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Algunos problemas de optimización para el p-Laplaciano

Del Pezzo, Leandro M.

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UNIVERSIDAD DE BUENOS AIRES
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Departamento de Matemática

Algunos Problemas de Optimización para el $p-$Laplaciano

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Leandro M. Del Pezzo

Director de tesis: Julián Fernández Bonder

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Algunos Problemas de Optimización para el $p-$Laplaciano

(Resumen)

Dentro de la teoría de autovalores para operadores elípticos diferenciales, un problema de especial importancia es el de optimización de estos autovalores con respecto a los diferentes parámetros considerados. En está tesis, nos dedicamos al estudio de algunos de estos problemas, considerando como operador no lineal modelo el $p-$Laplaciano que se define como

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

Palabras Claves: $p-$Laplaciano; primer autovalor; problemas de optimización; existencia; reordenamientos; derivada de forma.

Some Optimization Problems for the $p-$Laplacian

(Abstract)

Within the eigenvalues theory for elliptic differential operators, a relevant problem is the optimization of these eigenvalue with respect to the different parameters under consideration. In this thesis, we study some of this problems, we consider as a model of nonlineal operator we take the $p-$Laplacian, that is defined as

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

Key words: $p-$Laplacian; first eigenvalue; optimization problem; existence; rearrangements; shape derivative.

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Contents

Introduction

Eigenvalue problems for second order elliptic differential equations are one of the fundamental problems in mathematical physics and, probably, one of the most studied ones in the past years. See [DS1, DS2, DS3].

When studying eigenvalue problems for nonlinear homogeneous operators, the classical linear theory does not work, but some of its ideas can still be applied and partial results are obtained. See, for instance, [An, C, GAPA1, GAPA2].

For example, the eigenvalue problem for the p-Laplace operator subject to zero Dirichlet boundary condition, i.e., find λ and $u(x)$ such that

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \ \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$, and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian, have been studied extensively during the past two decades and many interesting results have been obtained. The investigations have principally relied on variational methods and the existence of a *principal eigenvalue* (i.e., the associated eigenspace has dimension one and the associated nonzero eigenfunction does not change sign) has been proved as a consequence of minimization results of appropriate functionals. Then, this principal eigenvalue λ_1 is the smallest of all possible eigenvalues λ. Moreover, λ_1 is isolated. On the other hand, the study of higher eigenvalues introduces complications which depend upon the boundary conditions in a significant way, and thus the existence proofs may differ significantly, as well.

In recent years, models involving the $p-$Laplace operator have been used in the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [E, T]), non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, etc. (see [ADT, AE, AC, Di]).

In the theory for eigenvalues of elliptic operators, a relevant problem is the optimization of these eigenvalues with respect to different parameters under consideration. Problems linking the shape of the domain or the coefficients of an elliptic operator to the sequence of its eigenvalues are among the most fascinating of mathematical analysis. In part, this is because they involve different fields of mathematics.

In this thesis, we focus on extremal problems for principal eigenvalues. For instance, we look for the optimization of the principal eigenvalue of the $p-$Laplace operator perturbed by a potential function $V(x)$ where the potential varies in an admissible class. This type of problems are nonlinear versions of Schröedinger operators (that is elliptic linear operators L under perturbations given by a potential V, in bounded regions). These operators appear in different fields of applications such as quantum mechanics, stability of bulk matter, scattering theory, etc. See Chapter 2. We investigate similar questions for other kind of eigenvalues and related elliptic operators, like the Steklov eigenvalue problem and nonlinear elastic membranes (see Chapter 3 and 6).

In [AsHa], for example, the authors consider Schröedinger operators, and the following problem is studied: Let L be a uniformly elliptic *linear* operator and assume that $\|V\|_{L^q(\Omega)}$ is constrained but otherwise the potential V is arbitrary. Can the maximal value of the first (fundamental) eigenvalue for the operator $L+V$ be estimated? And the minimal value? There exist *optimal* potentials? (i.e. potentials V^* and V_* such that the first eigenvalue for $L+V^*$ is maximal and the first eigenvalue for $L+V_*$ is minimal).

In [AsHa] these questions are answered in a positive way and, moreover, a characterization of these optimal potentials is given.

Other interesting example is given in [He]. In that article, the author studies a non-homogeneous membranes. He considers a membrane Ω in which non-homogeneity is characterized by a non-negative density function $g(x)$. The following eigenvalue problem is then analyzed:

$$\begin{cases} -\Delta u = \lambda g(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The first eigenvalue $\lambda(g)$ is characterized by the usual minimization formula:

$$\lambda(g) = \inf\left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega g(x) u^2 \, dx} : u \in H_0^1(\Omega) \right\}$$

The author is then interested in the following optimization problem:

$$\inf\{\lambda(g) : g \in \mathcal{G}\}, \quad \sup\{\lambda(g) : g \in \mathcal{G}\},$$

where

$$\mathcal{G} := \left\{ g \in L^\infty(\Omega) : \alpha \le g(x) \le \beta \text{ a.e. in } \Omega, \int_\Omega g \, dx = c \right\}$$

where α, β and c three real numbers such that $0 \le \alpha < \beta$ and $\alpha|\Omega| \le c \le \beta|\Omega|$.

In the case where $\Omega \subset \mathbb{R}^2$, this equation models the vibration of a non-homogeneous membrane Ω which is fixed along the boundary $\partial\Omega$. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials inside Ω so as to minimize or maximize the vibration of the corresponding membrane.

In the [He], the author proved that there exists a minimizer of $\lambda(g)$ in the class $\mathcal{G}$ and that there exists a unique maximizer of $\lambda(g)$ in the same class.

A very important issue in this type of problems is not only to establish the existence of optimal configurations but also to give some characterization of those optimal configurations or, at least, some necessary conditions that these optimal configurations must satisfy.

In order to deal with these issue, we compute the derivative of the functionals that we are trying to optimize with respect to perturbations of the parameters under consideration in the class of admissible parameters. This is achieve by means of suitable extentions of the Hadamard method of regular variations. See [HP].

This method has been proved to be extremely useful in order to perform actual computations of the optimal parameter configurations in many situations, see [P].

So, the computation of these derivatives will be extremely useful for designing numerical algorithms that compute the optimal configurations of the paremeters.

We perform this computations in most of the problems under consideration in this thesis (see Chapters 5, 6 and 7). We believe that the results in those chapters are the main contribution of this thesis.

Thesis outline

The rest of the thesis is organized as follows.

Chapter 1 contains the notation and some preliminary tools used throughout this thesis. Almost always, the results are not quoted in the most general form, but in a way that is appropriate to our purposes; nevertheless some of them are actually slightly more general than we strictly need. Most of these results are well known, but we include it here for the sake of completeness. We will not go into details, referring the reader to the corresponding literature.

The purpose of Chapter 2 is the extension of the results of [AsHa] to the nonlinear case. We are also interested in extending these results to degenerate/singular operators. As a model of these operators, we take the $p-$Laplacian.

We want to remark that the proofs are not straightforward extensions of those in [AsHa] since the proof there are not, in general, variational. Moreover, some new technical difficulties arise since solutions to a $p-$Laplace type equation are not regular and, mostly, since the eigenvalue problem for the $p-$Laplacian is far from being completely understood.

In Chapter 3, we study the first (nonlinear) Steklov eigenvalue, λ, of the following problem:

$$-\Delta_p u + |u|^{p-2}u + \alpha\phi|u|^{p-2}u = 0$$

in a bounded smooth domain Ω with

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda|u|^{p-2}u$$

on the boundary $\partial\Omega$. We analyze the dependence of this first eigenvalue with respect to the weight ϕ and with respect to the parameter α. We prove that for fixed α there exists an optimal ϕ_α that minimizes λ in the class of uniformly bounded measurable functions with fixed integral.

Next, we study the limit of these minima as the parameter α goes to infinity and we find that the limit is the first Steklov eigenvalue in Ω with a hole where the eigenfunction vanishes.

In Chapter 4, we compute the derivative of the norms $\|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{W^{1,p}(\Omega)}$ and $\|\cdot\|_{L^p(\partial\Omega)}$ with respect to perturbation in Ω. These computations are fundamental for the rest of this thesis.

Moreover, this chapter collects some general results on differential geometry that are needed in the course of our arguments.

In Chapter 5, we study the problem of minimizing the first eigenvalue of the $p-$Laplacian plus a potential with weights, when the potential and the weight are allowed to vary in the class of rearrangements of a given fixed potential V_0 and weight g_0.

More recently, in [CEP2], the authors analyze this problem but when the potential function is zero. In that work the authors prove the existence of a minimizing weight g_* in the class of rearrangements of a fixed function g_0 and, in the spirit of [Bu1] they found a sort of *Euler-Lagrange formula* for g_*. However, this formula does not appear to be suitable for use in actual computations of these minimizers.

In this chapter, we extend the results in [CEP2] to our problem. Also, the same type of Euler-Lagrange formula is proved for both the weight and potential. But, we go further and study the dependence of the first eigenvalue with respect to the weight and potential, and prove the continuous dependence in L^q norm and, moreover, the differentiability with respect to regular perturbations of the weight and the potential.

In the case when the perturbations are made inside the class of rearrangements, we exhibit a simple formula for the derivative of the eigenvalue with respect to the weight and the potential.

We believe that this formula can be used in actual computations of the optimal eigenvalue, weight and potential, since this type of formulas have been used in similar problems in the recent years with significant success, see [FBGR, H, O, P] and references therein.

In Chapter 6, we study some optimization problems for nonlinear elastic membranes. More precisely, we consider the problem of optimizing the cost functional

$$\mathcal{J}(u) = \int_{\partial\Omega} f(x)u \, d\mathcal{H}^{N-1}$$

over some admissible class of loads f where Ω is a bounded smooth domain, $\mathcal{H}^{N-1}$ is the $N-1-$dimensional Hausdorff measure and u is the (unique) solution to the problem

$$-\Delta_p u + |u|^{p-2}u = 0$$

in Ω with

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f$$

on $\partial\Omega$.

We have chosen three different classes of admissible functions $\mathcal{A}$ to work with.

- The class of rearrangements of a given function f_0.
- The (unit) ball in some L^q.
- The class of characteristic functions of sets of given surface measure.

Observe that this latter class is in fact a subclass of the first one. In fact, if we choose f_0 to be a characteristic function, then the class of rearrangements of f_0 is the class of characteristic functions of sets of given surface measure. Nevertheless, since we believe that this case is the most interesting one, we have chosen to treated separately from the others.

For each of these classes, we prove existence of a maximizing load (in the respective class) and analyze properties of these maximizers.

Then, in order to do that, we compute the first variation with respect to perturbations on the load.

Lastly, in Chapter 7, We study the Sobolev trace constant for functions defined in a bounded domain Ω that vanish in the subset A, i.e.,

$$S_q(A) := \inf\left\{ \frac{\int_\Omega |\nabla u|^p + |u|^p\,\mathrm{d}x}{\left(\int_{\partial\Omega} |u|^q\,\mathrm{d}\mathcal{H}^{N-1}\right)^{p/q}} : u \in W_A^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \right\},$$

with

$$W_A^{1,p}(\Omega) = \overline{C_0^\infty(\overline{\Omega} \setminus A)}$$

where where the closure is taken in $W^{1,p}$–norm.

We find a formula for the first variation of the Sobolev trace with respect to the hole. As a consequence of this formula, we prove that when Ω is a centered ball, the symmetric hole is critical when we consider deformations that preserve volume but is not optimal for some cases.

The results in this chapter generalize those in [FBGR] where the same problem was treated in the linear case $p = q = 2$. We want to remark that this extension is far from being elementary, since the arguments in [FBGR] uses the linearity in a crucial way. We have to develop a new method in order to consider the nonlinear setting that relates to that in [GMSL].

Included publications

The results in Chapters 2, 3, 5, 6 and 7 have appeared published as research articles. These results are readable as individuals contributions linked by a common theme and all of them are either published, accepted for publication or submitted for publication in refereed journals. The chapters contain the following papers:

Chapter 2

L. Del Pezzo and J. Fernández Bonder. *An optimization problem for the first eigenvalue of the p-Laplacian plus a potential.* Commun. Pure Appl. Anal., vol. 5 (2006), no. 4, pp. 675–690.

Chapter 3

L. Del Pezzo, J. Fernández Bonder and J. D. Rossi. *An optimization problem for the first Steklov eigenvalue of a nonlinear problem.* Differential Integral Equations, vol. 19 (2006), no. 9, pp. 1035–1046.

Chapter 5

L. Del Pezzo and J. Fernández Bonder. *An optimization problem for the first weighted eigenvalue problem plus a potential.* Submitted for publication.
arxiv.org/pdf/0906.2985v1.

Chapter 6

L. Del Pezzo and J. Fernández Bonder. *Some optimization problems for p-Laplacian type equations.* Appl. Math. Optim., vol. 59 (2009), no. 3, pp. 365–381.

L. Del Pezzo and J. Fernández Bonder. *Remarks on an optimization problem for the p−Laplacian.* Applied Mathematical Letters, vol. 23 (2010), no. 2, pp. 188–192.
doi:10.1016/j.aml.2009.09.010.

Chapter 7

L. Del Pezzo. *Optimization problem for extremals of the trace inequality in domains with holes.* Submitted for publication.
arxiv.org/pdf/0809.0246.

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Preliminaries

This chapter contains the notation and some preliminary tools used throughout this thesis. Almost always, the results are not quoted in the most general form, but in a way appropriated to our purposes; nevertheless some of them are actually slightly more general than we strictly need.

Section 1.1 fixes some notations. Section 1.2, 1.3, 1.4, 1.5, 1.6 and 1.7 collect some results regarding Banach spaces, measure theory, L^p−spaces, Sobolev spaces and spherical symmetrization, respectively. Section 1.8 consists in an overview of some results for the operator $H_V(u) := -\Delta_p u + V(x)|u|^{p-2}u$ with $V \in L^q(\Omega)$. Finally, in Section 1.9, we give some important results about functions of bounded variation.

Most of these results are well known, but we include it here for the sake of completeness. We will not go into details, referring the reader to the corresponding literature.

1.1 Notation

Throughout this thesis the term *domain* and the symbol Ω shall be reserved for an open set in the $N-$ dimensional, real Euclidean space $\mathbb{R}^N$.

A typical point in $\mathbb{R}^N$ is denoted by $x = (x_1, \ldots, x_N)$; its norm

$$|x| = \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}.$$

The inner product of x and y is $\langle x, y\rangle$ or $x \cdot y$, i.e.,

$$\langle x, y\rangle = x \cdot y = \sum_{i=1}^{N} x_i y_i.$$

If $u\colon \Omega \to \mathbb{R}$ is a continuous function, the *support* of u is defined by

$$\operatorname{supp} u = \Omega \cap \overline{\{x\colon u(x) \neq 0\}},$$

where the closure of a set $A \subset \mathbb{R}^N$ is denoted by $\overline{A}$. If $A \subset \Omega$, $\overline{A}$ compact and also $\overline{A} \subset \Omega$ we shall write $A \subset\subset \Omega$. The *boundary* of a set A is defined by

$$\partial A = \overline{A} \cap \overline{\mathbb{R}^N \setminus A}.$$

For $E \subset \mathbb{R}^N$ the characteristic function is denoted by χ_E and we write 2^E for the set of all subset of E.

The symbol

$$B(x, r) = \left\{y \in \mathbb{R}^N : |x - y| < r\right\}$$

denotes the open ball with center x and radius r, and

$$\overline{B}(x, r) = \left\{y \in \mathbb{R}^N : |x - y| \leq r\right\}$$

will stand for the closed ball.

We use the standard notation $C^k(\Omega; \mathbb{R}^m)$ for the k-times continuously differentiable functions on some domain Ω, for $m \in \mathbb{N}$ and $k = 0$ (continuous functions), $1, 2, \ldots, \infty$. We abbreviate $C^k(\Omega; \mathbb{R}) \equiv C^k(\Omega)$ and $C^0(\Omega) \equiv C(\Omega)$. The subspace $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consist of all those function in $C(\Omega)$ and $C^\infty(\Omega)$, respectively, which have compact support in Ω.

If $\alpha = (\alpha_1, \ldots, \alpha_N)$ is an N-tuple non-negative integers, α is called a *multi-index* and the *length* of α is

$$|\alpha| = \sum_{i=1}^{N} \alpha_i.$$

The higher order derivatives operators are defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

The *gradient* of $u \in C^1(\Omega)$ is

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right).$$

Let Ω be a open bounded subset of $\mathbb{R}^N$ and $0 < \gamma \leq 1$. We say that $f : \Omega \to \mathbb{R}$ is *Lipschitz continuous* if for all $x, y \in \Omega$,

$$|f(x) - f(y)| \leq C|x - y|,$$

for some constant C. It turns out to be useful to consider also functions f satisfying a variation of the above inequality, namely

$$|f(x) - f(y)| \leq C|x - y|^\gamma \quad \forall\, x, y \in \Omega,$$

for some constant C. Such a function is said to be *Hölder continuous with exponent* γ; and *locally Hölder continuous with exponent* γ if f is Hölder continuous with exponent γ on every compact subset of Ω.

Clearly if f is Lipschitz (Höder) continuous, then f is continuous.

Example 1.1.1. The function $f : B(0,1) \to \mathbb{R}$ given by $f(x) = |x|^\beta$, $0 < \beta < 1$ is Hölder continuous with exponent β, and is Lipschitz continuous when $\beta = 1$.

The *Hölder spaces* $C^{k,\gamma}(\overline{\Omega})$ ($C^{k,\gamma}_{loc}(\Omega)$) are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose $k-$th order partial derivatives are Hölder continuous (locally Hölder continuous) with exponent γ in Ω. For simplicity, we write

$$C^{0,\gamma}_{loc}(\Omega) \equiv C^{\gamma}_{loc}(\Omega) \quad \text{and} \quad C^{0,\gamma}(\overline{\Omega}) \equiv C^{\gamma}(\overline{\Omega}),$$

for each $0 < \gamma < 1$.

We will say that Ω is a Lipschitz (smooth) bounded domain when Ω is a bounded domain and its boundary is Lipschitz (smooth).

If Ω is a smooth bounded domain, ν and $\frac{\partial}{\partial \nu}$ denote the unit outer normal vector along $\partial\Omega$ and the outer normal derivative, respectively.

1.2 Banach spaces

Here, we give the functional analysis background that will be needed in this thesis.

Let E be a real linear space. A function $\|\cdot\|\colon E \to [0,+\infty]$ is called a *norm* if

(i) $\|x\| \geq 0$ for all $x \in E$, $\|x\| = 0$ if only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in E$, $\lambda \in \mathbb{R}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$, (triangle inequality).

A linear space E equipped with a norm is called a *normed linear space*. A normed linear space E is a metric space under metric ρ defined by

$$\rho(x,y) = \|x - y\| \quad \forall x, y \in E.$$

Hereafter, we assume that E is a normed linear space.

We say a sequence $\{x_n\}_{n\in\mathbb{N}}$ in E *converges to* $x \in E$, written

$$x_n \to x,$$

if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in E is called a *Cauchy sequence* if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \quad \forall\, n, m > n_0.$$

E is a *complete space* if each Cauchy sequence in E converges and E is called a *Banach space* if E is a complete normed linear space.

We say E is a *separable space* if E contains a countable dense subset.

Example 1.2.1. Euclidean space $\mathbb{R}^N$ is a Banach space under the standard norm.

Example 1.2.2. Let Ω be an open bounded subset of $\mathbb{R}^N$. The space of functions $C^{k,\gamma}(\overline{\Omega})$ is a Banach spaces under the norm:

$$||u||_{C^{k,\gamma}(\overline{\Omega})} = \sum_{j=0}^{k} |D^j u|_\Omega + [D^k u]_{\gamma;\Omega},$$

where

$$|D^j u|_\Omega = \sup_{|\beta|=j} \sup_{x\in\Omega} |D^\beta u(x)| \ (0 \le j \le k),$$

and

$$[D^k u]_{\gamma;\Omega} = \sup_{|\beta|=k} \sup_{\substack{x\neq y\\ x,y\in\Omega}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\gamma}.$$

1.2.1 Hilbert spaces

Let H be a real linear space.

A function $\langle , \rangle\colon H \times H \to \mathbb{R}$ is said to be an *inner product* if

(i) $\langle x, y\rangle = \langle x, y\rangle$ for each $x, y \in E$,

(ii) the function $x \to \langle x, y\rangle$ is linear for each $y \in E$,

(iii) $\langle x, u\rangle \ge 0$ for each $u \in E$,

(iv) $\langle x, x\rangle = 0$ if only if $x = 0$.

If $\langle , \rangle$ is an inner product, the associated norm is

$$||x|| := \langle x, x\rangle^{1/2} \quad \forall\, x \in H.$$

The *Cauchy–Schwarz inequality* states that

$$\langle x, y\rangle \le ||x||||y|| \quad \forall\, x, y \in H.$$

A *Hilbert space* H is a Banach space endowed with an inner product which generates the norm.

Example 1.2.3. Euclidean space $\mathbb{R}^N$ is a Hilbert space under the inner product

$$x \cdot y = \sum_{i=1}^{N} x_i y_i, \quad x = (x_1, \ldots, x_N),\ y = (y_1, \ldots, y_N).$$

1.2.2 Dual space

Let E and F be a Banach spaces.

A function $L : E \to F$ is a *linear operator* provided that

$$L(\lambda x + \gamma y) = \lambda L(x) + \gamma L(y) \quad \forall\, x, y \in E,\ \lambda, \gamma \in \mathbb{R}.$$

We say a linear operator $L : E \to F$ is *bounded* if

$$\|L\| = \sup\{\|L(x)\|_F : x \in E, \|x\|_E \leq 1\} < \infty.$$

A bounded linear operator $L : E \to \mathbb{R}$ is called a *bounded lineal functional*. We denote by E^* the set of all bounded linear functional on E. E^* is the *dual space* of E.

Observe that E^* is a Banach space with the norm

$$\|L\|_{E^*} = \sup\{\|L(x)\|_F : x \in E, \|x\|_E \leq 1\} \quad \forall\, L \in E^*.$$

The dual space of E^* is called the second dual of E and is denoted by E^{**}. Clearly, the mapping $\Psi : E \to E^{**}$ given by $\Psi(x)(f) = f(x)$ is a norm preserving, linear, one-to-one mapping of E into E^{**}. If $E^{**} = \Psi(E)$, then we call E *reflexive*.

1.2.3 Weak and weak* convergence

Let E be a Banach space.

Definition 1.2.4. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in E *converges weakly* to $x \in E$, written

$$x_n \rightharpoonup x,$$

if

$$L(x_n) \to L(x) \quad \forall\, L \in E^*.$$

Remark 1.2.5. It easy to check that

(i) if $x_n \to x$, then $x_n \rightharpoonup x$,

(ii) any weakly convergent sequence is bounded,

(iii) if $x_n \rightharpoonup x$ weakly in E and $L_n \to L$ strongly in E^*, then $L_n(x_n) \to L(x)$,

(iv) if $x_n \rightharpoonup x$, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

The proofs of the following theorems can be found in [Y].

Theorem 1.2.6. *Let E be a reflexive Banach space and let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in E. Then, there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and $x \in E$ such that*

$$x_{n_j} \rightharpoonup x.$$

In other words, bounded sequences in a reflexive Banach space are weakly precompact.

Theorem 1.2.7 (Mazur's Theorem)**.** *Let E be a reflexive Banach space. Then any convex, closed subset of E is weakly closed.*

Definition 1.2.8. A sequence $\{L_n\}_{n\in\mathbb{N}}$ in E^* *converges weakly∗* to $L \in E^*$, written

$$L_n \overset{*}{\rightharpoonup} L,$$

if

$$L_n(x) \to L(x) \quad \forall\, x \in E.$$

Remark 1.2.9. It easy to check that

(i) if $L_n \to L$, then $L_n \overset{*}{\rightharpoonup} L$,

(ii) if $L_n \rightharpoonup L$, then $L_n \overset{*}{\rightharpoonup} L$,

(iii) if $L_n \overset{*}{\rightharpoonup} L$, then $\{L_n\}_{n\in\mathbb{N}}$ is bounded and $\|L\|_{E^*} \le \liminf_{n\to\infty} \|L_n\|_{E^*}$,

(iv) if $L_n \overset{*}{\rightharpoonup} L$ weakly∗ in E^* and if $x_n \to x$ strongly in E, then $L_n(x_n) \to L(x)$.

1.3 Measure theory

This section provides a quick outline of some fundamentals of measure theory.

1.3.1 Measure

Let X be a nonempty subset of $\mathbb{R}^N$.

A *measure* μ is a function from 2^X into $[0, +\infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu(A) \le \sum_{n=1}^{\infty} \mu(A_n) \text{ whenever } A \subset \bigcup_{n=1}^{\infty} A_n.$$

Throughout the section, X and μ denote a nonempty of $\mathbb{R}^N$ and a measure on X, respectively.

If a set $A \subset X$ satisfies

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \quad \forall\, E \subset X,$$

then we say A is a $\mu-$*measurable*.

Remark 1.3.1. Observe that

(i) if $\mu(A) = 0$ then A is $\mu-$measurable,

(ii) A is measurable if and only if $X \setminus A$ is $\mu-$measurable,

(iii) if A is $\mu-$measurable and $B \subset X$, then $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

Some important examples,

Example 1.3.2 (Lebesgue Measure)**.** We consider the closed $N-$dimensional cube

$$Q = \{x \colon a_j \le x_j \le b_j, j = 1, \dots, N\},$$

and their volumes

$$v(Q) = \prod_{j=1}^{N} (b_j - a_j).$$

For any $A \subset \mathbb{R}^N$, we define its *Lebesgue measure* $|A|$ by

$$|A| = \inf \left\{ \sum_{n=1}^{\infty} v(Q_n) \colon A \subset \bigcup_{n=1}^{\infty} Q_n, Q_n \text{ is a cube } \forall\, n \in \mathbb{N} \right\}.$$

If μ is the Lebesgue measure, we say that $A \subset \mathbb{R}^N$ is measurable in place of $\mu-$measurable.

Example 1.3.3 ($d-$dimensional Hausdorff measure)**.** For any $A \subset \mathbb{R}^N$, let us denote by $\operatorname{diam}(A)$ the *diameter* of A, i.e.,

$$\operatorname{diam} A = \inf \left\{ \|x - y\| \colon x, y \in \mathbb{R}^N \right\}.$$

Now, fix $d > 0$ and let E be any subset of $\mathbb{R}^N$. Given $\varepsilon > 0$, let

$$\mathcal{H}^d_\varepsilon(E) = \inf \left\{ \sum_{n=1}^{\infty} \alpha(d) \left(\frac{\operatorname{diam} A_j}{2} \right)^d : A \subset \bigcup_{n=1}^{\infty} A_n, \operatorname{diam} C_j \le \varepsilon \right\}$$

where

$$\alpha(d) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}.$$

Here $\Gamma(d) = \int_0^\infty e^{-t} t^{s-1}\, \mathrm{d}t$, $(0 < d < \infty)$, is the usual gamma function.

Note that $\mathcal{H}^d_\varepsilon(E)$ is monotone decreasing in ε since the larger ε is, the more collections of sets are permitted. Thus, the limit $\lim_{\varepsilon \to 0} \mathcal{H}^d_\varepsilon(E)$ exists. Let

$$\mathcal{H}^d(E) = \sup \left\{ \mathcal{H}^d_\varepsilon(E) \colon \varepsilon > 0 \right\} = \lim_{\varepsilon \to 0} \mathcal{H}^d_\varepsilon(E).$$

$\mathcal{H}^d(E)$ is called *$d-$dimensional Hausdorff measure*.

Proposition 1.3.4. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of $\mu-$measurable sets.*

1. *The sets*
$$\bigcup_{n\in\mathbb{N}} A_n \quad \text{and} \quad \bigcap_{n\in\mathbb{N}} A_n$$
are $\mu-$ measurable.

2. *If the sets $\{A_n\}_{n\in\mathbb{N}}$ are pairwise disjoint, then*
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

3. *If $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}$, then*
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

4. *If $A_{n+1} \subset A_n$ for each $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then*
$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. See [EG]. □

A collection of subset $\Sigma \subset 2^X$ is a $\sigma-$*algebra* provided that

(i) $\emptyset, X \in \Sigma$;

(ii) $A \in \sum$ implies $X \setminus A \in \Sigma$;

(iii) $\{A_n\}_{n\in\mathbb{N}} \subset \Sigma$ implies $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$.

The collection of all $\mu-$measurable subset of X form a $\sigma-$algebra. The smallest $\sigma-$algebra of $\mathbb{R}^N$ that contains the open sets is called the *Borel $\sigma-$algebra* of $\mathbb{R}^N$.

Now we introduce certain classes of measures.

Definition 1.3.5. We say that

(i) μ is *regular* if for each set $A \subset X$ there exists a $\mu-$measurable set B such that $A \subset B$ and $\mu(A) = \mu(B)$;

(ii) μ is *Borel regular* if every open set is $\mu-$measurable and if for each $A \subset X$ there exists a Borel set $B \subset X$ such that $A \subset X$ and $\mu(A) = \mu(B)$;

(iii) μ is a *Radon measure* if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset X$.

Let $\mathcal{P}(x)$ be a statement or formula that contains a free variable $x \in X$. We say that $\mathcal{P}(x)$ holds for $\mu-$a.e. ($\mu-$almost every) $x \in X$ if

$$\mu(\{x \in X : \mathcal{P}(x) \text{ is false}\}) = 0.$$

If X is understood from context, then we simply say that $\mathcal{P}(x)$ holds μ-a. e. and when μ is the Lebesgue measure, a.e. is used in place of $\mu-$a.e.

Lastly, we give an important result about the Hausdorff measure, for the proof see [EG].

Theorem 1.3.6. *The $N-$ dimensional Hausdorff measure is equal to the Lebesgue measure on $\mathbb{R}^N$.*

1.3.2 Measurable function and integration

A function $f: X \to [-\infty, +\infty]$ is called *$\mu-$measurable* if for each open $U \subset \mathbb{R}$, $f^{-1}(U)$ is $\mu-$measurable. If μ is the Lebesgue measure, we say that f is measurable in place of $\mu-$measurable.

Proposition 1.3.7. *We have that*

1. *if $f, g: X \to \mathbb{R}$ are $\mu-$measurable, then so are $f+g$, fg, $|f|$, $\min\{f, g\}$ and $\max\{f, g\}$. the function f/g is also $\mu-$measurable, provided $g \neq 0$ on X;*
2. *if the functions $f_n: X \to [-\infty, +\infty]$ are $\mu-$measurable ($n \in \mathbb{N}$), then $\inf\{f_n : n \in \mathbb{N}\}$, $\sup\{f_n : n \in \mathbb{N}\}$, $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are also $\mu-$measurable;*
3. *if $f: X \to [0, +\infty]$ is $\mu-$measurable. Then there exist $\mu-$measurable sets $\{A_n\}_{n\in\mathbb{N}}$ in X such that*
$$f = \sum_{n=1}^{\infty} \frac{1}{n}\chi_{A_n}.$$

Proof. See [EG]. □

Let Ω be a domain in $\mathbb{R}^N$ and let μ a measure on Ω. The *support* of a $\mu-$measurable function f, supp f, is the complement of the largest open set which f vanishes $\mu-$a.e.

Observe that, if μ is the Lebesgue measure and f is continuous on Ω, this definition of support coincides with the definition that we gave in Section 1.1.

Given $f: X \to [-\infty, +\infty]$, we denote by

$$f^+ = \max\{f, 0\} \text{ and } f^- = \min\{f, 0\}.$$

Observe that $f = f^+ - f^-$.

A function $g\colon X \to [-\infty, +\infty]$ is called a *simple function* if the image of g is countable.

Our next task is to define the integrals of a $\mu-$measurable function.

We start with a non-negative simple $\mu-$measurable function g defined on X. We define the *integral of g over X* as

$$\int_X g \, \mathrm{d}\mu = \sum_{0 \le y \le +\infty} y\mu\left(g^{-1}(y)\right).$$

Then, for $f\colon X \to [0, +\infty]$ $\mu-$measurable we define the *integral of f* by

$$\int_X f \, \mathrm{d}\mu = \sup \int_X g \, \mathrm{d}\mu,$$

the supremum being taken over all simple $\mu-$measurable function g such that $0 \le g \le f$ $\mu-$ a.e.

Finally, a $\mu-$measurable function f is called $\mu-$*integrable* if

$$\int_X |f| \, \mathrm{d}\mu < +\infty,$$

in which case we write

$$\int_X f \, \mathrm{d}\mu = \int_X f^+ \, \mathrm{d}\mu - \int_X f^- \, \mathrm{d}\mu.$$

When μ is the Lebesgue measure, $\mathrm{d}x$ is used in place of $\mathrm{d}\mu$.

Given μ a Radon measure. We write

$$\mu \lfloor f$$

provided

$$\mu \lfloor f(K) = \int_K f \, \mathrm{d}\mu$$

holds for all compact sets K. Note $\mu \lfloor A = \mu \lfloor \chi_A$.

We now give the limit theorems (for the proofs, see [EG]).

Lemma 1.3.8 (Fatou's Lemma)**.** *If $f_n\colon X \to [0, +\infty]$ are $\mu-$measurable ($n \in \mathbb{N}$), then*

$$\int_X \liminf_{n\to\infty} f_n \, \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu.$$

Theorem 1.3.9 (Monotone Convergence Theorem)**.** *Let $f_n\colon X \to [0, +\infty]$ be $\mu-$measurable ($n \in \mathbb{N}$), with $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$. Then*

$$\int_X \lim_{n\to\infty} f_n \, \mathrm{d}\mu \le \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu.$$

Theorem 1.3.10 (Dominate Convergence Theorem)**.** *Suppose* $\{f_n\}_{n\in\mathbb{N}}$ *is a sequence of* $\mu-$*measurable functions such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists $\mu-$*a.e. If there is a function* g $\mu-$*integrable such that* $|f_n| \leq g$ $\mu-$ *a.e. for each* $n \in \mathbb{N}$, *then* f *is* $\mu-$*integrable and*

$$\lim_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0.$$

Theorem 1.3.11. *Assume* f *and* $\{f_n\}_{n\in\mathbb{N}}$ *are* $\mu-$*measurable and*

$$\lim_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0.$$

Then there exists a subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ *of* $\{f_n\}_{n\in\mathbb{N}}$ *such that*

$$\lim_{j\to\infty} f_{n_j}(x) = f(x) \quad \mu - a.\ e.$$

The following result can be easily deduced from [LL] (Theorem 1.14 p.28).

Theorem 1.3.12 (Bathtub Principle)**.** *Let* f *be a real-valued, measurable function on* X *such that* $\mu(\{x : f(x) > t\})$ *is finite for all* $t \in \mathbb{R}$. *Let the number* $G > 0$ *be given and define the class* C *of measurable functions on* X *by*

$$C = \left\{ g \colon 0 \leq g(x) \leq 1 \;\forall\, x \text{ and } \int_X g(x) \,\mathrm{d}\mu = G \right\}.$$

Then the maximization problem

$$I = \sup \left\{ \int_X f(x) g(x) \,\mathrm{d}\mu \colon g \in C \right\}$$

is solved by

$$g(x) = \chi_{\{y\colon f(y)>s\}}(x) + c\chi_{\{y\colon f(y)=s\}}(x), \tag{1.1}$$

where

$$s = \inf\{t : \mu(\{x\colon f(x) \geq t\}) \leq G\}$$

and

$$c\mu(\{x\colon f(x) = s\}) = G - \mu(\{x\colon f(x) > s\}).$$

The maximizer given in (1.1) *is unique if* $G = \mu(\{x\colon f(x) > s\})$ *or if* $G = \mu(\{x\colon f(x) \geq s\})$.

1.4 L^p−spaces

Throughout this section X is a nonempty subset of $\mathbb{R}^N$ and μ is positive measure on X.

Let p be a positive real number. We denote by $L^p(X,\mu)$ the class of all μ−measurable functions f, defined on X, for which

$$\int_X |f|^p \, d\mu < \infty.$$

When μ is the Lebesgue measure, $L^p(X)$ is used in place of $L^p(X,\mu)$.

The functional $\|\cdot\|_{L^p(X,\mu)}$ defined by

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$$

is a norm on $L^p(X,\mu)$ provided $1 \leq p < \infty$.

By $L^p_{loc}(X,\mu)$ we denote the set of all μ−measurable function f defined μ−a.e. on X, for which $f \in L^p(K,\mu)$ for every compact set $K \subset X$

A function f, μ−measurable on X, is said to be *essentially bounded* on Ω provided there exists a constant K for which $|f(x)| \leq K$ μ−a.e. on X. The greatest lower bound of such constants K is called the *essential supremum* of $|f|$ on X and is denoted

$$\operatorname{ess\,sup}\{|f(x)| : x \in X\}.$$

We denote by $L^\infty(X,\mu)$ the vector space consisting of all function f that are essentially bounded on X. The functional $\|\cdot\|$ defined by

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| : x \in X\}$$

is a norm on $L^\infty(X,\mu)$.

If μ is the Lebesgue measure, $L^\infty(X)$ is used in place of $L^\infty(X,\mu)$.

Let $1 \leq p \leq \infty$ we denote by p' the number

$$\begin{cases} \infty & \text{if } p = 1, \\ \dfrac{p}{p-1} & \text{if } 1 < p < \infty, \\ 1 & \text{if } p = \infty, \end{cases}$$

so that $1 \leq p' \leq \infty$ and ${}^1/_p + {}^1/_{p'} = 1$. p' is called the *exponent conjugate* to p.

For the proofs of the followings theorems, see [Ru1].

Theorem 1.4.1 (Höder's inequality)**.** *If $1 \leq p \leq \infty$ and $f \in L^p(X,\mu)$, $g \in L^{p'}(X,\mu)$ then $fg \in L^1(X,\mu)$ and*

$$\int_X |fg| \, d\mu \leq \|f\|_{L^p(X,\mu)} \|g\|_{L^{p'}(X,\mu)}.$$

Theorem 1.4.2. *Let $1 \leq p \leq \infty$, then $L^p(X,\mu)$ is a Banach space. $L^p(X,\mu)^* = L^{p'}(X,\mu)$ for all $1 \leq p < \infty$ and $L^1(X,\mu) \subset L^\infty(X,\mu)^*$.*

Corollary 1.4.3. *$L^2(X,\mu)$ is a Hilbert space with respect to the inner product*

$$\langle f, g\rangle = \int_\Omega fg\,\mathrm{d}\mu.$$

Theorem 1.4.4. *If $1 \leq p \leq \infty$, a Cauchy sequence in $L^p(X,\mu)$ has a subsequence converging pointwise almost everywhere on Ω.*

The proofs of the next theorems can be found in [B].

Theorem 1.4.5. *Let Ω be a domain in $\mathbb{R}^N$. Then $L^1(\Omega)$ is a separable Banach space and $L^p(X,\mu)$ is a separable, reflexive and uniformly convex Banch space for each $1 < p < \infty$.*

Theorem 1.4.6. *Let Ω be a domain in $\mathbb{R}^N$. Then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ if $1 \leq p < \infty$.*

Proposition 1.4.7. *Let Ω be a domain in $\mathbb{R}^N$ and let $A > 0$. The set*

$$\left\{\phi \in L^\infty(\Omega)\colon\ 0 \leq \phi \leq 1 \text{ and } \int_\Omega \phi(x)\,\mathrm{d}x = A\right\}$$

is the closure in the weak$*$ topology in $L^\infty(\Omega)$ of the set of characteristic functions

$$\{\chi_E\colon\ E \subset \Omega \text{ and } |E| = A\}.$$

1.5 Rearrangements of functions

Here, we recall some well-known facts concerning the rearrangements of functions. They can be found, for instance, in [Bu1, Bu2].

Throughout the section, Ω is a domain in $\mathbb{R}^N$, $\alpha \in \{N-1, N\}$ and

$$X_\alpha = \begin{cases} \partial\Omega & \text{if } \alpha = N-1 \\ \Omega & \text{if } \alpha = N \end{cases} \quad \text{and} \quad \mathcal{H}^\alpha = \begin{cases} \mathcal{H}^{N-1} & \text{if } \alpha = N-1 \\ \mathcal{H}^N & \text{if } \alpha = N. \end{cases}$$

Definition 1.5.1. Given two functions $f, g : X_\alpha \to \mathbb{R}$ $\mathcal{H}^\alpha-$measurable, we say that f is a *rearrangement of g* if

$$\mathcal{H}^\alpha(\{x \in X_\alpha\colon\ f(x) \geq t\}) = \mathcal{H}^\alpha(\{x \in X_\alpha\colon\ g(x) \geq t\}) \quad \forall\, t \in \mathbb{R}.$$

Lemma 1.5.2. *Let $f, g : X_\alpha \to \mathbb{R}$ be $\mathcal{H}^\alpha-$measurable functions, and suppose that f is a rearrangement of g. Then*

(i) For any Borel set $A \subset \mathbb{R}$, we have $\mathcal{H}^\alpha(f^{-1}(A)) = \mathcal{H}^\alpha(g^{-1}(A))$.

(ii) *If* $\phi : \mathbb{R} \to \mathbb{R}$ *is Borel measurable then* $\phi \circ f$ *is a rearrangement of* $\phi \circ g$.

(iii) *If* $f \in L^1(X_\alpha, \mathcal{H}^\alpha)$ *then* $g \in L^1(X_\alpha, \mathcal{H}^\alpha)$ *and*

$$\int_{X_\alpha} f \, \mathrm{d}\mathcal{H}^\alpha = \int_{X_\alpha} g \, \mathrm{d}\mathcal{H}^\alpha.$$

(iv) *If* $1 \le p < \infty$ *and* $f \in L^p(X_\alpha, \mathcal{H}^\alpha)$ *then* $f \in L^p(X_\alpha, \mathcal{H}^\alpha)$ $\|f\|_{L^p(X_\alpha, \mathcal{H}^\alpha)} = \|g\|_{L^p(X_\alpha, \mathcal{H}^\alpha)}$.

Proof. See [Bu2]. □

Now, given $f_0 \in L^p(X_\alpha, \mathcal{H}^\alpha)$ the set of all rearrangements of f_0 is denoted by $\mathcal{R}(f_0)$ and $\overline{\mathcal{R}(f_0)}$ denotes the closure of $\mathcal{R}(f_0)$ in $L^p(X_\alpha, \mathcal{H}^\alpha)$ with respect to the weak topology.

Theorem 1.5.3. *Let* $1 \le p < \infty$ *and let* p' *be the conjugate exponent of* p. *Let* $f_0 \in L^p(X_\alpha, \mathcal{H}^\alpha)$, $f_0 \not\equiv 0$ *and let* $g \in L^{p'}(X_\alpha, \mathcal{H}^\alpha)$. *Then, there exists* $f_*, f^* \in \mathcal{R}(f_0)$ *such that*

$$\int_{X_\alpha} f_* g \, \mathrm{d}\mathcal{H}^\alpha \le \int_{X_\alpha} f g \, \mathrm{d}\mathcal{H}^\alpha \le \int_{X_\alpha} f^* g \, \mathrm{d}\mathcal{H}^\alpha \quad \forall f \in \overline{\mathcal{R}(f)}.$$

Proof. The proof follows from Theorem 4 in [Bu1]. □

Theorem 1.5.4. *Let* $1 \le p \le \infty$ *and let* p' *be the conjugate of* p. *Let* $f_0 \in L^p(X_\alpha, \mathcal{H}^\alpha)$, $f_0 \not\equiv 0$ *and let* $g \in L^{p'}(X_\alpha, \mathcal{H}^\alpha)$.

If the linear functional $L(f) = \int_{X_\alpha} f g \, \mathrm{d}\mathcal{H}^\alpha$ *has a unique maximizer* f^* *relative to* $\mathcal{R}(f_0)$ *then there exists an increasing function* ϕ *such that* $f^* = \phi \circ g$ $\mathcal{H}^\alpha$*–a.e. in* Ω.

Furthermore, if the linear functional $L(f)$ *has a unique minimizer* f_* *relative to* $\mathcal{R}(f_0)$ *then there exists a decreasing function* ψ *such that* $f_* = \psi \circ g$ $\mathcal{H}^\alpha$*–a.e. in* Ω.

Proof. The proof follows from Theorem 5 in [Bu1]. □

1.6 Sobolev spaces

Let Ω be a domain in $\mathbb{R}^N$ and $u \in L^1_{loc}(\Omega)$. For α a multi-index, $|\alpha| \ge 1$, the function $v_\alpha \in L^1_{loc}(\mathbb{R}^N)$ is called *weak* (or *distributional*) *derivative* of u (of order α) if the identity

$$\int_\Omega v_\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, \mathrm{d}x.$$

holds for every $\phi \in C_0^\infty(\Omega)$. Then v_α is denoted by $D^\alpha u$.

We call a function *weakly derivative* if all its weak derivatives of first order exist. Let us denote the linear space of weakly derivative function by $W^1(\Omega)$. Observe that $C^1(\Omega)$ is included in $W^1(\Omega)$.

For the proof of the following lemmas, see [GT].

Lemma 1.6.1. *Let $u \in W^1(\Omega)$. Then u^+, u^- and $|u| \in W^1(\Omega)$.*

Lemma 1.6.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $u \in W^1(\Omega)$. Then $\nabla u = 0$ a.e. on any set where u is constant.*

For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we define the *Sobolev space* by

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \colon D^\alpha u \in L^p(\Omega),\ |\alpha| \leq n\}.$$

When $p = 2$, $H^k(\Omega)$ is used in place of $W^{k,2}(\Omega)$.

The space $W^{k,p}(\Omega)$ is a Banch space if equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

We denote by $W^{k,p}_{loc}(\Omega)$ the set of all functions u defined on Ω, for which $u \in W^{k,p}(K)$ for every compact $K \subset \Omega$.

Further, the space $W^{k,p}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ in the space $W^{k,p}(\Omega)$.

Theorem 1.6.3. *$W^{k,p}(\Omega)$ is separable if $1 \leq p < \infty$, and is reflexive and uniformly convex if $1 < p < \infty$. In particular, $H^k(\Omega)$ is a separable Hilbert space with inner product*

$$\langle u, v\rangle_k = \sum_{0\leq|\alpha|\leq k} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x.$$

Proof. See [A]. □

Theorem 1.6.4. *Assume $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in $W^{k,p}(\Omega) \cap C^\infty(\Omega)$ such that*

$$\phi_n \to u \quad \textit{strongly in } W^{k,p}(\Omega).$$

Proof. See [EG]. □

Theorem 1.6.5 (Rellich–Kondrachov Theorem)**.** *Let Ω be a Lipschitz bounded domain. Then,*

- *if $1 < p < N$, $W^{1,p}(\Omega)$ is embedded in $L^q(\Omega)$ for all $q \in [1, p^*)$ where $p^* = {}^{Np}/_{(N-p)}$,*
- *if $p = N$, $W^{1,p}(\Omega)$ is embedded in $L^q(\Omega)$ for all $q \in [1, +\infty)$,*
- *if $p > N$, $W^{1,p}(\Omega)$ is embedded in $C(\overline{\Omega})$.*

Moreover, all the embeddings are compact.

Proof. See [B]. □

Theorem 1.6.6 (Trace Theorem)**.** *Assume Ω is a Lipschitz bounded domain and $1 \leq p < \infty$. There exist a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial U)$ such that $Tu = u$ on $\partial\Omega$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Furthermore, for all $\phi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and $u \in W^{1,p}(\Omega)$,*

$$\int_\Omega u \operatorname{div} \phi \, \mathrm{d}x = -\int_\Omega \nabla u \cdot \phi \, \mathrm{d}x + \int_{\partial\Omega} \langle \phi, \nu \rangle T u \, \mathrm{d}\mathcal{H}^{N-1},$$

where ν denoting the unit outer normal to $\partial\Omega$.

Proof. See [EG]. □

The function Tu is called the *trace* of u on $\partial\Omega$.

Theorem 1.6.7 (Sobolev Trace Embedding Theorem)**.** *Let Ω be a Lipschitz bounded domain. Then $W^{1,p}(\Omega)$ is embedded in $L^q(\partial\Omega)$ for all $q \in [1, p_*)$ where*

$$\begin{cases} p_* = \frac{p(N-1)}{N-p} & \text{if } 1 < p < N, \\ p_* = +\infty & \text{if } p \geq N. \end{cases}$$

Moreover, the embedding is compact.

Proof. See [GT]. □

1.7 Spherical Symmetrization

In this section, we consider the case where Ω is the unit ball, $\Omega = B(0, 1)$.

Spherical symmetrization of a measurable set. Given a measurable set $A \subset \mathbb{R}^N$, the spherical symmetrization A^* of A is constructed as follows: for each r, take $A \cap \partial B(0, r)$ and replace it by the spherical cap of the same area and center re_N. This can be done for almost every r. The union of these caps is A^*.

Now, we define *spherical symmetrization of measurable function*. Given a measurable function $u \geq 0$. The spherical symmetrization u^* of u is constructed by symmetrizing the super-level sets so that, for all t, $\{u^* \geq t\} = \{u \geq t\}^*$. See [K, Sp].

The following theorem is proved in [K] (see also [Sp]).

Theorem 1.7.1. *Let $u \in W^{1,p}(B(0,1))$ and let u^* be its spherical symmetrization. Then $u^* \in W^{1,p}(B(0,1))$ and*

$$\begin{aligned}
\int_{B(0,1)} |\nabla u^*|^p \, \mathrm{d}x &\leq \int_{B(0,1)} |\nabla u|^p \, \mathrm{d}x, \\
\int_{B(0,1)} |u^*|^p \, \mathrm{d}x &= \int_{B(0,1)} |u|^p \, \mathrm{d}x, \\
\int_{\partial B(0,1)} |u^*|^p \, \mathrm{d}\mathcal{H}^{N-1} &= \int_{\partial B(0,1)} |u|^p \, \mathrm{d}\mathcal{H}^{N-1}, \\
\int_{B(0,1)} (\alpha\chi_D)_* |u^*|^p \, \mathrm{d}x &\leq \int_{B(0,1)} \alpha\chi_D |u|^p \, \mathrm{d}x,
\end{aligned} \tag{1.2}$$

where $D \subset B(0,1)$ and $(\alpha\chi_D)_* = -(-\alpha\chi_D)^*$.

1.8 $p-$Laplace equations

In this section we give some results regarding solutions of some $p-$Laplace type equations.

Given Ω a smooth bounded domain, $1 < p < \infty$ and $V \in L^q(\Omega)$ $(1 \leq q < \infty)$, consider the operator H_V, which has the form

$$H_V u := -\Delta_p u + V(x)|u|^{p-2}u, \tag{1.3}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the usual *$p-$Laplacian*. Suppose that $u \in W^{1,p}(\Omega)$ and V is a measurable function that satisfy the following assumptions:

$$V \in L^q(\Omega) \text{ where } \begin{cases} q > \frac{N}{p} & \text{if } 1 < p \leq N, \\ q = 1 & \text{if } p > N. \end{cases} \tag{H1}$$

We say u is a *weak solution of $H_V u = 0$* $(\geq 0, \leq 0)$ in Ω if

$$\mathcal{D}(u,v) := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, \mathrm{d}x + \int_\Omega V(x)|u|^{p-2}uw \, \mathrm{d}x = 0 \ (\leq 0, \ \geq 0), \tag{1.4}$$

for each $w \in C_0^1(\Omega)$. Let $f \in L^{p'}(\Omega)$, $u \in W^{1,p}(\Omega)$ is a *weak solution of the equation*

$$H_V u = f \quad \text{ in } \Omega, \tag{1.5}$$

if

$$\mathcal{D}(u,w) = \mathcal{J}(w) := \int_\Omega f w \, \mathrm{d}x \qquad \forall\, w \in C_0^1(\Omega). \tag{1.6}$$

The aim of this section is to study the Dirichlet problem for the equation (1.5).

We say $u \in W^{1,p}(\Omega)$ is a *weak solution of the Dirichlet problem*

$$\begin{cases} H_V(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

if u is a weak solution of (1.5) and $u \in W_0^{1,p}(\Omega)$.

Note that

$$\begin{aligned} |\mathcal{D}(u,w)| &\leq \|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla w\|_{L^p(\Omega)} + \int_\Omega (|V(x)|^{\frac{1}{p'}}|u|^{p-1})(|V(x)|^{\frac{1}{p}}|w|)\,\mathrm{d}x \\ &\leq \|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla w\|_{L^p(\Omega)} + \left(\int_\Omega |V(x)||u|^p\,\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_\Omega |V(x)||w|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla w\|_{L^p(\Omega)} + C\|V\|_{L^q(\Omega)}\|u\|_{W^{1,p}(\Omega)}^{p-1}\|w\|_{W^{1,p}(\Omega)} \\ &\leq (1 + C\|V\|_q)\|u\|_{W^{1,p}(\Omega)}^{p-1}\|w\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Hence, for fixed $u \in W^{1,p}(\Omega)$, the mapping $w \mapsto \mathcal{D}(u,w)$ is a bounded linear functional on $W_0^{1,p}(\Omega)$. Consequently the validity of the relations (1.4) for $w \in C_0^1(\Omega)$ imply their validity for $w \in W_0^{1,p}(\Omega)$. We remark that, for fixed $u \in W^{1,p}(\Omega)$, $H_V u$ may be defined as an element of the dual space of $W_0^{1,p}(\Omega)$, $W^{-1,p'}(\Omega)$, $H_V u(w) = \mathcal{D}(u,w)$, $w \in W_0^{1,p}(\Omega)$, and hence the Dirichlet problem (1.7) can be studied for $f \in W^{-1,p'}(\Omega)$.

1.8.1 Solvability of the Dirichlet problem

We need the following notation:

$$S_q := \inf\left\{ \frac{\int_\Omega |\nabla v|^p\,\mathrm{d}x}{\left(\int_\Omega |v|^q\,\mathrm{d}x\right)^{\frac{p}{q}}} : v \in W_0^{1,p}(\Omega) \right\}. \tag{1.8}$$

This constant S_q is positive and is the best (largest) constant in the Sobolev–Poincaré inequality

$$S\|v\|_{L^q(\Omega)}^p \leq \|\nabla v\|_{L^p(\Omega)}^p \qquad \forall\, v \in W_0^{1,p}(\Omega).$$

We have the following,

Theorem 1.8.1. *Let V be a measurable function that satisfy the assumptions (H1) and*

$$\|V^-\|_{L^q(\Omega)} < S_{pq'} \quad \text{or} \quad V \geq -S_p + \delta \quad \text{for some } \delta > 0. \tag{H2}$$

Then the Dirichlet problem (1.7) has a unique weak solution for any $f \in L^{p'}(\Omega)$.

Proof. The proof of this theorem is standard. First observe that the weak solutions of (1.7) are critical points of the functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\psi(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{p}\int_\Omega V(x)|u|^p \, dx - \int_\Omega f u \, dx.$$

Now, it is easy to see that ψ is bounded below, coercive, strictly convex and sequentially weakly lower semi continuous. Therefore it has a unique critical point which is a global minimum. □

It is proved in [GV] that solutions to (1.7) are bounded. We state the Theorem for future reference.

Theorem 1.8.2 ([GV], Proposition 1.3). *Assume $1 < p \le N$, $f \in L^q(\Omega)$ for some $q > N/p$ and $u \in W_0^{1,p}(\Omega)$ is a solution to (1.7). Then $u \in L^\infty(\Omega)$ and there exists a constant $C = C(N, p, |\Omega|)$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^q(\Omega)}^{1/(p-1)}.$$

1.8.2 The strong maximum principle

Here we recall the classical maximum principles for H_V.

Theorem 1.8.3 (Weak Maximum Principle). *Let V be a measurable function that satisfy the assumptions (H1) and (H2), $f \in L^{p'}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$ be the weak solution of (1.7). Then $f \ge 0$ implies $u \ge 0$ in Ω.*

Proof. The proof follows using u^- as a test function in the weak formulation of (1.7). See [GT] for the case $p = 2$. Here is analogous. □

For the strong maximum principle, we need the following result

Theorem 1.8.4 (Harnack's Inequality). *Let u be a weak solution of problem (1.7) in a cube $K = K(3\rho) \subset \Omega$, with $0 \le u < M$ in K. Then*

$$\max\{u(x) : x \in K(\rho)\} \le C \min\{u(x) : x \in K(\rho)\},$$

where $C = C(N, M, \rho)$.

Proof. See [Tr]. □

Now we can prove the strong maximum principle for weak solutions of (1.7).

Theorem 1.8.5 (Strong Maximum Principle). *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of problem (1.7). Then, if $f \ge 0$, $f \ne 0$,*

$$u > 0 \text{ in } \Omega.$$

Proof. It follows from Theorems 1.8.2, 1.8.3 and 1.8.4. □

1.8.3 The weighted eigenvalue problems

In this subsection we analyse the (nonlinear) weighted eigenvalue problems,

$$\begin{cases} H_V u = \lambda g |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

with $g \in L^q(\Omega)$.

Theorem 1.8.6. *Assume that V and g satisfy (H1). If V satisfies the assumption (H2) and $g^+ \not\equiv 0$, then there exists a unique positive principal weighted eigenvalue $\lambda(g, V)$ of (1.9) and it is characterized by*

$$\lambda(V, g) := \inf \left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V(x)|u|^p \,\mathrm{d}x \colon u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega g u \,\mathrm{d}x = 1 \right\}.$$

Proof. See [CRQ]. □

Obviously, if u is a minimizer, so is $|u|$; therefore we may assume $u \geq 0$.

When $g \equiv 1$, $\lambda(V)$ is used in place of $\lambda(1, V)$ and $\lambda(V)$ is called the first eigenvalue (or simply eigenvalue).

In the case $g \equiv 1$, we can relax the assumption for V.

Theorem 1.8.7. *Let V be a measurable functions that satisfy the assumptions (H1). Then there exists $u_0 \in W_0^{1,p}(\Omega)$ such that*

$$\begin{cases} \lambda(V) = \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V(x)|u_0|^p \,\mathrm{d}x \\ \|u\|_{L^p(\Omega)} = 1. \end{cases}$$

Moreover, u_0 is a weak solution of (1.9) with $\lambda = \lambda(V)$. Finally, $\lambda(V)$ is the lowest eigenvalue of (1.9) with $g \equiv 1$.

For the proof we need the following Lemma

Lemma 1.8.8. *Assume V be a measurable function that satisfy the assumptions (H1). Then, given $\varepsilon > 0$, there exists a constant $D_\varepsilon > 0$ such that*

$$\left| \int_\Omega V(x)|v|^p \,\mathrm{d}x \right| \leq \varepsilon \int_\Omega |\nabla v|^p \,\mathrm{d}x + D_\varepsilon \|V\|_{L^q(\Omega)} \int_\Omega |v|^p \,\mathrm{d}x,$$

for any $v \in W_0^{1,p}(\Omega)$.

Proof. First we assume that $1 \leq p \leq N$. Let us observe that $q > N/p$ implies that $pq' < p^*$. Now the Lemma follows from Hölder's inequality and the Sobolev embedding. In fact, let us see that if $1 < r < p^*$, there exists a constant M_ε such that

$$\|v\|_{L^r(\Omega)} \leq \varepsilon \|\nabla v\|_{L^p(\Omega)} + M_\varepsilon \|v\|_{L^p(\Omega)} \qquad \forall\, v \in W_0^{1,p}(\Omega). \tag{1.10}$$

Assume (1.10) does not hold, then there exists $\varepsilon_0 > 0$ and a sequence $\{v_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p}(\Omega)$ such that $\|v_n\|_{L^r(\Omega)} = 1$ and

$$\varepsilon_0\|\nabla v_n\|_{L^p(\Omega)} + n\|v_n\|_{L^p(\Omega)} < 1$$

for all $n \in \mathbb{N}$. But then $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ and $\|v_n\|_{L^p(\Omega)} \to 0$. Now, by the Rellich–Kondrachov Theorem, up to a subsequence, $v_n \to v$ strongly in $L^r(\Omega)$, and so $\|v\|_{L^r(\Omega)} = 1$. A contradiction.

Now, it is easy to check that (1.10) implies the lemma since $q > N/p$.

If $p > N$, the proof is similar to above case and is left to the reader. □

Proof of Theorem 1.8.7. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be a minimizing sequence for $\lambda(V)$, i.e.,

$$\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega V(x)|u_n|^p\,\mathrm{d}x \to \lambda(V) \quad \text{ and } \|u_n\|_{L^p(\Omega)} = 1\ \forall\, n \in \mathbb{N}.$$

Then there exists $C > 0$ such that

$$\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega V(x)|u_n|^p\,\mathrm{d}x \le C \quad \forall\, n \in \mathbb{N}.$$

Since V satisfies the assumptions (H1), by Lemma 1.8.8, given $\varepsilon > 0$ there exists D_ε such that

$$\left|\int_\Omega V(x)|u_n|^p\,\mathrm{d}x\right| \le \varepsilon\|\nabla u_n\|^p_{L^p(\Omega)} + D_\varepsilon\|V\|_{L^q(\Omega)}\|u_n\|^p_{L^p(\Omega)},$$

for any $n \in \mathbb{N}$. Then

$$(1-\varepsilon)\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - D_\varepsilon\|V\|_q \le \int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega V(x)|u_n|^p\,\mathrm{d}x \le C \quad \forall\, n \in \mathbb{N}.$$

Fixing $\varepsilon < 1$, we get

$$\int_\Omega |\nabla u_n|^p\,\mathrm{d}x \le \frac{C + D_\varepsilon\|V\|_{L^q(\Omega)}}{1-\varepsilon}, \quad \forall\, n \in \mathbb{N}.$$

Therefore $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$.

Now, by Rellich–Kondrachov Theorem, there exists a function $u_0 \in W_0^{1,p}(\Omega)$ such that, for a subsequence that we still call $\{u_n\}_{n\in\mathbb{N}}$,

$$u_n \rightharpoonup u_0, \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{1.11}$$

$$u_n \to u_0, \quad \text{strongly in } L^p(\Omega), \tag{1.12}$$

$$u_n \to u_0, \quad \text{strongly in } L^{pq'}(\Omega). \tag{1.13}$$

By (1.12), $\|u_0\|_{L^p(\Omega)} = 1$ so $u_0 \ne 0$ and by (1.11) and (1.13)

$$\lambda(V) = \lim_{n\to\infty}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega V(x)|u_n|^p\,\mathrm{d}x \ge \int_\Omega |\nabla u_0|^p\,\mathrm{d}x + \int_\Omega V(x)|u_0|^p\,\mathrm{d}x.$$

It is clear that u_0 is an eigenfunction of H_V with eigenvalue $\lambda(V)$.

Finally, let λ be an eigenvalue of problem (1.9) with associated eigenfunction $w \in W_0^{1,p}(\Omega)$. Then

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla w|^p \, \mathrm{d}x + \int_\Omega V(x)|w|^p \, \mathrm{d}x}{\displaystyle\int_\Omega |w|^p \, \mathrm{d}x} \geq \lambda(V).$$

This finishes the proof. □

Now, we prove that u_0 has constant sign in Ω.

Lemma 1.8.9 ([C], Proposition 3.2)**.** *Let g and V be two measurable functions that satisfy the assumption* (H1)*. If $u \in W_0^{1,p}(\Omega)$ is a non-negative weak solution to* (1.9) *then either $u \equiv 0$ or $u > 0$ for all $x \in \Omega$.*

Proof. The proof is a direct consequence of Harnack's inequality. See [S]. □

We therefore immediately obtain,

Corollary 1.8.10. *Under the assumptions of the previous Lemma, every eigenfunction associated to the principal positive eigenvalue has constant sign.*

Now, we recall a couple of results regarding the eigenvalue problem (1.9) when $g \equiv 1$. We do not use these results in the rest of the thesis, but we include them here for completeness.

Proposition 1.8.11. *If V satisfies the assumption* (H1) *and $g \equiv 1$, then there exists a increasing, unbounded sequence of eigenvalues for the problem* (1.9).

Proof. It is similar to [GAPA1, GAPA2]. □

Proposition 1.8.12. *If V satisfies the assumption* (H1) *and $g \equiv 1$, then $\lambda(V)$ is isolated in the spectrum.*

Proof. It is similar to [C]. □

Lastly, following [CRQ], we have that the principal eigenvalue $\lambda(g, V)$ is simple. This is, the only eigenfunction of H_V associated to $\lambda(V, g)$ are multiples of a single one, u_0.

Lemma 1.8.13. *Let g and V be two measurable functions that satisfy the assumption* (H1)*. Let u and v be two eigenfunction associated to $\lambda(g, V)$. Then, there exists a constant $c \in \mathbb{R}$ such that $u = cv$.*

1.9 BV functions

Throughout, this section Ω denote an open subset of $\mathbb{R}^N$.

We say that a function $f \in L^1(\Omega)$ has *bounded variation* in Ω if

$$\sup\left\{\int_\Omega f \operatorname{div}\varphi \,\mathrm{d}x \colon \varphi \in C_c^1(\Omega;\mathbb{R}^N) \text{ and } |\phi| \le 1\right\} < \infty.$$

The space of function of bounded is denoted by

$$BV(\Omega).$$

A measurable subset $E \subset \mathbb{R}^N$ has *finite perimeter* in Ω if

$$\chi_E \in BV(\Omega).$$

A function $f \in L^1_{loc}(\Omega)$ has *locally bounded variation* in Ω if for each open set $U \subset\subset \Omega$

$$\sup\left\{\int_\Omega f \operatorname{div}\varphi \,\mathrm{d}x \colon \varphi \in C_c^1(U;\mathbb{R}^N) \text{ and } |\phi| \le 1\right\} < \infty.$$

We write

$$BV_{loc}(\Omega).$$

A measurable subset $E \subset \mathbb{R}^N$ has *locally finite perimeter* in Ω if

$$\chi_E \in BV_{loc}(\Omega).$$

Now, we give the structure theorem.

Theorem 1.9.1 (Structure theorem for BV_{loc} functions)**.** *Let $f \in BV_{loc}(\Omega)$. Then there exists a Radon measure μ on Ω and $\mu-$measurable function $\sigma : \Omega \to \mathbb{R}^N$ such that*

1. *$|\sigma(x)| = 1$ $\mu-$a.e., and*
2. $\int_\Omega f \operatorname{div}\varphi \,\mathrm{d}x = -\int_\Omega \langle\varphi,\sigma\rangle \,\mathrm{d}\mu$

for all $\varphi \in C_c^1(\Omega;\mathbb{R}^N)$.

Proof. See [EG]. □

If $f \in BV_{loc}(\Omega)$, we will henceforth write

$$\|Df\|$$

for the measure μ, and

$$[Df] = \|Df\| \llcorner \sigma$$

Hence the assertion 2 in above theorem reads

$$\int_\Omega f \operatorname{div}\varphi \, \mathrm{d}x = -\int_\Omega \langle \varphi, \sigma \rangle \, \mathrm{d}\mu = -\int_\Omega \varphi \, \mathrm{d}[Df] \quad \forall \varphi \in C_c^1(\Omega; \mathbb{R}^N).$$

In the case that $f = \chi_E$, and E has locally finite perimeter in Ω, we will hereafter write

$$\|\partial E\|$$

for the measure μ, and

$$\nu_E \equiv -\sigma$$

Consequently,

$$\int_E \operatorname{div}\phi \, \mathrm{d}x = \int_\Omega \langle \phi, \nu_E \rangle \, \mathrm{d}\|\partial E\| \quad \forall \varphi \in C_c^1(\Omega; \mathbb{R}^N).$$

We now give a characterization of the measure $\|Df\|$, for the proof see [EG].

Theorem 1.9.2. *Given $f \in BV_{loc}(\Omega)$. For each $V \subset\subset \Omega$, we have*

$$\|Df\|(V) = \sup \left\{ \int_E \operatorname{div}\varphi \, \mathrm{d}x \colon \varphi \in C_c^1(V; \mathbb{R}^N), |\varphi| \leq 1 \right\}.$$

Example 1.9.3. Assume E is a smooth, open subset of $\mathbb{R}^N$ and $\mathcal{H}^{N-1}(\partial E \cap K) < \infty$ for each compact set $K \subset \Omega$. Then, for each $V \subset\subset \Omega$ and $\varphi \in C_c^1(V, \mathbb{R}^N)$, with $|\varphi| \leq 1$, we have that

$$\int_E \operatorname{div}\varphi \, \mathrm{d}x = -\int_{\partial E} \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{N-1},$$

ν denoting the outward unit normal along ∂E.

Hence

$$\int_E \operatorname{div}\varphi \, \mathrm{d}x = \int_{\partial E \cap V} \langle \varphi, \nu \rangle \, \mathrm{d}x \leq \mathcal{H}^{N-1}(\partial E \cap V) < \infty.$$

Thus $\chi_E \in BV_{loc}(\Omega)$. Moreover,

$$\|\partial E\|(\Omega) = \mathcal{H}^{N-1}(\partial E \cap \Omega)$$

and

$$\nu_E = \nu \quad \mathcal{H}^{N-1} - \text{a.e. on } \partial E \cap \Omega.$$

Our next aim is to give a Gauss-Green theorem for sets with locally finite perimeter in $\mathbb{R}^N$.

Let E be a set of locally finite perimeter in $\mathbb{R}^N$. The subset of the topological boundary ∂E defined by

$$\partial_* E := \left\{ x \in \mathbb{R}^N \colon \limsup_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} > 0 \text{ and } \limsup_{r \to 0} \frac{|B(x,r) \setminus E|}{|B(x,r)|} > 0 \right\}.$$

is called the *measure theoretic boundary* of E.

Remark 1.9.4. The measure theoretic boundary may differ from the topological boundary of a set of non-null $\mathcal{H}^{N-1}-$measure. Indeed, for example, if $N = 2$ we consider

$$E = B(0,1) \setminus \{(x,y)\colon\ x = 0, 0 \le y < 1\}.$$

Then, $\partial_* E$ is the sphere but ∂E is the union of the sphere and $\{(x,y)\colon\ x = 0, 0 \le y \le 1\}$.

Lastly, we give the generalized Gauss-Green theorem, for the proof [AGM].

Theorem 1.9.5 (Gauss-Green Theorem)**.** *Let $E \subset \mathbb{R}^N$ have locally finite perimeter.*

1. *Then $\mathcal{H}^{N-1}(\partial_* E \cap K) < \infty$ for each compact set $K \subset \mathbb{R}^N$.*
2. *Furthermore, for $\mathcal{H}^{N-1}-$a.e. $x \in \partial_* E$, there exist a unique unit vector $\nu_E(x)$, called the generalized outer normal vector to E at x, such that*

$$\int_E \operatorname{div} \phi \, \mathrm{d}x = \int_{\partial_* E} \langle \phi, \nu_E \rangle \, \mathrm{d}\mathcal{H}^{N-1}$$

for all $\phi \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$.

2

The first eigenvalue of the $p-$Laplacian plus a potential

Here, we consider Schröedinger operators, that is elliptic operators L under perturbations given by a potential V, in bounded regions. These operators appear in different fields of applications such as quantum mechanics, stability of bulk matter, scattering theory, etc.

In Ashbaugh–Harrell [AsHa] the following problem is studied: Let L be a uniformly elliptic linear operator and assume that $\|V\|_{L^q(\Omega)}$ is constrained but otherwise the potential V is arbitrary. Can the maximal value of the first (fundamental) eigenvalue for the operator $L + V$ be estimated? And the minimal value? There exists *optimal* potentials? (i.e. potentials V^* and V_* such that the first eigenvalue for $L + V^*$ is maximal and the first eigenvalue for $L + V_*$ is minimal).

In [AsHa] these questions are answered in a positive way and, moreover, a characterization of these optimal potentials is given.

The purpose of this first chapter is the extension of the results of [AsHa] to the nonlinear case. We are also interested in extending these results to degenerate/singular operators. As a model of these operators, we take the $p-$Laplacian. This operator has been intensively studied in recent years and is a model for the study of degenerated operators (if $p > 2$) and singular operators (if $1 < p < 2$). In the case $p = 2$ it agrees with the usual Laplacian. This operator also serves as a model in the study of non-Newtonian fluids. See [ADT, AE].

Here we prove that, if one consider perturbations of the $p-$Laplacian by a potential V with $\|V\|_{L^q(\Omega)}$ constrained, then there exists optimal potentials in the sense described above and a characterizations of these potentials are given.

We want to remark that the proofs are not straightforward extensions of those in [AsHa] since the proofs there are not, in general, variational. Moreover, some new technical difficulties arise since solutions to a $p-$Laplace type equation are not regular and, mostly, since the eigenvalue problem for the $p-$Laplacian is far from being completely understood.

The rest of the chapter is divided into four sections. In Section 2.1, we introduce the exact problem that we will study trough this chapter. Section 2.2, we prove some properties of the first eigenvalue of H_V respect to V. Finally, in Section 2.3 and 2.4, we analyze the existence and characterization problem for maximal potential and minimal potential, respectively.

2.1 The problems

Let $\Omega \subset \mathbb{R}^N$ be a connected smooth bounded domain. We consider the differential operator

$$H_V u := -\Delta_p u + V(x)|u|^{p-2}u$$

where $V \in L^q(\Omega)$ $(1 \le q \le +\infty)$ and $1 < p < +\infty$. Let $\lambda(V)$ be the lowest eigenvalue of H_V in $W_0^{1,p}(\Omega)$.

In this chapter we analyze the following problems: If $B \subset L^q(\Omega)$ is a convex, bounded and closed set,

1. find $\sup\{\lambda(V)\colon V \in B\}$ and $V \in B$, if any, where this value is attained,
2. find $\inf\{\lambda(V)\colon V \in B\}$ and $V \in B$, if any, where this value is attained.

Here, we answer these questions positively, following the approach of Ashbaugh–Harrell's work for the case $p = 2$ and $1 \le N \le 3$, see [AsHa, H].

2.2 Some properties of eigenvalue

We begin by proving some important properties of $\lambda(\cdot)$.

Lemma 2.2.1. *$\lambda : B \to \mathbb{R}$ is concave.*

Proof. Throughout the proof, $\mathcal{A}$ stand for the set

$$\{u \in W_0^{1,p}(\Omega)\colon \|u\|_{L^p(\Omega)} = 1\}.$$

Let $V_1, V_2 \in B$ and $0 \le t \le 1$. Then

$$\begin{aligned}
\lambda\,(tV_1 + (1-t)V_2) &= \inf\left\{\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega (tV_1 + (1-t)V_2)\,u\,\mathrm{d}x\colon u \in \mathcal{A}\right\}\\
&\ge t\inf\left\{\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega V_1 u\,\mathrm{d}x\colon u \in \mathcal{A}\right\}\\
&+ (1-t)\inf\left\{\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega V_2 u\,\mathrm{d}x\colon u \in \mathcal{A}\right\}\\
&= t\lambda(V_1) + (1-t)\lambda(V_2),
\end{aligned}$$

as we wanted to prove. □

Next we set M for which $\|V\|_{L^q(\Omega)} \le M$ for all $V \in B$.

Proposition 2.2.2. *There exists a constant $C > 0$, depending only on p, q, M and Ω such that*

$$\lambda(V) \le C \quad \forall\, V \in B.$$

Proof. Let $u_0 \in C_0^1(\Omega)$ be such that $\|u_0\|_{L^p(\Omega)} = 1$.

$$\begin{aligned}
\lambda(V) &\le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V(x)|u_0|^p \,\mathrm{d}x \\
&\le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \|u_0\|^p_{L^\infty(\Omega)} \int_\Omega V(x)\,\mathrm{d}x \\
&\le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \|u_0\|^p_{L^\infty(\Omega)} |\Omega|^{\frac{1}{q'}} \|V\|_q \\
&\le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \|u_0\|^p_{L^\infty(\Omega)} |\Omega|^{\frac{1}{q'}} M \\
&= C(p, q, M, \Omega).
\end{aligned}$$

□

2.3 Maximizing potentials

In this section we prove that there exists an unique $V^* \in B$ such that

$$\lambda(V^*) = \sup\{\lambda(V) \colon V \in B\}$$

and we characterize it.

Theorem 2.3.1. *Let $q > \max\{N/p, 1\}$. Then there exists $V^* \in B$ that maximizes $\lambda(V)$. Moreover if $V_i \in B$, $i = 1, 2$, are two maximizing potentials and $u_i \in W_0^{1,p}(\Omega)$, $i = 1, 2$, are the eigenfunction of H_{V_i} associated to $\lambda(V_i)$ respectively, then $u_1 = u_2$ a.e. in Ω and $V_1 = V_2$ a.e. in Ω.*

Proof. Let $\lambda^* = \sup\{\lambda(V) \colon V \in B\}$ and let $\{V_n\}_{n\in\mathbb{N}}$ be a maximizing sequence in B, i.e.,

$$\lim_{n\to\infty} \lambda(V_n) = \lambda^*.$$

Note that, by Proposition 2.2.2, λ^* is finite. As $\{V_n\}_{n\in\mathbb{N}} \subset B$ and B is bounded, there exists $V^* \in L^q(\Omega)$ and a subsequence of $\{V_n\}_{n\in\mathbb{N}}$, which we denote again by $\{V_n\}_{n\in\mathbb{N}}$, such that

$$V_n \rightharpoonup V^* \qquad \text{weakly in } L^q(\Omega).$$

By Mazur's Theorem, $V^* \in B$.

Let us see that $\lambda^* = \lambda(V^*)$. Given $\varepsilon > 0$, there exists $u_0 \in C_0^1(\Omega)$ such that

$$\lambda(V^*) \ge \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V^*(x)|u_0|^p \,\mathrm{d}x - \varepsilon.$$

Since Ω is bounded,

$$\lim_{n\to\infty}\int_\Omega V_n(x)|u_0|^p\,\mathrm{d}x = \int_\Omega V^*(x)|u_0|^p\,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
\lambda(V^*) + \varepsilon &\geq \int_\Omega |\nabla u_0|^p\,\mathrm{d}x + \int_\Omega V^*(x)|u_0|^p\,\mathrm{d}x \\
&= \int_\Omega |\nabla u_0|^p\,\mathrm{d}x + \lim_{n\to\infty}\int_\Omega V_n(x)|u_0|^p\,\mathrm{d}x \\
&= \lim_{n\to\infty}\int_\Omega |\nabla u_0|^p\,\mathrm{d}x + \int_\Omega V_n(x)|u_0|^p\,\mathrm{d}x \\
&\geq \lim_{n\to\infty}\lambda(V_n) \\
&= \lambda^*.
\end{aligned}$$

Then, as $V^* \in B$, $\lambda(V^*) = \lambda^*$.

We have just proved existence. Let us now show uniqueness.

Suppose that we have V_1 and V_2 two maximizing potentials and let $V_3 = \frac{V_1+V_2}{2}$. Since B is convex and $\lambda(\cdot)$ is concave, we have $V_3 \in B$ and

$$\lambda(V_3) \geq \frac{\lambda(V_1)+\lambda(V_2)}{2} = \lambda^*,$$

therefore V_3 is also a maximizing potential.

We denote the associated normalized, positive eigenfunction by u_1, u_2 and u_3 respectively. If $u_3 \neq u_1$ or $u_3 \neq u_2$, since, by Theorem 1.8.13, there exists only one normalized nonnegative eigenfunction,

$$\begin{aligned}
\lambda^* &= \lambda(V_3) \\
&= \int_\Omega |\nabla u_3|^p\,\mathrm{d}x + \int_\Omega V_3(x)|u_3|^p\,\mathrm{d}x \\
&= \frac{1}{2}\left(\int_\Omega |\nabla u_3|^p\,\mathrm{d}x + \int_\Omega V_1(x)|u_3|^p\,\mathrm{d}x + \int_\Omega |\nabla u_3|^p\,\mathrm{d}x + \int_\Omega V_2(x)|u_3|^p\,\mathrm{d}x\right) \\
&> \frac{\lambda(V_1)+\lambda(V_2)}{2} \\
&= \lambda^*,
\end{aligned}$$

a contradiction. Thus $u_1 = u_2 = u_3$. Now we have,

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u\nabla v\,\mathrm{d}x + \int_\Omega V_1(x)|u_1|^{p-2}u_1 v = \int_\Omega \lambda^*|u_1|^{p-2}u_1 v\,\mathrm{d}x, \tag{2.1}$$

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u\nabla v\,\mathrm{d}x + \int_\Omega V_2(x)|u_1|^{p-2}u_1 v = \int_\Omega \lambda^*|u_1|^{p-2}u_1 v\,\mathrm{d}x, \tag{2.2}$$

for all $v \in W_0^{1,p}(\Omega)$. Subtracting (2.2) from (2.1), we get

$$\int_\Omega (V_1(x) - V_2(x))\,|u_1|^{p-2}u_1 v\,\mathrm{d}x = 0 \qquad \forall\, v \in W_0^{1,p}(\Omega),$$

then

$$(V_1(x) - V_2(x))\,|u_1|^{p-2}u_1 = 0 \qquad \text{a.e. in } \Omega,$$

and therefore $V_1 = V_2$ a.e. in Ω. □

Remark 2.3.2. In the proof of Theorem 2.3.1 we only have used $q > \max\{{}^N\!/_p, 1\}$ to show the existence of an eigenfunction for the lowest eigenvalue.

Assume now that the convex set B is the ball in $L^q(\Omega)$. Then we can prove that

$$\lambda^*(M) := \max\left\{\lambda(V) \colon\ V \in L^q(\Omega) \text{ and } \|V\|_q \le M\right\}$$

is increasing in M. We will need this in the sequel.

Theorem 2.3.3. *Let* $\lambda^* : \mathbb{R}_{\ge 0} \to \mathbb{R}$

$$\lambda^*(M) := \max\left\{\lambda(V) \colon\ V \in L^q(\Omega) \textit{ and } \|V\|_q \le M\right\}.$$

Then $\lambda^*(\cdot)$ *increases monotonically.*

Proof. Let $0 \le M_1 < M_2$. Then, by Theorem 2.3.1, there exists $V_1 \in \overline{B(0, M_1)}$ such that $\lambda^*(M_1) = \lambda(V_1)$. Since $\|V_1\|_{L^q(\Omega)} \le M_1 < M_2$, there exists $t \in \mathbb{R}_{>0}$ such that

$$\|V_1 + t\|_{L^q(\Omega)} \le M_2.$$

Now, given $u \in W_0^{1,p}(\Omega)$, with $\|u\|_{L^p(\Omega)} = 1$, we have

$$\begin{aligned}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega (V_1(x) + t)|u|^p\,\mathrm{d}x &= \int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega V_1(x)|u|^p\,\mathrm{d}x + t \\ &\ge \lambda(V_1) + t.\end{aligned}$$

Thus

$$\lambda(V_1 + t) \ge \lambda(V_1) + t > \lambda(V_1).$$

As $(V_1 + t) \in \overline{B(0, M_2)}$,

$$\lambda^*(M_2) \ge \lambda(V_1 + t) > \lambda(V_1) = \lambda^*(M_1).$$

Then $\lambda^*(\cdot)$ increases monotonically. □

Remark 2.3.4. In the proof that $\lambda^*(\cdot)$ increases monotonically, what is actually proved is that $\lambda^*(M) \nearrow \infty$ as $M \nearrow \infty$.

Let $q > \max\{N/p, 1\}$ and consider the case $B = \overline{B(0, M)} \subset L^q(\Omega)$, for simplicity we take $M = 1$. Observe that B is a convex, closed and bounded set.

Let $V^* \in B$ be such that

$$\lambda(V^*) = \{\lambda(V) : V \in B\}$$

and

$$V_0 = \frac{|V^*|}{\|V^*\|_{L^q(\Omega)}} \in S := \partial B$$

.

Let $u_0 \in W_0^{1,p}(\Omega)$ be a normalized eigenfunction of H_{V_0} associated to $\lambda(V_0)$, i.e., $\|u_0\|_{L^p(\Omega)} = 1$ and

$$\lambda(V_0) = \int_\Omega |\nabla u_0|^p \, \mathrm{d}x + \int_\Omega \frac{|V^*(x)|}{\|V^*\|_{L^q(\Omega)}} |u_0|^p \, \mathrm{d}x.$$

Then

$$\lambda(V_0) \geq \int_\Omega |\nabla u_0|^p \, \mathrm{d}x + \int_\Omega V^*(x)|u_0|^p \, \mathrm{d}x \geq \lambda(V^*) = \lambda^*.$$

Thus, from uniqueness, $V_0 = V^*$, from where $\|V^*\|_{L^q(\Omega)} = 1$ and $V^* \geq 0$.

Therefore, if we take $S = \partial B(0, 1)$, there exists $V_0 \geq 0$ in S such that

$$\lambda(V_0) = \max\{\lambda(V) : V \in S\} = \max\{\lambda(V) : V \in B\}.$$

We now try to characterize V_0. For this, we need the following notation: For any $V \in S$, we denote by $T_V(S)$ the tangent space of S at V. It is well known that

$$T_V(S) = \left\{ W \in L^q(\Omega) : \int_\Omega |V|^{q-2} V W \, \mathrm{d}x = 0 \right\}.$$

Now, let $W \in T_{V_0}(S)$ and $\alpha : (-1, 1) \to L^q(\Omega)$ be a differentiable curve such that

$$\alpha(t) \in S \quad \forall\, t \in (-1, 1), \quad \alpha(0) = V_0 \quad \text{and} \quad \dot{\alpha}(0) = W.$$

We denote by $V_t = \alpha(t)$ and $\lambda(t) = \lambda(\alpha(t))$.

Let $u_t \in W_0^{1,p}(\Omega)$ be the nonnegative normalized eigenfunction of H_{V_t} with eigenvalue $\lambda(t)$, i.e., $\|u_t\|_{L^p(\Omega)} = 1$ and

$$\lambda(t) = \int_\Omega |\nabla u_t|^p \, \mathrm{d}x + \int_\Omega V_t(x)|u_t|^p \, \mathrm{d}x.$$

We have the following,

Lemma 2.3.5. *$\lambda(t)$ is continuous at $t = 0$, i.e.,*

$$\lim_{t \to 0} \lambda(t) = \lambda(0) = \lambda(V_0) = \lambda^*.$$

Proof. By Proposition 2.2.2, there exists $C = C(\Omega, q, p) > 0$ such that

$$C > \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x,$$

and as $q > {}^N\!/_p$, by Lemma 1.8.8, given $\varepsilon > 0$ there exists D_ε such that

$$\left|\int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x\right| \leq \varepsilon \|\nabla u_t\|^p_{L^p(\Omega)} + D_\varepsilon \|u_t\|^p_{L^p(\Omega)}$$

for any t. Thus if $\varepsilon < 1$

$$\|\nabla u_t\|_p^p \leq \frac{C + D_\varepsilon}{1 - \varepsilon}.$$

Then $\{u_t\}_{t\in(-1,1)}$ is bounded in $W_0^{1,p}(\Omega)$ and therefore it is bounded in $L^{pq'}(\Omega)$. Since

$$\lim_{t\to 0} V_t = V_0 \qquad \text{in } L^q(\Omega),$$

then

$$\lim_{t\to 0} \int_\Omega (V_t(x) - V_0(x))\, |u_t|^p \,\mathrm{d}x = 0.$$

Thus

$$\begin{aligned}
\lambda(t) &= \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x \\
&= \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_0(x)|u_t|^p \,\mathrm{d}x + \int_\Omega (V_t(x) - V_0(x))|u_t|^p \,\mathrm{d}x \\
&\geq \lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_t|^p \,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\lambda(0) &= \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V_0(x)|u_0|^p \,\mathrm{d}x \\
&= \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_0|^p \,\mathrm{d}x + \int_\Omega (V_0(x) - V_t(x))|u_0|^p \,\mathrm{d}x \\
&\geq \lambda(t) + \int_\Omega (V_0(x) - V_t(x))|u_0|^p \,\mathrm{d}x.
\end{aligned}$$

Therefore

$$\lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_0|^p \,\mathrm{d}x \geq \lambda(t) \geq \lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_t|^p \,\mathrm{d}x.$$

Hence,

$$\lim_{t\to 0} \lambda(t) = \lambda(0),$$

as we wanted to show. □

Lemma 2.3.6. *$\lambda(t)$ is differentiable at $t = 0$ and*

$$\frac{d\lambda}{dt}(0) = \int_\Omega W(x)|u_0|^p \, dx.$$

Proof. Let $\{t_n\}_{n\in\mathbb{N}}$ be such that $\lim_{n\to\infty} t_n = 0$. As $\{u_{t_n}\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence of $\{t_n\}_{n\in\mathbb{N}}$ (still denoted by $\{t_n\}_{n\in\mathbb{N}}$) and $u \in W_0^{1,p}(\Omega)$ such that

$$u_{t_n} \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{2.3}$$
$$u_{t_n} \to u \quad \text{strongly in } L^r(\Omega), \tag{2.4}$$

for any $1 < r < p^*$. Let us see that $u = u_0$.

In fact, by (2.4), we have $\|u\|_{L^p(\Omega)} = 1$ and, by (2.3), we have

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{t_n}|^p \, dx \geq \int_\Omega |\nabla u|^p \, dx.$$

Now, observe that, as in the proof of Lemma 2.3.5, $V_{t_n} \to V_0$ strongly in $L^q(\Omega)$, then, using again (2.4), we get

$$\lim_{n\to\infty} \int_\Omega V_{t_n}(x)|u_{t_n}|^p \, dx = \int_\Omega V_0(x)|u|^p \, dx.$$

Therefore,

$$\begin{aligned}
\lambda(0) &= \lim_{n\to\infty} \int_\Omega |\nabla u_{t_n}|^p \, dx + \int_\Omega V_{t_n}(x)|u_{t_n}|^p \, dx \\
&\geq \int_\Omega |\nabla u|^p \, dx + \int_\Omega V_0(x)|u|^p \, dx \\
&\geq \lambda(0).
\end{aligned}$$

Hence u is a nonnegative, normalized eigenfunction associated to $\lambda(0)$. By Theorem 1.8.13, we have that $u = u_0$. Since the limit u_0 is independent of the sequence $\{t_n\}_{n\in\mathbb{N}}$, it follows that (2.3)–(2.4) hold for the limit $t \to 0$.

By the differentiability of V_t and by (2.4) we obtain

$$\lim_{t\to 0} \int_\Omega \left(\frac{V_t(x) - V_0(x)}{t}\right) |u_t|^p \, dx = \int_\Omega W(x)|u_0|^p \, dx.$$

In the proof of Lemma 2.3.5, we have showed that

$$\lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_0|^p \, dx \geq \lambda(t) \geq \lambda(0) + \int_\Omega (V_t(x) - V_0(x))(x)|u_t|^p \, dx.$$

Thus, for $t > 0$,

$$\int_\Omega \left(\frac{V_t(x) - V_0(x)}{t}\right) |u_0|^p \, dx \geq \frac{\lambda(t) - \lambda(0)}{t} \geq \int_\Omega \left(\frac{V_t(x) - V_0(x)}{t}\right) |u_t|^p \, dx,$$

and an analogous inequality for $t < 0$. Then $\lambda(t)$ is differentiable at $t = 0$ and

$$\frac{d\lambda}{dt}(0) = \int_\Omega W(x)|u_0|^p \, dx.$$

The proof is now complete. □

Remark 2.3.7. Since λ has maximum at $t = 0$, we have

$$\int_\Omega W(x)|u_0|^p \, dx = 0 \qquad \forall\, W \in T_{V_0}S. \tag{2.5}$$

The following proposition characterize the support of the maximal potential.

Proposition 2.3.8. $\Omega \subseteq \operatorname{supp} V_0$.

Proof. Suppose not. Then, let $x \in \Omega$ such that $x \notin \operatorname{supp} V_0$. As $\operatorname{supp} V_0$ is closed there exists $r > 0$ such that

$$B(x,r) \subset \Omega \qquad \text{and} \qquad B(x,r) \cap \operatorname{supp} V_0 = \emptyset.$$

Then $W = \chi_{B(x,r)} \in T_{V_0}S$ and, by (2.5),

$$\int_{B(x,r)} |u_0|^p \, dx = 0.$$

Hence $u_0 = 0$ a.e. in $B(x,r)$, a contradiction. □

Finally, we arrive at the following characterization of the maximal potential.

Theorem 2.3.9. *Let V_0 be a maximal potential and let u_0 be the eigenfunction associated to $\lambda(V_0)$. Then, there exists a constant k such that*

$$|u_0|^p = k|V_0|^{q-1} \quad \text{in } \Omega. \tag{2.6}$$

Proof. Let T_1 and T_2 be subsets of $\operatorname{supp} V_0$. We denote

$$W(x) = \frac{\chi_{T_1}(x)}{\displaystyle\int_{T_1} |V_0|^{q-1} \, dx} - \frac{\chi_{T_2}(x)}{\displaystyle\int_{T_2} |V_0|^{q-1} \, dx}.$$

Let us see that $W \in T_{V_0}S$. In fact, as V_0 is a maximal potential, $V_0 \geq 0$. Then

$$\begin{aligned}
\int_\Omega |V_0|^{q-2} V_0 W \, dx &= \int_\Omega V_0^{q-1} W \, dx \\
&= \frac{\displaystyle\int_{T_1} V_0^{q-1} \, dx}{\displaystyle\int_{T_1} V_0^{q-1} \, dx} - \frac{\displaystyle\int_{T_2} V_0^{q-1} \, dx}{\displaystyle\int_{T_2} V_0^{q-1} \, dx} \\
&= 0.
\end{aligned}$$

Thus $W \in T_{V_0}S$, as we wanted to see.

By (2.5), we have

$$0 = \int_\Omega W|u_0|^p \, dx = \frac{\displaystyle\int_{T_1} |u_0|^p \, dx}{\displaystyle\int_{T_1} |V_0|^{q-1} \, dx} - \frac{\displaystyle\int_{T_2} |u_0|^p \, dx}{\displaystyle\int_{T_2} |V_0|^{q-1} \, dx}.$$

Then

$$\frac{\displaystyle\int_{T_1} |u_0|^p \, dx}{\displaystyle\int_{T_1} |V_0|^{q-1} \, dx} = \frac{\displaystyle\int_{T_2} |u_0|^p \, dx}{\displaystyle\int_{T_2} |V_0|^{q-1} \, dx}.$$

Therefore, there exists a constant k such that

$$\frac{\displaystyle\int_{T} |u_0|^p \, dx}{\displaystyle\int_{T} |V_0|^{q-1} \, dx} = k$$

for each $T \subset \operatorname{supp} V_0$. In particular, if we take

$$T = \left\{x \in \operatorname{supp} V_0 \colon k|V_0(x)|^{q-1} > |u_0(x)|^p\right\}$$

we get

$$\int_T |u_0|^p \, dx = k \int_T |V_0|^{q-1} \, dx,$$

thus

$$k \int_T |V_0|^{q-1} \, dx - \int_T |u_0|^p \, dx = 0.$$

Since $k|V_0(x)|^{q-1} > |u_0(x)|^p$ for any $x \in T$, the measure of T is zero. In the same way, we obtain that

$$\left\{x \in \operatorname{supp} V_0 \colon k|V_0(x)|^{q-1} < |u_0(x)|^p\right\}$$

has measure zero. Thus

$$|u_0|^p = k|V_0|^{q-1} \qquad \text{a.e. in} \operatorname{supp} V_0.$$

By Proposition 2.3.8,

$$|u_0|^p = k|V_0|^{q-1} \text{ in } \Omega.$$

This ends the proof. □

Equation (2.6) gives us purely algebraic relationship between the optimizing potentials and their associated eigenfunction. Since the eigenvalue equation is homogeneous of degree p in the eigenfunction, we can choose the constant in (2.6) to be equal to one, this can be obtained by taking $\frac{u_0}{k^p}$ as the eigenfunction instead of u_0. Replacing in equation (1.9), we see that the eigenfunction associated to the maximal eigenvalue satisfies

$$-\Delta_p u + u^\alpha = \lambda u^{p-1} \tag{2.7}$$

where λ is the maximal potential eigenvalue and the equation can be written in terms of the associated eigenfunction. An interesting consequence of Theorem 2.3.1 is, in this context, a proof of existence and certain properties of a solution of equation (2.7). More precisely, we have

Corollary 2.3.10. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 < p < \infty$ and $\alpha \in \mathbb{R}$. For any $\lambda > \lambda(0)$, where $\lambda(0)$ is the principal eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$, the nonlinear eigenvalue problem*

$$\begin{cases} -\Delta_p u + u^\alpha = \lambda u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

has a solution in the following cases:

1. *If $1 < p < N$, we take $p - 1 < \alpha < \frac{N(p-1)+p}{N-p}$.*
2. *If $p \geq N$, we take $\alpha > p - 1$.*

Proof. The existence of a potential V_0 maximizing of $-\Delta_p + V$ subject to $\|V\|_{L^q(\Omega)} = M$, for any $M > 0$ is known from Theorem 2.3.1, with $\alpha = \frac{pq-q+1}{q-1}$. If the maximized eigenvalue is $\lambda^* = \lambda(V_0)$, then the necessary condition (2.7) becomes (2.8) with $u = u_0$ and $\lambda = \lambda^*$.

The corollary will thus be proved if it is shown that λ^* increases continuously from $\lambda(0)$ to ∞ as M goes from 0 to ∞. By Remark 2.3.3, $\lambda^*(\cdot)$ is increases monotonically from $\lambda(0)$ to ∞ as $M \nearrow \infty$. It remains to prove the continuity.

We denote with V_0^M the maximal potential associated to $\lambda^*(M)$. If $t > 0$, then

$$\lambda(V_0^{M+t}) = \lambda^*(M+t) \geq \lambda^*(M).$$

Take $V = \frac{M}{M+t} V_0^{M+t}$, note that $\|V\|_q = M$, then $\lambda(V) \leq \lambda^*(M)$. Given $u \in W_0^{1,p}(\Omega)$, $\|u\|_p = 1$, we have

$$\begin{aligned} \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V(x)|u|^p \,\mathrm{d}x &= \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega \frac{M}{M+t} V_0^{M+t}(x)|u|^p \,\mathrm{d}x \\ &= \frac{M}{M+t}\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V_0^{M+t}(x)|u|^p \,\mathrm{d}x\right) \\ &+ \left(1 - \frac{M}{M+t}\right)\int_\Omega |\nabla u|^p \,\mathrm{d}x \\ &\geq \frac{M}{M+t}\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V_0^{M+t}(x)|u|^p \,\mathrm{d}x\right). \end{aligned}$$

Thus

$$\lambda(V) = \lambda\left(\frac{M}{M+t} V_0^{M+t}\right) \geq \frac{M}{M+t}\lambda\left(V_0^{M+t}\right) = \frac{M}{M+t}\lambda^*(M+t)$$

then, as $\lambda(V) \leq \lambda^*(M)$,

$$\frac{M}{M+t}\lambda^*(M+t) \leq \lambda^*(M) \leq \lambda^*(M+t). \tag{2.9}$$

Similarly,

$$\lambda^*(M-t) \leq \lambda^*(M) \leq \frac{M-t}{M}\lambda^*(M-t). \tag{2.10}$$

Then, taking limits in (2.9) and (2.10),

$$\lim_{t\to 0} \lambda^*(M+t) = \lambda^*(M).$$

This completes the proof. □

2.4 Minimizing potentials

In this section we present the results for minimizing potentials. Since the results and the proof are completely analogous to those of the previous subsection we only state the main results and point out only the significant differences.

Theorem 2.4.1. *If $q > \max\{N/p, 1\}$, there exists $V_* \in B$ that minimizes $\lambda(V)$.*

Proof. Is analogous to that of Theorem 2.3.1. □

As in the previous subsection, we consider the case $B = \overline{B(0, M)} \subset L^q(\Omega)$, and to simplify the computations, we take $M = 1$.

As a concave function defined over a convex set achieves its minimum at the extreme points of the convex, there exists $V_0 \in \partial B$ such that

$$\lambda(V_0) = \min\{\lambda(V) \colon V \in \partial B\} = \min\{\lambda(V) \colon V \in \partial B\}.$$

Moreover, since $-|V_0| \leq V_0$ and $\lambda(\cdot)$ is nondecreasing we may assume that $V_0 \leq 0$.

Let us now try to characterize V_0. As before, let $\alpha : (-1, 1) \to L^q(\Omega)$ be a differentiable curve such that

$$\alpha(t) \in S := \partial B, \qquad \alpha(0) = V_0 \qquad \text{and} \qquad \dot\alpha(0) = W \in T_{V_0}S.$$

We denote by $V_t = \alpha(t)$ and $\lambda(t) = \lambda(\alpha(t))$. Let u_t the normalized, nonnegative eigenfunction of H_{V_t} associated to $\lambda(t)$. Observe that Lemmas 2.3.5 and 2.3.6 apply. Hence, as λ has a minimum at $t = 0$ we have

$$\int_\Omega W(x)|u_0|^p \,\mathrm{d}x = 0 \qquad \forall W \in T_{V_0}S \tag{2.11}$$

As for maximizing potential, we have,

Proposition 2.4.2. $\Omega \subseteq \operatorname{supp} V_0$.

Proof. Analogous to that of Lemma 2.3.8. □

Proposition 2.4.3. *Let V_0 be a minimal potential and let u_0 be the normalized, nonnegative eigenfunction of H_{V_0} associated to $\lambda(V_0)$. Then, there exists a constant $k \in \mathbb{R}_+$ such that*

$$|u_0|^p = k|V_0|^{q-1} \quad \text{in } \Omega. \tag{2.12}$$

Proof. Analogous to that of Lemma 2.3.9. □

As before, from (2.12) we obtain a purely algebraic relationship between minimal potential and their associated eigenfunction. Using the homogeneity of the equation, we can choose the constant in (2.12) to be 1. Replacing in (1.9) we obtain that the eigenfunction associated to the minimal potential satisfies

$$-\Delta_p u - u^\alpha = \lambda u^{p-1} \tag{2.13}$$

where λ is the minimal eigenvalue and $\alpha = \frac{pq-q+1}{q-1}$.

Therefore, we obtain the following corollary

Corollary 2.4.4. *Let $\Omega \subset \mathbb{R}^N$ be a smooth open and bounded set, $1 < p < \infty$ and $\alpha \in \mathbb{R}$. For every $\lambda < \lambda(0)$, where $\lambda(0)$ is the principal eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, the nonlinear eigenvalue problem*

$$\begin{cases} -\Delta_p u - u^\alpha = \lambda u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.14}$$

has a solution in the cases

1. *If $1 < p < N$, we take $p - 1 < \alpha < \frac{N(p-1)+p}{N-p}$.*
2. *If $p \geq N$, we take $\alpha > p - 1$.*

Proof. Analogous to that of Corollary 2.3.10. □

3

The first Steklov eigenvalue of a nonlinear problem

Given a domain $\Omega \subset \mathbb{R}^N$ (bounded, connected, with smooth boundary), $\alpha > 0$ and $E \subset \Omega$ a measurable set, in this chapter, we want to study the eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \alpha\chi_E|u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda|u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

here λ stands for the eigenvalue and α is a positive parameter. We remark that in this problem the eigenvalue appears in the boundary condition. These type of problems are known as *Steklov eigenvalue problems*, see [St]. Observe that when $p = 2$ the problem becomes linear.

We denote the first eigenvalue by $\lambda(\alpha, E)$. The existence of this first eigenvalue and a positive associated eigenfunction follows easily from the variational characterization

$$\lambda(\alpha, E) := \inf\left\{\int_\Omega |\nabla v|^p + |v|^p \,dx + \alpha\int_E |v|^p \,dx \colon v \in \mathcal{W}\right\}, \tag{3.2}$$

where

$$\mathcal{W} = \left\{v \in W^{1,p}(\Omega) \colon \|v\|_{L^p(\partial\Omega)} = 1\right\},$$

and the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, see [FBR1].

Once the set E is fixed, it is not difficult to check that when $\alpha \to \infty$ the eigenvalues converge to the first eigenvalue of the problem with E as a hole (the eigenfunction vanishes on E). That is,

$$\lim_{\alpha\to\infty} \lambda(\alpha, E) = \lambda(\infty, E),$$

where

$$\lambda(\infty, E) := \inf\left\{\int_\Omega |\nabla v|^p + |v|^p \,dx \colon v \in \mathcal{W} \text{ and } v\,|_E \equiv 0\right\}.$$

The aim of this chapter is to study the following optimization problem: for a fixed α we want to optimize $\lambda(\alpha, E)$ with respect to E, that is, we want to look at the infimum,

$$\inf\{\lambda(\alpha, E) \colon E \subset \Omega \text{ and } |E| = A\} \tag{3.3}$$

for a fixed volume $A \in [0, |\Omega|]$. Moreover, we want to study the limit as $\alpha \to \infty$ in the above infimum. The natural limit problem for these infimum is

$$\lambda(\infty, A) := \inf \{\lambda(\infty, E) \colon E \subset \Omega \text{ and } |E| = A\}. \tag{3.4}$$

These kind of problems appear naturally in optimal design. They are usually formulated as problems of minimization of the energy, stored in the design under a prescribed loading. Solutions of these problems are unstable to perturbations of the loading. The stable optimal design problem is formulated as minimization of the stored energy of the project under the most unfavorable loading. This most dangerous loading is one that maximizes the stored energy over the class of admissible functions. The problem is reduced to minimization of Steklov eigenvalues. See [CC].

Also this limit problem (3.4) can be regarded as the study of the best Sobolev trace constant for functions that vanish in a subset of prescribed measure. The study of optimal constants in Sobolev embeddings is a very classical subject, see [DH]. Related problems for the best Sobolev trace constant can be found in [FBFR, FBR2]. In our case, the limit problem was studied in [FBRW2] where an optimal configuration is shown to exists and some properties of this optimal configuration are obtained. Among them it is proved that $\lambda(\infty, A)$ is strictly increasing with respect to A. In a companion paper [FBRW1] the interior regularity of the optimal hole is analyzed.

The rest of the chapter is organized as follows: in Section 3.1, we prove that there exists an optimal configuration; in Section 3.2, we analyze the limit $\alpha \to \infty$ and finally in Section 3.3 we study the symmetry properties of the optimal pairs in a ball.

3.1 Existence of an optimal configuration

In this section we prove that there exists an optimal configuration for the relaxed problem and find some properties of it.

To begin the study of our optimization problem (3.2), we prove that there exists an optimal configuration. To this end, it is better to relax the problem and consider $\phi \in L^\infty(\Omega)$, such that $0 \le \phi \le 1$ and $\int_\Omega \phi(x)\,\mathrm{d}x = A$ instead of χ_E. Hence we consider the problem,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \alpha\phi|u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

This relaxation is natural in the use of the direct method in the calculus of variations since

$$\mathcal{R} = \left\{ \phi \in L^\infty(\Omega) \colon\ 0 \le \phi \le 1 \text{ and } \int_\Omega \phi(x)\,\mathrm{d}x = A \right\}$$

is closed in the weak$*$ topology in $L^\infty(\Omega)$. In fact, by the Theorem 1.4.6, this set is the closure in this topology of the set of characteristic functions

$$\{\chi_E : |E| = A\}.$$

We denote by $\lambda(\alpha, \phi)$ the lowest eigenvalue of (3.5). This eigenvalue has the following variational characterization

$$\lambda(\alpha, \phi) := \inf \left\{ \int_\Omega |\nabla v|^p + |v|^p \, \mathrm{d}x + \alpha \int_\Omega \phi |v|^p \, \mathrm{d}x \colon v \in \mathcal{W} \right\}. \tag{3.6}$$

Again, as an immediate consequence of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, the above infimum is in fact a minimum. There exists $u = u_{\alpha,\phi} \in W^{1,p}(\Omega)$ such that $\|u\|_{L^p(\partial\Omega)} = 1$ and

$$\lambda(\alpha, \phi) = \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x + \alpha \int_\Omega \phi |u|^p \, \mathrm{d}x.$$

Moreover, u is a weak solution of (3.5), does not change sign (see [FBR1, FBR3, MR]) and hence, by Harnack's inequality (see Theorem 1.8.4), it can be assumed that u is strictly positive in $\overline{\Omega}$.

Define

$$\Lambda(\alpha, A) = \inf \{\lambda(\alpha, \phi) \colon \phi \in \mathcal{R}\}. \tag{3.7}$$

Any minimizer ϕ in (3.7) will be called an *optimal configuration* for the data (α, A). If ϕ is an optimal configuration and u satisfies (3.5) then (u, ϕ) will be called an *optimal pair (or solution)*.

By the direct method of the calculus of variations, it is not difficult to see that there exits an optimal pair. The main point of the following result is to show that we can recover a *classical* solution of our original problem (3.3). In fact, if (u, ϕ) is an optimal pair, then $\phi = \chi_D$ for some measurable set $D \subset \Omega$.

Theorem 3.1.1. *For any $\alpha > 0$ and $A \in [0, |\Omega|]$ there exists an optimal pair. Moreover, any optimal pair (u, ϕ) has the following properties:*

1. *$u \in C^{1,\delta}(\overline{\Omega})$ for some $0 < \delta < 1$.*
2. *There exists an optimal configuration $\phi = \chi_D$, where $\{u < t\} \subset D \subset \{u \le t\}$ with $t := \sup\{s \colon |\{u < s\}| \le A\}$.*

For the proof we use ideas from [CGIK, CGK] where a similar linear problem with homogeneous Dirichlet boundary conditions was studied.

Proof. To prove existence, fix α and A, and write $\Lambda = \Lambda(\alpha, A)$, $\lambda(\phi) = \lambda(\alpha, \phi)$ to simplify the notation. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a minimizing sequence, i.e., $0 \le \phi_n \le 1$, $\int_\Omega \phi_n \, \mathrm{d}x = A$ and $\lambda(\phi_n) \to \Lambda$ as $n \to \infty$.

Let $u_n \in W^{1,p}(\Omega)$, be a normalized eigenfunction associated to $\lambda(\phi_n)$, that is, u_n verifies $\|u_n\|_{L^p(\partial\Omega)} = 1$ and

$$\begin{aligned} \lambda(\phi_n) &= \int_\Omega |\nabla u_n|^p + |u_n|^p \, \mathrm{d}x + \alpha \int_\Omega \phi_n |u_n|^p \, \mathrm{d}x \\ &= \inf \left\{ \int_\Omega |\nabla v|^p + |v|^p \, \mathrm{d}x + \alpha \int_\Omega \phi_n |v|^p \, \mathrm{d}x \colon v \in \mathcal{W} \right\}. \end{aligned} \tag{3.8}$$

Then, u_n is a positive weak solution of

$$\begin{cases} -\Delta_p u_j + |u_n|^{p-2}u_n + \alpha\phi|u_n|^{p-2}u_n = 0 & \text{in } \Omega, \\ |\nabla u_n|^{p-2}\frac{\partial u_j}{\partial \nu} = \lambda(\phi_n)|u_n|^{p-2}u_n & \text{on } \partial\Omega, \\ \|u_n\|_{L^p(\partial\Omega)} = 1. \end{cases}$$

Since $\lambda(\phi_n)$ is bounded, the sequence u_n is bounded in $W^{1,p}(\Omega)$. Also $\{\phi_n\}$ is bounded in $L^\infty(\Omega)$. Therefore, we may choose a subsequence (again denoted u_n, ϕ_n) and $u \in W^{1,p}(\Omega)$, $\phi \in L^\infty(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.9}$$

$$u_n \to u \quad \text{strongly in } L^p(\Omega), \tag{3.10}$$

$$u_n \to u \quad \text{strongly in } L^p(\partial\Omega), \tag{3.11}$$

$$\phi_n \overset{*}{\rightharpoonup} \phi \quad \text{weakly}* \text{ in } L^\infty(\Omega). \tag{3.12}$$

By (3.10),

$$\|u\|_{L^p(\partial\Omega)} = 1,$$

and by (3.12)

$$0 \le \phi \le 1 \text{ and } \int_\Omega \phi \, \mathrm{d}x = A.$$

Now taking limits in (3.8), we get

$$\begin{aligned} \Lambda &= \lim_{n\to\infty} \lambda(\phi_n) \\ &\ge \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p + |u_n|^p \, \mathrm{d}x + \alpha \int_\Omega \phi_n |u_n|^p \, \mathrm{d}x \\ &\ge \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x + \alpha \int_\Omega \phi |u|^p \, \mathrm{d}x. \end{aligned} \tag{3.13}$$

Therefore, (u, ϕ) is an optimal pair and so u is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \alpha\phi|u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \Lambda|u|^{p-2}u & \text{on } \partial\Omega. \end{cases}$$

That (1) holds is a consequence of the regularity theory for quasilinear elliptic equations with bounded coefficients developed, for instance, in [L].

To prove (2), observe that, by the Theorem 1.3.12, the minimization problem

$$\inf\left\{ \int_\Omega \phi |u|^p \, \mathrm{d}x \colon \phi \in \mathcal{R} \right\}$$

has a solution $\phi = \chi_D$ where D is any set with $|D| = A$ and

$$\{x \colon u(x) < t\} \subset D \subset \{x \colon u(x) \le t\}, \quad t := \sup\{s \colon |\{u < s\}| \le A\}.$$

Therefore, we get from (3.13)

$$\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x + \alpha \int_\Omega \chi_D |u|^p \,\mathrm{d}x \le \Lambda.$$

By definition of Λ as a minimum, this must actually be an equality, and we conclude that (u, χ_D) is an optimal pair. □

Now, we find the derivative of $\lambda(\alpha, \phi)$ in an admissible direction $f \in F$, given by

$$F = \left\{ f \colon \; f \le 0 \text{ in } \{\phi = 1\}, \; f \ge 0 \text{ in } \{\phi = 0\}, \; \int_\Omega f \,\mathrm{d}x = 0 \right\}. \tag{3.14}$$

Proposition 3.1.2. *Let $f \in F$, then the derivative from the right of $\lambda(\alpha, \phi)$ in the direction of $f \in F$ is given by*

$$\lambda'(\alpha, \phi)(f) = \lim_{t \searrow 0} \frac{\lambda(\alpha, \phi + tf) - \lambda(\alpha, \phi)}{t} = \alpha \int_\Omega f |u|^p \, dx, \tag{3.15}$$

where u is an eigenfunction of $\lambda(\alpha, \phi)$.

Proof. Let us consider the curve

$$\phi_t = \phi + tf.$$

Note that since $f \in F$ and ϕ is admissible then ϕ_t is admissible for every $t \ge 0$ small enough. Therefore, we may compute $\lambda(\alpha, \phi_t)$.

Using an eigenfunction u_t of $\lambda(\alpha, \phi_t)$ in the variational formulation of $\lambda(\alpha, \phi)$ we get

$$\frac{\lambda(\alpha, \phi_t) - \lambda(\alpha, \phi)}{t} \le \alpha \int_\Omega f |u_t|^p \, dx. \tag{3.16}$$

On the other hand, using u in the variational formulation of $\lambda(\alpha, \phi_t)$ we get

$$\frac{\lambda(\alpha, \phi_t) - \lambda(\alpha, \phi)}{t} \ge \alpha \int_\Omega f |u|^p \, dx. \tag{3.17}$$

As before, using $v = 1$ as a test function in the definition of $\lambda(\alpha, \phi_t)$, we obtain that the family $\{u_t\}_{0 \le t \le t_0}$ is bounded in $W^{1,p}(\Omega)$. Then, by our previous arguments we have that

$$u_t \to u \qquad \text{strongly in } L^p(\Omega) \text{ when } t \to 0.$$

Hence, taking limits in (3.16) and (3.17) we conclude (3.15). □

Using this Proposition we can prove that the optimal set must be a sublevel set of u, i.e., there is a number $t \ge 0$ such that $\{x \colon u(x) \le t\}$ is the optimal set.

Corollary 3.1.3. *There exists a number $t \ge 0$ such that the optimal set D is*

$$D = \{x \colon u(x) \le t\}.$$

Proof. As χ_D realizes the minimum of $\lambda(\alpha, \phi)$, we have for all $f \in F$,

$$\lambda'(\alpha, \chi_D)(f) = \alpha \int_\Omega f|u|^p \, dx \geq 0, \tag{3.18}$$

Given two points $x_0 \in D$ of positive density (i.e., for every $\varepsilon > 0$, $|B(x_0, \varepsilon) \cap D| > 0$) and $x_1 \in (\Omega \setminus D)$ also with positive density we can take a function $f \in F$ of the form

$$f = M\chi_{T_0} - M\chi_{T_1},$$

with $T_0 \subset B(x_0, \varepsilon) \cap D$, $T_1 \subset B(x_1, \varepsilon) \cap (\Omega \setminus D)$ and $M^{-1} = |T_0| = |T_1|$. It is clear that $f \in F$. From our expression for the right derivative (3.15) and using that D is a minimizer, taking the limit as $\varepsilon \to 0$ and using the continuity of u we get $u(x_0) \leq u(x_1)$. We conclude that $D = \{x \colon u \leq t\}$. □

3.2 Limit as $\alpha \to \infty$

In this section, we analyze the limit as $\alpha \to \infty$ of the optimal configurations found in Theorem 3.1.1. We give a rigorous proof of the convergence of these optimal configurations to those of (3.4).

First, we need a result about the monotonicity of $\lambda(\infty, A)$ in A.

Lemma 3.2.1. *$\lambda(\infty, A)$ is strictly monotonically increasing in A.*

Proof. The prove of this lemma is found in [FBRW2]. We include here only by the sake of completeness.

We proceed in three steps.

Step 1. First, we show that

$$\begin{aligned}\lambda(\infty, A) &= \inf\{\lambda(\infty, E) \colon E \subset \Omega \text{ and } |E| = A\} \\ &= \inf\{\lambda(\infty, E) \colon E \subset \Omega \text{ and } |E| \geq A\}.\end{aligned}$$

It is clear that

$$\inf\{\lambda(\infty, E) \colon E \subset \Omega \text{ and } |E| = A\} \geq \inf\{\lambda(\infty, E) \colon E \subset \Omega \text{ and } |E| \geq A\}.$$

On the other hand, if v is a test function for a set of measure greater than or equal to A it is also a test function for a set of measure A. Then, the two infima coincide.

Step 2. we show that, if u is an extremal for $\lambda(\infty, A)$ then $|\{x \colon u(x) = 0\}| = A$.

Suppose by contradiction that u vanishes in a set E with $|E| > A$. By taking a subset we may assume that E is closed. Let us take a small ball B so that $|E \setminus B| > A$ with B centered at a point in $\partial E \cap \partial \Omega_1$, where Ω_1 is the connected component of $\Omega \setminus E$ such that $\partial\Omega \subset \partial\Omega_1$. We can pick the ball B in such a way that $|E \cap B| > 0$. In particular, $|\{x \colon u(x) = 0\} \cap B| > 0$.

Since u is an extremal for $\lambda(\infty, A)$ and $|E \setminus B| > A$, it is an extremal for $\lambda(\infty, E \setminus B)$. Thus, there holds that

$$-\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega \setminus (E \setminus B) = (\Omega \setminus E) \cup B.$$

Now, as $u \geq 0$, there holds that either $u \equiv 0$ or $u > 0$ in each connected component of $(\Omega \setminus A) \cup B$. Since $u \neq 0$ on $\partial\Omega$ there holds, in particular, that $u > 0$ in B. This is a contradiction to the choice of the ball B. Therefore,

$$|\{x\colon u(x) = 0\}| = A.$$

Step 3. Lastly, we show that $\lambda(\infty, A)$ is strictly monotonically increasing in A.

By the Step 1, we deduce that $\lambda(\infty, A)$ is nondecreasing respect to A. On the other hand, let $0 < A_1 < A_2 < |\Omega|$, such that $\lambda(\infty, A_1) = \lambda(\infty, A_2)$ and let u be an extremal for $\lambda(\infty, A_2)$ then, by step 2, $|\{u = 0\} = A_2$. But u is an admissible function for $\lambda(\infty, A_1)$, so that it is an extremal for $\lambda(\infty, A_1)$ with $|\{x\colon u(x) = 0\}| > A_1$. This is a contradiction with what has been proved in the step 2. Thus, $\lambda(\infty, A)$ is strictly monotonically increasing in A. □

Theorem 3.2.2. *For any sequence $\alpha_j \to \infty$ and optimal pairs (D_j, u_j) of (3.3) there exists a subsequence, that we still call α_j, and an optimal pair (D, u) of (3.4) such that*

$$\lim_{j\to\infty} \chi_{D_j} = \chi_D, \qquad \textit{weakly}{*} \textit{ in } L^\infty(\Omega),$$
$$\lim_{j\to\infty} u_j = u, \qquad \textit{strongly in } W^{1,p}(\Omega).$$

Moreover, $u > 0$ in $\Omega \setminus D$.

Proof. Let $(u_\alpha, \chi_{D_\alpha})$ be a solution to our minimization problem

$$\Lambda(\alpha, A) = \inf\left\{\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x + \alpha \int_\Omega \phi|u|^p \,\mathrm{d}x\colon u \in \mathcal{W} \text{ and } \phi \in \mathcal{R}\right\}.$$

Recall that u_α is a positive weak solution of

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \alpha\phi|u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \Lambda(\alpha, A)|u|^{p-2}u & \text{on } \partial\Omega, \\ \|u\|_{L^p(\partial\Omega)} = 1. \end{cases}$$

Let $u_0 \in W^{1,p}(\Omega)$ and $D_0 \subset \Omega$ be such that $|D_0| = A$ and $u_0\chi_{D_0} = 0$. Then, we have that

$$\begin{aligned}
\Lambda(\alpha, A) &\leq \frac{\displaystyle\int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega |u_0|^p \,\mathrm{d}x + \alpha \int_\Omega \chi_{D_0}|u_0|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u_0|^p \,\mathrm{d}\mathcal{H}^{N-1}} \\
&= \frac{\displaystyle\int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega |u_0|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u_0|^p \,\mathrm{d}\mathcal{H}^{N-1}} \\
&= K
\end{aligned}$$

with K independent of α.

Thus $\{\Lambda(\alpha, A)\}$ is a bounded sequence in $\mathbb{R}$ and it is clearly increasing. As a consequence, $\{u_\alpha\}$ is bounded in $W^{1,p}(\Omega)$. Moreover $\{\chi_{D_\alpha}\}$ is bounded in $L^\infty(\Omega)$. Therefore, we may choose a sequence α_j and $u_\infty \in W^{1,p}(\Omega)$, $\phi_\infty \in L^\infty(\Omega)$ such that

$$u_{\alpha_j} \rightharpoonup u_\infty \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.19}$$

$$u_{\alpha_j} \to u_\infty \quad \text{strongly in } L^p(\Omega), \tag{3.20}$$

$$u_{\alpha_j} \to u_\infty \quad \text{strongly in } L^p(\partial\Omega), \tag{3.21}$$

$$\chi_{D_{\alpha_j}} \overset{*}{\rightharpoonup} \phi_\infty \quad \text{weakly}* \text{ in } L^\infty(\Omega), \tag{3.22}$$

By (3.21) and as $\|u_{\alpha_j}\|_{L^p(\partial\Omega)} = 1$ for all $j \in \mathbb{N}$, we have that $\|u_\infty\|_{L^p(\partial\Omega)} = 1$ and, by (3.22), $0 \le \phi_\infty \le 1$ with $\int_\Omega \phi_\infty \,\mathrm{d}x = A$. Also, by (3.20) and (3.22), it holds

$$\int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \to \int_\Omega \phi_\infty |u_\infty|^p \,\mathrm{d}x.$$

As

$$0 \le \alpha_j \int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \le \Lambda_{\alpha_j} \le K \quad \forall\, j \in \mathbb{N},$$

we have

$$0 \le \int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \le \frac{K}{\alpha_j} \quad \forall\, j \in \mathbb{N},$$

then

$$\int_\Omega \chi_{D_{\alpha_j}} |u_{\alpha_j}|^p \,\mathrm{d}x \to 0.$$

Therefore

$$\int_\Omega \phi_\infty |u_\infty|^p \,\mathrm{d}x = 0,$$

and we conclude that

$$\phi_\infty u_\infty = 0 \quad \text{a.e. } \Omega.$$

Since $\{\Lambda(\alpha_j, A)\}$ is bounded and increasing, there exists the limit

$$\lim_{j\to\infty} \Lambda(\alpha_j, A) = \Lambda_\infty < +\infty.$$

Then

$$\begin{aligned} \Lambda_\infty &= \lim_{j\to\infty} \int_\Omega |\nabla u_{\alpha_j}|^p \,\mathrm{d}x + \int_\Omega |u_{\alpha_j}|^p \,\mathrm{d}x + \alpha_j \int_\Omega \chi_{\alpha_j} |u_{\alpha_j}|^p \,\mathrm{d}x \\ &\ge \liminf_{j\to\infty} \int_\Omega |\nabla u_{\alpha_j}|^p \,\mathrm{d}x + \int_\Omega |u_{\alpha_j}|^p \,\mathrm{d}x \\ &\ge \int_\Omega |\nabla u_\infty|^p \,\mathrm{d}x + \int_\Omega |u_\infty|^p \,\mathrm{d}x. \end{aligned}$$

Hence, we have

$$\begin{aligned}\Lambda_\infty &\geq \int_\Omega |\nabla u_\infty|^p \,\mathrm{d}x + \int_\Omega |u_\infty|^p \,\mathrm{d}x \\ &\geq \inf\left\{\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x \colon u \in \mathcal{W} \text{ and } \phi \in \mathcal{R}\right\}. \\ &\geq \Lambda(\alpha_j, A).\end{aligned}$$

foar all $j \in \mathbb{N}$.

Therefore

$$\begin{aligned}\Lambda_\infty &= \inf\left\{\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x \colon u \in \mathcal{W}, \phi \in \mathcal{R} \text{ and } u\phi = 0\right\} \\ &= \int_\Omega |\nabla u_\infty|^p + |u_\infty|^p \,\mathrm{d}x,\end{aligned}$$

and so the infimum in the above equation is achieved by (u_∞, ϕ_∞).

Now, if we take $D_\infty = \{\phi_\infty > 0\}$ we get that $|D_\infty| = B \geq A$. Hence

$$\lambda(\infty, B) \leq \lambda(\infty, D_\infty) = \Lambda_\infty \leq \lambda(\infty, A).$$

This implies that $|D_\infty| = A$ (otherwise, we have a contradiction with the strict monotonicity of $\lambda(\infty, A)$ in A). So, $\phi_\infty = \chi_{D_\infty}$.

We observe that $D_\infty \subset \{x \colon u_\infty(x) = 0\}$ and again, by the strict monotonicity of $\lambda(\infty, A)$ in A, $D_\infty = \{x \colon u_\infty(x) = 0\}$. □

3.3 Symmetry properties.

In this section, we consider the case where Ω is the unit ball, i.e., $\Omega = B(0,1)$.

Now, we study symmetry properties of the optimal configuration when Ω is the unit ball.

Theorem 3.3.1. *Fix $\alpha > 0$ and $0 < A < |B(0,1)|$, there exists an optimal pair of (3.5), (u, χ_D), such that u and D are spherically symmetric. Moreover, when $p = 2$, every optimal pair (u, χ_D) is spherically symmetric.*

Proof. Fix $\alpha > 0$ and A and assume (u, χ_D) is and optimal pair. Let u^* the spherical symmetrization of u. Define the set D^* by $\chi_{D^*} = (\chi_D)_*$. By Theorem 1.7.1, we get

$$\begin{aligned}\lambda(\alpha, D^*) &\leq \frac{\displaystyle\int_\Omega |\nabla u^*|^p \,\mathrm{d}x + \int_\Omega |u^*|^p \,\mathrm{d}x + \int_\Omega (\alpha\chi_D)_* |u^*|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u^*|^p \,\mathrm{d}\mathcal{H}^{N-1}} \\ &\leq \frac{\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}x + \alpha \int_\Omega \chi_D |u|^p \,\mathrm{d}x}{\displaystyle\int_{\partial\Omega} |u|^p \,\mathrm{d}\mathcal{H}^{N-1}} \\ &= \lambda(\alpha, D^*).\end{aligned}$$

Since we have $|D^*| = |D| = A$, optimality of (u, χ_D) implies that (u^*, χ_{D^*}) is also a minimizer.

Now consider $p = 2$. In this case, it is proved in [D] that if equality holds in (1.2) then for each $0 < r \leq 1$ there exists a rotation R_r such that

$$u\,|_{\partial B(0,r)} = (u^* \circ R_r)\,|_{\partial B(0,r)}\ . \tag{3.23}$$

We can assume that the axis of symmetry e_N was taken so that $R_1 = Id$. Therefore u and u^* coincide on the boundary of $B(0, 1)$. Then, the optimal sets D, D^* are sublevel sets of u and u^* with the same level, t. As u and u^* are solutions of a second order elliptic equation with bounded measurable coefficients they are C^1. Hence $\{x\colon u(x) > t\} \cap \{x\colon u^*(x) > t\}$ is an open neighborhood of $\partial\Omega \cap \{x\colon u(x) > t\}$. In that neighborhood both functions are solutions of the same equation, $\Delta v = v$ (which has a unique continuation property), and along $\partial\Omega \cap \{x\colon u(x) > t\}$ both coincide together with their normal derivatives. Thus they coincide in the whole neighborhood.

Now we observe that the set $\{x\colon u(x) > t\}$ is connected, because every connected component of $\{x\colon u(x) > t\}$ touches the boundary (since solutions of $\Delta v = v$ cannot have a positive interior maximum) and $\{x\colon u(x) > t\} \cap \partial\Omega$ is connected.

We conclude that $\{x\colon u(x) > t\} = \{x\colon u^*(x) > t\}$ and $u = u^*$ in that set. In the complement of this set both u and u^* satisfy the same equation with the same Dirichlet data, therefore they coincide. □

4

Differential calculus

In optimization problems, one of the aims is to obtain optimality conditions for the minimum. By example, a method for searching a minimum of a function $f : \mathbb{R}^N \to \mathbb{R}$ consists in solving the equation $\nabla f = 0$ in $\mathbb{R}^N$ and, then selecting between the solutions one corresponding to the minimum. In the following chapter, we will study the minima of some functionals defined on $W^{1,p}(\Omega)$ and we will give some optimality conditions for them. We will see how the minima depend respect to some perturbation of the domain and, then we will compute the derivatives of the functional respect to this perturbation for obtaining an equation similar to the case in that the functional is defined in $\mathbb{R}^N$. This approach for optimization problems has been used several times in the literature. For example, see [HP, DPFBR, FBRW2, KSS] and references therein.

In this kind of study, will be important calculate the derivative of the norms $\|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{W^{1,p}(\Omega)}$ and $\|\cdot\|_{L^p(\partial\Omega)}$.

The aim of this chapter is given some technical result, that we will use in the rest of this thesis.

Throughout this chapter, Ω will be a bounded domain in $\mathbb{R}^N$ with boundary of class C^2. ${}^T A$ and A^{-1} denote the transpose and the inverse of the matrix A, respectively. Let Φ be a C^1 field over $\mathbb{R}^N$, Φ' denotes the differential matrix of Φ and the Jacobian of Φ is denoted by $\mathrm{Jac}(\Phi)$.

The rest of the chapter is divided into three sections. In Section 4.1, we prove that the norms $\|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ are differentiation respect to perturbations in the domain Ω. The Section 4.2 collect some results regarding the differential geometry. Lastly, in Section 4.3, we show that $\|\cdot\|_{L^p(\partial\Omega)}$ is differentiable respect to perturbation in Ω.

4.1 Differentiation of the norms $\|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$

We begin by describing the kind of variations that we are going to consider. Let W be a regular (smooth) vector field, globally Lipschitz, with support in Ω and let $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$

be the flow defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x) = W(\varphi_t(x)) & t > 0, \\ \varphi_0(x) = x & x \in \mathbb{R}^N. \end{cases} \tag{4.1}$$

We have

$$\varphi_t(x) = x + tW(x) + o(t) \quad \forall\, x \in \mathbb{R}^N.$$

In [HP], the following asymptotic formulas were proved

$$[\psi_t']^{-1}(x) = Id - tW'(x) + o(t), \tag{4.2}$$

$$\mathrm{Jac}(\psi_t)(x) = 1 + t\,\mathrm{div}W(x) + o(t), \tag{4.3}$$

for all $x \in \mathbb{R}^N$.

Our first result of this section shows that $\|\cdot\|_{L^q(\Omega)}$ is differentiable with respect to t at $t = 0$.

Lemma 4.1.1. *Given $f \in L^q(\Omega)$ then*

$$f_t := f \circ \varphi_t^{-1} \to f \text{ strongly in } L^q(\Omega), \text{ as } t \to 0^+.$$

Moreover

$$\int_\Omega |f_t|^p \,\mathrm{d}x = \int_\Omega |f|^p \,\mathrm{d}x + t \int_\Omega |f|^q \,\mathrm{div}\, W \,\mathrm{d}x + o(t).$$

Proof. We proceed in two steps.

Step 1. First we show that

$$f_t \to f \text{ strongly in } L^q(\Omega), \text{ as } t \to 0^+.$$

Let $\varepsilon > 0$ and let $g \in C_c^\infty(\Omega)$ be fixed such that $\|f - g\|_{L^q(\Omega)} < \varepsilon$. By the usual change of variables formula, we have that

$$\|f_t - g_t\|_{L^q(\Omega)}^q = \int_\Omega |f - g|^q \,\mathrm{Jac}(\varphi_t) \,\mathrm{d}x,$$

where $g_t = g \circ \varphi_t^{-1}$.

Then

$$\|f_t - g_t\|_{L^q(\Omega)}^q = \int_\Omega |f - g|^q (1 + t\,\mathrm{div}\, W + o(t)) \,\mathrm{d}x.$$

Therefore, there exist $t_1 > 0$ such that if $0 < t < t_1$ then

$$\|f_t - g_t\|_{L^q(\Omega)} < C\varepsilon,$$

where C is a constant independent of t. Moreover, since $\varphi_t^{-1} \to Id$ in the C^1 topology when $t \to 0$ then $g_t = g \circ \varphi_t^{-1} \to g$ in the C^1 topology and therefore there exist $t_2 > 0$ such that if $0 < t < t_2$ then

$$\|g_t - g\|_{L^q(\Omega)} < \varepsilon.$$

Finally, for all $0 < t < t_0 = \min\{t_1, t_2\}$ we have that

$$\begin{aligned}\|f_t - f\|_{L^q(\Omega)} &\le \|f_t - g_t\|_{L^q(\Omega)} + \|g_t - g\|_{L^q(\Omega)} + \|f - g\|_{L^q(\Omega)}\\ &\le C\varepsilon,\end{aligned}$$

where C is a constant independent to t.

Step 2. Now we prove that

$$\int_\Omega |f_t|^p \,\mathrm{d}x = \int_\Omega |f|^p \,\mathrm{d}x + t\int_\Omega |f|^q \operatorname{div} W \,\mathrm{d}x + o(t).$$

Again, by the usual change of variables formula, we have

$$\begin{aligned}\int_\Omega |f_t|^q \,\mathrm{d}x &= \int_\Omega |f|^q \operatorname{Jac}(\varphi_t)\,\mathrm{d}x\\ &= \int_\Omega |f|^q(1 + t\operatorname{div} W + o(t))\,\mathrm{d}x\\ &= \int_\Omega |f|^p \,\mathrm{d}x + t\int_\Omega |f|^q \operatorname{div} W \,\mathrm{d}x + o(t),\end{aligned}$$

as we wanted to prove. □

Example 4.1.2. Let D be a locally finite perimeter set in Ω. If $D_t = \phi_t(\Omega)$, by Theorem 1.9.5 and the previous lemma, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|D_t|\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \chi_{D_t}\,\mathrm{d}x\Big|_{t=0}\\ &= \int_D \operatorname{div} V \,\mathrm{d}\,x\\ &= \int_D \langle V, \nu\rangle \,\mathrm{d}x.\end{aligned}$$

where ν is the generalized outer normal vector.

Now, we prove that $\|\cdot\|_{W^{1,p}(\Omega)}$ is differentiable with respect to t at $t = 0$. Note that, by the previous lemma, it is enough to prove that the L^p-norm of the gradient is differentiable.

Theorem 4.1.3. *Given $u \in W^{1,p}(\Omega)$*

$$u_t := u \circ \varphi_t^{-1} \to u \textit{ strongly in } W^{1,p}(\Omega), \textit{ as } t \to 0^+.$$

Moreover

$$\int_\Omega |\nabla u_t|^p \mathrm{d}x = \int_\Omega |\nabla u|^p \,\mathrm{d}x + t\int_\Omega |\nabla u|^p \operatorname{div} W \,\mathrm{d}x - pt\int_\Omega |\nabla u|^{p-2}\langle \nabla u,\, {}^T W' \nabla u^T\rangle \,\mathrm{d}x + o(t).$$

Proof. We proceed in three steps.

Step 1. First, we observe that, by the above lemma, we have that

$$u_t \to u \text{ strongly in } L^p(\Omega) \text{ as } t \to 0^+.$$

Then, it is enough to prove that

$$\nabla u_t \to \nabla u \text{ strongly in } (L^p(\Omega))^N \text{ as } t \to 0^+.$$

Step 2. We show that

$$\nabla u_t \to \nabla u \text{ strongly in } (L^p(\Omega))^N \text{ as } t \to 0^+,$$

Let $\varepsilon > 0$, by Theorem 1.6.4, there exists $g \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ such that

$$\|u - g\|_{W^{1,p}(\Omega)} < \varepsilon.$$

Let $g_t = g \circ \varphi_t^{-1}$, by the usual changes of variable formula, we get

$$\begin{aligned}\int_\Omega |\nabla u_t - \nabla g_t|^p \,\mathrm{d}x &= \int_\Omega |{}^T[\varphi_t']^{-1}(\nabla u - \nabla g)^T|^p \ \mathrm{Jac}(\phi_t)\,\mathrm{d}x \\ &= \int_\Omega |(Id - t^T W' + o(t))(\nabla u - \nabla g)^T|^p (1 + t \operatorname{div} W + o(t))\,\mathrm{d}x \\ &= \int_\Omega |\nabla u - \nabla g|^p \,\mathrm{d}x + t\int_\Omega |\nabla u - \nabla g|^p \ \operatorname{div} W\,\mathrm{d}x \\ &- tp\int_\Omega |\nabla u - \nabla g|^{p-2}\langle \nabla u - \nabla g, {}^T W'(\nabla u - \nabla g)^T\rangle\,\mathrm{d}x + o(t).\end{aligned}$$

Therefore, there exists $t_1 > 0$ such that if $0 < t < t_1$ then

$$\|\nabla u_t - \nabla g_t\|_{L^p(\Omega)} \le C\varepsilon,$$

where C is a constant independent of t.

As in the prove of the previous lemma, since $\varphi_t^{-1} \to Id$ in the C^1 topology when $t \to 0$ then $g_t = g \circ \varphi_t^{-1} \to g$ in the C^1 topology and therefore there exists $t_2 > 0$ such that if $0 < t < t_2$ then

$$\|\nabla g_t - \nabla g\|_{L^p(\Omega)} < \varepsilon.$$

Thus, for all $0 < t < t_0 = \min\{t_1, t_2\}$, we have

$$\begin{aligned}\|\nabla u_t - \nabla u\|_{L^p(\Omega)} &\le \|\nabla u_t - \nabla g_t\|_{L^p(\Omega)} + \|\nabla g_t - \nabla g\|_{L^p(\Omega)} + \|\nabla g - \nabla u\|_{L^p(\Omega)} \\ &< C\varepsilon,\end{aligned}$$

where C is a constant independent of t.

Step 3. Lastly, we show that

$$\int_\Omega |\nabla u_t|^p \mathrm{d}x = \int_\Omega |\nabla u|^p \mathrm{d}x + t\int_\Omega |\nabla u|^p \operatorname{div} W\,\mathrm{d}x - tp\int_\Omega |\nabla u|^{p-2}\langle \nabla u, {}^T W' \nabla u^T\rangle\,\mathrm{d}x + o(t).$$

Again, by the usual changes of variables formula, we have that

$$\begin{aligned}\int_\Omega |\nabla u_t|^p \,\mathrm{d}x &= \int_\Omega |^T[\varphi_t']^{-1}\nabla u^T|^p \ \mathrm{Jac}(\phi_t)\,\mathrm{d}x \\ &= \int_\Omega |(Id - t^T W' + o(t))\nabla u^T|^p(1 + t\,\mathrm{div}\, W + o(t))\,\mathrm{d}x \\ &= \int_\Omega |\nabla u|^p \,\mathrm{d}x + t\int_\Omega |\nabla u|^p \ \mathrm{div}\, W\,\mathrm{d}x - tp\int_\Omega |\nabla u|^{p-2}\langle \nabla u,^T W'\nabla u^T\rangle\,\mathrm{d}x + o(t).\end{aligned}$$

The prove is now complete. . □

Remark 4.1.4. By Lemma 4.1.1 and Theorem 4.1.3, we have that $\|u_t\|_{W^{1,p}(\Omega)}$ is differentiable with respect to t at $t = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^p_{W^{1,p}(\Omega)}\bigg|_{t=0} = \int_\Omega (|\nabla u|^p + |u|^p)\,\mathrm{div}\, W\,\mathrm{d}x - p\int_\Omega |\nabla u|^{p-2}\langle \nabla u,^T W'\nabla u^T\rangle\,\mathrm{d}x.$$

4.2 Results on differential geometry

Here, we state some results on differential geometry that will be used in the rest of this thesis. The proof of these results can be found, for instance, in [HP].

Definition 4.2.1 (Definition of the tangential Jacobian)**.** Let Φ be a C^1 field over $\mathbb{R}^N$. We call the tangential Jacobian of Φ

$$\mathrm{Jac}_\tau(\Phi) := |^T[\Phi']^{-1}\nu|\,\mathrm{Jac}(\Phi).$$

The definition of the tangential Jacobian is suited to state the following change of variables formula

Proposition 4.2.2. *Let f be a measurable function and let $\Omega_\Phi = \Phi(\Omega)$. Then $f \in L^1(\partial\Omega_\Phi)$ if only if $f \circ \Phi \in L^1(\partial\Omega)$ and*

$$\int_{\partial\Omega_\Phi} f\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial\Omega} (f\circ\Phi)\,\mathrm{Jac}_\tau(\Phi)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Definition 4.2.3 (Definition of the tangential divergence)**.** Let $V \in C^1(\partial\Omega, \mathbb{R}^N)$. The *tangential divergence* of V over $\partial\Omega$ is defined by

$$\mathrm{div}_\tau\, V := \mathrm{div}\,\widetilde{V} - \langle \widetilde{V}'\nu, \nu\rangle,$$

where $\widetilde{V} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $\widetilde{V}\big|_{\partial\Omega} = V$.

Observe that, the previous definition does not depend on the choice of $\widetilde{V}$.

Definition 4.2.4. The *mean curvature* of $\partial\Omega$ is defined by

$$H := \operatorname{div}_\tau \nu.$$

Definition 4.2.5. Let $g \in C^1(\partial\Omega, \mathbb{R})$. The *tangential gradient* is defined by

$$\nabla_\tau g := \nabla \widetilde{g} - \frac{\partial \widetilde{g}}{\partial \nu}\nu \quad \text{on } \partial\Omega,$$

where $\widetilde{g} \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\widetilde{g}\big|_{\partial\Omega} = g$.

This definition is also independent of the choice of the extension.

Remark 4.2.6. The Definitions 4.2.3 and 4.2.5 can be extended to $(W^{1,1}(\partial\Omega))^N$ and $W^{1,1}(\partial\Omega)$, respectively.

Proposition 4.2.7. *Let $g \in W^{1,1}(\partial\Omega)$ and $V \in C^1(\partial\Omega, \mathbb{R}^N)$. Then*

$$\langle V, \nabla_\tau g\rangle + g \operatorname{div}_\tau V = \operatorname{div}_\tau(gV).$$

Now, we give a version of the divergence Theorem.

Theorem 4.2.8 (Divergence Theorem)**.** *Let Ω be a bounded smooth open set of $\mathbb{R}^N$, $D \subset \partial\Omega$ be a (relatively) open smooth set. Let V be a $[W^{1,1}(\partial\Omega)]^N$ vector field. Then*

$$\int_D \operatorname{div}_\tau V \, d\mathcal{H}^{N-1} = \int_{\partial D} \langle V, \nu_\tau\rangle \, d\mathcal{H}^{N-2} + \int_D H\langle V, \nu\rangle \, d\mathcal{H}^{N-1},$$

where ν_τ is the outer unit normal vector to D along $\partial\Omega$.

4.3 Differentiation of the $L^q(\partial\Omega)-$norm

Now we are in condition to calculate the derivative of the norm $\|\cdot\|_{L^q(\partial\Omega)}$ with respect to perturbations in the domain.

Again, we begin by describing the kind of variations that we are considering. Let V be a regular (smooth) vector field, globally Lipschitz, with support in a neighbourhood of $\partial\Omega$ and let $\psi_t : \mathbb{R}^N \to \mathbb{R}^N$ be defined as the unique solution to

$$\begin{cases} \frac{d}{dt}\psi_t(x) = V(\psi_t(x)) & t > 0, \\ \psi_0(x) = x & x \in \mathbb{R}^N. \end{cases} \tag{4.4}$$

We have

$$\psi_t(x) = x + tV(x) + o(t) \quad \forall x \in \mathbb{R}^N.$$

Lemma 4.3.1. *Given $f \in L^q(\partial\Omega)$ then*

$$f_t = f \circ \psi_t^{-1} \to f \text{ strongly in } L^q(\partial\Omega), \text{ as } t \to 0.$$

Moreover

$$\int_{\partial\Omega} |f_t|^q \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial\Omega} |f|^q \, \mathrm{d}\mathcal{H}^{N-1} + t \int_{\partial\Omega} |f|^q \operatorname{div}_\tau V \, \mathrm{d}\mathcal{H}^{N-1} + o(t).$$

Proof. We proceed in two steps.

Steps 1 First, we show that

$$f_t \to f \text{ strongly in } L^q(\partial\Omega), \text{ as } t \to 0.$$

Let $\varepsilon > 0$, and let $g \in C_c^\infty(\partial\Omega)$ fixed such that $\|f - g\|_{L^q(\partial\Omega)} < \varepsilon$. By the Theorem 4.2.2, we have,

$$\|f_t - g_t\|^q_{L^q(\partial\Omega)} = \int_{\partial\Omega} |f - g|^q \operatorname{Jac}_\tau(\psi_t) \mathrm{d}\mathcal{H}^{N-1},$$

where $g_t = g \circ \psi_t^{-1}$. We also know that

$$\operatorname{Jac}_\tau(\psi) := 1 + t \operatorname{div}_\tau V + o(t). \tag{4.5}$$

Then

$$\|f_t - g_t\|^q_{L^q(\partial\Omega)} = \int_{\partial\Omega} |f - g|^q (1 + t \operatorname{div}_\tau V + o(t)) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Therefore, there exists $t_1 > 0$ such that if $0 < t < t_1$ then

$$\|f_t - g_t\|_{L^q(\partial\Omega)} \le C\varepsilon.$$

where C is a constant independent of t. Moreover, since $\psi_t^{-1} \to Id$ in the C^1 topology when $t \to 0$ then $g_t = g \circ \psi_t^{-1} \to g$ in the C^1 topology and therefore there exists $t_2 > 0$ such that if $0 < t < t_2$ then

$$\|g_t - g\|_{L^q(\partial\Omega)} < \varepsilon.$$

Finally, we have for all $0 < t < t_0 = \min\{t_1, t_2\}$ then

$$\begin{aligned} \|f_t - f\|_{L^q(\partial\Omega)} &\le \|f_t - g_t\|_{L^q(\partial\Omega)} + \|g_t - g\|_{L^q(\partial\Omega)} + \|g - f\|_{L^q(\partial\Omega)} \\ &\le C\varepsilon, \end{aligned}$$

where C is a constant independent of t.

Step 2 Now, we prove that

$$\int_{\partial\Omega} |f_t|^q \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial\Omega} |f|^q \, \mathrm{d}\mathcal{H}^{N-1} + t \int_{\partial\Omega} |f|^q \operatorname{div}_\tau V \, \mathrm{d}\mathcal{H}^{N-1} + o(t).$$

Again, by the Theorem 4.2.2 and (4.5), we have

$$\begin{aligned} \int_{\partial\Omega} |f_t|^q \, \mathrm{d}\mathcal{H}^{N-1} &= \int_{\partial\Omega} |f|^q (1 + t \operatorname{div}_\tau V + o(t)) \, \mathrm{d}\mathcal{H}^{N-1} \\ & \int_{\partial\Omega} |f|^q \, \mathrm{d}\mathcal{H}^{N-1} + t \int_{\partial\Omega} |f|^q \operatorname{div}_\tau V \, \mathrm{d}\mathcal{H}^{N-1} + o(t), \end{aligned}$$

as we wanted to prove. □

5

The first weighted eigenvalue problem plus a potential

In this chapter we consider the following eigenvalue problem with weights

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda g(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a smooth bounded open subset of $\mathbb{R}^N$. Here V is a potential function and g is a weight.

Our aim is to study the following optimization problems:

$$I := \inf \{\lambda(g, V) \colon g \in \mathcal{R}(g_0), V \in \mathcal{R}(V_0)\}, \tag{5.2}$$

where V_0 and g_0 are fixed potential and weight functions respectively that satisfy the following assumptions

$$V_0, g_0 \in L^q(\Omega) \text{ where } \begin{cases} q > \frac{N}{p} & \text{if } 1 < p \leq N, \\ q = 1 & \text{if } p > N. \end{cases} \tag{H1}$$

$$\|V_0^-\|_{L^q(\Omega)} < S_{pq'} \quad \text{or} \quad V_0 \geq -S_p + \delta \quad \text{for some } \delta > 0. \tag{H2}$$

(observe that, this assumptions are the same that we use in Section 1.8), and $\mathcal{R}(V_0)$, $\mathcal{R}(g_0)$ are the classes of rearrangements of V_0 and g_0 respectively.

A related minimization problem when the minimization parameter was allowed to vary in the class of rearrangements of a fixed function, was first considered by [CEP1].

More recently, in [CEP2], the authors analyze problem (5.2) but when the potential function is zero. In that work the authors prove the existence of a minimizing weight g_* in the class of rearrangements of a fixed function g_0 and, in the spirit of [Bu1] they found a sort of *Euler-Lagrange formula* for g_*. However, this formula does not appear to be suitable for use in actual computations of these minimizers.

In this chapter, we first extend the results in [CEP2] to (5.1) and prove the existence of a minimizing weight and potential for (5.2). Also the same type of Euler-Lagrange formula is proved for both the weight and potential. But, we go further and study the dependence of the eigenvalue $\lambda(g, V)$ with respect to g and V and prove the continuous dependence in L^q−norm and, moreover, the differentiability with respect to regular perturbations of the weight and the potential.

In the case when the perturbations are made inside the class of rearrangements, we exhibit a simple formula for the derivative of the eigenvalue with respect to g and V.

We believe that this formula can be used in actual computations of the optimal eigenvalue, weight and potential, since this type of formulas have been used in similar problems in the past with significant success, see [FBGR, H, O, P] and references therein.

The chapter is organized as follows. In Section 5.1, we prove the existence of a unique minimizer and give a characterization of it, similar to the one found in [CEP2] for the problem without potential. In Section 5.2, we study the dependence of the eigenvalue with respect to the weight and the potential and prove, first the continuous dependence in the L^q−topology (Proposition 5.2.1), and finally we show a simple formula for the derivative of the eigenvalue with respect to regular variations of the weight and the potential within the class of rearrangements (Theorem 5.2.11).

5.1 Minimization and characterization

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$ and $1 < p < \infty$.

Definition 5.1.1. Given g and V measurable functions, we say that g and V satisfy the asumption (H) if

$$\begin{cases} g \text{ satisfies the assumption (H1)}, \\ V \text{ satisfies the assumption (H1) and (H2)}. \end{cases} \tag{H}$$

Given g_0 and V_0 measurable functions that satisfy the asumption (H) our aim in this section is to analyze the following problem

$$I = \inf\{\lambda(g, V) \colon g \in \mathcal{R}(g_0),\ V \in \mathcal{R}(V_0)\},$$

where $\mathcal{R}(g_0)$ (resp. $\mathcal{R}(V_0)$) is the set of all rearrangements of g_0 (resp. V_0) and $\lambda(g, V)$ is the first positive principal eigenvalue of problem (5.1) and it is characterized by

$$\lambda(V, g) := \inf\left\{J_V(u) \colon u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega gu\,\mathrm{d}x = 1\right\} \tag{5.3}$$

where

$$J_V(u) := \int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega V(x)|u|^p\,\mathrm{d}x,$$

see Theorem 1.8.6.

Remark 5.1.2. Observe that if $g \in \mathcal{R}(g_0)$ and $V \in \mathcal{R}(V_0)$ then g and V satisfy (H).

We first need a lemma to show that, under hypotheses (H1) and (H2), the functionals $J_V(\cdot)$ are uniformly coercive for $V \in \mathcal{R}(V_0)$.

Lemma 5.1.3. *Let V_0 satisfies (H1) and (H2). Then, there exists $\delta_0 > 0$ such that*

$$J_V(u) \geq \delta_0 \int_\Omega |\nabla u|^p \,\mathrm{d}x, \quad \forall\, V \in \mathcal{R}(V_0).$$

Proof. We prove the lemma assuming that $\|V_0^-\|_{L^q(\Omega)} < S_{pq'}$. Also, we assume that $1 < p \leq N$. The other cases are easier and are left to the reader.

First, observe that

$$J_V(u) \geq \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V^-(x)|u|^p \,\mathrm{d}x.$$

On the other hand, $q > N/p$ implies that $pq' < p^*$. So

$$\int_\Omega |V^-(x)||u|^p \,\mathrm{d}x \leq \|V^-\|_{L^q(\Omega)}\|u\|^p_{L^{pq'}(\Omega)} = \|V_0^-\|_{L^q(\Omega)}\|u\|^p_{L^{pq'}(\Omega)}.$$

Then, by (H2), there exists δ_0 such that

$$\|V_0^-\|_{L^q(\Omega)} \leq (1-\delta_0)S_{pq'}.$$

Therefore

$$J_V(u) \geq \delta_0 \int_\Omega |\nabla u|^p \,\mathrm{d}x,$$

as we wanted to prove. □

Remark 5.1.4. We remark that is actually needed the uniform coercitivity of the functionals J_V for $V \in \mathcal{R}(V_0)$. Hypotheses (H1) and (H2) are a simple set of hypotheses that guaranty that.

We now prove that the infimum is achieved.

Theorem 5.1.5. *Let g_0 and V_0 be measurable functions that satisfy the assumption (H), and let $\mathcal{R}(g_0)$ and $\mathcal{R}(V_0)$ be the sets of all rearrangements of g_0 and V_0 respectively. Then there exists $g^* \in \mathcal{R}(g_0)$ and $V_* \in \mathcal{R}(V_0)$ such that*

$$I = \lambda(g^*, V_*).$$

Proof. Let $\{(g_n, V_n)\}_{n\in\mathbb{N}}$ be a minimizing sequence, i.e.,

$$g_n \in \mathcal{R}(g_0) \text{ and } V_n \in \mathcal{R}(V_0) \quad \forall\, n \in \mathbb{N}$$

and

$$I = \lim_{n\to\infty} \lambda(g_n, V_n).$$

Let u_n be the positive eigenfunction corresponding to $\lambda(g_n, V_n)$ then

$$\int_\Omega g_n(x)u_n^p = 1 \quad \forall\, n \in \mathbb{N}, \tag{5.4}$$

and

$$\lambda(g_n, V_n) = \int_\Omega |\nabla u_n|^p \, \mathrm{d}x + \int_\Omega V_n(x)u_n^p \, \mathrm{d}x \quad \forall\, n \in \mathbb{N}.$$

Hence

$$I = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \, \mathrm{d}x + \int_\Omega V_n(x)u_n^p \, \mathrm{d}x. \tag{5.5}$$

Thus, by Lemma 5.1.3, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ and therefore there exists $u \in W_0^{1,p}(\Omega)$ and some subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (still denoted by $\{u_n\}_{n\in\mathbb{N}}$) such that

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \tag{5.6}$$

$$u_n \to u \quad \text{strongly in } L^{pq'}(\Omega). \tag{5.7}$$

Recall that our assumptions on q imply that $pq' < p^*$.

On the other hand, $g_n \in \mathcal{R}(g_0)$ and $V_n \in \mathcal{R}(V_0)$ for all $n \in \mathbb{N}$ then

$$\|g_n\|_{L^q(\Omega)} = \|g_0\|_{L^q(\Omega)} \text{ and } \|V_n\|_{L^q(\Omega)} = \|V_0\|_{L^q(\Omega)} \quad \forall\, n \in \mathbb{N}.$$

Therefore, there exists f, $W \in L^q(\Omega)$ and subsequence of $\{g_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ (still call by $\{g_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$) such that

$$g_n \rightharpoonup f \quad \text{weakly in } L^q(\Omega), \tag{5.8}$$

$$V_n \rightharpoonup W \quad \text{weakly in } L^q(\Omega). \tag{5.9}$$

Thus, by (5.5), (5.6), (5.7) and (5.9), we have that

$$I \geq \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega W(x)|u|^p \, \mathrm{d}x,$$

and by (5.4), (5.7) and (5.8) we get

$$\int_\Omega f(x)|u|^p \, \mathrm{d}x = 1.$$

Now, since $f \in \overline{\mathcal{R}(g_0)}$ and $W \in \overline{\mathcal{R}(V_0)}$, by Theorem 1.5.3, there exists $g^* \in \mathcal{R}(g_0)$ and $V_* \in \mathcal{R}(V_0)$ such that

$$\alpha = \int_\Omega g^*(x)|u|^p \, \mathrm{d}x \geq \int_\Omega f(x)u^p \, \mathrm{d}x = 1$$

and

$$\int_\Omega V_*(x)u^p \, \mathrm{d}x \leq \int_\Omega W(x)|u|^p \, \mathrm{d}x.$$

Let $v = \alpha^{-1/p}|u|$, then
$$\int_\Omega g^*(x)v^p \mathrm{d}x = 1,$$
and
$$\begin{aligned}\int_\Omega |\nabla v|^p \,\mathrm{d}x + \int_\Omega V_*(x)v^p \,\mathrm{d}x &= \frac{1}{\alpha}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V_*(x)|u|^p \,\mathrm{d}x \\ &\le \frac{1}{\alpha}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega W(x)|u|^p \,\mathrm{d}x.\end{aligned}$$

Consequently
$$\lambda(g^*, V_*) \le I,$$
then
$$I = \lambda(g^*, V_*).$$
The proof is now complete. □

Now we give a characterization of g^* and V_*.

Theorem 5.1.6. *Let g_0 and V_0 be measurable functions that satisfy the assumption* (H). *Let $g^* \in \mathcal{R}(g_0)$ and $V_* \in \mathcal{R}(V_0)$ be such that $\lambda(g^*, V_*) = I$ are the ones given by Theorem 5.1.5. Then there exist an increasing function ϕ and a decreasing function ψ such that*
$$\begin{aligned} g^* &= \phi(u_*) \quad \text{a.e. in } \Omega, \\ V_* &= \psi(u_*) \quad \text{a.e. in } \Omega,\end{aligned}$$
where u_* is the positive eigenfunction associated to $\lambda(g^*, V_*)$.

Proof. We proceed in four steps.

Step 1. First we show that V_* is a minimizer of the linear functional
$$L(V) := \int_\Omega V(x)u_*^p \,\mathrm{d}x$$
relative to $V \in \overline{\mathcal{R}(V_0)}$.

We have that
$$\int_\Omega g^*(x)u_*^p \,\mathrm{d}x = 1$$
and
$$I = \lambda(g^*, V_*) = \int_\Omega |\nabla u_*|^p \,\mathrm{d}x + \int_\Omega V_*(x)u_*^p \,\mathrm{d}x,$$
then, for all $V \in \mathcal{R}(V_0)$,
$$\int_\Omega |\nabla u_*|^p \,\mathrm{d}x + \int_\Omega V_*(x)u_*^p \,\mathrm{d}x \le \lambda(g^*, V) \le \int_\Omega |\nabla u_*|^p \,\mathrm{d}x + \int_\Omega V(x)u_*^p \,\mathrm{d}x$$

and therefore

$$\int_\Omega V_*(x)u_*^p \,\mathrm{d}x \le \int_\Omega V(x)u_*^p \,\mathrm{d}x \quad \forall\, V \in \mathcal{R}(V_0).$$

Thus, we can conclude that

$$\int_\Omega V_*(x)u_*^p \,\mathrm{d}x = \inf\left\{L(V) : V \in \overline{\mathcal{R}(V_0)}\right\}.$$

Step 2. We show that V_* is the unique minimizer of $L(V)$ relative to $\mathcal{R}(V_0)$.

Suppose that W is another minimizer of $L(V)$ relative to $\mathcal{R}(V_0)$, then

$$\int_\Omega V_*(x)u_*^p \,\mathrm{d}x = \int_\Omega W(x)u_*^p \,\mathrm{d}x.$$

Thus

$$\begin{aligned} I &= \lambda(g^*, V_*) \\ &= \int_\Omega |\nabla u_*|^p \,\mathrm{d}x + \int_\Omega V_*(x)u_*^p \,\mathrm{d}x \\ &= \int_\Omega |\nabla u_*|^p \,\mathrm{d}x + \int_\Omega W(x)u_*^p \,\mathrm{d}x \\ &\ge \lambda(g^*, W) \\ &\ge I. \end{aligned}$$

Hence u_* is the positive eigenfunction associated to $\lambda(g^*, V_*) = \lambda(g^*, W)$. Then

$$-\Delta_p u_* + V_*(x)u_*^{p-1} = \lambda(g^*, V_*)g^*(x)u_*^{p-1} \quad \text{in } \Omega, \tag{5.10}$$

$$-\Delta_p u_* + W(x)u_*^{p-1} = \lambda(g^*, V_*)g^*(x)u_*^{p-1} \quad \text{in } \Omega. \tag{5.11}$$

Subtracting (5.11) from (5.10), we get

$$(V_*(x) - W(x))u_*^{p-1} = 0 \quad \text{a.e. in } \Omega,$$

then $V_* = W$ a.e. in Ω.

Thus, by Theorem 1.5.4, there exists decreasing function ψ such that

$$V_* = \psi(u_*) \quad \text{a.e. in } \Omega.$$

Step 3. Now, we show that g^* is a maximizer of the linear functional

$$H(g) := \int_\Omega g(x)u_*^p \,\mathrm{d}x$$

relative to $g \in \overline{\mathcal{R}(g_0)}$.

We argue by contradiction, so assume that there exists $g \in \mathcal{R}(g_0)$ such that

$$\alpha = \int_\Omega g(x)u_*^p \,\mathrm{d}x > \int_\Omega g^*(x)u_*^p \,\mathrm{d}x = 1$$

and take $v = \alpha^{-1/p} u_*$. Then

$$\int_\Omega g(x) v^p \, \mathrm{d}x = 1$$

and

$$\int_\Omega |\nabla v|^p \, \mathrm{d}x + \int_\Omega V_*(x) v^p \, \mathrm{d}x = \frac{1}{\alpha} \int_\Omega |\nabla u_*|^p \, \mathrm{d}x + \int_\Omega V_*(x) u_*^p \, \mathrm{d}x = \frac{1}{\alpha} \lambda(g^*, V_*) < \lambda(g^*, V_*).$$

Therefore

$$\lambda(g, V_*) < \lambda(g^*, V_*),$$

which contradicts the minimality of $\lambda(g^*, V_*)$.

Step 4. Lastly, we show that g^* is the unique maximizer of $H(g)$ relative to $\mathcal{R}(g_0)$.

Assume that there exists another maximizer f of $H(g)$ relative to $\mathcal{R}(g_0)$. Then

$$\int_\Omega f(x) u_*^p \, \mathrm{d}x = \int_\Omega g^*(x) u_*^p \, \mathrm{d}x = 1$$

and therefore

$$I = \lambda(g^*, V_*) \leq \lambda(f, V_*) \leq \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega V_*(x) u_*^p \, \mathrm{d}x = I,$$

then $\lambda(g^*, V_*) = \lambda(f, V_*)$ and hence u_* is the eigenfunction associated to $\lambda(g^*, V_*) = \lambda(f, V_*)$. Thus

$$\begin{aligned} -\Delta_p u_* + V_*(x) u_*^{p-1} &= \lambda(g^*, V_*) g^*(x) u_*^{p-1} \quad \text{in } \Omega, && (5.12) \\ -\Delta_p u_* + V_*(x) u_*^{p-1} &= \lambda(g^*, V_*) f(x) u_*^{p-1} \quad \text{in } \Omega. && (5.13) \end{aligned}$$

Subtracting (5.13) from (5.12), we get

$$\lambda(g^*, V_*)\,(g^*(x) - f(x))\, u_*^p = 0 \quad \text{a.e. in } \Omega,$$

thus $g^* = f$ a.e. in Ω.

Then, by Theorem 1.5.4, there exist increasing function ϕ such that

$$g^* = \phi(u_*) \quad \text{a.e. in } \Omega.$$

This finishes the proof. □

5.2 Differentiation of the eigenvalue

The first aim of this section is prove the continuity of the first positive eigenvalue $\lambda(g, V)$ with respect to g and V. Then we proceed further and compute the derivative of $\lambda(g, V)$ with respect to perturbations in g and V.

Proposition 5.2.1. *The first positive eigenvalue* $\lambda(g, V)$ *of* (5.1) *is continuous with respect to* $(g, V) \in \mathcal{A}$ *where*

$$\mathcal{A} := \{(g, V) \in L^q(\Omega) \times L^q(\Omega) \colon g \text{ and } V \text{ satisfy (H)} \}.$$

i.e.,

$$\lambda(g_n, V_n) \to \lambda(g, V),$$

when $(g_n, V_n) \to (g, V)$ *strongly in* $L^q(\Omega) \times L^q(\Omega)$ *and* $(g_n, V_n), (g, V) \in \mathcal{A}$.

Proof. We know that

$$\lambda(g_n, V_n) = \int_\Omega |\nabla u_n|^p \, \mathrm{d}x + \int_\Omega V_n(x) u_n^p \, \mathrm{d}x$$

and

$$\lambda(g, V) = \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega V(x) u^p \, \mathrm{d}x,$$

with

$$\int_\Omega g_n(x) u_n^p \, \mathrm{d}x = \int_\Omega g(x) u^p \, \mathrm{d}x = 1,$$

where u_n and u are the positive eigenfunction associated to $\lambda(g_n, V_n)$ and $\lambda(g, V)$ respectively.

We begin by observing that

$$H(g_n) := \int_\Omega g_n(x) u^p \, \mathrm{d}x = \int_\Omega (g_n(x) - g(x)) u^p \, \mathrm{d}x + 1 \to 1,$$

as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that

$$H(g_n) > 0 \quad \forall\, n \geq n_0.$$

Thus we take $v_n := H(g_n)^{-1/p} u$, and by (5.3) we have

$$\lambda(g_n, V_n) \leq \int_\Omega |\nabla v_n|^p \, \mathrm{d}x + \int_\Omega V_n(x) v_n^p \, \mathrm{d}x = \frac{1}{H(g_n)} \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega V_n(x) u^p \, \mathrm{d}x.$$

Therefore, taking limits when $g_n \to g$ and $V_n \to V$ in $L^q(\Omega)$, we get that

$$\limsup_{n\to\infty} \lambda(g_n, V_n) \leq \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega V(x) u^p \, \mathrm{d}x = \lambda(g, V).$$

On the other hand, as $V_n \to V$ strongly in $L^q(\Omega)$ it is easy to see that there exist $n_1 \in \mathbb{N}$ and $\delta_1 > 0$ such that

$$\|V_n^-\|_{L^q(\Omega)}, \|V^-\|_{L^q(\Omega)} < S_{pq'}(1 - \delta_1) \qquad \forall n \geq n_1,$$

or there exist a subsequence of $\{V_n\}_{n\in\mathbb{N}}$, wich we denote again by $\{V_n\}_{n\in\mathbb{N}}$, and $\delta_2 > 0$ such that

$$V_n, V > -S_p + \delta_2 \qquad \forall n \in \mathbb{N}.$$

Therefore, as $\{\lambda(g_n, V_n)\}_{n\in\mathbb{N}}$ is bounded, arguing as in Lemma 5.1.3, we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore there exists $v \in W_0^{1,p}(\Omega)$ and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (that we still denote by $\{u_n\}_{n\in\mathbb{N}}$) such that

$$u_n \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{5.14}$$
$$u_n \to v \quad \text{strongly in } L^{pq'}(\Omega). \tag{5.15}$$

By (5.15) and as $g_n \to g$ in $L^q(\Omega)$, we have that

$$1 = \lim_{n\to\infty} \int_\Omega g_n(x)|u_n|^p \,\mathrm{d}x = \int_\Omega g(x)|v|^p \,\mathrm{d}x.$$

Finally, by (5.14), (5.15) and, as $V_n \to V$ in $L^q(\Omega)$, we arrive at

$$\begin{aligned}
\liminf_{n\to\infty} \lambda(g_n, V_n) &= \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega V_n(x) u_n^p \,\mathrm{d}x \\
&\geq \int_\Omega |\nabla v|^p \,\mathrm{d}x + \int_\Omega V(x)|v|^p \,\mathrm{d}x \\
&\geq \lambda(g, V)
\end{aligned}$$

and the result follows. □

Remark 5.2.2. Observe that if instead of (H2) we required *only* that $V > -S_p + \delta$, the same proof of Proposition 5.2.1 gives the continuity of $\lambda(g, V)$ with respect to weak convergence.

Now we arrive at the main result of this section, namely we compute the derivative of the first positive eigenvalue $\lambda(g, V)$ with respect to perturbations in g and V.

We begin by describing the kind of variations that we are going to consider. Let W be a regular (smooth) vector field, globally Lipschitz, with support in Ω and let $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$ be the flow defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x) = W(\varphi_t(x)) & t > 0, \\ \varphi_0(x) = x & x \in \mathbb{R}^N. \end{cases} \tag{5.16}$$

We have

$$\varphi_t(x) = x + tW(x) + o(t) \quad \forall x \in \mathbb{R}^N.$$

Thus, if g and V are measurable functions that satisfy the assumption (H), we define $g_t := g \circ \varphi_t^{-1}$ and $V_t := V \circ \varphi_t^{-1}$. Now, let

$$\lambda(t) := \lambda(g_t, V_t) = \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x,$$

with

$$\int_\Omega g_t(x) u_t^p \,\mathrm{d}x = 1,$$

where u_t is the eigenfunction associated to $\lambda(t)$.

Remark 5.2.3. In order for this approach to be useful for the optimization problem of the previous section, we need to guaranty that $g_t \in \mathcal{R}(g_0)$ and $V_t \in \mathcal{R}(V_0)$ whenever $g \in \mathcal{R}(g_0)$ and $V \in \mathcal{R}(V_0)$.

It is not difficult to check that this is true for incompressible deformation fields, i.e., for those W's such that

$$\operatorname{div} W = 0.$$

By Proposition 5.2.1 and Lemma 4.1.1, we have that

Theorem 5.2.4. *Let g and V be measurable functions that satisfy the assumption* (H). *Then, with the previous notation, $\lambda(t)$ is continuous at $t = 0$, i.e.,*

$$\lambda(t) \to \lambda(0) = \lambda(g, V) \quad \text{as } t \to 0^+.$$

Lemma 5.2.5. *Let g and V be measurable functions that satisfy the assumption* (H). *Let u_t be the normalized positive eigenfunction associated to $\lambda(t)$ with $t > 0$. Then*

$$\lim_{t \to 0^+} u_t = u_0 \quad \text{strongly in } W_0^{1,p}(\Omega).$$

where u_0 is the unique normalized positive eigenfunction associated to $\lambda(g, V)$.

Proof. From the previous theorem, we deduce that $\lambda(t)$ is bounded and, as in the proof of Proposition 5.2.1, we further deduce that $\{u_t\}$ is bounded in $W_0^{1,p}(\Omega)$.

So, given $\{t_n\}_{n\in\mathbb{N}}$, we have that $\{u_{t_n}\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ and therefore there exists $u_0 \in W_0^{1,p}(\Omega)$ and some subsequence (still denoted by $\{u_{t_n}\}_{n\in\mathbb{N}}$) such that

$$u_{t_n} \rightharpoonup u_0 \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{5.17}$$

$$u_{t_n} \to u_0 \quad \text{strongly in } L^{pq'}(\Omega). \tag{5.18}$$

Since $(g_{t_n}, V_{t_n}) \to (g, V)$ strongly in $L^q(\Omega) \times L^q(\Omega)$ as $n \to \infty$ and by (5.18) we get

$$1 = \lim_{n\to\infty} \int_\Omega g_{t_n}(x)|u_{t_n}|^p \,dx = \int_\Omega g(x)|u_0|^p \,dx$$

and

$$\lim_{n\to\infty} \int_\Omega V_{t_n}(x)|u_{t_n}|^p \,dx = \int_\Omega V(x)|u_0|^p \,dx.$$

Thus, using (5.17),

$$\begin{aligned}
\lambda(0) &= \lim_{n\to\infty} \lambda(t_n) \\
&= \lim_{n\to\infty} \int_\Omega |\nabla u_{t_n}|^p \,dx + \int_\Omega V_{t_n}(x)|u_{t_n}|^p \,dx \\
&\geq \int_\Omega |\nabla u_0|^p \,dx + \int_\Omega V(x)|u_0|^p \,dx \\
&\geq \lambda(0),
\end{aligned}$$

then u_0 is the a normalized eigenfunction associated to $\lambda(0)$ and, as $\{u_{t_n}\}_{n\in\mathbb{N}}$ are positive, it follows that u_0 is positive.

Moreover

$$\|\nabla u_{t_n}\|_{L^p(\Omega)} \to \|\nabla u_0\|_{L^p(\Omega)} \quad \text{as } n \to \infty.$$

Then, using again (5.17), we have

$$u_{t_n} \to u_0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.$$

as we wanted to show. □

Remark 5.2.6. It is easy to see that, as $\varphi_t \to Id$ in the C^1 topology, then from Lemma 5.2.5 it follows that

$$u_t \circ \varphi_t \to u_0 \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ as } t \to 0,$$

when $u_t \to u_0$ strongly in $W_0^{1,p}(\Omega)$.

Now, we arrive at the main result of the section

Theorem 5.2.7. *With the previous notation, if g and V are measurable functions that satisfy the assumption* (H)*, we have that $\lambda(t)$ is differentiable at $t = 0$ and*

$$\begin{aligned}\frac{d\lambda(t)}{dt}\Big|_{t=0} = &\int_\Omega (|\nabla u_0|^p + V(x)|u_0|^p)\operatorname{div} W\,dx - p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W'\nabla u_0^T\rangle\,dx\\ &- \lambda(0)\int_\Omega g(x)|u_0|^p \operatorname{div} W\,dx,\end{aligned}$$

where u_0 is the eigenfunction associated to $\lambda(0) = \lambda(g, V)$.

Proof. First we consider $v_t := u_0 \circ \varphi_t^{-1}$. Then, by the Lemma 4.1.1, we get

$$\begin{aligned}\int_\Omega g_t(x)|v_t|^p\,dx &= 1 + t\int_\Omega g(x)|u_0|^p \operatorname{div} W\,dx + o(t),\\ \int_\Omega V_t(x)|v_t|^p\,dx &= \int_\Omega V(x)|u_0|^p\,dx + t\int_\Omega V(x)|u_0|^p \operatorname{div} W\,dx + o(t)\end{aligned}$$

and, by Theorem 4.1.3,

$$\begin{aligned}\int_\Omega |\nabla v_t|^p\,dx = &\int_\Omega |\nabla u_0|^p\,dx + t\int_\Omega |\nabla u_0|^p \operatorname{div} W\,dx\\ &- tp\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W'\nabla u_0^T\rangle\,dx + o(t).\end{aligned}$$

Then, for t small enough,

$$\int_\Omega g_t(x)|v_t|^p\,dx > 0,$$

and therefore

$$\lambda(t) \leq \frac{\int_\Omega |\nabla v_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|v_t|^p \,\mathrm{d}x}{\int_\Omega g_t(x)|v_t|^p \,\mathrm{d}x}.$$

So

$$\lambda(t) \int_\Omega g_t(x)|v_t|^p \,\mathrm{d}x \leq \int_\Omega |\nabla v_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|v_t|^p \,\mathrm{d}x,$$

then, we have that

$$\begin{aligned}
\lambda(t)\left(1 + t\int_\Omega g(x)|u_0|^p \operatorname{div} W \,\mathrm{d}x\right) \leq{}& \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V(x)|u_0|^p \,\mathrm{d}x \\
&+ t\int_\Omega (|\nabla u_0|^p + V(x)|u_0|^p) \operatorname{div} W \,\mathrm{d}x \\
&- tp\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W' \nabla u_0^T\rangle \,\mathrm{d}x + o(t) \\
={}& \lambda(0) + t\int_\Omega (|\nabla u_0|^p + V(x)|u_0|^p) \operatorname{div} W \,\mathrm{d}x \\
&- tp\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W' \nabla u_0^T\rangle \,\mathrm{d}x + o(t),
\end{aligned}$$

and we get that

$$\begin{aligned}
\frac{\lambda(t)-\lambda(0)}{t} \leq{}& \int_\Omega (|\nabla u_0|^p + V(x)|u_0|^p) \operatorname{div} W \,\mathrm{d}x - p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W' \nabla u_0^T\rangle \,\mathrm{d}x \\
&- \lambda(t)\int_\Omega g(x)|u_0|^p \operatorname{div} W \,\mathrm{d}x + o(1).
\end{aligned}$$

In a similar way, if we take $w_t = u_t \circ \varphi_t$ we have that

$$\begin{aligned}
\frac{\lambda(t)-\lambda(0)}{t} \geq{}& \int_\Omega (|\nabla w_t|^p + V(x)|w_t|^p) \operatorname{div} W \,\mathrm{d}x - p\int_\Omega |\nabla w_t|^{p-2}\langle \nabla w_t,^T W' \nabla w_t^T\rangle \,\mathrm{d}x \\
&- \lambda(0)\int_\Omega g(x)|w_t|^p \operatorname{div} W \,\mathrm{d}x + o(1).
\end{aligned}$$

Thus, taking limit in the two last inequalities as $t \to 0^+$, by the Theorem 5.2.4 and Remark 5.2.6, we get that

$$\begin{aligned}
\lim_{t\to 0^+} \frac{\lambda(t)-\lambda(0)}{t} ={}& \int_\Omega (|\nabla u_0|^p + V(x)|u_0|^p) \operatorname{div} W \,\mathrm{d}x - p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W' \nabla u_0^T\rangle \,\mathrm{d}x \\
&- \lambda(0)\int_\Omega g(x)|u_0|^p \operatorname{div} W \,\mathrm{d}x.
\end{aligned}$$

This finishes the proof. □

Remark 5.2.8. When we work in the class of rearrangements of a fixed pair (g_0, V_0), as was mentioned in Remark 5.2.3, we need the deformation field W to verified $\operatorname{div} W = 0$. So, in this case, the formula for $\lambda'(0)$ reads,

$$\left.\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\right|_{t=0} = -p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W' \nabla u_0^T\rangle \,\mathrm{d}x.$$

In order to improve this expression for $\lambda'(0)$, we need a lemma that will allow us to regularize problem (5.1) since solutions to (5.1) are $C^{1,\delta}$ for some $\delta > 0$ but are not C^2 nor $W^{2,q}$ in general (see [L]).

Lemma 5.2.9. *Let V, g be measurable functions that satisfy the assumption (H), and let $V_\varepsilon, g_\varepsilon \in C_0^\infty(\Omega)$ be such that $V_\varepsilon \to V$ and $g_\varepsilon \to g$ in $L^q(\Omega)$. Let*

$$\lambda_\varepsilon := \min\left\{J_\varepsilon(v)\colon\, v \in W_0^{1,p}(\Omega),\, \int_\Omega g_\varepsilon(x)|v|^p\,\mathrm{d}x = 1\right\}$$

where

$$J_\varepsilon(v) := \int_\Omega (|\nabla v|^2 + \varepsilon^2)^{(p-2)/2}|\nabla v|^2\,\mathrm{d}x + \int_\Omega V_\varepsilon(x)|v|^p\,\mathrm{d}x.$$

Finally, let u_ε be the unique normalized positive eigenfunction associated to λ_ε.

Then, $\lambda_\varepsilon \to \lambda(g, V)$ and $u_\varepsilon \to u_0$ strongly in $W_0^{1,p}(\Omega)$, where u_0 is the unique normalized positive eigenfunction associated to $\lambda(g, V)$.

Proof. First, observe that, as $g_\varepsilon \to g$ in $L^q(\Omega)$ if u_0 is the normalized positive eigenfunction associated to $\lambda(g, V)$, we have that

$$\int_\Omega g_\varepsilon(x)|u_0|^p\,\mathrm{d}x > 0.$$

for all $\varepsilon > 0$ small enough. Then, for all $\varepsilon > 0$ small enough, taking

$$v_\varepsilon = \frac{u_0}{\int_\Omega g_\varepsilon(x)|u_0|^p\,\mathrm{d}x}$$

in the characterization of λ_ε, we get

$$\lambda_\varepsilon \leq \int_\Omega (|\nabla v_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2}|\nabla v_\varepsilon|^2 + V_\varepsilon(x)|v_\varepsilon|^p\,\mathrm{d}x.$$

Hence, passing to the limit as $\varepsilon \to 0^+$, since $\int_\Omega g_\varepsilon(x)|u_0|^p\,\mathrm{d}x \to \int_\Omega g(x)|u_0|^p\,\mathrm{d}x = 1$ as $\varepsilon \to 0^+$, we arrive at

$$\limsup_{\varepsilon\to 0^+} \lambda_\varepsilon \leq \lambda(g, V).$$

Now, for any $v \in W_0^{1,p}(\Omega)$ normalized such that

$$\int_\Omega g_\varepsilon(x)|v|^p\,\mathrm{d}x = 1,$$

we have that

$$\begin{aligned}\int_\Omega (|\nabla v|^2 + \varepsilon^2)^{(p-2)/2}|\nabla v|^2\,\mathrm{d}x + \int_\Omega V_\varepsilon(x)|v|^p\,\mathrm{d}x &\geq \int_\Omega |\nabla v|^p\,\mathrm{d}x + \int_\Omega V_\varepsilon(x)|v|^p\,\mathrm{d}x \\ &\geq \lambda(g_\varepsilon, V_\varepsilon),\end{aligned}$$

therefore $\lambda_\varepsilon \geq \lambda(g_\varepsilon, V_\varepsilon)$.

Now, by Proposition 5.2.1, we have that $\lambda(g_\varepsilon, V_\varepsilon) \to \lambda(g, V)$ as $\varepsilon \to 0^+$. So

$$\liminf_{\varepsilon \to 0^+} \lambda_\varepsilon \geq \lambda(g, V).$$

Finally, from the convergence of the eigenvalues, it is easy to see that the normalized eigenfunction u_ε associated to λ_ε are bounded in $W_0^{1,p}(\Omega)$ uniformly in $\varepsilon > 0$. Therefore, there exists a sequence, that we still call $\{u_\varepsilon\}$, and a function $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega), \\ u_\varepsilon &\to u \qquad \text{strongly in } L^{pq'}(\Omega). \end{aligned}$$

Recall that our assumptions on q imply that $pq' < p^*$.

Hence,

$$\int_\Omega g(x)|u|^p \, dx = \lim_{\varepsilon \to 0^+} \int_\Omega g_\varepsilon(x)|u_\varepsilon|^p \, dx = 1,$$

and so

$$\begin{aligned} \lambda(g, V) &= \lim_{\varepsilon \to 0^+} \lambda_\varepsilon \\ &= \lim_{\varepsilon \to 0^+} \int_\Omega (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2} |\nabla u_\varepsilon|^2 \, dx + \int_\Omega V_\varepsilon(x)|u_\varepsilon|^p \, dx \\ &\geq \int_\Omega |\nabla u|^p \, dx + \int_\Omega V(x)|u|^p \, dx \\ &\geq \lambda(g, V). \end{aligned}$$

These imply that $u = u_0$ the unique normalized positive eigenfunction associated to $\lambda(g, V)$ and that $\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \to \|u\|_{W_0^{1,p}(\Omega)}$ as $\varepsilon \to 0^+$. So

$$u_\varepsilon \to u_0 \qquad \text{strongly in } W_0^{1,p}(\Omega).$$

This finishes the proof. □

Remark 5.2.10. Observe that the eigenfunctions u_ε are weak solutions to

$$\begin{cases} -\operatorname{div}((|\nabla u_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2} \nabla u_\varepsilon) + V_\varepsilon(x)|u_\varepsilon|^{p-2} u_\varepsilon = \lambda_\varepsilon g_\varepsilon(x)|u_\varepsilon|^{p-2} u_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.19}$$

Therefore, by the classical regularity theory (see [LU]), the functions u_ε are $C^{2,\delta}$ for some $\delta > 0$.

With these preparatives we can now prove the following Theorem.

Theorem 5.2.11. *With the assumptions and notations of Theorem 5.2.7, we have that*

$$\left.\frac{d\lambda(t)}{dt}\right|_{t=0} = \lambda'(0) = \int_\Omega (V(x) - \lambda(0) g(x)) \operatorname{div}(|u_0|^p W) \, dx,$$

for every field W such that $\operatorname{div} W = 0$.

Proof. During the proof of the Theorem, we will require that the eigenfunction u_0 to be C^2. As it is well known (see [L]), this is not true.

In order to overcome this difficulty, we regularize the problem and work with the regularized eigenfunction u_ε defined in Lemma 5.2.9.

Since in the resulting formula only appears up to the first derivatives of u_ε and $u_\varepsilon \to u_0$ strongly in $W_0^{1,p}(\Omega)$ the result will follows.

Given $W \in C_0^1(\Omega;\mathbb{R}^N)$ such that $\operatorname{div} W = 0$, by the Theorem 5.2.7 and the Lemma 5.2.9, we have that

$$\begin{aligned}\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\Big|_{t=0} &= -p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0,^T W'\nabla u_0^T\rangle \,\mathrm{d}x\\ &= \lim_{\varepsilon\to 0} -p\int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon,^T W'\nabla u_\varepsilon^T\rangle \,\mathrm{d}x\end{aligned}$$

Since $W \in C_0^1(\Omega;\mathbb{R}^N)$,

$$\int_\Omega \operatorname{div}((|\nabla u_\varepsilon|^2+\varepsilon^2)^{p/2}W)\,\mathrm{d}x = 0.$$

So,

$$\begin{aligned}\int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon,^T W'\nabla u_\varepsilon^T\rangle \,\mathrm{d}x &= \int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon,^T W'\nabla u_\varepsilon^T\rangle \,\mathrm{d}x\\ &+\frac{1}{p}\int_\Omega \operatorname{div}((|\nabla u_0|^2+\varepsilon^2)^{p/2}W)\,\mathrm{d}x\\ &= \int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon,^T W'\nabla u_\varepsilon^T\rangle \,\mathrm{d}x\\ &+\int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon, D^2u_\varepsilon W^T\rangle \,\mathrm{d}x\\ &= \int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon,^T W'\nabla u_\varepsilon^T + D^2u_\varepsilon W^T\rangle \,\mathrm{d}x\\ &= \int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon, \nabla\langle\nabla u_\varepsilon, W\rangle\rangle \,\mathrm{d}x.\end{aligned}$$

Now, we use the fact that u_ε is a weak solution to (5.19) to get

$$\begin{aligned}\int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon,^T W'\nabla u_\varepsilon^T\rangle \,\mathrm{d}x &= \int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon, \nabla\langle\nabla u_\varepsilon, W\rangle\rangle \,\mathrm{d}x\\ &= \int_\Omega (\lambda_\varepsilon g_\varepsilon(x) - V_\varepsilon)|u_\varepsilon|^{p-2}u_\varepsilon\langle\nabla u_\varepsilon, W\rangle \,\mathrm{d}x\end{aligned}$$

Now, using again the Lemma 5.2.9, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega \|\nabla u_\varepsilon|^2+\varepsilon^2|^{(p-2)/2}\langle \nabla u_\varepsilon,^T W'\nabla u_\varepsilon^T\rangle \,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega (\lambda_\varepsilon g_\varepsilon(x) - V_\varepsilon)|u_\varepsilon|^{p-2}u_\varepsilon\langle\nabla u_\varepsilon, W\rangle \,\mathrm{d}x\\ &= \int_\Omega (\lambda(0)g(x) - V(x))\operatorname{div}(|u_0|^pW)\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda'(0) &= -p \lim_{\varepsilon\to 0} \int_\Omega ||\nabla u_\varepsilon|^2 + \varepsilon^2|^{(p-2)/2} \langle \nabla u_\varepsilon,^T W' \nabla u_\varepsilon^T \rangle \, \mathrm{d}x \\
&= p \int_\Omega (V(x) - \lambda(0) g(x)) |u_0|^{p-2} u_0 \langle \nabla u_0, W \rangle \, \mathrm{d}x \\
&= \int_\Omega (V(x) - \lambda(0) g(x)) \operatorname{div}(|u_0|^p W) \, \mathrm{d}x.
\end{aligned}$$

The proof is now complete. □

6

Some optimization problems for $p-$Laplacian type equations

In this chapter we analyze the following optimization problem: Consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and some class of admissible loads $\mathcal{A}$. Then, we want to maximize the cost functional

$$\mathcal{J}(f) := \int_{\partial\Omega} f(x)u \,\mathrm{d}\mathcal{H}^{N-1},$$

for $f \in \mathcal{A}$, where u is the (unique) solution to the nonlinear membrane problem with load f

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

These types of optimization problems have been considered in the literature due to many applications in science and engineering, specially in the linear case $p = 2$. See for instance [CC].

We have chosen three different classes of admissible functions $\mathcal{A}$ to work with.

- The class of rearrangements of a given function f_0.
- The (unit) ball in some L^q.
- The class of characteristic functions of sets of given surface measure.

This latter case is what we believe is the most interesting one and where our main results are obtained.

For each of these classes, we prove existence of a maximizing load (in the respective class) and analyze properties of these maximizers.

The approach to the class of rearrangements follows the lines of [CEP1], where a similar problem was analysed, namely, the maximization of the functional

$$\bar{\mathcal{J}}(g) := \int_\Omega gu\,\mathrm{d}x,$$

where u is the solution to $-\Delta_p u = g$ in Ω with Dirichlet boundary conditions.

When we work in the unit ball of L^q the problem becomes trivial and we explicitly find the (unique) maximizer for $\mathcal{J}$, namely, the first eigenfunction of a Steklov-like nonlinear eigenvalue problem (see Section 6.2).

Finally, we arrive at the main part of this chapter, namely, the analysis of the problem for the class of characteristic functions of sets of given boundary measure. In order to work within this class, we first relax the problem and work with the weak$*$ closure of the characteristic functions (i.e. bounded functions of given L^1 norm), prove existence of a maximizer within this relaxed class and then prove that this optimizer is in fact a characteristic function.

Then, in order to analyze properties of this maximizers, we compute the first variation with respect to perturbations on the load.

This approach for optimization problems has been used several times in the literature. Just to cite a few, see [DPFBR, FBRW2, KSS] and references therein. Also, our approach to the computation of the first variation borrows ideas from [GMSL].

The chapter is organized as follows. First, in Section 6.1, we study the problem when the admissible class of loads $\mathcal{A}$ is the class of rearrangements of a given function f_0. In Section 6.2, we study the simpler case when $\mathcal{A}$ is the unit ball in L^q. In Section 6.3, we analyze the case where $\mathcal{A}$ is the class of characteristic functions of sets with given surface measure. Lastly, in Section 6.4, we compute the first variation with respect to the load.

6.1 Maximizing in the class of rearrangements

Given a domain $\Omega \subset \mathbb{R}^N$ (bounded, connected, with smooth boundary), first we want to study the following problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Here $1 < p < \infty$ and f is a measurable function that satisfy the assumption

$$f \in L^q(\Omega) \text{ where } \begin{cases} q > \frac{N'}{p'} & \text{if } 1 < p < N, \\ q > 1 & \text{if } p \geq N. \end{cases} \tag{A1}$$

We say $u \in W^{1,p}(\Omega)$ is a *weak solution of* (6.2) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\,\mathrm{d}x = \int_{\partial\Omega} fv\,\mathrm{d}\mathcal{H}^{N-1} \quad \forall\, v \in W^{1,p}(\Omega).$$

The assumption (A1) is related to the fact that ${}^{p'}/_{N'} = p'_*$ if $1 < p < N$, and $q' < \infty$ if $p \geq N$. So, in order for that the right side of last equality to make sense for $f \in L^q(\partial\Omega)$, we need v to belong to $L^{q'}(\partial\Omega)$. This is achieved by the assumption (A1) and the Sobolev Trace Embedding Theorem.

It is a standard result that (6.2) has a unique weak solution u_f, for which the following equations hold

$$\int_{\partial\Omega} f u_f \, d\mathcal{H}^{N-1} = \sup\left\{\mathcal{I}(u) : u \in W^{1,p}(\Omega)\right\}, \tag{6.3}$$

where

$$\mathcal{I}(u) = \frac{1}{p-1}\left\{p\int_{\partial\Omega} f u \, d\mathcal{H}^{N-1} - \int_{\Omega} |\nabla u|^p + |u|^p \, dx\right\}.$$

Let f_0 be a measurable function that satisfy the assumption (A1), we are interested in finding

$$\sup\left\{\int_{\partial\Omega} f u_f \, d\mathcal{H}^{N-1} : f \in \mathcal{R}(f_0)\right\}. \tag{6.4}$$

Theorem 6.1.1. *There exists $\hat{f} \in \mathcal{R}(f_0)$ such that*

$$\begin{aligned}\mathcal{J}(\hat{f}) &= \int_{\partial\Omega} \hat{f}\hat{u} \, d\mathcal{H}^{N-1} \\ &= \sup\{\mathcal{J}(f) : f \in \mathcal{R}(f_0)\} \\ &= \sup\left\{\int_{\partial\Omega} f u_f \, d\mathcal{H}^{N-1} : f \in \mathcal{R}(f_0)\right\},\end{aligned}$$

where $\hat{u} = u_{\hat{f}}$.

Proof. Let

$$I = \sup\left\{\int_{\partial\Omega} f u_f \, d\mathcal{H}^{N-1} : f \in \mathcal{R}(f_0)\right\}.$$

We first show that I is finite. Let $f \in \mathcal{R}(f_0)$. By Hölder's inequality and the Sobolev Trace Embedding Theorem, we have

$$\int_{\Omega} |\nabla u_f|^p + |u_f|^p \, dx \leq C\|f\|_{L^q(\partial\Omega)}\|u_f\|_{W^{1,p}(\Omega)},$$

then

$$\|u_f\|_{W^{1,p}(\Omega)} \leq C \quad \forall f \in \mathcal{R}(f_0) \tag{6.5}$$

since $\|f\|_{L^q(\partial\Omega)} = \|f_0\|_{L^q(\partial\Omega)}$ for all $f \in \mathcal{R}(f_0)$. Therefore I is finite.

Now, let $\{f_n\}_{n\in\mathbb{N}}$ be a maximizing sequence and let $u_n = u_{f_n}$. From (6.5) it is clear that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$, then there exists a function $u \in W^{1,p}(\Omega)$ such that, for a subsequence that we still call $\{u_n\}_{n\in\mathbb{N}}$,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \\ u_n &\to u \quad \text{strongly in } L^p(\Omega), \\ u_n &\to u \quad \text{strongly in } L^{q'}(\partial\Omega). \end{aligned}$$

On the other hand, since $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(\partial\Omega)$, we may choose a subsequence, still denoted by $\{f_n\}_{n\in\mathbb{N}}$, and $f \in L^q(\partial\Omega)$ such that

$$f_n \rightharpoonup f \quad \text{weakly in } L^q(\partial\Omega).$$

Then

$$\begin{aligned} I &= \lim_{n\to\infty} \int_{\partial\Omega} f_n u_n \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \frac{1}{p-1} \lim_{n\to\infty} \left\{ p \int_{\partial\Omega} f_n u_n \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u_n|^p + |u_n|^p \, \mathrm{d}x \right\} \\ &\leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} f u \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x \right\}. \end{aligned}$$

Furthermore, by Lemma 1.5.3, there exists $\hat{f} \in \mathcal{R}(f_0)$ such that

$$\int_{\partial\Omega} f u \, \mathrm{d}\mathcal{H}^{N-1} \leq \int_{\partial\Omega} \hat{f} u \, \mathrm{d}\mathcal{H}^{N-1}.$$

Thus

$$I \leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} \hat{f} u \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x \right\}.$$

As a consequence of (6.3), we have that

$$\begin{aligned} I &\leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} \hat{f} u \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x \right\} \\ &\leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} \hat{f} \hat{u} \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla \hat{u}|^p + |\hat{u}|^p \, \mathrm{d}x \right\} \\ &= \int_{\partial\Omega} \hat{f} \hat{u} \, \mathrm{d}\mathcal{H}^{N-1} \\ &\leq I. \end{aligned}$$

Recall that $\hat{u} = u_{\hat{f}}$. Therefore $\hat{f}$ is a solution to (6.4). This completes the proof. □

Remark 6.1.2. With a similar proof we can prove a slighter stronger result. Namely, we can consider the functional

$$\mathcal{J}_1(f, g) := \int_\Omega g u \, \mathrm{d}x + \int_{\partial\Omega} f u \, \mathrm{d}\mathcal{H}^{N-1},$$

where u is the (unique, weak) solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = g & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = f & \text{on } \partial\Omega, \end{cases}$$

and consider the problem of maximizing $\mathcal{J}_1$ over the class $\mathcal{R}(g_0) \times \mathcal{R}(f_0)$ for some fixed g_0 and f_0.

We leave the details to the reader.

Now, we give characterization of a maximizer function in the spirit of [CEP1].

Theorem 6.1.3. *Let $\hat{f} \in \mathcal{R}(f_0)$ such that*

$$\int_{\partial\Omega} \hat{f}\hat{u}\, d\mathcal{H}^{N-1} = \sup\left\{\int_{\partial\Omega} f u_f\, d\mathcal{H}^{N-1} \colon f \in \mathcal{R}(f_0)\right\},$$

where $\hat{u} = u_{\hat{f}}$. Then $\hat{f}$ is the unique maximizer of linear functional

$$L(f) := \int_{\partial\Omega} f\hat{u}\, d\mathcal{H}^{N-1},$$

relative to $f \in \mathcal{R}(f_0)$.

Therefore, there is an increasing function ϕ such that $\hat{f} = \phi \circ \hat{u}$ $\mathcal{H}^{N-1}-$a.e.

Proof. We proceed in three steps.

Step 1. First we show that $\hat{f}$ is a maximizer of $L(f)$ relative to $f \in \mathcal{R}(f_0)$.

In fact, let $h \in \mathcal{R}(f_0)$, since

$$\int_{\partial\Omega} \hat{f}\hat{u}\, d\mathcal{H}^{N-1} = \sup\left\{\int_{\partial\Omega} f u_f\, d\mathcal{H}^{N-1} \colon f \in \mathcal{R}(f_0)\right\},$$

we have that

$$\begin{aligned}
\int_{\partial\Omega} \hat{f}\hat{u}\, d\mathcal{H}^{N-1} &\geq \int_{\partial\Omega} h u_h\, d\mathcal{H}^{N-1} \\
&= \frac{1}{p-1} \sup\left\{p\int_{\partial\Omega} hu\, d\mathcal{H}^{N-1} - \int_{\partial\Omega} |\nabla u|^p + |u|^p\, dx \colon u \in W^{1,p}(\Omega)\right\} \\
&\geq \frac{1}{p-1}\left\{p\int_{\partial\Omega} h\hat{u}\, d\mathcal{H}^{N-1} - \int_{\partial\Omega} |\nabla \hat{u}|^p + |\hat{u}|^p\, dx\right\},
\end{aligned}$$

and, since

$$\int_{\partial\Omega} \hat{f}\hat{u}\, d\mathcal{H}^{N-1} = \frac{1}{p-1}\left\{p\int_{\partial\Omega} \hat{f}\hat{u}\, d\mathcal{H}^{N-1} - \int_{\partial\Omega} |\nabla \hat{u}|^p + |\hat{u}|^p\, dx\right\},$$

we have

$$\int_{\partial\Omega} \hat{f}\hat{u}\, d\mathcal{H}^{N-1} \geq \int_{\partial\Omega} h\hat{u}\, d\mathcal{H}^{N-1}.$$

Therefore,

$$\int_{\partial\Omega} \hat{f}\hat{u}\, d\mathcal{H}^{N-1} = \sup\left\{L(f) \colon f \in \mathcal{R}(f_0)\right\}.$$

Step 2. Now, we show that $\hat{f}$ is the unique maximizer of $L(f)$ relative to $f \in \mathcal{R}(f_0)$.

We suppose that g is another maximizer of $L(f)$ relative to $f \in \mathcal{R}(f_0)$. Then

$$\int_{\partial\Omega} \hat{f}\hat{u}\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial\Omega} g\hat{u}\,\mathrm{d}\mathcal{H}^{N-1}.$$

Thus

$$\begin{aligned}\int_{\partial\Omega} g\hat{u}\,\mathrm{d}\mathcal{H}^{N-1} &= \int_{\partial\Omega} \hat{f}\hat{u}\,\mathrm{d}\mathcal{H}^{N-1}\\ &\geq \int_{\partial\Omega} gu_g\,\mathrm{d}\mathcal{H}^{N-1}\\ &=\frac{1}{p-1}\sup\left\{p\int_{\partial\Omega} gu\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Omega}|\nabla u|^p + |u|^p\,\mathrm{d}x\colon u \in W^{1,p}(\Omega)\right\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\partial\Omega} g\hat{u}\,\mathrm{d}\mathcal{H}^{N-1} &= \int_{\partial\Omega} \hat{f}\hat{u}\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \frac{1}{p-1}\left\{p\int_{\partial\Omega} \hat{f}\hat{u}\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Omega}|\nabla\hat{u}|^p + |\hat{u}|^p\,\mathrm{d}x\right\}\\ &= \frac{1}{p-1}\left\{p\int_{\partial\Omega} g\hat{u}\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Omega}|\nabla\hat{u}|^p + |\hat{u}|^p\,\mathrm{d}x\right\}.\end{aligned}$$

Then

$$\int_{\partial\Omega} g\hat{u}\,\mathrm{d}\mathcal{H}^{N-1} = \frac{1}{p-1}\sup\left\{p\int_{\partial\Omega} gu\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Omega}|\nabla u|^p + |u|^p\,\mathrm{d}x\colon u \in W^{1,p}(\Omega)\right\}.$$

Therefore $\hat{u} = u_g$. Then $\hat{u}$ is the unique weak solution to

$$\begin{cases}-\Delta_p\hat{u} + |\hat{u}|^{p-2}\hat{u} = 0 & \text{in } \Omega,\\ |\nabla\hat{u}|^{p-2}\frac{\partial\hat{u}}{\partial\nu} = g & \text{on } \partial\Omega.\end{cases}$$

Furthermore, we now that $\hat{u}$ is the unique weak solution to

$$\begin{cases}-\Delta_p\hat{u} + |\hat{u}|^{p-2}\hat{u} = 0 & \text{in } \Omega,\\ |\nabla\hat{u}|^{p-2}\frac{\partial\hat{u}}{\partial\nu} = \hat{f} & \text{on } \partial\Omega.\end{cases}$$

Therefor $\hat{f} = g$ $\mathcal{H}^{N-1}$–a.e.

Step 3. Lastly, we have that there is an increasing function ϕ such that $\hat{f} = \phi\circ\hat{u}$ $\mathcal{H}^{N-1}$–a.e.

This is a direct consequence of Steps 1, 2 and Theorem 1.5.4.

This completes the proof of theorem. □

6.2 Maximizing in the unit ball of L^q

In this section we consider the optimization problem

$$\max \mathcal{J}(f)$$

where the maximum is taken over the unit ball in $L^q(\partial\Omega)$.

In this case, the answer is simple and we find that the maximizer can be computed explicitly in terms of the extremal of the Sobolev trace embedding.

So, we let f be a measurable function that satisfy the assumptions (A1) and

$$\|f\|_{L^q(\partial\Omega)} \leq 1,$$

we consider the problem

$$\sup\left\{\int_{\partial\Omega} fu_f \, d\mathcal{H}^{N-1} : f \in L^q(\partial\Omega) \text{ and } \|f\|_{L^q(\partial\Omega)} \leq 1\right\}, \tag{6.6}$$

where u_f is the weak solution of

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases} \tag{6.7}$$

The assumption (A1) is taken by the same reason that in the previous section.

In this case it is easy to see that the solution becomes

$$\hat{f} = v_{q'}^{q'-1}$$

where $v_{q'} \in W^{1,p}(\Omega)$ is a nonnegative extremal for $S_{q'}$ normalized such that $\|v_{q'}\|_{L^{q'}(\partial\Omega)} = 1$, and $S_{q'}$ is the Sobolev trace constant. Furthermore

$$\hat{u} = u_{\hat{f}} = \frac{1}{S_{q'}^{1/(p-1)}} v_{q'}.$$

Observe that, as f satisfies the assumption (A1), there exists an extremal for $S_{q'}$. See [FBR1] and references therein.

In fact

$$\begin{aligned} \mathcal{J}(\hat{f}) &= \int_{\partial\Omega} \hat{f}\hat{u} \, d\mathcal{H}^{N-1} \\ &= \int_{\Omega} |\nabla \hat{u}|^p + |\hat{u}|^p \, dx \\ &= \frac{1}{S_{q'}^{p/(p-1)}} \int_{\Omega} |\nabla v_{q'}|^p + |v_{q'}|^p \, dx \\ &= \frac{1}{S_{q'}^{1/(p-1)}}. \end{aligned}$$

On the other hand, given $f \in L^q(\partial\Omega)$, such that $\|f\|_{L^q(\partial\Omega)} \le 1$, we have

$$\begin{aligned}\mathcal{J}(f) &= \int_{\partial\Omega} f u_f \,\mathrm{d}\mathcal{H}^{N-1}\\ &\le \|f\|_{L^q(\partial\Omega)}\|u_f\|_{L^{q'}(\partial\Omega)}\\ &\le \left(\frac{1}{S_{q'}}\int_\Omega |\nabla u_f|^p + |u_f|^p \,\mathrm{d}x\right)^{1/p}\\ &= \frac{1}{S_{q'}^{1/p}}\left(\int_{\partial\Omega} f u_f \,\mathrm{d}\mathcal{H}^{N-1}\right)^{1/p},\end{aligned}$$

from which it follows that

$$\mathcal{J}(f) \le \frac{1}{S_{q'}^{1/(p-1)}}.$$

This completes the characterization of the optimal load in this case.

6.3 Maximizing in L^∞

Now we consider the problem

$$\sup\left\{\int_{\partial\Omega} \phi u_\phi \,\mathrm{d}\mathcal{H}^{N-1} \colon \phi \in \mathbf{B}\right\}, \tag{6.8}$$

where

$$\mathbf{B} := \left\{\phi : 0 \le \phi(x) \le 1 \text{ for all } x \in \partial\Omega \text{ and } \int_{\partial\Omega} \phi \,\mathrm{d}\mathcal{H}^{N-1} = A\right\},$$

for some fixed $0 < A < \mathcal{H}^{N-1}(\partial\Omega)$, and u_ϕ is the weak solution of

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \phi & \text{on } \partial\Omega. \end{cases} \tag{6.9}$$

We believe that this is the most interesting case considered in this chapter.

In this case, we have the following theorem:

Theorem 6.3.1. *There exists $D \subset \partial\Omega$ with $\mathcal{H}^{N-1}(D) = A$ such that*

$$\int_{\partial\Omega} \chi_D u_D \,\mathrm{d}\mathcal{H}^{N-1} = \sup\left\{\int_{\partial\Omega} \phi u_\phi \,\mathrm{d}\mathcal{H}^{N-1} \colon \phi \in \mathbf{B}\right\},$$

where $u_D = u_{\chi_D}$.

Proof. Let

$$I = \sup\left\{\int_{\partial\Omega} \phi u_\phi \,\mathrm{d}\mathcal{H}^{N-1} \colon \phi \in \mathbf{B}\right\}.$$

Arguing as in the first part of the proof for Theorem 6.1.1, we have that I is finite.

Next, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a maximizing sequence and let $u_n = u_{\phi_n}$. It is clear that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$, then there exists a function $u \in W^{1,p}(\Omega)$ such that, for a subsequence that we still call $\{u_i\}_{n\in\mathbb{N}}$,

$$\begin{array}{lcll} u_n & \rightharpoonup & u & \text{weakly in } W^{1,p}(\Omega), \\ u_n & \to & u & \text{strongly in } L^p(\Omega), \\ u_n & \to & u & \text{strongly in } L^p(\partial\Omega). \end{array}$$

On the other hand, since $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\partial\Omega)$, we may choose a subsequence, again denoted $\{\phi_n\}_{n\in\mathbb{N}}$, and $\phi \in L^\infty(\partial\Omega)$ and such that

$$\phi_n \overset{*}{\rightharpoonup} \phi \quad \text{weakly}* \text{ in } L^\infty(\partial\Omega).$$

Then

$$\begin{aligned} I &= \lim_{n\to\infty} \int_{\partial\Omega} \phi_n u_n \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \frac{1}{p-1} \lim_{n\to\infty} \left\{ p \int_{\partial\Omega} \phi_n u_n \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u_n|^p + |u_n|^p \, \mathrm{d}x \right\} \\ &\leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} \phi u \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x \right\}. \end{aligned}$$

Furthermore, by Theorem 1.3.12, there exists $D \subset \partial\Omega$ with $\mathcal{H}^{N-1}(D) = A$ such that

$$\int_{\partial\Omega} \phi u \, \mathrm{d}\mathcal{H}^{N-1} \leq \int_{\partial\Omega} \chi_D u \, \mathrm{d}\mathcal{H}^{N-1},$$

and

$$\{t < u\} \subset D \subset \{t \leq u\}, \quad t := \inf \left\{ s : \mathcal{H}^{N-1}(\{s < u\}) < A \right\}.$$

Thus

$$I \leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} \chi_D u \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x \right\}.$$

As a consequence of (6.3), we have that

$$\begin{aligned} I &\leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} \chi_D u \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x \right\} \\ &\leq \frac{p}{p-1} \left\{ p \int_{\partial\Omega} \chi_D u_D \, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u_D|^p + |u_D|^p \, \mathrm{d}x \right\} \\ &= \int_{\partial\Omega} \chi_D u_D \, \mathrm{d}\mathcal{H}^{N-1} \\ &\leq I. \end{aligned}$$

Recall that $u_D = u_{\chi_D}$. Therefore χ_D is a solution to (6.8). This completes the proof. □

Remark 6.3.2. Note that in arguments in the proof of Theorem 6.3.1, using again the Theorem 1.3.12, we can prove that

$$\{t < u_D\} \subset D \subset \{t \leq u_D\}$$

where $t := \inf\left\{s : \mathcal{H}^{N-1}(\{s < u_D\}) < A\right\}$. Therefore u_D is constant on ∂D.

6.4 Derivate with respect to the load

Now we compute the derivate of the functional $\mathcal{J}(\hat{f})$ with respect to perturbations in $\hat{f}$. We will consider regular perturbations and assume that the function $\hat{f}$ has bounded variation in $\partial\Omega$.

We begin by describing the kind of variations that we are considering. Let V be a regular (smooth) vector field, globally Lipschitz, with support in a neighborhood of $\partial\Omega$ such that $\langle V, \nu\rangle = 0$ and let $\psi_t : \mathbb{R}^N \to \mathbb{R}^N$ be defined as the unique solution to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = V(\psi_t(x)) & t > 0, \\ \psi_0(x) = x & x \in \mathbb{R}^N. \end{cases} \tag{6.10}$$

We have

$$\psi_t(x) = x + tV(x) + o(t) \quad \forall x \in \mathbb{R}^N.$$

Thus, if f satisfies the assumption (A1), we define $f_t = f \circ \psi_t^{-1}$. Now, let

$$I(t) := \mathcal{J}(f_t) = \int_{\partial\Omega} u_t f_t \mathrm{d}\mathcal{H}^{N-1}$$

where $u_t \in W^{1,p}(\Omega)$ is the unique solution to

$$\begin{cases} -\Delta_p u_t + |u_t|^{p-2} = 0 & \text{in } \Omega, \\ |\nabla u_t|^{p-2}\frac{\partial u_t}{\partial \nu} = f_t & \text{on } \partial\Omega. \end{cases} \tag{6.11}$$

Lemma 6.4.1. *Let u_0 and u_t be the solution of (6.11) with $t = 0$ and $t > 0$, respectively. Then*

$$u_t \to u_0 \text{ in } W^{1,p}(\Omega), \text{ as } t \to 0^+.$$

Proof. The proof follows exactly as the one in Lemma 4.2 in [CEP1]. The only difference being that we use the trace inequality instead of the Poincaré inequality.

In fact, as

$$C\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq \begin{cases} |x-y|^p & \text{if } p \geq 2, \\ \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } p \leq 2 \end{cases} \tag{6.12}$$

for all $x, y \in \mathbb{R}^N$, where C is a positive constant (see, for example,[T]). We consider two cases

Case 1. Let $p \geq 2$. Using (6.12) we have

$$\frac{1}{C}\|u_t - u_0\|_{W^{1,p}(\Omega)} \leq \int_\Omega \langle |\nabla u_t|^{p-2}\nabla u_t - |\nabla u_0|^{p-2}\nabla u_0, \nabla u_t - \nabla u_0\rangle \, \mathrm{d}x$$
$$+ \int_\Omega \left(|u_t|^{p-2}u_t - |u_0|^{p-2}u_0\right)(u_t - u_0)\, \mathrm{d}x.$$

By (6.11), we can rewrite the above inequality as

$$\|u_t - u_0\|^p_{W^{1,p}(\Omega)} \leq C \int_{\partial\Omega} (f_t - f)(u_t - u)\, \mathrm{d}\mathcal{H}^{N-1}.$$

So by applying the Hölder's inequality followed by the trace inequality we obtain

$$\|u_t - u_0\|^{p-1}_{W^{1,p}(\Omega)} \leq \widetilde{C}\|f_t - f\|_{L^q(\partial\Omega)}.$$

From the above inequality and Lemma 4.3.1, the assertion of the lemma follows.

Case 2. Let $p \leq 2$. Let us begin with the following observation

$$\begin{aligned}
\|u_t - u_0\|^p_{W^{1,p}(\Omega)} &= \int_\Omega \frac{|\nabla u_t - \nabla u_0|^p}{(|\nabla u_t| + |\nabla u_0|)^{p(2-p)/2}} (|\nabla u_t| + |\nabla u_0|)^{p(2-p)/2}\, \mathrm{d}x \\
&+ \int_\Omega \frac{|u_t - u_0|^p}{(|u_t| + |u_0|)^{p(2-p)/2}} (|u_t| + |u_0|)^{p(2-p)/2}\, \mathrm{d}x \\
&\leq \left(\int_\Omega \frac{|\nabla u_t - \nabla u_0|^2}{(|\nabla u_t| + |\nabla u_0|)^{(2-p)}}\, \mathrm{d}x\right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_t| + |\nabla u_0|)^p\, \mathrm{d}x\right)^{\frac{2-p}{2}} \\
&+ \left(\int_\Omega \frac{|u_t - u_0|^2}{(|u_t| + |u_0|)^{(2-p)}}\, \mathrm{d}x\right)^{\frac{p}{2}} \left(\int_\Omega (|u_t| + |u_0|)^p\, \mathrm{d}x\right)^{\frac{2-p}{2}},
\end{aligned}$$

which follows from the Hölder inequality, since ${}^2\!/_p > 1$. Note that $\{u_t\}_{t>o}$ is bounded in $W^{1,p}(\Omega)$. Thus from the above inequality

$$\frac{1}{C}\|u_t - u_0\|^p_{W^{1,p}(\Omega)} \leq \left(\int_\Omega \frac{|\nabla u_t - \nabla u_0|^2}{(|\nabla u_t| + |\nabla u_0|)^{(2-p)}}\, \mathrm{d}x\right)^{\frac{p}{2}} + \left(\int_\Omega \frac{|u_t - u_0|^2}{(|u_t| + |u_0|)^{(2-p)}}\, \mathrm{d}x\right)^{\frac{p}{2}}.$$

Now, applying (6.12) to the right hand side of the last inequality, the assertion of the lemma can be confirmed using similar arguments as in the ending part of Case 1. □

Remark 6.4.2. It is easy to see that, as $\psi_t \to Id$ in the C^1 topology, then from Lemma 6.4.1 it follows that

$$w_t := u_t \circ \psi_t \to u_0 \quad \text{strongly in } W^{1,p}(\Omega).$$

Now, we can prove that $I(t)$ is differentiable at $t = 0$ and give a formula for the derivative.

Theorem 6.4.3. *With the previous notation, we have that $I(t)$ is differentiable at $t = 0$ and*

$$\left.\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right|_{t=0} = \frac{p}{p-1}\int_{\partial\Omega} u_0 f \operatorname{div}_\tau V \,\mathrm{d}\mathcal{H}^{N-1} + \frac{1}{p-1}\int_\Omega \Big[p|\nabla u_0|^{p-2}\langle \nabla u_0,^T V'\nabla u_0^T\rangle - (|\nabla u_0|^p + |u_0|^p)\operatorname{div} V\Big]\,\mathrm{d}x,$$

where u_0 is the solution of (6.11) *with $t = 0$.*

Proof. By (6.3) we have that

$$I(t) = \sup \frac{1}{p-1}\left\{p\int_{\partial\Omega} v f_t \,\mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla v|^p + |v|^p\,\mathrm{d}x \colon v\in W^{1,p}(\Omega)\right\}.$$

Given $v \in W^{1,p}(\Omega)$, we consider $u = v\circ\psi_t \in W^{1,p}(\Omega)$. Then, by the Lemma 4.1.1, we have

$$\int_\Omega |v|^p\,\mathrm{d}x = \int_\Omega |u|^p\,\mathrm{d}x + t\int_\Omega |u|^p \operatorname{div} V\,\mathrm{d}x + o(t).$$

and, by the Theorem 4.1.3,

$$\int_\Omega |\nabla v|^p\,\mathrm{d}x = \int_\Omega |\nabla u|^p\,\mathrm{d}x + t\int_\Omega |\nabla u|^p \operatorname{div} V\,\mathrm{d}x - tp\int_\Omega |\nabla u|^{p-2}\langle\nabla u,^T V'\nabla u^T\rangle \mathrm{d}x + o(t),$$

Also, by Lemma 4.3.1, we have

$$\int_{\partial\Omega} v f_t\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial\Omega} u f\,\mathrm{d}\mathcal{H}^{N-1} + t\int_{\partial\Omega} u f \operatorname{div}_\tau V\,\mathrm{d}\mathcal{H}^{N-1} + o(t),$$

Then, for all $v \in W^{1,p}(\Omega)$ we have that

$$p\int_{\partial\Omega} v f_t\,\mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla v|^p + |v|^p\,\mathrm{d}x = \varphi(u) + t\phi(u) + o(t),$$

where

$$\varphi(u) = p\int_{\partial\Omega} u f\,\mathrm{d}\mathcal{H}^{N-1} - \int_\Omega |\nabla u|^p + |u|^p\,\mathrm{d}x$$

and

$$\phi(u) = p\int_{\partial\Omega} u f \operatorname{div}_\tau V \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega \Big[p|\nabla u|^{p-2}\langle\nabla u,^T V'\nabla u^T\rangle - (|\nabla u|^p + |u|^p)\operatorname{div} V\Big]\,\mathrm{d}x.$$

Therefore, we can rewrite $I(t)$ as

$$I(t) = \sup\left\{\frac{1}{p-1}\left[\varphi(u) + t\phi(u)\right] + o(t) \colon u\in W^{1,p}(\Omega)\right\}.$$

If we define $w_t = u_t\circ\psi_t$ for all $t > 0$, we have that $w_0 = u_0$ and

$$I(t) = \frac{1}{p-1}\left[\varphi(w_t) + t\phi(w_t)\right] + o(t) \quad \forall\, t.$$

Thus
$$I(t)-I(0)\geq \frac{1}{p-1}\left[\varphi(u_0)+t\phi(u_0)\right]+o(t)-\frac{1}{p-1}\varphi(u_0),$$
then
$$\liminf_{t\to 0^+}\frac{I(t)-I(0)}{t}\geq \frac{1}{p-1}\phi(u_0). \tag{6.13}$$

On the other hand
$$I(t)-I(0)\leq \frac{1}{p-1}\left[\varphi(w_t)+t\phi(w_t)\right]+o(t)-\frac{1}{p-1}\varphi(w_t),$$
hence,
$$\frac{I(t)-I(0)}{t}\leq \frac{1}{p-1}\phi(w_t)+\frac{1}{t}o(t).$$
By Remark 6.4.2,
$$\phi(w_t)\to\phi(u_0)\quad \text{as } t\to 0^+,$$
therefore,
$$\limsup_{t\to 0^+}\frac{I(t)-I(0)}{t}\leq \frac{1}{p-1}\phi(u_0). \tag{6.14}$$

From (6.13) and (6.14) we deduced that there exists $I'(0)$ and
$$\begin{aligned} I'(0) &= \frac{1}{p-1}\phi(u_0)\\ &= \frac{p}{p-1}\int_{\partial\Omega} u_0 f \operatorname{div}_\tau V\, \mathrm{d}\mathcal{H}^{N-1}\\ &+\frac{1}{p-1}\int_{\Omega}\Big[p|\nabla u_0|^{p-2}\langle \nabla u_0,^T V'\nabla u_0^T\rangle-(|\nabla u_0|^p+|u_0|^p)\operatorname{div} V\Big]\mathrm{d}x. \end{aligned}$$
The prove is now complete. □

Now we try to find a more explicit formula for $I'(0)$. For This, we consider
$$f\in L^q(\partial\Omega)\cap BV(\partial\Omega).$$

Theorem 6.4.4. *If $f\in L^q(\partial\Omega)\cap BV(\partial\Omega)$, we have that*
$$\left.\frac{\partial I(t)}{\partial t}\right|_{t=0}=\frac{p}{p-1}\int_{\partial\Omega} u_0 V\,\mathrm{d}[Df].$$
where u_0 is the solution of (6.11) *with $t=0$.*

Proof. In the course of the computations, we require the solution u_0 to
$$\begin{cases} -\Delta u_0+|u_0|^{p-2}u_0=0 & \text{in } \Omega,\\ |\nabla u_0|^{p-2}\frac{\partial u_0}{\partial \nu}=f & \text{on } \partial\Omega,\end{cases}$$

to be C^2. However, this is not true. As it is well known (see, for instance, [L]), u_0 belongs to the class $C^{1,\delta}$ for some $0 < \delta < 1$.

In order to overcome this difficulty, we proceed as follows. We consider the regularized problems

$$\begin{cases} -\operatorname{div}\left(\left(|\nabla u_0^\varepsilon|^2 + \varepsilon^2\right)^{(p-2)/2} \nabla u_0^\varepsilon\right) + |u_0^\varepsilon|^{p-2} u_0^\varepsilon = 0 & \text{in } \Omega, \\ \left(|\nabla u_0^\varepsilon|^2 + \varepsilon^2\right)^{(p-2)/2} \frac{\partial u_0^\varepsilon}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases} \tag{6.15}$$

It is well known that the solution u_0^ε to (6.15) is of class $C^{2,\rho}$ for some $0 < \rho < 1$ (see [LSU]).

Then, we can perform all of our computations with the functions u_0^ε and pass to the limit as $\varepsilon \to 0+$ at the end.

We have chosen to work formally with the function u_0 in order to make our arguments more transparent and leave the details to the reader. For a similar approach, see the proof of the Theorem 5.2.11.

Now, by Theorem 6.4.3 and since

$$\operatorname{div}(|u_0|^p V) = p|u_0|^{p-2} u_0 \langle \nabla u_0, V\rangle + |u_0|^p \operatorname{div} V,$$
$$\operatorname{div}(|\nabla u_0|^p V) = p|\nabla u_0|^{p-2} \langle \nabla u_0 D^2 u_0, V\rangle + |\nabla u_0|^p \operatorname{div} V,$$

we obtain

$$\begin{aligned} I'(0) &= \frac{p}{p-1} \int_{\partial\Omega} u_0 f \operatorname{div}_\tau V \, d\mathcal{H}^{N-1} \\ &+ \frac{1}{p-1} \int_\Omega \left[p|\nabla u_0|^{p-2} \langle \nabla u_0,{}^T V' \nabla u_0^T\rangle - (|\nabla u_0|^p + |u_0|^p) \operatorname{div} V \right] dx \\ &= \frac{p}{p-1} \left\{ p \int_{\partial\Omega} u_0 f \operatorname{div}_\tau V \, d\mathcal{H}^{N-1} + \int_\Omega |\nabla u_0|^{p-2} \langle \nabla u_0,{}^T V' \nabla u_0^T\rangle \, dx \right\} \\ &+ \frac{1}{p-1} \int_\Omega \left\{ p|\nabla u_0|^{p-2} \langle \nabla u_0 D^2 u_0, V\rangle - \operatorname{div}([|\nabla u_0|^p + |u_0|^p] V) \right\} dx \\ &+ \frac{p}{p-1} \int_\Omega |u_0|^{p-2} u_0 \langle \nabla u_0, V\rangle \, dx. \end{aligned}$$

Hence, using that $\langle V, \nu\rangle = 0$ in the right hand side of the above equality, we find

$$\begin{aligned} I'(0) &= \frac{p}{p-1} \left\{ \int_{\partial\Omega} u_0 f \operatorname{div}_\tau V \, d\mathcal{H}^{N-1} + \int_\Omega |\nabla u_0|^{p-2} \langle \nabla u_0,{}^T V' \nabla u_0^T + D^2 u_0 V^T\rangle \, dx \right\} \\ &+ \frac{p}{p-1} \int_\Omega |u_0|^{p-2} u_0 \langle \nabla u_0, V\rangle \, dx \\ &= \frac{p}{p-1} \left\{ \int_{\partial\Omega} u_0 f \operatorname{div}_\tau V \, d\mathcal{H}^{N-1} + \int_\Omega |\nabla u_0|^{p-2} \langle \nabla u_0, \nabla(\langle \nabla u_0, V\rangle)\rangle \, dx \right\} \\ &+ \frac{p}{p-1} \int_\Omega |u_0|^{p-2} u_0 \langle \nabla u_0, V\rangle \, dx. \end{aligned}$$

Since u_0 is a week solution of (6.11) with $t = 0$, we have

$$I'(0) = \frac{p}{p-1}\left\{\int_{\partial\Omega} u_0 f \operatorname{div}_\tau V \, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \langle \nabla u_0, V\rangle f \, d\mathcal{H}^{N-1}\right\}$$
$$= \frac{p}{p-1}\int_{\partial\Omega} \operatorname{div}_\tau(u_0 V) f \, d\mathcal{H}^{N-1}.$$

Finally, since $f \in BV(\partial\Omega)$ and $V \in C^1(\partial\Omega; \mathbb{R}^N)$,

$$I'(0) = \frac{p}{p-1}\int_{\partial\Omega} \operatorname{div}_\tau(u_0 V) f \, d\mathcal{H}^{N-1}$$
$$= \frac{p}{p-1}\int_{\partial\Omega} u_0 V \, d[Df],$$

as we wanted to prove. □

Lastly, we consider the case that $f = \chi_D$. Observe that, in this case,

$$\mathcal{R}(\chi_D) = \{\chi_E : |E| = |D|\},$$

and therefore we find in the case studied in Section 6.3.

Corollary 6.4.5. *Let D be a locally finite perimeter set in $\partial\Omega$. If $f = \chi_D$, with the previous notation, we have that*

$$\left.\frac{d}{dt} I(t)\right|_{t=0} = \frac{p}{p-1}\int_{\partial D} u_0 \langle V, \nu_\tau\rangle \, d\mathcal{H}^{N-2},$$

where u_0 is the solution of (6.11) *with $t = 0$.*

Proof. Since D has locally finite perimeter in $\partial\Omega$, it follows that

$$f = \chi_D \in L^q(\partial\Omega) \cap BV(\partial\Omega).$$

Then, by the previous theorem and Theorem 1.9.5, we have

$$\left.\frac{d}{dt} I(t)\right|_{t=0} = \frac{p}{p-1}\int_{\partial\Omega} u_0 V \, d[D\chi_D]$$
$$= \frac{p}{p-1}\int_{\partial D} u_0 \langle V, \nu_\tau\rangle \, d\mathcal{H}^{N-2},$$

where u_0 is the solution of (6.11) with $t = 0$.

This completes the proof. □

The following theorem is a result that we have already observed, actually under weaker assumptions on D, in Remark 6.3.2.

Nevertheless, we have chosen to include this remark as a direct application of the Lemma 4.3.1 and Corollary 6.4.5.

Theorem 6.4.6. *Let χ_D be a maximizer for $\mathcal{J}$ over the class* **B** *and assume that D has locally finite perimeter in $\partial\Omega$. Let u_D be the solution to the associated state equation*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \chi_D & \text{on } \partial\Omega. \end{cases}$$

Then, u_D is constant along ∂D.

Proof. Let D be a critical point of I and, with the previous notation, $D_t = \psi_t(D)$. Then, by Theorem 1.9.5 and Lemma 4.3.1, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{N-1}(D_t)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\chi_{D_t}\,\mathrm{d}\mathcal{H}^{N-1}\Big|_{t=0} \\ &= \int_D \operatorname{div}_\tau V\,\mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{\partial D}\langle V, \nu_\tau\rangle\,\mathrm{d}\mathcal{H}^{N-2}. \end{aligned}$$

Thus, the fact that D is a critical point of I and by Corollary 6.4.5, we derive

$$I'(0) = c\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{N-1}(D_t)\Big|_{t=0} \iff u = \mathit{constant}, \text{ on } \partial D.$$

As we wanted to prove. $\square$

7

Extremals of the trace inequality in domains with holes

Throughout this chapter, Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$ and $1 < p < \infty$.

For any $A \subset \overline{\Omega}$, which is a smooth open subset, we define the space

$$W_A^{1,p}(\Omega) = \overline{C_0^\infty(\overline{\Omega} \setminus A)},$$

where the closure is taken in $W^{1,p}$–norm. By the Sobolev Trace Embedding Theorem, there is a compact embedding

$$W_A^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega), \tag{7.1}$$

for all $1 \leq q < p_*$.

Thus, given $1 < q < p_*$, there exist a constant $C = C(q, p)$ such that

$$C\left(\int_{\partial\Omega} |u|^q \, d\mathcal{H}^{N-1}\right)^{p/q} \leq \int_\Omega |\nabla u|^p + |u|^p \, dx.$$

The best (largest) constant in the above inequality is given by

$$S_q(A) := \inf\left\{\frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left(\int_{\partial\Omega} |u|^q \, d\mathcal{H}^{N-1}\right)^{p/q}} : u \in W_A^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)\right\}. \tag{7.2}$$

By (7.1), there exist an extremal for $S_q(A)$. Moreover, an extremal for $S_q(A)$ is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \setminus \overline{A}, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega \setminus \partial\overline{A}, \\ u = 0 & \text{on } \partial A, \end{cases} \tag{7.3}$$

where λ depends on the normalization of u. When $\|u\|_{L^q(\partial\Omega)} = 1$, we have that $\lambda = S_q(A)$. Moreover, when $p = q$ the problem (7.3) becomes homogeneous, and therefore it is a nonlinear eigenvalue problem. In this case, the first eigenvalue of (7.3) coincides with the best Sobolev trace constant $S_q(A) = \lambda_1(A)$ and it is shown in [MR], that it is simple (see also [FBR3]). Therefore, if $p = q$, the extremal for $S_p(A)$ is unique up to constant factor. In the linear setting, i.e., when $p = q = 2$, this eigenvalue problem is known as the Steklov eigenvalue problem, see [St].

We say that hole A^* is *optimal* for the parameter α, $0 < \alpha < |\Omega|$, if $|A^*| = \alpha$ and

$$S_q(A^*) = \inf\left\{S_q(A)\colon A \subset \overline{\Omega} \text{ and } |A| = \alpha\right\}.$$

In [FBRW2], the authors proof the existence of an optimal hole. Moreover, in the case that Ω is a ball, they prove the following result

Theorem 7.0.7. *Let $\Omega = B(0,1)$ and $0 < \alpha < |B(0,1)|$. Then there exists an optimal hole of measure α which is spherically symmetric.*

The aim of this chapter is to analyze the dependence of the Sobolev trace constant $S_q(A)$ with respect to variations on the set A. To this end, we compute the so-called shape derivative of $S_q(A)$ with respect to regular perturbations of the *hole* A.

In [FBGR], this problem is analyzed in the linear case $p = q = 2$. There, the authors consider the following kind of variation. Let $V : \mathbb{R}^N \to \mathbb{R}^N$ be a regular (smooth) vector filed, globally Lipschitz, with support in Ω and let $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$ be defined as the unique solution to

$$\begin{cases} \frac{d}{dt}\varphi_t(x) = V(\varphi_t(x)) & t > 0 \\ \varphi_0(x) = x & x \in \mathbb{R}^N. \end{cases}$$

Then, they define $A_t := \varphi_t(A) \subset \Omega$ for all $t > 0$ and

$$S_2(t) = \inf\left\{\frac{\int_\Omega |\nabla v|^2 + |v|^2\, dx}{\int_{\partial\Omega} |v|^2\, d\mathcal{H}^{N-1}} : v \in W^{1,p}_{A_t}(\Omega) \setminus W^{1,p}_0(\Omega)\right\}.$$

Observe that $A_0 = A$ and therefore $S_2(0) = S_2(A)$. The authors prove that $S_2(t)$ is differentiable with respect to t at $t = 0$ and it holds

$$S_2'(0) = \frac{d}{dt} S_2(t)\Big|_{t=0} = -\int_{\partial A} \left(\frac{\partial u}{\partial \nu}\right)^2 \langle V, \nu\rangle\, d\mathcal{H}^{N-1},$$

where u is a normalized eigenfunction for $S_2(A)$ and ν is the exterior normal vector to $\Omega \setminus \overline{A}$.

Furthermore, in the case that Ω is the ball B_R with center 0 and radius $R > 0$ the authors show that a centered ball $A = B_r$, $r < R$, is critical in the sense that $S_2'(A) = 0$ when considering deformations that preserves volume but this configuration is not optimal.

Therefore there is a lack of symmetry in the optimal configuration.

Here, we extend these results to the more general case $1 < p < \infty$ and $1 < q < p_*$. Our method differs from the one in [FBGR] in order to deal with the nonlinear character of the problem.

The rest of the chapter is organized as follows: in Section 7.1, we compute the derivative of $S_q(\cdot)$ with respect to theregular perturbation of the hole and in Section 7.2, we study the lack of symmetry in the case that Ω is a ball.

7.1 Differentiation of the extremal

In this section, we compute the shape derivative of $S_q(\cdot)$ with respect to the regular perturbations of the hole.

As in Section 5.2, we consider the following variation. Let $V : \mathbb{R}^N \to \mathbb{R}^N$ be a regular (smooth) vector filed, globally Lipschitz, with support in Ω , and let $\varphi_t : \mathbb{R}^N \to \mathbb{R}^N$ be defined as the unique solution to

$$\begin{cases} \frac{d}{dt}\varphi_t(x) = V(\varphi_t(x)) & t > 0 \\ \varphi_0(x) = x & x \in \mathbb{R}^N. \end{cases}$$

Given $A \subset \partial\Omega$, we define $A_t := \varphi_t(A) \subset \Omega$ for all $t > 0$ and

$$S_q(t) = \inf\left\{ \frac{\int_\Omega |\nabla v|^p + |v|^p \,\mathrm{d}x}{\left(\int_{\partial\Omega} |v|^q \,\mathrm{d}\mathcal{H}^{N-1}\right)^{p/q}} : v \in W^{1,p}_{A_t}(\Omega) \setminus W^{1,p}_0(\Omega) \right\}. \tag{7.4}$$

The aim of this section is show that $S_q(t)$ is differentiable to t at $t = 0$. For this we require some previous results. Here, we use some ideas from [GMSL].

We begin by observing that if $v \in W^{1,p}_{A_t}(\Omega) \setminus W^{1,p}_0(\Omega)$, then

$$u = v \circ \varphi_t \in W^{1,p}_A(\Omega) \setminus W^{1,p}_0(\Omega).$$

Thus, by the Lemma 4.1.1, we have that

$$\int_\Omega |u|^p \,\mathrm{d}x = \int_\Omega |u|^p \,\mathrm{d}x + t\int_\Omega |u|^p \operatorname{div} V \,\mathrm{d}x + o(t),$$

and by the Theorem 4.1.3,

$$\int_\Omega |\nabla v|^p \,\mathrm{d}x = \int_\Omega |\nabla u|^p \,\mathrm{d}x + t\int_\Omega |\nabla u|^p \operatorname{div} V \,\mathrm{d}x - pt\int_\Omega |\nabla u|^{p-2}\langle \nabla u, {}^T V' \nabla u^T \rangle \,\mathrm{d}x + o(t).$$

Moreover, since $\operatorname{supp} V \subset \Omega$, we have that

$$\int_{\partial\Omega} |v|^q \,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial\Omega} |u|^q \,\mathrm{d}\mathcal{H}^{N-1}.$$

Therefore, we can rewrite (7.4) as

$$S_q(t) = \inf\left\{\rho(u) + t\gamma(u) \colon v \in W_A^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)\right\} \tag{7.5}$$

where

$$\rho(u) = \frac{\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x}{\left\{\int_{\partial\Omega} |u|^q \,\mathrm{d}\mathcal{H}^{N-1}\right\}^{p/q}},$$

and

$$\gamma(u) = \frac{\int_\Omega \{|\nabla u|^p + |u|^p\} \operatorname{div} V \,\mathrm{d}x - p\int_\Omega |\nabla u|^{p-2}\langle \nabla u, {}^T V' \nabla u^T\rangle \,\mathrm{d}x}{\left\{\int_{\partial\Omega} |u|^q \,\mathrm{d}\mathcal{H}^{N-1}\right\}^{p/q}} + o(1).$$

Given $t \geq 0$, let $u_t \in W_A^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$ such that $\|u_t\|_{L^q(\partial\Omega)} = 1$ and

$$S_q(t) = \psi(t) + t\phi(t),$$

where

$$\psi(t) = \rho(u_t) \quad \text{and} \quad \phi(t) = \gamma(u_t) \quad \forall\, t \geq 0.$$

We observe that $\psi, \phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and

Lemma 7.1.1. *The function ϕ is nonincreasing.*

Proof. Let $0 \leq t_1 \leq t_2$. By (7.5), we have that

$$\psi(t_2) + t_1\phi(t_2) \geq S_q(t_1) = \psi(t_1) + t_1\phi(t_1) \tag{7.6}$$

$$\psi(t_1) + t_2\phi(t_1) \geq S_q(t_2) = \psi(t_2) + t_2\phi(t_2). \tag{7.7}$$

Subtracting (7.6) from (7.7), we get

$$(t_2 - t_1)\phi(t_1) \geq (t_2 - t_1)\phi(t_2).$$

Since $t_2 - t_1 \geq 0$, we obtain

$$\phi(t_1) \geq \phi(t_2).$$

This ends the proof. □

Remark 7.1.2. Since ϕ is nonincreasing, we have

$$\phi(t) \leq \phi(0) \quad \forall\, t \geq 0,$$

and there exists

$$\phi(0^+) = \lim_{t \to 0^+} \phi(t).$$

Corollary 7.1.3. *The function ψ is nondecreasing.*

Proof. Let $0 \le t_1 \le t_2$. Again, by (7.5), we have that

$$\psi(t_2) + t_1\phi(t_2) \ge S_q(t_1) = \psi(t_1) + t_1\phi(t_1) \tag{7.8}$$

so

$$\psi(t_2) - \psi(t_1) \ge t_1(\phi(t_1) - \phi(t_2)).$$

Since $0 \le t_1 \le t_2$, by Lemma 7.1.1, we have that $\phi(t_1) - \phi(t_2) \ge 0$. Then

$$\psi(t_2) - \psi(t_1) \ge 0,$$

that is what we wished to prove. □

Now we can prove that $S_q(t)$ is continuous at $t = 0$.

Theorem 7.1.4. *The function $S_q(t)$ is continuous at $t = 0$, i.e.,*

$$\lim_{t\to 0^+} S_q(t) = S_q(0).$$

Proof. Given $t \ge 0$ so, by Corollary 7.1.3,

$$S_q(t) - S_q(0) = \psi(t) + t\phi(t) - \psi(0) \ge t\phi(t).$$

On the other hand, by (7.5), we have that

$$S_q(t) \le \psi(0) + t\phi(0) = S_q(0) + t\phi(0).$$

Then

$$t\phi(t) \le S_q(t) - S_q(0) \le t\phi(0).$$

Thus, by Remark 7.1.2,

$$\lim_{t\to 0^+} S_q(t) - S_q(0) = 0.$$

This finishes the proof. □

Thus, from Remark 7.1.2 and Theorem 7.1.4, we obtain the following corollary:

Corollary 7.1.5. *The function ψ is continuous at $t = 0$, i.e.,*

$$\lim_{t\to 0^+} \psi(t) = \psi(0).$$

Proof. We observe that

$$\psi(t) - \psi(0) = S_q(t) - S_q(0) - t\phi(t)$$

then, by Remark 7.1.2 and Theorem 7.1.4,

$$\lim_{t\to 0^+} \psi(t) - \psi(0) = 0.$$

That proves the result. □

Theorem 7.1.6. *The function ψ is differentiable at $t = 0$ and*

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(0) = 0.$$

Proof. Let $0 < r < t$. By (7.5), we get

$$S_q(r) = \psi(r) + r\phi(r) \leq \psi(t) + r\phi(t),$$

and

$$S_q(t) = \psi(t) + t\phi(t) \leq \psi(r) + t\phi(r).$$

So

$$\frac{r}{t}(\phi(r) - \phi(t)) \leq \frac{\psi(t) - \psi(r)}{t} \leq \phi(r) - \phi(t)$$

hence, taking limits when $r \to 0^+$, by Remark 7.1.2 and Corollary 7.1.5, we have that

$$0 \leq \frac{\psi(t) - \psi(0)}{t} \leq \phi(0^+) - \phi(t).$$

Now, taking limits when $t \to 0^+$, and again, by Remark 7.1.2, we get

$$\lim_{t\to 0^+} \frac{\psi(t) - \psi(0)}{t} = 0$$

as we wanted to show. □

Now, we are in condition to prove the main result of this section.

Theorem 7.1.7. *Suppose $A \subset \overline{\Omega}$ is a smooth open subset and let $1 < q < p_*$. Then, with the previous notation, we have that $S_q(t)$ is differentiable at $t = 0$ and*

$$\frac{d}{dt}S_q(t)\bigg|_{t=0} = (1 - p)\int_{\partial A} \left|\frac{\partial u_0}{\partial \nu}\right|^p \langle V, \nu\rangle \, \mathrm{d}\mathcal{H}^{N-1},$$

where u_0 is a normalized extremal for $S_q(A)$ and ν is the exterior normal vector to $\Omega \setminus \overline{A}$.

Remark 7.1.8. If u_0 is an extremal for $S_q(A)$ we have that $|u_0|$ is also an extremal associated to $S_q(A)$. Then, in the above theorem, we can suppose that $u_0 \geq 0$ in Ω. Moreover, by [L], we have that $u_0 \in C^{1,\rho}(\overline{\Omega \setminus A})$ if $\Omega \setminus \overline{A}$ is an open smooth open and if $\Omega \setminus \overline{A}$ satisfies the interior ball condition for all $x \in \partial\Omega \setminus \partial\overline{A}$ then $u_0 > 0$ on $\partial\Omega \setminus \partial\overline{A}$, see [V].

Proof of Theorem 7.1.7. We proceed in three steps.

Step 1. We show that $S_q(t)$ is differentiable at $t = 0$ and

$$\frac{d}{dt}S_q(t)\bigg|_{t=0} = \phi(0^+).$$

We have that
$$\frac{S_q(t)-S_q(0)}{t}=\frac{\psi(t)-\psi(0)}{t}-\phi(t).$$
Then, by Remark 7.1.2 and Theorem 7.1.6,
$$\frac{d}{dt}S_q(t)\Big|_{t=0}=S_q'(0)=\lim_{t\to 0^+}\frac{S_q(t)-S_q(0)}{t}=\phi(0^+).$$

Step 2. We show that there exists u extremal for $S_q(A)$ such that $\|u\|_{L^q(\partial\Omega)}=1$ and
$$\phi(0^+)=\int_\Omega(|\nabla u|^p+|u|^p)\,\mathrm{div}\,V\,\mathrm{d}x-p\int_\Omega|\nabla u|^{p-2}\langle\nabla u,{}^TV'\nabla u\rangle\,\mathrm{d}x.$$

By Theorem 7.1.5,
$$\|v_t\|^p_{W^{1,p}(\Omega)}=\psi(t)\to\psi(0)=S_q(0)\text{ as }t\to 0^+.\tag{7.9}$$
Then, there exists $u\in W^{1,p}(\Omega)$ and $t_n\to 0^+$ as $n\to\infty$ such that
$$v_{t_n}\rightharpoonup u\text{ weakly in }W^{1,p}(\Omega),\tag{7.10}$$
$$v_{t_n}\to u\text{ strongly in }L^q(\partial\Omega),\tag{7.11}$$
$$v_{t_n}\to u\text{ a.e. in }\Omega.\tag{7.12}$$
By (7.11) and (7.12), $u\in W^{1,p}_A(\Omega)$ and $\|u\|_{L^q(\partial\Omega)}=1$ and by (7.10)
$$S_q(0)=\lim_{n\to\infty}\|v_{t_n}\|^p_{W^{1,p}(\Omega)}\geq\|u\|^p_{W^{1,p}(\Omega)}\geq S_q(0),$$
then
$$S_q(0)=\|u\|^p_{W^{1,p}(\Omega)}.\tag{7.13}$$
Moreover, by (7.9), (7.10) and (7.13), we have that
$$v_{t_n}\to u\text{ strongly in }W^{1,p}(\Omega).$$

Therefore
$$\begin{aligned}\phi(0^+)&=\lim_{n\to\infty}\phi(v_{t_n})\\&=\int_\Omega(|\nabla u|^p+|u|^p)\,\mathrm{div}\,V\,\mathrm{d}x-p\int_\Omega|\nabla u|^{p-2}\langle\nabla u,{}^TV'\nabla u^T\rangle\,\mathrm{d}x.\end{aligned}$$

Step 3. Lastly, we show that
$$\begin{aligned}S_q'(0)&=\int_\Omega(|\nabla u|^p+|u|^p)\,\mathrm{div}\,V\,\mathrm{d}x-p\int_\Omega|\nabla u|^{p-2}\langle\nabla u,{}^TV'\nabla u^T\rangle\,\mathrm{d}x\\&=-\int_{\partial A}\left|\frac{\partial u}{\partial\nu}\right|^p\langle V,\nu\rangle\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

To show this we would require that $u \in C^2$. However, this is not true. Since u is an esxtremal for $S_q(A)$ and $\|u\|_{L^q(\Omega)} = 1$, we known that u is weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \setminus \overline{A}, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = S_q(A)|u|^{q-2}u & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial A, \end{cases}$$

and by [L] we get that u belongs to the class $C^{1,\delta}$ for some $0 < \delta < 1$.

In order to overcome this difficulty, we proceed as follows. We consider the regularized problems

$$S_\varepsilon := \inf \left\{ \frac{\int_\Omega (|\nabla v|^2 + \varepsilon^2)^{\frac{p-2}{2}}|\nabla v|^2 + |v|^p \, dx}{\left\{\int_{\partial\Omega} |v|^q \, dS\right\}^{\frac{p}{q}}} : v \in W^{1,p}_A(\Omega) \setminus W^{1,p}_0(\Omega) \right\}. \tag{7.14}$$

Let u_ε be a normalized positive eigenvalue associated to S_ε. Observe that the eigenfunction is weak solution to

$$\begin{cases} -\mathrm{div}(|\nabla u_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2}\nabla u_\varepsilon) + |u_\varepsilon|^{p-2}u_\varepsilon = 0 & \text{in } \Omega \setminus \overline{A}, \\ (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2}\frac{\partial u_\varepsilon}{\partial \nu} = S_\varepsilon |u_\varepsilon|^{q-2}u_\varepsilon & \text{on } \partial\Omega, \\ u_\varepsilon = 0 & \text{on } \partial A. \end{cases} \tag{7.15}$$

It is well known that the solution u_ε to (7.15) is of class $C^{2,\rho}$ for some $0 < \rho < 1$ (see [LSU]).

Then, we can perform all of our computations with the functions u_ε and pass to the limit as $\varepsilon \to 0$ at the end.

We have chosen to work formally with the function u in order to make our arguments more transparent and leave the details to the reader. For a similar approach, see Lemma 5.2.9, Remark 5.2.10 and proof of the Theorem 5.2.11.

Since

$$\begin{aligned} \mathrm{div}(|u|^p V) &= |u|^p \,\mathrm{div}\, V + p|u|^{p-2}u\langle \nabla u, V\rangle, \\ \mathrm{div}(|\nabla u|^p V) &= |\nabla u|^p \,\mathrm{div}\, V + p|\nabla u|^{p-2}\langle \nabla u D^2 u, V\rangle, \end{aligned}$$

we have that

$$\begin{aligned} \int_\Omega (|\nabla u|^p + |u|^p)\,\mathrm{div}\, V \, dx = \int_\Omega & \mathrm{div}(|u|^p V + |\nabla u|^p V)\, dx \\ & -p \int_\Omega \{|u|^{p-2}u_0\langle \nabla u, V\rangle + |\nabla u|^{p-2}\langle \nabla u D^2 u, uV\rangle\} dx. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_\Omega \operatorname{div}(|u|^p V + |\nabla u|^p V)\,\mathrm{d}x &= \int_{\partial\Omega} (|u|^p + |\nabla u|^p)\langle V, \nu\rangle\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad - \int_{\partial A} (|u|^p + |\nabla u|^p)\langle V, \nu\rangle\,\mathrm{d}\mathcal{H}^{N-1}\\ &= -\int_{\partial A} |\nabla u|^p \langle V, \nu\rangle\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

where the las equality follows from the fact that supp $V \subset \Omega$ and $u = 0$ on ∂A.

Thus

$$\begin{aligned}S_q'(0) &= -\int_{\partial A} |\nabla u|^p \langle V, \nu\rangle\,\mathrm{d}\mathcal{H}^{N-1} - p\int_\Omega |u|^{p-2}u\langle \nabla u_0, V\rangle \mathrm{d}x\\ &\quad -p\int_\Omega |\nabla u|^{p-2}\langle \nabla u, {}^T V'\nabla u +^T D^2 u V^T\rangle\,\mathrm{d}x\\ &= -\int_{\partial A} |\nabla u|^p \langle V, \nu\rangle\,\mathrm{d}\mathcal{H}^{N-1} - p\int_\Omega |u|^{p-2}u\langle \nabla u, V\rangle \mathrm{d}x\\ &\quad -p\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla(\langle \nabla u, V\rangle)\rangle\,\mathrm{d}x.\end{aligned}$$

Since u is a week solution of (7.3) as $\lambda = S_q(0)$ and supp $V \subset \Omega$ we have

$$S_q'(0) = \int_{\partial A} |\nabla u|^p \langle V, \nu\rangle\,\mathrm{d}\mathcal{H}^{N-1} - p\int_{\partial A} |\nabla u|^{p-2}\langle \nabla u, \nu\rangle\langle \nabla u, V\rangle\,\mathrm{d}\mathcal{H}^{N-1}.$$

Then, noticing that $\nabla u = \frac{\partial u}{\partial \nu}\nu$, the proof is complete. □

7.2 Lack of Symmetry in the Ball

In this section, we consider the case where $\Omega = B(0, R)$ and $A = B(0, r)$ with $0 < r < R$. The proofs of this section are based on the argument of [FBGR] and [LDT] adapted to our problem. In order to simplify notations, we write B_s and $S_q(r)$ instead $B(0, s)$ and $S_q(B(0, r))$, respectively.

First, we prove that the nonnegative solution of (7.3) is unique in this case.

Proposition 7.2.1. *Let $1 < q < p$. The nonnegative solution of (7.3) is unique.*

Proof. Suppose that there exist two nonnegative solutions u and v of (7.3). By Remark 7.1.8, it follows that $u, v > 0$ on $\partial\Omega$. Let $v_n = v + \frac{1}{n}$ with $n \in \mathbb{N}$, using first Piccone's identity (see [AH]) and the weak formulation of (7.3), we have

$$
\begin{aligned}
0 &\le \int_{B_R} |\nabla u|^p \,\mathrm{d}x - \int_{B_R} |\nabla v_n|^{p-2}\nabla v_n \nabla\left(\frac{u^p}{v_n^{p-1}}\right)\mathrm{d}x \\
&= \int_{B_R} |\nabla u|^p \,\mathrm{d}x - \int_{B_R} |\nabla v|^{p-2}\nabla v \nabla\left(\frac{u^p}{v_n^{p-1}}\right)\mathrm{d}x \\
&= -\int_{B_R} u^p \,\mathrm{d}x + \lambda \int_{\partial B_R} u^q \,\mathrm{d}\mathcal{H}^{N-1} + \int_{B_R} v^{p-1}\frac{u^p}{v_n^{p-1}}\,\mathrm{d}x - \lambda \int_{\partial B_R} v^{q-1}\frac{u^p}{v_n^{p-1}}\,\mathrm{d}\mathcal{H}^{N-1} \\
&\le \lambda \int_{\partial B_R} u^q \,\mathrm{d}\mathcal{H}^{N-1} - \lambda \int_{\partial B_R} v^{q-1}\frac{u^p}{v_n^{p-1}}\,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned}
$$

Thus, by the Monotone Convergence Theorem,

$$
0 \le \int_{\partial B_R} u^q \,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial B_R} v^{q-1}\frac{u^p}{v^{p-1}}\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial B_R} u^p(u^{q-p} - v^{q-p})\,\mathrm{d}\mathcal{H}^{N-1}.
$$

Note that the role of u and v in the above equation are exchangeable. Therefore, adding we get

$$
0 \le \int_{\partial B_R} (u^p - v^p)(u^{q-p} - v^{q-p})\,\mathrm{d}\mathcal{H}^{N-1}.
$$

Since $q < p$, we have that $u \equiv v$ on ∂B_R. Then, by uniqueness of solution to the Dirichlet problem, we get $u \equiv v$ in B_R. □

Remark 7.2.2. As the problem (7.3) is rotationally invariant, by uniqueness we obtain that the nonnegative solution of (7.3) must be radial. Therefore, if $\Omega = B_R$, $A = B_r$ and $1 < q \le p$ we can suppose that the extremal for $S_q(r)$ found in the Theorem 7.1.7 is nonnegative and radial.

Now, we can prove that this kind of configuration is critical.

Theorem 7.2.3. *Let $\Omega = B_R$ and let the hole be a centered ball $A = B_r$. Then, if $1 < q \le p$, this configuration is critical in the sense that $S_q'(r) = 0$ for all deformations V that preserve the volume of B_r.*

Proof. We consider $\Omega = B_R$, $A = B_r$ and $1 < q \le p$. By Theorem 7.3 and Remark 7.2.2, there exist a nonnegative and radial normalized extremal for $S_q(r)$ such that

$$
S_q'(0) = (1-p)\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^p \langle V, \nu\rangle \,\mathrm{d}\mathcal{H}^{N-1}.
$$

Since u is radial,

$$
\frac{\partial u}{\partial \nu} \equiv c \text{ on } \partial B_r,
$$

where c is a constant.

Thus, using that we are dealing with deformations V that preserves the volume of the B_r, we have that

$$S_q'(0) = (1-p)c^p \int_{\partial B_r} \langle V, \nu\rangle \,\mathrm{d}\mathcal{H}^{N-1} = (p-1)c^p \int_{B_r} \operatorname{div}(V)\,\mathrm{d}x = 0.$$

□

But, if q is sufficiently large, the symmetric hole with a radial extremal is not an optimal configuration. To prove this, we need two previous results.

Proposition 7.2.4. *Let $r > 0$ fixed. Then, there exists a positive radial function u_0 such that*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \mathbb{R}^N \setminus B_r, \\ u = 0 & \text{on } \partial B_r. \end{cases} \tag{7.16}$$

This u_0 is unique up to a constant factor and for any $R > r$ the restriction of u_0 to B_R is the first eigenfunction of (7.3) with $q = p$.

Proof. For $R > r$, let u_R be the unique solution of the Dirichlet problem

$$\begin{cases} \Delta_p u_R = |u_R|^{p-2}u_R & \text{in } B_R \setminus \overline{B_r}, \\ u(R) = 1, \\ u(r) = 0. \end{cases}$$

Then, by uniqueness, u_R is a nonnegative and radial function. Moreover, by the regularity theory and maximum principle we have $\frac{\partial u_R}{\partial \nu}(r) \neq 0$ (see [L, V]). Thus, for any $R > r$, we define the restriction of u_0 by

$$u_0 = \frac{u_R}{\dfrac{\partial u_R}{\partial \nu}(r)}.$$

By uniqueness of the Dirichlet problem, it is easy to check that u_0 is well defined and is a nonnegative radial solution of (7.16). Furthermore, by the simplicity of $S_p(r)$, u_0 is the eigenfunction associated to $S_p(r)$ for every $R > r$. □

Proposition 7.2.5. *Let v be a radial solution of (7.3). Then v is a multiple of u_0. In particular, any radial minimizer of (7.2) is a multiple of u_0.*

Proof. Let $a > 0$ be such that $v = au_0$ on $\partial B(0, R)$. Then v and au_0 are two solutions to the Dirichlet problem $\Delta_p w = w^{p-1}$ and $w = v$ on $\partial\left(B_R \setminus \overline{B_r}\right)$. Hence, by uniqueness, we have that $v = au_0$ in B_R. □

Remark 7.2.6. If $1 < q < p$ then the solution of (7.3), by Remark 7.2.2 and Proposition 7.2, is a multiple of u_0.

Now, we are in condition to prove that he symmetric hole with a radial extremal is not an optimal configuration if q is sufficiently large.

Theorem 7.2.7. *Let $r > 0$ and $1 < p < \infty$ be fixed. Let $R > r$ and*

$$Q(R) = \frac{1}{S_p(B_r)^{\frac{p}{p-1}}}\left(1 - \frac{N-1}{R}S_p(B_r)\right) + 1. \tag{7.17}$$

If $q > Q(R)$ then the centered hole B_r is not optimal.

Proof. Let $R > r$ be fixed and consider u_0 to be the nonnegative radial function given by Proposition 7.2.4 such that that $u_0 = 1$ on ∂B_R. Then, by Proposition 7.2.5, it is enough to prove that u_0 is not a minimizer for $S_q(r)$ when $q > Q(R)$.

First, let us move this symmetric configuration in the x_1 direction. For any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, we denote $x_t = (x_1 - t, x_2, \dots, x_N)$ and define

$$U(t)(x) = u_0(x_t)$$

Observe that U vanishes in $A_t := B(te_1, r)$ a subset of B_R of the same measure of B_r for all t small.

Consider the function

$$h(t) = \frac{f(t)}{g(t)}$$

where

$$f(t) = \int_{B_R} |\nabla U|^p + U^p \,\mathrm{d}x \quad \text{and} \quad g(t) = \left(\int_{\partial B_R} U^q \mathrm{d}\mathcal{H}^{N-1}\right)^{p/q}.$$

We observe that $h(0) = 0$ and since h is an even function, we have $h'(0) = 0$. Now,

$$h''(0) = \left.\frac{f''g^2 - fgg'' - 2f'gg' - 2fgg'}{g^3}\right|_{t=0}.$$

Next we compute these terms. First, since u_0 is the first eigenfunction of (7.3) with $q = p$ and $u_0 = 1$ on ∂B_R we get

$$f(0) = S_p(r)|\partial B_R| \quad \text{and} \quad g(0) = |\partial B_R|^{\frac{p}{q}}.$$

Thus, by Gauss-Green's Theorem and using the fact that u_0 is radial, we get

$$f'(0) = -\int_{B_R} \frac{\partial}{\partial x_1}\left(|\nabla u_0|^p + u_0^p\right) \mathrm{d}x = \int_{\partial B_R} (|\nabla u_0|^p + u_0^p)\nu_1 \mathrm{d}\mathcal{H}^{N-1} = 0.$$

Again, since u_0 is radial,

$$g'(0) = \frac{p}{q}\left(\int_{\partial B_R} u^q \mathrm{d}\mathcal{H}^{N-1}\right)^{\frac{p}{q}-1}\left(\int_{\partial B_R} \frac{\partial u^q}{\partial x_1}\mathrm{d}\mathcal{H}^{N-1}\right) = 0.$$

Finally, using that $u_0 = 1$ on ∂B_R, we obtain

$$g''(0) = p|\partial B_R|^{\frac{p}{q}-1}\int_{\partial B_R} (q-1)\left(\frac{\partial u_0}{\partial x_1}\right)^2 + \frac{\partial^2 u_0}{\partial x_1^2}\,\mathrm{d}\mathcal{H}^{N-1},$$

and by the Gauss–Green's Theorem,

$$f''(0) = p\int_{B_R} \frac{\partial}{\partial x_1}\left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial x_1} + \frac{1}{p}\frac{\partial u_0^p}{\partial x_1}\right) \mathrm{d}x$$
$$= p\int_{\partial B_R}\left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial x_1} + \frac{1}{p}\frac{\partial u_0^p}{\partial x_1}\right)\nu_1 \,\mathrm{d}\mathcal{H}^{N-1}.$$

Then

$$h''(0) = \frac{p}{|\partial B_R(0)|^{p/q}}\Bigg[\int_{\partial B_R}\left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial x_1} + \frac{1}{p}\frac{\partial u_0^p}{\partial x_1}\right)\nu_1 \,\mathrm{d}\mathcal{H}^{N-1}$$
$$- S_p(r)\int_{\partial B_R}(q-1)\left(\frac{\partial u_0}{\partial x_1}\right)^2 + \frac{\partial^2 u_0}{\partial x_1^2}\,\mathrm{d}\mathcal{H}^{N-1}\Bigg].$$

Thus, since u_0 is radial, we get

$$h''(0) = \frac{p}{N|\partial B_R(0)|^{p/q}}\Bigg[\int_{\partial B_R}\left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial \nu} + \frac{1}{p}\frac{\partial u_0^p}{\partial \nu}\right) \mathrm{d}\mathcal{H}^{N-1}$$
$$- S_p(r)\int_{\partial B_R}(q-1)|\nabla u_0|^2 + \Delta u_0 \,\mathrm{d}\mathcal{H}^{N-1}\Bigg].$$

Now, by definition, $u_0(x) = u_0(|x|)$ satisfies

$$(s^{N-1}|u_0'|^{p-1}u_0')' = s^{N-1}u_0^{p-1} \quad \forall\, s > r$$

with $u_0(R) = 1$ and $u_0(r) = 0$. Moreover, by Proposition 7.2.4, we have

$$u_0'(s)^{p-1} = S_p(r)u_0(s)^{p-1} \quad \forall\, s > r.$$

Then

$$\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial \nu} + \frac{1}{p}\frac{\partial u_0^p}{\partial \nu} = \frac{S_p(r)^{\frac{1}{p-1}}}{p-1}\left(1 - \frac{N-1}{R}S_p(r)\right) + S_p(r)^{\frac{1}{p-1}},$$

and

$$S_p(r)\left[(q-1)|\nabla u_0|^2 + \Delta u_0\right] = (q-1)S_p(r)^{\frac{p+1}{p-1}} + \frac{S_p(r)^{\frac{1}{p-1}}}{p-1}\left(1 - \frac{N-1}{R}S_p(r)\right)$$
$$+ \frac{N-1}{R}S_p(r)^{\frac{p}{p-1}}.$$

Therefore

$$h''(0) = \frac{pS_p^{\frac{1}{p-1}}}{N|\partial B_R|^{\frac{p}{q}-1}}\left[1 - (q-1)S_p(r)^{\frac{p}{p-1}} - \frac{N-1}{R}S_p(r)\right].$$

Thus, if $q > Q(R)$, we get that $h''(0) < 0$ and so 0 is a strict local maxima of h. So we have proved that

$$S_q(r) = h(0) > h(t) \geq S_q(B(te_1, r))$$

for all t small. Therefore a symmetric configuration is not optimal. □

Lastly, to study the asymptotic behavior of $Q(R)$

Proposition 7.2.8. *The function $Q(R)$ has the following asymptotic behavior*

$$\lim_{R\to r} Q(R) = 1^- \quad \textit{and} \quad \lim_{R\to+\infty} Q(R) = p.$$

Remark 7.2.9. Observe that $Q(R) < 1$ for R close to r, and therefore the symmetric hole with a radial extremal is not an optimal configuration for R close to r.

Proof of Proposition 7.2.8. We proceed in two step.

Step 1. First we show that, for $R > r$, $S_p(R, r) = S_p(r)$ verifies the differential equation

$$\frac{\partial S_p}{\partial R} = -\frac{N-1}{R} S_p + 1 - (p-1) S_p^{\frac{p}{p-1}} \tag{7.18}$$

with the condition

$$S_p|_{R=r} = +\infty.$$

Again, we consider $u_0(x) = u_0(|x|)$ the nonnegative radial function given by Proposition 7.2.4. Thus, for all $R > r$, we get

$$\begin{cases} (p-1)\left(u_0'\right)^{p-2} u_0'' + \dfrac{N-1}{R}(u_0')^{p-1} = u_0^{p-1}, \\ u_0'(R)^{p-1} = S_p u_0(R)^{p-1}, \\ u_0(r) = 0. \end{cases}$$

Then

$$S_p = \left(\frac{u_0'(R)}{u_0(R)}\right)^{p-1}.$$

Thus

$$\begin{aligned} \frac{\partial S_p}{\partial R} &= (p-1)\left(\frac{u_0'(R)}{u_0(R)}\right)^{p-2} \frac{u_0''(R)u_0(R) - u_0'(R)^2}{u_0(R)^2} \\ &= (p-1)\left(\frac{u_0'(R)}{u_0(R)}\right)^{p-2} \frac{u_0''(R)}{u_0(R)} - (p-1)S_p^{\frac{p}{p-1}} \\ &= (p-1)\frac{u_0'(R)^{p-2}u_0''(R)}{u_0(R)^{p-1}} - (p-1)S_p^{\frac{p}{p-1}} \\ &= 1 - \frac{N-1}{R}S_p - (p-1)S_p^{\frac{p}{p-1}}. \end{aligned}$$

On the other hand, since (by definition) $\frac{\partial u_0}{\partial \nu} \equiv 1$ on ∂B_r, we get that $u'(r) = 1$. Then

$$\lim_{R\to r} S_p = \lim_{R\to r}\left(\frac{u_0'(R)}{u_0(R)}\right)^{p-1} = +\infty.$$

Now, it is easy to check that $\lim_{R\to r} Q(R) = 1^-$.

Step 2. Finally, we prove that

$$\lim_{R\to+\infty} Q(R) = p.$$

We begin differentiating (7.18) to obtain

$$\frac{\partial^2 S_p}{\partial R^2} = \frac{N-1}{R^2} S_p - \frac{N-1}{R}\frac{\partial S_p}{\partial R} - p S_p^{\frac{1}{p-1}} \frac{\partial S_p}{\partial R}.$$

Then, since $S_p > 0$, at any critical point ($S_p' = 0$) we have that $S_p'' > 0$. Thus, S_p has at most one critical point, which is a minimum. If S_p has a minimum, then there exist $R_0 > r$ such that $S_p'(R_0) = 0$. Moreover, since $S_p'(R) \neq 0$ for any $R \neq R_0$ and $S_p \to +\infty$ as $R \to r$ and by (7.18), we get that $S_p' < 0$ for all $r < R < R_0$ and $S_p' > 0$ for all $R > R_0$. Thus, using again (7.18), we have that

$$S_p^{\frac{p}{p-1}} < \frac{1}{p-1} \qquad \forall\, R > R_0.$$

Then S_p is strictly increasing as a function of R and bonded for all $R > R_0$. Consequently $S_p' \to 0$ as $R \to +\infty$. It follows, by (7.18), that

$$S_p^{\frac{p}{p-1}} \to \frac{1}{p-1} \qquad \text{as } R \to +\infty.$$

On the other hand using (7.17) and (7.18) we see that

$$S_p = (Q(R) - p) S_p^{\frac{p}{p-1}}. \tag{7.19}$$

So, if S_p has a minimum, we get that $Q(R) > p$ for all $R > R_0$ and $Q(R) \to p^+$ as $R \to +\infty$. Now, If S_p has not critical points so $S_p' \neq 0$ for all $R > r$ and using that $S_p \to +\infty$ as $R \to r$, and (7.18) we get that $S_p' < 0$ for all $R > r$. Consequently, in this case, S_p is strictly decreasing, and therefore $S_p' \to 0$ as $R \to +\infty$. By (7.18) we have that

$$S_p \to \frac{1}{p-1} \qquad \text{as } R \to +\infty.$$

Then, if S_p has not critical points, we get $Q(R) < p$ and $Q(R) \to p^-$ as $R \to +\infty$. □

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