $V''(a_0)$ of the potential (4.22) or (5.1) evaluated at the constant solutions $a = a_0$ changes sign, which is a measure of (in)stability under perturbations. The constants solutions for which $V''(a_0) > 0$ are depicted in figure 5.4.

The remaining two lines (yellow lines) are symmetric around the horizontal line $u = 0$. Each curve corresponds to solutions such that one of the mass parameters vanishes i.e. one of the bulk regions is pure vacuum. Note that they exist only for $w = 1 + \frac{4\alpha\Lambda}{3} > 0$. These configurations are discussed in sections 3.2 and 3.3 as an interesting example of certain non-trivial features $C^0$ metrics acquire when Einstein gravity is supplemented by the Gauss-Bonnet term is five dimensions.

Figure 5.3: In the upper half plain, the shaded region is where $M_{(+)} > 0$. The inequality $M_{(-)} > 0$ has been plotted in the lower half plain, making use of Lemma 22 (it should be remembered that in the lower half plane we have actually plotted $M_{(-)}(-u)$)). We note that $M_{(+)}$ and $M_{(-)}$ are both negative for all $w < 0$, which is the left half of the diagram.



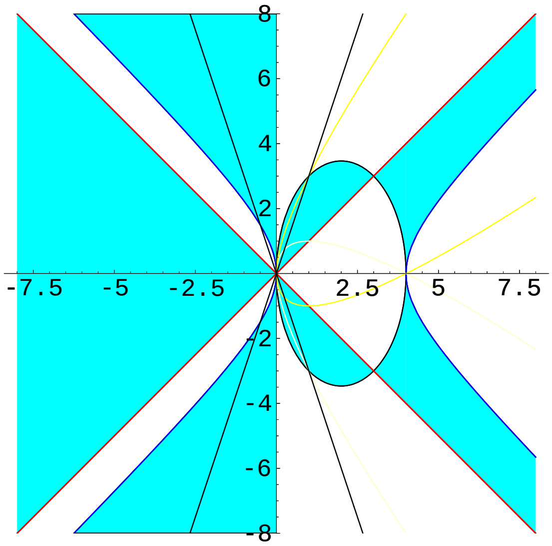



Figure 5.4: The shaded regions are where $V''(a_0) > 0$. For the timelike shells (outside of the ellipse), the unshaded regions correspond to solutions unstable with respect to radial perturbations.

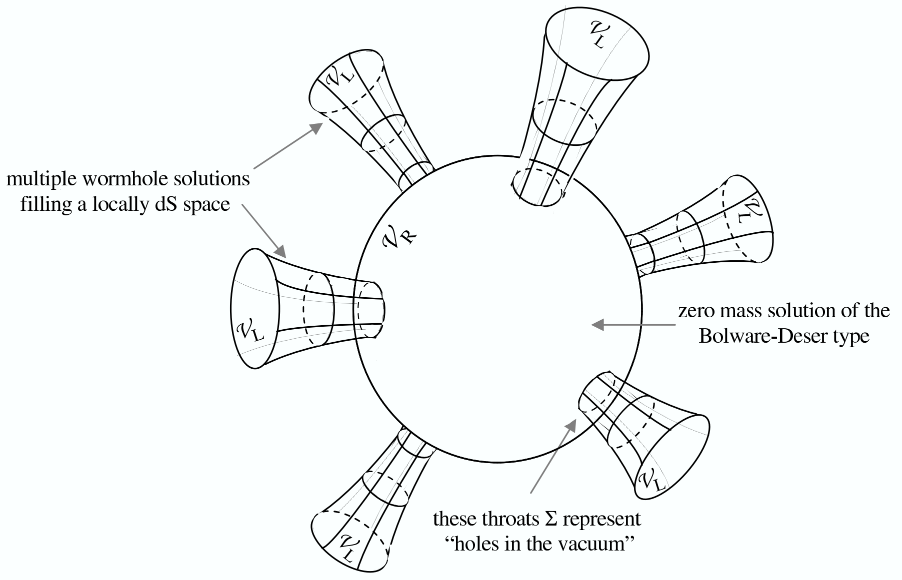



Figure 5.5: $K > 0$. When $\alpha > 0$ the solution is a multi-soliton, with multiple asymptotic massive regions joined to a de Sitter space. A spatial slice of a multi-soliton is sketched above. When $\alpha < 0$ and $3/8|\alpha| < K < 3/4|\alpha|$ the solutions are multi-instantons. The radii of the instantons have a lower bound: $a_0^2 > 16|\alpha|/3$.

# Chapter 6

# Naked singularities

## 1 Singularity and Quantum Mechanics

As we have seen in the previous chapters, there are many features of Lovelock solutions that are not present in GR. Eternal black holes and wormholes are remarkable examples. Another example is the existence of positive mass solutions with naked singularities[1]. In fact, naked singularities appear in all the catalog of solutions, for both spherically symmetric and extended objects, for both solutions with a suitable GR limit and solutions without it. But, what kind of naked singularities are these? For instance, we could ask whether these are stable under gravitational perturbations [83, 86]; or whether these turn out to be "bad" singularities when probed with wave functions [97].

Regarding the question about the stability, this issue was studied recently within the framework of the Kodama-Ishibashi formalism, and some evidence of instabilities was found [77]. On the other hand, here we will address the second question, the one about how these naked singularities look like when analyzed with quantum probes. To do this we will employ the method developed by Horowitz and Marolf in Ref. [97], based on the pioneer work of Wald [159]. The basic idea es the following: Unlike what happens in the classical regime, where a singular space is defined by the concept of geodesic incompleteness, in the quantum mechanical regime the singular character of the space-time is defined in terms of the ambiguity in the definition of the Hamiltonian evolution of wave functions on it [97]. More specifically, the

[1] For a discussion on the formation of naked singularities, see[117, 139].

singular nature of a given space is determined in terms of the ambiguity when trying to find a self-adjoint extension of the Hamiltonian operator to the whole space. When such self-adjoint extension exists and is unique, then it is said that the space is quantum mechanically regular, in spite of the singularities it might present at classical level. Notice that this is not matter of deforming the space or somehow resolving it, but it is rather a reconsideration of what is the relevant physical dynamics on it. In fact, a space can be classically singular but still regular when it is analyzed with quantum probes.

# 2 Quantum Probes Method

Here, we will apply the concept of quantum probes to the singular solutions of Lovelock theory discussed above. But, first, let us review the method developed in [97, 100]. Consider the quantum dynamics of a scalar field $\varphi$ on the spherically symmetric space (3.2), which is governed by the Klein-Gordon equation

$$\left(\nabla_\mu \nabla^\mu - m^2 - 2\xi R\right)\varphi = 0. \tag{6.1}$$

This equation can be written as follows

$$\partial_t^2 \varphi + \mathcal{H}^2 \varphi = 0, \quad \text{with} \quad \mathcal{H}^2 = -V_{(r)} \nabla^i \left(V_{(r)} \nabla_i \varphi\right) + V_{(r)}^2 m^2 \varphi + 2V_{(r)}^2 \xi R \varphi \tag{6.2}$$

where $\nabla^i$ is the covariant derivative on the spacelike hypersurfaces defined by constant $t$ foliations, and where the metric function $V^2(r)$ is given by (3.3). The piece $V_{(r)} \nabla^i \left(V_{(r)} \nabla_i \varphi\right)$ in (6.2) involves the Laplacian operator on the unitary 3-sphere, whose eigenvalues are known to be given by $-l(l+2)$ with positive integers $l = 0, 1, 2, 3, ...$

Now, equation (6.2) can be written in its Schrödinger-like form, schematically,

$$i\partial_t \varphi = \mathcal{H}\varphi,$$

and then the problem to deal with is to decide whether the Hamiltonian operator $\mathcal{H}$ admits a unique self-adjoint extension in spite of the fact the space is singular at the origin $r = 0$. As mentioned, in the quantum mechanical context the existence of singularity is associated to the non-existence of a unique self-adjoint extension of the Hamiltonian operator rather than to a geodesical completeness. Then, the problem of determining whether the space is regular is translated into the problem of verifying whether $\mathcal{H}^2$ admits a unique self-adjoint extension $\mathcal{H}_E^2$. If such extended operator exists, then the Hamiltonian evolution of the wave function in this space would be given by

$$\varphi(t) = \exp(-it\ \mathcal{H}_E)\ \varphi(0),$$

and it would be well-defined.

It turns out that a sufficient condition for $\mathcal{H}_E^2$ to exist and be unique is that at least one of the solutions of the differential equation

$$\partial_r^2 \phi_{(r)} + \partial_r \log\left(r^3 V_{(r)}^2\right) \partial_r \phi_{(r)} - V_{(r)}^{-2}\left(r^{-2} l(l+2) - m^2 + \xi R \pm i V_{(r)}^{-2}\right) \phi_{(r)} = 0 \tag{6.3}$$

fails to be of finite norm near the origin for any value of $l$ and for any of the two possible signs $\pm$ in (6.3); see [97] for details. In other words, for the space to be considered regular quantum mechanically it is necessary to see that at least one solution $\phi$ to (6.3) is non-normalizable around the origin. This criterion strongly depends on which norm $||\phi||$ is considered.

The well-posedness of an initial value problem requires not only the existence and uniticity of conditions, but also continuous dependence of solutions on initial data. Then, the norm $||\phi||$ to be considered should select a the function space that fulfills these requirements. A sensitive norm in this sense is the Sobolev norm [99].

To see how the method works in the case we are interested in, let us consider again the five-dimensional Boulware-Deser space (3.2)-(3.3). The branch $\sigma = +1$ of this space presents a naked singularity for all positive values of $M$, while the branch $\sigma = -1$ only presents naked singularities within the range $0 < M < \alpha$. Then, let us solve the wave equation for these spaces. To analyze the solutions of (6.3) near the singular point $r = 0$ it is convenient to write this equation as $\partial_r^2 \phi + r^{-1} p_{(r)} \partial_r \phi + r^{-2} q_{(r)} \phi = 0$, with $p(r)$ and $q(r)$ being two functions analytic at the origin. This is a Fuchsian equation and so it admits solutions with the form $\phi(r) = r^\eta f(r)$ for certain analytic function $f(r)$ and a complex number $\eta$ that is known to solve the indicial equation $\eta^2 + (p_{(r=0)} - 1)\eta + q_{(r=0)} = 0$. Then, replacing (3.3) in (6.3) we find $p_{(r=0)} = 3$, $q_{(r=0)} = -l(l+2)/(1+\sigma\sqrt{M/\alpha})$, and two independent solutions to (6.3) are then given by the two values of $\eta$ that solve $(\eta+1)^2 = 1 + l(l+2)/(1+\sigma\sqrt{M/\alpha})$. Therefore, we find that one of the solutions to (6.3) always diverges at least as rapidly as $|\phi|^2 \simeq r^{-2}$, and so it fails to be integrable with respect to the Sobolev norm.

Summarizing, there exists a unique self-adjoint extension $\mathcal{H}_E^2$, from what we conclude that five-dimensional Boulware-Deser metric turns out to be regular when tested by quantum probes. It is remarkable that the positive (but small) mass solutions of five-dimensional black holes are in a sense regular quantum mechanically, despite the naked curvature singularity they exhibit at the origin [80].

# 3 Concluding Remarks

Before concluding, we wish to make a remark about the consistency of studying naked singularities in this way. The reason why we find convenient to discuss this point is that the reader could be concerned about whether probing naked singularities in a theory with a finite higher curvature expansion makes sense or not. For instance, in string inspired models, as soon as one approaches the singularity, neglecting higher order corrections seems to be impossible since higher and higher order terms start to dominate as we go close enough to the singularity. However, let us argue here that, even though this is true, this is not necessarily an obstruction for testing singularities with quantum probes up to certain order in the higher curvature expansion. The argument is the following: Let us be reminded of what we do when we solve the Schrödinger equation for the Coulombian potential (e.g. for hydrogen atom in quantum mechanics). In fact, the analogy is quite direct since such problem also corresponds to solving a wave function equation in presence of a central potential whose classical counterpart breaks down at the origin. The key argument is that, even though the Coulombian potential diverges at the origin, we know that the quantum problem still makes sense, and we do solve the wave equation without complaining about the fact that other corrections to the potential (e.g. effective screening due to quantum effects, or short distance corrections to the Coulombian potential) could in principle appear at very short distances. Heuristically speaking, what one really has to do to make sure the whole procedure makes sense is comparing the typical size of the wave packet with the length scale where the terms that were neglected would dominate. For example, above we were dealing with the EGB action, and the terms $\mathcal{R}^3$ were certainly neglected, and so the analysis carried out could still make sense as long as the Compton length of the wave packet is small enough in comparison with the length scale imposed by the coupling constant $\alpha_n$ with $n > 2$, and provided the fact higher curvature terms act as a perturbation.

For some particular models where the couplings $\alpha_n$ are given in terms of the same fundamental scale (like the models inspired in string theory where the scale is given by $l_s^2 \sim \alpha'$) the story could be a little more subtle, and so the argument above would not be valid since neglecting higher order contributions near the singularity in that case would be impossible. However, it is likely the case that higher order terms would contribute by smoothing out the singularity even more, although not necessarily resolving it in a classical sense.

# Chapter 7

# Conclusions

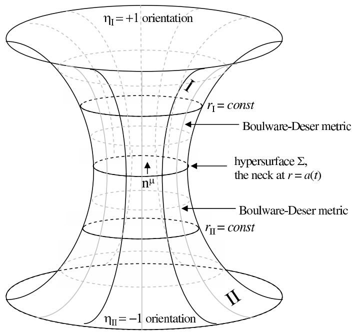
In this Thesis we studied black hole physics in the Lovelock theory of gravity. As we argued throughout the discussion, this theory is the natural extension of Einstein's General Relativity to higher dimensions. In fact, this is the most general theory of gravity yielding second order conserved equations of motion. Unlike Einstein theory, Lovelock theory is non-linear in the second derivatives of the metric, and this ultimately yields a richer family of solutions, exhibiting different brances and exotic causal structure.

In this Thesis we have explicitly constructed a large family of vacuum solutions which can be thought of as gravitational solitons of the theory. These correspond to localized solutions in vacuum for which a notion of finite energy can be accomplished.

Even though Lovelock theory of gravity is interesting in its own right, one finds the additional motivation that five-dimensional Lovelock Lagrangian emerges in low energy effective actions of heterotic string theory and in M-theory compactifications. Nevertheless, we were not concerned with this stringy origin herein. Instead, we focussed our attention on the black hole content of the theory and on the solitons we could construct starting from them.

First, we exhaustively review the black hole solutions in Lovelock theory. These correspond to exact solutions to a theory of gravity that contains higher-curvature corrections, and were first found by Boulware and Deser. Is remarkable that, in spite of the non-trivial structure of the theory, the spherically symmetric sector of theory is exactly solvable.

As mentioned above, the space of solutions present different branches. More precisely, if the Lagrangian contains $\mathcal{O}(R^n)$ terms, then $n$ different




Vacuum solution representing an Einstein-Rosen like geometry. The spacetime is given by two copies of the Boulware-Deser metric connected through the hypersurface Σ. The wormhole-type orientation corresponds to $\eta_{I}\eta_{II} = -1$.

Figure 7.1: Einstein-Rosen bridge geometry as a vacuum solution.

branches of static spherically symmetric solutions exist, and more than one of them may be considered *physical*. Apart from that, a Birkhoff like theorem was proven for this theory, which states that, at least locally, the spherically symmetric solution to Einstein-Gauss-Bonnet Lagrangian in vacuum in $D$ dimensions, with arbitrary $D$, and with coupling constant not corresponding to those of the Chern-Simons gravity in five-dimensions, is: ($a$) static, ($b$) unique, and ($c$) given by the two branches of the Boulware-Deser metric[1].

In this Thesis we have proven that the Birkhoff theorem doesn't hold globally, and this is the first counterexample found which does not require a fine tuning of the parameters to circunvent the uniqueness argument. The way we proved it was by explicitly constructing spherically symmetric solutions in vacuum which, still being of the Boulware-Deser form in patches, globally corresponds to the junction of two different solutions which obey junction conditions of the theory. This is one of the original result of this Thesis.

Unlike the Israel junction conditions of General Relativity, the boundary conditions in Lovelock theory admit the existence of vacuum thin shells. Solutions with non-trivial topology also arise, as wormholes in vacuum. In five dimensions, we studied the existence and stability of these solutions under perturbations that preserve the spherical symmetry. We explored the

[1]Here, we are rephrasing this Birkhoff theorem in such a way that the statement contains all the hypotesis that were missing in the original statement. That is, our version of the theorem incorporates hypotesis that were shown to be strictly necessary just recently.

space of solutions exhaustively, considering different orientations, different signs for the curvature of the base manifold, etc. This is the first analysis of this kind performed for Lovelock theory.

We also studied solutions with naked singularity. By using the method of quantum probes, we tested the singular nature of these spaces in the quantum mechanical context. The analogous discussion for the case of negative mass Schwarzschild black hole gave raise to intense debate in the context of General Relativity recently. Unlike the case of General Relativity, naked singularities exist for positive mass objects in Lovelock theory. For such solutions we have proven that, while singular in the classical context, the spaces can be thought of as regular solutions when tested with quantum probes. That is, spaces which are timelike geodesically incomplete turn out to be quantum mechanically regular in Lovelock theory. This observation is also an original result of this Thesis.

More recently, five-dimensional Lovelock theory was considered as a working example to study the effects of including higher-curvature terms in AdS/CFT correspondence. This turns out to be a very active line of research. Just recently, papers discussing the interplay between causality and higher-curvature terms in the context of AdS/CFT appeared. Also, holographic superconductors in five-dimensional Lovelock gravity were considered, showing that higher-curvature corrections affect the condensation phenomenon. In turn, the applications of Lovelock theory to holographic duality seems to be a very interesting line of research to follow our investigations.

# Chapter 8

# Appendix

## 1 Horizon Structure of the Boulware-Deser Metric

An interesting issue about the Boulware-Deser solutions is that it contains a square root, whose reality imposes constraints. From (4.9) we see that: when $w < 0$ there is a *maximum* radius; when $M/\alpha < 0$ there is a *minimum* radius in spacetime. At those finite radii there exists curvature singularities, known as branch singularities [78]. We call them outer and inner branch singularities, respectively to the cases above. These unusual spacetimes can also have horizons behind which the singularities are hidden.

We turn now to discuss the horizon structure of the Boulware-Deser spacetimes. The following does not intend to be an exhaustive analysis, it is rather a list of general formulas in our notation useful for our purposes. We will use the dimensionless parameters $w$ and $\bar{M}$. Recall the Boulware-Deser metric function $f(r)$ given in (4.9) and define $r_H$ by $f(r_H) = 0$. One finds that if $w \neq 1$

$$r^2_{H\pm} = 4\alpha \, \frac{1 \pm \sqrt{\bar{M}(w)}}{w-1} \; . \tag{1.1}$$

We have defined the useful quantity

$$\bar{M}(w) = w + (1-w)\bar{M} \; , \tag{1.2}$$

which looks an interpolation between $\bar{M}$ and 1.

From the definition of $r_{H+}$ we see that $r_{H+} > 0$ if:

$$\frac{3}{\Lambda} = \frac{4\alpha}{w-1} > 0 \ . \tag{1.3}$$

That is $\Lambda > 0$. Also $r_{H-} > 0$ one finds that it is equivalent to $M > \alpha$. Therefore we have:

**Remark 41.** *Elementary conditions for the existence of $r_{H+}$ is $\Lambda > 0$ and for the existence of $r_{H-}$ the condition $M > \alpha$.*

When $0 < |\alpha| < \infty$, $w = 1 \Leftrightarrow \Lambda = 0$. So the previous formula holds for non-zero $\Lambda$. When $\Lambda = 0$, the correct result can be obtained as the limit $w \to 1$ of the previous formula for $r_{H-}$. It reads

$$r_{H-}^2 = 2\alpha\,(\bar{M} - 1) \ . \tag{1.4}$$

We must substitute (1.1) back to $f(r_H) = 0$ to solve for the signs. We have:

$$-\xi = \text{sign} \ \ \frac{w \pm \sqrt{\bar{M}(w)}}{1 \pm \sqrt{\bar{M}(w)}}\Bigg) \ , \tag{1.5}$$

for $r_{H\pm}$ respectively. Again the case $w = 1$ i.e. $\Lambda = 0$ can be correctly obtained from the limit $w \to 1$ for $r_{H-}$. Explicitly it reads

$$-\xi = \text{sign}\left(\frac{\bar{M}+1}{\bar{M}-1}\right) \ . \tag{1.6}$$

We have used the sign function defined by $\text{sign}(x) = x/|x|$. When $x = 0$ it is ambiguous.

Before continuing note the following. One implicit inequality that should be respected for horizons to exist is

$$\bar{M}(w) \geq 0 \ . \tag{1.7}$$

This is related to the reality of the square root of the Boulware-Deser metric function (4.9). $\bar{M}$ and $w$ cannot be both negative. That is, if $w\,\bar{M} \geq 0$ then it must be $w + \bar{M} \geq 0$. This is precisely what is guarantied by (1.7).

From remark 41 we have

**Remark 42.** *$r_{H+} > 0$ is equivalent to* $\mathrm{sign}(\alpha) = \mathrm{sign}(w-1)$*. $r_{H-} > 0$ is equivalent to* $\mathrm{sign}(\alpha) = \mathrm{sign}(\bar{M}-1)$*.*

Note that, as we will solve the problem of existence for the real numbers $r_{H\pm}^2/4\alpha$ the positivity conditions above essentially restrict the sign of $\alpha$.

Now

$$r_{H+}^2 - r_{H-}^2 = \frac{4\alpha}{w-1} 2\sqrt{\bar{M}(w)} \ , \tag{1.8}$$

and (1.3) tell us that

**Remark 43.** *If $r_{H+}$ exists then $r_{H+} \geq r_{H-}$.*

A solution $r_H$ corresponds to a horizon if $r_H > 0$ and there exist $r_1 < r_H < r_2$ such that $f(r_1)f(r_2) < 0$.

Recall (1.5). We have some *Special cases*:

*i*). $w \pm \sqrt{\bar{M}(w)} = 0 \Leftrightarrow \bar{M} = -w$. (Note again that the correct result for $w = 1$ is obtained as the limit). Then one of the two solutions $r_{H\pm}$ coincides with the branch singularity $r_E$: $Y(r_E) = 0 = f(r_E)$. I.e. in this case the branch singularity is *null.* [This is possible for $\alpha < 0$ otherwise this solution doesn't exist.]

There is a single horizon solution given by

$$r_H^2 = 4\alpha \, \frac{w+1}{w-1} \ , \tag{1.9}$$

It is a horizon of the branch $\xi$ according to

$$-\xi = \mathrm{sign}(w\,(w+1)) \ . \tag{1.10}$$

[Of course the case $w = 0 = \bar{M}$ does not have two branches.] The case $w = -1$ i.e. $\bar{M} = 1$ does not have a horizon as $r_H$ vanishes (if $\xi = -\mathrm{sign}(\alpha)$), or $f$ which reads

$$f = 1 + \frac{r^2}{4\alpha} + \xi \mathrm{sign}(\alpha) \sqrt{\left(1 - \frac{r^2}{4\alpha}\right)\left(1 + \frac{r^2}{4\alpha}\right)} \ , \tag{1.11}$$

can vanish only at the branch singularity when $\alpha < 0$.

Finally one should bear in mind that $r_H = r_{H-}$ when $\bar{M} = -w > 0$ and $r_H = r_{H+}$ when $\bar{M} = -w < 0$.

*ii*). $1 - \sqrt{\bar{M}(w)} = 0 \Leftrightarrow \bar{M}(w) = \bar{M} = 1$. (Note again that the correct result for $w = 1$ is obtained as the limit). We just learned that when we also we have $w = -1$ there are no horizons. So we assume that $w \neq -1$. We observe that $r_{H-} = 0$. This actually happens if $\xi = -\mathrm{sign}(\alpha)$ otherwise this solution doesn't exist.

The single horizon solution is

$$r_{H+}^2 = 4\alpha \, \frac{2}{w-1} = \frac{6}{\Lambda} \; . \tag{1.12}$$

It is a horizon of the branch $\xi$ according to

$$-\xi = \mathrm{sign}(w+1) \; . \tag{1.13}$$

*iii*). $\bar{M}(w) = 0$. This is the saturated case where the two radii coincide: $r_{H\pm}^2 = 4\alpha/(w-1) = 3/\Lambda$. Condition (1.5) works well in this case: $\xi = -\mathrm{sign}(w)$. Also from $r_H^2 > 0$ we have $\mathrm{sign}(\alpha) = \mathrm{sign}(w-1)$.

In this case $r_H$ is not a horizon radius. It is the (single) zero of $f$ which has the same sign everywhere else. There are three non-trivial cases. $w < 0$. Then there is an outer branch singularity and $f(r) \geq 0$. $0 < w < 1$. Then there is an inner branch singularity and $f(r) \leq 0$. $w > 1$. Then $0 < r < \infty$ and $f(r) \leq 0$. $\Box$

Recall (1.5).

**Proposition 44.** *With the exception of cases covered in i) and ii) we have: The radius $r_{H+}$ is a horizon of the branch $\xi$ if*

$$-\xi = \mathrm{sign}\left( w + \sqrt{\bar{M}(w)} \right) \; ; \tag{1.14}$$

*the radius $r_{H-}$ is a horizon of the branch $\xi$ if*

$$-\xi = \mathrm{sign}((\bar{M}+w)(\bar{M}-1))\,\mathrm{sign}\left( w + \sqrt{\bar{M}(w)} \right) \; . \tag{1.15}$$

The type of the horizon, i.e. whether it is black hole, inner or cosmological horizon, can be determined by the sign of the first derivative of $f(r)$ (combined with Remark 43). We have

$$r_{H\pm} f'(r_{H\pm}) = \mp 2 \sqrt{\bar{M}(w)} \cdot \frac{1 \pm \sqrt{\bar{M}(w)}}{w \pm \sqrt{\bar{M}(w)}} \; . \tag{1.16}$$

Therefore for $\bar{M}(w) > 0$, when $r_{H-}$ or $r_{H+}$ does correspond to a horizon, the type is determined by

$$\text{sign}(f'(r_{H\pm})) = \pm\xi \; . \tag{1.17}$$

Remarks 42 and 43, Proposition 44, and formula (1.17) provide criteria for the existence and the type of horizons for each branch $\xi$ of the Boulware-Deser metric.

For the exotic branch ($\xi = +1$) a black hole horizon must be and $r_{H+}$. This is not possible by (1.14). Thus there no black holes in the exotic branch. For the good branch ($\xi = -1$) a black hole horizon must be an $r_{H-}$. From (41), this is possible only for $M > \alpha$.

# 2 The Junction conditions

For our purposes, a singular shell $\Sigma$ is a submanifold of codimension one at which the metric is continuous but the extrinsic curvature has a finite discontinuity. The field equations of Einstein-Gauss-Bonnet theory are given by (4.6). Integrating the field equations across $\Sigma$, one obtains the junction condition [53, 87, 88].

$$(\mathfrak{Q}^+)^a{}_b - (\mathfrak{Q}^-)^a{}_b = -\kappa^2 S^a_b \; . \tag{2.1}$$

Where the symmetric tensor $\mathfrak{Q}^a{}_b$ is given by[1]

$$\mathfrak{Q}^a{}_b = -\delta^{ac}_{bd} K^d_c + \alpha\, \delta^{acde}_{bfgh} \Big( - K^f_c R^{gh}{}_{de} + \frac{2}{3} K^f_c K^g_d K^h_e \Big) \tag{2.3}$$

[1]The notation of Ref. [88] has been used. However in that reference there was an unconventional sign convention used (in equation A3) for the definition of extrinsic curvature. Although none of the results of that paper were not affected by this, unfortunately the formulae B13-B17 for the Einstein-Gauss-Bonnet in the appendix were a mixture of inconsistent sign conventions. Here we correct this sign error by choosing the standard sign convention as in Refs. [101] and [53]. The developed expression is:

$$\Big[\varsigma(K^a_b - \delta^a_b K) + 2\alpha \left(3J^a_b - \delta^a_b J - 2\varsigma P^{ac}{}_{bd} K^d_c\right)\Big]^+_- = -\kappa^2 S^a_b \, , \tag{2.2}$$

where $\varsigma$ is $+1$ for a timelike shell and $-1$ for a spacelike shell, $P_{abcd} := R_{abcd} + 2R_{b[c}g_{d]a} - 2R_{a[c}g_{d]b} + Rg_{a[c}g_{d]b}$ is the trace-free part of the intrinsic curvature and $J_{ab} := (2KK_{ac}K^c_b + K_{cd}K^d K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2 K_{ab})/3$. In the case of a timelike shell ($\varsigma = +1$), this expression agrees with that given in Ref. [53, 87].

for the timelike case, and by

$$\mathfrak{Q}^a{}_b = \delta^{ac}_{bd} K^d_c + \alpha\, \delta^{acde}_{bfgh} \Big( K^f_c R^{gh}{}_{de} + \frac{2}{3} K^f_c K^g_d K^h_e \Big) \tag{2.4}$$

for the spacelike case. Here, lower case Roman letters from the beginning of the alphabet $a$, $b$ etc. represent four-dimensional tensor indices on the tangent space of the world-volume of the shell. The $R^{ab}{}_{cd}$ appearing in the junction condition is the four-dimensional intrinsic curvature. The antisymmetrized Kronecker delta is defined as $\delta^{a_1 \ldots a_p}_{b_1 \ldots b_p} \equiv p!\, \delta^{a_1}_{[b_1} \cdots \delta^{a_p}_{b_p]}$.

Now we calculate the intrinsic curvature of the world-volume of a spherical shell of radius $a(\tau)$ and the extrinsic curvature (which takes a diagonal form). There are two cases: For the timelike case the components are

$$R^{\tau\phi}{}_{\tau\phi} = \frac{\ddot{a}}{a}\,, \qquad R^{\phi\theta}{}_{\phi\theta} = R^{\theta\chi}{}_{\theta\chi} = R^{\chi\phi}{}_{\chi\phi} = \frac{(k + \dot{a}^2)}{a^2}\,,$$

$$K^\tau_\tau = \eta \frac{\ddot{a} + \frac{1}{2} f'}{\sqrt{\dot{a}^2 + f}}\,, \qquad K^\theta_\theta = K^\phi_\phi = K^\chi_\chi = \frac{\eta}{a} \sqrt{\dot{a}^2 + f}\,;$$

while for the spacelike case these are

$$R^{\tau\phi}{}_{\tau\phi} = -\frac{\ddot{a}}{a}\,, \qquad R^{\phi\theta}{}_{\phi\theta} = \frac{(k - \dot{a}^2)}{a^2}\,,$$

$$K^\tau_\tau = \eta \frac{\ddot{a} - \frac{1}{2} f'}{\sqrt{\dot{a}^2 - f}}\,, \qquad K^\theta_\theta = K^\phi_\phi = K^\chi_\chi = \frac{\eta}{a} \sqrt{\dot{a}^2 - f}\,.$$

In this paper we are interested in pure vacuum shells, i.e. when $S^a_b = 0$. It is clear that in this case one can pull out a factor of $\Delta K^d_c \equiv (K_+ - K_-)^d_c$, which is the jump in the extrinsic curvature across the shell.

$$\Delta K^d_c \,(\cdots) = S^a_b = 0\,. \tag{2.5}$$

In the case of interest in this paper, the extrinsic curvature is diagonal. Thus, one expects each component of the junction conditions to factorize conveniently.

Using the above formulae, we derive $\mathfrak{Q}^\tau_\tau$ given in (4.17). The angular components are, for the timelike case:

$$\mathfrak{Q}^\theta_\theta = -2!\; a^{-2} \left\{ \eta\, \frac{\frac{1}{2} f' \{a^2 + 4\alpha(k - f)\}}{\sqrt{\dot{a}^2 + f}} + \eta\; 2a\sqrt{\dot{a}^2 + f} + \eta\; 4\alpha\, \frac{\ddot{a}}{\sqrt{\dot{a}^2 + f}} \left( k + f + 2\dot{a}^2 + \frac{a^2}{4\alpha} \right) \right\} \tag{2.6}$$

# 3 The derivatives of the potential

As before, let us denote the derivative with respect to $a$ by a prime. In analysing dynamical shells and the stability of static shells it is useful to calculate the derivatives of $V(a)$ with respect to $a$, $V'$, $V''$ etc. First we recall the definition of $Y(a)$; namely

$$f(a) \equiv k + \frac{a^2}{4\alpha}\left(1 + \xi Y(a)\right) , \qquad Y := \sqrt{w + \frac{16M\alpha}{a^4}} . \tag{3.1}$$

Note that $Y$ obeys the simple differential equation:

$$(Ya^2)' = \frac{2wa}{Y}, \tag{3.2}$$

where we recall that $w := 1 + \frac{4\alpha\Lambda}{3}$.

In terms of $Y_R$ and $Y_L$, the effective potential defined in (4.22) takes the form:

$$\sigma V = \left(k + \frac{a^2}{4\alpha}\right) - \frac{a^2}{12\alpha}\left(\xi_R Y_R + \xi_L Y_L - \frac{\xi_R \xi_L Y_R Y_L}{\xi_R Y_R + \xi_L Y_L}\right) . \tag{3.3}$$

This can be also written as

$$V(a) = \sigma\left(k + \frac{a^2}{4\alpha}\right) - \frac{\sigma a^2}{4\alpha}\left(\frac{3(\xi_R Y_R + \xi_L Y_L)^2 + (\xi_R Y_R - \xi_L Y_L)^2}{12(\xi_R Y_R + \xi_L Y_L)}\right) . \tag{3.4}$$

By repeated application of the differential equation (3.2) we obtain:

$$\sigma V' = \frac{a}{2\alpha}\left(1 - \frac{w}{\xi_R Y_R + \xi_L Y_L}\right) , \tag{3.5}$$

$$\sigma V'' = \frac{1}{2\alpha}\left(1 - \frac{3w}{\xi_R Y_R + \xi_L Y_L} + \frac{2w^2}{\xi_R \xi_L Y_R Y_L (\xi_R Y_R + \xi_L Y_L)}\right) , \tag{3.6}$$

Note that the second derivative of $V$ depends on $a$ only implicitly through $Y(a)$.

Let $a_e$ be the radius at which $V$ is an extremum, $V'(a_e) = 0$. From (3.5) we have

$$\xi_R Y_R(a_e) + \xi_L Y_L(a_e) = w . \tag{3.7}$$

It is of interest to know whether the extremum is minimum or maximum. The second derivative evaluated at the extremum is:

$$V''(a_e) = \frac{\sigma}{\alpha}\left(\frac{w}{\xi_R\xi_L Y_R(a_e)Y_L(a_e)} - 1\right), \tag{3.8}$$

If the right hand side of (3.8) is positive, the extremum is a minimum.

Let us look for a solution where the minimum of the potential coincides with $V = 0$. Imposing at some radius $a_0$ that $V(a_0) = V'(a_0) = 0$ implies:

$$\xi_R Y_R + \xi_L Y_L = w\,, \tag{3.9}$$

$$\xi_R\xi_L Y_R Y_L = w^2 - \left(3 + \frac{12k\alpha}{a_0^2}\right) w\,. \tag{3.10}$$

One can verify as a consistency check that the static and instantaneous shell solutions of section 4 are recovered. In terms of the metric functions $f$ the above two equations are:

$$f_R + f_L = \left(\frac{3}{4\alpha} + \frac{\Lambda}{3}\right) a_0^2 + 2k\,, \qquad f_R f_L = \left(\frac{\Lambda a_0^2}{3} - k\right)^2,$$

c.f. the junction conditions for static and instantaneous shells in proposition 6. Upon imposing the inequalities (4.23-4.25) we recover exactly the solutions of that section.

It is important in analyzing the stability of the static ($\sigma = +1$) vacuum shells to know the sign of $V''$ evaluated at the static radius $a_0$.

$$V''(a_0) = \frac{1}{\alpha}\left(\frac{w}{w^2 - \left(3 + \frac{12k\alpha}{a_0^2}\right)} - 1\right), \tag{3.11}$$

Note that this can also be written

$$V''(a_0) = -\frac{1}{\alpha}\left.\quad 1 + \frac{\frac{ka_0^2}{4\alpha}}{3 + (2 - \frac{4\alpha\Lambda}{3})\frac{ka_0^2}{4\alpha}}\right) = -\frac{1}{\alpha}\frac{xy - 3kx - 3y}{xy - 3kx - 2y}$$

in terms of the original variables and of the variables of section 4 respectively.

# 4 Diagrams of the moduli space

Here we collect the diagrams referred to in section 4.

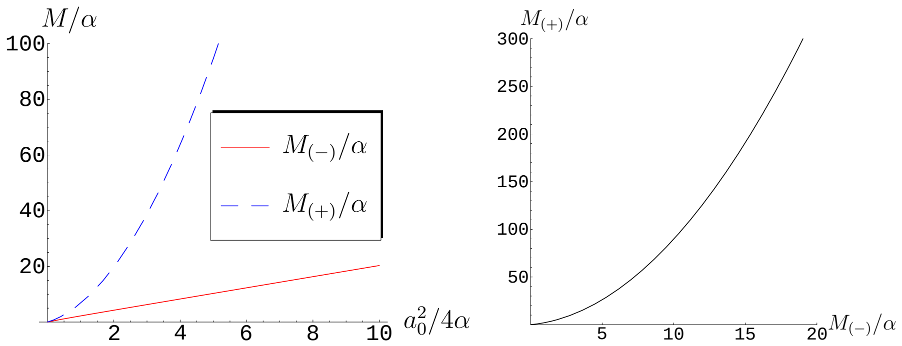



Figure 8.1: For $\Lambda = 0$, spherically symmetric shells exist only with standard orientation and for $\alpha > 0$. Masses $M_{(-)}$, $M_{(+)}$ and shell radius $a_0$ are measured in units of the Gauss-Bonnet coupling, $\alpha$.

# 5 Adding matter: A working example

So far, we have discussed junction conditions for spherical thin shells in Einstein-Gauss-Bonnet gravity. We focussed our attention on vacuum wormholes and bubble-type solutions; the latter were called vacuum-shells. Now, let us consider the presence of matter; that is, let us consider matter on the thin-shell. This will enable us to discuss an application of the junction conditions to the black hole thermodynamics.

In this Appendix we will analyze the effect of "bubble absortion" by a black hole. This will allow us to conclude the consistency between the junction conditions we worked out in this Thesis and black hoe thermodynamics.

Consider again a Bolware-Deser black hole, whose metric takes the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_3^2.$$

with

$$f(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha}\sqrt{1 + \frac{16M\alpha}{r^4} + \frac{4\Lambda\alpha}{3}}.$$

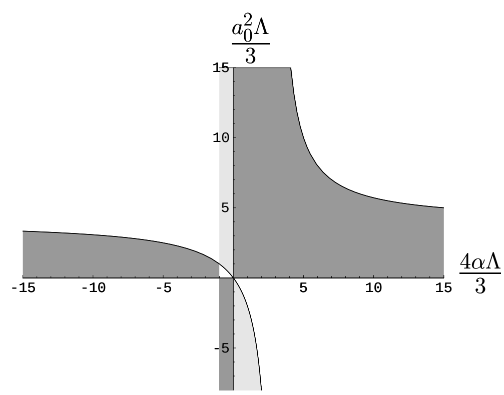



Figure 8.2: Static vacuum shells exist in the dark grey region. Instantaneous vacuum shells with $a = a_0$ exist in the light grey region.

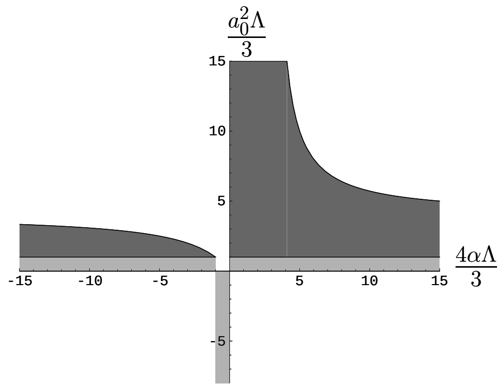



Figure 8.3: The static vacuum shells can have the standard orientation $\eta_L\eta_R > 0$ (light grey) or wormhole orientation $\eta_L\eta_R < 0$ (dark grey).

As we discussed, the large $r^2/\alpha$ limit of this metric mimics the (A)dS-Schwarzschild black hole in five dimensions. The radius of the event horizon $r_+$ is located at

$$r_+^2 = \frac{3}{\Lambda}\left(1 - \sqrt{1 + \frac{4}{3}\Lambda(\alpha - M)}\right),$$

while the cosmological horizon $r_{++}$ is located at

$$r_{++}^2 = \frac{3}{\Lambda}\left(1 + \sqrt{1 + \frac{4}{3}\Lambda(\alpha - M)}\right).$$

We also mentioned that the Hawking temperature of this solution is given by

$$T = \frac{\hbar}{2\pi}\frac{r_+}{4\alpha + r_+},$$

while the entropy goes like

$$S \propto r_+^3 + 12\alpha r_+.$$

In this Appendix, let us consider the positive sign for $\alpha$, and positive $\Lambda$ as well. From the expression for the horizon radius, we can find the following expression by differenciating it

$$dM = r_+ dr_+ \sqrt{1 + \frac{4}{3}\Lambda(\alpha - M)} - \frac{r_+^4}{12} d\Lambda \tag{5.1}$$

Now, supose that our configuration is such that outside the black hole, at a fixed radius $a > r_+$, a thin shell is located, creating a bubble outside which the space is full of a cosmological constant[2] of value $\Lambda$, while in the interior its value is $\Lambda + \delta\Lambda$ (inside such bubble). This kind of configurations was considered in four-dimensional general relativity by Teitelboim in the 1980's. The scenario for providing $\Lambda$ of such a non-constant behaviour is thinking that the effective value of $\Lambda$ is actually given by a field strengh $F_{\mu\nu\rho\delta\sigma}$ of a 4-form field $A_{\mu\nu\rho\sigma}$ which couples to the thin shell in a non-trivial way. In such a framework the thin-shell (which we will thought of as having tension $T$) represents an spherical 3-brane charged under the 4-form field $F_{\mu\nu\rho\delta\sigma}$, so that the bubble would tend to collapse for $\delta\Lambda > 0$ due to the effect of gravity and the negative presure.

Then, the question arises as to how the laws of black hole thermodynamics work in such a scenario having $\Lambda$ as a thermodynamics parameter. More precisely, what happens with the law that states the "increassing of the entropy" when the bubble colapses inside the black hole? If the bubble collapses, and then $\Lambda$ changes, can the entropy decrease?

Giving an answer to this question requires an analysis of the thin-shell dynamics in the background described above. In four-dimensional GR, Teitelboim argued that the laws of black hole thermodynamics still appear to hold when $\Lambda$ is considered as a black hole parameter, and so one can wonder whether such a conclusion also can be obtained for the case of five-dimensional black holes in Einstein-Gauss-Bonnet theory too, where both the classical solutions and the junctions conditions are quite different. We will show that the same mechanism found by Teitelboim works here, and so the laws of thermodynamics still hold when $\Lambda$ is considered as a black hole parameter in the Lovelock theory of gravity.

To show this, we first have to consider the junction conditions

$$8\pi T = -3 < K_\phi^\phi > +4\alpha < (K_\phi^\phi)^3 - 3K_\phi^\phi P_{\tau\phi}^{\tau\phi} >,$$

where we recall that $T$ represents the tension of the 3-brane (henceforth called thin-shell), and where the symbol $< \mathcal{O} >$ means that the quantity $\mathcal{O}$

[2]Notice that, strictly speacking, $\Lambda$ is not the cosmological constant when $\alpha$ is not zero, but it contributes to it.

is evaluated computing the difference between the value it takes inside the region defined by the shell and the value it takes outide it. To evaluate the above equation, one has to notice that the metric inside and outside the shell takes different values, given by

$$f_{\text{inside}}(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha}\sqrt{1 + \frac{16(M+\delta M)\alpha}{r^4} + \frac{4(\Lambda + \delta\Lambda)\alpha}{3}}.$$

and

$$f_{\text{outside}}(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha}\sqrt{1 + \frac{16M\alpha}{r^4} + \frac{4\Lambda\alpha}{3}},$$

respectively. Notice that the difference between the masses $M_{\text{inside}} = M$ and $M_{\text{outside}} = M + \delta M$, denoted by $\delta M$, is a constant of motion and turns out to be given by the (kinetic and tension) energy that the thin shell has at the moment of collapsing into the black hole. Hence, with these metrics, at first order in $\delta M$ and $\delta\Lambda$ the junction conditions read

$$8\pi T = \frac{3}{2a\sqrt{f(a) + \dot{a}^2}}\left(\frac{2\delta M}{a^2} + \frac{\delta\Lambda a^2}{6}\right),$$

where $f(a)$ refers to the value that $f_{\text{inside}}(r)$ takes on the radius where the shell is located. It is remarkable that this expression agrees with the one for General Relativity, up to the explicit form of the function $f(r)$, despite the rather different junction conditions. Then, the next step is writting the last equation in a convenient way; namely

$$\delta M = \frac{8\pi T a^3}{3}\sqrt{f(a) + \dot{a}^2} - \frac{\delta\Lambda a^4}{12}. \qquad (5.2)$$

The quantity $\delta M$ corresponds to the increassing of the black hole mass due to the colapse of the thin shell. The first term in this expression for $\delta M$ represents the kinetic energy of the thin shell at the (proper) moment of the colapse, which is proportional to the product between the tension $T$ and the world-volume of the 3-brane $\sim a^3$. On the other hand, the second term is a sort of "potential energy" due to the varying $\Lambda$. Since $M$ is a constant of motion, we can evaluate the right hand side of the last equation in proper time $\tau_+$, when the bubble cross the horizon. Then, we find

$$\delta M = \frac{8\pi T r_+^3}{3}|\dot{r}(\tau_+)| - \frac{\delta\Lambda r_+^4}{12},$$

with $r(\tau_+) = r_+$. Thus, as in the case of Einstein theory, the remarkable occurrence is now that the $\delta\Lambda$ term in the last equation exactly cancels the contribution proportional to $d\Lambda$ appearing explicitly in (5.1), so that one has[3]

$$dr_+ = \frac{8\pi T |\dot{r}(\tau_+)| r_+^2}{1 - \frac{\Lambda r_+^2}{3}} > 0,$$

where it is worth noticing that the black hole horizon radius $r_+^2 < 3/\Lambda$ (unlike the cosmological horizon, which is located beyond the distance $3/\Lambda$). Therefore, exactly the same arguments that works for GR work here, and even in presence of a changing $\Lambda$ the change in the horizon radius comes solely from the inertial of the bubble and thus the general laws of black hole thermodynamics stating that $dS \sim 3(r_+^2 + 4\alpha)dr_+ > 0$, still appear to hold.

This can be regarded as a consistency check of the junction conditions we derived in this paper.

[3]This argument exactly parallels the one by Teitelboim for the case of four-dimensional Einstein theory.

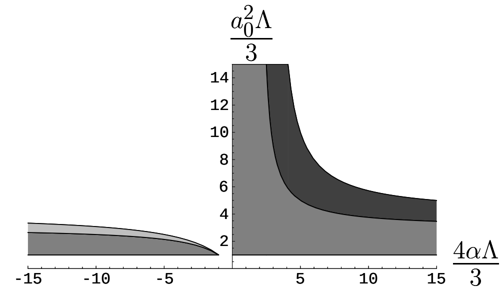



Figure 8.4: There are three types of static wormholes according to the branch signs $(\xi_L, \xi_R)$ in each bulk region: $(-,-)$ lightest grey; $(-,+)$ medium grey; $(+,+)$ dark grey.

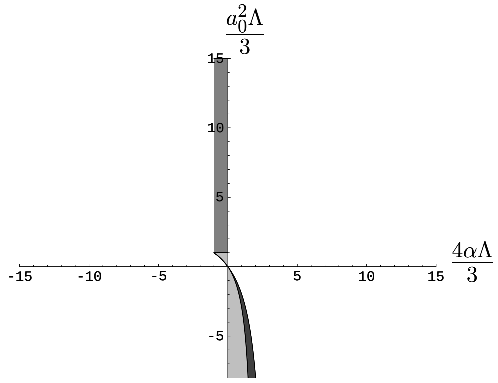



Figure 8.5: The different types of constant $a$ instantaneous shells are: $(-,+)$ branch standard orientation (light grey); $(-,+)$ branch wormhole orientation (medium grey); $(+,+)$ branchstandard orientation (dark grey).

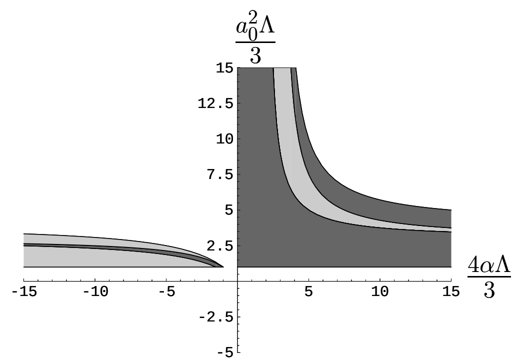



Figure 8.6: The stable region $V''(a_0) > 0$ for wormholes is shown in light grey. For positive $\alpha$ this is a region of the (+,+) branch soluions. For negative $\alpha$ it includes all except a small region of the $(-,+)$ branch.

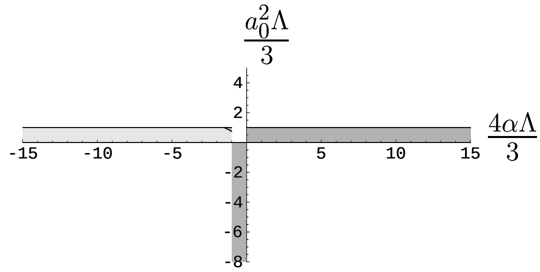



Figure 8.7: The stability of the standard shells. The stable regions are shown in light grey and the unstable regions in dark grey. All standard shells are $(-,+)$.

# Chapter 9

# Objetivos y Logros

## 1 Objetivos

Este trabajo de tesis de doctorado pretende estudiar las soluciones de agujero negro y solitones gravitatorios (localmente) esféricamente simétricos en la teoría de gravedad de Lovelock, la cual es la generalización natural de la Relatividad General de Einstein a dimensión $D$ mayor que cuatro. El tipo de solitones gravitatorios que se pretende estudiar son generalizaciones de la solución de Boulware-Deser, la cual corresponde a la solución general de tal simetría en la teoría de gravedad definida por la acción que resulta de suplementar a la acción de Einstein-Hilbert con términos cuadráticos en la curvatura, los cuales coinciden con la extensión dimensional de la densidad de Euler en $D = 4$. Esta acción remeda la acción efectiva de bajas energías de ciertas teorías de cuerdas, debido a lo cual ha recibido mucha atención recientemente en el marco de la correspondencia AdS/CFT.

Una de las motivaciones originales de nuestra investigación fue la de responder a la pregunta acerca de la existencia de soluciones de vacío con estructura causal no-trivial, del tipo *wormhole*, en la teoría de Lovelock. Esta pregunta se origina en la no-existencia de este tipo de soluciones en la teoría de Einstein.

## 2 Logros

Nuestro estudio nos llevó al descubrimiento de una gran familia de nuevas soluciones de vacío en cinco dimensiones, las cuales pueden ser consideradas solitones gravitatorios, y se construyen mediante un método de cirugía geométrica a partir de las soluciones de Boulware-Deser. La motivación original de nuestro trabajo ha sido, no sólo el descubrir nuevas soluciones, sino el de efectuar una clasificación exhaustiva de este tipo de solitones gravitatorios. Esta clasificación nos llevó, como primer resultado, a demostrar la no-validez de los teoremas tipo Birkhoff en la teoría de Lovelock. Nuestros contraejemplos explícitos son los primeros obtenidos para las teorías de Lovelock fuera de los llamados *Chern-Simons points*.

La enorme gama de nuevas soluciones que encontramos en nuestra clasificación de geometrías de vacío contiene soluciones tipo wormhole, soluciones del tipo burbujas de vacío, universos cerrados, estructuras causales no-triviales, agujeros negros topológicos, y otras geometrías de topología muy variada.

Aunque nuestras soluciones presentan un salto en las derivadas de la métrica, son inconsútiles en el sentido de que satisfacen las condiciones de juntura de la teoría de Lovelock. En particular, descubrimos la primera solución de *wormhole* (esféricamente simétrica) en una teoría gravitatoria en vacío. Nuestra solución es asintóticamente de-Sitter y localmente Boulware-Deser. Para cierta región del espacio de parámetro, la solución resulta, además, estable ante perturbaciones que respetan la simetría. Encontramos también soluciones inestables y estudiamos también la comunión entre las dos diferentes ramas de la solución de Boulware-Deser cuando se unen éstas con diferentes orientaciones.

Nuestros resultados de clasificación de nuevas soluciones fueron reportados en las siguientes publicaciones

- C. Garraffo, G. Giribet, E. Gravanis y S. Willison, J. Math. Phys. **49** (2008) 042502, [arXiv:0711.2992].

- C. Garraffo, G. Giribet, E. Gravanis y S. Willison, Proceeding of the XII Marcel Grossmann Meeting, París (2009), por aparecer.

El descubrimiento de soluciones estables que conectan una rama asintóticamente de-Sitter con una burbuja de vacío cuyo interior corresponde a una rama con singularidades desnudas, nos hizo replantear el significado de este tipo de singularidades en la teoría de Lovelock. Básicamente, surgió la pregunta acerca de la peligrosidad de este tipo de singularidades en el contexto físico. A efectos de responder esta cuestión, nos dispusimos a analizar las soluciones de masa positiva y singularidad desnuda en la teoría de Lovelock con el método de *quantum probes*. Nuestro análisis develó que, mientras las soluciones de masa positiva con singularidad desnuda en la teoría de Lovelock en cinco dimensiones corresponden a geometrías de curvatura singular en el contexto clásico, desde el punto de vista cuántico corresponden a geometrías regulares.

Nuestros resultados sobre el estudio de singularidades desnudas de masa positiva en la teoría de Lovelock fueron publicados en

- C. Garraffo y G. Giribet, Mod. Phys. Lett. **A23** (2008) 1801, [arXiv:0805.3575],

y fueron reportados en una serie de seminarios dictados en el Martin A. Fisher Physics Department de Brandeis University, MA, USA, y en otros centros de investigación en Chile, Francia y los Estados Unidos.

# Acknowledgements

Agradezco muy especialmente a Gaston Giribet por su inmenso compromiso con la dirección de esta tesis, y cuya generosa dedicación fue fundamental en mi formación. Agradezco también al Instituto de Astronomía y Física del Espacio, al Centro de Estudios Científicos CECS de Valdivia, y al Martin A. Fisher Physics Department de Brandeis University por su hospitalidad. A su vez, han sido de gran importancia para este trabajo las contribuciones de Rafael Ferraro, Elias Gravanis, H. Maeda, R. Troncoso, Steve Willison, J. Zanelli y Mauricio Leston, como también las discusiones con los estudiantes de doctorado del IAFE y del Departamento de Física de la UBA. Este trabajo fue financiado con una beca del Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET, Argentina.

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