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PROBLEMAS DE FRONTERA LIBRE EN ESPACIOS DE ORLICZ

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Problemas de frontera libre en espacios de Orlicz

Resumen

En esta tesis, se estudia el siguiente problema de frontera libre: Para un dominio Ω de $\mathbb{R}^N$, hallar $u \geq 0$ tal que

$$(B) \qquad \begin{cases} \mathcal{L}u := \operatorname{div}\Big(\dfrac{g(|\nabla u|)}{|\nabla u|}\nabla u\Big) = 0 & \text{en } \{u > 0\} \cap \Omega \\ |\nabla u| = \lambda^* & \text{en } \partial\{u > 0\} \cap \Omega \end{cases}$$

Se denomina Problema de Frontera Libre ya que no se conoce a priori la ubicación de $\partial\{u > 0\}$. La segunda ecuación en (B) se conoce como "condición de frontera libre".

Este problema aparece en numerosas aplicaciones. En este trabajo discutiremos tres de ellas.

Primero, estudiamos el problema de "chorros" (jets). Para un dominio suave y acotado en $\mathbb{R}^N$, consideramos primero el siguiente problema, minimizar el funcional,

$$\mathcal{J}(v) = \int_\Omega G(|\nabla v|)\, dx + \lambda |\{v > 0\}|$$

con $v - \varphi_0 \in W_0^{1,G}(\Omega)$ para una $\varphi_0 \geq 0$, $\varphi_0 \in L^\infty(\Omega)$ y $\int_\Omega G(|\nabla \varphi_0|)\, dx < \infty$. $W^{1,G}(\Omega)$ es la clase de funciones débilmente diferenciables con $\int_\Omega G(|v|)\, dx < \infty$ y $\int_\Omega G(|\nabla v|)\, dx < \infty$. Aquí denominamos $G' = g$.

El segundo, es un problema de diseño óptimo. Más precisamente, minimizar

$$\mathcal{J}(v) = \int_\Omega G(|\nabla v|)\, dx$$

con $v - \varphi_0 \in W_0^{1,G}(\Omega)$ y tal que $|\{v > 0\}| = \alpha \in (0, |\Omega|)$ fijo, para una función acotada, nonnegativa y no idénticamente nula φ_0 tal que $\int_\Omega G(|\nabla \varphi_0|)\, dx < \infty$.

Como tercera aplicación estudiamos un problema de perturbación singular de interés en combustión. Para $\varepsilon > 0$, tomamos u^ε una solución débil de $\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon)$ con $u^\varepsilon \geq 0$. Aquí $\beta \in \mathrm{Lip}(\mathbb{R})$, es positiva en $(0,1)$, cero fuera de $[0,1]$ y tal que $\int_0^1 \beta(s)\, ds = M$ y $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$.

En todos estos problemas, imponemos condiciones sobre la función g de forma tal que se puede comportar distinto en 0 y en infinito. Más precisamente, pedimos que existan constantes $\delta, g_0 > 0$ tales que,

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \qquad \forall t > 0.$$

Es fácil ver que, el conjunto de funciones que cumplen nuestras condiciones incluye funciones no homogéneas. Estas condiciones fueron introducidas por Lieberman en [**22**] y generalizan las llamadas condiciones naturales de Ladyzhenskaya y

Ural'tseva (ver [**18**]). En dicho trabajo el autor estudia la regularidad de soluciones de $\mathcal{L}u = f$, donde f es una función acotada.

Para el primer problema, probamos las siguientes propiedades de los minimizantes: Primero la existencia, luego la continuidad Hölder y con ésto probamos la continuidad Lipschitz uniforme (i.e: el $|\nabla u|$ está acotado en cada compacto de Ω por una constante independiente del minimizante u). Además tenemos que los minimizantes satisfacen, en un sentido débil, el problema de frontera libre (B), con λ^* una constante tal que $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$.

Además probamos una cierta propiedad de nodegeneración de los minimizantes en cualquier punto de la frontera libre, y finalmente obtenemos que la misma tiene medida de Hausdorff $N-1$ dimensional finita; por lo tanto $\{u > 0\} \cap \Omega$ tiene perímetro localmente finito en Ω.

También definimos dos nociones distintas de solución débil (en sentido distribucional y en sentido puntual) del problema (B) y probamos, para las primeras, que estas soluciones tienen casi todas las mismas propiedades que tienen los minimizantes.

Probamos la regularidad de la frontera libre de las soluciones débiles de (B), es decir que $\partial_{red}\{u > 0\} \cap \Omega$ es una superficie $C^{1,\alpha}$ y que en el caso de los minimizantes (y para las soluciones débiles en sentido distribucional) el complemento tiene medida de Hausdorff $N-1$ dimensional nula. Para ello tomamos ideas del trabajo pionero [**4**]. También probamos, para un subclase de funciones g, y cuando $N = 2$, que toda la frontera libre es regular.

Para trabajar con este problema, debimos lidiar con la degeneración del problema y con la falta de homogeneidad al mismo tiempo.

En el segundo problema, probamos la regularidad de los minimizantes estudiando un problema de penalización asociado a éste. Probamos que los minimizantes del problema penalizado son soluciones débiles de (B) en sentido distribucional (de tipo I). Los resultados de regularidad para el problema de penalización son consecuencia de los resultados que tenemos para soluciones débiles de (B). La ventaja del método es que no es necesario pasar al límite para volver al problema original. Esto es, si el parámetro en el problema de perturbación es suficientemente chico, tenemos que los minimizantes son soluciones del problema de optimización. Nuevamente, para tratar este problema, debimos lidiar con la no linealidad y la no homogeneidad del operador.

En el tercer problema, probamos que bajo, ciertas hipótesis sobre las soluciones, una función límite es una solución débil en el sentido puntual del problema (B) (de tipo II). Por lo tanto, todos los resultados de regularidad de soluciones de tipo II se aplican a límites de soluciones del problema de perturbación singular.

Palabras clave: Problemas de frontera libre, problemas de minimización, espacios de Orlicz, regularidad, optimización, perturbación singular.

Free boundary problems in Orlicz spaces

In this thesis, we study the following free boundary problem: For a domain Ω in $\mathbb{R}^N$, find $u \geq 0$ such that

$$(B) \qquad \begin{cases} \mathcal{L}u := \operatorname{div}\left(\dfrac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 & \text{in } \{u > 0\} \cap \Omega \\ |\nabla u| = \lambda^* & \text{on } \partial\{u > 0\} \cap \Omega \end{cases}$$

We call it a Free Boundary Problem because we do not know a priori the location of $\partial\{u > 0\}$. The second equation in (B) is known as the "free boundary condition".

This problem appears in many applications. In this thesis, we will discuss three of them.

First, we study the problem of jets. For a bounded smooth domain in $\mathbb{R}^N$, we consider the following problem: Minimize the functional,

$$\mathcal{J}(v) = \int_\Omega G(|\nabla v|)\, dx + \lambda|\{v > 0\}|$$

with $v - \varphi_0 \in W_0^{1,G}(\Omega)$ for a function $\varphi_0 \geq 0$, $\varphi_0 \in L^\infty(\Omega)$ with $\int_\Omega G(|\nabla \varphi_0|)\, dx < \infty$. $W^{1,G}(\Omega)$ is the class of weakly differentiable functions with $\int_\Omega G(|v|)\, dx < \infty$ and $\int_\Omega G(|\nabla v|)\, dx < \infty$. Here we denote $G' = g$.

The second one, is a shape optimization problem. More precisely, to minimize

$$\mathcal{J}(v) = \int_\Omega G(|\nabla v|)\, dx$$

with $v - \varphi_0 \in W_0^{1,G}(\Omega)$ and such that $|\{v > 0\}| = \alpha \in (0, |\Omega|)$ fixed, for a bounded nonnegative function φ_0 and not identically zero such that $\int_\Omega G(|\nabla \varphi_0|)\, dx < \infty$.

As a third application we study a singular perturbation problem, of interest in combustion. For $\varepsilon > 0$, take u^ε a weak solution of $\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon)$ with $u^\varepsilon \geq 0$. Here $\beta \in \mathrm{Lip}(\mathbb{R})$, is positive in $(0,1)$, zero outside $[0,1]$ and is such that $\int_0^1 \beta(s)\, ds = M$ and $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$.

In all these problems we impose conditions on the function g such that allow to have different behaviors at 0 and at infinity. More precisely, we assume that there exist constants $\delta, g_0 > 0$ such that,

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \qquad \forall t > 0.$$

It is easy to see that the set of functions that satisfy these conditions includes non homogeneous functions. These conditions were introduced by Lieberman in [**22**] and generalize the so called natural conditions of Ladyzhenskaya and Ural'tseva (see

[**18**]). In that paper the author studies the regularity of solutions of $\mathcal{L}u = f$, where f is a bounded function.

For the first problem, we prove the following properties of the minimizers: First the existence, then the Hölder continuity and finally we prove the uniform Lipschitz continuity (i.e: the $|\nabla u|$ is bounded in any compact subset of Ω by a constant independent of the minimizer u). Moreover, we have that the minimizers satisfy, in a weak sense, the free boundary problem (B) with λ^* a constant such that $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$.

Moreover, we prove some properties of non-degeneracy of the the minimizers at every point of the free boundary. Finally, we obtain that the free boundary has finite $N-1$ dimensional Hausdorff measure. Therefore, $\{u > 0\} \cap \Omega$ has finite perimeter locally in Ω.

We also define two different notions of weak solutions of the problem (B) (in a distributional sense and in a pointwise sense). We prove, for the first ones, that they have almost all the properties that minimizers have.

Then, we prove the regularity of the free boundary of weak solutions of (B). That is, $\partial_{red}\{u > 0\} \cap \Omega$ is a $C^{1,\alpha}$ surface and, in the case of minimizers (and for the weak solutions in the distributional sense), the remainder has zero $N-1$ dimensional Hausdorff measure. To this end, we take ideas from the paper [**4**]. We also prove, for a subclass of functions g, and when $N = 2$, that the whole free boundary is regular.

In order to get our results, we have to deal with the degeneracy of the problem and with the loss of homogeneity at the same time.

In the second problem, we prove the regularity of minimizers by studying an associated penalization problem. We prove that the minimizers of the penalized problem are weak solutions of (B) in the distributional sense (of type I). The regularity results for the penalized problem, are a consequence of the results that we have for weak solutions of (B). The advantage of this method is that in order to return to the original problem, it is not necessary to pass to the limit. That is, if the penalization parameter is sufficiently small, then we have that the minimizers are solutions of the optimization problem. Again, to treat this problem, we had to deal with the degeneracy and the non homogeneity of the operator.

In the third problem we prove that, under some hypothesis on the solutions, a limiting function is a weak solution in the pontwise sense of the problem (B) (of type II). Therefore, all the regularity results of solutions of type II can be applied to the limit of solutions of the singular perturbation problem.

Key words : Free boundary problems, minimization problems, Orlicz spaces, regularity, optimization, singular perturbation.

Contents

Introducción

En esta tesis, se estudia el siguiente problema de frontera libre: Para un dominio Ω de $\mathbb{R}^N$, hallar u tal que

$$
\begin{cases} \mathcal{L}u := \operatorname{div}\left(\dfrac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 & \text{en } \{u > 0\} \cap \Omega \\ |\nabla u| = \lambda^* & \text{en } \partial\{u > 0\} \cap \Omega. \end{cases} \tag{0.0.1}
$$

Se denomina Problema de Frontera Libre ya que no se conoce a priori la ubicación de $\partial\{u > 0\}$. La segunda ecuación en (0.0.1) se conoce como "condición de frontera libre".

Este problema aparece en numerosas aplicaciones. En este trabajo discutiremos tres de ellas a saber, el problema de "chorros" (jets) que consiste en minimizar,

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|) + \lambda\chi_{\{u>0\}}\,dx$$

en la clase de funciones

$$\mathcal{K} = \Big\{v \in W^{1,G}(\Omega) : \quad v = \varphi_0 \text{ en } \partial\Omega\Big\},$$

donde φ_0 es una función no negativa con $\varphi_0 \in L^\infty(\Omega)$, $\int_\Omega G(|\nabla\varphi_0|)\,dx < \infty$ y G es tal que $g' = G$. $W^{1,G}(\Omega)$ es un espacio de Sobolev-Orlicz (ver Capítulo 1).

El segundo, es un problema de diseño óptimo. A saber, minimizar

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|)\,dx$$

en el conjunto

$$\mathcal{K}_\alpha = \{v \in W^{1,G}(\Omega) : \ |\{v > 0\}| = \alpha,\, v = \varphi_0 \ \text{ en } \partial\Omega\},$$

para una función φ_0 nonnegativa, acotada con $\int_\Omega G(|\nabla\varphi_0|)\,dx < \infty$ y no idénticamente nula.

Finalmente, el tercer problema tiene origen en la teoría de combustión en el llamado "límite para energía de activación tendiendo a infinito" y consiste en lo siguiente: Para $\varepsilon > 0$, tomamos u^ε una solución débil de,

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon), \quad u^\varepsilon \geq 0 \tag{P_ε}$$

con $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$, $\beta \in \operatorname{Lip}(\mathbb{R})$, positiva en $(0,1)$, y cero fuera de $[0,1]$

En este caso, estamos interesados en el estudio de propiedades uniformes de las soluciones y el estudio del problema límite, cuando $\varepsilon \to 0$.

En este caso probamos, bajo condiciones adecuadas, que para cualquier sucesión $\varepsilon_n \to 0$ existe una subsucesión ε_{n_k} y una función límite u, tal que $u = \lim u^{\varepsilon_{n_k}}$, y u es una solución del problema de frontera libre (0.0.1) para alguna constante λ^* dependiendo de g y M.

En todos los casos vemos que la solución del problema que nos interesa resulta ser solución de (0.0.1) en un sentido débil. Por lo tanto, resulta de gran interés estudiar la regularidad de las soluciones débiles de (0.0.1) y la de sus fronteras libres.

En este sentido, el primer paso es dar una buena definición de solución débil del problema (0.0.1) que englobe todas las aplicaciones que tenemos en mente. El segundo paso, es ver cual es la regularidad óptima que van a tener estas soluciones (observar que, si queremos que la función tenga derivada normal constante en la frontera de $\{u > 0\}$, la regularidad óptima no podrá ser más que Lipschitz). Finalmente, sería deseable obtener la regularidad $C^{1,\alpha}$ de la frontera libre ya que ésta va a implicar que la solución débil satisface la condición de frontera libre en sentido clásico.

Todos estos temas se encuentran bien estudiados en el caso en que el operador $\mathcal{L}$ es el laplaciano (ver por ejemplo, [**2, 4, 5, 6, 7, 8, 20**]). En particular, para este caso se han desarrollado diversas teorías para el estudio de la regularidad de las soluciones débiles de (0.0.1). En estos trabajos se ha demostrado que las soluciones débiles son localmente Lipschitz y que la frontera libre $\partial\{u > 0\}$ es una superficie $C^{1,\alpha}$ cuando $N = 2$, y tiene esta regularidad en un entorno de todo punto donde es “chata” (flat), en dimensiones mayores. Estos resultados se han obtenido tanto para soluciones distribucionales como para soluciones viscosas. En el primer caso, la condición de frontera libre aparece en forma integral, y esta definición es más apropiada para el problema de “chorros” y el de optimización. El concepto de solución viscosa ha sido el utilizado en problemas a dos fases y para el problema de combustión.

Estos resultados se han extendido a operadores cuasilineales o fuertemente no lineales independientes de la variable espacial, y a operadores lineales con coeficientes variables. En todos estos casos el operador $\mathcal{L}$ se supone uniformemente elíptico.

Recientemente, algunos de los resultados fueron demostrados también en el caso en que $\mathcal{L}$ es el p–laplaciano. Es decir, $\mathcal{L}u = \Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ que es un operador elíptico degenerado en el caso $p > 2$ y singular en el caso $p < 2$ (en el caso $p = 2$ coincide con el laplaciano) (ver, por ejemplo, [**10, 11**]). En [**10**] se estudia el problema de jets y se prueban los resultados de regularidad que fueran mencionados arriba para el caso del laplaciano usual. En [**11**] se estudia el problema de combustión (en el límite para energías de activación tendiendo a infinito) y se prueba que en el límite se encuentra una solución viscosa del problema (0.0.1). No se obtienen resultados de regularidad de la frontera libre para soluciones viscosas en este caso degenerado o singular.

El objetivo de esta tesis es el estudio de este problema –incluyendo la regularidad de la frontera libre– para operadores que puedan ser elípticos degenerados o singulares, posiblemente no homogéneos (el p–laplaciano es homogéneo y esto simplifica algunas demostraciones). Aquí se admiten, además, funciones g en el operador $\mathcal{L}$ con un comportamiento diferente en 0 y en infinito. Clásicamente, las suposiciones sobre el comportamiento de g en 0 e infinito han sido siempre similares al caso del p–laplaciano. Aquí, en cambio, se adoptan las condiciones introducidas por G. Lieberman en [**22**] para el estudio de la regularidad de soluciones débiles de la ecuación elíptica (posiblemente degenerada o singular) $\mathcal{L}u = f$ con f acotada.

Estas condiciones aseguran que la ecuación $\mathcal{L}u = 0$ es equivalente a una ecuación uniformemente elíptica en forma de no divergencia con constantes de elipticidad independientes de la solución u en conjuntos donde $\nabla u \neq 0$. Es más, estas condiciones no implican ningún tipo de homogeneidad en las función g y además permiten diferente comportamiento de la función g cuando $|\nabla u|$ está cerca de cero o de infinito. A saber, asumimos que g satisfase

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \qquad \forall t > 0 \tag{0.0.2}$$

para ciertas constantes $0 < \delta \leq g_0$.

Observemos que $\delta = g_0 = p - 1$ cuando $G(t) = t^p$, y recíprocamente, si $\delta = g_0$ entonces G es una potencia.

Otro ejemplo de función g que satisface (0.0.2) es la función $g(t) = t^a \log(bt + c)$ con $a, b, c > 0$. En este caso se satisface (0.0.2) con $\delta = a$ y $g_0 = a + 1$.

Otro caso interesante es el de funciones $G \in C^2([0, \infty))$ con $G'(t) = g(t) = c_1 t^{a_1}$ para $t \leq s$, $g(t) = c_2 t^{a_2} + d$ para $t \geq s$. En este caso g satisface (0.0.2) con $\delta = \min(a_1, a_2)$ y $g_0 = \max(a_1, a_2)$.

Más aún, cualquier combinación lineal con coeficientes positivos de funciones satisfaciendo (0.0.2) también satisface (0.0.2). Por otro lado, si g_1 y g_2 satisfacen (0.0.2) con constantes δ^i y g_0^i, $i = 1, 2$, la función $g = g_1 g_2$ satisface (0.0.2) con $\delta = \delta^1 + \delta^2$ y $g_0 = g_0^1 + g_0^2$, y la función $g(t) = g_1\big(g_2(t)\big)$ satisface (0.0.2) con $\delta = \delta^1\delta^2$ y $g_0 = g_0^1 g_0^2$.

Esta observación muestra que existe un amplio rango de funciones g bajo las hipótesis de esta tesis.

Con respecto a la noción de solución débil considerada en este trabajo, resaltamos que si bien la noción que podríamos llamar “distribucional” daría lugar, en principio, a mejores resultados, ésta no es adecuada para el problema de perturbación singular de interés en combustión que estudiamos en el último capítulo. Por lo tanto, en esta tesis se introduce una nueva noción de solución débil de (0.0.1) en la que la condición de frontera libre se pide que se satisfaga en un sentido puntual y no integral como es el caso de las soluciones distribucionales. A pesar de ésto, para estas soluciones débiles la demostración de la regularidad de la frontera libre en un entorno de cada punto donde es “chata” es muy similar a la que se puede encontrar para soluciones distribucionales en el caso $\mathcal{L} = \Delta$ en [**4**].

Por otro lado, probamos que las soluciones distribucionales son soluciones débiles en este sentido puntual. Por lo tanto, los resultados se aplican también al caso distribucional. En este último caso, los resultados resultan más fuertes ya que, como en el caso de ecuaciones uniformemente elípticas, se prueba en esta tesis que la frontera libre de una solución distribucional es "chata" en casi todo punto (respecto de la medida de Hausdorff $N-1$ dimensional).

Como se ha comentado anteriormente, en esta tesis se aplican los resultados de regularidad a las soluciones de tres problemas de interés en aplicaciones. Para el problema de jets y el de diseño óptimo se prueba que las soluciones son soluciones distribucionales del problema (0.0.1). Por lo tanto, en estas dos aplicaciones se obtiene regularidad $C^{1,\alpha}$ salvo medida de Hausdorff $N-1$ dimensional nula y, en el caso de dimensión 2, se prueba que no hay singularidades.

En lo que sigue, vamos a describir con detalle los tres problemas estudiados en esta tesis.

1. El problema de minimización y soluciones débiles

En la primera parte de la tesis estudiamos el siguiente problema de minimización. Para Ω un dominio suave en $\mathbb{R}^N$ y φ_0 una función no negativa con $\varphi_0 \in L^\infty(\Omega)$ y $\int_\Omega G(|\nabla\varphi_0|)\,dx < \infty$, consideramos el problema de minimizar el funcional,

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|) + \lambda\chi_{\{u>0\}}\,dx \tag{0.1.3}$$

en la clase de funciones

$$\mathcal{K} = \Big\{ v \in W^{1,G}(\Omega):\ v = \varphi_0 \text{ en } \partial\Omega \Big\}.$$

Esta clase de problemas de minimización fueron estudiados extensivamente para diferentes funciones G. De hecho, el primer trabajo donde este problema fue estudiado es [**4**]. Los autores consideran el caso $G(t) = t^2$ y prueban que estos minimizantes son soluciones débiles del problema de frontera libre,

$$\begin{cases} \Delta u = 0 & \text{en } \Omega\cap\{u>0\} \\ u = 0,\ |\nabla u| = \sqrt{\lambda} & \text{en } \Omega\cap\partial\{u>0\} \end{cases} \tag{0.1.4}$$

y prueban la regularidad Lipschitz de soluciones y la regularidad $C^{1,\alpha}$ de sus fronteras libres ($\Omega\cap\partial\{u>0\}$) localmente alrededor de $\mathcal{H}^{N-1}$–casi todo punto en $\Omega\cap\partial\{u>0\}$.

Los resultados principales en el Capítulo 2 son los siguientes:

Teorema 0.1.5. *Si g satisface* (0.0.2)*, existe un minimizante de $\mathcal{J}$ en $\mathcal{K}$ y cualquier minimizante u es no negativo y pertenece a $C^{0,1}_{loc}(\Omega)$. Además, para cualquier dominio $D \subset\subset \Omega$ conteniendo un punto de la frontera libre, la constante de Lipschitz de u en D está controlada en términos de $N, g_0, \delta, dist(D,\partial\Omega)$ y λ.*

También probamos que $\mathcal{L}\,u = 0$ en el conjunto $\{u > 0\}$ y que $\{u > 0\}$ tiene perímetro localmente finito en Ω. Como es usual, definimos la frontera reducida por $\partial_{red}\{u > 0\} := \{x \in \Omega \cap \partial\{u > 0\} \,/\, |\nu_u(x)| = 1\}$, donde $\nu_u(x)$ es la normal unitaria exterior en el sentido de la medida (ver Capítulo 1), cuando existe, y $\nu_u(x) = 0$ en otro caso. Luego, podemos probar que $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$.

También probamos que los minimizantes tiene un desarrollo asintótico cerca de cualquier punto en la frontera reducida. Es decir,

Teorema 0.1.6. *Sea u un minimizante, entonces para cualquier $x_0 \in \partial_{red}\{u > 0\}$,*

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|) \qquad \text{cuando } x \to x_0 \tag{0.1.7}$$

donde λ^* es tal que $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$. $\big($ Aqui $\langle \cdot, \cdot \rangle$ denota el producto escalar en $I\!R^N$ y $v^- = -\min(v, 0)\big)$.

Por lo tanto, en un sentido débil los minimizantes satisfacen,

$$\begin{cases} \mathcal{L}u = 0 & \text{en} \quad \Omega \cap \{u > 0\}, \\ u = 0, \quad |\nabla u| = \lambda^* & \text{en} \quad \Omega \cap \partial\{u > 0\}. \end{cases} \tag{0.1.8}$$

Estos resultados sugieren que consideremos soluciones débiles del problema (0.1.8). Damos dos definiciones de solución débil (Definición 2.6.1 y Definición 2.6.2). Los minimizantes del funcional $\mathcal{J}$ verifican ambas definiciones de solución débil. La diferencia principal entre estas dos definiciones es que para las funciones que satisfacen la Definición 2.6.1 tenemos que $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$, mientras que para funciones satisfaciendo la Definición 2.6.2 podemos tener que $\partial_{red}\{u > 0\} = \emptyset$. La Definición 2.6.2 es más apropiada para límites de soluciones de problemas de perturbación singular.

Probamos el siguiente teorema,

Teorema 0.1.9. *Supongamos que g satisface* (0.0.2). *Sea u una solución débil. Entonces, $\mathcal{H}^{N-1}$ casi todo punto en la frontera libre reducida $\partial_{red}\{u > 0\}$ tiene un entorno donde la frontera libre es una superficie $C^{1,\alpha}$. Más aún, si u es una solución débil en el sentido de la definición 2.6.1, el resto de la frontera libre tiene medida $\mathcal{H}^{N-1}$ cero, y si u es un minimizante, entonces toda la frontera libre reducida es regular.*

Recalcamos que lo que probamos es que si u es una solución débil, la frontera libre es una superficie $C^{1,\alpha}$ en un entorno de cualquier punto donde u tiene el desarrollo asintótico (0.1.7) para algún vector unitario ν. Probamos que ese es el caso para cualquier punto en la frontera libre reducida cuando u es un minimizante (ver Teorema 2.5.5). Por lo tanto, si u es un minimizante, la frontera libre reducida es una superficie $C^{1,\alpha}$ y el resto de la frontera libre tiene medida $\mathcal{H}^{N-1}$ cero.

También mejoramos el resultado de regularidad para el caso $N = 2$, para una subclase de funciones que satisfacen (0.0.2). Probamos que, en este caso, toda la frontera libre es regular.

Resultados de regularidad de toda la frontera libre en dimensión 2 fueron probados en [**2**], [**5**], y en [**12**] si $2 - \delta \leq p < \infty$ para $\delta > 0$ chico.

Probamos el siguiente,

TEOREMA 0.1.10. *Sea $N = 2$, g satisfaciendo* (0.0.2) *y* (2.7.57) *y $u \in \mathcal{K}$ un minimizante de* (0.1.3). *Entonces $\partial\{u > 0\}$ es una superficie $C^{1,\alpha}$ localmente en Ω.*

2. El problema de optimización de dominio

Estudiamos, como segunda aplicación, un problema de optimización de dominio. Empezaremos primero con algunos observaciones históricas de este problema. En el trabajo [**2**], Aguilera, Alt y Caffarelli estudian un problema de diseño óptimo con restricción en el volumen. Los autores prueban la regularidad de los minimizantes introduciendo un término de penalización en el funcional de energía (la integral de Dirichlet) y minimizando sin la restricción en el volumen. Los pasos que realizan, son los siguientes. Primero, los autores observan que, para valores fijos del parámetro de penalización, el funcional es muy similar al considerado en el trabajo [**4**], por lo tanto los resultados de regularidad de los minimizantes del problema de penalización salen casi sin ninguna modificación como en [**4**]. Finalmente, ellos prueban que para valores chicos del parámetro de penalización, el volumen prefijado es alcanzado. De esta manera, los resultados de regularidad se aplican para soluciones del problema de diseño óptimo.

Este método fue aplicado a otros problemas con similar éxito. En [**3, 16, 19, 23**], donde la ecuación diferencial que satisfacen los minimizantes es no degenerada, uniformemente elíptica y en [**15**], donde la ecuación involucrada podría llegar a ser degenerada o singular, pero todavía tiene la propiedad de ser homogénea.

En el Capítulo 3 probamos que el mismo tipo de resultados se pueden obtener si estudiamos un problema tal que la ecuación diferencial que satisfacen los minimizantes es no lineal, degenerada o singular, y posiblemente no homogénea. O sea, cuando el operador tiene la forma $\mathcal{L}u = \operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)$ y g satisface las condiciones (0.0.2).

A continuación damos más precisamente la descripción del problema que estudiamos,

Sea Ω un dominio suave y acotado en $\mathbb{R}^N$ y $\varphi_0 \in W^{1,G}(\Omega)$, un dato de Dirichlet, con $\varphi_0 \geq c_0 > 0$ en $\bar{A}$, donde A es un subconjunto abierto relativo y no vacio de $\partial\Omega$ tal que $A \cap \partial\Omega$ es C^2. Acá $W^{1,G}(\Omega)$ es un espacio de Sobolev-Orlicz (ver Capítulo 1). Sea

$$\mathcal{K}_\alpha = \{u \in W^{1,G}(\Omega) \,/\, |\{u > 0\}| = \alpha,\, u = \varphi_0 \text{ en } \partial\Omega\}.$$

Nuestro problema es minimizar $\mathcal{J}(u) = \int_\Omega G(|\nabla u|)\,dx$ en $\mathcal{K}_\alpha$, con $g = G'$ satisfaciendo (0.0.2).

Una de las dificultades de estos problemas es probar la regularidad de los minimizantes, ya que no es fácil hacer perturbaciones que preserven el volumen sin saber previamente la regularidad de $\partial\{u > 0\}$.

Para poder resolver el problema original, de manera que nos permita perturbaciones que no preserven el volumen, seguimos las ideas de [**2**] y consideramos el siguiente problema de penalización. Sea

$$\mathcal{K} = \{u \in W^{1,G}(\Omega) \,/\, u = \varphi_0 \ \text{ en } \partial\Omega\}$$

y

$$\mathcal{J}_\varepsilon(u) = \int_\Omega G(|\nabla u|)\,dx + F_\varepsilon(|\{u > 0\}|), \tag{0.2.1}$$

donde

$$F_\varepsilon(s) = \begin{cases} \varepsilon(s-\alpha) & \text{si} s < \alpha \\ \frac{1}{\varepsilon}(s-\alpha) & \text{si } s \geq \alpha. \end{cases}$$

Luego el problema de penalización es el siguiente

$$(P_\varepsilon) \qquad \text{Encontrar } u_\varepsilon \in \mathcal{K} \quad \text{tal que} \quad \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v\in\mathcal{K}} \mathcal{J}_\varepsilon(v).$$

Para poder probar la existencia de los minimizantes usamos los teoremas de inmersión en espacios de Sobolev-Orlicz, y el resultado sale fácilmente por minimización directa. La regularidad de los minimizantes y de sus fronteras libres $\partial\{u_\varepsilon > 0\}$ salen probando que cualquier minimizante u_ε es una solución del siguiente problema,

$$\begin{cases} \mathcal{L}u_\varepsilon = 0 & \text{en } \{u_\varepsilon > 0\} \cap \Omega, \\ u_\varepsilon = 0, \quad \dfrac{\partial u_\varepsilon}{\partial \nu} = \lambda_\varepsilon & \text{en } \partial\{u_\varepsilon > 0\} \cap \Omega, \end{cases}$$

en el sentido de la Definición 2.6.1, donde λ_ε es una constante positiva.

Las propiedades de la definición de solución débil no son difíciles de establecer ya que el problema de minimización estudiado en el Capitulo II es muy similar a (P_ε). La única diferencia es que $\mathcal{J}$ es lineal en $|\{u > 0\}|$ y acá el término F_ε es lineal a trozos y cero en α.

Con este resultado tenemos que para $\mathcal{H}^{N-1}-$ casi todo punto, la frontera libre es una superficie $C^{1,\beta}$ en un entorno (ver Corolario 2.7.56 del Capítulo 2).

También mejoramos el resultado de regularidad para el caso $N = 2$, para una subclase de funciones satisfaciendo (0.0.2). Como en el Capítulo 2, probamos que en este caso, toda la frontera libre es regular. Acá tenemos que lidiar con la no homogeneidad y el término de penalización a la vez. El primer término lo tratamos como en el Capítulo 2, y para tratar el segundo término, tomamos ideas de [**19**].

Como en [**2**], la razón por la cual es tan útil este método es que no es necesario pasar al limite en el término de penalización ε donde serían necesarias estimaciones uniformes en ε. De hecho, probamos que para valores chicos de ε el volumen prefijado es alcanzado. Esto es, $|\{u_\varepsilon > 0\}| = \alpha$ para ε chico. Este es el paso donde la demostración se aparta de trabajos previos en problemas similares, debido a que acá podemos no tener la homogeneidad de la función g (ver Lema 3.2.6).

Finalmente, el hecho de que, para ε chico, cualquier minimizante de $\mathcal{J}_\varepsilon$ satisface $|\{u_\varepsilon > 0\}| = \alpha$ implica que cualquier minimizante de nuestro problema es un minimizante de $\mathcal{J}_\varepsilon$, por lo tanto es localmente Lipschitz y la frontera libre es suave.

3. El problema de perturbación singular

Estudiamos, como última aplicación el siguiente problema de perturbación singular: Para $\varepsilon > 0$, tomamos $u^\varepsilon \geq 0$ una solución de,

$$(P_\varepsilon) \qquad \mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon).$$

Una solución de (P_ε) es una función $u^\varepsilon \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$ tal que

$$(0.3.2) \qquad \int_\Omega g(|\nabla u^\varepsilon|)\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\nabla\varphi\, dx = -\int_\Omega \varphi\beta_\varepsilon(u^\varepsilon)\, dx$$

para toda $\varphi \in C_0^\infty(\Omega)$.

Aquí $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$, para $\beta \in \mathrm{Lip}(\mathbb{R})$, positiva en $(0,1)$ y cero fuera de $[0,1]$. Llamamos M a $\int_0^1 \beta(s)\, ds$.

Estamos interesados en estudiar propiedades uniformes de las soluciones, y ver qué pasa con el problema límite, cuando $\varepsilon \to 0$. La idea es probar que para cualquier sucesión $\varepsilon_n \to 0$ existe una subsucesión ε_{n_k} y una función límite u, tal que $u = \lim u^{\varepsilon_{n_k}}$, y que u es una solución débil del problema de frontera libre (0.1.8) para alguna constante λ^* dependiendo de g y de M.

Los primeros en plantear el paso al límite en este problema de perturbación singular en el caso de evolución fueron Zeldovich y Frank-Kamenetski en 1938, [**24**]. En dicho trabajo, los autores proponen hacer un análisis del límite para energias de activación altas para el estudio de la propagación de llamas. El estudio riguroso matemático recién fue realizado en 1990 por Berestycki, Caffarelli y Nirenberg en el caso de ondas viajeras (ver [**6**]) y posteriormente en [**9**] para el caso general de evolución a una fase.

Más específicamente, en [**6**] los autores consideran una familia uniformemente acotada de soluciones de

$$(0.3.3) \qquad \mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{en } \Omega$$

donde $\mathcal{L}u = \sum a_{ij}u_{x_ix_j} + \sum b_i u_{x_i} + cu$ es un operador lineal uniformemente elíptico con coeficientes regulares y ven qué pasa cuando $\varepsilon \to 0$. Prueban que para toda sucesión $\varepsilon_n \to 0$ existe una subsucesión ε_{n_k} y una función límite u Lipschitz que resuelve el siguiente problema de frontera libre,

$$\begin{cases} \mathcal{L}u = 0 & \text{en } \Omega \cap \{u > 0\}, \\ \sum a_{ij} u_{x_i} \eta_j = \sqrt{2M} & \text{en } \Omega \cap \partial\{u > 0\} \end{cases} \tag{0.3.4}$$

donde η es la normal interior a $\partial\{u > 0\}$.

En dicho trabajo prueban que la condición de frontera libre se satisface en toda porción suave de la misma.

Por otro lado, en los trabajos [**7**] y [**8**] se prueba que las soluciones viscosas de (0.3.4) con $\mathcal{L} = \Delta$ tienen frontera libre $C^{1,\alpha}$ alrededor de puntos donde es "chata" (flat). Esto permitió obtener resultados de regularidad de la frontera libre en el caso $\mathcal{L} = \Delta$ en [**20**].

Más recientemente, el caso elíptico no lineal para el p–laplaciano fue considerado en [**11**]. Los autores estudiaron el problema (P_ε) cuando el operador $\mathcal{L}$ es el p–laplaciano (i.e $g(t) = t^{p-1}$). Como en el caso uniformemente elíptico, para una familia uniformemente acotada de soluciones u^ε encontraron estimaciones Lipschitz uniformes en ε y probaron que el límite de u^ε es una solución viscosa de (0.0.1) para $\mathcal{L} = \Delta_p$ y $\lambda^* = \left(\frac{p}{p-1}M\right)^{1/p}$.

En dicho trabajo no se obtienen resultados sobre la regularidad de la frontera libre ya que no hay ninguna teoría de regularidad para soluciones viscosas en el caso degenerado o singular.

En este trabajo, para nuestro problema P_ε, podemos probar primero la continuidad Lipschitz uniforme a saber,

Teorema 0.3.5. *Supongamos que g satisface* (0.0.2). *Sea u^ε una solución de*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{en } \Omega,$$

con $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq L$. Entonces, para $\Omega' \subset\subset \Omega$ se tiene que

$$|\nabla u^\varepsilon(x)| \leq C \quad \text{en } \Omega'$$

con $C = C(N, \delta, g_0, L, \|\beta\|_\infty, g(1), dist(\Omega', \partial\Omega))$, si $\varepsilon \leq \varepsilon_0(\Omega, \Omega')$.

Con este resultado, tenemos que, via una subsucesión, existe una función límite u.

El siguiente paso, es probar que la función u es una solución débil en el sentido de la definición 2.6.2 del Capítulo II del problema de frontera libre (0.0.1) para una constante λ^* dependiendo de g y M. Para ello, tenemos que probar que tenemos un desarrollo asintótico de u en cada punto de la frontera reducida.

Aquí encontramos diversas dificultades técnicas asociadas con la falta de homogeneidad del operador $\mathcal{L}$ y con el hecho de estar trabajando en espacios de Orlicz. Por ejemplo, para probar que $\mathcal{L}u = 0$ en $\{u > 0\}$ debemos probar que $\nabla u_\varepsilon \to \nabla u$ en casi todo punto. Esto lo logramos probando que se tiene convergencia en L^{g_0+1}.

En otro punto nos encontramos con la necesidad de agregar la siguiente hipótesis sobre g; existen $-1 < \alpha_1 \leq \alpha_2$ tales que para toda $s, t \geq 0$ tenemos que,

$$g'(ts) \geq \text{mín}\{s^{\alpha_1}, s^{\alpha_2}\} g'(t). \tag{0.3.6}$$

Todos los resultados probados en esta sección valdrán por lo tanto, cuando además, g' satisface la condición (0.3.6).

Finalmente probamos,

TEOREMA 0.3.7. *Sea u^{ε_j} una solución de (P_{ε_j}) en un dominio $\Omega \subset \mathbb{R}^N$ tal que $u^{\varepsilon_j} \to u$ uniformemente en compactos de Ω y $\varepsilon_j \to 0$. Sea $x_0 \in \Omega \cap \partial\{u > 0\}$ tal que $\partial\{u > 0\}$ tiene una normal interior η en sentido de la medida en x_0, y supongamos que u es no degenerada en x_0. Bajo estas hipótesis, tenemos que*

$$u(x) = \Phi^{-1}(M)\langle x - x_0, \eta\rangle^+ + o(|x - x_0|).$$

Finalmente, podemos aplicar la teoría del Capítulo 2. Tenemos que u es una solución débil en el sentido de la definición 2.6.2 del problema de frontera libre. Tenemos el siguiente,

TEOREMA 0.3.8. *Supongamos que g satisface* (0.0.2) *y* (0.3.6). *Sea u^{ε_j} una solución de (P_{ε_j}) en un dominio $\Omega \subset \mathbb{R}^N$ tal que $u^{\varepsilon_j} \to u$ uniformemente en compactos de Ω cuando $\varepsilon_j \to 0$. Sea $x_0 \in \Omega \cap \partial\{u > 0\}$, tal que tiene una normal interior η en el sentido de la medida en x_0. Supongamos que u es uniformemente no degenerada en la frontera libre en un entorno de x_0 (ver Definición 4.3.1). Entonces, existe $r > 0$ tal que $B_r(x_0) \cap \partial\{u > 0\}$ es una superficie $C^{1,\alpha}$.*

4. Notación

A lo largo de la tesis N denota la dimensión y,

$$\begin{aligned} B_r(x) &= \{x \in \mathbb{R}^N, |x - x_0| < r\}, \\ B_r^+(x) &= \{x \in \mathbb{R}^N, x_N > 0,\ |x - x_0| < r\}, \\ B_r^-(x) &= \{x \in \mathbb{R}^N, x_N < 0,\ |x - x_0| < r\}. \end{aligned}$$

Para $v, w \in \mathbb{R}^N$, $\langle v, w\rangle$ denota el producto escalar standard.

Para una función escalar f, $f^+ = \text{máx}(f, 0)$ y $f^- = \text{máx}(-f, 0)$.

Para la función G definida en el Capítulo 1, denotamos,

$$\begin{aligned} g(t) &= G'(t), \\ F(t) &= g(t)/t, \\ \Phi(t) &= g(t)t - G(t), \\ A(p) &= F(|p|)p \qquad \text{para } p \in \mathbb{R}^N, \\ a_{ij} &= \frac{\partial A_i}{\partial p_j} \qquad \text{para } 1 \leq i, j \leq N. \end{aligned}$$

5. Estructura de la tesis

Estructura del Capítulo 1. El Capítulo 1 está organizado de la siguiente manera: En la Sección 1 damos algunas propiedades de la función g y definimos los espacios de Orlicz y los de Sobolev-Orlicz, y probamos algunos teoremas de inclusión. Estos espacios serán usados para probar la existencia de los minimizantes.

En la Sección 2 damos algunas propiedades analíticas de las funciones con $\int_\Omega G(|\nabla u|)\,dx$ finita (generalización de la desigualdad de Poincaré y el Teorema de Morrey). Luego, enunciamos algunas propiedades de las soluciones, subsoluciones y supersoluciones de $\mathcal{L}v = 0$ (desigualdad de Harnack, cotas $C^{1,\alpha}$, principio de comparación, principio fuerte del máximo y una desigualdad importante que usaremos a lo largo de la tesis (Teorema 1.2.38)). También probamos una desigualdad de tipo Cacciopoli válida para estas funciones (Lema 1.2.12). Finalmente, mostramos una familia explícita de subsoluciones de $\mathcal{L}u = 0$ (Lema 1.2.47) que usamos como barreras en varios puntos de esta tesis.

En la Sección 3 enunciamos la definición de medida y distancia de Hausdorff.

En la Sección 4 damos un Teorema de Representación que usaremos en esta tesis. También damos la definición de conjuntos de perímetro localmente finito y algunas de sus propiedades.

En la Sección 5 probamos algunos resultados de $\mathcal{L}-$ soluciones con crecimiento lineal.

En la Sección 6 probamos algunos resultados de los límites de sucesiones de blow up.

En la Sección 7 damos algunos resultados sobre simetrización de Schwartz.

Estructura del Capítulo 2. En la Sección 1 probamos la existencias de minimizantes y que los mismos son subsoluciones de $\mathcal{L}v = 0$. También probamos que los minimizantes son no negativos. La demostración de existencia de minimizantes, que es standard en su forma, hace uso fuertemente de los espacios de Orlicz y la segunda desigualdad en la condición (0.0.2).

En la Sección 2 probamos que cualquier minimizante u es Hölder continuo (Teorema 2.2.1), $\mathcal{L}u = 0$ en $\{u > 0\}$ (Lema 2.2.12) y finalmente probamos la continuidad Lipschitz local (Teorema 2.2.25). La demostración de la continuidad Hölder de los minimizantes es un paso crucial en nuestro análisis y es una de las demostraciones principales de este capítulo en donde juegan todas las propiedades de la función G a través de la desigualdad del Teorema 1.2.38.

En la Sección 3 probamos que los minimizantes satisfacen una propiedad de no degeneración cerca de la frontera libre $\Omega \cap \partial\{u > 0\}$ (Teorema 2.3.5). También probamos que los conjuntos $\{u > 0\}$ y $\{u = 0\}$ tienen la propiedad de densidad uniforme positiva en la frontera libre. En este teorema usamos fuertemente las propiedades de la función G y los correspondientes espacios de Orlicz.

En la Sección 4 probamos que la frontera libre tiene dimensión Hausdorff $N-1$ finita y obtenemos un teorema de representación para los minimizantes (Teorema 2.4.5). Esto implica que $\{u>0\}$ tiene perímetro localmente finito en Ω. Finalmente probamos que $\mathcal{H}^{N-1}(\partial\{u>0\}\setminus\partial_{\rm red}\{u>0\})=0$.

En la Sección 5 damos algunas propiedades de las sucesiones de blow up de los minimizantes. Probamos que cualquier límite de una sucesión de blow up es otra vez un minimizante (Lema 2.5.1) y finalmente probamos el desarrollo asintótico de los minimizantes en cualquier punto de la frontera libre reducida (Teorema 2.5.5).

En la Sección 6 damos la definición de solución débil (Definición 2.6.1 y Definición 2.6.2). Probamos que casi todas los propiedades que probamos para los minimizantes también valen para soluciones débiles que corresponden a la Definición 2.6.1, y mencionamos las diferencias entre las dos definiciones (Observación 2.6.8 y Observación 2.6.21).

En la Sección 7 probamos la regularidad de la frontera libre de las soluciones débiles, cerca de puntos "flat" de la frontera (Teorema 2.7.54) y luego deducimos la regularidad de la frontera libre de soluciones débiles cerca de casi todo punto en la frontera reducida. En el caso de los minimizantes, se obtiene la regularidad de toda la frontera reducida (Teorema 2.7.56). También probamos, para cierta clase de funciones satisfaciendo (0.0.2) que, en el caso $N=2$ toda la frontera libre es regular (Corolario 2.7.66).

Estructura del Capítulo 3. El Capítulo 3 esta organizado de la siguiente manera: En la Sección 1 empezamos nuestro análisis del problema (P_ε) para ε fijo. Primero probamos la existencia de minimizantes, la regularidad Lipschitz local y la no degeneración cerca de la frontera libre (Teorema 3.1.2) y probamos que los minimizantes son soluciones débiles del problema de frontera libre, como fue definitdo en el Capítulo 2 (Observación 3.1.14). Luego tenemos que, para $\mathcal{H}^{N-1}-$ casi todo punto, la frontera libre es localmente una superficie $C^{1,\beta}$ (Corolario 3.1.15). Probamos que para el caso $N=2$, para una subclase de funciones satisfaciendo (0.0.2) toda la frontera libre es regular (Corolario 3.1.22). En la Sección 2 probamos que para valores chicos de ε recuperamos nuestro problema original.

Estructura del Capítulo 4. El Capítulo 4 está organizado de la siguiente manera: En la Sección 1 probamos la continuidad Lipschitz uniforme de las soluciones de (P_ε) (Corolario 4.1.8).

En la Sección 2, probamos que si u es una función límite, entonces $\mathcal{L}u$ es una medida de Radon soportada en la frontera libre (Teorema 4.2.1). Luego, probamos la Proposición 4.2.18, que dice que si u es un semiplano, entonces la pendiente es 0 o $\Phi^{-1}(M)$ donde $\Phi(t)=tg(t)-G(t)$, y la Proposición 4.2.20 que dice que si u es una suma de dos semiplanos, entonces las pendientes tienen que ser iguales y a lo sumo $\Phi^{-1}(M)$.

En la sección 3 probamos el desarrollo asintótico de u (Teorema 4.3.3).

En la sección 4 aplicamos los resultados del Capítulo 2 para probar la regularidad de la frontera libre (Teorema 4.4.7).

Introduction

In this thesis, we study a free boundary problem: For a domain Ω in $\mathbb{R}^N$, find u such that

$$\begin{cases} \mathcal{L}u := \operatorname{div}\Big(\dfrac{g(|\nabla u|)}{|\nabla u|}\nabla u\Big) = 0 & \text{in } \{u > 0\} \cap \Omega \\ |\nabla u| = \lambda^* & \text{on } \partial\{u > 0\} \cap \Omega. \end{cases} \tag{0.0.1}$$

It is called a Free Boundary Problem because the location of $\partial\{u > 0\}$ is not known a priori. The second equation in (0.0.1) is known as the "free boundary condition".

This problem appears in many applications. In this work we will discuss three of them. First, the problem of jets which consist on minimizing,

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|) + \lambda\chi_{\{u>0\}}\, dx$$

in the class of functions

$$\mathcal{K} = \Big\{v \in W^{1,G}(\Omega) : \quad v = \varphi_0 \text{ on } \partial\Omega\Big\},$$

where φ_0 is a nonnegative function with $\varphi_0 \in L^\infty(\Omega)$, $\int_\Omega G(|\nabla\varphi_0|)\, dx < \infty$ and G is such that $g = G'$. $W^{1,G}(\Omega)$ is a Sobolev-Orlicz space (see Chapter 1).

The second one, is an optimal design problem. To minimize

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the set

$$\mathcal{K}_\alpha = \{v \in W^{1,G}(\Omega) : \ |\{v > 0\}| = \alpha,\, v = \varphi_0 \ \text{ on } \partial\Omega\},$$

for a bounded nonnegative and not identically zero φ_0 with $\int_\Omega G(|\nabla\varphi_0|)\, dx < \infty$.

Finally, the third problem is originated in the theory of combustion in the so-called "limit for activation energy going to infinity" and consist on the following: For $\varepsilon > 0$, take u^ε a weak solution of,

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon), \quad u^\varepsilon \geq 0 \tag{P_ε}$$

with $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$, $\beta \in \operatorname{Lip}(\mathbb{R})$, positive in $(0, 1)$, and zero outside $[0, 1]$

In this case, we are interested in the study of the uniform properties of the solutions and the study of the limit problem, when $\varepsilon \to 0$.

In this case we prove, under some adequate conditions, that for any sequence $\varepsilon_n \to 0$ there exists a subsequence ε_{n_k} and a limit function u, such that $u = \lim u^{\varepsilon_{n_k}}$, and u is a solution of the free boundary problem (0.0.1) for some constant λ^* depending on g and M.

In all these cases we see that the solutions of the problem we are interested in turn out to be solutions of (0.0.1) in a weak sense. Therefore, it is of a grate interest to study the regularity of the weak solutions (0.0.1) and of their free boundaries.

In this sense, the first step is to give a good definition of weak solution of the problem (0.0.1) that can be applied to all the applications that we have in mind. The second step, is to see which is the optimal regularity that these solutions are going to have (observe that, if we want the function to have constant normal derivative on the boundary of $\{u > 0\}$, the optimal regularity can not be more than Lipschitz). Finally, it would be desirable to obtain the $C^{1,\alpha}$ regularity of the free boundary since this will imply that the weak solution satisfies the free boundary condition in a classical sense.

All these topics are well studied when the operator $\mathcal{L}$ is the laplacian (see for example, [**2, 4, 5, 6, 7, 8, 20**]). In particular in this case several theories were developed for the study of the regularity of the weak solutions of (0.0.1). In these works it has been proved that the weak solutions u are locally Lipschitz and that their free boundaries $\partial\{u > 0\}$ are $C^{1,\alpha}$ surfaces when $N = 2$, and that they have this regularity in a neighborhood of any point where they are flat, in higher dimensions. These results have been obtained both for distributional solutions and for viscosity solutions. In the first case, the free boundary condition appears in an integral form, and the definition is more appropriate for the problem of jets and of optimization. The concept of viscosity solution was used in two phases problems and for the combustion problem.

These results have been extended to quasilinear operators or fully nonlinear operators independent of the spatial variable, and to linear operators with variable coefficients. In all these cases the operator $\mathcal{L}$ is supposed to be uniformly elliptic.

Recently, some of the results were proved also in the case where $\mathcal{L}$ is the p–laplacian. That is, $\mathcal{L}u = \Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ which is a degenerate elliptic operator when $p > 2$ and singular when $p < 2$ (in the case $p = 2$ it coincides with the laplacian) (see, for example, [**10, 11**]). In [**10**] the authors study the problem of jets and prove the regularity results that were mentioned above for the case of the usual laplacian. In [**11**] the authors study the combustion problem (in the limit for activation energy going to infinity) and prove that, in the limit, they get a viscosity solution of the problem (0.0.1). There are no regularity results of the free boundary for viscosity solutions in the degenerate or singular cases.

The aim of this thesis is the study of this problem –including the regularity of the free boundary– for operators that can be either degenerate or singular elliptic, possibly non homogeneous (the p–laplacian is homogeneous and this simplifies some of the proofs). Here we admit, moreover, functions g in the operator $\mathcal{L}$ with a different behavior at 0 and at infinity. Classically, the assumptions on the behavior of g at 0 and at infinity were always similar to the case of the p–laplacian. Here, instead, we adopt the conditions introduced by G. Lieberman in [**22**] for the study of the regularity of weak solutions of the elliptic equation (possibly degenerate or singular) $\mathcal{L}u = f$ with f bounded.

These conditions ensures that the equation $\mathcal{L}u = 0$ is equivalent to a uniformly elliptic equation in nondivergence form with ellipticity constants independent of the solution u in sets where $\nabla u \neq 0$. Furthermore, these conditions do not imply any type of homogeneity of the function g, and moreover they allow for a different behavior of the function g when $|\nabla u|$ is near zero or infinity. That is, we assume that g satisfies

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0 \qquad \forall t > 0 \tag{0.0.2}$$

for certain constants $0 < \delta \leq g_0$.

Let us observe that $\delta = g_0 = p - 1$ when $G(t) = t^p$, and reciprocally, if $\delta = g_0$ then G is a power.

Another example of a function g which satisfies (0.0.2) is the function $g(t) = t^a \log(bt + c)$ with $a, b, c > 0$. In this case (0.0.2) is satisfied with $\delta = a$ and $g_0 = a + 1$.

Another interesting case is the one of a function $G \in C^2([0,\infty))$ with $G'(t) = g(t) = c_1 t^{a_1}$ for $t \leq s$, $g(t) = c_2 t^{a_2} + d$ for $t \geq s$. In this case g satisfies (0.0.2) with $\delta = \min(a_1, a_2)$ and $g_0 = \max(a_1, a_2)$.

Furthermore, any linear combination with positive coefficients of functions satisfying (0.0.2) also satisfies (0.0.2). On the other hand, if g_1 and g_2 satisfy (0.0.2) with constants δ^i and g_0^i, $i = 1, 2$, the function $g = g_1 g_2$ satisfies (0.0.2) with $\delta = \delta^1 + \delta^2$ and $g_0 = g_0^1 + g_0^2$, and the function $g(t) = g_1\big(g_2(t)\big)$ satisfies (0.0.2) with $\delta = \delta^1\delta^2$ and $g_0 = g_0^1 g_0^2$.

This observation shows that there is a wide range of functions g under the hypotheses of this thesis.

With regards to the notion of weak solution considered in this work, we emphasize that, although the notion that we can call "distributional" would give, in principle, a better result, this is not the suitable notion for the singular perturbation problem that we study in the last chapter. Therefore, in this thesis we introduce a new notion of weak solution of (0.0.1) where the free boundary condition holds in a pointwise sense and not in integral one as in the case of weak distributional solutions. Anyhow, for these weak solutions the proof of the regularity of the free boundary in a neighborhood of every point where it is flat, is similar to the one that can be fond for distributional solutions in the case $\mathcal{L} = \Delta$ in [**4**].

On the other hand, we prove that the distributional solutions are weak solutions in this pointwise sense. Therefore, the results can also by applied to the distributional solutions. In this last case, the results turn out to be stronger since, as in the case of uniformly elliptic equations, it is proved in this thesis that the free boundary of distributional solution is flat at almost every point (respect to the $N-1$ dimensional Hausdorff measure).

As was mentioned before, in this thesis we apply the regularity results to solutions of the three problems of interest in applications. For the problem of jets and of optimal design we prove that the solutions are distributional solutions of the problem (0.0.1). Therefore, in these two applications we obtain the $C^{1,\alpha}$ regularity up to zero $N-1$ Hausdorff measure. In the case of dimension 2, we prove that there are no singularities.

In what follows, we will describe in details the three problems studied in this thesis.

1. The minimization problem and weak solutions

In the first part of the thesis we study the following minimization problem: For Ω a smooth bounded domain in $\mathbb{R}^N$ and φ_0 a nonnegative function with $\varphi_0 \in L^\infty(\Omega)$ and $\int_\Omega G(|\nabla\varphi_0|)\,dx < \infty$, we consider the problem of minimizing the functional,

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|) + \lambda\chi_{\{u>0\}}\,dx \tag{0.1.3}$$

in the class of functions

$$\mathcal{K} = \Big\{ v \in W^{1,G}(\Omega) : \ v = \varphi_0 \text{ on } \partial\Omega \Big\}.$$

This minimization problem has been widely studied for different functions G. In fact, the first paper in which this problem was studied is [**4**]. The authors considered the case $G(t) = t^2$. They proved that minimizers are weak solutions to the free boundary problem,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap \{u>0\} \\ u = 0, \ |\nabla u| = \sqrt{\lambda} & \text{on } \Omega \cap \partial\{u>0\} \end{cases} \tag{0.1.4}$$

and proved the Lipschitz regularity of the solutions and the $C^{1,\alpha}$ regularity of their free boundaries ($\Omega \cap \partial\{u>0\}$) locally around $\mathcal{H}^{N-1}$ almost every point on $\Omega \cap \partial\{u>0\}$.

The main results in Chapter 2 are the following:

THEOREM 0.1.5. *If g satisfies* (0.0.2)*, there exists a minimizer of $\mathcal{J}$ in $\mathcal{K}$ and any minimizer u is nonnegative and belongs to $C^{0,1}_{loc}(\Omega)$. Moreover, for any domain $D \subset\subset \Omega$ containing a free boundary point, the Lipschitz constant of u in D is controlled in terms of $N, g_0, \delta, dist(D, \partial\Omega)$ and λ.*

We also prove that $\mathcal{L}u = 0$ in the set $\{u > 0\}$ and that $\{u > 0\}$ has finite perimeter locally in Ω. As usual, we define the reduced boundary by $\partial_{red}\{u > 0\} := \{x \in \Omega \cap \partial\{u > 0\} \,/\, |\nu_u(x)| = 1\}$, where $\nu_u(x)$ is the unit outer normal in the measure theoretic sense (see Chapter 1), when it exists, and $\nu_u(x) = 0$ otherwise. Then, we prove that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$.

We also prove that minimizers have an asymptotic development near any point in their reduced free boundary. Namely,

THEOREM 0.1.6. *Let u be a minimizer, then for every $x_0 \in \partial_{red}\{u > 0\}$,*

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|) \quad \text{as } x \to x_0 \tag{0.1.7}$$

where λ^* is such that $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$. (Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $I\!R^N$ and $v^- = -\min(v, 0)$).

So that, in a weak sense minimizers satisfy,

$$\begin{cases} \mathcal{L}u = 0 & \text{in} \quad \{u > 0\}, \\ u = 0, \quad |\nabla u| = \lambda^* & \text{on} \quad \Omega \cap \partial\{u > 0\}. \end{cases} \tag{0.1.8}$$

These results suggest that we consider weak solutions of the problem (0.1.8). We give two different definitions of weak solution (Definition 2.6.1 and Definition 2.6.2). Minimizers of the functional $\mathcal{J}$ verify both definitions of weak solution. The main difference between these two definitions is that for functions satisfying Definition 2.6.1 we have that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) = 0$, whereas for functions satisfying Definition 2.6.2 we may have $\partial_{red}\{u > 0\} = \emptyset$. Definition 2.6.2 is more suitable for limits of singular perturbation problems.

We prove the following theorem,

THEOREM 0.1.9. *Suppose that g satisfies* (0.0.2). *Let u be a weak solution. Then, $\mathcal{H}^{N-1}$ almost every point in the reduced free boundary $\partial_{red}\{u > 0\}$ has a neighborhood where the free boundary is a $C^{1,\alpha}$ surface. Moreover, if u is a weak solution according to Definition 2.6.1, the remainder of the free boundary has $\mathcal{H}^{N-1}-$ measure zero, and if u is a minimizer, then the whole reduced free boundary is regular.*

We point out that we prove that if u is a weak solution, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of every point where u has the asymptotic development (0.1.7) for some unit vector ν. We prove that this is the case for every point in the reduced free boundary when u is a minimizer (see Theorem 2.5.5).

We also improve the regularity result for the case $N = 2$, for a subclass of functions satisfying (0.0.2). We prove that, in this case, the whole free boundary is regular.

Full regularity of the free boundary in dimension 2 was prove in [**4**] and [**5**], and in [**12**] if $2-\delta \leq p < \infty$ for a small $\delta > 0$.

We prove the following,

THEOREM 0.1.10. *Let* $N = 2$, *g satisfying conditions* (0.0.2) *and* (2.7.57) *and* $u \in \mathcal{K}$ *be a minimizer of* (2.0.1). *Then* $\partial\{u > 0\}$ *is a* $C^{1,\alpha}$ *surface locally in* Ω.

2. The shape optimization problem

We study, as a second application, an optimization problem. We begin with a few historical remarks on this problem. In the paper [**2**], Aguilera, Alt and Caffarelli study an optimal design problem with a volume constrain. The authors prove the regularity of minimizers by introducing a penalization term in the energy functional (the Dirichlet integral) and minimizing without the volume constrain. The steps that they follow are the following. First, the authors observe that, for fixed values of the penalization parameter, the penalized functional is very similar to the one considered in the paper [**4**], then the regularity results for minimizers of the penalized problem follow almost without change as in [**4**]. Finally, they prove that for small values of the penalization parameter, the constrained volume is attained. In this way, all the regularity results apply to the solution of the optimal design problem.

This method has been applied to other problems with similar success. See for instance, [**3, 16, 19, 23**], where the differential equation satisfied by the minimizers is nondegenerate, uniformly elliptic and [**15**], where the equation involved may be degenerate or singular elliptic, but it still has the property of being homogeneous.

In Chapter 3 we show that the same kind of results can be obtained if we study a problem such that the differential equation satisfied by the minimizers is nonlinear degenerate or singular, and possibly non homogeneous. More precisely, the operator here has the form $\mathcal{L}u = \operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)$ where g satisfies the conditions (0.0.2).

We give now, more precisely the description of the problem that we study.

Take Ω a smooth bounded domain in $\mathbb{R}^N$ and $\varphi_0 \in W^{1,G}(\Omega)$, a Dirichlet datum, with $\varphi_0 \geq c_0 > 0$ in $\bar{A}$, where A is a nonempty relatively open subset of $\partial\Omega$ such that $A \cap \partial\Omega$ is C^2. Here $W^{1,G}(\Omega)$ is a Sobolev-Orlicz space (see Chapter 1). Let

$$\mathcal{K}_\alpha = \{u \in W^{1,G}(\Omega) \,/\, |\{u > 0\}| = \alpha,\, u = \varphi_0 \text{ on } \partial\Omega\}.$$

Our problem is to minimize in $\mathcal{K}_\alpha$, the functional $\mathcal{J}(u) = \int_\Omega G(|\nabla u|)\, dx$, with $g = G'$ satisfying (0.0.2).

One of the difficulties of these kind of problems is to get regularity results for the minimizers, since it is hard to make enough volume preserving perturbations without the previous knowledge of the regularity of $\partial\{u > 0\}$.

In order to solve our original problem, in a way that allows us to perform non volume preserving perturbations, we follow the idea of [**2**] and consider instead the following penalized problem: We let

$$\mathcal{K} = \{u \in W^{1,G}(\Omega) \,/\, u = \varphi_0 \ \text{ on } \partial\Omega\}$$

and

$$\mathcal{J}_\varepsilon(u) = \int_\Omega G(|\nabla u|)\,dx + F_\varepsilon(|\{u > 0\}|), \tag{0.2.11}$$

where

$$F_\varepsilon(s) = \begin{cases} \varepsilon(s - \alpha) & \text{if } s < \alpha \\ \frac{1}{\varepsilon}(s - \alpha) & \text{if } s \geq \alpha. \end{cases}$$

Then, the penalized problem is

$$\text{Find } u_\varepsilon \in \mathcal{K} \quad \text{such that} \quad \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v \in \mathcal{K}} \mathcal{J}_\varepsilon(v). \tag{P_ε}$$

In order to prove the existence of minimizers we use some compact immersion theorems in Sobolev-Orlicz spaces, and the result follows easily by direct minimization. The regularity of the minimizers and of their free boundaries $\partial\{u_\varepsilon > 0\}$ follows by showing that any minimizer u_ε is a solution of the following free boundary problem,

$$\begin{cases} \mathcal{L}u_\varepsilon = 0 & \text{in } \{u_\varepsilon > 0\} \cap \Omega, \\ u_\varepsilon = 0, \quad \dfrac{\partial u_\varepsilon}{\partial \nu} = \lambda_\varepsilon & \text{on } \partial\{u_\varepsilon > 0\} \cap \Omega, \end{cases}$$

in the sense of Definition 2.6.1, where λ_ε is a positive constant.

The properties of the definition of weak solution are not difficult to establish since the minimization problem studied in Chapter 2 is very similar to (P_ε). The only difference is that $\mathcal{J}$ is linear in $|\{u > 0\}|$ and here the term F_ε is piecewise linear and zero at α.

With these results we have that the free boundary is a $C^{1,\beta}$ surface in a neighborhood of $\mathcal{H}^{N-1}-$ almost every point (see Corollary 2.7.56 in Chapter 2).

We also improve the regularity result in the case $N = 2$, for a subclass of functions satisfying (0.0.2). As in Chapter 2, we prove that, in this case, the whole free boundary is regular. Here we have to deal with the non homogeneity and the penalization term at the same time. In the first case we proceed as in Chapter 2, and in order to deal with the penalization term, we take ideas from [**19**].

As in [**2**], the reason why this penalization method is so useful is that there is no need to pass to the limit in the penalization parameter ε for which uniform, in ε, regularity estimates would be needed. In fact, we show that for small values of ε the right volume is already attained. This is, $|\{u_\varepsilon > 0\}| = \alpha$ for ε small. This is the step where the proof parts from previous work on similar problems, since here we may not have the homogeneity of the function g (see Lemma 3.2.6).

Finally, the fact that, for small ε, any minimizer of $\mathcal{J}_\varepsilon$ satisfies $|\{u_\varepsilon > 0\}| = \alpha$ implies that any minimizer of our original optimization problem is also a minimizer of $\mathcal{J}_\varepsilon$, so that it is locally Lipschitz continuous with smooth free boundary.

3. The singular perturbation problem

We study, as a third application the following singular perturbation problem:

For any $\varepsilon > 0$, take u^ε a solution of,

$$(P_\varepsilon) \qquad \mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon), \quad u^\varepsilon \geq 0.$$

A solution to (P_ε) is a function $u^\varepsilon \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega g(|\nabla u^\varepsilon|)\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\nabla\varphi\,dx = -\int_\Omega \varphi\beta_\varepsilon(u^\varepsilon)\,dx \tag{0.3.12}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Here $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$, for $\beta \in \mathrm{Lip}(\mathbb{R})$, positive in $(0,1)$ and zero outside $[0,1]$. We call $M = \int_0^1 \beta(s)\,ds$.

We are interested in studying the uniform properties of solutions, and limit problem, as $\varepsilon \to 0$. The idea is to prove that, for every sequence $\varepsilon_n \to 0$, there exists a subsequence ε_{n_k} and a limit function u such that $u = \lim u^{\varepsilon_{n_k}}$ and u is a weak solution of the free boundary problem (0.1.8) for some constant λ^* depending on g and M.

The idea of passing to the limit in this singular perturbation problems in the evolution case was first proposed by Zeldovich and Frank-Kamenetski in 1938, [**24**]. However, a rigorous mathematical analysis of the limiting process was not performed until 1990, when Berestycki, Caffarelli and Nirenberg studied the case of travelling waves (see [**6**]). Next, in [**9**] the general evolution problem in the one phase case was analyzed.

More precisely, in [**6**] the authors consider a uniformly bounded family of solutions of

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } \Omega \tag{0.3.13}$$

where $\mathcal{L}u = \sum a_{ij}u_{x_ix_j} + \sum b_i u_{x_i} + cu$ is a linear uniformly elliptic operator with smooth coefficients. The authors study the limit as $\varepsilon \to 0$. They prove that for any sequence $\varepsilon_n \to 0$ there exists a subsequence ε_{n_k} and a limit function u which is Lipschitz and a solution to the following free boundary problem,

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \cap \{u > 0\}, \\ \sum a_{ij}u_{x_i}\eta_j = \sqrt{2M} & \text{on } \Omega \cap \partial\{u > 0\} \end{cases} \tag{0.3.14}$$

where η is the inward normal to $\partial\{u > 0\}$.

In that work the authors prove that the free boundary condition it is satisfied on any portion of the free boundary where it is smooth.

On the other hand, in the papers [**7**] and [**8**] Caffarelli proved that the viscosity solutions of (0.3.14) with $\mathcal{L} = \Delta$ have $C^{1,\alpha}$ free boundaries around points where they are flat. By using this theory, regularity results of the free boundary for the limit $u = \lim u^\varepsilon$ were obtained in the case $\mathcal{L} = \Delta$ in [**20**].

More recently, the nonlinear elliptic case of the p–laplacian was considered in [**11**]. The authors study the problem (P_ε) when the operator $\mathcal{L}$ is the p–laplacian (i.e $g(t) = t^{p-1}$). As in the uniformly elliptic case, they find for a uniformly bounded family of solutions u^ε, uniform in ε Lipschitz estimates and proved that the limit of u^ε is a viscosity solution of (0.0.1) for $\mathcal{L} = \Delta_p$ and $\lambda^* = \left(\frac{p}{p-1}M\right)^{1/p}$.

In that work the authors do not obtain any regularity of the free boundary since there is no regularity theory for viscosity solutions in the degenerate or singular case.

In this chapter, for our problem P_ε, we first prove the uniform Lipschitz continuity,

Theorem 0.3.15. *Suppose that g satisfies condition* (0.0.2). *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \qquad in\ \Omega,$$

with $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq L$. Then, for $\Omega' \subset\subset \Omega$ we have,

$$|\nabla u^\varepsilon(x)| \leq C \qquad in\ \Omega'$$

with $C = C(N, \delta, g_0, L, \|\beta\|_\infty, g(1), dist(\Omega', \partial\Omega))$, if $\varepsilon \leq \varepsilon_0(\Omega, \Omega')$.

And with this estimate we have, via a subsequence that there exists a limit function u.

The next step, is to prove that the function u is a weak solution in the sense of definition 2.6.2 of Chapter 2 of the free boundary problem (0.0.1) for a constant λ^* depending on g and M. To this end, we prove that that we have an asymptotic development of u at any point in the reduced free boundary.

At this point we find several technical difficulties associated to the loss of homogeneity of the operator $\mathcal{L}$ and of the fact that we are working with the Orlicz spaces. For example, in order to prove that $\mathcal{L}u = 0$ in $\{u > 0\}$ we have to prove that $\nabla u_\varepsilon \to \nabla u$ in almost every point. This is obtained by proving the convergence in L^{g_0+1}.

At another point, we need to add the following hypothesis over g: There exist $-1 < \alpha_1 \leq \alpha_2$ such that for all $s, t \geq 0$ we have that,

$$g'(ts) \geq \min\{s^{\alpha_1}, s^{\alpha_2}\} g'(t). \tag{0.3.16}$$

Thus, all the results proved in in this chapter hold when moreover, g' satisfies condition (0.3.16).

Finally we prove,

Theorem 0.3.17. *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has an inward unit normal η at x_0 in the measure theoretic* *sense, and suppose that u is non-degenerate at x_0. Under these assumptions, we have*

$$u(x) = \Phi^{-1}(M)\langle x - x_0, \eta\rangle)^+ + o(|x - x_0|).$$

Finally, we can apply the theory developed in Chapter 2. We have that, u is a weak solution in the sense of Definition 2.6.2 of the free boundary problem.

Thus, we have the following,

THEOREM 0.3.18. *Suppose that g satisfies* (0.0.2) *and* (0.3.16). *Let u^{ε_j} be a solution of (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω as $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$, such that there is an inward unit normal η in the measure theoretic sense at x_0. Suppose that u is uniformly non-degenerate at the free boundary in a neighborhood of x_0 (see Definition 4.3.1). Then, there exists $r > 0$ such that $B_r(x_0) \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface.*

4. Notation

Throughout the thesis N will note the dimension and,

$$\begin{aligned} B_r(x) &= \{x \in \mathbb{R}^N, |x - x_0| < r\}, \\ B_r^+(x) &= \{x \in \mathbb{R}^N, x_N > 0,\ |x - x_0| < r\}, \\ B_r^-(x) &= \{x \in \mathbb{R}^N, x_N < 0,\ |x - x_0| < r\}. \end{aligned}$$

For $v, w \in \mathbb{R}^N$, $\langle v, w\rangle$ notes the standard scalar product.

For a scalar function f, $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

For the function G defined in Chapter 1, we denote,

$$\begin{aligned} g(t) &= G'(t), \\ F(t) &= g(t)/t, \\ \Phi(t) &= g(t)t - G(t), \\ A(p) &= F(|p|)p \qquad \text{for } p \in \mathbb{R}^N, \\ a_{ij} &= \frac{\partial A_i}{\partial p_j} \qquad \text{for } 1 \le i, j \le N. \end{aligned}$$

5. Outline of the thesis

Outline of Chapter 1. Chapter 1 is organized as follows: In Section 1 we give some properties of the function g and define the Orlicz and Sobolev-Orlicz spaces, and prove some inclusion Theorems. These spaces will be used to prove existence of minimizers.

In Section 2 we state some real analytic properties for functions with finite $\int_\Omega G(|\nabla u|)\,dx$ (generalization of Poincaré's inequality and Morrey's Theorem). Then, we state some properties of solutions, subsolution and supersolutions of $\mathcal{L}v = 0$; Harnack inequality, $C^{1,\alpha}$ bounds, comparison principle, strong maximum principle and an important inequality that will be use throughout the thesis (Theorem 1.2.38). We also prove a Cacciopoli type inequality valid for these functions (Lemma 1.2.12). Finally, we show an explicit family of subsolutions of $\mathcal{L}u = 0$ (Lemma 1.2.47), that will be used as a barrier at many points of this thesis.

In Section 3 we give the definition of Hausdorff measure and Hausdorff distance.

In Section 4 we state a Representation Theorem that will be used in this thesis. We also give the definition of sets of locally finite perimeter, and state some of their properties.

In section 5 we prove a result of $\mathcal{L}-$ solutions with linear growth.

In section 6 we prove some results of limits of blow up sequences.

In section 7 we give some results of the Schwartz symmetrized function.

Outline of Chapter 2. In Section 1 we prove the existence of minimizers of (0.1.3) and that they are subsolutions of $\mathcal{L}v = 0$. We also prove that the minimizers are nonnegative. The existence of minimizers, while standard in its form, makes strong use of the Orlicz spaces and the second inequality in condition (0.0.2).

In Section 2 we prove that any local minimizer u is Hölder continuous (Theorem 2.2.1), $\mathcal{L}u = 0$ in $\{u > 0\}$ (Lemma 2.2.12) and finally we prove the local Lipschitz continuity (Theorem 2.2.25). The proof of the Hölder continuity of the minimizers is a key step in our analysis and is one of the main proofs in this paper in which all the properties of the function G come into play through the inequality of Theorem 1.2.38.

In Section 3 we prove that minimizers satisfy a nondegeneracy property near the free boundary $\Omega \cap \partial\{u > 0\}$. We also prove that the sets $\{u > 0\}$ and $\{u = 0\}$ have locally uniform positive density at the free boundary (Theorem 2.3.5). In this theorem we make strong use of the properties of G and the corresponding Orlicz space.

In Section 4 we prove that the free boundary has finite $N-1$ dimensional Hausdorff measure and we obtain a representation theorem for minimizers (Theorem 2.4.5). This implies that $\{u > 0\}$ has locally finite perimeter in Ω. Finally we prove that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$.

In Section 5 we give some properties of blow up sequences of minimizers. We prove that any limit of a blow up sequence of minimizers is again a minimizer (Lemma 2.5.1) and we finally prove the asymptotic development of minimizers at every point in their reduced free boundary (Theorem 2.5.5).

In Section 6 we give the definition of weak solution (Definition 2.6.1 and Definition 2.6.2). We show that most of the properties that we proved for minimizers also hold for weak solutions according to Definition 2.6.1, and we mention the differences between the two definitions (Remark 2.6.8 and Remark 2.6.21).

In Section 7 we prove the regularity of the free boundary of weak solutions near "flat" free boundary points (Theorem 2.7.54) and then, we deduce the regularity of the free boundary of weak solutions near almost every point in their reduced free boundary and, in the case of minimizers, we obtain the regularity of the whole reduced free boundary (Theorem 2.7.56). We also prove for a certain class of functions satisfying (0.0.2) that, in the case $N = 2$, the whole free boundary is regular (Corollary 2.7.66).

Outline of Chapter 3. Chapter 3 is organized as follows: In Section 1 we begin our analysis of problem (P_ε) for fixed ε. First we prove the existence of a minimizer, the local Lipschitz regularity and nondegeneracy near the free boundary (Theorem 3.1.2) and we prove that minimizers are weak solutions of a free boundary problem as defined in Chapter 2 (Remark 3.1.14). Then we have that for $\mathcal{H}^{N-1}-$ almost every free boundary point, the free boundary is locally $C^{1,\beta}$ surface (Corollary 3.1.15). We prove that, for the case $N = 2$ and for a subclass of functions satisfying (0.0.2), the whole free boundary is regular (Corollary 3.1.22).

In Section 2 we prove that for small values of ε we recover our original optimization problem.

Outline of Chapter 4. Chapter 4 is organized as follows: In Section 1 we prove the uniform Lipschitz continuity of solutions of (P_ε) (Corollary 4.1.8).

In Section 2 we prove that if u is a limiting function, then $\mathcal{L}u$ is a Radon measure supported on the free boundary (Theorem 4.2.1). Then we prove Proposition 4.2.18, that says that if u is a half plane, then the slope is 0 or $\Phi^{-1}(M)$, and Proposition 4.2.20 that says that if u is a sum of two half plane, then the slopes must be equal and at most $\Phi^{-1}(M)$.

In Section 3 we prove the asymptotic development of u (Theorem 4.3.3).

In Section 4 we apply the results of Chapter 2 to prove the regularity of the free boundary (Theorem 4.4.7).

CHAPTER 1

Preliminaries

This Chapter we will state some results that will be used throughout the thesis. We also prove some new facts that are going to be needed in many proofs. From now on we will assume that the function g satisfies condition (0.0.2).

1. Properties of the function G

In Section 1 we state and prove some properties of the function G and its derivative g that are used throughout the thesis. We give the definition of Orlicz space and Sobolev-Orlicz space, and some properties of these spaces. All these results can be found in [**1**].

LEMMA 1.1.1. *The function g satisfies the following properties,*

(g1) $\min\{s^\delta, s^{g_0}\}g(t) \le g(st) \le \max\{s^\delta, s^{g_0}\}g(t)$

(g2) *G is convex and C^2*

(g3) $\dfrac{tg(t)}{1+g_0} \le G(t) \le tg(t) \quad \forall\ t \ge 0$

PROOF. For the proofs of (g1)–(g3) see [**22**]. $\square$

REMARK 1.1.2. *By (g1) and (g3) we have a similar inequality for G,*

(G1) $\min\{s^{\delta+1}, s^{g_0+1}\}\dfrac{G(t)}{1+g_0} \le G(st) \le (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t)$

and, then using the convexity of G and this last inequality we have,

(G2) $G(a+b) \le 2^{g_0}(1+g_0)(G(a)+G(b))\ \forall\ a, b > 0.$

As g is strictly increasing we can define g^{-1}. Now we prove that g^{-1} satisfies a condition similar to (0.0.2). That is,

LEMMA 1.1.3. *The function g^{-1} satisfies the inequalities*

$$\frac{1}{g_0} \le \frac{t(g^{-1})'(t)}{g^{-1}(t)} \le \frac{1}{\delta} \qquad \forall t > 0. \tag{1.1.4}$$

Moreover, g^{-1} satisfies,

$(\widetilde{g}1)$ $\qquad \min\{s^{1/\delta}, s^{1/g_0}\}g^{-1}(t) \le g^{-1}(st) \le \max\{s^{1/\delta}, s^{1/g_0}\}g^{-1}(t)$

and if $\widetilde{G}$ is such that $\widetilde{G}'(t) = g^{-1}(t)$ then,

$$(\widetilde{g}2) \qquad \frac{\delta t g^{-1}(t)}{1+\delta} \le \widetilde{G}(t) \le t g^{-1}(t) \quad \forall\, t \ge 0$$

$$(\widetilde{G}1) \quad \frac{(1+\delta)}{\delta}\min\{s^{1+1/\delta}, s^{1+1/g_0}\}\widetilde{G}(t) \le \widetilde{G}(st) \le \frac{\delta}{1+\delta}\max\{s^{1+1/\delta}, s^{1+1/g_0}\}\widetilde{G}(t)$$

$$(\widetilde{g}3) \qquad ab \le \varepsilon G(a) + C(\varepsilon)\widetilde{G}(b) \quad \forall\, a, b > 0 \text{ and } \varepsilon > 0 \text{ small}$$

$$(\widetilde{g}4) \qquad \widetilde{G}(g(t)) \le g_0 G(t)$$

Proof. Let $s = g^{-1}(t)$, then

$$\frac{t(g^{-1})'(t)}{g^{-1}(t)} = \frac{g(s)}{g'(s)s}$$

and using (0.0.2) we have the desired inequalities.

Now $(\widetilde{g}1)$ follows by property (g1) applied to g^{-1}, and $(\widetilde{g}2)$ by property (g3). $(\widetilde{G}1)$ follows by $\widetilde{g}1$ and $\widetilde{g}2$.
By Young's inequality we have that $ab \le G(a) + \widetilde{G}(b)$ and then, for $0 < \varepsilon' < 1$ such that $\varepsilon = (1+g_0)\varepsilon'^{(1+\delta)}$,

$$\varepsilon' a \frac{b}{\varepsilon'} \le G(\varepsilon' a) + \widetilde{G}\Big(\frac{b}{\varepsilon'}\Big) \le \varepsilon\, G(a) + C(\varepsilon)\widetilde{G}(b).$$

In the last inequality we have used $(G1)$ and $(\widetilde{G}1)$. Thus $(\widetilde{g}3)$ follows.
As g is strictly increasing we have that $\widetilde{G}(g(t)) + G(t) = tg(t)$ (see equation (5), Section 8.2 in [**1**]) and applying (g3), we get

$$\widetilde{G}(g(t)) = tg(t) - G(t) \le g_0 G(t).$$

Thus, $(\widetilde{g}4)$ follows. ☐

In order to prove the existence of minimizers we will use some compact embedding results (all these results are included in [**1**]). To this end, we have to define some Orlicz and Orlicz-Sobolev spaces. We recall that the functional

$$\|u\|_G = \inf\Big\{k > 0 : \int_\Omega G\Big(\frac{|u(x)|}{k}\Big)\,dx \le 1\Big\}$$

is a norm in the Orlicz space $L^G(\Omega)$ which is the linear hull of the Orlicz class

$$K_G(\Omega) = \Big\{u \text{ measurable } : \int_\Omega G(|u|)\,dx < \infty\Big\}.$$

Observe that this set is convex, since G is also convex (property (g2)). The Orlicz-Sobolev space $W^{1,G}(\Omega)$ consists of those functions in $L^G(\Omega)$ whose distributional derivatives ∇u also belong to $L^G(\Omega)$. We have that $\|u\|_{W^{1,G}} = \max\{\|u\|_G, \|\nabla u\|_G\}$ is a norm for this space.

LEMMA 1.1.5. *There exists a constant $C = C(g_0, \delta)$ such that,*

$$\|u\|_G \leq C \max\Big\{\Big(\int_\Omega G(|u|)\,dx\Big)^{1/(\delta+1)}, \Big(\int_\Omega G(|u|)\,dx\Big)^{1/(g_0+1)}\Big\}$$

PROOF. □

If $\displaystyle\int_\Omega G(|u|)\,dx = 0$ then $u = 0$ a.e and the result follows. If $\int_\Omega G(|u|)\,dx \neq 0$, take $k = \max\Big\{\Big(2(1+g_0)\int_\Omega G(|u|)\,dx\Big)^{1/(\delta+1)}, \Big(2(1+g_0)\int_\Omega G(|u|)\,dx\Big)^{1/(g_0+1)}\Big\}$. By (G1) we have,

$$\int_\Omega G\Big(\frac{|u|}{k}\Big)\,dx \leq (1+g_0)\max\Big\{\frac{1}{k^{\delta+1}}, \frac{1}{k^{g_0+1}}\Big\}\int_\Omega G(|u|)\,dx \leq 1$$

therefore $\|u\|_G \leq k$ and the result follows.

DEFINITION 1.1.6. *A function G is said to satisfy a global Δ_2- condition if there exists a positive constant k such that for every $t \geq 0$,*

$$G(2t) \leq kG(t).$$

Similarly G is said to satisfy a Δ_2- condition near infinity if there exists $t_0 > 0$ such that

$$G(2t) \leq kG(t)$$

holds for all $t \geq t_0$.

DEFINITION 1.1.7. *We call a pair (G, Ω) $\Delta-$ regular if either,*

1. *G satisfies a global Δ_2- condition, or*
2. *G satisfies a Δ_2- condition near infinity and Ω has finite volume.*

THEOREM 1.1.8. *$L^{\widetilde{G}}(\Omega)$ is the dual of $L^G(\Omega)$. Moreover, $L^G(\Omega)$ and $W^{1,G}(\Omega)$ are reflexive.*

PROOF. As G satisfies property $(G1)$ and $\widetilde{G}$ property $(\widetilde{G}1)$, we have that both pairs (G, Ω) and $(\widetilde{G}, \Omega)$ are $\Delta-$ regular. Therefore we are in the hypothesis of Theorem 8.19 and Theorem 8.28 in [**1**], and the result follows. □

THEOREM 1.1.9. *$L^G(\Omega) \hookrightarrow L^{1+\delta}(\Omega)$ continuously.*

PROOF. By theorem 8.12 of [**1**] we only have to prove that G dominates $t^{1+\delta}$ near infinity. That is, there exits constants k, t_0 such that $t^{1+\delta} \leq G(kt) \quad \forall t \ \geq t_0$. But this is true by property (G1). So the result follows. □

2. Properties of $\mathcal{L}$- solutions and subsolutions

In Section 2, we state some real analytic properties for functions with finite $\int_\Omega G(|\nabla u|)\,dx$ like a form of Poincaré inequality, a Cacciopoli type inequality, the Hölder continuity of functions in a kind of Morrey space, properties of weak solutions to $\mathcal{L}u = 0$ and a comparison principle for sub and supersolutions. We also prove that if u is a continuous function which is an $\mathcal{L}$ solution in the set $\{u > 0\}$ then is a $\mathcal{L}-$ subsolution. At the end of this section we give an explicit family of subsolutions and supersolutions in an annulus. All these properties will be thoroughly used throughout the thesis. Some of them have been proved in [**22**]. We only write down the proof of statements not contained in [**22**].

The following result is a Poincaré type inequality.

Lemma 1.2.10. *If $u \in W^{1,1}(\Omega)$ with $u = 0$ on $\partial\Omega$ and $\int_\Omega G(|\nabla u|)\,dx$ is finite, then*
$$\int_\Omega G\Big(\frac{|u|}{R}\Big)\,dx \le \int_\Omega G(|\nabla u|)\,dx \qquad \textit{for } R = diam\,\Omega.$$

Proof. See Lemma 2.2 of [**22**]. □

Now we state a generalization of Morrey's Theorem. Let
$$[u]_{0,\alpha,\Omega} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

We have the following result,

Lemma 1.2.11. *Let $u \in L^\infty(\Omega)$ such that for some $0 < \alpha < 1$ and $r_0 > 0$,*
$$\int_{B_r} G(|\nabla u|)\,dx \le C r^{N+\alpha-1} \quad \forall\ 0 < r \le r_0$$
with $B_r \subset \Omega$. Then, $u \in C^\alpha(\Omega)$ and there exists a constant $C_1 = C_1(C, \alpha, N, g_0, G(1))$ such that $[u]_{0,\alpha,\Omega} \le C_1$.

Proof. The proof of this lemma is included in the proof of Theorem 1.7 (pag. 346) in [**22**]. □

The next lemma is a Cacciopoli type inequality for subsolutions of $\mathcal{L}v = 0$.

Lemma 1.2.12. *Let v be a nonnegative weak subsolution of $\mathcal{L}v = 0$. That is,*
$$0 \ge \int_\Omega g(|\nabla v|)\frac{\nabla v}{|\nabla v|}\nabla\phi\,dx \quad \forall\ \phi \in C_0^\infty(\Omega) \textit{ such that } \phi \ge 0. \tag{1.2.13}$$
Then, there exists $C = C(N, \delta, g_0) > 0$ such that
$$\int_{B_r} G(|\nabla v|)\,dx \le C \int_{B_{\frac{3}{2}r}} G\Big(\frac{|v|}{r}\Big)\,dx$$
for all $r > 0$, such that $B_{\frac{3}{2}r} \subset \Omega$.

PROOF. Let $\phi = v\eta^{g_0+1}$, where $0 \leq \eta \in C_0^1(B_{\frac{3}{2}r})$, with $|\nabla \eta| \leq \frac{C}{r}$, $\eta \leq 1$, $\eta \equiv 1$ in B_r. Then, $\nabla\phi = \eta^{g_0+1}\nabla v + v\nabla\eta(g_0+1)\eta^{g_0}$ and replacing in (1.2.13) we have,

$$0 \geq \int_{B_{\frac{3}{2}r}} g(|\nabla v|)|\nabla v|\eta^{g_0+1}\,dx + (g_0+1)\int_{B_{\frac{3}{2}r}} g(|\nabla v|)\frac{\nabla v}{|\nabla v|}\nabla\eta\; v\,\eta^{g_0}\,dx.$$

Then,

$$\int_{B_{\frac{3}{2}r}} g(|\nabla v|)|\nabla v|\eta^{g_0+1}\,dx \leq (g_0+1)\int_{B_{\frac{3}{2}r}} g(|\nabla v|)|\nabla\eta||v|\eta^{g_0}\,dx,$$

By property ($\widetilde{g}3$) we have,

$$g(|\nabla v|)|\nabla\eta||v|\eta^{g_0} \leq \varepsilon\widetilde{G}(g(|\nabla v|)\eta^{g_0}) + C(\varepsilon)G(|\nabla\eta||v|).$$

Then, by property ($\widetilde{G}1$) and as $\eta \leq 1$, we have,

$$\widetilde{G}(g(|\nabla v|)\eta^{g_0}) \leq C\eta^{g_0\left(1+\frac{1}{g_0}\right)}\widetilde{G}(g(|\nabla v|)) \leq C\eta^{1+g_0}G(|\nabla v|),$$

where the last inequality holds by ($\widetilde{g}4$). Summing up, and using property (g3), we obtain

$$\int_{B_{\frac{3}{2}r}} G(|\nabla v|)\eta^{g_0+1}\,dx \leq C\varepsilon\int_{B_{\frac{3}{2}r}} G(|\nabla v|)\eta^{g_0+1}\,dx + C(\varepsilon)\int_{B_{\frac{3}{2}r}} G(|\nabla\eta||v|)\,dx,$$

and if we take ε small and use the bound for $|\nabla\eta|$ we have,

$$\int_{B_{\frac{3}{2}r}} G(|\nabla v|)\eta^{g_0+1}\,dx \leq C\int_{B_{\frac{3}{2}r}} G(|\nabla\eta||v|)\,dx \leq C\int_{B_{\frac{3}{2}r}} G\Big(\frac{|v|}{r}\Big)\,dx.$$

Finally, if we use that $\eta \equiv 1$ in B_r the result follows.

$\square$

The following lemmas are a generalization of the weak Harnack and Harnack inequality, and the proofs are all include in [**22**],

LEMMA 1.2.14. *Let $R > 0$, $u \in L^\infty \cap W^{1,G}(B_R)$ and such that $\mathcal{L}u \geq 0$ in B_R. Set $m = 2N(g_0+1)$. Then for $s > 0$, $\sigma \in (0,1)$, there is a constant C depending on g_0 and N such that*

$$\sup_{B_{\sigma R}} u^+ \leq \frac{C}{(1-\sigma)^{m/2s}}\Big(R^{-N}\int_{B_R}(u^+)^s\,dx\Big)^{1/s}.$$

LEMMA 1.2.15. *Let $R \leq 1$, $0 \leq u \in W^{1,G}(B_R)$ and such that $\mathcal{L}u \leq 0$ in B_R. Then there exist constants p_0 and C depending on g_0, δ and N such that*

$$\inf_{B_{R/2}} u \geq C\Big(R^{-N}\int_{B_R}(u^+)^{p_0}\,dx\Big)^{1/p_0}.$$

If we choose $s = p_0$ we infer the usual Harnack inequality, Theorem 1.2.16. *Let $R \leq 1$, $0 \leq u \in W^{1,G}(B_R)$ and such that $\mathcal{L}u = 0$ in B_R. Then there exist a constant C depending on g_0, δ and N such that*

$$\sup_{B_{R/2}} u \leq C \inf_{B_{R/2}} u.$$

A consequence of the weak Harnack inequalities is the following strong maximum and minimum principles,

Theorem 1.2.17. *Let $u \in W^{1,G}(\Omega)$,*

1. *if u satisfy $\mathcal{L}u \geq 0$. And if, for some ball $B \subset\subset \Omega$ we have,*

$$\sup_B u = \sup_\Omega u \geq 0,$$

or

2. *if u satisfy $\mathcal{L}u \leq 0$. And if, for some ball $B \subset\subset \Omega$ we have,*

$$\inf_B u = \inf_\Omega u \geq 0,$$

then the function u must be constant in Ω. Moreover, if u is continuous this implies that the strong classical maximum and minimum principle hold.

Proof. First, (1) follows as in Theorem 8.19 of [**17**] by using Lemma 1.2.15. Finally (2) follow by replacing u by $-u$. $\square$

Lemma 1.2.18. *Let v be a weak solution of $\mathcal{L}v = 0$, that is*

$$\int_\Omega g(|\nabla u|)\frac{\nabla v}{|\nabla v|}\nabla\phi\, dx = 0 \quad \forall\ \phi \in C_0^\infty(\Omega).$$

Then $v \in C^{1,\alpha}(\Omega)$ for some $\alpha = \alpha(N, \delta, g_0)$. Moreover, there exists positive constant $C = C(N, \delta, g_0)$ such that for every ball $B_r \subset \Omega$,

$$\sup_{B_{r/2}} G(|\nabla v|) \leq \frac{C}{r^N}\int_{B_{\frac{2}{3}r}} G(|\nabla v|)\, dx \tag{1}$$

$$\sup_{B_{r/2}} |\nabla v| \leq \frac{C}{r}\sup_{B_r}|v| \tag{2}$$

For every $\beta \in (0, N)$, there exists $C = C(N, \beta, \delta, g_0, \|v\|_{L^\infty(\frac{2}{3}r)}) > 0$ such that,

$$\int_{B_{r/2}} G(|\nabla v|) \leq Cr^\beta. \tag{3}$$

If $v = u$ on ∂B_r, with $u \in W^{1,G}(B_r)$ then ,

$$\int_{B_r} G(|\nabla v|) \leq C\int_{B_r}(1 + G(|\nabla u|)). \tag{4}$$

PROOF. For the proof of (1) and (4) see Lemma 5.1 of [**22**] and for the proof of (3) see (5.9) page 346 of [**22**]. Let us prove (2). By using (1) and then Lemma 1.2.12 we have,

$$\sup_{B_{r/2}} G(|\nabla v|) \leq \frac{C}{r^N}\int_{B_{\frac{2}{3}r}} G(|\nabla v|)\,dx \leq \frac{C}{r^N}\int_{B_r} G\Big(\frac{|v|}{r}\Big)\,dx \quad \leq G\Big(\frac{C}{r}\|v\|_{L^\infty(B_r)}\Big).$$

Then

$$|\nabla v(y_0)| \leq \frac{C}{r}\|v\|_{L^\infty(B_r)} \quad \forall y_0 \in B_{r/2}$$

and the result follows. □

We finally state the regularity result of [**22**] for weak solutions of $\mathcal{L}u = 0$.

THEOREM 1.2.19. *Any $u \in W^{1,G}(\Omega)$ solution of $\mathcal{L}u = 0$ in Ω, is in $C^{1,\beta}(\Omega)$ for some positive β depending on δ, g_0, N and*

$$\|u\|_{1,\beta;\Omega'} \leq C(\delta, g_0, N, dist(\Omega', \partial\Omega), \|u\|_{L^\infty(\Omega)})$$

for open $\Omega' \subset\subset \Omega$.

THEOREM 1.2.20. *Let Ω a bounded domain with $C^{1,\alpha}$ boundary, $\phi \in C^{1,\alpha}(\partial\Omega)$ with $\|\phi\|_{C^{1,\alpha}} \leq a$. Then any $u \in W^{1,G}(\Omega)$ solution of $\mathcal{L}u = 0$ in Ω, $u = \phi$ on $\partial\Omega$, is in $C^{1,\beta}(\bar\Omega)$ for some positive β depending on δ, g_0, N, α; moreover*

$$\|u\|_{1,\beta,\bar\Omega} \leq C(\delta, g_0, N, \Omega, a, \|u\|_{L^\infty(\Omega)}).$$

REMARK 1.2.21. *The results that we mentioned before are proved in [**22**] for a class of more general operators. We are stating here only the cases that we are going to use. For example, we also have similar results for the operator $Qv(x) = \mathcal{L}v(x) - f(x)$ where $f \geq 0$. In this case, the inequality in Theorem 1.2.16 holds with the following version,*

$$\sup_{B_{R/2}} u \leq C(\inf_{B_{R/2}} + \|f\|_\infty)$$

with $C = C(N, \delta, g_0)$, and Theorem 1.2.19 also holds, but with a constant $C = C(N, \delta, g_0, g(1), \|f\|_\infty, dist(\Omega', \partial\Omega))$.

Now, we will give some properties of subsolutions and solutions of

$$\mathcal{L}v = \operatorname{div}(A(\nabla v)) = 0, \tag{1.2.22}$$

where $A(p) = g(|p|)\frac{p}{|p|}$. First, let us observe that if $a_{ij}(p) = \frac{\partial A_i}{\partial p_j}$ then,

$$a_{ij} = \Big(g'(|p|) - \frac{g(|p|)}{|p|}\Big)\frac{p_i p_j}{|p|^2} + \frac{g(|p|)}{|p|}\delta_{ij}. \tag{1.2.23}$$

Then, for $\xi \in \mathbb{R}^N$ we have,

$$a_{ij}\xi_i\xi_j = \Big(g'(|p|) - \frac{g(|p|)}{|p|}\Big)\frac{(p,\xi)^2}{|p|^2} + \frac{g(|p|)}{|p|}|\xi|^2$$

and using (0.0.2), we get

$$\min\{\delta, 1\}\frac{g(|p|)}{|p|}|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \max\{g_0, 1\}\frac{g(|p|)}{|p|}|\xi|^2, \tag{1.2.24}$$

which means that the equation $\mathcal{L}v = 0$ is uniformly elliptic for $\frac{g(|p|)}{|p|}$ bounded and bounded away from zero.

REMARK 1.2.25. *Observe that in any ring $\mathcal{R} = \{c \leq |p| \leq C\}$, $A(p)$ satisfies the natural conditions of Ladyzhenskaya and Ural'tseva and therefore in any open set contained in $\{C \geq |\nabla u| \geq c\}$, $u \in W^{2,1}(U)$ (see Chapter 4 in* [**18**]*). Then as a_{ij} is in $L^\infty(\mathcal{R})$, we can differentiate equation* (1.2.22)*, and we obtain by* (1.2.23) *that u satisfies a linear nondivergence uniformly elliptic equation, $\mathcal{T}u = 0$ in U, where*

$$\mathcal{T}v = b_{ij}(\nabla u)D_{ij}v = 0, \tag{1.2.26}$$

$$b_{ij}(\nabla u) = \delta_{ij} + \Big(\frac{g'(|\nabla u|)|\nabla u|}{g(|\nabla u|)} - 1\Big)\frac{D_iuD_ju}{|\nabla u|^2}, \tag{1.2.27}$$

and by (1.2.24) *we have that the matrix $b_{ij}(\nabla u)$ is β-elliptic, with* $\beta = \max\{\max\{g_0, 1\}\max\{1, 1/\delta\}\}$.

LEMMA 1.2.28. *Let u by a solution of* (1.2.22) *in an open set U, and such that $c \leq |\nabla u| \leq C$. Then, for any unit vector e the function $w = \frac{\partial u}{\partial e}$ satisfies the uniformly elliptic equation $D_i(a_{ij}(\nabla u)D_jw) = 0$*

PROOF. By Remark 1.2.25 and Theorem 1.2.19, $u \in W^{2,1}(U) \cap C^{1,\alpha}(U)$ and then the result follows as in Section 13.1 in [**17**]. □

THEOREM 1.2.29. *Let w be a solution to the following uniformly elliptic operator*

$$c_{ij}D_{ij}w = 0 \quad \text{in } \Omega$$

where the coefficients $c_{ij} \in C^0(\Omega)$, $|c_{ij}| \leq \Lambda$, and with ellipticity constant β. Then,

1. *For any $0 < \alpha < 1$ and $\Omega' \subset \Omega$ there exists a constant $C = C(\beta, \alpha, N, \|u\|_{L^\infty(\Omega)}, dist(\Omega', \partial\Omega))$ such that*
$$\|w\|_{C^{1,\alpha}(\Omega')} \leq C.$$
2. *For any $q > 1$ there exists a constant $C = C(\beta, N, \|u\|_{L^\infty(\Omega)})$ such that,*
$$\|w\|_{W^{2,q}(\Omega)} \leq C.$$

PROOF. See [**17**]. □

REMARK 1.2.30. *Suppose that $\mathcal{L}u = 0$ and $|\nabla u| > c$ in an open set U. We have by Theorem 1.2.19 that $u \in C^{1,\beta}(U)$ and by Remark 1.2.30 we also have that $\mathcal{T}u = 0$. As $g \in C^1$ we have that the coefficients $c_{ij} = b_{ij}(\nabla u)$ are continuous. So, we can apply Theorem 1.2.29 to a solution of $\mathcal{T}w = 0$ in U.*

LEMMA 1.2.31. *Suppose that conditions* (0.0.2) *are satisfied and suppose that there exist a constant c such that*

$$F(t) \geq c \quad \forall t > 0. \tag{1.2.32}$$

Let u be a bounded weak solution of $\mathcal{L}u = 0$ in $B(4R)$, with

$$\int_{B(4R)} F(|\nabla u|)(1 + |\nabla u|)^2 \, dx < \infty, \tag{1.2.33}$$

where $F(t) = g(t)/t$. Then $v = G(|\nabla u|)$ is a weak subsolution of $D_i(b_{ij}(\nabla u)D_j v) = 0$ in B_{3R}, where b_{ij} is defined in (1.2.27).

PROOF. See page 1205 in [**21**]. □

LEMMA 1.2.34. *With the same hypothesis of Lemma 1.2.31, if f is a bounded function, and u is a solution of $\mathcal{L}u = f$ we have that $u \in W^{2,2}(\Omega)$.*

PROOF. The proof follows as the proof of Lemma 1 in [**21**] where the authors considered the case for $f = 0$. In our case we proceed in a similar way using, as in [**21**] ideas from [**18**], Section 4.5; 17, (4.4).. □

Now we prove the comparison principle,

LEMMA 1.2.35. *Let U be an open subset, u a weak subsolution and w a weak supersolution of $\mathcal{L}u = 0$ in U. If $w \geq u$ on ∂U then, $w \geq u$ in U. If w is a solution to $\mathcal{L}w = 0$ and $w = u$ on ∂U then, w is uniquely determined.*

PROOF.

$$\begin{aligned}
0 &\geq \int_U \Big(g(|\nabla u|)\frac{\nabla u}{|\nabla u|} - g(|\nabla w|)\frac{\nabla w}{|\nabla w|}\Big).\nabla(u-w)^+ \, dx \\
&= \int_{U\cap\{u>w\}} \Big(g(|\nabla u|)\frac{\nabla u}{|\nabla u|} - g(|\nabla w|)\frac{\nabla w}{|\nabla w|}\Big).\nabla(u-w) \, dx \\
&= \int_{U\cap\{u>w\}} \int_0^1 a_{ij}(\nabla u + (1-t)(\nabla w - \nabla u))(u_{x_i} - w_{x_i})(u_{x_j} - w_{x_j}) \, dt \, dx
\end{aligned}$$

And using (1.2.24) we have that the right hand side is grater than or equal to

$$C \int_{U\cap\{u>w\}} \int_0^1 F(|\nabla u + (1-t)(\nabla w - \nabla u)|)|\nabla w - \nabla u|^2 \, dt \, dx,$$

where $F(t) = \frac{g(t)}{t}$. Now, we take the following subsets of U

$$S_1 = \{x \in U : |\nabla u - \nabla w| \leq 2|\nabla u|\}, \; S_2 = \{x \in U : |\nabla u - \nabla w| > 2|\nabla u|\}$$

Then $S_1 \cup S_2 = U$ and

(1.2.36)
$$\frac{1}{2}|\nabla u| \le |\nabla u + (1-t)(\nabla w - \nabla u)| \le 3|\nabla u| \qquad \text{in } S_1 \text{ for } t \ge \frac{3}{4}$$

(1.2.37)
$$\frac{1}{4}|\nabla u - \nabla w| \le |\nabla u + (1-t)(\nabla w - \nabla u)| \le 3|\nabla u - \nabla w| \qquad \text{in } S_2 \text{ for } t \le \frac{1}{4}.$$

In S_1, and for $t \ge 3/4$ we have using (1.2.36), that

$$\begin{aligned} F(|\nabla u + (1-t)(\nabla w - \nabla u)|) &= \frac{g(|\nabla u + (1-t)(\nabla w - \nabla u)|)}{|\nabla u + (1-t)(\nabla w - \nabla u)|} \\ &\ge \frac{g(\frac{1}{2}|\nabla u|)}{3|\nabla u|} \ge \frac{1}{2^{g_0}3} F(|\nabla u|) \end{aligned}$$

where in the last inequality we have used (g1).

In S_2, and for $t \le 1/4$ we have using (g3) and then (1.2.37) that,

$$\begin{aligned} &F(|\nabla u + (1-t)(\nabla w - \nabla u)|)|\nabla u - \nabla w|^2 \\ &\ge \frac{G(|\nabla u + (1-t)(\nabla w - \nabla u)|)}{|\nabla u + (1-t)(\nabla w - \nabla u)|^2}|\nabla u - \nabla w|^2 \\ &\ge \frac{G(\frac{1}{4}|\nabla u - \nabla w|)}{9|\nabla u - \nabla w|^2}|\nabla u - \nabla w|^2 \ge \frac{G(|\nabla u - \nabla w|)}{4^{g_0+1}9(1+g_0)} \end{aligned}$$

where in the last inequality we have used (G1).

Therefore, we have that

$$0 \ge C\Big(\int_{S_1} F(|\nabla u|)|\nabla(u-w)^+|^2\,dx + \int_{S_2} G(|\nabla(u-w)^+|)\,dx\Big).$$

Hence $\nabla(u-w)^+ = 0$ in S_2 and $\nabla(u-w)^+ = 0$, or $F(|\nabla u|) = 0$ in S_1 in which case $\nabla u = 0$ and, by the definition of S_1, this implies that $\nabla(u-w) = 0$ in S_1. Therefore, $\nabla(u-w)^+ = 0$ in U, then $(u-w)^+ = 0$, which implies $u \le w$. □

The following inequality will be a key tool in the proof of the Hölder continuity of minimizers of the problem in Chapter 2. Also, it will be used to prove that in the optimization problem, we don't need to pass to the limit with the penalization parameter. As an observation, we mention that the following result is a generalization of well known integral inequalities for the p- Laplacian (see, for example, pag.4 in [**10**]). Here the difference is that we obtain a unique inequality for any δ and g_0, (for the p-Laplacian the inequalities were separated in two cases $p \ge 2$ and $1 < p < 2$).

Theorem 1.2.38. *Let $u \in W^{1,G}(\Omega)$, $B_r \subset\subset \Omega$ and v be a solution of*

$$\mathcal{L}v = 0 \quad \textit{in } B_r, \qquad v - u \in W_0^{1,G}(B_r).$$

then

$$\int_{B_r} (G(|\nabla u|) - G(|\nabla v|))\,dx \ge C\Big(\int_{A_2} G(|\nabla u - \nabla v|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx\Big),$$

where $F(t) = g(t)/t$,

$A_1 = \{x \in B_r : |\nabla u - \nabla v| \le 2|\nabla u|\}$ *and* $A_2 = \{x \in B_r : |\nabla u - \nabla v| > 2|\nabla u|\}$

and $C = C(g_0, \delta)$.

PROOF. Let $u^s = su + (1-s)v$. Using the integral form of the mean value theorem and the fact that v is an $\mathcal{L}$– solution, we have,

$$\begin{aligned}\mathcal{I} := \int_{B_r} (G(|\nabla u|) - G(|\nabla v|))\,dx &= \int_0^1 \int_{B_r} g(|\nabla u^s|)\frac{\nabla u^s}{|\nabla u^s|}.\nabla(u-v)\,dx\,ds\\ &= \int_0^1 \frac{1}{s}\int_{B_r} \Big(g(|\nabla u^s|)\frac{\nabla u^s}{|\nabla u^s|} - g(|\nabla v|)\frac{\nabla v}{|\nabla v|}\Big).\nabla(u^s - v)\,dx\,ds\\ &= \int_0^1 \frac{1}{s}\int_{B_r}\int_0^1 a_{ij}(\nabla u^s + (1-t)(\nabla v - \nabla u^s))(u^s_{x_i} - v_{x_i})(u^s_{x_j} - v_{x_j})\,dt\,dx\,ds.\end{aligned}$$

And, by (1.2.24) we have that the right hand side is grater than or equal to

$$C\int_0^1 \frac{1}{s}\int_{B_r}\int_0^1 F(|\nabla u^s + (1-t)(\nabla v - \nabla u^s)|)|\nabla v - \nabla u^s|^2\,dt\,dx\,ds.$$

where F was defined in Lemma 1.2.35 and $C = C(\delta)$.

Now, we take the following subsets of B_r

$$S_1 = \{x \in B_r : |\nabla u^s - \nabla v| \le 2|\nabla u^s|\}, \quad S_2 = \{x \in B_r : |\nabla u^s - \nabla v| > 2|\nabla u^s|\}$$

Then $S_1 \cup S_2 = B_r$ and

(1.2.39)
$$\frac{1}{2}|\nabla u^s| \le |\nabla u^s + (1-t)(\nabla v - \nabla u^s)| \le 3|\nabla u^s| \qquad \text{on } S_1 \text{ for } t \ge \frac{3}{4}$$

(1.2.40)
$$\frac{1}{4}|\nabla u^s - \nabla v| \le |\nabla u^s + (1-t)(\nabla v - \nabla u^s)| \le 3|\nabla u^s - \nabla v| \qquad \text{on } S_2 \text{ for } t \le \frac{1}{4}.$$

Proceeding as in Lemma 1.2.35, we get

$$F(|\nabla u^s + (1-t)(\nabla v - \nabla u^s)|) \ge \frac{1}{2^{g_0}3}F(|\nabla u^s|)$$

in S_1 and

$$F(|\nabla u^s + (1-t)(\nabla v - \nabla u^s)|)|\nabla u^s - \nabla v|^2 \ge \frac{G(|\nabla u^s - \nabla v|)}{4^{g_0+1}9(1+g_0)}$$

in S_2.

Therefore, we have that

$$\mathcal{I} \ge C\Big(\int_0^1 \frac{1}{s}\int_{S_1} F(|\nabla u^s|)|\nabla v - \nabla u^s|^2\,dx\,ds + \int_0^1 \frac{1}{s}\int_{S_2} G(|\nabla u^s - \nabla v|)\,dx\,ds\Big)$$

Now, let

$$A_1 = \{x \in B_r : |\nabla u - \nabla v| \le 2|\nabla u|\}, \quad A_2 = \{x \in B_r : |\nabla u - \nabla v| > 2|\nabla u|\},$$

then $B_r = A_1 \cup A_2$, and

$$\frac{1}{2}|\nabla u| \leq |\nabla u^s| \leq 3|\nabla u| \qquad \text{on } A_1 \text{ for } s \geq \frac{3}{4} \tag{1.2.41}$$

$$\frac{1}{4}|\nabla u - \nabla v| \leq |\nabla u^s| \leq 3|\nabla u - \nabla v| \qquad \text{on } A_2 \text{ for } s \leq \frac{1}{4}. \tag{1.2.42}$$

Therefore

$$\begin{aligned}
\mathcal{I} \geq C\Big(& \int_0^{1/4} \frac{1}{s} \int_{S_1 \cap A_2} F(|\nabla u^s|)|\nabla v - \nabla u^s|^2 \, dx \, ds \\
& + \int_{3/4}^1 \frac{1}{s} \int_{S_1 \cap A_1} F(|\nabla u^s|)|\nabla v - \nabla u^s|^2 \, dx \, ds \\
& + \int_0^{1/4} \frac{1}{s} \int_{S_2 \cap A_2} G(|\nabla u^s - \nabla v|) \, dx \, ds \\
& + \int_{3/4}^1 \frac{1}{s} \int_{S_2 \cap A_1} G(|\nabla u^s - \nabla v|) \, dx \, ds\Big) = I + II + III + IV.
\end{aligned}$$

Let us estimate these four terms,

In $S_1 \cap A_2$, for $s \leq 1/4$ we have by (1.2.42) and (g1), that

$$F(|\nabla u^s|) \geq \frac{1}{4^{g_0} 3} F(|\nabla u - \nabla v|).$$

Therefore,

$$\begin{aligned}
I &\geq C \int_0^{1/4} \frac{1}{s} \int_{S_1 \cap A_2} F(|\nabla u - \nabla v|)|\nabla v - \nabla u^s|^2 \, dx \, ds \\
&= C \int_0^{1/4} s \int_{S_1 \cap A_2} F(|\nabla u - \nabla v|)|\nabla v - \nabla u|^2 \, dx \, ds \\
&\geq C \int_0^{1/4} s \int_{S_1 \cap A_2} G(|\nabla u - \nabla v|) \, dx \, ds
\end{aligned}$$

where in the last inequality we are using (g3).

In $S_1 \cap A_1$, for $s \geq 3/4$ we have by (1.2.41) and (g1), that

$$F(|\nabla u^s|) \geq \frac{1}{2^{g_0} 3} F(|\nabla u|).$$

Therefore,

$$\begin{aligned}
II &\geq C \int_{3/4}^1 s \int_{S_1 \cap A_1} F(|\nabla u|)|\nabla v - \nabla u|^2 \, dx \, ds \\
&\geq C \int_{3/4}^1 \int_{S_1 \cap A_1} F(|\nabla u|)|\nabla v - \nabla u|^2 \, dx \, ds.
\end{aligned}$$

In $S_2 \cap A_2$, for $s \le 1/4$ we have by definition of S_2, by (1.2.42) and (G1), that

$$G(|\nabla u^s - \nabla v|) \ge \frac{1}{2^{g_0+1}(g_0+1)} G(|\nabla u - \nabla v|),$$

therefore

$$\begin{aligned} III &\ge C \int_0^{1/4} \frac{1}{s} \int_{S_2 \cap A_2} G(|\nabla u - \nabla v|)\,dx\,ds \\ &\ge C \int_0^{1/4} s \int_{S_2 \cap A_2} G(|\nabla u - \nabla v|)\,dx\,ds. \end{aligned}$$

In $S_2 \cap A_1$, for $s \ge 3/4$ we have, by definition of S_2 and by (1.2.41)

$$|\nabla u^s - \nabla v| > 2|\nabla u^s| \ge |\nabla u| \tag{1.2.43}$$

By (g3), using (1.2.43) and the definition of A_1 we have,

$$\begin{aligned} G(|\nabla u^s - \nabla v|) &\ge \frac{1}{g_0+1} g(|\nabla u^s - \nabla v|)|\nabla u^s - \nabla v| \ge \frac{1}{g_0+1} g(|\nabla u|)|\nabla u^s - \nabla v| \\ &= \frac{1}{g_0+1} F(|\nabla u|) s |\nabla u - \nabla v||\nabla u| \ge \frac{s}{2(g_0+1)} F(|\nabla u|)|\nabla u - \nabla v|^2. \end{aligned}$$

Therefore,

$$IV \ge C \int_{3/4}^1 \int_{S_2 \cap A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx\,ds.$$

If we sum $I + III$, we obtain

$$\begin{aligned} I + III &\ge \int_0^{1/4} Cs\Big(\int_{S_1 \cap A_2} G(|\nabla u - \nabla v|)\,dx + \int_{S_2 \cap A_2} G(|\nabla u - \nabla v|)\,dx\,ds \Big) \\ &= C \int_0^{1/4} s \int_{A_2} G(|\nabla u - \nabla v|)\,dx\,ds = C \int_{A_2} G(|\nabla u - \nabla v|)\,dx \end{aligned}$$

and if we sum $II + IV$, we obtain

$$\begin{aligned} II + IV \ge C \int_1^{3/4} \Big(&\int_{S_1 \cap A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx \\ &+ \int_{S_2 \cap A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx \Big)\,ds = C \int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx. \end{aligned}$$

Therefore,

$$\mathcal{I} \ge C\Big(\int_{A_2} G(|\nabla u - \nabla v|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx \Big), \tag{1.2.44}$$

where $C = C(g_0, \delta)$.

□

LEMMA 1.2.45. *Let u be a continuous and nonnegative function in $\mathbb{R}^N$, such that $\mathcal{L}u = 0$ in $\{u > 0\}$. Then u is in $W^{1,G}_{loc}(\Omega)$ and $\Lambda := \mathcal{L}u$ is a nonnegative Radon measure with support in $\Omega \cap \partial\{u > 0\}$ (in particular, u is an $\mathcal{L}-$ subsolution in Ω).*

Proof. Since $\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$, then u is in $C^{1,\alpha}$ in $\Omega \cap \{u > 0\}$. For $s > 0$, take $v = (u-s)^+$. Let $\eta \in C_0^\infty(\Omega)$ with $0 \le \eta \le 1$. We have,

$$\begin{aligned}0 &= \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \nabla(\eta^{g_0+1}v)\,dx\\ &= \int_{\Omega\cap\{u>s\}} \eta^{g_0+1} g(|\nabla u|)|\nabla u| + (g_0+1)\int_\Omega \eta^{g_0}\, v\,\frac{g(|\nabla u|)}{|\nabla u|}\nabla u \nabla \eta\,dx.\end{aligned}$$

Therefore,

$$\int_{\Omega\cap\{u>s\}} \eta^{g_0+1} g(|\nabla u|)|\nabla u|\,dx \le (g_0+1)\int_{\Omega\cap\{u>s\}} g(|\nabla u|)\,v\,|\eta|^{g_0}|\nabla\eta|\,dx. \tag{1.2.46}$$

By $(\widetilde{g}3)$, $(\widetilde{G}1)$ and $(\widetilde{g}4)$ we obtain,

$$\begin{aligned}g(|\nabla u|)|\eta|^{g_0}|v||\nabla \eta| &\le \varepsilon \widetilde{G}(g(|\nabla u|)|\eta|^{g_0}) + C(\varepsilon)G(|v||\nabla\eta|)\\ &\le C\varepsilon\eta^{g_0+1}\widetilde{G}(g(|\nabla u|)) + C(\varepsilon)G(|v||\nabla\eta|)\\ &\le C\varepsilon G(|\nabla u|)\eta^{g_0+1} + C(\varepsilon)G(|v||\nabla\eta|).\end{aligned}$$

Then, using $(g3)$, (1.2.46) and choosing ε small enough, we have that

$$\int_{\Omega\cap\{u>s\}} \eta^{g_0+1}G(|\nabla u|)\,dx \le C\int_{\Omega\cap\{u>s\}} G(|v||\nabla\eta|)\,dx \le C\int_\Omega G(|u||\nabla\eta|)\,dx.$$

Then, letting $s \to 0$ yields the first assertion.

To prove the second part, take $\xi \in C_0^\infty(\Omega)$ nonnegative, $\varepsilon > 0$ and $v = \max\Big(\min\big(1, 2 - \frac{u}{\varepsilon}\big), 0\Big)$. As $\mathcal{L}u = 0$ in $\{u > 0\}$ and using that $supp(1-v) \subset \{u > \varepsilon\}$, we have that,

$$\begin{aligned}&\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla\xi\,dx\\ &\quad= \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla\big(\xi(1-v)\big)\,dx + \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla(\xi v)\,dx\\ &\quad= \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla(\xi v)\,dx = \int_{\Omega\cap\{0<u<2\varepsilon\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla(\xi v)\,dx\\ &\quad= \int_{\Omega\cap\{\varepsilon<u<2\varepsilon\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla(\xi v)\,dx + \int_{\Omega\cap\{0<u<\varepsilon\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla\xi\,dx\\ &\quad\le 2\int_{\Omega\cap\{\varepsilon<u<2\varepsilon\}} g(|\nabla u|)|\nabla\xi|\,dx + \int_{\Omega\cap\{0<u<\varepsilon\}} g(|\nabla u|)|\nabla\xi|\,dx\\ &\quad\le 2\int_{\Omega\cap\{0<u<2\varepsilon\}} g(|\nabla u|)|\nabla\xi|\,dx,\end{aligned}$$

which tends to zero when $\varepsilon \to 0$ yielding the desired result. □

In several points on this thesis we will need an explicit family of subsolutions and supersolutions in an annulus. We state here the required lemma.

LEMMA 1.2.47. *Let $w_\mu = \varepsilon e^{-\mu|x|^2}$, for $\varepsilon > 0$ and $r_1 > r_2 > 0$ then there exists $\mu > 0$ such that*

$$\mathcal{L}w_\mu > 0 \text{ in } B_{r_1} \setminus B_{r_2}$$

and μ depends only on r_2, g_0, δ and N.

PROOF. First note that

$$\mathcal{L}w = \frac{g(|\nabla w|)}{|\nabla w|^3}\Big\{\Big(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w| - 1\Big)\sum_{i,j} w_{x_i}w_{x_j}w_{x_ix_j} + \triangle w|\nabla w|^2\Big\}.$$

Computing, we have

(1.2.48)

$$w_{x_i} = -2\mu\varepsilon x_i, e^{-\mu|x|^2},\ w_{x_ix_j} = \varepsilon(4\mu^2 x_ix_j - 2\mu\delta_{ij})e^{-\mu|x|^2},\ |\nabla w| = 2\varepsilon\mu|x|e^{-\mu|x|^2},$$

therefore using (1.2.48) and (0.0.2) we obtain,

$$\begin{aligned}
&e^{3\mu|x|^2}\mathcal{L}w\\
&\quad= \varepsilon^3\frac{g(|\nabla w|)}{|\nabla w|^3}\Big\{\Big(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w| - 1\Big)(16\mu^4|x|^4 - 8\mu^3|x|^2)\\
&\qquad + (4\mu^2|x|^2 - 2\mu N)4\mu^2|x|^2\Big\}\\
&\quad= \varepsilon^3\frac{g(|\nabla w|)}{|\nabla w|^3}4\mu^3|x|^2\Big\{\Big(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w| - 1\Big)(4\mu|x|^2 - 2) + (4\mu|x|^2 - 2N)\Big\}\\
&\quad= \varepsilon^3\frac{g(|\nabla w|)}{|\nabla w|^3}4\mu^3|x|^2\Big\{\Big(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w|\Big)4\mu|x|^2 - \Big(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w| - 1\Big)2 - 2N\Big\}\\
&\quad\geq \varepsilon^3\frac{g(|\nabla w|)}{|\nabla w|^3}4\mu^3|x|^2(4\mu|x|^2\delta - K) \geq \varepsilon^3\frac{g(|\nabla w|)}{|\nabla w|^3}4\mu^3{r_2}^2(4\mu{r_2}^2\delta - K)
\end{aligned}$$

where $K = 2N$ if $g_0 < 1$ and $K = 2(g_0 - 1) + 2N$ if $g_0 > 1$. Therefore if μ is big enough, depending only on δ, g_0, r_2 and N we have $\mathcal{L}w > 0$. $\square$

3. Hausdorff measure and Hausdorff distance

DEFINITION 1.3.49. *For $A \subset \mathbb{R}^N$, $k > 0$ and $\alpha > 0$, let*

$$\mathcal{H}^k_\alpha(A) = \omega_k 2^{-k}\inf\Big\{\sum_{j=1}^\infty (diam\, S_j)^k : A \subset \bigcup_{j=1}^\infty S_j,\ diam\, S_j < \alpha\Big\},$$

where ω_k is a positive constant, such that, when $k \in \mathbb{N}$ is the volume of the unit ball in $\mathbb{R}^k$.

As $\mathcal{H}^k_\alpha(A)$ increases, when α decrease, the following limit exists and then we define the k dimensional Hausdorff measure, by

$$\mathcal{H}^k(A) = \lim_{\alpha\to 0}\mathcal{H}^k_\alpha(A) = \sup_{\alpha>0}\mathcal{H}^k_\alpha(A).$$

REMARK 1.3.50. *If $k \in \mathbb{N}$, $k \leq n$, $\mathcal{H}^k(E)$ coincides with the notion of k- dimensional area in $\mathbb{R}^N$, under suitable hypotheses on E.*

DEFINITION 1.3.51. *We define the Hausdorff distance $d(A, B)$ between two sets $A, B \subset \mathbb{R}^N$ by,*

$$d(A, B) = \inf\{\varepsilon > 0 : A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\}$$

where we denote for a set $D \subset \mathbb{R}^N$ and $\varepsilon > 0$,

$$D^\varepsilon = \{x \in \mathbb{R}^N : d(x, D) < \varepsilon\}.$$

THEOREM 1.3.52. *Let $K \subset \mathbb{R}^N$ be compact. Then the family of all the compact subsets of K is a complete metric space with the Hausdorff distance.*

4. Representation Theorem and sets of locally finite perimeter

4.1. Representation Theorem. The following result is a generalization of Theorem 4.5 in [**4**], and will be used throughout the thesis. Its proof follows exactly as the one in [**4**].

Let Λ be an application from $C_0^\infty(\Omega)$ into $\mathbb{R}$, that defines a nonnegative Radon measure Λ with support on ∂A, where A is an open subset of Ω. Assume that Λ satisfies that for any domain $D \subset\subset \Omega$ there exist constants c_0, C_0, depending on D, such that, for every $B_r \subset \Omega$, centered on ∂A

$$c_0 r^{N-1} \leq \int_{B_r} d\Lambda \leq C_0 r^{N-1}. \tag{1.4.53}$$

Then we have,

THEOREM 1.4.54 (Representation Theorem). *For Λ, A satisfying* (1.4.53) *we have,*

1. *$\mathcal{H}^{N-1}(D \cap \partial A) < \infty$ for every $D \subset\subset \Omega$.*
2. *There exists a Borel function q such that*
$$\Lambda = q\, \mathcal{H}^{N-1} \lfloor \partial A.$$
i.e
$$\Lambda(\varphi) = \int_{\Omega \cap \partial A} \varphi q \, d\mathcal{H}^{N-1} \quad \forall\ \varphi \in\ C_0^\infty(\Omega).$$
3. *For $D \subset\subset \Omega$ there are constants $0 < c \leq C < \infty$ depending on c_0, C_0, Ω, D such that for $B_r(x) \subset D$ and $x \in \partial A$,*
$$c \leq q_u(x) \leq C, \quad c\, r^{N-1} \leq \mathcal{H}^{N-1}(B_r(x) \cap \partial A) \leq C\, r^{N-1}.$$

PROOF. It follows as in Theorem 4.5 in [**4**]. □

4.2. Sets of locally finite perimeter (see [14], [13]). Let $\Omega \subset \mathbb{R}^N$, an open set, and let $f \in L^1(\Omega)$. We define,

$$\int_\Omega |\nabla f| = sup\Big\{ \int_\Omega f \mathrm{div} g \, dx : g \in C_0^1(\Omega;\mathbb{R}^N), |g(x)| \leq 1 \ \forall x \in \Omega \Big\}.$$

If $\int_\Omega |\nabla f| < \infty$ we say that f has bounded total variation and we denote,

$$BV(\Omega) = \Big\{ f \in L^1(\Omega) : \int_\Omega |\nabla f| < \infty \Big\}.$$

We can observe that if $f \in BV(\Omega)$ the derivatives of f in the distributional sense are Radon measures in Ω.

Definition 1.4.55. *Given $E \subset \mathbb{R}^N$, a borel set. We define the perimeter of E in the open set Ω as*

$$P(E,\Omega) = \int_\Omega |\nabla \chi(E)|,$$

and we say that E is a set of finite perimeter in Ω if $\chi(E) \in BV(\Omega)$.

If $P(E,\Omega) < \infty$ for all open set Ω we say that E is a set of locally finite perimeter. In this case, we have that $\mu := -\nabla \chi_E$ is a Borel measure, and the total variation $|\mu|$ is a Radon measure. We have that, for any $\varphi \in C_0^1(\Omega;\mathbb{R}^N)$

$$\int_E \mathrm{div}\varphi \, dx = \int_\Omega \varphi \, d\mu. \tag{1.4.56}$$

Theorem 1.4.57. *Let $E \subset \mathbb{R}^N$ be a set of locally finite perimeter. Then,*

$$\int_A |\nabla \chi(E)| \leq \mathcal{H}^{N-1}(A \cap \partial E),$$

for all $A \subset \mathbb{R}^N$.

Theorem 1.4.58. *Let $E \subset \mathbb{R}^N$ be a borelian set and suppose that $\mathcal{H}^{N-1}(K \cap \partial E) < +\infty$ for any compact subset $K \subset \mathbb{R}^N$. Then E has locally finite perimeter.*

Definition 1.4.59. *Let E be a Lebesgue measurable set. Given $x \in \partial E$, we say that the unit vector ν is a normal exterior to E at x in the measure theoretical sense:*

$$\int_{B_r(x)} |\chi_E - \chi_{\{y/\langle y-x, \nu_u(x)\rangle < 0\}}| = o(r^N). \tag{1.4.60}$$

If such an ν exists, it is unique and we denote it by $\nu(x,E)$. Then, we define,

$$\partial_{red} E := \{ x \in \Omega \cap \partial E / \textit{ if } \nu(x,E) \textit{exists } \}.$$

We have, by the results in [**14**] Theorem 4.5.6 that,

$$\mu = \nu \mathcal{H}^{N-1} \lfloor \partial_{red} E.$$

Therefore, by 1.4.56 we have that, for any $\varphi \in C_0^1(\Omega;\mathbb{R}^N)$

$$\int_E \mathrm{div}\varphi \, dx = \int_{\partial_{red} E} \varphi \cdot \nu \, d\mathcal{H}^{N-1} \tag{1.4.61}$$

REMARK 1.4.62. *If A, Λ satisfy the conclusions of Theorem 1.4.54, then by Theorem 1.4.58 the set $\Omega \cap A$ has finite perimeter locally in Ω.*

THEOREM 1.4.63. *Let $E \subset \mathbb{R}^N$ a set of finite local perimeter, and suppose that for all $x_0 \in \partial E$, it holds that*

$$\limsup_{r\to 0} \frac{|B_r(x_0) \cap E|}{|B_r(x_0)|} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{|B_r(x_0) \cap \mathbb{R}^N \setminus E|}{|B_r(x_0)|} > 0.$$

Then, $\mathcal{H}^{N-1}(\partial E \setminus \partial_{red} E) = 0$.

THEOREM 1.4.64. *For $\mathcal{H}^{N-1}-$ almost every point $x_0 \in \partial E \setminus \partial_{red} E$*

$$|\mu|(B_r(x_0)) = o(r^{N-1}).$$

THEOREM 1.4.65. *Let $E \subset \mathbb{R}^N$ a set of finite local perimeter then,*

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(B_r(x_0) \cap \partial E)}{\omega_{N-1} r^{N-1}} = 1 \text{ for } \mathcal{H}^{N-1} - \text{ almost every point } x_0 \in \partial_{red} E, \tag{1.4.66}$$

where w_{N-1} is the $\mathcal{H}^{N-1}-$ surface measure of the unit sphere in $\mathbb{R}^N$.

THEOREM 1.4.67. *Let $E \subset \mathbb{R}^N$ a set of finite local perimeter then, and $x_0 \in \partial_{red} E$ satisfying 1.4.66. Let $\rho_k \to 0$ and $E_k = \frac{E - x_0}{\rho_k}$, such that*

$$\begin{aligned} \chi_{E_k} &\to \chi_{\{x_N<0\}} && \text{in } L^1_{loc}(\mathbb{R}^N) \\ \partial E_k &\to \{x_N = 0\} && \text{locally in Hasdorff distance.} \end{aligned}$$

Then, for every $D \subset\subset \{x_N < 0\}$ we have

$$\mathcal{H}^{N-1}(\partial E_k \cap D) \to 0.$$

And for,

$$\xi(x) = \min\Big(2\big(1 - \frac{|x_N|}{2}, 1\big)\Big)\eta(x_1, ..., x_{N-1})$$

where $0 \le \eta \in C_0^\infty(B'_r)$, ($B'_r$ is a $(N-1)$ dimensional ball with radius r) we have,

$$\int_{\partial E_k} \xi d\mathcal{H}^{N-1} \to \int_{\mathbb{R}^{N-1}} \eta d\mathcal{H}^{N-1}. \tag{1.4.68}$$

PROOF. See the proof of Theorem 4.8 in [**4**]. □

5. A result on $\mathcal{L}$-solutions with linear growth

In this section we state some properties of $\mathcal{L}$-subsolutions. From now on, we denote $B_r^+ = B_r(0) \cap \{x_N > 0\}$. We take ideas from [**19**]. The proofs in this section are more involved than the ones in [**19**] because of the lack of homogeneity of the function g.

THEOREM 1.5.1. *Let u be a Lipschitz function in $I\!R^N$ with Lipschitz constant L and such that*

1. *$u \ge 0$ in $I\!R^N$, $\mathcal{L}u = 0$ in $\{u > 0\}$.*
2. *$\{x_N < 0\} \subset \{u > 0\}$, $u = 0$ in $\{x_N = 0\}$.*

3. *There exists* $0 < \lambda_0 < 1$ *such that* $\dfrac{|\{u=0\} \cap B_R(0)|}{|B_R(0)|} > \lambda_0$, $\forall R > 0$.

Then $u = 0$ *in* $\{x_N > 0\}$.

In order to prove this theorem we need a couple of lemmas.

Lemma 1.5.2. *Let* u *be a* $\mathcal{L}-$ *subsolution in* B_r^+ *such that,* $0 \leq u \leq \alpha x_N$ *in* B_r^+, $u = 0$ *on* $\overline{B_r^+} \cap \{x_N = 0\}$, $u \leq \delta_0 \alpha x_N$ *on* $\partial B_r^+ \cap B_{r_0}(\bar{x})$ *with* $\bar{x} \in \partial B_r^+$, $\bar{x}_N > 0$ *and* $0 < \delta_0 < 1$.

Then there exists $0 < \gamma < 1$ *and* $0 < \varepsilon \leq 1$, *depending only on* $\delta_0, r_0, \bar{x}$ *and* N, *such that if* $0 < r \leq 1$, $u(x) \leq \gamma \alpha x_N$ *in* $B_{\varepsilon r}^+$.

Proof. By the invariance of $\mathcal{L}$- solutions under the rescaling $\bar{u}(x) = u(rx)/r$ and since $r \leq 1$, we can suppose that $r = 1$.

Let ψ^α be an $\mathcal{L}_\alpha$-solution in B_1^+, with smooth boundary data, such that

$$\begin{cases} \psi^\alpha = x_N & \text{on } \partial B_1^+ \setminus B_{r_0}(\bar{x}) \\ \delta_0 x_N \leq \psi^\alpha \leq x_N & \text{on } \partial B_1^+ \cap B_{r_0}(\bar{x}) \\ \psi^\alpha = \delta_0 x_N & \text{on } \partial B_1^+ \cap B_{r_0/2}(\bar{x}), \end{cases}$$

where $\mathcal{L}_\alpha v = \operatorname{div}\Big(\frac{g_\alpha(|\nabla v|)}{|\nabla v|} \nabla v\Big)$ and $g_\alpha(t) = g(\alpha t)$.

Therefore $\mathcal{L}(\alpha\psi^\alpha) = 0$ and, by the comparison principle (Theorem 1.2.35), $u \leq \alpha\psi^\alpha$ in B_1^+. If we see that there exist $0 < \gamma < 1$ and $\varepsilon > 0$, independent of α, such that $\psi^\alpha \leq \gamma x_N$ in B_ε^+, the result follows.

First, observe that,

$$\delta \leq \frac{g_\alpha'(t) t}{g_\alpha(t)} \leq g_0. \tag{1.5.3}$$

Then, by Theorems 1.2.16 and 1.2.20,

$$\begin{aligned} &\psi^\alpha \in C^{1,\beta}(\overline{B_1^+}) \text{ for some } \beta > 0 \text{ independent of } \alpha, \\ &\text{the } C^{1,\beta} \text{ norm is bounded by a constant independent of } \alpha \text{ and} \\ &\text{the constant in Harnack inequality is independent of } \alpha. \end{aligned} \tag{1.5.4}$$

If $|\nabla\psi^\alpha| \geq \mu > 0$ in some open set U, we have by Remark 1.2.30 that $\psi^\alpha \in W^{2,p}(U)$ and it is a solution of the linear uniformly elliptic equation,

$$\mathcal{T}_\alpha \psi^\alpha = \sum_{i,j=1}^N b_{ij}^\alpha \psi_{x_i x_j}^\alpha = 0 \quad \text{in } U, \tag{1.5.5}$$

where b_{ij}^α was define in (1.2.27), and the constant of ellipticity depends only on g_0 and δ.

Now, we divide the proof in several steps,

Step 1

Let $w^\alpha = x_N - \psi^\alpha$ then $w^\alpha \in C^{1,\beta}(\overline{B_1^+})$ and is a solution of $\mathcal{T}_\alpha w^\alpha = 0$ in any open set U where $|\nabla \psi^\alpha| \geq \mu > 0$.

On the other hand, as $\psi^\alpha \leq x_N$ on ∂B_1^+ and both functions are $\mathcal{L}^\alpha$-solutions we have, by comparison, that $\psi^\alpha \leq x_N$ in B_1^+. Therefore $w^\alpha \geq 0$ in B_1^+.

Step 2

Let us prove that there exist ρ and $\tilde{c}$ independent of α, such that $|\nabla\psi^\alpha| \geq \tilde{c}$ in B_ρ^+.

First, let as see that there exists $c > 0$ independent of α such that

$$\psi^\alpha\Big(\frac{1}{2}e_N\Big) \geq c. \tag{1.5.6}$$

If not, there exists a sequences of $\alpha_k \to 0$ such that $\psi^{\alpha_k}(1/2e_N) \to 0$. But, since the constant in Harnack inequality is independent of α (see (1.5.4)), we have that, $\psi^{\alpha_k} \to 0$ uniformly in compact sets of B_1^+. On the other hand, using that ψ^α are uniformly bounded in $C^\beta(\overline{B_1^+})$, we have that there exists $\psi \in C(\overline{B_1^+})$ such that, for a subsequence $\psi^{\alpha_k} \to \psi$ uniformly in $\overline{B_1^+}$. Therefore $\psi = 0$ in $\overline{B_1^+}$, but we have that $\psi = \delta_0 x_N$ on $B_{r_0/2}(\bar{x}) \cap \partial B_1^+$, which is a contradiction.

Let $x_1 \in \{x_N = 0\} \cap B_{1/2}$, take $x_0 = x_1 + \frac{e_N}{4}$. By (1.5.4) we have that there exists a constant c_1 independent of α such that, $\psi^\alpha(x) \geq c_1\psi^\alpha(1/2e_N)$ for any $x \in \partial B_{1/8}(x_0)$, and therefore by (1.5.6) $\psi^\alpha \geq \bar{c}$ in $\partial B_{1/8}(x_0)$.
Take $v = s(e^{-\lambda/16} - e^{-\lambda|x-x_0|^2})$ with $s > 0$, and choose λ such that $\mathcal{L}v > 0$ in $B_{1/4}(x_0) \setminus B_{1/8}(x_0)$ and s such that $v = \bar{c}$ on $\partial B_{1/8}(x_0)$ (observe that, by Lemma 1.2.47 λ and s can be chosen independent of α). Since $\psi^\alpha \geq 0$ and $\psi^\alpha \geq v$ on $\partial B_{1/8}(x_0)$ we have by comparison that $\psi^\alpha \geq v$ in $B_{1/4}(x_0) \setminus B_{1/8}(x_0)$. On the other hand $v_{x_N}(x_1) = s2\lambda(x_1 - x_0)_N = s\frac{\lambda}{2}$, and therefore $\psi^\alpha_{x_N}(x_1) \geq \frac{\lambda s}{2}$. As ψ^α are uniformly Lipschitz, we have that there exists ρ independent of α such that $\psi^\alpha_{x_N}(x) \geq \frac{\lambda s}{2}$ in B_ρ^+ .

Step 3

Since $|\nabla\psi^\alpha| \geq \frac{\lambda s}{2}$ in B_ρ^+, we have that, $\mathcal{T}_\alpha w^\alpha = 0$ there. Suppose that $w^\alpha(1/2e_N\rho) \geq c > 0$, with c independent of α. Then by Harnack inequality we have that there exists σ_1 independent of α such that, $w^\alpha \geq \sigma_1 w^\alpha(1/2e_N\rho) \geq \sigma_2$ in $B_{\rho/4}(1/2e_N\rho)$, where σ_2 is a constant independent of α. Therefore, $w^\alpha_{x_N}(0) \geq \sigma_3 > 0$ with σ_3 independent of α. Since $w^\alpha_{x_N}$ are $C^\beta(\overline{B_{1/2}^+})$ with norm controlled independently of α there holds that $w^\alpha_{x_N}(x) \geq \sigma > 0$ independent of α in B_ε^+, if ε is small independent of α. So that, $w^\alpha_{x_N}(x) \geq \sigma x_N$ in B_ε^+ and we have that $\psi^\alpha_{x_N}(x) \leq (1-\sigma)x_N$ in B_ε^+. Thus, the result would be true with $\gamma = 1-\sigma$ if we have that $w^\alpha(1/2e_N\rho) \geq c > 0$ independent of α.

$w^\alpha \geq \sigma_2 2\rho^{-1}x_N$ in $B_{\rho/2}^+$, then taking $\gamma = 1 - 2\rho^{-1}\sigma_2$ and $\varepsilon = \rho/2$, we obtain the desired result.

Step 4

Let as see that $w_\alpha(1/2e_N\rho) \geq c > 0$ where c is independent of α. Suppose, by contradiction that for a subsequence, $w_{\alpha_k}(1/2e_N\rho) \to 0$. We know that in B_ρ^+ $\mathcal{T}_\alpha w_\alpha = 0$. Then, applying Harnack inequality, we have that for any compact subset $K \subset\subset B_\rho^+$ we have that $w_\alpha \to 0$ uniformly in K. On the other hand, ψ_α are uniformly bounded in $C^\beta(\overline{B_1^+})$. Thus, there exists $\bar{w} \in C(\overline{B_1^+})$ such that, for a subsequence $w^{\alpha_k} \to \bar{w}$ in $C(\overline{B_1^+})$. Let

$$\mathcal{A} = \{x \in B_1^+ \ / \ \bar{w}(x) = 0\},$$

and suppose that, there exist a point $x_1 \in \partial A \cap B_1^+$, then as $w^\alpha \geq 0$ we have that $\bar{w}$ has a minimum at x_1. Therefore, $\nabla \bar{w}(x_1) = 0$. As $\nabla w^{\alpha_k} \to \nabla \bar{w}$ uniformly on compact subsets of B_1^+, we have that for some $\tau > 0$ independent of α_k, $|\nabla \psi_{\alpha_k}| \geq 1/2$ in $B_\tau(x_1)$. Thus, in this ball, w^{α_k} satisfy $\mathcal{T}_{\alpha_k} w^{\alpha_k} = 0$. We can apply Harnack inequality in $B_\tau(x_1)$ and, passing to the limit we obtain that $\bar{w} = 0$ in $B_{\tau/2}(x_1)$, which is a contradiction. Hence $\bar{w} = 0$ in $\overline{B_1^+}$. But on the other hand, we have $\bar{w} = x_N - \delta_0 x_N > 0$ on $\partial B_1 \cap \partial B_{r_0/2}(\bar{x})$, which is a contradiction.

□

Now we are ready to proceed with the proof of the theorem.

Proof of Theorem 1.5.1. The proof will be divided into several steps.

Step 1 Let $u_0(x) = \frac{u(Tx)}{T}$, with $T > 0$ to be chosen later.

Then, the function u_0 satisfies the same properties as u with the same constants L and λ_0.

Let $\beta = \frac{\lambda_0}{2^{N-1}} < 1$, then by properties (2) and (3) with $R = 1$ we have that there exists $x_0 \in B_1(0)$, with $x_{0,N} > \beta$ such that $u_0(x_0) = 0$. Since u_0 is Lipschitz, with constant L, we have that $u_0(x) \leq L|x - x_0|$. Thus, if we take $r_0 = \frac{\beta}{4}$, we have that $u_0(x) \leq \frac{L\beta}{4}$ for $|x - x_0| < r_0$. There holds that $x_N \geq \frac{3\beta}{4}$ in this ball. Hence, we have that

$$u_0(x) \leq \frac{Lx_N}{3} \qquad \text{on } \partial B_{R_1}^+ \cap B_{r_0}(x_0),$$

where $R_1 = |x_0| > \beta$.

By property (1), and by Lemma (1.2.45) u_0 is an $\mathcal{L}$- subsolution. By property 2 $0 \leq u_0(x) \leq Lx_N$.

Taking in Lemma 1.5.2 $\delta_0 = 1/3$, $\bar{x} = x_0$, $\alpha = L$ and $r = R_1$ we have that there exist $0 < \gamma_1 < 1$ and $0 < \varepsilon_1 \leq 1$, depending only on r_0 and x_0 such that

$$0 \leq u_0(x) \leq \gamma_1 L x_N \qquad \text{in } B_{R_1\varepsilon_1}^+. \tag{1.5.7}$$

Observe that since $x_{0,N} > \beta$ then γ_1 and ε_1 depend only on λ_0.

Now, take $u_1(x) = \frac{u_0(R_1\varepsilon_1 x)}{R_1\varepsilon_1}$. Then, again u_1 satisfies the properties of u with the same constants L and λ_0.

Therefore, there exists $x_1 \in B_1(0)$, with $x_{1,N} > \beta$ such that $u_1(x_1) = 0$. By (1), $u_1(x) \leq L|x - x_1|$. Thus, if we take $r_1 = \frac{\gamma_1\beta}{4}$, we have $u_1(x) \leq \frac{\gamma_1 L\beta}{4}$ for $|x - x_1| < r_1$. As $\gamma_1 \leq 1$, in that ball there holds that $x_N \geq \frac{3\beta}{4}$. Thus, we have that

$$u_1(x) \leq \frac{\gamma_1 L x_N}{3} \qquad \text{on } \partial B_{R_2}^+ \cap B_{r_1}(x_1),$$

where $R_2 = |x_1| > \beta$.

By property (1), u_1 is an $\mathcal{L}$- subsolution. And by (1.5.7), we have $0 \leq u_1(x) \leq \gamma_1 L x_N$ in B_1^+.

Taking in Lemma 1.5.2 $\delta_0 = 1/3$, $\bar{x} = x_1$, $\alpha = \gamma_1 L$ and $r = R_2$ we have that there exist $0 < \gamma_2 < 1$ and $0 < \varepsilon_2 \leq 1$, depending only on λ_0 such that $u_1(x) \leq \gamma_2 \gamma_1 L x_N$ in $B^+_{R_2 \varepsilon_2}$.

Inductively, we construct a sequence u_k such that, u_k satisfies the same hypotheses as u with the same constants L and λ_0 and such that,

$$0 \leq u_{k-1} \leq \alpha_k x_N \quad \text{in } B^+_{R_k \varepsilon_k} \tag{1.5.8}$$

where $\alpha_k = L \prod_{i=1}^k \gamma_i$, and $0 < \gamma_i, \varepsilon_i < 1$ depend only on λ_0. In the construction we have $u_k(x) = \frac{u_{k-1}(R_k \varepsilon_k x)}{R_k \varepsilon_k}$.

Therefore, for any $k \geq 1$

$$u_0 \leq \alpha_k x_N \quad \text{in } B^+_{\delta_k} \tag{1.5.9}$$

where, $\delta_k = \prod_{i=1}^k R_i \varepsilon_i$.

Step 2 Let us see that $\alpha_k \to 0$ when $k \to \infty$. Suppose by contradiction that this does not hold. Then, since α_k is decreasing, there exists $\alpha_0 > 0$ such that $\alpha_k \geq \alpha_0$ for all $k \geq 1$. We have that $\alpha_{k+1} = \gamma_{k+1} \alpha_k$, and $r_k = \frac{\beta}{4} \alpha_k \geq \frac{\beta}{4} \alpha_0$. Thus, in Lemma (1.5.2) we can take, for the function u_k, r_0 as $\frac{\beta}{4} \alpha_0$ and γ_0 the corresponding γ. We can think that γ_{k+1} was taken as the minimum over the γ's such that the conclusion of the lemma is satisfied. Therefore $\gamma_{k+1} \leq \gamma_0 < 1$ for all k. Then, $\alpha_k \leq L \gamma^k$ for all $k \geq 1$. Therefore $\alpha_k \to 0$; a contradiction.

Step 3 Now, we can prove that if $x_N > 0$ then $u(x_N) = 0$. Suppose that, there exists ξ with $\xi_N > 0$ such that $u(\xi) > 0$. Then, since $\alpha_k \to 0$, there exists $k \geq 1$ such that $u(\xi) > \alpha_k \xi_N$. Now, for this fixed k, take $T > |\xi| \beta^{-k} \prod_{i=1}^k \varepsilon_i$. Then, since $R_i > \beta$ we have that $|\xi| < T \delta_k$. Thus, if we take $\bar{\xi} = \frac{\xi}{T}$ we have that $u_0(\bar{\xi}) > \alpha_k \bar{\xi_N}$. But, on the other hand, by (1.5.9), since $|\bar{\xi}| < \delta_k$, we have that $u_0(\bar{\xi}) \leq \alpha_k \bar{\xi_N}$, which is a contradiction.

$\square$

As a remark we mention that with Lemma 1.5.2 we can also prove the asymptotic development of $\mathcal{L}-$ solutions.

Lemma 1.5.10. *Let u be Lipschitz continuous in $\overline{B_1^+}$, $u \geq 0$ in B_1^+, $\mathcal{L}$-solution in $\{u > 0\}$ and vanishing on $B_1^+ \cap \{x_N = 0\}$. Then, in B_1^+, u has the asymptotic development*

$$u(x) = \alpha x_N + o(|x|),$$

with $\alpha \geq 0$.

Proof. Let

$$\alpha_j = \inf\{l \;/\; u \leq l x_n \text{ in } B^+_{2^{-j}}\}.$$

Let $\alpha = \lim_{j \to \infty} \alpha_j$.

Given $\varepsilon_0 > 0$ there exists j_0 such that for $j \geq j_0$ we have $\alpha_j \leq \alpha + \varepsilon_0$. From here, we have $u(x) \leq (\alpha + \varepsilon_0)x_N$ in $B^+_{2^{-j}}$. So that,

$$u(x) \leq \alpha x_N + o(|x|) \text{ in } B^+_1.$$

If $\alpha = 0$ the result follows. Assume that $\alpha > 0$ and let us suppose that $u(x) \neq \alpha x_N + o(|x|)$. Then, there exist $x_k \to 0$ and $\bar{\delta} > 0$ such that

$$u(x_k) \leq \alpha x_{k,N} - \bar{\delta}|x_k|.$$

Let $r_k = |x_k|$ and $u_k(x) = r_k^{-1}u(r_k x)$. Then, there exists u_0 such that, for a subsequence that we still call u_k, $u_k \to u_0$ uniformly in $\overline{B^+_1}$ and

$$\begin{aligned} &u_k(\bar{x}_k) \leq \alpha \bar{x}_{k,N} - \bar{\delta} \\ &u_k(x) \leq (\alpha + \varepsilon_0)x_N \text{ in } B^+_1, \end{aligned}$$

where $\bar{x}_k = \frac{x_k}{r_k}$. Also, we can assume that $\bar{x}_k \to x_0$.

In fact, $u(x) \leq (\alpha + \varepsilon_0)x_N$ in $B^+_{2^{-j_0}}$. Therefore, $u_k(x) \leq (\alpha + \varepsilon_0)x_N$ in $B^+_{r_k^{-1}2^{-j_0}}$, and the estimate follows if k is big enough so that $r_k^{-1}2^{-j_0} \geq 1$.

If we take $\bar{\alpha} = \alpha + \varepsilon_0$ we have

$$\begin{cases} \mathcal{L}u_k \geq 0 & \text{in } B^+_1 \\ u_k = 0 & \text{on } \{x_N = 0\} \\ 0 \leq u_k \leq \bar{\alpha}x_N & \text{on } \partial B^+_1 \\ u_k \leq \delta_0 \bar{\alpha} x_N & \text{on } \partial B^+_1 \cap B_{\bar{r}}(\bar{x}), \end{cases}$$

for some $0 < \delta_0 < 1$, $\bar{x} \in \partial B^+_1$, $\bar{x}_N > 0$ and some small $\bar{r} > 0$.

In fact, as u_k are continuous with uniform modulus of continuity, we have

$$u_k(x_0) \leq \alpha x_{0,N} - \frac{\bar{\delta}}{2}, \text{ if } k \geq \bar{k}.$$

Moreover, there exists $r_0 > 0$ such that $u_k(x) \leq \alpha x_N - \frac{\bar{\delta}}{4}$ in $B_{2r_0}(x_0)$. If $x_{0,N} > 0$ we take $\bar{x} = x_0$, if not, we take $\bar{x} \in B_{2r_0}(x_0)$ with $\bar{x}_N > 0$ and

$$u_k(x) \leq \alpha x_N - \frac{\bar{\delta}}{4}, \text{ in } B_{r_0}(\bar{x}) \subset\subset \{x_N > 0\}.$$

As $B_{r_0}(\bar{x}) \subset\subset \{x_N > 0\}$, there exists δ_0 such that $\alpha x_N - \frac{\bar{\delta}}{4} \leq \delta_0 \alpha x_N \leq \delta_0 \bar{\alpha} x_N$ in $B_{\bar{r}}(\bar{x})$ for some small $\bar{r}$, and the claim follows.

Now, by Lemma 1.5.2, there exist $0 < \gamma < 1$, $\varepsilon > 0$ independent of ε_0 and k, such that $u_k(x) \leq \gamma(\alpha + \varepsilon_0)x_N$ in B^+_ε. As γ and ε are independent of k and ε_0, taking $\varepsilon_0 \to 0$, we have

$$u_k(x) \leq \gamma \alpha x_N \text{ in } B^+_\varepsilon.$$

So that,

$$u(x) \leq \gamma \alpha x_N \text{ in } B^+_{r_k \varepsilon}.$$

Now if j is big enough we have $\gamma\alpha < \alpha_j$ and $2^{-j} \leq r_k\varepsilon$. We get a contradiction to the definition of α_j. Therefore,

$$u(x) = \alpha x_N + o(|x|),$$

as we wanted to prove. $\square$

6. Blow up limits

We give here the definition of blow up sequence, and give some properties of the blow up limits.

DEFINITION 1.6.11. *Let $D \subset \mathbb{R}^N$. We say that u satisfies hypothesis (H1) if,*

$$\begin{cases} u \text{ Lipschitz in } D \text{ with constant } L > 0 \\ u \geq 0 \text{ in } D \\ \mathcal{L}u = 0 \text{ in } D \cap \{u > 0\} \end{cases}$$

$$\text{(H1)} \qquad \begin{cases} \text{Given } 0 < \kappa < 1, \text{ there exist } c_\kappa \text{ and } r_\kappa \text{ such that for} \\ B_r(x_0) \text{ with } 0 < r < r_\kappa \text{ we have} \\ \frac{1}{r}\fint_{B_r(x_0)} u \leq c_\kappa \Longrightarrow u \equiv 0 \text{ in } B_{\kappa r}(x_0) \end{cases}$$

$$\begin{cases} \text{There exist constants } r_0 \text{ and } 0 < \lambda_1 < \lambda_2 < 1 \text{ such that} \\ \text{for } B_r(x_0) \subset D \text{ with } x_0 \in \partial\{u > 0\} \text{ and } 0 < r < r_0 \\ \lambda_1 \leq \dfrac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} \leq \lambda_2 \end{cases}$$

DEFINITION 1.6.12. *Let $B_{\rho_k}(x_k) \subset \Omega$ be a sequence of balls with $\rho_k \to 0$, $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$. Let*

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

We call u_k a blow-up sequence with respect to $B_{\rho_k}(x_k)$.

Since u is locally Lipschitz continuous, there exists a blow-up limit $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that, for a subsequence,

$$\begin{aligned} &u_k \to u_0 \quad \text{in} \quad C^\alpha_{\text{loc}}(\mathbb{R}^N) \quad \text{for every} \quad 0 < \alpha < 1, \\ &\nabla u_k \to \nabla u_0 \quad * -\text{weakly in} \quad L^\infty_{\text{loc}}(\mathbb{R}^N), \end{aligned}$$

and u_0 is Lipschitz in $I\!R^N$ with constant L.

LEMMA 1.6.13. *If u satisfies hypothesis (H1), then*

1. *$\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ locally in Hausdorff distance,*
2. *$\chi_{\{u_k > 0\}} \to \chi_{\{u_0 > 0\}}$ in $L^1_{\text{loc}}(\mathbb{R}^N)$,*
3. *If $D \subset\subset \{u_0 > 0\} \cup \{u_0 = 0\}^\circ$, then $\nabla u_k \to \nabla u_0$ uniformly in D,*

4. $\nabla u_k \to \nabla u_0$ *a.e in* $\mathbb{R}^N$,
5. *If* $x_k \in \partial\{u > 0\}$, *then* $0 \in \partial\{u_0 > 0\}$
6. $\mathcal{L}u_0 = 0$ *in* $\{u_0 > 0\}$.
7. *There exist* $0 < \lambda < 1$ *such that* $\frac{|B_r(x_0)\cap\{u_0=0\}|}{|B_r(x_0)|} > \lambda$ $\forall R > 0$, *and* $\forall y_0 \in \partial\{u_0 > 0\}$.

Proof. Is clear that $u_0 \geq 0$. If $B_r \subset\subset \{u_0 > 0\}$, we will have that $B_r \subset\subset \{u_k > 0\}$ for k large and then $\mathcal{L}u_k = 0$ in B_r. Then there exists a constant $C = C(g_0, \delta, N, \{u_0 > 0\}, r, \|u_k\|_{L^\infty(B_r)})$ such that, $\|u_k\|_{C^{1,\beta}(\overline{B_r})} \leq C$, but $|u_k(x)| \leq Lr$, and then the constant C is independent of k. Therefore for a subsequence $\nabla u_k \rightrightarrows \nabla u_0$ in B_r therefore $\mathcal{L}u_0 = 0$ in B_r. Then (6) is proved.

Let us see (1). Observe that if $B_r(y) \cap \partial\{u_0 > 0\} = \emptyset$ and $u_0 = 0$ in $B_r(y)$, then the u_k are uniformly small in $B_r(y)$ for k large and by the second property in (H1) $u_k = 0$ in $B_{r/2}(y)$. If $u_0 > 0$ in $B_r(y)$ then $u_k > 0$ in $B_{r/2}(y)$, therefore $B_{r/2}(y) \cap \partial\{u_k > 0\} = \emptyset$ for k large.

On the other hand, if $B_r(y) \cap \partial\{u_k > 0\} = \emptyset$ for k large, and for a subsequence $u_k = 0$ in $B_r(y)$, then $u_0 = 0$ in $B_r(y)$. If not, we will have $u_k > 0$ in $B_r(y)$ for k large and then $\mathcal{L}u_0 = 0$ in $B_r(y)$, then by the strong maximum principle (see Theorem 1.2.17), $u_0 = 0$ in $B_r(y)$ or $u_0 > 0$ in $B_r(y)$. In both cases $B_r(y) \cap \partial\{u_0 > 0\} = \emptyset$.

Let us see (2). By the second property in (H1) we have that, given a compact set K, $|K \cap \partial\{u_k > 0\}| = 0$ for k large, and then by (1) we have that

$$|\partial\{u_0 > 0\}| = 0. \tag{1.6.14}$$

On the other hand, if we argue as before, for R and r fixed we have, for k large,

$$\int_{B_R} |\chi_{\{u_k>0\}} - \chi_{\{u>0\}}| \leq |\{x \in B_R / d(x, \partial\{u_0 > 0\}) \leq r\}|$$

and then by (1.6.14) the result follows.

Using property 2 of (H1) we derive (3). (4) follows by (2) and (1.6.14).

Finally we prove (7). Note that, by (1) given $y_0 \in \partial\{u_0 > 0\}$, there exists $y_k \in \partial\{u_k > 0\}$ such that $y_k \to y_0$. Therefore, by property 3 in (H1), if we fix $R > 0$, we will have for k large

$$\lambda_1 \leq \frac{|B_R(y_k) \cap \{u_k > 0\}|}{|B_R(y_k)|} \leq \lambda_2$$

and applying (2) we obtain (7). □

7. Schwartz symmetrization

Definition 1.7.15. *Let* Ω *a bounded domain. We define the symmetrized domain* Ω^* *as the ball* $\{x/|x| < \rho\}$ *with the same volume as* Ω. *Let* u *be a function define on the bounded domain* $D \subset \mathbb{R}^N$ *and*

$$D(\mu) := \{x \in D / u(x) \geq \mu\}.$$

We define $u^* : D^* \to \mathbb{R}$ *by,*

$$u^*(x) := \sup\{\mu / x \in D(\mu)^*\} \qquad \textit{if } x \in D^*.$$

We call the function u^* *the Schwartz symmetrized of* u.

PROPOSITION 1.7.16.

1. u^* *is radially symmetric, i.e,* $u^*(x) = u^*(|x|)$, *and* $u^*(|x|)$ *is a nondecreasing function of* $|x|$.
2. $|\{x \in D / u(x) \geq \mu\}| = |\{x \in D^* / u^*(x) \geq \mu\}|$, *for all* $\mu \in \mathbb{R}$.
3. *If* Ω *is smooth, i.e,* $\partial\Omega$ *piecewise analytic,* u *is non negative, and* $u : \Omega \to \mathbb{R}_0^+$ *is in* $W_0^{1,G}(\Omega)$, *and* $G : \mathbb{R}_0^+ \to \mathbb{R}$ *is monotone nondecreasing and convex, then*

$$\int_\Omega G(|\nabla u|)\, dx \geq \int_{\Omega^*} G(|\nabla u^*|)\, dx. \tag{1.7.17}$$

4. *If in addition* G *is increasing and strictly convex then equality holds in* (1.7.17) *if and only if* $u = u^*$.

CHAPTER 2

A minimization problem and weak solutions

In the first part of this chapter, we will study the following minimization problem. For Ω a smooth domain in $\mathbb{R}^N$ and φ_0 a non negative function with $\varphi_0 \in L^\infty(\Omega)$ and $\int_\Omega G(|\nabla \varphi_0|)\,dx < \infty$, we consider the problem, of minimizing the functional

$$\mathcal{J}(v) = \int_\Omega G(|\nabla v|) + \lambda \chi_{\{v>0\}}\,dx \tag{2.0.1}$$

in the class of functions

$$\mathcal{K} = \Big\{ v \in W^{1,G}(\Omega) : \, v = \varphi_0 \text{ on } \partial\Omega \Big\}.$$

Here $G' = g$ and we will assume all the time, that g satisfies condition (0.0.2).

1. Existence of minimizers

In this section we look for minimizers of the functional $\mathcal{J}$. We begin by discussing the existence of extremals. Next, we prove that any minimizer is a subsolution to the equation $\mathcal{L}u = 0$ and finally, we prove that $0 \le u \le \sup \varphi_0$.

Theorem 2.1.1. *If $\mathcal{J}(\varphi_0) < \infty$, then there exists a minimizer of $\mathcal{J}$.*

Proof. The proof of existence is standard. We write it here for the reader's convenience and in order to show how the Orlicz spaces and the condition (0.0.2) on the function G come into play.

Take a minimizing sequence $(u_n) \subset \mathcal{K}$, then $\mathcal{J}(u_n)$ is bounded, so $\int_\Omega G(|\nabla u_n|)$ and $|\{u_n > 0\}|$ are bounded. As $u_n = \varphi_0$ in $\partial\Omega$, we have by Lemma 1.1.5 that $\|\nabla u_n - \nabla \varphi_0\|_G \le C$ and by Lemma 1.2.10 we also have $\|u_n - \varphi_0\|_G \le C$. Therefore, by Theorem 1.1.8 there exists a subsequence (that we still call u_n) and a function $u_0 \in W^{1,G}(\Omega)$ such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W^{1,G}(\Omega),$$

and by Theorem 1.1.9

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W^{1,\delta+1}(\Omega),$$

and by the compactness of the immersions $W^{1,\delta+1}(\Omega) \hookrightarrow L^{\delta+1}(\Omega)$ and $W^{1,\delta+1}(\Omega) \hookrightarrow L^{\delta+1}(\partial\Omega)$ we have that,

$$\begin{aligned} u_n \to u_0 \quad & \text{a.e. } \Omega. \\ u_0 = \varphi_0 \quad & \text{on} \quad \partial\Omega, \end{aligned}$$

Thus,

$$|\{u_0 > 0\}| \leq \liminf_{n\to\infty} |\{u_n > 0\}| \quad \text{ and}$$

$$\int_\Omega G(|\nabla u_0|)\,dx \leq \liminf_{n\to\infty} \int_\Omega G(|\nabla u_n|)\,dx.$$

In fact,

$$\int_\Omega G(|\nabla u_n|)\,dx \geq \int_\Omega G(|\nabla u_0|)\,dx + \int_\Omega g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\cdot(\nabla u_n - \nabla u_0)\,dx. \tag{2.1.2}$$

Recall that ∇u_n converges weakly to ∇u_0 in L^G. Now, since by property $(\widetilde{g}4)$

$$\widetilde{G}\big(g(|\nabla u_0|)\big) \leq CG(|\nabla u_0|),$$

there holds that $g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|} \in L^{\widetilde{G}}$ so that, by Theorem 1.1.8 and passing to the limit in (2.1.2) we get

$$\liminf_{n\to\infty} \int_\Omega G(|\nabla u_n|)\,dx \geq \int_\Omega G(|\nabla u_0|)\,dx.$$

Hence $u_0 \in \mathcal{K}$ and

$$\mathcal{J}(u_0) \leq \liminf_{n\to\infty} \mathcal{J}(u_n) = \inf_{v\in\mathcal{K}} \mathcal{J}(v).$$

Therefore, u_0 is a minimizer of $\mathcal{J}$ in $\mathcal{K}$. □

Lemma 2.1.3. *Let u be a minimizer of $\mathcal{J}$. Then, u is an $\mathcal{L}-$ subsolution.*

Proof. Let $\varepsilon > 0$ and $0 \leq \xi \in C_0^\infty$. Using the minimality of u and the convexity of G we have

$$\begin{aligned} 0 &\leq \frac{1}{\varepsilon}(\mathcal{J}(u-\varepsilon\xi) - \mathcal{J}(u)) \leq \frac{1}{\varepsilon}\int_\Omega G(|\nabla u - \varepsilon\nabla\xi|) - G(|\nabla u|)\,dx \\ &\leq \int_\Omega -g(|\nabla u - \varepsilon\nabla\xi|)\frac{\nabla u - \varepsilon\nabla\xi}{|\nabla u - \varepsilon\nabla\xi|}\nabla\xi\,dx \end{aligned}$$

and if we take $\varepsilon \to 0$ we obtain

$$0 \leq \int_\Omega -g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\xi\,dx$$

□

Lemma 2.1.4. *Let u be a minimizer of $\mathcal{J}$. Then $0 \leq u \leq \sup_\Omega \varphi_0$.*

Proof. Let $M = \sup \varphi_0$, $\varepsilon > 0$ and $v = \min(M - u, 0)$, then

$$
\begin{aligned}
0 &\le \frac{1}{\varepsilon}\Big(\mathcal{J}(u+\varepsilon v) - \mathcal{J}(u)\Big)\\
&= \frac{1}{\varepsilon}\Big(\int_\Omega G(|\nabla u + \varepsilon \nabla v|) - G(|\nabla u|) + \lambda \chi_{\{u+\varepsilon v>0\}} - \lambda \chi_{\{u>0\}}\, dx\Big)\\
&\le \frac{1}{\varepsilon}\Big(\int_\Omega \Big(G(|\nabla u + \varepsilon \nabla v|) - G(|\nabla u|)\Big)\, dx\Big) \le \int_\Omega g(|\nabla u + \varepsilon \nabla v|)\frac{\nabla u + \varepsilon \nabla v}{|\nabla u + \varepsilon \nabla v|}\nabla v\, dx
\end{aligned}
$$

where in the last inequality we are using the convexity of G.

Now, takeing $\varepsilon \to 0$, using the definition of v and $(g3)$ we have that,

$$
\begin{aligned}
0 \le \int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla v\, dx &= -\int_{\{u>M\}} g(|\nabla u|)|\nabla u|\, dx \le -\int_{\{u>M\}} G(|\nabla u|)\, dx\\
&= -\int_{\{u>M\}} G(|\nabla v|)\, dx,
\end{aligned}
$$

therefore $\nabla v = 0$ in Ω and as $v = 0$ on $\partial\Omega$ we have that $v = 0$ in Ω and then $u \le M$.

To prove that $u \ge 0$ we argue in a similar way. Take $v = \min(u, 0)$, then we have that,

$$
0 \le \frac{1}{\varepsilon}\mathcal{J}(u - \varepsilon v) - \mathcal{J}(u) \le -\int_\Omega g(|\nabla u - \varepsilon \nabla v|)\frac{\nabla u - \varepsilon \nabla v}{|\nabla u - \varepsilon \nabla v|}\nabla v\, dx.
$$

Therefore taking $\varepsilon \to 0$, using the definition of v and $(g3)$ we have that

$$
0 \ge \int_\Omega G(|\nabla v|)\, dx.
$$

As in the first part, we conclude that $u \ge 0$. □

2. Lipschitz continuity

In this section we study the regularity of the minimizers of $\mathcal{J}$. The main result is the local Lipschitz continuity of a minimizer. This result, together with the rescaling invariance of the minimization problem, is a key step in the analysis. Once this regularity is proven, a blow up process (passage to the limit in linear rescalings) at points of $\partial\{u > 0\}$ allows to simplify the analysis by assuming that u is a plane solution.

As a first step, we prove that minimizers are Hölder continuous. We use ideas of [**10**], here all the properties of the function G come into play.

Theorem 2.2.1. *For every $0 < \alpha < 1$, any minimizer u is in $C^\alpha(\Omega)$ and for $\Omega' \subset\subset \Omega$, $\|u\|_{C^\alpha(\Omega')} \le C$, where $C = C(g_0, \delta, \lambda, \|u\|_\infty, \alpha, dist(\Omega', \partial\Omega), G(1))$.*

Proof. We will see that, for every $0 < \alpha < 1$ and $\Omega' \subset\subset \Omega$ there exists ρ_0 such that if $y \in \Omega'$, $0 < \rho < \rho_0$ we have that

$$
\frac{1}{\rho^N}\int_{B_\rho(y)} G(|\nabla u|)\, dx \le C\rho^{\alpha-1},
$$

for a constant $C(N, \delta, g_0, \|u\|_{L^\infty(\Omega)}, \rho_0, G(1))$.

In fact, let $r > 0$ such that, $B_r(y) \subset \Omega$. We can suppose that $y = 0$. Then if v is the solution of

$$\mathcal{L}v = 0 \quad \text{in } B_r, \qquad v - u \in W_0^{1,G}(B_r),$$

we have, therefore by Theorem 1.2.38 that

$$\int_{B_r} (G(|\nabla u|) - G(|\nabla v|))\,dx \geq C\Big(\int_{A_2} G(|\nabla u - \nabla v|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx\Big), \tag{2.2.2}$$

where

$$A_1 = \{x \in B_r : |\nabla u - \nabla v| \leq 2|\nabla u|\}, \quad A_2 = \{x \in B_r : |\nabla u - \nabla v| > 2|\nabla u|\},$$

and $C = C(g_0, \delta)$.

On the other hand, by the minimality of u, we have

$$\int_{B_r} (G(|\nabla u|) - G(|\nabla v|))\,dx \leq \lambda(|\{v > 0 \cap B_r\} - |\{u > 0 \cap B_r\}|) \leq \lambda r^N C_N. \tag{2.2.3}$$

Combining (2.2.2) and (2.2.3) we obtain

$$\int_{A_2} G(|\nabla u - \nabla v|)\,dx \leq C\lambda r^N \tag{2.2.4}$$

$$\int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx \leq C\lambda r^N \tag{2.2.5}$$

Let $\varepsilon > 0$ and suppose that $r^\varepsilon \leq 1/2$. Then, using (g3), Hölder's inequality, the definition of A_1 and (2.2.5) we obtain,

$$\begin{aligned}\int_{A_1 \cap B_{r^{1+\varepsilon}}} & G(|\nabla u - \nabla v|)\,dx \\ &\leq C\Big(\int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx\Big)^{1/2}\Big(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\Big)^{1/2} \\ &\leq C\lambda^{1/2} r^{N/2}\Big(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\Big)^{1/2}.\end{aligned} \tag{2.2.6}$$

Therefore, by (2.2.4) and (2.2.6), we get,

$$\int_{B_{r^{1+\varepsilon}}} G(|\nabla u - \nabla v|)\,dx \leq C\lambda^{1/2}\Big(\lambda^{1/2} r^N + r^{N/2}\Big(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\Big)^{1/2}\Big). \tag{2.2.7}$$

On the other hand by property (3) of Lemma 1.2.18 we have for every $\beta \in (0, N)$, that there exists a constant $C = C(\delta, g_0, N, \beta, \|v\|_{L^\infty(B_r)})$ such that

$$\int_{B_{r/2}} G(|\nabla v|)\,dx \leq Cr^\beta. \tag{2.2.8}$$

By the maximum principle we have,

$$\|v\|_{L^\infty(B_r)} \le \|v\|_{L^\infty(\partial B_r)} = \|u\|_{L^\infty(\partial B_r)} \le \|u\|_{L^\infty(B_r)} \le \|v\|_{L^\infty(B_r)} \tag{2.2.9}$$

where in the last inequality we are using Lemma 1.2.35. Then $\|v\|_{L^\infty(B_r)} = \|u\|_{L^\infty(B_r)}$. This means that the constant C depends on δ, g_0, N, β and $\|u\|_{L^\infty(B_r)}$.

By (G2) we have, $G(|\nabla u|) \le C(G(|\nabla u - \nabla v|) + G(|\nabla v|))$. Therefore by (2.2.7) and (2.2.8), and for $r \le 1$ we have,

$$\begin{aligned}\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx &\le C\Big(r^\beta(1+\lambda) + \lambda^{1/2} r^{N/2}\Big(\int_{B_r^{1+\varepsilon}} G(|\nabla u|)\,dx\Big)^{1/2}\Big)\\ &\le C\Big(r^\beta(1+\lambda) + r^{\beta/2}(1+\lambda)^{1/2}\Big(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\Big)^{1/2}\Big).\end{aligned}$$

If we call $A = \int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx$, we have

$$\begin{aligned}A &\le C\Big((1+\lambda)r^\beta + (1+\lambda)^{1/2} r^{\beta/2} A^{1/2}\Big) \le C\Big((1+\lambda)r^\beta + 2(1+\lambda)^{1/2} r^{\beta/2} A^{1/2}\Big)\\ &= C\Big(\big(r^{\beta/2}(1+\lambda)^{1/2} + A^{1/2}\big)^2 - A\Big),\end{aligned}$$

therefore

$$\begin{aligned}(C+1)A &\le C\Big(r^{\beta/2}(1+\lambda)^{1/2} + A^{1/2}\Big)^2\\ \Rightarrow (C+1)^{1/2}A^{1/2} &\le C^{1/2}\Big(r^{\beta/2}(1+\lambda)^{1/2} + A^{1/2}\Big)\\ \Rightarrow ((C+1)^{1/2} - C^{1/2})A^{1/2} &\le C^{1/2} r^{\beta/2}(1+\lambda)^{1/2}.\end{aligned}$$

Thus, we have the inequality

$$\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx \le ((C+1)^{1/2} + C^{1/2})^2 C(1+\lambda) r^\beta \tag{2.2.10}$$

Let now, $0 < \alpha < 1$, and take $\varepsilon > 0$ such that $\beta := (1+\varepsilon)\big(N - (1-\alpha)\big) < N$. Take $\rho_0 = \big(\frac{1}{2}\big)^{1+1/\varepsilon}$. Then, if $0 < \rho < \rho_0$, taking $r = \rho^{1/(1+\varepsilon)}$, we have that $r^\varepsilon < 1/2$. And therefore replacing in (2.2.10) we have,

$$\int_{B_\rho} G(|\nabla u|) \le ((C+1)^{1/2} + C^{1/2}) C(1+\lambda)\rho^{N-(1-\alpha)} \tag{2.2.11}$$

and by Lemma 1.2.11 we conclude that for all $0 < \alpha < 1$, $u \in C^\alpha(B_\rho)$ for $0 < \rho \le \rho_0$ and $\|u\|_{C^\alpha(B_\rho)} \le \overline{C}$ where $\overline{C} = \overline{C}(N, \alpha, g_0, \delta, \lambda, \rho_0, \|u\|_{L^\infty(\Omega)})$. □

We then have that u is continuous. Therefore, $\{u > 0\}$ is open. We can prove the following property for minimizers.

Lemma 2.2.12. *Let u be a minimizer of $\mathcal{J}$. Then u is an $\mathcal{L}$–solution in $\{u > 0\}$.*

PROOF. Let $B \subset \{u > 0\}$ and v such that

$$\begin{cases} \mathcal{L}v = 0 & \text{in } B, \\ v = u & \text{in } B^c. \end{cases}$$

By the comparison principle we have that $v \geq u$ in B. Thus,

$$0 \geq \int_\Omega G(|\nabla u|) - G(|\nabla v|)\, dx + \lambda|\{u > 0\}| - \lambda|\{v > 0\}| = \int_\Omega G(|\nabla u|) - G(|\nabla v|)\, dx$$
$$\geq C\Big(\int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\, dx + \int_{A_2} G(|\nabla u - \nabla v|)\, dx\Big)$$

where we are using Theorem 1.2.38 and A_1, A_2, and F are as define therein.

Therefore

$$\int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\, dx = 0.$$

Thus, $F(|\nabla u|)|\nabla u - \nabla v|^2 = 0$ in A_1 and, by the definition of A_1, we conclude that $|\nabla u - \nabla v| = 0$ in this set.

On the other hand, we also have

$$\int_{A_2} G(|\nabla u - \nabla v|)\, dx = 0$$

so that $|\nabla u - \nabla v| = 0$ everywhere in B.

Hence, as $u = v$ on ∂B we have that $u = v$. Thus, $\mathcal{L}u = 0$ in B.

□

In order to get the Lipschitz continuity we first prove the following estimate for minimizers.

LEMMA 2.2.13. *For all $x \in \Omega$, with $5d(x) < d(x, \partial\Omega)$ we have $u(x) \leq Cd(x)$, where $d(x) = dist(x, \{u = 0\})$. The constant C depends only on N, δ, g_0 and λ.*

To prove Lemma 2.2.13 it is enough to prove the following lemma. In this proof it is essential that the class of functions G satisfying condition (0.0.2) is closed under the rescaling

$$G_s(t) := \frac{G(st)}{sg(s)}.$$

LEMMA 2.2.14. *If u is a minimizer in B_1 with $u(0) = 0$, there exists a constant C such that $\|u\|_{L^\infty(B_{1/4})} \leq C$, and C depends only on N, λ, δ and g_0.*

PROOF. Suppose that there exists a sequence $u_k \in \mathcal{K}$ of minimizers in $B_1(0)$ such that

$$u_k(0) = 0 \qquad \text{and} \qquad \max_{\overline{B}_{1/4}} u_k(x) > k.$$

Let $d_k(x) = \operatorname{dist}(x, \{u_k = 0\})$ and $\mathcal{O}_k = \left\{x \in B_1 : d_k(x) \le \dfrac{1-|x|}{3}\right\}$. Since $u_k(0) = 0$ there holds that $\overline{B}_{1/4} \subset \mathcal{O}_k$, therefore

$$m_k := \sup_{\mathcal{O}_k}(1-|x|)u_k(x) \ge \max_{\overline{B}_{1/4}}(1-|x|)u_k(x) \ge \frac{3}{4}\max_{\overline{B}_{1/4}} u_k(x) > \frac{3}{4}k.$$

For each fixed k, u_k is bounded. Thus $(1-|x|)u_k(x) \to 0$ when $|x| \to 1$ which means that there exists $x_k \in \mathcal{O}_k$ such that $(1-|x_k|)u_k(x_k) = \sup_{\mathcal{O}_k}(1-|x|)u_k(x)$, and then

$$u_k(x_k) = \frac{m_k}{1-|x_k|} \ge m_k > \frac{3}{4}k.$$

Let $\delta_k := d_k(x_k) \le \frac{1-|x_k|}{3}$ and $y_k \in \partial\{u_k > 0\} \cap B_1$ such that $|y_k - x_k| = \delta_k$. Then,

(1) $B_{2\delta_k}(y_k) \subset B_1$,

since if $y \in B_{2\delta_k}(y_k) \Rightarrow |y| < 3\delta_k + |x_k| \le 1$,

(2) $B_{\frac{\delta_k}{2}}(y_k) \subset \mathcal{O}_k$,
since if $y \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow |y| \le \dfrac{3}{2}\delta_k + |x_k| \le 1 - \dfrac{3}{2}\delta_k \Rightarrow d_k(y) \le \dfrac{\delta_k}{2} \le \dfrac{1-|y|}{3}$ and

(3) if $z \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow 1-|z| \ge 1-|x_k| - |x_k - z| \ge 1 - |x_k| - \dfrac{3}{2}\delta_k \ge \dfrac{1-|x_k|}{2}$.

By (2) we have

$$\max_{\mathcal{O}_k}(1-|x|)u_k(x) \ge \max_{B_{\frac{\delta_k}{2}}(y_k)}(1-|x|)u_k(x) \ge \max_{B_{\frac{\delta_k}{2}}(y_k)} \frac{(1-|x_k|)}{2}u_k(x),$$

where in the last inequality we are using (3). Then,

$$2u_k(x_k) \ge \max_{B_{\frac{\delta_k}{2}}(y_k)} u_k(x). \tag{2.2.15}$$

As $B_{\delta_k}(x_k) \subset \{u_k > 0\}$ there holds that $\mathcal{L}u_k = 0$ in $B_{\delta_k}(x_k)$, and by Harnack inequality (Theorem 1.2.16) we have

$$\min_{B_{\frac{3}{4}\delta_k}(x_k)} u_k(x) \ge c u_k(x_k). \tag{2.2.16}$$

As $\overline{B_{\frac{3}{4}\delta_k}}(x_k) \cap \overline{B_{\frac{\delta_k}{4}}}(y_k) \ne \emptyset$ we have by (2.2.16)

$$\max_{B_{\frac{\delta_k}{4}}(y_k)} u_k(x) \ge c u_k(x_k). \tag{2.2.17}$$

Let $w_k(x) = \dfrac{u_k(y_k + \frac{\delta_k}{2}x)}{u_k(x_k)}$. Then, $w_k(0) = 0$ and, by (2.2.15) and (2.2.17) we have,

$$\max_{B_1} w_k \le 2 \qquad \max_{B_{1/2}} w_k \ge c > 0. \tag{2.2.18}$$

Let now

$$J_k(w) = \int_{B_1} \frac{G(|\nabla w| c_k)}{g(c_k)c_k}\,dx + \frac{\lambda}{g(c_k)c_k}\int_{B_1} \chi_{\{w>0\}}(x)\,dx$$

where $c_k = \frac{2u_k(x_k)}{\delta_k}$ so that $c_k \to \infty$.

Let us prove, that w_k is a minimizer of J_k. In fact, for any $v \in W^{1,G}(B_1)$ with $v = w_k$ on ∂B_1, define $v_k(y) = v\big(\dfrac{y-y_k}{\delta_k/2}\big)u_k(x_k)$. Thus, $v_k = u_k$ on $\partial B_{\delta_k/2}(y_k)$. Then,

$$\begin{aligned}
J_k(w_k) &= \frac{2^N}{\delta_k^N}\Big(\int_{B_{\frac{\delta_k}{2}}(y_k)} \frac{G(|\nabla u_k|)}{g(c_k)c_k}\,dy + \frac{\lambda}{g(c_k)c_k}\int_{B_{\frac{\delta_k}{2}}} \chi_{\{u_k>0\}}(y)\,dy\Big)\\
&\le \frac{2^N}{\delta_k^N}\Big(\int_{B_{\frac{\delta_k}{2}}(y_k)} \frac{G(|\nabla v_k|)}{g(c_k)c_k}\,dy + \frac{\lambda}{g(c_k)c_k}\int_{B_{\frac{\delta_k}{2}}(y_k)} \chi_{\{v_k>0\}}(y)\,dy\Big)\\
&= \int_{B_1} \frac{G(|\nabla v|c_k)}{g(c_k)c_k}\,dx + \frac{\lambda}{g(c_k)c_k}\int_{B_1} \chi_{\{v>0\}}(y)\,dx = J_k(v).
\end{aligned}$$

Let $g_k(t) := \frac{g(tc_k)}{g(c_k)}$, where the primitive of g_k is $G_k(t) = \frac{G(tc_k)}{g(c_k)c_k}$ and $\lambda_k = \frac{\lambda}{g(c_k)c_k} \to 0$. Then,

$$J_k(w) = \int_{B_1} G_k(|\nabla w|)\,dx + \lambda_k \int_{B_1} \chi_{\{w>0\}}(x)\,dx.$$

Observe that for all k, g_k satisfies the inequality (0.0.2), with the same constants δ and g_0. In fact,

$$\frac{g_k'(t)t}{g_k(t)} = \frac{g'(c_kt)c_kt}{g_k(c_kt)},$$

and then by (0.0.2) applied to tc_k we have the desired inequality.

Let us take $v_k \in W^{1,G}(B_{3/4})$ such that,

$$\begin{aligned}
\mathcal{L}_k v_k &= 0 \qquad \text{in } B_{3/4}\\
v_k &= w_k \qquad \text{on } \partial B_{3/4}
\end{aligned} \tag{2.2.19}$$

where $\mathcal{L}_k$ is the operator associated to g_k. By (2.2.7), (2.2.11) and the fact that $\lambda_k \to 0$, we have that

$$\int_{B_{3/4}} G_k(|\nabla w_k - \nabla v_k|)\,dx \le C\lambda_k^{1/2},$$

where C depends on δ, g_0, N and $\|w_k\|_{L^\infty(B_1)}$. We have used that $\{\lambda_k\}$ is bounded when applying (2.2.7) and (2.2.11). We also have, by (2.2.18) that C depends only on δ, g_0 and N. On the other hand, by (G1) and (g3) we have

$$G_k(t) = \frac{G(tc_k)}{g(c_k)c_k} \ge \frac{G(c_k)}{(1+g_0)g(c_k)c_k}\min\{t^{g_0+1}, t^{\delta+1}\} \ge \frac{1}{(1+g_0)^2}\min\{t^{g_0+1}, t^{\delta+1}\}.$$

Therefore,

$$C\lambda_k^{1/2} \geq \int_{B_{3/4}} G_k(|\nabla w_k - \nabla v_k|)\,dx \geq \int_{B_{3/4}\cap\{|\nabla w_k-\nabla v_k|<1\}} \frac{|\nabla w_k - \nabla v_k|^{g_0+1}}{(1+g_0)^2}\,dx + \int_{B_{3/4}\cap\{|\nabla w_k-\nabla v_k|\geq 1\}} \frac{|\nabla w_k - \nabla v_k|^{\delta+1}}{(1+g_0)^2}\,dx.$$

Hence

$$\begin{aligned} A_k &:= \int_{B_{3/4}\cap\{|\nabla w_k-\nabla v_k|\geq 1\}} |\nabla w_k - \nabla v_k|^{\delta+1}\,dx \to 0 \quad \text{and} \\ B_k &:= \int_{B_{3/4}\cap\{|\nabla w_k-\nabla v_k|<1\}} |\nabla w_k - \nabla v_k|^{g_0+1}\,dx \to 0. \end{aligned} \tag{2.2.20}$$

By Hölder inequality and (2.2.20) we have,

$$C_k := \int_{B_{3/4}\cap\{|\nabla w_k-\nabla v_k|<1\}} |\nabla w_k - \nabla v_k|^{\delta+1}\,dx \leq B_k^{\frac{\delta+1}{g_0+1}} |B_{3/4}|^{\frac{g_0-\delta}{g_0+\delta}} \to 0,$$

therefore,

$$\int_{B_{3/4}} |\nabla w_k - \nabla v_k|^{\delta+1}\,dx = A_k + C_k \to 0. \tag{2.2.21}$$

As $w_k = v_k$ on $\partial B_{3/4}$ then $p_k = w_k - v_k \in W_0^{1,\delta+1}(B_{3/4})$ and by (2.2.21) we have

$$p_k \to 0 \quad \text{in } W_0^{1,\delta+1}(B_{3/4}). \tag{2.2.22}$$

On the other hand by Theorem 2.2.1 we have that,

(2.2.23)

$$\|w_k\|_{C^\alpha(B')} \leq C(\|w_k\|_{L^\infty(B_{3/4})}, g_0, \delta, B', G_k(1)) \leq C(g_0, \delta, B') \quad \forall B' \subset\subset B_{3/4}.$$

(Here again we may suppose that the constant C dose not depend on λ_k, since $\lambda_k \to 0$). Also, recall that $\|w_k\|_{L^\infty(B_1)} \leq 2$, and that $G_k(1) = \frac{G(c_k)}{c_k g(c_k)} \leq 1$.

As v_k are solutions of (2.2.19) by Theorem 1.2.19, we have for $B' \subset\subset B_{3/4}$

$$\|v_k\|_{C^{1,\alpha}(B')} \leq C(N, \delta, g_0, \operatorname{dist}(B', \partial B_{3/4}), \|v_k\|_{L^\infty(B_{3/4})}). \tag{2.2.24}$$

By (g3) and $\|v_k\|_{L^\infty(B_{3/4})} \leq \|w_k\|_{L^\infty(\partial B_{3/4})} \leq 2$. Then, this constant only depends on N, δ, g_0 and B'.

Therefore, by (2.2.23) and (2.2.24) we have that there exist subsequences, that we call for simplicity v_k and w_k, and functions w_0, $v_0 \in C^\alpha(B')$ for every $B' \subset\subset B_{3/4}$, such that

$$\begin{aligned} w_k &\to w_0 \quad \text{uniformly in } B_{3/4}, \\ v_k &\to v_0 \quad \text{uniformly in } B', \end{aligned}$$

Then,

$$p_k = w_k - v_k \to w_0 - v_0 \quad \text{uniformly in } B'.$$

But by (2.2.22) we have $p_k \to 0$ in $W^{1,\delta+1}(B')$. Thus, $v_0 = w_0$.

Using Harnack inequality (see Theorem 1.2.16), we have that

$$\sup_{B_{1/2}} v_k \leq C \inf_{B_{1/2}} v_k$$

where the constant C depends only on g_0, δ, N. Then, passing to the limit and using that $v_0 = w_0$ we have that

$$\sup_{B_{1/2}} w_0 \leq C \inf_{B_{1/2}} w_0.$$

But by (2.2.18), passing to the limit again, we have that $\sup_{B_{1/2}} w_0 > c > 0$ and $\inf_{B_{1/2}} w_0 = 0$ since $w_k(0) = 0$ for every k, this is a contradiction. $\square$

Proof of Lemma 2.2.13. Let $x_0 \in \{u > 0\}$ with $5d(x_0) < d(x_0, \partial\Omega)$. Take $\widetilde{u}(x) = \frac{u(y_0+4d_0x)}{4d_0}$, where $d_0 = dist(x_0, \partial\{u > 0\}) = dist(x_0, y_0)$ with $y_0 \in \partial\{u > 0\}$. If we prove that $\widetilde{u}$ is a minimizer in $B_1(0)$, as $\widetilde{u}(0) = 0$ and $\frac{|x_0-y_0|}{4d_0} = 1/4$, by Lemma 2.2.14 we have

$$C \geq \widetilde{u}\Big(\frac{x_0 - y_0}{4d_0}\Big) = \frac{u(x_0)}{4d_0}$$

and the result follows.

So, let us prove that $\tilde{u}$ is a minimizer in $B_1(0)$. As $5d(x_0) < d(x_0, \partial\Omega)$ we have, $B_{4d_0}(y_0) \subset \Omega$. Let $\widetilde{v} \in W^{1,G}(B_1(0))$ and v such that $\widetilde{v}(x) = \frac{v(y_0+4d_0x)}{4d_0}$. Then, changing variables we have,

$$\int_{B_1} G(|\nabla\widetilde{v}|)\, dx = \int_{B_1} G(|\nabla v(y_0 + 4d_0x)|)\, dx = \int_{B_{4d_0}(y_0)} \frac{G(|\nabla v(y)|)}{d_0^N 4^N}\, dy$$

and

$$|\{\widetilde{v} > 0 \cap B_1\}| = \frac{|\{\widetilde{v} > 0 \cap B_{4d_0}(y_0)\}|}{d_0^N 4^N}.$$

As u is a minimizer of $\mathcal{J}$ in $B_{4d_0}(y_0)$ we have, if $\widetilde{v} = \widetilde{u}$ on $\partial B_1(0)$,

$$\begin{aligned}
&\int_{B_1(0)} G(|\nabla\widetilde{u}(x)|)\, dx + \lambda|\{\widetilde{u} > 0 \cap B_1(0)\}| \\
&\qquad = \int_{B_{4d_0}(y_0)} \frac{G(|\nabla u(y)|)}{d_0^N 4^N}\, dy + \frac{\lambda|\{u > 0 \cap B_{4d_0}(y_0)\}|}{d_0^N 4^N} \\
&\qquad \leq \int_{B_{4d_0}(y_0)} \frac{G(|\nabla v(y)|)}{d_0^N 4^N}\, dy + \frac{\lambda|\{v > 0 \cap B_{4d_0}(y_0)\}|}{d_0^N 4^N} \\
&\qquad = \int_{B_1(0)} G(|\nabla\widetilde{v}(x)|)\, dx + \lambda|\{\widetilde{v} > 0 \cap B_1(0)\}|.
\end{aligned}$$

Therefore, $\widetilde{u}$ is a minimizer of $\mathcal{J}$ in $B_1(0)$.

$\square$

Now we can prove the uniform Lipschitz continuity of minimizers of $\mathcal{J}$.

THEOREM 2.2.25. *Let u be a minimizer. Then u is locally Lipschitz continuous in Ω. Moreover, for any connected open subset $D \subset\subset \Omega$ containing free boundary points, the Lipschitz constant of u in D is estimated by a constant C depending only on $N, g_0, \delta, dist(D, \partial\Omega)$ and λ.*

PROOF. First, take x such that $d(x) < \frac{1}{5}\text{dist}(x, \partial\Omega)$ and $\widetilde{u}(y) = \frac{1}{d(x)}u(x + d(x)y)$ for $y \in B_1(0)$. By Lemma 2.2.14 we have $\widetilde{u}(0) \le C$ in B_1, where C depends only on N, λ, δ and g_0. Since $u > 0$ in $B_{d(x)}(x)$, $\mathcal{L}u = 0$ in this ball. Thus $\mathcal{L}\widetilde{u} = 0$ in $B_1(0)$. By Harnack inequality $\widetilde{u}(y) \le C$ in $B_{1/2}(0)$ where C depends only on N, λ, δ and g_0. Now, by property (2) in Lemma 1.2.18, $|\nabla\widetilde{u}(0)| \le C\|\widetilde{u}\|_{L^\infty(B_{1/2})} \le C$ where C depends only on N, λ, δ and g_0. Since $\nabla u(x) = \nabla\widetilde{u}(0)$, the result follows in the case $d(x) < \frac{1}{5}\text{dist}(x, \partial\Omega)$.

Let r_1 such that $\mathrm{dist(x, \partial\Omega) \geq r_1} > 0\ \forall x \in D$, take D', satisfying $D \subset\subset D' \subset\subset \Omega$ given by

$$D' = \{x \in \Omega / \mathrm{dist(x, D) < r_1/2}\}.$$

If $d(x) \le \frac{1}{5}\text{dist}(x, \partial\Omega)$ we proved that $|\nabla u(x)| \le C$. If $d(x) > \frac{1}{5}\text{dist}(x, \partial\Omega)$, there holds that $u > 0$ in $B_{\frac{r_1}{5}}(x)$ and $B_{\frac{r_1}{5}}(x) \subset D'$ so that $|\nabla u(x)| \le \frac{C}{r_1}\|u\|_{L^\infty(D')}$.

To prove the second part of the theorem, consider now any domain D that contains a free boundary point, and D' as in the previous paragraph. Let us see that $\|u\|_{L^\infty(D')}$ is bounded by a constant that depends only on $N, D, r_1, \lambda, \delta$, and g_0 (we argue as in [**4**] Theorem 4.3). Let $x_0 \in D$ and $r_0 = \frac{r_1}{5}$, since D' is connected and not contained in $\{u > 0\} \cap \Omega$, there exists $x_0, ..., x_k \in D'$ such that $x_j \in B_{\frac{r_0}{2}}(x_{j-1})$ $j = 1, ..., k$, $B_{r_0}(x_j) \subset \{u > 0\}$ $j = 0, ..., k-1$ and $B_{r_0}(x_k) \not\subseteq \{u > 0\}$. By Lemma 2.2.14 $u(x_k) \le Cr_0$ and by Harnack inequality we have $u(x_{j+1}) \ge cu(x_j)$. Inductively we obtain. Therefore, the supremum of u over D' can be estimated by a constant depending only on N, r_1, λ, δ, and g_0. $\square$

Observe that, if we don't use Lemma 2.2.13, then we obtain that the Lipschitz constant depends also on $\|u\|_{L^\infty(\Omega)}$ (that is, depends also on the Dirichlet datum φ_0).

3. Nondegeneracy

In this section we prove the nondegeneracy of a minimizer at the free boundary and the locally uniform positive density of the sets $\{u > 0\}$ and $\{u = 0\}$.

LEMMA 2.3.1. *Let $\gamma > 0$, $D \subset\subset \Omega$ and C the constant in Theorem 2.2.25. Then, if $C_1 > C$, $B_r \subset \Omega$ and u is a minimizer, there holds that*

$$\frac{1}{r}\Big(\fint_{B_r} u^\gamma\Big)^{1/\gamma} \ge C_1 \quad \textit{implies} \quad u > 0 \textit{ in } B_r$$

Proof. If B_r contains a free boundary point, if u vanishes at some point $x_0 \in B_r$, and since $|\nabla u(x)| \le C$ in B_r, then $|u(x) - u(x_0)| \le Cr$. That is, $u(x) \le Cr$ in B_r and then $\frac{1}{r}\left(⨍_{B_r} u^\gamma\right)^{1/\gamma} \le C$ which is a contradiction.

□

Lemma 2.3.2. *For any $\gamma > 1$ and for any $0 < \kappa < 1$ there exists a constant c_κ such that, for any minimizer u and for every $B_r \subset \Omega$, we have*

$$\frac{1}{r}\left(⨍_{B_r} u^\gamma\right)^{1/\gamma} \le c_\kappa \quad \textit{implies} \quad u = 0 \textit{ in } B_{\kappa r},$$

where c_κ depends also on N, λ, g_0, δ and γ.

Proof. We may suppose that $r = 1$ and that B_r is centered at zero, (if not, we take the rescaled function $\tilde{u} = \frac{u(x_0+rx')}{r}$). By Theorem 1.2.14 we have

$$\varepsilon := \sup_{B_{\sqrt{\kappa}}} u < C\left(⨍_{B_1} u^\gamma\right)^{1/\gamma}$$

where $C = C(\kappa, \gamma)$. Now chose v such that

$$v = \begin{cases} C_1\varepsilon(e^{-\mu|x|^2} - e^{-\mu\kappa^2}) & \text{in } B_{\sqrt{\kappa}} \setminus B_\kappa, \\ 0 & \text{in } B_\kappa. \end{cases}$$

Here the constants $\mu > 0$ and $C_1 < 0$ with $C_1 = C_1(\mu, \kappa)$, are chosen so that $\mathcal{L}v < 0$ in $B_{\sqrt{\kappa}} \setminus B_\kappa$ (see Lemma 1.2.47) and $v = \varepsilon$ on $\partial B_{\sqrt{\kappa}}$. Hence, $v \ge u$ on $\partial B_{\sqrt{\kappa}}$, and therefore if

$$w = \begin{cases} \min(u, v) & \text{in} \quad B_{\sqrt{\kappa}}, \\ u & \text{in} \quad \Omega \setminus B_{\sqrt{\kappa}}, \end{cases}$$

w is an admissible function for the minimizing problem. Thus, using the convexity of G, we find that

$$
\begin{aligned}
&\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}| \\
&\quad = \mathcal{J}(u) - \int_{\Omega\setminus B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}| - \lambda|\Omega \cap \{u>0\}| \\
&\quad \le \mathcal{J}(w) - \int_{\Omega\setminus B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}| - \lambda|\Omega \cap \{u>0\}| \\
&\quad \le \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} G(|\nabla w|)\,dx - \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} G(|\nabla u|)\,dx \\
&\quad \le \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} g(|\nabla w|)\frac{\nabla w}{|\nabla w|}(\nabla w - \nabla u)\,dx \\
&\quad = -\int_{B_{\sqrt{\kappa}}\setminus B_\kappa} g(|\nabla w|)\frac{\nabla w}{|\nabla w|}\nabla(u-v)^+\,dx \\
&\quad = -\int_{(B_{\sqrt{\kappa}}\setminus B_\kappa)\cap\{u>v\}} g(|\nabla v|)\frac{\nabla v}{|\nabla v|}\nabla(u-v)^+\,dx
\end{aligned}
$$

and as v is a supersolution we have,

$$
\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}| \le -\int_{\partial B_\kappa} g(|\nabla v|)\frac{\nabla v}{|\nabla v|} u\,\nu\,d\mathcal{H}^{N-1}.
$$

And, as $|\nabla v| \le C\varepsilon$ we have that

$$
\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}| \le g(C\varepsilon)\int_{\partial B_\kappa} u\,d\mathcal{H}^{N-1}.
$$

By Sobolev's trace inequality and by $(\widetilde{g}3)$, for $\widetilde{G}(\alpha) = \lambda$ we have,

$$
\begin{aligned}
\int_{\partial B_\kappa} u &\le C(N,\kappa)\int_{B_\kappa} |\nabla u| + u\,dx \\
&\le C(N,\kappa)\Big(\int_{B_\kappa} G\Big(\frac{|\nabla u|}{\alpha}\Big) + \int_{B_\kappa\cap\{u>0\}} \widetilde{G}(\alpha) + \int_{B_\kappa} u\,dx\Big) \\
&\le C(N,\kappa,\lambda)(1+\varepsilon)\Big(\int_{B_\kappa} G(|\nabla u| + \lambda|\{u>0\}\cap B_\kappa|\Big)
\end{aligned}
$$

where in the last inequality we are using that $\int_{B_\kappa} u\,dx \le \varepsilon|\{u>0\}\cap B_\kappa|$. Therefore,

$$
\begin{aligned}
&\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}| \\
&\qquad\qquad \le g(C\varepsilon)C(1+\varepsilon)\Big(\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}|\Big).
\end{aligned}
$$

So that, if ε is small enough

$$\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda|B_\kappa \cap \{u>0\}| = 0.$$

Then, $u=0$ in B_κ and the result follows.

□

As a corollary we have,

COROLLARY 2.3.3. *Let $D \subset\subset \Omega$, $x \in D \cap \partial\{u>0\}$. Then*

$$\sup_{B_r(x)} u \geq cr,$$

where c is the constant in Lemma 2.3.2 corresponding to $\kappa = 1/2$ and γ fixed.

COROLLARY 2.3.4. *For any domain $D \subset\subset \Omega$ there exist constants c, C depending on N, g_0, δ, D and λ, such that, for any minimizer u and for every $B_r(x) \subset D \cap \{u > 0\}$, touching the free boundary we have*

$$cr \leq u(x) \leq Cr$$

PROOF. It follows by Lemma 2.2.13 and Lemma 2.3.2. □

THEOREM 2.3.5. *For any domain $D \subset\subset \Omega$ there exists a constant c, with $0 < c < 1$ depending on N, g_0, δ, D and λ, such that, for any minimizer u and for every $B_r \subset \Omega$, centered on the free boundary we have,*

$$c \leq \frac{|B_r \cap \{u>0\}|}{|B_r|} \leq 1-c$$

PROOF. First, by Corollary 2.3.3 we have that there exists $y \in B_r$ such that $u(y) > cr$ and as u is a subsolution we have by Lemma 1.2.14 that

$$\Big(\fint_{B_{\kappa r}} u^\gamma\,dx\Big)^{1/\gamma} \geq Cu(y).$$

Therefore

$$\frac{1}{\kappa r}\Big(\fint_{B_{\kappa r}} u^\gamma\,dx\Big)^{1/\gamma} \geq \frac{C}{\kappa}.$$

Now, if κ is small enough, we have

$$\frac{1}{\kappa r}\Big(\fint_{B_{\kappa r}} u^\gamma\,dx\Big)^{1/\gamma} \geq C_1,$$

so that by Lemma 2.3.1, we have that $u>0$ in $B_{\kappa r}$, where $\kappa = \kappa(C_1, C)$. Thus,

$$\frac{|B_r \cap \{u>0\}|}{|B_r|} \geq \frac{|B_{\kappa r}|}{|B_r|} = \kappa^N,$$

and $\kappa = \kappa(C_1, C)$.

In order to prove the other inequality, we may assume that $r=1$. Let us suppose by contradiction that, there exists a sequence of minimizers u_k in B_1, such that, $0 \in \partial\{u_k > 0\}$, with $|\{u_k = 0\} \cap B_1| = \varepsilon_k \to 0$. Let us take $v_k \in W^{1,G}(B_{1/2})$ such that,

$$\begin{aligned} \mathcal{L}v_k &= 0 \qquad \text{in } B_{1/2} \\ v_k &= u_k \qquad \text{in } \partial B_{1/2} \end{aligned} \tag{2.3.6}$$

Let A_1 and A_2 as in the proof of Theorem 2.2.1, for $r = 1/2$. Then we have, by Theorem 1.2.38 and (2.2.3) that

$$\begin{aligned} &\int_{A_2} G(|\nabla u_k - \nabla v_k|)\,dx \leq C\varepsilon_k \qquad \text{and} \\ &\int_{A_1} F(|\nabla u_k|)|\nabla u_k - \nabla v_k|^2\,dx \leq C\varepsilon_k, \end{aligned}$$

where $C = C(g_0)$. By (2.2.6) we have,

$$\int_{A_1} G(|\nabla u_k - \nabla v_k|)\,dx \leq C\Big(\int_{A_1} F(|\nabla u_k|)|\nabla u_k - \nabla v_k|^2\,dx\Big)^{1/2}\Big(\int_{A_1} G(|\nabla u_k|)\Big)^{1/2}.$$

Therefore, by (2.2.11), there exists C independent of k such that

$$\int_{B_{1/2}} G(|\nabla u_k - \nabla v_k|)\,dx \leq C\varepsilon_k^{1/2} \to 0.$$

As $u_k = v_k$ on $\partial B_{1/2}$, $w_k = u_k - v_k \in W_0^{1,\delta+1}(B_{1/2})$. Thus,

$$w_k \to 0 \qquad \text{in } W_0^{1,\delta+1}(B_{1/2}). \tag{2.3.7}$$

By Theorem 2.2.1 and Theorem 1.2.19, we have

$$\begin{aligned} &\|u_k\|_{C^\alpha(\overline{B_{1/2}})} \leq C(N, \delta, g_0, \|u_k\|_{L^\infty(B_1)}, \alpha) \qquad (\text{for } \varepsilon_k \text{ small}), \\ &\|v_k\|_{C^{1,\alpha}(B')} \leq C(N, \delta, g_0, \|u_k\|_{L^\infty(B_{1/2})}, B', \alpha) \qquad \text{for } B' \subset\subset B \qquad (\text{see } (2.2.9)). \end{aligned}$$

Therefore, there exist subsequences, that we call for simplicity u_k and v_k, and functions $v_0 \in C^1(B')$, $u_0 \in C(B')$ for all $B' \subset\subset B_{1/2}$ such that

$$\begin{aligned} &u_k \to u_0 \qquad \text{uniformly in } B_{1/2} \\ &v_k \to v_0 \qquad \text{uniformly in } B' \\ &w_k = u_k - v_k \to 0 \qquad \text{uniformly in } B'. \end{aligned}$$

Thus, $v_0 = u_0$. By Lemma 2.3.2 we have that

$$\Big(\fint_{B_{1/4}} u_k^\gamma\Big)^{1/\gamma} \geq C > 0.$$

Therefore, passing to the limit, we have

$$\Big(\fint_{B_{1/4}} u_0^\gamma\Big)^{1/\gamma} \geq C > 0.$$

On the other hand, by Harnack inequality $\sup_{B_{1/4}} v_k \leq C \inf_{B_{1/4}} v_k$ and again, passing to the limit we have, $\sup_{B_{1/4}} u_0 \leq C \inf_{B_{1/4}} u_0$. As $u_0(0) = 0$, then $u_0 \equiv 0$ in $B_{1/4}$, which is a contradiction.

□

REMARK 2.3.8. *Theorem 2.3.5 implies that the free boundary has Lebesgue measure zero. Moreover, it implies that for every $D \subset\subset \Omega$, the intersection $\partial\{u > 0\} \cap D$ has Hausdorff dimension less than N. In fact, to prove these statements, it is enough to use the left hand side estimate in Theorem 2.3.5. In fact, this estimate says that the set of Lebesgue points of $\chi_{\{u>0\}}$ in $\partial\{u > 0\} \cap D$ is empty. On the other hand almost every point $x_0 \in \partial\{u > 0\} \cap D$ is a Lebesgue point, therefore $|\partial\{u > 0\} \cap D| = 0$.*

4. The measure $\Lambda = \mathcal{L}u$

In this section we prove that $\{u > 0\} \cap \Omega$ is locally of finite perimeter. Then, we study the measure $\Lambda = \mathcal{L}u$ and prove that it is absolutely continuous with respect to the $\mathcal{H}^{N-1}$ measure on the free boundary. This result gives rice to a representation theorem for the measure Λ. Finally, we prove that almost every point in the free boundary belongs to the reduced free boundary.

THEOREM 2.4.1. *For every $\varphi \in C_0^\infty(\Omega)$ such that* $\operatorname{supp}(\varphi) \subset \{u > 0\}$,

$$\int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\varphi = 0. \tag{2.4.2}$$

Moreover, the application

$$\Lambda(\varphi) := -\int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\varphi\, dx$$

from $C_0^\infty(\Omega)$ into $\mathbb{R}$ defines a nonnegative Radon measure $\Lambda = \mathcal{L}u$ with support on $\Omega \cap \partial\{u > 0\}$.

PROOF. We know that u is an $\mathcal{L}-$ subsolution. Then by the Riesz Representation Theorem, there exists a nonnegative Radon measure Λ, such that $\mathcal{L}u = \Lambda$. And as $\mathcal{L}u = 0$ in $\{u > 0\}$, then for any $\varphi \in C_0^\infty(\Omega \setminus \partial\{u > 0\})$

$$\Lambda(\varphi) = -\int_{\{u>0\}} \nabla\varphi\; g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\, dx = 0,$$

and the result follows.

□

Now we want to prove that $\Omega \cap \partial\{u > 0\}$, has Hausdorff dimension $N - 1$. First we need the following lemma,

LEMMA 2.4.3. *If u_k is a sequence of minimizers in compact subsets of B_1, such that $u_k \to u_0$ uniformly in B_1, then*

1. $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in Hausdorff distance,*
2. $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$,
3. *If* $0 \in \partial\{u_k > 0\}$, *then* $0 \in \partial\{u_0 > 0\}$.

PROOF. Here we only have to use Lemma 2.3.2 and Theorem 2.3.5 and the fact that $u_k \to u_0$ uniformly in compacts subsets of B_1. Then the proof follows as in Lemma 1.6.13. □

Now, we prove the following theorem,

THEOREM 2.4.4. *For any domain $D \subset\subset \Omega$ there exist constants c, C, depending on N, g_0, δ, D and λ, such that, for any minimizer u and for every $B_r \subset D$, centered on the free boundary we have*

$$cr^{N-1} \leq \int_{B_r} d\Lambda \leq Cr^{N-1}$$

PROOF. Let $\xi \in C_0^\infty(\Omega)$, $\xi \geq 0$. Then,

$$\Lambda(\xi) = -\int_{\{u>0\}} g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\xi\, dx.$$

Approximating χ_{B_r} from below by a sequence $\{\xi_n\}$ such that $\xi_n = 1$ in $B_{r-\frac{1}{n}}$ and $|\nabla \xi_n| \leq C_N n$ and using that u is Lipschitz we have that,

$$\Big|\int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\xi_n\, dx\Big| \leq Cn\Big|B_r \setminus B_{r-\frac{1}{n}}\Big| \leq C(r^{N-1} + O(1/n)).$$

Then, as

$$\int_\Omega \xi_n d\Lambda \to \int_{B_r} d\Lambda,$$

the bound from above holds.

In order to prove the other inequality, we will suppose that $r = 1$. Arguing by contradiction we assume that there exists a sequence of minimizers u_k in B_1, with $0 \in \partial\{u_k > 0\}$, and $\Lambda_k = \mathcal{L}u_k$, such that $\int_{B_1} d\Lambda_k = \varepsilon_k \to 0$. As the $u_k's$ are uniformly Lipschitz, we can assume that $u_k \to u_0$ uniformly in $B_{1/2}$. Let $h_k = g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}$. Then, there exists a subsequence and a function h_0 such that $h_k \rightharpoonup h_0$ $*-$ weakly in $L^\infty(B_{1/2})$. We claim that $h_0 = g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}$. In fact, if $B_\rho \subset\subset \{u_0 > 0\}$ then, by $C^{1,\alpha}$ estimates, there exists a subsequence such that $u_k \to u_0$ strongly in $C^{1,\alpha}(B_\rho)$. So that $h_0 = g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}$. If $B_\rho \subset \{u_0 = 0\}$, then by Lemma 2.3.2 we have that $u_k = 0$ in $B_{\rho\kappa}$ for $k \geq k_0(\kappa)$. Thus $h_0 = 0 = g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}$ also in this case. Finally $\partial\{u_0 > 0\} \cap B_{1/2}$ has zero Lebesgue measure. In fact, by (1) in Lemma 2.4.3, every point $x_0 \in \partial\{u_0 > 0\} \cap B_{1/2}$ is a limit point of $x_k \in \partial\{u_k > 0\} \cap B_{1/2}$. Thus,

$$\Big(\fint_{B_r(x_0)} u_0^\gamma\Big)^{1/\gamma} \geq cr$$

for any ball $B_r(x_0) \subset B_{1/2}$. Using this fact, and the Lipschitz continuity we have that $|B_r(x_0) \cap \{u_0 > 0\}| \geq c|B_r(x_0)|$ with $c > 0$. This implies that $|\partial\{u_0 > 0\} \cap B_{1/2}| = 0$ (see Remark 2.3.8).

Therefore, for all $\xi \in C_0^\infty(B_{1/2})$, $\xi \geq 0$ we have

$$\int_{B_{1/2}} \xi d\Lambda_0 := \int_{B_{1/2}} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla \xi = \lim_{k\to\infty}\int_{B_{1/2}} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla \xi.$$

On the other hand,

$$\int_{B_{1/2}} \xi \, d\Lambda_0 = \lim_{k\to\infty}\int_{B_{1/2}} \xi \, d\Lambda_k \leq \|\xi\|_{L^\infty(B_{1/2})} \lim_{k\to\infty} \varepsilon_k = 0.$$

Therefore $\Lambda_0 = 0$ in $B_{1/2}$. That is, $\mathcal{L}u_0 = 0$ in $B_{1/2}$. But $u_0 \geq 0$ and $u_0(0) = 0$, so that by Harnack inequality we have $u_0 = 0$ in $B_{1/2}$.

On the other hand, $0 \in \partial\{u_k > 0\}$, and by the nondegeneracy, we have

$$\Big(\int_{B_{1/4}} u_k^\gamma\Big)^{1/\gamma} \geq c > 0.$$

Thus,

$$\Big(\int_{B_{1/4}} u_0^\gamma\Big)^{1/\gamma} \geq c > 0$$

which is a contradiction.

□

Therefore, we have the following representation theorem

THEOREM 2.4.5 (Representation Theorem). *Let u be a minimizer. Then,*

1. $\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ *for every* $D \subset\subset \Omega$.
2. *There exists a Borel function q_u such that*

$$\mathcal{L}u = q_u \, \mathcal{H}^{N-1} \lfloor \partial\{u > 0\}.$$

i.e

$$-\int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\varphi \, dx = \int_{\Omega\cap\partial\{u>0\}} \varphi q_u \, d\mathcal{H}^{N-1} \quad \forall\ \varphi \in \ C_0^\infty(\Omega).$$

3. *For $D \subset\subset \Omega$ there are constants $0 < c \leq C < \infty$ depending on $N, g_0, \delta, \Omega, D$ and λ such that for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,*

$$c \leq q_u(x) \leq C, \quad c\, r^{N-1} \leq \mathcal{H}^{N-1}(B_r(x) \cap \partial\{u > 0\}) \leq C\, r^{N-1}.$$

PROOF. See Theorem 1.4.54. □

REMARK 2.4.6. *As u satisfies the conclusions of Theorem 1.4.54, the set $\Omega \cap \{u > 0\}$ has finite perimeter locally in Ω (see Remark 1.4.62). That is, $\mu_u := -\nabla\chi_{\{u>0\}}$ is a Borel measure, and the total variation $|\mu_u|$ is a Radon measure. Moreover we have,*

$$\mu_u = \nu_u \mathcal{H}^{N-1} \lfloor \partial_{red}\{u > 0\},$$

where $\nu_u(x)$ is the normal exterior to $\{u > 0\} \cap \Omega$. See Definition 1.4.59 and the results in that section.

LEMMA 2.4.7. $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$.

PROOF. This is a consequence of the density property of Theorem 2.3.5 and Theorem 1.4.63. □

5. Asymptotic development and identification of the function q_u

In this section we give some properties of blow up sequences of minimizers, we prove that any limit of a blow up sequence is a minimizer. We prove the asymptotic development of minimizers near points in their reduced free boundary. We finally identify the function q_u for almost every point in the reduced free boundary.

LEMMA 2.5.1. *If $u(x_m) = 0$, $x_m \to x_0$ in Ω. Then, any blow up limit u_0 respect to $B_{\rho_m}(x_m)$ is a minimizer of $\mathcal{J}$ in any ball.*

PROOF. Let u_m, u_0 be as is Lemma 1.6.13, $R > 0$ and v such that $v - u_0 \in W_0^{1,G}(B_R(0))$. Let $\eta \in C_0^\infty(B_R(0))$, $0 \le \eta \le 1$ and $v_m = v + (1-\eta)(u_m - u_0)$ then $v_m = u_m$ in $\partial B_R(0)$. Therefore

$$\int_{B_R(0)} (G(|\nabla u_m|) + \lambda\chi_{\{u_m>0\}})\, dx \le \int_{B_R(0)} (G(|\nabla v_m|) + \lambda\chi_{\{v_m>0\}})\, dx.$$

As $|\nabla u_m| \le C$ and $\nabla u_m \to \nabla u_0$ a.e, we have

$$\int_{B_R(0)} G(|\nabla u_m|)\, dx \to \int_{B_R(0)} G(|\nabla u_0|)\, dx,$$
$$\int_{B_R(0)} G(|\nabla v_m|)\, dx \to \int_{B_R(0)} G(|\nabla v|)\, dx$$

and

$$\chi_{\{v_m>0\}} \le \chi_{\{v>0\}} + \chi_{\{\eta<1\}}.$$

Therefore,

$$\int_{B_R(0)} (G(|\nabla u_0|)+\lambda\chi_{\{u_0>0\}})\, dx \le \int_{B_R(0)} (G(|\nabla v|)+\lambda\chi_{\{v>0\}})\, dx + \lambda|B_R(0)\cap\{\eta<1\}|.$$

Taking η such that $|\{\eta < 1\} \cap B_R(0)| \to 0$ we have the desired result.

□

Let λ^* be such that, $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$. Then we have,

LEMMA 2.5.2. *Let u be a local minimizer in $\mathbb{R}^N$ such that $u = \lambda_0\langle x, \nu_0\rangle^-$ in B_{r_0}, with $r_0 > 0$, $0 < \lambda_0 < \infty$ and ν_0 a unit vector. Then, $\lambda_0 = \lambda^*$.*

PROOF. Let $\tau_\varepsilon(x) = x + \varepsilon\eta(x)$ with $\eta \in C_0^\infty(B_{r_0}))$, and let $u_\varepsilon(\tau_\varepsilon(x)) = u(x)$. Then,

$$0 \le \mathcal{J}(u_\varepsilon) - \mathcal{J}(u),$$

$$\begin{aligned}|B_{r_0}\cap\{u_\varepsilon>0\}| &= \int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} |\text{det}D\tau_\varepsilon|\,dx\\ &= \int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} (1+\varepsilon\,\text{div}\,\eta+o(\varepsilon))\,dx\end{aligned}$$

and

$$\begin{aligned}&\int_{B_{r_0}\cap\{u_\varepsilon>0\}} G(|\nabla u_\varepsilon|)\,dy\\ &\quad=\int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \Big(G(|\nabla u|)+\varepsilon\Big(G(|\nabla u|)\text{div}\eta-\frac{g(|\nabla u|)}{|\nabla u|}\nabla u D\eta\nabla u\Big)\Big)\,dx+o(\varepsilon).\end{aligned}$$

Therefore, since $u_\varepsilon=u$ in $I\!R^N\setminus B_{r_0}$,

$$0\le\varepsilon\int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \Big((G(|\nabla u|)+\lambda)\text{div}\eta-\frac{g(|\nabla u|)}{|\nabla u|}\nabla u D\eta\nabla u\Big)\,dx+o(\varepsilon).$$

Thus,

$$\int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \Big((G(|\nabla u|)+\lambda)\text{div}\eta-\frac{g(|\nabla u|)}{|\nabla u|}\nabla u D\eta\nabla u\Big)\,dx\ge 0.$$

If we change η by $-\eta$ and recall that $\nabla u=-\lambda_0\,\nu_0$ in $\{\langle x,\nu_0\rangle<0\}$ we obtain,

$$\int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \Big((G(\lambda_0)+\lambda)\text{div}\eta-g(\lambda_0)\lambda_0\,\nu_0\,D\eta\,\nu_0\Big)\,dx=0$$

for all $\eta\in C_0^\infty(B_{r_0})$.

Take $\eta(x)=\phi(|x|)\nu_0$ with $\text{supp}\,\phi\subset(-r_0,r_0)$. Then,

$$\begin{aligned}\text{div}\,\eta(x)&=\frac{\phi'(|x|)}{|x|}\langle x,\nu_0\rangle\\ \nu_0\,D\eta\,\nu_0&=\nu_{0i}\frac{\partial\eta_j}{\partial x_i}\nu_{0j}=\langle x,\nu_0\rangle\frac{\phi'(|x|)}{|x|}=\text{div}\,\eta.\end{aligned}$$

Hence

$$\begin{aligned}0&=\int_{\{\langle x,\nu_0\rangle<0\}\cap B_{r_0}}\big(G(\lambda_0)+\lambda-g(\lambda_0)\lambda_0\big)\text{div}\eta\,dx\\ &=\big(G(\lambda_0)+\lambda-g(\lambda_0)\lambda_0\big)\int_{\{\langle x,\nu_0\rangle=0\}\cap B_{r_0}}\eta\,\nu_0\,d\mathcal{H}^{N-1}(x)\\ &=\big(G(\lambda_0)+\lambda-g(\lambda_0)\lambda_0\big)\int_{\{\langle x,\nu_0\rangle=0\}\cap B_{r_0}}\phi(|x|)\,d\mathcal{H}^{N-1}(x)\end{aligned}$$

for all $\phi\in C_0^\infty(-r_0,r_0)$.

Therefore, $g(\lambda_0)\lambda_0-G(\lambda_0)=\lambda$.

$\square$

LEMMA 2.5.3. *Let $u \in \mathcal{K}$ be minimizer. Then, for every $x_0 \in \Omega \cap \partial\{u > 0\}$*

$$\limsup_{\substack{x \to x_0 \\ u(x)>0}} |\nabla u(x)| = \lambda^*. \tag{2.5.4}$$

PROOF. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let

$$l := \limsup_{\substack{x \to x_0 \\ u(x)>0}} |\nabla u(x)|.$$

Then there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0, \qquad |\nabla u(z_k)| \to l.$$

Let y_k be the nearest point to z_k on $\Omega \cap \partial\{u > 0\}$ and let $d_k = |z_k - y_k|$. Consider the blow up sequence with respect to $B_{d_k}(y_k)$ with limit u_0, such that there exists

$$\nu := \lim_{k \to \infty} e_k,$$

where $e_k = \frac{y_k - z_k}{d_k}$, and suppose that $\nu = e_N$. Then, by Lemma 1.6.13(1), $0 \in \partial\{u_0 > 0\}$. By Lemma 1.6.13(2) and by Lemma 2.5.1 we have that u_0 satisfies Theorem 2.3.5. Then, $B_1(-e_N) \subset \{u_0 > 0\}$. By Lemma 1.6.13(3) we obtain,

$$|\nabla u_0| \le l \text{ in } \{u_0 > 0\} \quad \text{and} \quad |\nabla u_0(-e_N)| = l.$$

Then, $0 < l < \infty$ and since, by Lemma 1.6.13 (6), we have that u_0 is an $\mathcal{L}$ solution in $\{u_0 > 0\}$ then, we have that u_0 is locally $C^{1,\alpha}$ there. Thus, there exists $\mu > 0$ such that $|\nabla u_0| > l/2$ in $B_\mu(-e_N)$. Let $e = \frac{\nabla u_0(-e_N)}{|\nabla u_0(-e_N)|}$ and $v = \frac{\partial u_0}{\partial e}$, then by Lemma (1.2.28) v satisfies the uniformly elliptic equation, $D_i(a_{ij} D_j v) = 0$.

Then, by the strong maximum principle we have $D_e u_0 = l$ in $B_\mu(-e_N)$ so that, $\nabla u_0 = le$ in $B_\mu(-e_N)$. By continuation we can prove that this is true in $B_1(-e_N)$. Then, $u_0(x) = l\langle x, e\rangle + C$ in $B_1(-e_N)$. As $u_0(0) = 0$ and $u_0 > 0$ in $B_1(-e_N)$, we have $u_0(x) = l\langle x, e\rangle$ and $e = -e_N$. Therefore $u_0(x) = -lx_N$ in $B_1(-e_N)$. Using again a continuation argument we have that $u_0(x) = -lx_N$ in $\{x_N < 0\}$.

Now, we want to prove that $u_0 = 0$ in $\{0 < x_N < \varepsilon_0\}$ for some $\varepsilon_0 > 0$.

We argue by contradiction. Let

$$s := \limsup_{\substack{x_N \to 0^+ \ x' \in \mathbb{R}^{N-1} \\ u_0(x', x_N) > 0}} D_N u_0(x', x_N),$$

and suppose that $s > 0$ ($s < \infty$ since u_0 is uniformly Lipschitz). Let (z_k, h_k) such that, $h_k \to 0^+$ and $D_N u_0(z_k, h_k) \to s$, and take a blow up sequence with respect to $B_{h_k}(z_k, 0)$ with limit u_{00}. Arguing as before, we have that $u_{00} = sx_N$ for $x_N > 0$. On the other hand, we have $u_{00} = -lx_N$ for $x_N < 0$. By Lemma 2.5.1 u_{00} is a minimizer, and as all the points of the form $(x', 0)$ belong to the free boundary, we get a contradiction to the positive density property of the set $\{u_{00} = 0\}$(Theorem 2.3.5).

Therefore, $s = 0$. But this implies that $u_0(x', x_N) = o(x_N)$ as $x_N \searrow 0^+$. Thus, for all $\varepsilon > 0$, $h_0 > 0$,

$$\frac{1}{r}\Big(\fint_{B_r(x_0)} u_0^\gamma\Big)^{1/\gamma} < \varepsilon \quad \text{if } x_0 = (y_0, h_0) \text{ and } r = h_0$$

for r small enough independent of y_0. Then, by the nondegeneracy property, Lemma 2.3.2, we have that $u_0 = 0$ in $\{0 < x_N < \varepsilon_0\}$.

Now, by Lemmas 2.5.1 and 2.5.2 we conclude that $l = \lambda^*$, and the result follows. □

Now we prove the asymptotic development of minimizers.

THEOREM 2.5.5. *Let u be a minimizer. Then, at every $x_0 \in \partial_{red}\{u > 0\}$, u has the following asymptotic development*

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|). \tag{2.5.6}$$

where $\nu(x_0)$ is the outer unit normal to $\partial\{u > 0\}$ at x_0.

PROOF. Take $B_{\rho_k}(x_0)$ balls with $\rho_k \to 0$ and u_k be a blow up sequence with respect to these balls with limit u_0. Suppose that $\nu_u(x_0) = e_N$, and $x_0 = 0$.

First we prove that

$$\begin{cases} u_0 = 0 & \text{in } \{x_N \geq 0\}, \\ u_0 > 0 & \text{in } \{x_N < 0\}. \end{cases}$$

In fact, by Lemma 1.6.13, $\chi_{\{u_k > 0\}}$ converges to $\chi_{\{u_0 > 0\}}$ in L^1_{loc}. On the other hand, $\chi_{\{u_k > 0\}}$ converges to $\chi_{\{x_N < 0\}}$ in L^1_{loc} by (1.4.60). It follows that $u_0 = 0$ in $\{x_N \geq 0\}$ and $u_0 > 0$ a.e in $\{x_N < 0\}$.

If u_0 were zero somewhere in $\{x_N < 0\}$ there should exist a point $\bar{x}$ in $\{x_N < 0\} \cap \partial\{u_0 > 0\}$. But, as u_0 is a minimizer, for $0 < r < |\bar{x}_N|$,

$$\frac{|B_r(\bar{x}) \cap \{u_0 = 0\} \cap \{x_N < 0\}|}{|B_r(\bar{x})|} \geq c > 0.$$

Since this is a contradiction we conclude that $u_0 > 0$ in $\{x_N < 0\}$ and therefore $\mathcal{L}u_0 = 0$ in this set. Since $u_0 = 0$ on $\{x_N = 0\}$, we conclude that $u_0 \in C^{1,\alpha}(\{x_N \leq 0\})$ (see Theorem 1.2.20). Thus, there exists $0 \leq \lambda_0 < \infty$ such that

$$u_0(x) = \lambda_0 x_N^- + o(|x|).$$

By the nondegeneracy of u at every free boundary point (Lemma 2.3.2) we deduce that $\lambda_0 > 0$.

Now, let u_{00} be a blow up limit of u_0. This is, $u_{00}(x) = \lim \frac{u_0(r_n x)}{r_n}$ with $r_n \to 0$. Then, $u_{00} = \lambda_0 x_N^-$. Since u_{00} is again a minimizer, Lemma 2.5.2 gives that $\lambda_0 = \lambda^*$.

Let us see that actually $u_0 = \lambda^* x_N^-$. In fact, by applying Lemma 2.5.3 we see that $|\nabla u_0| \le \lambda^*$ and thus, $u_0 \le \lambda^* x_N^-$. Since the function $w = \lambda^* x_N^-$ is a solution to

$$Tw = \sum_{i,j} b_{ij} w_{x_i x_j} = 0 \quad \text{in } \{x_N < 0\}$$

with b_{ij} as in (1.2.27) and u_0 is a classical solution of the same equation in a neighborhood of any point where $|\nabla u_0| > 0$, and since $u_0 \le w$ in $\{x_N < 0\}$, $u_0 = w$ in $\{x_N = 0\}$, there holds that either $u_0 \equiv w$ or $u_0 < w$. In the latter case, there exists $\delta_0 > 0$ such that

$$(w - u_0)(x) \ge -\delta_0\, x_N + o(|x|).$$

But $(w - u_0)(x) = o(|x|)$. Thus, $u_0 \equiv w = \lambda^* x_N^-$.

Finally, since the blow up limit u_0 is independent of the blow up sequence ρ_k, we deduce that

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|).$$

□

LEMMA 2.5.7. *For $\mathcal{H}^{N-1}-$ almost every point x_0 in $\partial_{red}\{u > 0\}$ there holds that,*

$$\int_{B_r(x_0)\cap\partial\{u>0\}} |q_u - q_u(x_0)| d\mathcal{H}^{N-1} = o(r^{N-1}), \qquad \textit{as } r \to 0$$

PROOF. It follows by Theorem 2.4.5 (3) that q_u is locally integrable in $\mathbb{R}^{N-1}$ and therefore almost every point is a Lebesgue point. Moreover, Theorem 2.4.5 (3) also implies that $o(\mathcal{H}^{N-1}(B_r(x_0) \cap \partial\{u > 0\})) = o(r^{N-1})$. □

Moreover, we have the following result that holds at points $x_0 \in \partial_{red}\{u > 0\}$ that are Lebesgue points of the function q_u and are such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^{N-1}(\partial\{u > 0\} \cap B(x_0, r))}{\mathcal{H}^{N-1}(B'(x_0, r))} \le 1. \tag{2.5.8}$$

Recall that $\mathcal{H}^{N-1}$ – *a.e.* point in $\partial_{red}\{u > 0\}$ satisfies (2.5.8) (see Theorem 1.4.65).

LEMMA 2.5.9. *Let u be a minimizer, then for $\mathcal{H}^{N-1}$ a.e $x_0 \in \partial_{red}\{u > 0\}$,*

$$q_u(x_0) = g(\lambda^*).$$

PROOF. Let u_0 be as in Theorem 2.5.5. Now let

$$\xi(x) = \min\Big(2\big(1 - \frac{|x_N|}{2}, 1\big)\Big)\eta(x_1, ..., x_{N-1})$$

where $\eta \in C_0^\infty(B'_r)$, (where B'_r is a $N-1$- dimensional ball with radius r) and $\eta \ge 0$. By Lemma 1.4.67 and using Lemmas 1.6.13 and 2.5.7, we get for almost every point

$x_0 \in \partial_{red}\{u > 0\}$ (satisfying (2.5.8)) and $u_0 = \lim_{k\to\infty} \frac{u(x_0+\rho_k x)}{\rho_k}$ that

$$\begin{aligned} -\int_{\mathbb{R}^N} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi\,dx &\leftarrow -\int_{\mathbb{R}^N} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla\xi\,dx \\ &= \int_{\partial\{u_k>0\}} \xi(x) q_u(x_0+\rho_k x)\,d\mathcal{H}^{N-1} \\ &\to q_u(x_0)\int_{\mathbb{R}^{N-1}} \xi(x',0)\,d\mathcal{H}^{N-1}, \end{aligned}$$

where we have assumed that $\nu(x_0) = e_N$. Therefore, $\forall\ \xi \in\ C_0^1(B_r)$, we have

$$-\int_{B_r\cap\{x_N<0\}} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi\,dx = q_u(x_0)\int_{B'_r} \xi(x',0)\,d\mathcal{H}^{N-1} \tag{2.5.10}$$

By Lemma 2.5.5, $u_0 = \lambda^* x_N^-$. Substituting in (2.5.10) we get

$$g(\lambda^*)\int_{B'_r} \xi(x',0)\,d\mathcal{H}^{N-1} = q_u(x_0)\int_{B'_r} \xi(x',0)\,d\mathcal{H}^{N-1} \quad \forall\ \xi \in\ C_0^\infty(B_r).$$

Thus, $q_u(x_0) = g(\lambda^*)$. □

As a corollary we have

THEOREM 2.5.11. *Let u be a minimizer, then for $\mathcal{H}^{N-1}$ a.e $x_0 \in \partial\{u > 0\}$, the following properties hold,*

$$q_u(x_0) = g(\lambda^*)$$

and

$$u(x) = \lambda^*\langle x - x_0, \nu_u(x_0)\rangle^- + o(|x - x_0|)$$

where λ^* is such that, $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$.

PROOF. The result follows by Lemma 2.4.7 and by Theorem 2.5.5. □

6. Weak solutions

In this section we introduce the notion of weak solution. The idea, as in [**4**], is to identify the essential properties that minimizers satisfy and that may be found in applications in which minimization does not take place. For instance, in Chapter 3 we study an optimization problem, and prove that minimizers of the penalization problem are weak solutions in the sense of Definition 2.6.1. On the other hand, in Chapter 4 we study a singular perturbation problem for the operator $\mathcal{L}$ and prove that limits of this singular perturbation problem are weak solutions in the sense of Definition 2.6.2. In the next section, we will prove that weak solutions have smooth free boundaries. In this way, the regularity results may be applied both to minimizers and to limits of singular perturbation problems.

With these applications in mind, we introduce two notions of weak solution. Definition 2.6.1 is similar to the one in [**4**] for the case $\mathcal{L} = \Delta$. On the other hand, as stated before, Definition 2.6.2 is more suitable for limits of the singular perturbation problem.

Since we want to ask as little as possible for a function u to be a weak solution, some properties already proved for minimizers need a new proof. We keep these proofs as short as possible by sending the reader to the corresponding proofs for minimizers as soon as possible.

One of the main differences between these two definitions of weak solution is that for weak solutions according to Definition 2.6.1 almost every free boundary point is in the reduced free boundary. Instead, weak solutions according to Definition 2.6.2 may have an empty reduced boundary (see, for instance, example 5.8 in [**4**]).

In the sequel λ^* will be a fixed positive constant.

DEFINITION 2.6.1 (Weak solution I). *We call u a weak solution (I), if*

1. *u is continuous and non-negative in Ω and $\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$.*
2. *For $D \subset\subset \Omega$ there are constants $0 < c_{min} \leq C_{max}$, $\gamma \geq 1$, such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$*
$$c_{min} \leq \frac{1}{r}\Big(\fint_{B_r(x)} u^\gamma dx\Big)^{1/\gamma} \leq C_{max}$$
3. $$\mathcal{L}u = g(\lambda^*)\,\mathcal{H}^{N-1}\lfloor\partial_{red}\{u > 0\}.$$
i.e
$$-\int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\varphi\,dx = \int_{\Omega\cap\partial_{red}\{u>0\}} \varphi g(\lambda^*)\,d\mathcal{H}^{N-1} \quad \forall\ \varphi \in\ C_0^\infty(\Omega)$$
4. $$\limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)| \leq \lambda^*, \qquad \text{for every } x_0 \in \Omega\cap\partial\{u > 0\}$$

DEFINITION 2.6.2 (Weak solution II). *We call u a weak solution (II), if*

1. *u is continuous and non-negative in Ω and $\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$.*
2. *For $D \subset\subset \Omega$ there are constants $0 < c_{min} \leq C_{max}$, $\gamma \geq 1$, such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$*
$$c_{min} \leq \frac{1}{r}\Big(\fint_{B_r(x)} u^\gamma dx\Big)^{1/\gamma} \leq C_{max}$$
3. *For $\mathcal{H}^{N-1}$ a.e $x_0 \in \partial_{red}\{u > 0\}$, u has the asymptotic development*
$$u(x) = \lambda^*\langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|)$$
where $\nu(x_0)$ is the unit exterior normal to $\partial\{u > 0\}$ at x_0 in the measure theoretic sense.

4.
$$\limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)| \le \lambda^*, \qquad \textit{for every } x_0 \in \Omega\cap\partial\{u>0\}$$

and for any ball B in $\{u=0\}$ touching $\Omega\cap\partial\{u>0\}$ at x_0,we have,
$$\limsup_{x\to x_0} \frac{u(x)}{dist(x,B)} \ge \lambda^*$$

Remark 2.6.3. *Any minimizer is a weak solution in the sense of Definitions 2.6.1 and 2.6.2. In fact, (1) follows from Lemma 2.2.1, (2) from Lemmas 2.3.2 and 2.3.1, (3) from Theorem 2.5.11 and finally (4) from Lemma 2.5.3.*

Remark 2.6.4. *Observe that by hypothesis (1) of Definitions* (2.6.1) *and* (2.6.2) *we have by Lemma* (1.2.45) *that u is in $W^{1,G}_{loc}(\Omega)$ and $\Lambda := \mathcal{L}u$ is a nonnegative Radon measure with support in $\Omega\cap\partial\{u>0\}$ (in particular, u is an $\mathcal{L}-$ subsolution in Ω)*

Now we will prove as in Theorem 2.3.5, the density property of the set $\{u>0\}$ at free boundary points. It is not true in general, for weak solutions satisfying only properties (1) and (2) of Definitions 2.6.1 or 2.6.2 that the set $\{u=0\}$ has positive density at $\mathcal{H}^{N-1}-$ almost every free boundary point (see examples in [**4**]).

Theorem 2.6.5. *For any domain $D\subset\subset\Omega$ there exists a constant c, with $0<c<1$ depending on N,γ,g_0,δ,D, c_{min} and C_{max}, such that, for any function u satisfying (1) and (2) of Definitions 2.6.1 and 2.6.2 and for every $B_r\subset D$, centered at the free boundary we have*
$$\frac{|B_r\cap\{u>0\}|}{|B_r|} \ge c$$

Proof. The proof follows as in Theorem 2.3.5, the only difference here is that, instead of using Lemma 2.3.1 and 2.3.2, we use property (2) of Definitions 2.6.1 and 2.6.2. □

Remark 2.6.6. *Now, by Remark 2.3.8 we have that the free boundary has Lebesgue measure zero. Moreover, for every $D\subset\subset\Omega$, the intersection $\partial\{u>0\}\cap D$ has Hausdorff dimension less than N.*

Lemma 2.6.7. *If u satisfies hypothesis* (1) *and* (2) *of Definitions* (2.6.1) *and* (2.6.2) *then*

1. *u is Lipschitz and for any domain $D\subset\subset\Omega$, the Lipschitz constant depends only on $N,\gamma,g_0,\delta,dist(D,\partial\Omega)$ and C_{max}, provided D contains a free boundary point.*
2. *For any domain $D\subset\subset\Omega$ there exist constants c,C depending on N,γ,g_0,δ, D, c_{min} and C_{max}, such that, for every $B_r\subset D$ centered at the free boundary we have*
$$cr^{N-1} \le \int_{B_r} d\Lambda \le Cr^{N-1}.$$

PROOF. The proof of (1) is similar to the one in Theorem 2.2.25. The only change that we have to make here is the following, instead of using Lemma 2.2.13 we have to use property (2) of Definitions 2.6.1 and 2.6.2. We give the proof for the readers convenience.

Let $d(x) = \text{dist}(x, \Omega \cap \partial\{u > 0\})$. First, take x such that $d(x) < \frac{1}{5}\text{dist}(x, \partial\Omega)$. Let $y \in \partial\{u > 0\} \cap \partial B_{d(x)}(x)$. As $u > 0$ in $B_{d(x)}(x)$, $\mathcal{L}u = 0$ in that ball and u is an $\mathcal{L}-$ subsolution in $B_{3d(x)}(y)$. By using the gradient estimates and Harnack inequality (see Lemmas 1.2.18 and 1.2.14) and property (2) of Definitions 2.6.1 and 2.6.2 we have,

$$\begin{aligned}|\nabla u(x)| &\leq C\frac{1}{d(x)} \sup_{B_{d(x)}(x)} u \leq C\frac{1}{d(x)} \sup_{B_{2d(x)}(y)} u \\ &\leq C\frac{1}{d(x)}\Big(\fint_{B_{3d(x)}(y)} u^\gamma dx\Big)^{1/\gamma} \leq CC_{max}.\end{aligned}$$

So, the result follows in the case $d(x) < \frac{1}{5}\text{dist}(x, \partial\Omega)$.

Let r_1 such that $dist(x, \partial\Omega) \geq r_1 > 0\ \forall x \in D$, take D', satisfying $D \subset\subset D' \subset\subset \Omega$ given by

$$D' = \{x \in \Omega / dist(x, D) < r_1/2\}.$$

Let $x \in D$. If $d(x) \leq \frac{1}{5}\text{dist}(x, \partial\Omega)$ we have proved that $|\nabla u(x)| \leq C$.

If $d(x) > \frac{1}{5}\text{dist}(x, \partial\Omega)$, $u > 0$ in $B_{\frac{r_1}{5}}(x)$ and $B_{\frac{r_1}{5}}(x) \subset D'$ so that $|\nabla u(x)| \leq \frac{C}{r_1}\|u\|_{L^\infty(D')}$.

To prove the second part of (1), consider now a connected domain D that contains a free boundary point and let D' as in the previous paragraph. Let us see that $\|u\|_{L^\infty(D')}$ is bounded by a constant that depends only on $N, \gamma, D, r_1, \lambda, \delta$, and g_0. Let $r_0 = \frac{r_1}{4}$ and $x_0 \in D'$. Since D' is connected and not contained in $\{u > 0\} \cap \Omega$, there exists $x_1, ..., x_k \in D'$ such that $x_j \in B_{\frac{r_0}{2}}(x_{j-1})$ $j = 1, ..., k$, $B_{r_0}(x_j) \subset \{u > 0\}$ $j = 0, ..., k-1$ and $B_{r_0}(x_k) \not\subseteq \{u > 0\}$. Let $y_0 \in \partial\{u > 0\} \cap B_{r_0}(x_k)$. As u is an $\mathcal{L}-$ subsolution, by Lemma 1.2.14 there exists C depending on N, γ, δ, g_0 such that,

$$u(x_k) \leq C\Big(\fint_{B_{2r_0}(y_0)} u^\gamma dx\Big)^{1/\gamma} \leq CC_{max}r_0,$$

where in the last inequality we have used property (2) of Definitions 2.6.1 and 2.6.2. By Harnack inequality (Theorem 1.2.16) we have $u(x_{j+1}) \geq cu(x_j)$. Inductively we obtain $u(x_0) \leq Cr_0\ \forall x_0 \in D'$. Therefore, the supremum of u over D' can be estimated by a constant depending only on $N, \gamma, r_1, \lambda, \delta$, and g_0.

In order to prove (2) we use that Lemma 2.4.3 holds if u_k is a sequence of functions satisfying properties (1) and (2) of Definitions 2.6.1 and 2.6.2 with the same constants c_{min} and C_{max}. Then, the rest of the proof follows as in Theorem 2.4.4. □

REMARK 2.6.8. *Now, we are under the conditions used in the proof of Theorem 2.4.5 and therefore this result applies to functions u satisfying properties* (1) *and* (2) *of Definition 2.6.1 and 2.6.2. This is, $\Omega \cap \partial\{u > 0\}$ has finite perimeter and there exists a Borel function q_u defined on $\Omega \cap \partial\{u > 0\}$ such that $\mathcal{L}u = q_u \mathcal{H}^{N-1} \lfloor \partial\{u > 0\}$.*

As u satisfies the conclusions of Theorem 2.4.5, then Remark 1.4.62 also holds. We also have that any blow up sequence satisfies the properties of Lemma 1.6.13.

Moreover, we have the following result that holds at points $x_0 \in \partial_{red}\{u > 0\}$ that are Lebesgue points of the function q_u and are such that

$$\limsup_{r \to 0} \frac{\mathcal{H}^{N-1}(\partial\{u > 0\} \cap B(x_0, r))}{\mathcal{H}^{N-1}(B'(x_0, r))} \leq 1. \tag{2.6.9}$$

(Here $B'(x_0, r) = \{x' \in I\!R^{N-1} \,/\, |x'| < r\}$).

Recall that $\mathcal{H}^{N-1} - a.e.$ point in $\partial_{red}\{u > 0\}$ satisfies (2.6.9) (see Theorem 1.4.65).

Lemma 2.6.10. *If u is a function satisfying properties (1), (2) and (3) of Definition 2.6.1 or 2.6.2 we have that $q_u(x_0) = g(\lambda^*)$ for $\mathcal{H}^{N-1}$ a.e $x_0 \in \partial_{red}\{u > 0\}$.*

Proof. Clearly, we only have to prove the statement for weak solutions (II).

If u satisfies (3) of Definition 2.6.2, take $x_0 \in \partial_{red}\{u > 0\}$ such that

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|).$$

Take $\rho_k \to 0$ and $u_k(x) = \frac{1}{\rho_k} u(x_0 + \rho_k x)$. If $\xi \in C_0^\infty(\Omega)$ we have

$$-\int_{\{u>0\}} g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \nabla \xi \, dx = \int_{\partial\{u>0\}} q_u(x) \xi d\mathcal{H}^{N-1},$$

and if we replace ξ by $\xi_k(x) = \rho_k \xi(\frac{x - x_0}{\rho_k})$ with $\xi \in C_0^\infty(B_R)$, $k \geq k_0$ and we change variables we obtain,

$$-\int_{\{u_k>0\}} g(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \nabla \xi \, dx = \int_{\partial\{u_k>0\}} q_u(x_0 + \rho_k x) \xi d\mathcal{H}^{N-1}.$$

Now, recall that for a subsequence, $\chi_{\{u_k>0\}} \to \chi_{\{x_N<0\}}$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ and $g(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \rightharpoonup g(|\nabla u_0|) \frac{\nabla u_0}{|\nabla u_0|}$ $*-$ weakly in $L^\infty_{loc}(I\!R^N)$. Thus,

$$\int_{\{u_k>0\}} g(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \nabla \xi \, dx \to \int_{\{x_N<0\}} g(|\nabla u_0|) \frac{\nabla u_0}{|\nabla u_0|} \nabla \xi \, dx$$

On the other hand, $\partial\{u_k > 0\} \to \{x_N = 0\}$ locally in Hausdorff distance. Then, if x_0 is a Lebesgue point of q_u satisfying (2.6.9),

$$\int_{\partial\{u_k>0\}} q_u(x_0 + \rho_k x) \xi \, d\mathcal{H}^{N-1} \to q_u(x_0) \int_{\{x_N=0\}} \xi \, d\mathcal{H}^{N-1}. \tag{2.6.11}$$

As, $\nabla u_0 = -\lambda^* e_N \chi_{\{x_N<0\}}$, we deduce that for almost every point $x_0 \in \partial_{red}\{u > 0\}$, $q_u(x_0) = g(\lambda^*)$. $\square$

Now we prove the asymptotic development for weak solutions satisfying Definition 2.6.1.

LEMMA 2.6.12. *If u satisfies* (1), (2), (3) *and* (4) *of Definition 2.6.1, then for $x_0 \in \partial_{red}\{u>0\}$ satisfying* (2.6.9)*, u has the following asymptotic development*

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x-x_0|) \tag{2.6.13}$$

where $\nu(x_0)$ is the unit outer normal to the free boundary at x_0.

PROOF. Let $x_0 \in \partial_{red}\{u>0\}$ and let $\rho_k \to 0$. Let $u_k(x) = \frac{1}{\rho_k} u(x_0 + \rho_k x)$ be a blow up sequence (observe that u_k is again a weak solution in the rescaled domain). Assume that $u_k \to u_0$ uniformly on compact subsets of $I\!R^N$. Also assume that $\nu(x_0) = e_N$. As in the proof of Theorem 2.5.5 we deduce that

$$\begin{aligned} u_0 &\ge 0 \quad \text{in } \{x_N < 0\} \\ u_0 &= 0 \quad \text{in } \{x_N \ge 0\}. \end{aligned}$$

Let us see that $u_0 > 0$ in $\{x_N < 0\}$. To this end, let $D \subset\subset \{x_N < 0\}$ and let $\xi \in C_0^\infty(D)$. For k large enough,

$$-\int_{\{u_k>0\}} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla\xi\,dx = \int_{\partial_{red}\{u_k>0\}} g(\lambda^*)\xi(x)\,d\mathcal{H}^{N-1}. \tag{2.6.14}$$

By Lemma 1.4.67, we have that for every $x_0 \in \partial_{red}\{u>0\}$ satisfying (2.6.9),

$$\mathcal{H}^{N-1}(\partial\{u_k>0\}\cap D) \to 0 \quad \text{as } k \to \infty.$$

Thus, the right hand side of (2.6.14) goes to zero as $k \to \infty$. Since the left hand side goes to

$$-\int g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi\,dx$$

we deduce that $\mathcal{L}u_0 = 0$ in $\{x_N<0\}$. Thus, $u_0 > 0$ in $\{x_N < 0\}$.

As in Theorem 2.5.5 we have that there exists $0 < \lambda_0 < \infty$ such that

$$u_0(x) = \lambda_0 x_N^- + o(|x|).$$

By property (2) of Lemma 1.6.13 we have that

$$\chi_{\{u_k>0\}} \to \chi_{\{x_N<0\}} \quad \text{in } L^1_{loc}(I\!R^N) \quad \text{as } k \to \infty.$$

Let now $\xi \in C_0^\infty(I\!R^N)$ in (2.6.14). Passing to the limit as $k \to \infty$ and using Lemma 1.6.13 (1) we get,

$$-\int_{\{x_N<0\}} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi\,dx = \int_{\{x_N=0\}} g(\lambda^*)\xi(x)\,d\mathcal{H}^{N-1}.$$

Replacing ξ by $r\xi(x/r)$ with $r \to 0$, using the fact that $\frac{1}{r}u_0(rx) \to \lambda_0 x_N^-$ uniformly on compact sets of $I\!R^N$, changing variables and passing to the limit we get

$$g(\lambda_0)\int_{\{x_N<0\}} \xi_N\,dx = g(\lambda^*)\int_{\{x_N=0\}} \xi(x)\,d\mathcal{H}^{N-1}.$$

Thus, $\lambda_0 = \lambda^*$.

At this point we proceed as in Theorem 2.5.5 to deduce that actually $u_0(x) = \lambda^* x_N^-$ (observe that here we are using property (4) of Definition 2.6.1). As the blow up limit u_0 is independent of the blow up sequence ρ_k we conclude that u has the asymptotic development (2.6.13). $\square$

Now we prove the property that we mentioned in the introduction to this section. The following lemma only holds for weak solutions satisfying Definition 2.6.1.

LEMMA 2.6.15. *If u satisfies* (1), (2) *and* (3) *of Definition 2.6.1,*

1. $\mathcal{H}^{N-1}(\partial\{u>0\} \setminus \partial_{red}\{u>0\}) = 0$
2. $|D \cap \{u=0\}| > 0$ *for every open set $D \subset \Omega$ containing a point of $\{u=0\}$.*
3. *For any ball B in $\{u=0\}$ touching $\Omega \cap \partial\{u>0\}$ at x_0, there holds that,*

$$\limsup_{x\to x_0} \frac{u(x)}{dist(x,B)} \geq \lambda^* \tag{2.6.16}$$

PROOF. By Theorem 1.4.64 we have,

$$|\mu_u|(B_r(x_0)) = o(r^{N-1}) \quad \text{for } r \to 0 \tag{2.6.17}$$

for $\mathcal{H}^{N-1}$ almost all points $x_0 \in \partial\{u>0\}\setminus\partial_{red}\{u>0\}$ (Recall that $\mu_u = -\nabla\chi_{\{u>0\}}$). Let $x_0 \in \partial\{u>0\} \setminus \partial_{red}\{u>0\}$ satisfying (2.6.17). Then, if u_0 is a blow up limit with respect to balls $B_{\rho_k}(x_0)$, we obtain for $\xi \in C_0^\infty(B_1)$ that,

$$\begin{aligned}
-\int_{\mathbb{R}^N} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi\,dx \leftarrow -\int_{\mathbb{R}^N} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla\xi\,dx&\\
&= \rho_k^{1-N} g(\lambda^*) \int_{\partial_{red}\{u>0\}\cap B_{\rho_k}(x_0)} \xi\Big(\frac{y-x_0}{\rho_k}\Big)\,d\mathcal{H}^{N-1}\\
&= \rho_k^{1-N} g(\lambda^*) \int_{B_{\rho_k}(x_0)} \xi\Big(\frac{y-x_0}{\rho_k}\Big)\,d|\mu_u|(x)\\
&\leq C\rho_k^{1-N}|\mu_u|(B_{\rho_k}(x_0)) \to 0,
\end{aligned}$$

therefore $\mathcal{L}u_0 = 0$. Since $u_0(0) = 0$, we must have $u_0 = 0$, but this contradicts the nondegeneracy property (2) of the Definition 2.6.1. Therefore (1) holds.

To prove (2), suppose that $\chi_{\{u>0\}} = 1$ almost everywhere in D, hence the reduced boundary must be outside of D. Then, by Definition 2.6.1 (3), $\mathcal{L}u = 0$ in D, and therefore u is positive. Hence $D \cap \{u=0\} = \emptyset$.

In order to prove (3), Let l be the finite limit on the left of (2.6.16), and $y_k \to x_0$ with $u(y_k) > 0$ and

$$\frac{u(y_k)}{d_k} \to l, \quad d_k = \text{dist}(y_k, B).$$

Consider the blow up sequence u_k with respect to $B_{d_k}(x_k)$, where $x_k \in \partial B$ are points with $|x_k - y_k| = d_k$, and choose a subsequence with blow up limit u_0, such that

$$e := \lim_{k\to\infty} \frac{x_k - y_k}{d_k}$$

exists. Then by construction, since $l > 0$ by nondegenaracy, $u_0(-e) = l$, and $u_0(x) \le -l\langle x, e\rangle$ for $x \cdot e \le 0$, $u_0(x) = 0$ for $x \cdot e \ge 0$. Both, u_0 and $l\langle x, e\rangle^-$ are $\mathcal{L}$ solutions in $\{u_0 > 0\}$, and coincide in $-e$. Since $l > 0$, and $|\nabla u_0| > l/2$ in a neighborhood of $-e$, we have that $\mathcal{L}$ is uniformly elliptic there. Then we can apply the strong maximum principle to conclude that they must coincide in that neighborhood of $-e$. By a continuation argument, we have that $u_0 = l\langle x, e\rangle^-$.

By the Representation Theorem, $\forall\ \varphi \in\ C_0^\infty(B_1)$, $\varphi \ge 0$

(2.6.18)
$$\int_{\partial\{u_k>0\}} \varphi q_{u_k}\, d\mathcal{H}^{N-1} = -\int_{\mathbb{R}^N} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla\varphi\, dx \to -\int_{\mathbb{R}^N} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\varphi\, dx$$
$$= g(l)\int_{\{\langle x,e\rangle=0\}} \varphi\, d\mathcal{H}^{N-1}$$

and

$$\begin{aligned}\int_{\partial\{u_k>0\}} \varphi\, d\mathcal{H}^{N-1} &\ge \int_{\partial_{red}\{u_k>0\}} \varphi\langle e.\nu_{u_k}\rangle\, d\mathcal{H}^{N-1} \\ &= \int \varphi e.d\mu_{u_k} = \int_{\{u_k>0\}} \partial_e\varphi\, dx \to \int_{\{\langle x,e\rangle<0\}} \partial_e\varphi\, dx \\ &= \int_{\{\langle x,e\rangle=0\}} \varphi d\mathcal{H}^{N-1}.\end{aligned} \tag{2.6.19}$$

Therefore, for weak solutions of type I and II we have,

$$g(l) \ge \liminf_{x\to x_0} q_u(x).$$

Now, if u is a weak solution of type I we have, that $q_u(x) = g(\lambda^*)$ for $\mathcal{H}^{N-1}-$ a.e $x \in \Omega\cap\partial\{u > 0\}$. Thus, $g(l) \ge g(\lambda^*)$ and $l \ge \lambda^*$. □

We then conclude,

Theorem 2.6.20. *If u satisfies* (1), (2) (3) *and* (4) *of Definition 2.6.1, then for $\mathcal{H}^{N-1}$ a.e $x_0 \in \partial\{u > 0\}$, u has the asymptotic development* (2.6.13)

Proof. It follows by Lemmas 2.6.12 and 2.6.15. □

Remark 2.6.21. *Now we have that with the additional hypothesis* (4)*, weak solutions (I) satisfy the same properties that we proved in the previous section for minimizers (with the only difference that in (4) we have a less than or equal instead of an equal). Observe that minimizers have the asymptotic development* (2.6.13) *at every point in their reduced free boundary, but we only proved that this development holds at almost every point of $\partial_{red}\{u > 0\}$ when u is a weak solution.*

Remark 2.6.22. *We have proved that weak solutions I are also weak solutions II.*

7. Regularity of the free boundary

In this section we prove the regularity of the free boundary of a weak solution u in a neighborhood of every "flat" free boundary point. In particular, we prove the regularity in a neighborhood of every point in $\partial_{red}\{u > 0\}$ where u has the asymptotic development (2.6.13). Then, if u is a minimizer, $\partial_{red}\{u > 0\}$ is smooth and the remainder of the free boundary has $\mathcal{H}^{N-1}-$ measure zero.

The proof of the regularity of the free boundary is based on the works [**4**] and [**10**]. The main differences with [**10**] come from the fact that we don't assume the locally uniform positive density of the set $\{u \equiv 0\}$ at the free boundary. This is a property satisfied by minimizers that is not know to hold, in principle, for weak solutions that appear in a different context. This uniform density property implies, in particular, that $\mathcal{H}^{N-1}-$ almost every point on the free boundary belongs to the reduced free boundary and this is a very strong assumption that we don't want to make.

The proof will be done in a series of steps.

7.1. Flatness and nondegeneracy of the gradient.

DEFINITION 2.7.1 (Flat free boundary points). *Let $0 < \sigma_+, \sigma_- \leq 1$ and $\tau > 0$. We say that u is of class*

$$F(\sigma_+, \sigma_-; \tau) \quad in \quad B_\rho = B_\rho(0)$$

if

1. $0 \in \partial\{u > 0\}$ *and*

$$\begin{array}{lll} u = 0 & for & x_N \geq \sigma_+\rho, \\ u(x) \geq -\lambda^*(x_N + \sigma_-\rho) & for & x_N \leq -\sigma_-\rho. \end{array}$$

2. $|\nabla u| \leq \lambda^*(1 + \tau)$ *in* B_ρ.

If the origin is replaced by x_0 and the direction e_N by the unit vector ν we say that u is of class $F(\sigma_+, \sigma_-; \tau)$ in $B_\rho(x_0)$ in direction ν.

REMARK 2.7.2. *First, observe that we may suppose that $x_0 = 0$ and $\rho = 1$, if not we replace u by $v_1(x) = u(x_0 + x\rho)/\rho$. We also may suppose that $\lambda^* = 1$. In fact, take the function $g^*(t) = g(\lambda^* t)$. Then, g^* satisfies condition* (0.0.2) *with the same δ and g_0. If $v_2 = v_1/\lambda^*$, then v_2 satisfies all the properties of weak solution where the constants in (2) are replaced by C_{max}/λ^* and c_{min}/λ^*, and in (3) and (4) we have a one instead of λ^*. Finally, if we take $v_3(x) = v_2(Tx)$ where T is a rotation with $T(e_n) = \nu$ then u is a weak solution in $B_\rho(x_0)$ with $u \in F(\sigma_+, \sigma_-; \tau)$ in $B_\rho(x_0)$ in direction ν, if only if v_3 is a weak solution in B_1 associated to the function g^*, $\lambda^* = 1$ and with $v_3 \in F(\sigma_+, \sigma_-; \tau)$ in B_1 in direction e_N. In this section we will then suppose that $\rho = 1$, $x_0 = 0$, $\lambda^* = 1$ and $\nu = e_N$ but, by this observation, we will have that all the following results hold also in the general case.*

We will prove the following results,

THEOREM 2.7.3. *There exists $\sigma_0 > 0$ and $C_0 > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \text{ implies } u \in F(2\sigma, C_0\sigma; \sigma) \text{ in } B_{1/2}$$

for $0 < \sigma < \sigma_0$.

THEOREM 2.7.4. *For every $\delta > 0$ there exist $\sigma_\delta > 0$ and $C_\delta > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \text{ implies } |\nabla u| \geq 1 - \delta \text{ in } B_{1/2} \cap \{x_N \leq -C_\delta \sigma\}$$

for $0 < \sigma < \sigma_\delta$.

We first prove the following weak forms of the theorems.

LEMMA 2.7.5. *For every $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \text{ implies } u \in F(2\sigma, \varepsilon; \sigma) \text{ in } B_{1/2}$$

for $0 < \sigma < \sigma_\varepsilon$.

PROOF. We develop the proof of the Lemma in several steps.
Step I We use the following construction from [**4**] and [**5**]. Let

$$\eta(y) = \exp\left(\frac{-9|y|^2}{1 - 9|y|^2}\right)$$

for $|y| < 1/3$ and $\eta(y) = 0$ for $|y| > 1/3$, and chose $s \geq 0$ maximal such that

$$B_1 \cap \{u > 0\} \subset D := \{x \in B_1 : x_N < \sigma - s\eta(x')\},$$

where $x = (x', x_N)$. Hence, there exists a point

$$z \in B_{1/2} \cap \partial D \cap \partial\{u > 0\}.$$

Observe also that $s \leq \sigma$ since $0 \in \partial\{u > 0\}$. Now, let $\xi \in \partial B_{3/4}$ with $\xi_N \leq -1/2$. We want to prove an estimate for $u(\xi)$ from below. Consider the solution $v = v_{\kappa,\rho}$ of,

$$\begin{cases} \mathcal{L}v = 0 & \text{in} \quad D \setminus \overline{B_\rho(\xi)}, \\ v = 0, & \text{on} \quad \partial D \cap B_1, \\ v = (1 + \sigma)(\sigma - x_N), & \text{on} \quad \partial D \setminus B_1, \\ v = -(1 - \kappa\sigma)x_N, & \text{on} \quad \partial B_\rho(\xi), \end{cases}$$

where $\kappa > 0$ is large and $\rho > 0$ is a small constant to be chosen later.
Step II The function $v = v_{\kappa,\rho}$ constructed above satisfies.

$$\partial_{-\nu} v(z) \leq 1 + C\sigma - c\kappa\sigma \quad \text{for } z \in B_{1/2} \cap \partial D$$

for some positive constants $C = C(\rho)$ and $c = c(\rho)$, if $0 < \sigma < \sigma(\kappa, \rho)$.
The idea of the proof is to construct an explicit $\mathcal{L}-$ supersolution w in $D \setminus B_\rho(\xi)$ in order to to estimate v. We construct w of the form $w = v_1 - \kappa\sigma v_2$, where v_1 and v_2 are defined below.

First, let v_1 and v_2 be defined as follows.

$$v_1 = \frac{\gamma_1}{\mu_1}(1 - \exp(-\mu_1 d_1)) \quad \text{in } D,$$

where

$$d_1(x) = -x_N + \sigma - s\eta(x'),$$

and μ_1, γ_1 depending on σ are positive constants. Then,

$$1 \leq |\nabla d_1| \leq 1 + C\sigma, \quad |D^2 d_1| \leq C\sigma.$$

Hence, if b_{ij} is the matrix defined in Remark 1.2.25 with ellipticity β we have,

(2.7.6)

$$b_{ij}D_{ij}v_1 = \gamma_1 \exp(-\mu_1 d_1) b_{ij}(D_{ij}d_1 - \mu_1 D_i d_1 D_j d_1) \leq \gamma_1 \exp(-\mu_1 d_1)(C\sigma - \beta^{-1}\mu_1) < 0,$$

if we choose $\mu_1 = C_1\sigma$ for C_1 large. Next,

$$|\nabla v_1(x)| = \gamma_1 \exp(-\mu_1 d_1)|\nabla d_1| \geq \gamma_1 \exp(-\mu_1 d_1) \geq \gamma_1(1 - 2C_1\sigma) \geq 1, \tag{2.7.7}$$

if $\sigma < 2$, $\gamma_1 = 1 + C_2\sigma$ for $C_2 = 3C_1$ and $\sigma \leq \sigma_0(C_1)$.

Hence by Remark 1.2.25 v_1 is an $\mathcal{L}-$ supersolution in D.

Moreover, if $\sigma \leq 1$ then,

$$v_1(x) \geq \gamma_1 d_1(x)(1 - 2\mu_1) \geq (1 + \frac{C_1}{2}\sigma)d_1(x) \tag{2.7.8}$$

if $\sigma \leq \sigma_0(C_1)$.

If $x \in \partial D \setminus B_1$ and if $|x'| \geq 1/3$ then $\eta(x') = 0$ and we have $v(x) = 0$. If $x \in \partial D \setminus B_1$ and if $|x'| \leq 1/3$ then $x_N \geq -\sqrt{2/3}$ and by (2.7.8) we have,

$$\begin{aligned} v_1(x) &\geq (1 + C_1/2)d_1(x) = \sigma - x_N + \frac{C_1}{2}\sigma(\sigma - x_N) - (1 + \frac{C_1}{2}\sigma)s\eta \\ &\geq \sigma - x_N + \sigma(\frac{C_1}{2}(\sigma + \sqrt{2/3}) - (1 + \frac{C_1}{2}\sigma)) = \sigma - x_N + (\frac{C_1}{2}\sqrt{2/3} - 1)\sigma \\ &\geq \sigma - x_N + \sigma(\sigma + 1) \geq \sigma - x_N + \sigma(\sigma - x_N) = v(x), \end{aligned}$$

if $\sigma \leq \sigma_0(C_1)$.

If $x \in \partial B_\rho(\xi)$ and if $\rho \leq -1/4$ then $x_N \leq -1/4$ and by (2.7.8) we have,

$$\begin{aligned} v_1(x) &\geq (1 + C_1/2)d_1(x) = -x_N - \frac{C_1}{2}\sigma x_N + (\sigma - s\eta)(1 + \frac{C_1}{2}\sigma) \geq -x_N - \frac{C_1}{2}\sigma x_N \\ &\geq -x_N + \kappa\sigma x_N = v(x). \end{aligned}$$

Therefore, for $x \in \partial(D \setminus B_\rho(\xi))$

$$v_1(x) \geq (1 + \frac{C_1}{2}\sigma)d_1(x) \geq v(x),$$

if σ is sufficiently small. By the maximum principle,

$$v_1(x) \geq v \quad \text{in } D \setminus B_\rho(\xi).$$

We also have that at $z \in B_{1/2} \cap \partial D$

$$|\nabla v_1(z)| = \gamma_1 |\nabla d_1| \leq 1 + C\sigma. \tag{2.7.9}$$

Next, we define v_2 depending on $B_\rho(\xi)$ by

$$v_2(x) = \frac{\gamma_2}{\mu_2}(\exp(\mu_2 d_2) - 1) \quad \text{in } \widetilde{D} \setminus B_\rho(\xi),$$

with constants γ_2, μ_2, and $\widetilde{D} \subset D$ a domain with smooth boundary containing $D \setminus B_{1/10}(\partial B_1' \times \{0\})$, and d_2 is a function in $C^2(D \setminus B_\rho(\xi))$ satisfying,

$$\begin{aligned} d_2 &= 0 \quad \text{on } \partial\tilde{D}, \\ d_2 &= 1 \quad \text{on } \partial B_\rho(\xi), \\ C \geq |\nabla d_2| \geq c &> 0 \quad \text{in } \widetilde{D} \setminus \overline{B_\rho(\xi)}. \end{aligned}$$

Thus,

(2.7.10)

$$b_{ij} D_{ij} v_2 = \gamma_2 \exp(\mu_2 d_2) b_{ij}(D_{ij} d_2 + \mu_2 D_i d_2 D_j d_2) \geq \gamma_2 \exp(\mu_2 d_2)(-C + c\mu_2) > 0,$$

if μ_2 is large enough. Then choosing γ_2 such that $v_2 = 1$ on $\partial B_\rho(\xi)$ we have that in $\widetilde{D} \setminus \overline{B_\rho(\xi)}$ we have

$$|\nabla v_2| = \gamma_2 \exp(\mu_2 d_2)|\nabla d_2| \leq C, \tag{2.7.11}$$

and at the point z

$$|\nabla v_2(z)| = \gamma_2 |\nabla d_2(z)| \geq c > 0. \tag{2.7.12}$$

Thus by (2.7.6) and (2.7.10), the function $w = v_1 - \kappa\sigma v_2$ satisfies

$$b_{ij} D_{ij} w \leq 0 \text{ in } \widetilde{D} \setminus \overline{B_\rho(\xi)} \tag{2.7.13}$$

with

$$w = v_1 \geq v \quad \text{on } \partial\widetilde{D}.$$

If $x \in \partial B_\rho(\xi)$ and $\rho \leq 1/4$, then $-x_N \leq -1/4$ and using (2.7.8) with $C_3 = \frac{C_2}{2} \geq 3\kappa$ we have that,

$$\begin{aligned} w(x) &\geq (1 + C_3\sigma) d_1(x) - k\sigma = -x_N - C_3 \sigma x_N + (1 + C_3\sigma)(\sigma - s\eta) - \kappa\sigma \\ &\geq -x_N - 3\kappa\sigma x_N - \kappa\sigma \geq -x_N - \sigma\kappa x_N = v(x). \end{aligned}$$

Therefore

$$w(x) \geq v(x) \quad \text{on } \partial(\widetilde{D} \setminus B_\rho(\xi)) \tag{2.7.14}$$

The functions v_1, v_2 are C^2 and we proved in (2.7.7) and (2.7.11) that $|\nabla v_1| > 1$ and $|\nabla v_2| \leq C$. We obtain that $|\nabla w| \geq |\nabla v_1| - \kappa\sigma |\nabla v_2| \geq 1 - C\kappa\sigma > 0$ if $\sigma < \sigma(\kappa)$. This fact and (2.7.13) will imply that w is an $\mathcal{L}$- supersolution. On the other hand, by (2.7.14) $w \geq v$ on $\partial(\widetilde{D} \setminus \overline{B_\rho(\xi)})$, then by comparison we have that $w \geq v$ in $\widetilde{D} \setminus \overline{B_\rho(\xi)}$. Therefore, by (2.7.9) and (2.7.12) we have,

$$\partial_{-\nu} v(z) \leq \partial_{-\nu} w(z) = |\nabla v_1(z)| - \kappa\sigma |\nabla v_2(z)| \leq 1 + C\sigma - c\kappa\sigma.$$

Step III By (4) in Definition 2.6.2 and (3) in Lemma 2.6.15 we have that, if B is a ball in $\{u = 0\}$ touching the free boundary at x_0 then

$$\limsup_{x \to x_0} \frac{u(x)}{\operatorname{dist}(x, B)} \geq 1.$$

We want to prove that, for large $\kappa = \kappa(\rho)$

$$u(x_\xi) \geq v(x_\xi) \quad \text{for some } x_\xi \in \partial B_\rho(\xi). \tag{2.7.15}$$

Indeed, otherwise $u \leq v$ on $\partial B_\rho(\xi)$; also $u \leq v$ on ∂D (by 2.7.1 with (2) $\tau = \sigma$). Therefore $u \leq v$ on $\partial(D \setminus B_\rho(\xi))$ and, by comparison principle, also $u \leq v$ in $\overline{D} \setminus \overline{B_\rho(\xi)}$. By step III applied to z we have that

$$1 \leq \limsup_{x \to z} \frac{u(x)}{\operatorname{dist}(x, \partial D)} \leq \partial_{-\nu} v(z)$$

and the contradiction follows by step II choosing κ large enough .

Now, for κ large, we have

$$\begin{aligned} u(\xi) &\geq u(x_\xi) - \rho(1+\sigma) \geq v(x_\xi) - \rho(1+\sigma) = -(1-\kappa\sigma)(x_\xi)_N - \rho(1+\sigma) \\ &\geq -(x_\xi)_N - \kappa\sigma - 2\rho \geq -\xi_N - 4\rho \end{aligned}$$

for $\sigma < \sigma(\rho)$ sufficiently small. That is,

$$u(\xi) \geq -\xi_N - 4\rho \quad \text{on } \{\xi \in \partial B_{3/4}, \xi_N \leq -1/2\}. \tag{2.7.16}$$

Integrating along vertical lines and using that $|\nabla u| \leq 1 + \sigma$, we obtain for $\alpha > 0$,

$$u(\xi + \alpha e_N) \geq u(\xi) - \alpha(1+\sigma) \geq -\xi_N - 4\rho - \alpha - \alpha\sigma \geq -(\xi_N + \alpha) - 5\rho$$

for $\sigma < \sigma(\rho)$. Choosing $\rho = \varepsilon/10$, we complete the proof of the Lemma. □

LEMMA 2.7.17. *For every $\varepsilon > 0$ and $\delta > 0$ there exists $\sigma_{\varepsilon,\delta} > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad |\nabla u| \geq 1 - \delta \text{ in } B_{1/2} \cap \{x_N \leq -\varepsilon\}$$

for $0 < \sigma < \sigma_{\varepsilon,\delta}$.

PROOF. Assume the contrary. Then there exists a sequence $u_k \in F(1/k, 1; 1/k)$ such that

$$|\nabla u_k| \leq 1 - \delta \text{ for some } x_k \in B_{1/2} \cap \{x_N \leq -\varepsilon\}.$$

From Lemma 2.7.5 we have that $u_k \in F(2/k, 1/k; 1/k)$ and letting $k \to \infty$ we obtain,

$$u_k(x) \to u_0(x) = x_N^- \quad \text{uniformly on } \overline{B_{1/4}}.$$

Moreover on the set $\{u_0 > 0\} = \{x_N < 0\}$ the convergence is locally in $C^{1,\alpha}$. This implies that if $x_k \to x_0 \in B_{1/2} \cap \{x_N \leq -\varepsilon\}$, then $|\nabla u(x_0)| \leq 1 - \delta$ which contradicts the fact that $|\nabla u_0| = 1$. □

PROOF OF THEOREM 2.7.3. We revisit the proof of Lemma 2.7.5. Choose $\rho = 1/10$ and $\kappa = \kappa(\rho)$ such that (2.7.15) holds. We can refine the estimate (2.7.16) as follows. Set,

$$w(x) = (1+\sigma)(\sigma - x_N) - u(x).$$

Then $u \in F(\sigma, 1; \sigma)$ implies $w(x) \geq 0$ in $B_{2\rho}(\xi)$ and

$$w(x_\xi) \leq -(x_\xi)_N - v(x_\xi) + C\sigma \leq C\sigma.$$

From σ sufficiently small, we know from Lemma 2.7.17 that $|\nabla u| \geq 1/2$, hence u will satisfy $\mathcal{T}u = 0$, where $\mathcal{T}$ was define in (1.2.26). As a consequence, w will also satisfies $\mathcal{L}w = 0$. Then, applying the Harnack inequality we obtain that

$$w(\xi) \leq Cw(x_\xi) \leq C\sigma$$

or

$$u(\xi) \geq -\xi_N - C\sigma \quad \text{ on } \{\xi \in \partial B_{3/4},\, \xi_N \leq -1/2\}.$$

Integrating along vertical lines and using that $|\nabla u| \leq 1 + \sigma$, we conclude that

$$u(\xi + \alpha e_N) \geq u(\xi) \geq -(1+\sigma)\alpha \geq -(\xi_N + \alpha) - C\sigma,$$

which implies that $u \in F(2\sigma, C\sigma; \sigma)$ in $B_{1/2}$. $\square$

Proof of Theorem 2.7.4. Assume the contrary. Then there exists a sequence $\sigma_k \to 0$ and $u_k \in F(\sigma_k, 1; \sigma_k)$ such that

$$|\nabla u_k(x^k)| < 1 - \delta \text{ for some } x^k \in B_{1/2} \cap \{x_N \leq -k\sigma_k\}.$$

Let $d_k = \operatorname{dist}(x^k, \partial\{u_k > 0\})$ and $y^k \in \partial\{u_k > 0\}$ be such that $|x^k - y^k| = d_k$. From Theorem 2.7.3 it follows that $d_k \geq (k - C_0)\sigma_k$. Define now

$$\widetilde{u}_k(x) = \frac{u_k(y^k + 2d_k)}{2d_k}, \quad \tilde{x}_k = \frac{x^k - y^k}{2d_k}.$$

Then one easily verify that

$$\widetilde{u}_k \in F\Big(\frac{(C_0+1)}{2(k-C_0)}, 1; \sigma_k\Big) \quad \text{ in } B_1,$$

$$|\tilde{x}_k| = 1/2, (\tilde{x}_k)_N \leq -\frac{1}{2}\Big(1 - \frac{C_0+1}{k-C_0}\Big),$$

and $|\nabla \widetilde{u}_k(\tilde{x}_k)| < 1 - \delta$. This is a contradiction to Lemma 2.7.17. $\square$

7.2. Nonhomogeneous blow-up. We shall denote points in $\mathbb{R}^N$ by (y, h) with $y \in \mathbb{R}^{N-1}$ and $h \in \mathbb{R}$. Balls in $\mathbb{R}^{N-1}$ by $B'_\rho(y)$ and $B_1^- = B_1 \cap \{h < 0\}$.

Lemma 2.7.18. *Let $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ in B_{ρ_k} with $\sigma_k \to 0$, $\tau_k\sigma_k^{-2} \to 0$. For $y \in B'_1$, set*

$$f_k^+(y) = \sup\{h : (\rho_k y, \sigma_k\rho_k h) \in \partial\{u_k > 0\}\},$$
$$f_k^-(y) = \inf\{h : (\rho_k y, \sigma_k\rho_k h) \in \partial\{u_k > 0\}\}.$$

Then, for a subsequence,

1. $f(y) = \limsup_{\substack{z\to y\\k\to\infty}} f_k^+(z) = \liminf_{\substack{z\to y\\k\to\infty}} f_k^-(z)$ *for all* $y \in B'_1$.

   Further, $f_k^+ \to f$, $f_k^- \to f$ *uniformly,* $f(0) = 0$, $|f| \leq 1$ *and f is continuous.*

2. *f is subharmonic.*

PROOF. Define,

$$D_k := \{(y,h)/(y,\sigma_k h) \in B_1 \cap \{u_k > 0\}\}.$$

We can choose a subsequence u_k such that the set $\bar{D}_k$ is convergent in Hausdorff distance, and for a subsequence we define f as in (1). Therefore, for $y_0 \in B_1'$ there exist points y_k for all k such that

$$y_k \to y_0, \quad \text{ and } f_k^+(y_k) \to f(y_0).$$

As f is upper semicontinuous, given $\delta > 0$ there exists $\alpha > 0$ such that for k big,

$$\Big(\bar{B}_\alpha'(y_k) \times [f_k^+(y_k) + \delta, \infty)\Big) \cap \bar{D}_k = \emptyset$$

and then

$$\Big(\bar{B}_\alpha'(y_k) \times [(f_k^+(y_k) + \delta)\sigma_k, \infty)\Big) \cap B_1 \cap \{u_k > 0\} = \emptyset.$$

As $(y_k, \sigma_k f^+(y_k)) \in \partial\{u_k > 0\}$, then $u_k \in F(\sigma_k\delta/\alpha, 1; \tau_k)$ in $B_\alpha((y_k, \sigma_k f_k^+(y_k)).$

Since $\sigma_k \to 0$ $\tau_k = O(\sigma_k)$ we can apply Theorem 2.7.3 for k large, and obtain $u_k \in F(2\sigma_k\delta/\alpha, C\sigma_k\delta/\alpha; \tau_k)$ in $B_{\alpha/2}((y_k, \sigma_k f_k^+(y_k))$, which implies that for large k the set

$$\{(y,h) \in B_1/|y - y_k| < \alpha/4 \quad \text{and} \quad h < \sigma_k(f_k^+(y_k) - C\delta)\}$$

is contained in $\{u_k > 0\}$, that is

$$f_k^-(y) \geq f_k^+(y_k) - C\delta \quad \text{ for } y \in B_{\alpha/4}'(y_k),$$

then $\liminf_{\substack{y\to y_0 \\ k\to\infty}} f_k^-(y) \geq f(y_0) - 2C\delta$ which proves the assertion.

We may assume by replacing u_k by $\widetilde{u}_k = \frac{1}{\rho_k} u_k(\rho_k x)$, that $\rho_k = 1$. Let us assume, by contradiction that there is a ball $B_\rho'(y_0) \subset B_1'$ and a harmonic function g in a neighborhood of this ball, such that

$$g > f \text{ on } \partial B_\rho'(y_0) \quad \text{ and } \quad f(y_0) > g(y_0).$$

Let

$$Z = B_\rho'(y_0) \times \mathbb{R}, \qquad Z^+ = \{(y,h) \in Z,\, h > \sigma_k g(y)\},$$
$$Z^- = \{(y,h) \in Z,\, h < \sigma_k g(y)\}, \qquad Z^0 = \{(y,h) \in Z,\, h = \sigma_k g(y)\}.$$

Take $d_\delta(A)(x) = \min\{(1/\delta)\mathrm{dist}(x, \mathbb{R}^N \setminus A), 1\}$, then by the Representation Theorem 2.4.5 (see Remark 2.6.8) we arrive at,

$$\int_{\{u_k>0\}} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla(d_\delta(Z^+))\,dx = \int_{\partial\{u_k>0\}} d_\delta(Z^+) q_{u_k}(x)\,d\mathcal{H}^{N-1},$$

taking $\delta \to 0$ and assuming that $\mathcal{H}^{N-1}(Z^0 \cap \partial\{u_k > 0\}) = 0$ (if this is not true we replace g by $g + c_0$ for a small constant c_0) we have that

$$\int_{\{u_k>0\}\cap Z_0} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \cdot \nu\, d\mathcal{H}^{N-1} = \int_{\partial\{u_k>0\}\cap Z^+} q_{u_k}(x)\,d\mathcal{H}^{N-1}. \tag{2.7.19}$$

As $u_k \in F(\sigma_k, \sigma_k, \tau_k)$ we have that $|\nabla u_k| \leq \lambda^*(1+\tau_k)$ and, by Lemma 2.6.10, there holds that $q_{u_k}(x) = g(\lambda^*)$ for $\mathcal{H}^{N-1} - a.e$ point in $\partial_{red}\{u_k > 0\}$. Then, by (2.7.19) we have,

$$g(\lambda^*)\mathcal{H}^{N-1}(\partial_{red}\{u_k > 0\} \cap Z^+) \leq g(\lambda^*(1+\tau_k))\mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0). \tag{2.7.20}$$

On the other hand, using the fact that $f(y_0) > g(y_0)$, we can prove the following excess area type estimate,

$$\mathcal{H}^{N-1}(\partial_{red}E_k \cap Z) \geq \mathcal{H}^{N-1}(Z_0) + c\sigma_k^2, \tag{2.7.21}$$

where $E_k = \{u_k > 0\} \cup Z^-$. We leave the proof for a moment.

We also have,

$$\mathcal{H}^{N-1}(\partial_{red}E_k \cap Z) \leq \mathcal{H}^{N-1}(Z^+ \cap \partial_{red}\{u_k > 0\}) + \mathcal{H}^{N-1}(Z_0 \cap \{u_k = 0\}).$$

Using this inequality, (2.7.21) and the fact that $\mathcal{H}^{N-1}(Z_0 \cap \partial\{u_k > 0\}) = 0$ (if this is not true we replace g by $g + c_0$ for a small constant c_0) we have that,

$$\mathcal{H}^{N-1}(\partial_{red}\{u_k > 0\} \cap Z^+) \geq \mathcal{H}^{N-1}(Z_0 \cap \{u_k > 0\}) + c\sigma_k^2. \tag{2.7.22}$$

Finally by (2.7.20) and (2.7.22) we have that,

$$g(\lambda^*)\big[\mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0) + c\sigma_k^2\big] \leq g(\lambda^*(1+\tau_k))\mathcal{H}^{N-1}(\{u_k > 0\} \cap Z_0).$$

Therefore, for some positive constant c we have

$$c \leq \frac{g(\lambda^*(1+\tau_k)) - g(\lambda^*)}{\sigma_k^2}$$

and this contradicts the fact that $\frac{\tau_k}{\sigma_k^2} \to 0$ as $k \to \infty$.

So, we only have to prove (2.7.21). Let as take, for $\kappa > 0$, the solution of,

$$\begin{cases} \Delta\eta = -\varphi_\kappa & \text{in } B'_\rho(y_0), \\ \eta = g & \text{on } \partial B'_\rho(y_0), \end{cases}$$

Where $\varphi_\kappa \in C_0^\infty(B'_\rho(y_0))$, $0 \leq \varphi_\kappa \leq 1$ and is supported in $B'_\kappa(y_0)$. By the uniform estimates of η, we have that, when $\kappa \to 0$, η converges uniformly to g. Then since $f(y_0) > g(y_0)$, we can choose κ sufficiently small such that η is less that f in $B'_{2\kappa}(y_0)$.

Let $\widetilde{Z}^+ = \{(y,h) \in Z,\ h > \sigma_k\eta(y)\}$, and define analogously $\widetilde{Z}^-$ and $\widetilde{Z}_0$. As before, we can assume that the sets $\widetilde{Z}_0 \cap \partial E_k$ have $\mathcal{H}^{N-1}$ measure zero.

First observe that, since $u_k \in F(\sigma_k, \sigma_k; \tau_k)$, then $\partial\{u_k > 0\}$ lies in the strip $\{|x| \leq \sigma_k\}$. Therefore, we have,

$$|\widetilde{Z}^+ \cap E_k| \leq |\widetilde{Z}^+ \cap \{u_k > 0\}| + |\widetilde{Z}^+ \cap Z^+| \leq c\sigma_k \tag{2.7.23}$$

$$|\widetilde{Z}^- \setminus E_k| \leq |\widetilde{Z}^- \setminus \{u_k > 0\}| \leq c\sigma_k \tag{2.7.24}$$

Now, take the vector field,

$$V_k(x) = \frac{(-\sigma_k\nabla\eta(x'), 1)}{\sqrt{1 + |\sigma_k\nabla\eta(x')|^2}}.$$

We see that,

$$\mathrm{div} V_k = -\frac{\sigma_k \Delta\eta}{\sqrt{1+|\sigma_k \nabla\eta(x')|^2}} + \frac{\sigma_k^3}{(1+|\sigma_k\nabla\eta(x')|^2)^{3/2}} \sum_{1\le i,j\le N-1} \frac{\partial^2\eta}{\partial x_i \partial x_j}\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j},$$

then, as $D^2\eta$ is bounded and $\Delta\eta \le 0$ we have that

$$\mathrm{div} V_k \ge -C\sigma_k^3. \tag{2.7.25}$$

Moreover, if $x \in B'_\rho(y_0) \setminus B'_\kappa(y_0)$ then $\Delta\eta = 0$ and we have that $|\mathrm{div} V_k| \le C\sigma_k^3$. Now, observe that, by construction of η we have that, if k is large enough

$$B'_\kappa(y_0) \times \mathbb{R} \cap \widetilde{Z}^- \subset E_k,$$

therefore

$$|\mathrm{div} V_k| \le C\sigma_k^3 \quad \text{in } \widetilde{Z}^- \setminus E_k. \tag{2.7.26}$$

On the other hand we have,

$$\int_{E_k} \mathrm{div}(d_\delta(\widetilde{Z}^+)V_k)\, dx \le \int_{\partial_{red}E_k} |d_\delta((\widetilde{Z}^+)V_k)|\, d\mathcal{H}^{N-1} \le \mathcal{H}^{N-1}(\widetilde{Z}^+ \cap \partial_{red}E_k),$$

and since

$$\int_{E_k} \mathrm{div}(d_\delta(\widetilde{Z}^+)V_k)\, dx \to -\int_{\partial\widetilde{Z}^+\cap E_k} V_k\nu\, d\mathcal{H}^{N-1} + \int_{\widetilde{Z}^+\cap E_k} \mathrm{div} V_k\, dx$$

we have using (2.7.23) and (2.7.25) that,

$$\mathcal{H}^{N-1}(\widetilde{Z}^+ \cap \partial_{red}E_k) \ge \mathcal{H}^{N-1}(\widetilde{Z}^0 \cap E_k) - C\sigma_k^4.$$

Analogously, we get,

$$\mathcal{H}^{N-1}(\widetilde{Z}^- \cap \partial_{red}E_k) \ge \int_{\partial\widetilde{Z}^-\setminus E_k} V_k\nu\, d\mathcal{H}^{N-1} - \int_{\widetilde{Z}^-\setminus E_k} \mathrm{div} V_k\, dx,$$

and using (2.7.24) and (2.7.26) we obtain,

$$\mathcal{H}^{N-1}(\widetilde{Z}^- \cap \partial_{red}E_k) \ge \mathcal{H}^{N-1}(\widetilde{Z}^0 \setminus E_k) - C\sigma_k^4.$$

Thus, we proved

$$\mathcal{H}^{N-1}(Z \cap \partial_{red}E_k) \ge \mathcal{H}^{N-1}(\widetilde{Z}^0) - C\sigma_k^4. \tag{2.7.27}$$

Finally, using that g is harmonic in $B'_\rho(y_0)$ and $\eta = g$ on $\partial B'_\rho(y_0)$ we get,

$$\begin{aligned} \mathcal{H}^{N-1}(\widetilde{Z}^0) - \mathcal{H}^{N-1}(Z^0) &= \int_{B'_\rho(y_0)} \left(\sqrt{1+|\sigma_k\nabla\eta|^2} - \sqrt{1+|\sigma_k\nabla g|^2}\right) d\mathcal{H}^{N-1} \\ &\ge c\sigma_k^2 \int_{B'_\rho(y_0)} \left(|\nabla\eta|^2 - |\nabla g|^2\right) d\mathcal{H}^{N-1} - C\sigma_k^4 \\ &= c\sigma_k^2 \int_{B'_\rho(y_0)} |\nabla\eta - \nabla g|^2\, d\mathcal{H}^{N-1} - C\sigma_k^4. \end{aligned}$$

Combining this last inequality with inequality (2.7.27), we obtain the desired estimate.

□

The following Lemma was proved in [**5**],

LEMMA 2.7.28. *Let w be a function satisfying:*

$c_{ij}D_{ij}w = 0$ *in* $B_1 \cap \{h < 0\}$, *where* c_{ij} *is an elliptic constant matrix.*

$w(y,0) = g(y)$ *in the sence that* $w(y,h)$ *converges in* L^1 *to* g *as* $h \uparrow 0$

g *is subharmonic and continuous in* B_1', $g(0) = 0$.

$w(0,h) \le C|h|$,

$w \ge -C$.

Then

$$\int_0^{1/2} \frac{1}{r^2}\Big(\fint_{\partial B_r'(y)} g(y)\, d\mathcal{H}^{N-2}\Big)\, dr \le C_0,$$

where C_0 *is a constant depending only on* C.

PROOF. See Lemma 5.5 in [**5**]. □

LEMMA 2.7.29. *There exists a positive constant* $C = C(N)$ *such that, for any* $y_0 \in B_{r/2}'$,

$$\int_0^{1/4} \frac{1}{r^2}\Big(\fint_{\partial B_r'(y)} f - f(y_0)\, d\mathcal{H}^{N-2}\Big)\, dr \le C_1.$$

PROOF. It follows as in Lemma 8.3 at [**10**]. Without loss of generality we assume $\rho_k = 1$. Also, it is sufficient to prove it for $y_0 = 0$, since $u_k \in F(8\sigma_k, 8\sigma_k; \tau_k)$ in $B_{1/4}(y, \sigma_k f_k^+(y_0))$.

STEP I. Set $w_k(y,h) = \dfrac{u_k(y,h) + h}{\sigma_k}$. Then for subsequence,

$$\lim_{k\to\infty} w_k = w \text{ exists everywhere in } B_1^-.$$

The convergence is uniform in compact subsets of B_1^-, and w satisfies

$$c_{ij}D_{ij}w = \sum_{i=1}^{N-1} D_{ii}w + \frac{g'(1)}{g(1)} D_{NN}w = 0 \quad \text{in } B_1^-, \tag{2.7.30}$$

$$w(0,h) \le 0, \tag{2.7.31}$$

$$w(y,0) = f(y) \text{ in the sense that } \lim_{h\to 0^-} w(y,h) = f(y), \tag{2.7.32}$$

$$|w| \le C, \tag{2.7.33}$$

$$w(y,h) - w(y,0) \le 0 \text{ for all } (y,h) \in B_1^-. \tag{2.7.34}$$

In fact, since the free boundary of u_k lies in the strip $|x_N| \le \sigma_k$, $|\nabla u_k| \le 1 + \tau_k$, $\tau_k \le \sigma_k$, and we have $w_k \le C$ in B_1^-. The flatness assumption also implies $w_k \ge -C$ in B_1^-, and thus

$$|w_k| \le C \text{ in } B_1^-.$$

By Theorem 2.7.4 we have that

$$|\nabla u_k| \ge 1/2 \quad \text{in } B_1 \cap \{h \le -C_0\sigma_k\} \tag{2.7.35}$$

for σ_k sufficiently small. Then by Remark 1.2.25, u_k satisfies

$$b_{ij}(\nabla u_k) D_{ij} u_k = 0$$

where b_{ij} was define in (1.2.27), for $h \le -C_0\sigma_k$. Therefore, we have

$$b_{ij}(\nabla u_k) D_{ij} w_k = 0 \quad \text{in } B_1 \cap \{h \le -C_0\sigma_k\}. \tag{2.7.36}$$

From the flatness assumption and by Theorem 1.2.19 we have that for $D \subset B_1^-$ $\|u_k\|_{C^{1,\beta}(D)}$ is bounded, and therefore, for a subsequence, $u_k \to u$ in $C^1(D)$. Again, by the flatness assumption, we have that $u = -h$.

On the other hand, as w_k satisfies the uniformly elliptic equation (2.7.36) with continuous coefficients, and as the w_k are bounded we have by Remark 1.2.30 and Theorem 1.2.29 that for any compact set $D \subset B_1^-$ and any $q > 1$, $\|w_k\|_{C^{1,\beta}(D)}$ and $\|w_k\|_{W^{2,q}(D)}$ are bounded, therefore we may assume

$$\begin{aligned} w_k &\to w && \text{in } C^1 \text{ in compact subsets of } B_1^- \\ D^2 w_k &\to D^2 w && \text{in } L^q_{loc}(B_1^-) \text{ for any } pq > 1. \end{aligned} \tag{2.7.37}$$

Therefore, passing to the limit in (2.7.36) we obtain (2.7.30). Clearly, (2.7.33) is valid.

Since,

$$-D_N w_k = -\frac{1}{\sigma_k}(D_N u_k + 1) \le \frac{|\nabla u_k| - 1}{\sigma_k} \le \frac{\tau_k}{\sigma_k} \tag{2.7.38}$$

and $w_k(0,0) = 0$ we obtain, for $h \le 0$

$$w_k(0,h) \le |h| \frac{\tau_k}{\sigma_k} \to 0,$$

thus $w(0,h) \le 0$ and (2.7.31) follows.

It only remains to prove (2.7.32). First we show that for small δ and large K

$$w_k(y,h) \to f(y) \text{ uniformly in } D, \tag{2.7.39}$$

where $D := B'_{1-\delta} \times [-K,-1]$. By Lemma 2.7.18, it is sufficient to prove

$$w_k(y,h) - f_k^+(y) \to 0. \tag{2.7.40}$$

From (2.7.38) and by the definition of f_k^+ we obtain

$$
\begin{aligned}
w_k(y, h\sigma_k) - f_k^+(y) &\leq w_k(y, \sigma_k f_k^+(y)) - f_k^+(y) + (f_k^+(y) - h)\frac{\tau_k}{\sigma_k} \\
&= (f_k^+(y) - h)\frac{\tau_k}{\sigma_k} \leq (1+K)\frac{\tau_k}{\sigma_k} \to 0.
\end{aligned} \tag{2.7.41}
$$

To show (2.7.40) take a sequence $y_k \in B'_{1-\delta}$, $-K \leq h_k \leq -1$ and consider u_k in $B_{R\sigma_k}(x_k)$, where x_k is the free boundary point $x_k = (y_k, \sigma_k f_k^+(y_k))$ and R is a large constant. If we define,

$$
\widetilde{\delta}_k = \frac{1}{R} \sup_{y \in B'_{R\sigma_k}(y_k)} (f_k^+(y) - f_k^+(y_k)),
$$

we have that $x_k \in \partial\{u_k > 0\}$ and

$$
u_k(y,h) = 0 \quad \text{if } (y,h) \in B_{R\sigma_k}(x_k), \quad h - \sigma_k f_k^+(y_k) > \widetilde{\delta}_k R\sigma_k,
$$

and that means that,

$$
u_k \in F(\widetilde{\delta}_k, 1; \tau_k) \quad \text{in } B_{R\sigma_k}(x_k).
$$

Observe that, by Lemma 2.7.18, $\widetilde{\delta}_k \to 0$, then we can apply Theorem 2.7.3 and obtain that,

$$
u_k \in F(2\delta_k, C\delta_k; \tau_k) \quad \text{in } B_{(R/2)\sigma_k}(x_k),
$$

for $\delta_k = \max\{\widetilde{\delta}_k, \tau_k\}$. Hence, for $-R/2 \leq h \leq -C(R/2)\delta_k$,

$$
u_k(x_k + h\sigma_k e_N) \geq -(h\sigma_k + C\delta_k(R/2)\sigma_k).
$$

In other words,

$$
w_k(x_k + h\sigma_k e_N) - f_k^+(y_k) = \frac{u_k(x_k + h\sigma_k e_N) + h\sigma_k}{\sigma_k} \geq -C\delta_k(R/2) \to 0. \tag{2.7.42}
$$

For any $-K \leq h_k \leq -1$ we have that $0 \leq f_k^+(y_k) - h_k \leq 1 + K \leq R/2$ if we choose R large (depending only on K). Therefore, by taking $h = -f_k^+(y_k) + h_k$ in (2.7.42) and by (2.7.41) we obtain

$$
w_k(y_k, \sigma_k h_k) - f_k^+(y_k) \to 0
$$

and this holds for any $(y_k, h_k) \in D$, therefore (2.7.40) holds.

We now use a barrier argument. Let Ω_δ be a C^∞ domain with regular boundary that approximates $B^-_{1-\delta}$ in such a way that

$$
B^-_{1-2\delta} \subset \Omega_\delta \subset B^-_{1-\delta}.
$$

For small $\varepsilon > 0$, let also g_ε be a C^∞ function on $\partial\Omega_\delta$, which satisfies,

$$
\begin{aligned}
f - 2\varepsilon \leq g_\varepsilon &\leq f - \varepsilon \quad \text{on } \partial\Omega_\delta \cap \partial B^-_{1-3\delta} \cap \{h = 0\} \\
g_\varepsilon &\leq f - \varepsilon \quad \text{on } \partial\Omega_\delta \cap \{h = 0\} \\
g_\varepsilon &\leq w - \varepsilon \quad \text{on } \partial\Omega_\delta \cap \{h < 0\}.
\end{aligned} \tag{2.7.43}
$$

Let now, Φ_ε solve the Dirichlet problem,

$$b_{ij}(e_N)D_{ij}\Phi_\varepsilon = 1 \quad \text{in } \Omega_\delta \quad \Phi_\varepsilon = g_\varepsilon \quad \text{on } \partial\Omega_\delta.$$

Take $(y,h) \in \Omega_\delta \cap \{h = -K\sigma_k\}$. By continuity of Φ_ε, by (2.7.43) and (2.7.39) we have that,

$$\Phi_\varepsilon(y, -\sigma_k K) < \Phi_\varepsilon(y,0) + \varepsilon/2 \leq f(y) - \varepsilon/2 < w_k(y, -\sigma_k K),$$

for $k < k(K,\delta,\varepsilon)$.

On the other hand, if $(y,h) \in \partial\Omega_\delta \cap \{h < -K\sigma_k\}$ we have by (2.7.43), (2.7.37) that

$$\Phi_\varepsilon \leq w - \varepsilon < w_k,$$

for $k > k(\delta, K, \varepsilon)$.

Therefore,

$$w_k > \Phi_\varepsilon \quad \text{on } \partial(\Omega_\delta \cap \{h < -K\sigma_k\}) \tag{2.7.44}$$

for large $k \geq k(\varepsilon,\delta,K)$.

We may assume that $K > 2C_0$ where C_0 is the constant in (2.7.35). The function w_k is bounded and satisfies (2.7.36) which is uniformly elliptic, with elipticity constant β. Then, by interior gradient estimates, we deduce that

$$|\nabla w_k| \leq \frac{C}{(K - C_0)\sigma_k} \quad \text{in } \Omega_\delta \cap \{h \leq -K\sigma_k\},$$

where C is independent of k, K. In particular

$$|\nabla u_k - (-e_N)| \leq 2\frac{C}{K} \quad \text{in } \Omega_\delta \cap \{h \leq -K\sigma_k\}. \tag{2.7.45}$$

Hence, if $x \in V = \Omega_\delta \cap \{h \leq -K\sigma_k\}$ then

$$\begin{aligned} b_{ij}(\nabla u_k)D_{ij}\Phi_\varepsilon &= (b_{ij}(\nabla u_k) - b_{ij}(-e_N))D_{ij}\Phi_\varepsilon + 1 \\ &\geq 1 - \|\Phi_\varepsilon\|_{C^{1,1}(\Omega_\delta)}\|b_{ij}(\nabla u_k) - b_{ij}(-e_N)\|_{L^\infty(V)}. \end{aligned} \tag{2.7.46}$$

The function

$$H_{ij}(p) = \Big(\frac{g'(|p|)|p|}{g(|p|)} - 1\Big)\frac{p_i p_j}{|p|^2}$$

is uniformly continuous in any ring $\{c \leq |p| \leq C\}$. Therefore, as $1/2 \leq |\nabla u_k| \leq 1 + \tau_k$ in $\Omega_\delta \cap \{h \leq -K\sigma_k\}$, by (2.7.45) we can choose $K = K(\varepsilon,\delta)$ sufficiently large to make the right hand side of (2.7.46) positive. Thus,

$$b_{ij}(\nabla u_k)D_{ij}\Phi_\varepsilon > b_{ij}(\nabla u_k)D_{ij}w_k \quad \text{in } \Omega_\delta \cap \{h < -K\sigma_k\}$$

and combining with (2.7.44) we deduce $\Phi_\varepsilon \leq w_k$ in $\Omega_\delta \cap \{h < -K\sigma_k\}$, for sufficiently large k. Letting $k \to \infty$ we obtain $w(y,h) \geq \Phi_\varepsilon(y,h)$ in Ω_δ and consequently,

$$\liminf_{h\to 0^-} w(y,h) \geq f(y) - 2\varepsilon,$$

for $y \in B'_{1-3\delta}$. Similarly, by constructing a barrier from above, we obtain

$$\limsup_{h \to 0^-} w(y,h) \le f(y) + 2\varepsilon,$$

for $y \in B'_{1-3\delta}$. Since ε and δ were arbitrary, the proof of (2.7.32) is completed.

Now, take $h < h' < 0$ then by (2.7.38) we have that

$$w_k(y,h) - w_k(y,h') \le \frac{\tau_k}{\sigma_k}|h - h'|,$$

and taking limit we have,

$$w(y,h) - w(y,h') \le 0.$$

Using (2.7.32) we get, taking $h' \to 0$ that $w(y,h) - w(y,0) \le 0$, and the proof of (2.7.34) is completed.

STEP II. Fix $\bar{y} \in B'_{1/2}(0)$ and define,

$$w^*(y,h) = w(\frac{y}{2} + \bar{y}, \frac{h}{2}) - w(\bar{y}, 0), \quad (y,h) \in B_1^-.$$

Then if

$$g^*(y) = f(\frac{y}{2} + \bar{y}) - f(\bar{y}),$$

we have by Step I and Lemma 2.7.28 that

$$\int_0^{1/2} \frac{1}{r^2} \Big(\fint_{\partial B'_r(y)} g^*(y) \, d\mathcal{H}^{N-2} \Big) \, dr \le C_0,$$

and by definition we have,

$$\int_0^{1/4} \frac{1}{r^2} \Big(\fint_{\partial B'_r(y)} (f(y) - f(\bar{y}) \, d\mathcal{H}^{N-2} \Big) \, dr \le C_1(N),$$

and the result follows. $\square$

The following Lemma was proved in [**4**],

LEMMA 2.7.47. *Let g be a function satisfying,*

g is subharmonic and continuous in B'_1
$g(0) = 0, \quad |g| \le 1,$
there exists $C_1 > 0$ such that if $y \in B'_{1/2}$

$$\int_0^{1/4} \frac{1}{r^2} \Big(\fint_{\partial B'_r(y)} (g - g(y)) \, d\mathcal{H}^{N-2} \Big) \, dr \le C_1.$$

Then

1. *g is Lipschitz in $\bar{B}'_{1/4}$ with Lipschitz constant depending on C_1 and N.*
2. *There exists a constant $C = C(N) > 0$ and for $0 < \theta < 1$, there exists $c_\theta = c(\theta, N) > 0$, such that we can find a ball B'_r and a vector $l \in \mathbb{R}^{N-1}$ with*

$$c_\theta \le r \le \theta, \quad |l| \le C, \quad \text{and} \quad g(y) \le l.y + \frac{\theta}{2} r \quad \text{for } |y| \le r.$$

PROOF. See Lemma 7.7 and Lemma 7.8 in [**4**]. □

Now by Lemmas 2.7.29 and 2.7.47 we have the following,

LEMMA 2.7.48.

1. *f is Lipschitz in $\bar{B}'_{1/4}$ with Lipschitz constant depending on C_1 and N.*
2. *There exists a constant $C = C(N) > 0$ and for $0 < \theta < 1$, there exists $c_\theta = c(\theta, N) > 0$, such that we can find a ball B'_r and a vector $l \in \mathbb{R}^{N-1}$ with*

$$c_\theta \leq r \leq \theta, \quad |l| \leq C, \quad \text{and} \quad f(y) \leq l.y + \frac{\theta}{2}r \quad \text{for} \quad |y| \leq r.$$

Now, we can improve the flatness,

LEMMA 2.7.49. *Let θ, C, c_θ as in Lemma 2.7.48. There exists a positive constants σ_θ, such that*

$$u \in F(\sigma, \sigma; \tau) \text{ in } B_\rho \text{ in direction } \nu \tag{2.7.50}$$

with $\sigma \leq \sigma_\theta$, $\tau \leq \sigma_\theta \sigma^2$, implies

$$u \in F(\theta\sigma, 1; \tau) \text{ in } B_{\bar{\rho}} \text{ in direction } \bar{\nu}$$

for some $\bar{\rho}$ and $\bar{\nu}$ with $c_\theta \rho \leq \bar{\rho} \leq \theta\rho$ and $|\bar{\nu} - \nu| \leq C\sigma$, where $\sigma_\theta = \sigma_\theta(\theta, N)$.

PROOF. Let u_k a sequence as in Lemma 2.7.18. That is, $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ in $B_{\rho_k}(x_k)$ in direction ν_k with $\sigma_k \leq 1/k$, $\tau_k \leq \sigma_k^2/k$ and $C^*\rho_k \leq \sigma_k$.

For simplicity, we assume that $x_k = 0$ and $\nu_k = e_N$ for all k. Then if we define f as in Lemma 2.7.18 we have by Lemma 2.7.48 that

$$f(y) \leq l \cdot y + \frac{\theta}{2}r \quad \text{for } |y| \leq r,$$

with r, l as in that lemma. Therefore, again by Lemma 2.7.18 we have for k large depending on θ that,

$$f_k^+(y) \leq l \cdot y + \theta r \quad \text{for } |y| \leq r.$$

This means, by the definition of f_k^+, that

$$u_k(\rho_k y, \rho_k h) = 0 \quad \text{if } (y, h) \in B_r \quad \text{with } h \geq \sigma_k l \cdot y + \theta\sigma_k r.$$

But this means that u_k is of class $F(\bar{\sigma}_k, 1; \bar{\sigma}_k)$ in $B_{\bar{\rho}_k}$ in direction $\bar{\nu}_k$ with

$$\bar{\rho}_k := \rho_k r, \quad \bar{\sigma}_k := \frac{\theta\sigma_k}{\sqrt{1 + |\sigma_k l|^2}}, \quad \bar{\nu}_k := \frac{(-\sigma_k l, 1)}{\sqrt{1 + |\sigma_k l|^2}}.$$

As $\bar{\sigma}_k \leq \theta\sigma_k$, $c_\theta\rho_k \leq \bar{\rho}_k \leq \theta\rho_k$, and $|\bar{\nu}_k - e_N| \leq C\sigma_k$, the conclusion of the lemma is fulfilled for u_k.

For the case x_k and ν_k arbitrary, we take $v_k(x) = u_k(x_k + \rho_k T_k x)/\rho_k$ where T_k is a rotation, with $T_k e_N = \nu_k$. □

LEMMA 2.7.51. *Given $0 < \theta < 1$, there exist positive constants σ_θ, c_θ and C such that*

$$u \in F(\sigma, 1; \tau) \text{ in } B_\rho \text{ in direction } \nu \tag{2.7.52}$$

with $\sigma \leq \sigma_\theta$ and $\tau \leq \sigma_\theta \sigma^2$, then

$$u \in F(\theta\sigma, \theta\sigma; \theta^2\tau) \text{ in } B_{\bar\rho} \text{ in direction } \bar\nu$$

for some $\bar\rho$ and $\bar\nu$ with $c_\theta \rho \leq \bar\rho \leq \frac{1}{4}\rho$ and $|\bar\nu - \nu| \leq C\sigma$, where $c_\theta = c_\theta(\theta, N)$, $C = C(N, \delta, g_0)$, $\sigma_\theta = \sigma_\theta(\theta, N)$.

PROOF. Assume that $\rho = 1$. If σ_θ is small enough, we can apply Theorem 2.7.3 and obtain

$$u \in F(C\sigma, C\sigma; \tau) \text{ in } B_{1/2} \text{ in direction } \nu.$$

Then for $0 < \theta_1 \leq \frac{1}{2}$ we can apply Lemma 2.7.49, if again σ_θ is small, and we obtain

$$u \in F(C\theta_1\sigma, C\sigma; \tau) \text{ in } B_{r_1} \text{ in direction } \nu_1 \tag{2.7.53}$$

for some r_1, ν_1 with

$$c_{\theta_1} \leq 2r_1 \leq \theta_1, \text{ and } |\nu_1 - \nu| \leq C\sigma.$$

We obtain the improvement of the value τ inductively. In order to improve τ, we consider the functions $U_\varepsilon = \big(G(|\nabla u|) - G(\lambda^*) - \varepsilon\big)^+$ and $U_0 = \big(G(|\nabla u|) - G(\lambda^*)\big)^+$ in B_{2r_1}. By Lemma 2.5.3, and (4) in Definitions 2.6.1 and 2.6.2 we know that U_ε vanishes in a neighborhood of the free boundary. Since $U_\varepsilon > 0$ implies $G(|\nabla u|) > G(\lambda^*) + \varepsilon$, the closure of $\{U_\varepsilon > 0\}$ is contained in $\{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\}$.

Since $|\nabla u|$ is bounded from above in B_{2r_1}, and from below in the set $\{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\}$, we have that $F(|\nabla u|) \geq c$ in the set $\{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\} \cap B_{2r_1}$. Then hypothesis (1.2.32) and (1.2.33) of Lemma 1.2.31 are satisfied, and we have that $v = G(|\nabla u|)$ satisfies,

$$Mv = D_i(b_{ij}(\nabla u) D_j v) \geq 0 \quad \text{in } \{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\} \cap B_{2r_1},$$

where b_{ij} is defined in (1.2.26), and is β-elliptic in $\{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\}$.

Extending the operator M with the uniformly elliptic divergence-form operator

$$\widetilde{M} w = D_i(\widetilde{b}_{ij}(x) D_j w) \quad \text{in } B_{2r_1}$$

with measurable coefficients such that

$$\widetilde{b}_{ij}(x) = b_{ij}(\nabla u) \quad \text{in } \{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\},$$

we obtain

$$\widetilde{M} U_\varepsilon \geq 0 \quad \text{in } B_{2r_1}.$$

Moreover, by (2.7.52) we have that $U_\varepsilon \leq G(\lambda^*(1+\tau)) - G(\lambda^*)$ and by (2.7.53) $U_\varepsilon = 0$ in $B = B_{r_1/4}\big(\frac{r_1}{2}\nu_1\big)$, if $C\sigma \leq 1/2$.

Take now, V such that,

$$\begin{cases} \widetilde{M}V = 0 & \text{in } B_{2r_1} \setminus \bar{B}, \\ V = G(\lambda^*(1+\tau)) - G(\lambda^*) & \text{on } \partial B_{2r_1}, \\ V = 0 & \text{on } \partial B. \end{cases}$$

Then, there exists $0 < c(N, \beta) < 1$ such that $V \leq c\big(G(\lambda^*(1+\tau)) - G(\lambda^*)\big)$ in B_{r_1}. Applying the maximum principle we have that, $U_\varepsilon \leq c(G(\lambda^*(1+\tau)) - G(\lambda^*))$ in B_{r_1}. Taking $\varepsilon \to 0$ we obtain,

$$G(|\nabla u|) \leq cG(\lambda^*(1+\tau)) + G(\lambda^*)(1-c) \quad \text{in } B_{r_1}.$$

Since, $G(\lambda^*(1+\tau)) = G(\lambda^*) + g(\lambda^*)\lambda^*\tau + o(\tau)$ we have that

$$cG(\lambda^*(1+\tau)) + G(\lambda^*)(1-c) = G(\lambda^*) + cg(\lambda^*)\lambda^*\tau + o(\tau),$$

and since G is strictly increasing, we have,

$$\begin{aligned} |\nabla u| &\leq G^{-1}(G(\lambda^*) + cg(\lambda^*)\lambda^*\tau + o(\tau)) \\ &= \lambda^* + \frac{1}{g(\lambda^*)}(g(\lambda^*)\lambda^*\tau c + o(\tau)) + o(\tau) \\ &= \lambda^*\Big(1 + \tau\big(c + \frac{o(\tau)}{\tau}\big)\Big) \leq \lambda^*\Big(1 + \tau\frac{(c+1)}{2}\Big), \end{aligned}$$

if we choose τ small enough. And we see that if we choose θ_1 small enough (depending on N), we have

$$u \in F(\theta_0\sigma, 1; \theta_0^2\tau) \text{ in } B_{r_1} \text{ in direction } \nu_1,$$

where $\theta_0 = \sqrt{\frac{c+1}{2}}$.

We can repeat this argument a finite number of times, and we obtain

$$u \in F(\theta_0^m\sigma, 1; \theta_0^{2m}\tau) \text{ in } B_{r_1 \ldots r_m} \text{ in direction } \nu_m,$$

with

$$c_{\theta_j} \leq 2r_j \leq \theta_j, \text{ and } |\nu_m - \nu| \leq \frac{C}{1-\theta_0}\sigma.$$

Finally we choose m large enough and use Theorem 2.7.3. $\square$

7.3. Smoothness of the free boundary.

THEOREM 2.7.54. *Suppose that u is a weak solution, and $D \subset\subset \Omega$. Then there exist positive constants $\bar{\sigma}_0$, C and α such that if*

$$u \in F(\sigma, 1; \infty) \quad \textit{in } B_\rho(x_0) \subset D \textit{ in direction } \nu$$

with $\sigma \leq \bar{\sigma}_0$, $\rho \leq \bar{\rho}_0(\bar{\sigma}_0, \sigma)$, then

$$B_{\rho/4}(x_0) \cap \partial\{u > 0\} \textit{ is a } C^{1,\alpha} \textit{ surface,}$$

more precisely, a graph in direction ν of a $C^{1,\alpha}$ function, and, for any x_1, x_2 on this surface

$$|\nu(x_1) - \nu(x_2)| \leq C\sigma\Big|\frac{x_1 - x_2}{\rho}\Big|^\alpha$$

PROOF. By property (4) in Definitions 2.6.1 and 2.6.2 we have that, for every $\rho-$ neighborhood D_ρ of $D \cap \partial\{u > 0\}$,

$$|\nabla u(x)| \leq \lambda^* + \tau(\rho), \qquad \text{for every } x \in D_\rho$$

where $\tau(\rho) \to 0$ when $\rho \to 0$.

Therefore,

$$u \in F(\sigma, 1; \tau) \quad \textit{in } B_\rho(x_0) \text{ in direction } \nu.$$

Applying Theorem 2.7.3 we have that

$$u \in F(C_0\sigma, C_0\sigma; \tau) \quad \textit{in } B_{\rho/2}(x_0) \text{ in direction } \nu$$

if $\sigma \leq \sigma_0$ and $\tau \leq \sigma$.

Let $x_1 \in B_{\rho/2}(x_0) \cap \partial\{u > 0\}$ then

$$u \in F(C_0\sigma, 1; \tau) \quad \textit{in } B_{\rho/2}(x_1) \text{ in direction } \nu$$

and applying again Theorem 2.7.3 we have,

$$u \in F(C_0^2\sigma, C_0^2\sigma; \tau) \quad \textit{in } B_{\rho/4}(x_1) \text{ in direction } \nu$$

if $C_0\sigma \leq \sigma_0$ and $\tau \leq C_0\sigma$.

Let $0 < \theta < 1$, take $\rho_0 = \rho/4$, $\nu_0 = \nu$, $C = C_0^2$, $\sigma \leq \frac{\sigma_\theta}{C}$ and $\tau \leq \sigma_\theta C^2\sigma^2$. Now, by Lemma 2.7.51 and iterating we get that there exist sequences ρ_m and ν_m such that,

$$u \in F(\theta^m C\sigma, \theta^m C\sigma; \theta^{2m}\tau) \quad \textit{in } B_{\rho_m}(x_1) \text{ in direction } \nu_m$$

with $c_\theta\rho_m \leq \rho_{m+1} \leq \rho_m/4$ and $|\nu_{m+1} - \nu_m| \leq \theta^m C\sigma$.

Thus, we have that $|\langle x - x_1, \nu_m\rangle| \leq \theta^m C\sigma\rho_m$ for $x \in B_{\rho_m}(x_1) \cap \partial\{u > 0\}$.

We also have that there exists $\nu(x_1) = \lim_{m\to\infty} \nu_m$ and

$$|\nu(x_1) - \nu_m| \leq \frac{C\theta^m}{1-\theta}\sigma.$$

Now let $x \in B_{\rho/4}(x_1) \cap \partial\{u > 0\}$ and choose m such that $\rho_{m+1} \leq |x - x_1| \leq \rho_m$. Then

$$|\langle x - x_1, \nu(x_1)\rangle| \leq C\theta^m\sigma\Big(\frac{|x - x_1|}{1-\theta} + \rho_m\Big) \leq C\theta^m\sigma\Big(\frac{1}{1-\theta} + \frac{1}{c_\theta}\Big)|x - x_1|$$

and since $|x - x_1| \geq c_\theta^{m+1}\rho_0$ we have

$$\theta^{m+1} \leq \Big(\frac{|x - x_1|}{\rho_0}\Big)^\alpha \quad \text{with } \alpha = \frac{log(\theta)}{log(c_\theta)},$$

and we conclude that

$$|\langle x - x_1, \nu(x_1)\rangle| \leq \frac{C\sigma}{\rho^\alpha}|x - x_1|^{1+\alpha}.$$

Finally, observe that the result follows if we take, $\bar\sigma_0 = \min\{\sigma_0, \frac{\sigma_0}{C_0}, \frac{\sigma_\theta}{C}\}$ and if we choose $\bar\rho_0$ small enough such that if $\rho \leq \bar\rho_0$, $\tau(\rho) \leq \min\{\sigma, C_0\sigma, \sigma_\theta C^2\sigma^2\}$. $\square$

REMARK 2.7.55. *By Lemma 2.6.12 and Definition 2.6.2 we have that there exists a set* $A \subset \partial_{red}\{u > 0\}$*, with* $\mathcal{H}^{N-1}(\partial_{red}\{u > 0\} \setminus A) = 0$*, such that for* $x_0 \in A$ *we have that* $u \in F(\sigma_\rho, 1; \infty)$ *in* $B_\rho(x_0)$ *in direction* $\nu_u(x_0)$*, with* $\sigma_\rho \to 0$ *for* $\rho \to 0$*. Observe that by Theorem 2.5.5 when* u *is a minimizer* $A = \partial_{red}\{u > 0\}$*. Hence applying Theorem 2.7.54 we have,*

THEOREM 2.7.56. *Suppose that* g *satisfies* (0.0.2)*. If* u *is a weak solution then there exists a subset* $A \subset \partial_{red}\{u > 0\}$ *with* $\mathcal{H}^{N-1}(\partial_{red}\{u > 0\} \setminus A) = 0$ *such that for any* $x_0 \in A$ *there exists* $r > 0$ *so that* $B_r(x_0) \cap \partial\{u > 0\}$ *is a* $C^{1,\alpha}$ *surface. Moreover, if* u *satisfies Definition 2.6.1 then the remainder of* $\partial\{u > 0\}$ *has* $\mathcal{H}^{N-1}$*–measure zero. Finally, if* u *is a minimizer,* $\partial_{red}\{u > 0\}$ *is a* $C^{1,\alpha}$ *surface and* $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{red}\{u > 0\} = 0$*.*

7.4. Full regularity of minimizers in the case $N = 2$. We will prove, for minimizers of 2.0.1 that in dimension two, for a subclass of functions satisfying (0.0.2), their whole free boundary is a $C^{1,\alpha}$ surface.

The class that we consider consists of those functions satisfying condition (0.0.2) and such that,

(2.7.57) There exist constants t_0 and $k > 0$ so that $g(t) \leq kt$ for $t \leq t_0$.

Observe that this condition is satisfied for example, if $\delta \geq 1$ or when $g_0 \geq 1$ and there exists a constant C such that $\lim_{t\to 0} \dfrac{g(t)}{t^{g_0}} = C$.

In order to prove the full regularity, we first need two lemmas, the first one holds for any dimension and for any g satisfying (0.0.2).

LEMMA 2.7.58. *Let* $u \in \mathcal{K}$ *be a local minimizer. Given* $D \subset\subset \Omega$*, there exist constants* $C = C(N, D, \lambda^*)$*,* $r_0 = r_0(N, D) > 0$ *and* $\gamma = \gamma(N, D) > 0$ *such that, if* $x_0 \in D \cap \partial\{u > 0\}$ *and* $r < r_0$*, then*

$$\sup_{B_r(x_0)} |\nabla u| \leq \lambda^* + Cr^\gamma.$$

PROOF. The proof is similar to the proof of Theorem 7.1 in [**10**] but here we make a little modification by using a result of [**21**]. This result allows us to avoid having to add any new hypothesis to the function g.

Let U_ε, U_0, M and $\widetilde{M}$ be as in Lemma 2.7.51. Then $\widetilde{M}U_\varepsilon \geq 0$ in Ω.

For any $r > 0$ set

$$h_\varepsilon(r) = \sup_{B_r(x_0)} U_\varepsilon, \qquad h_0(r) = \sup_{B_r(x_0)} U_0,$$

for any $r < r_0 = \operatorname{dist}(D, \partial\Omega)$ and $x_0 \in D \cap \partial\{u > 0\}$.

Then, $h_\varepsilon(r) - U_\varepsilon$ is a $\widetilde{M}$- supersolution in the ball $B_r(x_0)$ and

$$\begin{aligned} h_\varepsilon(r) - U_\varepsilon \ &\geq 0 && \text{in } B_r(x_0) \\ &= h_\varepsilon(r) && \text{in } B_r(x_0) \cap \{u = 0\}. \end{aligned}$$

Applying the weak Harnack inequality (see [**17**] Theorem 8.18) with $1 \leq p < N/(N-2)$, we get

$$\inf_{B_{r/2}(x_0)} \big(h_\varepsilon(r) - U_\varepsilon\big) \geq cr^{-N/p}\|h_\varepsilon(r) - U_\varepsilon\|_{L^p(B_r(x_0))} \geq ch_\varepsilon(r),$$

since, by Theorem 2.1.1, $|B_r(x_0) \cap \{u = 0\}| \geq cr^N$. Taking now $\varepsilon \to 0$ we obtain

$$\inf_{B_{r/2}(x_0)} \big(h_0(r) - U_0\big) \geq ch_0(r),$$

for some $0 < c < 1$, which is the same as

$$\sup_{B_{r/2}(x_0)} U_0 \leq (1-c)h_0(r).$$

Therefore

$$h_0\left(\frac{r}{2}\right) \leq (1-c)h_0(r),$$

from which it follows that $h_0(r) \leq Cr^\gamma$ for some $C > 0$, $0 < \gamma < 1$. That is,

$$G(|\nabla u|) \leq G(\lambda^*) + Cr^\gamma$$

and therefore

$$|\nabla u| \leq \lambda^* + Cr^\gamma$$

and now the conclusion of the lemma follows. □

In the following Lemma is where we need to impose condition (2.7.57).

LEMMA 2.7.59. *Let $\Phi(t) = g(t)t - G(t)$, and g satisfying condition* (2.7.57). *Let x_0 be a free boundary point, $D \subset\subset \Omega$ and $B_\mu(x_0) \subset D$. Take $v = \max(u - t\eta, 0)$, where $t > 0$, $\eta \in C_0^\infty(\Omega)$, $\eta = 0$ in $\Omega \setminus B_{\mu(x_0)}$, $\eta \geq 0$ and $|\nabla\eta| \leq C/t$. Then,*

$$\int_{B_\mu(x_0)\cap\{u>0\}} (G(|\nabla v|) - G(|\nabla u|))\,dx \leq \int_{B_\mu(x_0)\cap\{0<u\leq t\eta\}} \Phi(|\nabla u|)\,dx + C_0 t^2 \int_{B_\mu(x_0)\cap\{u>t\eta\}} |\nabla\eta|^2\,dx,$$

for $C_0 = C_0(N, \delta, g_0, dist(D, \partial\Omega), C)$.

PROOF. The Lemma follows as in Theorem 4.3 in [**5**] by making some modifications.

First, observe that for $0 \leq s \leq 1$ we have $|\nabla u - ts\nabla\eta| \leq |\nabla u| + C \leq C_1 + C$, where C_1 is the constant in Theorem 2.2.25 . On the other hand, if g satisfies (2.7.57), and if $F(a) = \frac{g(a)}{a}$ then for $0 \leq a \leq C_1 + C$, there exists a constant C_0 such that $F(a) \leq C_0$. Therefore we have that,

$$F(|\nabla u - st\nabla\eta|) \leq C_0 \quad \text{for all } 0 \leq s \leq 1. \tag{2.7.60}$$

Set $f(p) = G(|p|)$, then

(2.7.61)
$$f(\nabla v) - f(\nabla u) = \int_0^1 f_p(\nabla u - t\nabla(u-v))\nabla(v-u)\,dt = f_p(\nabla u)\nabla(v-u) + \int_0^1\int_0^t \nabla(v-u) f_{pp}(\nabla u - s\nabla(u-v))\nabla(v-u)\,ds\,dt.$$

On the set $\{u > t\eta\}$ we have $v - u = -t\eta$, hence by (1.2.24) and (2.7.60) we have,

(2.7.62)
$$\int_{\{u>t\eta\}\cap B_\mu(x_0)} \int_0^1\int_0^t \nabla(v-u) f_{pp}(\nabla u - s\nabla(u-v))\nabla(v-u)\,ds\,dt\,dx \leq C\int_{\{u>t\eta\}\cap B_\mu(x_0)} |\nabla v - \nabla u|^2\,dx = Ct^2 \int_{\{u>t\eta\}\cap B_\mu(x_0)} |\nabla\eta|^2\,dx.$$

On the set $\{u \leq t\eta\}$ we have $v = 0$ and thus,

$$\int_{\{u\leq t\eta\}\cap B_\mu(x_0)} \int_0^1\int_0^t \nabla(v-u) f_{pp}(\nabla u - s\nabla(u-v))\nabla(v-u)\,ds\,dt = \int_{\{u\leq t\eta\}\cap B_\mu(x_0)} \int_0^1\int_0^t \nabla u f_{pp}((1-s)\nabla u)\nabla u\,ds\,dt\,dx.$$

Next,

$$0 = f(0) = f(\nabla u) - \int_0^1 f_p((1-t)\nabla u)\nabla u\,dt = f(\nabla u) - f_p(\nabla u)\nabla u + \int_0^1\int_0^t \nabla u f_{pp}((1-s)\nabla u)\nabla u\,ds\,dt$$

(2.7.63)
$$\int_{\{u\leq t\eta\}\cap B_\mu(x_0)} \int_0^1\int_0^t \nabla(v-u) f_{pp}(\nabla u - s\nabla(u-v))\nabla(v-u)\,ds\,dt \leq \int_{\{u\leq t\eta\}\cap B_\mu(x_0)} \Phi(|\nabla u|)\,dx$$

Therefore by (2.7.61), (2.7.62) and (2.7.63) we have

$$\int_{B_\mu(x_0)\cap\{u>0\}} (G(|\nabla v|) - G(|\nabla u|))\,dx \leq \int_{\{u\leq t\eta\}} \Phi(|\nabla u|)\,dx + Ct^2\int_{\{u>t\eta\}} |\nabla\eta|^2\,dx + \int_\Omega f_p(\nabla u)\nabla(v-u)\,dx.$$

In the last integral the integrand vanishes on the set $\{u = 0\}$. Since also $v - u = 0$ on $\partial\{u > 0\}$ and $\mathcal{L}u = 0$ on $\{u > 0\}$, this integral vanishes. (For a rigorous proof approximate $v - u$ by $-min(u - \delta, t\eta)$ for $\delta > 0$). The result follows. $\square$

Now, following ideas from [**2**], using Lemmas 2.7.58 and 2.7.59, we prove, for $N = 2$ and g satisfying (2.7.57) the following, THEOREM 2.7.64. *Let $N = 2$, g satisfying* (2.7.57) *and u a minimizer. Then, for any ball B_r centered at the free boundary we have,*

$$\fint_{B_r\cap\{u>0\}} (\Phi(\lambda^*) - \Phi(|\nabla u|))^+ \to 0 \text{ as } r \to 0,$$

where $\Phi(t) = g(t)t - G(t)$ and $\Phi(\lambda^*) = \lambda$.

PROOF. Let $1 \geq \mu > r$, and take v, as in Lemma 2.7.59, then $\mathcal{J}(u) \leq \mathcal{J}(v)$ and

$$\int_{B_\mu(x_0)\cap\{0<u\leq t\eta\}} \lambda \leq \int_{B_\mu(x_0)\cap\{u>0\}} (G(|\nabla v|) - G(|\nabla u|))\, dx. \tag{2.7.65}$$

Let $t = Cr$, where C is the constant such that $u \leq Cr$ in B_r. Now choose

$$\eta(x) = \begin{cases} \frac{\log(\mu/|x-x_0|)}{\log(\mu/r)} & \text{in } B_\mu(x_0)\setminus B_r(x_0),\\ 1 & \text{in } B_r(x_0)\\ 0 & \text{in } \Omega\setminus B_\mu(x_0),\end{cases}$$

observe that the condition $|\nabla\eta| \leq C/t$ is satisfied if we choose μ such that $\mu > 2r$.

Observe that,

$$\begin{aligned}\int_{B_\mu(x_0)\cap\{0<u\leq t\eta\}} (\Phi(\lambda^*) - \Phi(|\nabla u|))\, dx = &\int_{B_\mu(x_0)\cap\{0<u\leq t\eta\}} (\Phi(\lambda^*) - \Phi(|\nabla u|))^+\, dx\\ &- \int_{B_\mu(x_0)\cap\{0<u\leq t\eta\}} (\Phi(|\nabla u|) - \Phi(\lambda^*))^+\, dx.\end{aligned}$$

On the other hand by our election of t and η, we have that in B_r, $u \leq t\eta$ and then by Lemma 2.7.59, (2.7.65) and the definition of λ^*, we have,

$$\begin{aligned}\int_{B_r(x_0)\cap\{u>0\}} (\Phi(\lambda^*) - \Phi(|\nabla u|))^+\, dx \leq &\int_{B_\mu(x_0)} (\Phi(|\nabla u|)) - (\Phi(\lambda^*))^+\, dx\\ &+ \frac{Cr^2}{\log(\mu/r)}.\end{aligned}$$

By Lemma 2.7.58, we have that $\Phi(|\nabla u|) - \Phi(\lambda^*) \leq \Phi(\lambda^* + Cr^\gamma) - \Phi(\lambda^*) = \Phi'(\xi)Cr^\gamma$, for $\lambda^* \leq \xi \leq \lambda^* + Cr^\gamma$. As $\Phi'(t) = g'(t)t \leq g_0 g(t)$, and as g is nondecreasing we have that $\Phi'(\xi) \leq g_0 g(\xi) \leq g_0 g(\lambda^* + Cr^\gamma)$. Therefore we have,

$$\frac{1}{r^2}\int_{B_r(x_0)\cap\{u>0\}} (\Phi(\lambda^*) - \Phi(|\nabla u|))^+\, dx \leq C\Big(\frac{\mu^{\gamma+2}}{r^2} + \frac{r^2}{\log(\mu/r)}\Big).$$

Taking $r = \mu^{1+\beta}$, with $\beta < \min\{\gamma/2, 1/2\}$, we have the desired result. $\square$

COROLLARY 2.7.66. *Let $N = 2$, suppose that g satisfies* (0.0.2) *and moreover satisfies* (2.7.57). *Let $u \in \mathcal{K}$ be a solution to* (2.0.1). *Then $\partial\{u > 0\}$ is a $C^{1,\alpha}$ surface locally in Ω.*

Proof. The proof follows now as in [**4**], we give the proof here for the readers convenience. Let u_k be a blow up sequence converging to u_0. Since, $\nabla u_k \to \nabla u_0$ a.e in $\mathbb{R}^N$, we conclude from Theorem 2.5.3 and Theorem 2.7.64 that $|\nabla u_0| = \lambda^*$ in $B_1 \cap \{u_0 > 0\}$, and then

$$0 = \mathcal{L}u_0 = div\Big(\frac{g(|\nabla u_0|)}{|\nabla u_0|}\nabla u_0\Big) = \frac{g(\lambda^*)}{\lambda^*}\triangle u_0 \quad \text{ in } \{u_0 > 0\}.$$

Therefore u_0 is harmonic in $\{u_0 > 0\}$, and if we take $v = |\nabla u_0|^2$, we have $0 = \triangle v = |D^2 u_0|^2$ and that means that ∇u_0 is constant in each connected component of this set. Therefore, by Lemma 1.6.13 (5) and (7) we have,

$$u_0 = \lambda^* \max\langle x, \nu_0, 0\rangle + q \max\langle -x, \nu_0, s\rangle,$$

for some ν_0 and $q, s \geq 0$. Since $\{u_0 = 0\}$ has positive density at the origin, we have that $s > 0$ or $q = 0$. Therefore, we have proved that any blow up sequence has a subsequences that converges to a half linear function $u_0 = \lambda^* \max\langle x, \nu_0, 0\rangle$ in some neighborhood of the origin, then applying Theorem 2.7.54 we have the desired result. $\square$

CHAPTER 3

The optimization problem

In this Chapter we study, the following optimization problem. Take Ω a smooth bounded domain in $\mathbb{R}^N$ and $\varphi_0 \in W^{1,G}(\Omega)$, a Dirichlet datum, with $\varphi_0 \geq c_0 > 0$ in $\bar{A}$, where A is a nonempty relatively open, C^2 subset of $\partial\Omega$. Here $W^{1,G}(\Omega)$ is a Sobolev-Orlicz space (see Chapter 1). Let

$$\mathcal{K}_\alpha = \{u \in W^{1,G}(\Omega) \,/\, |\{u > 0\}| = \alpha,\, u = \varphi_0 \ \text{ on } \partial\Omega\}.$$

Our problem is to minimize in $\mathcal{K}_\alpha$, the functional $\mathcal{J}(u) = \int_\Omega G(|\nabla u|)\, dx$, with $g = G'$ satisfying (0.0.2).

1. The penalized problem

In order to solve our original problem in a way that allows us to perform non volume preserving perturbations we follow the idea of [**2**] and consider instead the following penalized problem: We let

$$\mathcal{K} = \{u \in W^{1,G}(\Omega) \,/\, u = \varphi_0 \ \text{ on } \partial\Omega\}$$

and

$$\mathcal{J}_\varepsilon(u) = \int_\Omega G(|\nabla u|)\, dx + F_\varepsilon(|\{u > 0\}|), \tag{3.1.1}$$

where

$$F_\varepsilon(s) = \begin{cases} \varepsilon(s-\alpha) & \text{if } s < \alpha \\ \frac{1}{\varepsilon}(s-\alpha) & \text{if } s \geq \alpha. \end{cases}$$

Then, the penalized problem is

$$\text{Find } u_\varepsilon \in \mathcal{K} \quad \text{such that} \quad \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v\in\mathcal{K}} \mathcal{J}_\varepsilon(v). \tag{P_ε}$$

In the next section we will study the properties of solutions to (P_ε).

1.1. Existence, regularity of minimizers and their free boundaries. We begin by discussing the existence of extremals and the regularity. We are going to give some properties of the minimizers, but as the functional $\mathcal{J}_\varepsilon$ is very similar to the one in Chapter 2, we are only going to state the results and avoid any proof. The only difference between the two functionals is that in Chapter 2 the functional is lineal in $|\{u > 0\}|$ and here it is piecewise linear. Next, we prove that any minimizer of $\mathcal{J}_\varepsilon$ is a weak solution, as defined in Chapter 2. Therefore we will have, by the results therein that the free boundary is smooth.

THEOREM 3.1.2. *Let $\Omega \subset \mathbb{R}^N$ be bounded. Then there exists a solution to the problem (P_ε). Moreover, any such solution u_ε has the following properties:*

1. *u_ε is locally Lipschitz continuous in Ω, and for $D \subset\subset \Omega$ we have that, $\|\nabla u\|_{L^\infty}(D) \leq C$ with $C = C(N, g_0, \delta, dist(\partial\Omega, D))$.*
2. *$\mathcal{L}u_\varepsilon = 0$ in $\{u_\varepsilon > 0\}$.*
3. *There are constants $0 < c_{min} \leq C_{max}$, $\gamma \geq 1$, such that for balls $B_r(x) \subset D$ with $x \in \partial\{u_\varepsilon > 0\}$*
$$c_{min} \leq \frac{1}{r}\Big(\fint_{B_r(x)} u_\varepsilon^\gamma dx\Big)^{1/\gamma} \leq C_{max}$$
4. *For every $D \subset\subset \Omega$, there exist constants $C, c > 0$ such that for every $x \in D \cap \{u_\varepsilon > 0\}$,*
$$c\,\mathrm{dist}(x, \partial\{u_\varepsilon > 0\}) \leq u_\varepsilon(x) \leq C\,\mathrm{dist}(x, \partial\{u_\varepsilon > 0\}).$$
5. *For every $D \subset\subset \Omega$, there exists a constant $c > 0$ such that for $x \in \partial\{u_\varepsilon > 0\}$ and $B_r(x) \subset D$,*
$$c \leq \frac{|B_r(x) \cap \{u_\varepsilon > 0\}|}{|B_r(x)|} \leq 1 - c.$$

The constants may depend on ε.

PROOF. Observe that since F_ε satisfies,
$$\text{if } A \leq B, \text{ then } \varepsilon(A - B) \leq F_\varepsilon(A) - F_\varepsilon(B) \leq \frac{1}{\varepsilon}(A - B),$$
then if u_ε is a minimizer, $B_r \subset\subset \Omega$ and v is a solution of
$$\mathcal{L}v = 0 \quad \text{in } B_r, \qquad v - u_\varepsilon \in W_0^{1,G}(B_r).$$
we have the following inequality
$$\begin{aligned}\varepsilon(|B_r \cap \{v > 0\}| - |B_r \cap \{u_\varepsilon > 0\}|) &\leq F_\varepsilon(|B_r \cap \{v > 0\}|) - F_\varepsilon(|B_r \cap \{u_\varepsilon > 0\}|)\\ &\leq \frac{1}{\varepsilon}(|B_r \cap \{v > 0\}| - |B_r \cap \{u_\varepsilon > 0\}|).\end{aligned}$$
Therefore all the proofs of sections 3, 4 and 5 of Chapter 2 can be modify using this fact. Observe that here, all the constants may depend on ε.

□

From now on we drop the subscript ε and denote by u instead of u_ε a solution to (P_ε).

THEOREM 3.1.3 (Representation Theorem). *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then,*

1. *$\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ for every $D \subset\subset \Omega$.*
2. *There exists a Borel function q_u such that*
$$\mathcal{L}u = q_u\,\mathcal{H}^{N-1}\lfloor\partial\{u > 0\}.$$

3. *For $D \subset\subset \Omega$ there are constants $0 < c \le C < \infty$ depending on N, Ω, D and ε such that for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,*
$$c \le q_u(x) \le C, \quad c\, r^{N-1} \le \mathcal{H}^{N-1}(B_r(x) \cap \partial\{u > 0\}) \le C\, r^{N-1}.$$
4. $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}) = 0.$

PROOF. See Theorem 1.4.54. Observe that $D \cap \partial\{u > 0\}$ has finite perimeter, thus, the reduce boundary $\partial_{\mathrm{red}}\{u > 0\}$ is defined as well as the measure theoretic normal $\nu(x)$ for $x \in \partial_{\mathrm{red}}\{u > 0\}$ (see Chapter 1 Section 4.2). $\square$

We are going to prove, that any solution to (P_ε), u is a weak solution in the sense of Definition 2.6.1. First we prove that, if we take two points on the free boundary such that the blow up sequences with respect to these points have blow up limits that are half planes, then the slopes must coincide. Finally we will prove that the function q_u is constant and that property (4) of Definition 2.6.1 is satisfied with $\lambda^* = g^{-1}(q_u)$. We will divide the proof into several lemmas.

LEMMA 3.1.4. *Let $x_0, x_1 \in \partial\{u > 0\}$ and $\rho_k \to 0^+$. For $i = 0, 1$ let $x_{i,k} \to x_i$ with $u(x_{i,k}) = 0$ such that $B_{\rho_k}(x_{i,k}) \subset \Omega$ and such that the blow-up sequence*
$$u_{i,k}(x) = \frac{1}{\rho_k} u(x_{i,k} + \rho_k x)$$
has a limit $u_i(x) = \lambda_i \langle x, \nu_i\rangle^-$, with $0 < \lambda_i < \infty$ and ν_i a unit vector. Then $\lambda_0 = \lambda_1$.

PROOF. Assume that $\lambda_1 < \lambda_0$ then we will perturb the minimizer u near x_0 and x_1 and get an admissible function with less energy. To this end, we take a nonnegative C_0^∞ function ϕ supported in the unit interval. For k large, define
$$\tau_k(x) = \begin{cases} x + \rho_k^2 \phi\Big(\dfrac{|x - x_{0,k}|}{\rho_k}\Big)\nu_0 & \text{for } x \in B_{\rho_k}(x_{0,k}), \\[2ex] x - \rho_k^2 \phi\Big(\dfrac{|x - x_{1,k}|}{\rho_k}\Big)\nu_1 & \text{for } x \in B_{\rho_k}(x_{1,k}), \\[2ex] x & \text{elsewhere,} \end{cases}$$
which is a diffeomorphism if k is big enough. Now let
$$v_k(x) = u(\tau_k^{-1}(x)),$$
then, v_k is an admissible function. Let us also define
$$\eta_i(y) = (-1)^i \phi(|y|)\nu_i. \tag{3.1.5}$$
We have
$$F_\varepsilon(|\{v_k > 0\}|) - F_\varepsilon(|\{u > 0\}|) = o(\rho_k^{N+1}). \tag{3.1.6}$$

In order to estimate the other term in $\mathcal{J}_\varepsilon$ we make a change of variables and get

$$\rho_k^{-N}\int_{B_{\rho_k}(x_i)} (G(|\nabla v_k|) - G(|\nabla u|))\ dx$$
$$= \int_{B_1(0)\cap\{u_{i,k}>0\}} \rho_k\Big[G(|\nabla u_{i,k}|)\operatorname{div}(\eta_i) - \ F(|\nabla u_{i,k}|)(\nabla u_{i,k})^t D\eta_i \nabla u_{i,k}\Big] + o(\rho_k)\ dy.$$

On the other hand, by Lemma 1.6.13, we have

$$B_1(0)\cap\{u_{i,k}>0\} \to B_1(0)\cap\{y\cdot\nu_i<0\},\ \text{as } k\to 0,\ \text{and}$$
$$\nabla u_{i,k} \to \nabla u_i = -\lambda_i\nu_i\chi_{\{\langle y,\nu_i\rangle<0\}}\ \text{a.e in } B_1(0).$$

Therefore

$$\rho_k^{-N-1}\int_{B_{\rho_k}(x_i)} (G(|\nabla v_k|) - G(|\nabla u|))\, dx \to$$
$$\int_{B_1(0)\cap\{\langle y,\nu_i\rangle<0\}} \big(G(\lambda_i)\operatorname{div}(\eta_i) - g(\lambda_i)\lambda_i\,\nu_i^t\, D\eta_i\,\nu_i\big)\, dy$$

Using that

$$\begin{aligned}\operatorname{div}(\eta_i) - \frac{g(\lambda_i)\lambda_i}{G(\lambda_i)}\,\nu_i^t\, D\eta_i\,\nu_i &= (-1)^i\Big(1-\frac{g(\lambda_i)\lambda_i}{G(\lambda_i)}\Big)\frac{\phi'(|y|)}{|y|}\langle y,\nu_i\rangle \\ &= (-1)^i\Big(1-\frac{g(\lambda_i)\lambda_i}{G(\lambda_i)}\Big)\operatorname{div}(\eta_i),\end{aligned}$$

we obtain

$$\rho_k^{-N-1}\int_{B_{\rho_k}(x_i)} (G(|\nabla v_k|) - G(|\nabla u_\varepsilon|))\, dx \to$$
$$(-1)^{i+1}\Phi(\lambda_i)\int_{B_1(0)\cap\{y\cdot\nu_i=0\}} \phi(|y|)\, d\mathcal{H}^{N-1}(y),$$

where $\Phi(t) = g(t)t - G(t)$. Hence

$$\begin{aligned}&\int_\Omega G(|\nabla v_k|)\, dx - \int_\Omega G(|\nabla u|)\, dx = \\ &\quad = \rho_k^{N+1}(\Phi(\lambda_1)-\Phi(\lambda_0))\int_{B_1(0)\cap\{y_1=0\}}\phi(|y|)\, d\mathcal{H}^{N-1}(y) \\ &\quad + o(\rho_k^{N+1}).\end{aligned} \tag{3.1.7}$$

If we take k large enough we get

$$\mathcal{J}_\varepsilon(v_k) < \mathcal{J}_\varepsilon(u),$$

a contradiction.

$\square$

LEMMA 3.1.8. *Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let*

$$\lambda = \lambda(x_0) := \limsup_{\substack{x \to x_0 \\ u(x)>0}} |\nabla u(x)|.$$

Then, there exist sequences y_k, d_k and ν such that the blow up sequence with respect to $B_{d_k}(y_k)$ has limit,

$$u_0(x) = \lambda \langle x, \nu\rangle^-.$$

PROOF. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let

$$\lambda = \lambda(x_0) := \limsup_{\substack{x \to x_0 \\ u(x)>0}} |\nabla u(x)|.$$

Then there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0, \qquad |\nabla u(z_k)| \to \lambda.$$

Let y_k be the nearest point to z_k on $\Omega \cap \partial\{u > 0\}$ and let $d_k = |z_k - y_k|$. Consider the blow up sequence with respect to $B_{d_k}(y_k)$ with limit u_0, such that there exists

$$\nu := \lim_{k\to\infty} \frac{y_k - z_k}{d_k},$$

and suppose that $\nu = e_N$. As in Lemma 2.5.3 we can prove that $0 < \lambda < \infty$ and

$$u_0(x) = -\lambda x_N \text{ in } \{x_N \le 0\}.$$

Finally by Lemma 1.6.13 we have that $0 \in \partial\{u_0 > 0\}$ and then, using Lemma 1.6.13 we see that u_0 satisfies the hypotheses of Theorem 1.5.1. Therefore $u_0 = 0$ in $\{x_N > 0\}$. Then $u_0 = \lambda\langle x, \nu\rangle^-$. ☐

LEMMA 3.1.9. *For $\mathcal{H}^{N-1}$–a.e. $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$, there exist a sequence γ_n such that if u_n is the blow up sequence with respect to $B_{\gamma_n}(x_0)$ we have that,*

$$u_n \to \lambda^* \langle x, \nu(x_0)\rangle^-$$

with $\nu(x_0)$ the outward unit normal of $\partial\{u > 0\}$ at x_0 in the measure theoretic sense and $\lambda^* = g^{-1}(q_u(x_0))$.

PROOF. Take $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$ and suppose that $\nu(x_0) = e_N$. We consider a blow up sequence with respect to balls $B_{\rho_k}(x_0)$, with blow up limit u_0. As in Theorem 2.5.5 we have,

$$\begin{cases} u_0 = 0 & \text{in } \{x_N \ge 0\}, \\ u_0 > 0 & \text{in } \{x_N < 0\}. \end{cases}$$

And, as in Lemma 2.5.9 we have that for a.e $\mathcal{H}^{N-1}$ $x_0 \in \partial_{red}\{u > 0\}$, $q_{u_0}(x) = q_u(x_0)$ in the sense that for all $\xi \in C_0^1(B_r)$

$$(3.1.10) \qquad -\int_{B_r \cap \{x_N<0\}} g(|\nabla u_0|) \frac{\nabla u_0}{|\nabla u_0|} \nabla \xi \, dx = q_u(x_0) \int_{B'_r} \xi(x', 0) \, d\mathcal{H}^{N-1}.$$

From the boundary regularity we have, that this is satisfied in the classical sense, Therefore , $u_0(x) = \lambda^* x_N^- + o(|x|)$.

Take now $u_{0,j}$, a blow up sequence of u_0, with respect to balls $B_{\mu_j}(0)$. Then,

$$u_{0,j} \to u_{00} = \lambda^* x_N^-.$$

Now, we want to construct a blow up sequence of u with limit u_{00}. Observe, that

$$\Big|\frac{1}{\rho_k \mu_j} u(x_0 + \rho_k \mu_j x) - u_{00}(x)\Big| \le \frac{1}{\mu_j} |u_k(\mu_j x) - u_0(\mu_j x)| + |u_{0,j}(x) - u_{00}(x)|,$$

and since $u_k \to u_0$ uniformly over compacts sets we have that for $j \ge j_n$, $|u_{0,j}(x) - u_{00}(x)| < 1/n$ and for $k \ge k_n$, $|u_k(\mu_j x) - u_0(\mu_j x)| < \mu_j/n$ if $|x| \le n$. We may suppose that $j_n \ge n$ and $k_n \ge n$. Taking $\gamma_n = \rho_{k_n}\mu_{j_n}$, we have that $\gamma_n \to 0$ and $|u_{\gamma_n}(x) - u_{00}(x)| < 2/n$ in B_n. Replacing ξ by $\xi(x) = \xi\Big(\frac{x}{\gamma_n}\Big)\gamma_n$ in (3.1.10), changing variables and passing to the limit we get

$$\int_{B_1 \cap \{x_N < 0\}} g(\lambda^*) \xi_{x_N}\, dx = \int_{B_1'} \xi(x', 0)\, \mathcal{H}^{N-1},$$

and the result follows. $\square$

THEOREM 3.1.11. *Let $u \in \mathcal{K}$ be a solution to (P_ε) and q_u the function in Theorem 3.1.3. Then, there exists a positive constant λ_u such that*

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)| = \lambda_u, \qquad \textit{for every } x_0 \in \Omega \cap \partial\{u > 0\} \tag{3.1.12}$$

$$q_u(x_0) = g(\lambda_u), \qquad \mathcal{H}^{N-1} - a.e\ x_0 \in \Omega \cap \partial\{u > 0\}. \tag{3.1.13}$$

PROOF. Let $x_1 \in \partial_{red}\{u > 0\}$ satisfying the properties of Lemma 3.1.9. Set $\lambda_u := g^{-1}(q_u(x_1))$. Then, there exists a blow up sequence $u_n \to \lambda_u \langle x, \nu(x_1)\rangle^-$. For any $x_0 \in \partial\{u > 0\}$, we have by Lemma 3.1.8, that there exists for a certain unit vector $\nu(x_0)$ a blow up sequence $u_k \to \lambda(x_0)\langle x, \nu(x_0)\rangle^-$. Then, by Lemma 3.1.4, we have that $\lambda(x_0) = \lambda_u$ and then (3.1.12) follows. If we apply Lemmas 3.1.9 and 3.1.4 again, we obtain (3.1.13) for almost every point in $\partial_{red}\{u > 0\}$, and the result follows by Theorem 3.1.3 (4). $\square$

REMARK 3.1.14. *Now we have, by properties (1), (2), (3) in Theorem 2.1.1, (2), (4) in Theorem 3.1.3 and Theorem 3.1.11 that any minimizer satisfies all the properties of the definition of weak solution I in Chapter 2. Therefore we have by Theorem 2.7.56 and Remark 2.7.55 in Chapter 2 the following regularity result for the free boundary $\partial\{u > 0\}$.*

COROLLARY 3.1.15. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then $\partial_{red}\{u > 0\}$ is a $C^{1,\beta}$ surface locally in Ω and the remainder of the free boundary has $\mathcal{H}^{N-1}-$measure zero.*

1.2. Full regularity for the case $N = 2$. We will prove, that in dimension two, for the subclass of functions satisfying (0.0.2) and (2.7.57), their whole free boundary is a $C^{1,\beta}$ surface.

As in Section 7.4 of Chapter 2, the following lemma holds for any dimension and for any δ and g_0,

LEMMA 3.1.16. *Let $u \in \mathcal{K}$ be a local minimizer. Given $D \subset\subset \Omega$, there exist constants $C = C(N, D, \lambda_u)$, $r_0 = r_0(N, D) > 0$ and $\gamma = \gamma(N, D) > 0$ such that, if $x_0 \in D \cap \partial\{u > 0\}$ and $r < r_0$, then*

$$\sup_{B_r(x_0)} |\nabla u| \leq \lambda_u + Cr^\gamma.$$

PROOF. The proof follows as in Lemma 2.7.58 by using the density property of the set $\{u = 0\}$ (Theorem 3.1.2 (5)).

□

Now, we have to do some changes on the proof of Theorem 2.7.64, since here there is no explicit relation between the constant λ_u and the parameter ε of the functional $\mathcal{J}_\varepsilon$ (recall that for the minimization problem in Chapter 2, we use that we have the relation $\Phi(\lambda^*) = \lambda$). In order to prove the full regularity of the free boundary, we need the following lemma, that also holds for any dimension and for any δ and g_0.

LEMMA 3.1.17. *Let x_1 be regular free boundary point.*

Take

$$\tau_\rho(x) = \begin{cases} x + \rho^2 \phi\left(\dfrac{|x - x_1|}{\rho}\right) \nu_u(x_1) & \text{for } x \in B_\rho(x_1), \\ x & \text{elsewhere,} \end{cases}$$

where $\phi \in C_0^\infty(-1, 1)$ with $\phi'(0) = 0$.

Let

$$\delta = \rho^2 \int_{B_\rho(x_1) \cap \partial\{u>0\}} \phi\left(\frac{|x - x_1|}{\rho}\right) d\mathcal{H}^{N-1}. \tag{3.1.18}$$

Take $v_\delta(x) = v_\rho(x) = u(\tau_\rho^{-1}(x))$, then

$$\int_{B_\rho(x_1)} (G(|\nabla v_\rho|) - G(|\nabla u|))\, dx = -l\rho^{N+1}\Phi(\lambda_u) + o(\rho^{N+1}), \tag{3.1.19}$$

where $l = \lim_{\rho \to 0} \frac{\delta}{\rho^{N+1}}$ and $\Phi(t) = g(t)t - G(t)$.

PROOF. The proof is included in the proof of Lemma 3.1.4.. □

The following lemma is the place where, as in Lemma 2.7.59, we need to impose condition (2.7.57).

LEMMA 3.1.20. *Let* $\Phi(t) = g(t)t - G(t)$*, and* g *satisfying condition* (2.7.57)*. Let* x_0 *be a free boundary point ,* $D \subset\subset \Omega$ *and* $B_\mu(x_0) \subset D$*. Take* $v = \max(u - t\eta, 0)$*, where* $t > 0$*,* $\eta \in C_0^\infty(\Omega)$*,* $\eta = 0$ *in* $\Omega \setminus B_{\mu(x_0)}$ *and* $|\nabla\eta| \leq C/t$*. Then,*

$$\int_{B_\mu(x_0)\cap\{u>0\}} (G(|\nabla v|) - G(|\nabla u|))\, dx \leq \int_{B_\mu(x_0)\cap\{0<u\leq t\eta\}} \Phi(|\nabla u|)\, dx$$
$$+C_0 t^2 \int_{B_\mu(x_0)\cap\{u>t\eta\}} |\nabla\eta|^2\, dx,$$

for $C_0 = C_0(N, \delta, g_0, dist(\partial\Omega, D), C)$*.*

PROOF. It follows as in Lemma 2.7.59, by using Theorem 3.1.2 (1). □

Now, following ideas of [**19**], using Lemmas 3.1.16, 3.1.17 and 3.1.20, we prove, for $N = 2$ and g satisfying (2.7.57) the following

THEOREM 3.1.21. *Let* $N = 2$*,* g *satisfying* (2.7.57) *and* u *a minimizer. Then, for any ball* B_r *centered at the free boundary we have,*

$$⨍_{B_r\cap\{u>0\}} (\Phi(\lambda_u) - \Phi(|\nabla u|))^+ \to 0 \text{ when } r \to 0,$$

where $\Phi(t) = g(t)t - G(t)$*.*

PROOF. Let $0 < r < \mu$, $t > 0$ and v_0 be the function defined in Lemma 3.1.20. By Theorem 2.1.1 $u \leq Cr$ in $B_r(x_0)$. Take $t = Cr$ and let $\delta_t = |\{0 < u \leq t\eta\} \cap B_\mu(x_0)|$.

Now, let us take x_1 far from x_0 and such that $\partial\{u > 0\} \cap B_{r_1}(x_1)$ is regular, for r_1 small. Let ρ be such that (3.1.18) is satisfied for $\delta = \delta_t$, and consider $v_1 = v_{\delta_t}$ defined in $B_{r_1}(x_1)$ as in Lemma 3.1.17. Then, the function

$$v = \begin{cases} v_0 & \text{in } B_\mu(x_0) \\ v_1 & \text{in } B_{r_1}(x_1) \\ u & \text{elsewhere} \end{cases}$$

is admissible for our minimization problem and $|\{v > 0\}| = |\{u > 0\}|$. Therefore, by Lemmas 3.1.17 and 3.1.20 we have

$$\begin{aligned} 0 &\leq \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u) \\ &= \int_{B_\rho(x_0)} (G(|\nabla v|) - G(|\nabla u|))\, dx + \int_{B_{r_1}(x_1)} (G(|\nabla v|) - G(|\nabla u|))\, dx \\ &\leq \int_{B_\mu(x_0)\cap\{u\leq t\eta\}} \Phi(|\nabla u|) + Ct^2 \int_{B_\mu(x_0)\cap\{u>t\eta\}} |\nabla\eta|^2\, dx - l\rho^3\Phi(\lambda_u) + o(\rho^3). \end{aligned}$$

By the definition of δ_t we have,

$$\begin{aligned} \int_{B_\mu(x_0)\cap\{0<u\leq t\eta\}} (\Phi(\lambda_u) - \Phi(|\nabla u|))\, dx \leq & Ct^2 \int_{B_\mu(x_0)\cap\{u>t\eta\}} |\nabla\eta|^2\, dx + o(\rho^3) \\ & + (\delta_t - l\rho^3)\Phi(\lambda_u). \end{aligned}$$

Now choose

$$\eta(x) = \begin{cases} \dfrac{\log(\mu/|x-x_0|)}{\log(\mu/r)} & \text{in } B_\mu(x_0) \setminus B_r(x_0), \\ 1 & \text{in } B_r(x_0) \\ 0 & \text{in } \Omega \setminus B_\mu(x_0), \end{cases}$$

observe that the condition $|\nabla \eta| \leq C/t$ is satisfied if we choose μ such that $\mu > 2r$.

By our election of t and η we have,

$$\int_{B_r(x_0)\cap\{u>0\}} (\Phi(\lambda_u) - \Phi(|\nabla u|))^+ \, dx \leq \int_{B_\mu(x_0)} (\Phi(|\nabla u|)) - (\Phi(\lambda_u))^+ \, dx + \frac{Cr^2}{\log(\mu/r)} + o(\rho^3) + (\delta_t - l\rho^3)\Phi(\lambda_u).$$

By Lemma 3.1.16, we have that $\Phi(|\nabla u|) - \Phi(\lambda_u) \leq \Phi(\lambda_u + Cr^\gamma) - \Phi(\lambda_u) = \Phi'(\xi)Cr^\gamma$, for $\lambda_u \leq \xi \leq \lambda_u + Cr^\gamma$. As $\Phi'(t) = g'(t)t \leq g_0 g(t)$, and as g is nondecreasing we have that $\Phi'(\xi) \leq g_0 g(\xi) \leq g_0 g(\lambda_u + Cr^\gamma)$.

Therefore by the definition of l we have

$$⨍_{B_r(x_0)\cap\{u>0\}} (\Phi(\lambda_u) - \Phi(|\nabla u|))^+ \, dx \leq \frac{(\mu^{\gamma+2} + o(\rho^3))}{r^2} + \frac{C}{\log(\mu/r)}.$$

As $\delta_t \leq c\mu^2$ we have that $o(\rho^3) = o(\mu^2)$. Taking $r = \mu h(\mu)^\beta$, where $h(\mu) = \max\left(\mu, \frac{o(\mu^2)}{\mu^2}\right)$ with $\beta < \min\{\gamma/2, 1/2\}$, we have the desired result. $\square$

Now we have, as in Corollary 2.7.66 the following,

Corollary 3.1.22. *Let $N = 2$, g satisfying* (2.7.57) *and $u \in \mathcal{K}$ be a solution to* (P_ε). *Then $\partial\{u > 0\}$ is a $C^{1,\beta}$ surface locally in Ω.*

2. Behavior of the minimizer for small ε .

In this section, since we want to analyze the dependence of the problem with respect to ε, we will again denote by u_ε a solution to problem (P_ε).

To complete the analysis of the problem, we will now show that if ε is small enough, then

$$|\{u_\varepsilon > 0\}| = \alpha.$$

To this end, we need to prove that the constant $\lambda_\varepsilon := \lambda_{u_\varepsilon}$ is bounded from above and below by positive constants independent of ε. We perform this task in a series of lemmas.

Lemma 3.2.1. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε). Then, there exists a constant $C > 0$ independent of ε such that*

$$\lambda_\varepsilon \leq C.$$

Proof. First we will prove that there exist $C, c > 0$, independent of ε, such that

$$c \le |\{u_\varepsilon > 0\}| \le C\varepsilon + \alpha.$$

In fact, by taking u_0 such that $|\{u_0 > 0\}| \le \alpha$ we have that $\mathcal{J}_\varepsilon(u_0) \le C$ then we have that $F_\varepsilon(|\{u_\varepsilon > 0\}|) \le C$ thus obtaining the bound from above. We also have that $\int_\Omega G(|\nabla u_\varepsilon|)$ is bounded. As $u_\varepsilon = \varphi_0$ on $\partial\Omega$, we have, by Lemma 1.1.5, that $\|\nabla u_\varepsilon - \nabla\varphi_0\|_G \le C$ and, by Lemma 1.2.10, we also have $\|u_\varepsilon - \varphi_0\|_G \le C$. Then, $\|u_\varepsilon\|_{W^{1,G}(\Omega)} \le C$. Using the Sobolev trace Theorem, the Hölder inequality and the embedding Theorem 1.1.9 we have, for $q < \delta + 1$

$$\begin{aligned}\int_{\partial\Omega} \varphi_0^q \, d\mathcal{H}^{N-1} &\le C|\{u_\varepsilon > 0\}|^{\frac{\delta+1-q}{\delta+1}} \|u_\varepsilon\|^q_{W^{1,\delta+1}(\Omega)} \\ &\le C|\{u_\varepsilon > 0\}|^{\frac{\delta+1-q}{\delta+1}} \|u_\varepsilon\|^q_{W^{1,G}(\Omega)} \le C|\{u_\varepsilon > 0\}|^{\frac{\delta+1-q}{\delta+1}},\end{aligned}$$

and thus we obtain the bound from below.

Take $D \subset\subset \Omega$ smooth, such that $\theta = |D| > \alpha$ and $|\Omega \setminus D| < c$. Then,

$$|D \cap \{u_\varepsilon > 0\}| \le \alpha + C\varepsilon < \theta$$

for ε small enough. On the other hand,

$$|D \cap \{u_\varepsilon > 0\}| \ge |\{u_\varepsilon > 0\}| - |\Omega \setminus D| \ge c - |\Omega \setminus D| > 0,$$

Therefore by the relative isoperimetric inequality (see [**13**] 5.6.2) we have

$$\begin{aligned}\mathcal{H}^{N-1}(D \cap \partial\{u_\varepsilon > 0\}) &\ge c \min\Big\{|D \cap \{u_\varepsilon > 0\}|, |D \cap \{u_\varepsilon = 0\}|\Big\}^{\frac{N-1}{N}} \\ &\ge c > 0.\end{aligned}$$

Now let w be the $\mathcal{L}-$ solution in Ω with boundary data equal to φ_0. Using Theorem 3.1.3 and Theorem 2.5.3 we have,

$$\begin{aligned}C &\ge \int_\Omega F(|\nabla u_\varepsilon|)\nabla u_\varepsilon \nabla(u_\varepsilon - w)\, dx = \int_{\Omega\cap\partial\{u_\varepsilon>0\}} w g(\lambda_\varepsilon)\, d\mathcal{H}^{N-1} \\ &\ge \int_{D\cap\partial\{u_\varepsilon>0\}} w g(\lambda_\varepsilon)\, d\mathcal{H}^{N-1} \ge g(\lambda_\varepsilon)(\inf_D w)\mathcal{H}^{N-1}(D \cap \partial\{u_\varepsilon > 0\}) \ge c g(\lambda_\varepsilon).\end{aligned}$$

Now the result follows.

□

Lemma 3.2.2. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε), $B_r \subset\subset \Omega$ and v a solution to*

$$\mathcal{L}v = 0 \quad \text{in } B_r, \qquad v = u_\varepsilon \quad \text{on } \partial B_r.$$

Then, there exists a positive constant $\gamma = \gamma(\delta, g_0, N)$ such that

$$\int_{B_r} |\nabla(u_\varepsilon - v)|^q \, dx \ge C|B_r \cap \{u_\varepsilon = 0\}| \left(\frac{1}{r}\left(⨍_{B_r} u_\varepsilon^\gamma \, dx\right)^{1/\gamma}\right)^q$$

for all $q \ge 1$ and where C is a constant independent of ε.

PROOF. First let us assume that $B_r = B_1(0)$. For $|z| \leq \frac{1}{2}$ we consider the change of variables from B_1 into itself such that z becomes the new origin. We call $u_z(x) = u\big((1-|x|)z + x\big)$, $v_z(x) = v\big((1-|x|)z + x\big)$ and define

$$r_\xi = \inf\left\{r \,/\, \frac{1}{8} \leq r \leq 1 \quad \text{and} \quad u_z(r\xi) = 0\right\},$$

if this set is nonempty. Observe that this change of variables leaves the boundary fixed.

Now, for almost every $\xi \in \partial B_1$ we have

(3.2.3)

$$v_z(r_\xi \xi) = \int_{r_\xi}^1 \frac{d}{dr}(u_z - v_z)(r\xi)\,dr \leq (1-r_\xi)^{1/q'}\left(\int_{r_\xi}^1 |\nabla(u_z - v_z)(r\xi)|^q\,dr\right)^{1/q}.$$

Let us assume that the following inequality holds

$$v_z(r_\xi \xi) \geq C(N,\Omega)(1-r_\xi)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}. \tag{3.2.4}$$

Then, using (3.2.3) and (3.2.4), integrating first over ∂B_1 and then over $|z| \leq 1/2$ we obtain as in [**4**],

$$\int_{B_1} |\nabla(u-v)|^q\,dx \geq C|B_1 \cap \{u = 0\}|\left(\fint_{B_1} u^\gamma\,dx\right)^{q/\gamma}.$$

If we take $u_r(x) = \frac{1}{r}u(x_0 + rx)$ (where x_0 is the center of the ball B_r) then

$$\int_{B_1} |\nabla(u_r - v_r)|^q\,dx = r^{-N}\int_{B_r} |\nabla(u-v)|^q\,dy,$$

$$|B_1 \cap \{u_r = 0\}| = r^{-N}|B_r \cap \{u = 0\}| \quad \text{and}$$

$$\left(\fint_{B_1} u_r^\gamma\,dx\right)^{1/\gamma} = \frac{1}{r}\left(\fint_{B_r} u^\gamma\,dy\right)^{1/\gamma},$$

so we have the desired result.

Therefore we only have to prove (3.2.4). If $|(1-r_\xi)z + r_\xi\xi| \leq \frac{3}{4}$, by Harnack inequality,

$$v_z(r_\xi\xi) \geq C_N v(0).$$

By the weak Harnack inequality 1.2.15 we have

$$v(0) \geq \alpha(N,\Omega)\left(\fint_{B_1} v^\gamma\,dx\right)^{1/\gamma} \geq \alpha(N,\Omega)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}. \tag{3.2.5}$$

If $|(1-r_\xi)z + r_\xi\xi| \geq \frac{3}{4}$ we prove by a comparison argument that inequality (3.2.4) also holds. In fact, again by Harnack inequality,

$$v \geq C_N\alpha\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma} \quad \text{in } B_{3/4}.$$

Let $w(x)=(e^{-\lambda|x|^2}-e^{-\lambda})\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}$. There exists $\lambda=\lambda(N,\alpha)$ such that

$$\begin{cases} \mathcal{L}w\geq 0 & \text{in } B_1\setminus B_{3/4},\\ w\leq C_N\alpha\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma} & \text{on } \partial B_{3/4},\\ w=0 & \text{on } \partial B_1,\end{cases}$$

(see Lemma 1.2.47) so that,

$$v\geq w\geq C(1-|x|)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}\quad \text{in}\quad B_1\setminus B_{3/4}.$$

Therefore

$$v_z(r_\xi\xi)\geq C\Big(1-|(1-r_\xi)z+r_\xi\xi|\Big)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}\geq C(1-r_\xi)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}$$

since $|z|\leq\frac12$. So that (3.2.4) holds for every $r_\xi\geq\frac18$.

This completes the proof. □

Without loss of generality, from now on we will suppose that $g_0\geq 1$.

Lemma 3.2.6. *Let u_ε and v be as in Lemma 3.2.2, and B_r a ball centered on the free boundary, then if r is small enough (depending on ε) we have,*

$$\int_{B_r}(G(|\nabla u|)-G(|\nabla v|))\,dx\geq C\int_{B_r}|\nabla u_\varepsilon-\nabla v|^{g_0+1}\,dx \tag{3.2.7}$$

for some constant C independent of ε.

Proof. Let,

$$A_1=\{x\in B_r:|\nabla u_\varepsilon-\nabla v|\leq 2|\nabla u_\varepsilon|\},\quad A_2=\{x\in B_r:|\nabla u_\varepsilon-\nabla v|>2|\nabla u_\varepsilon|\},$$

then $B_r=A_1\cup A_2$ and by Theorem 1.2.38 we have that,

$$\begin{aligned}\int_{B_r}(G(|\nabla u_\varepsilon|)-G(|\nabla v|))\,dx\geq C\Big(&\int_{A_2}G(|\nabla u_\varepsilon-\nabla v|)\,dx\\ &+\int_{A_1}F(|\nabla u_\varepsilon|)|\nabla u_\varepsilon-\nabla v|^2\,dx\Big).\end{aligned} \tag{3.2.8}$$

Therefore we have, using that $g_0\geq 1$ and (g1) in Lemma 1.1.1, that when $|\nabla u_\varepsilon|\leq 1$ and $|\nabla v-\nabla u_\varepsilon|\leq 1$,

$$\begin{aligned}&G(|\nabla u_\varepsilon-\nabla v|)\geq C|\nabla u_\varepsilon-\nabla v|^{g_0+1}\\ &\qquad F(|\nabla u_\varepsilon|)\geq C|\nabla u_\varepsilon|^{g_0-1}\geq C|\nabla u_\varepsilon-\nabla v|^{g_0-1}\quad \text{in } A_1.\end{aligned} \tag{3.2.9}$$

On the other hand, by Lemma 3.2.1 and (3.1.12), we have that for small r (depending on ε), $|\nabla u_\varepsilon|$ is bounded by a constant independent of ε. By Lemma 1.2.18 there exist $C_0, C_1 = C_0, C_1(N, g_0, \delta)$ such that,

$$\sup_{B_r} G(|\nabla v|) \leq \frac{C_0}{r^N} \int_{B_{2r}} G(|\nabla v|)\, dx \leq \frac{C_1}{r^N} \int_{B_{2r}} (1 + G(|\nabla u_\varepsilon|))\, dx \leq C,$$

if we choose r small (depending on ε) and where C is independent of ε. Therefore, (3.2.9) holds for all $x \in B_r$. Combining (3.2.8) and (3.2.9) we obtain the desired result.

$\square$

LEMMA 3.2.10. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε). Then*

$$\lambda_\varepsilon \geq c > 0,$$

where c is independent of ε

PROOF. We will use the following fact that we prove in Lemma 3.2.16 bellow: For every $\varepsilon > 0$ there is a neighborhood of A in Ω where $u_\varepsilon > 0$.

Let $y_0 \in A$ and let D_t with $0 \leq t \leq 1$ be a family of open sets with smooth boundary and uniformly (in ε and t) bounded curvatures such that D_0 is an exterior tangent ball at y_0, D_1 contains a free boundary point, $D_0 \subset\subset D_t$ for $t > 0$ and $D_t \cap \partial\Omega \subset A$.

Let $t \in (0, 1]$ be the first time such that D_t touches the free boundary and let $x_0 \in \partial D_t \cap \partial\{u_\varepsilon > 0\} \cap \Omega$. Now, take w such that $\mathcal{L}w = 0$ in $D_t \setminus \overline{D}_0$ with $w = c_0$ on ∂D_0 and $w = 0$ on ∂D_t. Thus $w \leq u_\varepsilon$ in $D_t \cap \Omega$ and $\partial_{-\nu} w(x_0) \geq c$ with $c > 0$ independent of ε. Therefore, for r small enough,

$$\left(\fint_{B_r(x_0)} u_\varepsilon^\gamma\, dx \right)^{1/\gamma} \geq \left(\fint_{B_r(x_0)} w^\gamma\, dx \right)^{1/\gamma} \geq r\bar{c}, \tag{3.2.11}$$

with $\bar{c}$ is independent of ε.

If v_0 is the solution to

$$\begin{cases} \mathcal{L}v_0 = 0 & \text{in } B_r(x_0) \\ v_0 = u_\varepsilon & \text{on } \partial B_r(x_0), \end{cases}$$

then, by Lemma 3.2.2, we have

$$\int_{B_r} |\nabla(u_\varepsilon - v_0)|^q\, dx \geq C|B_r \cap \{u_\varepsilon = 0\}| \left(\frac{1}{r} \left(\fint_{B_r} u_\varepsilon^\gamma\, dx \right)^{1/\gamma} \right)^q,$$

Then be Lemma 3.2.6 we obtain,

$$\int_{B_r} (G(|\nabla u_\varepsilon|) - G(|\nabla v_0|))\, dx \geq C|B_r \cap \{u_\varepsilon = 0\}| \left(\frac{1}{r} \left(\fint_{B_r} u_\varepsilon^\gamma\, dx \right)^{1/\gamma} \right)^{g_0+1}. \tag{3.2.12}$$

Then by (3.2.11)

$$\int_{B_r(x_0)} (G(|\nabla u_\varepsilon|) - G(|\nabla v_0|))\, dx \geq c\delta_r \tag{3.2.13}$$

where $\delta_r = |B_r(x_0) \cap \{u_\varepsilon = 0\}|$ and c is a constant independent of ε.

Consider now a free boundary point x_1 away from x_0. We can choose x_1 to be regular.

Let us take

$$\tau_\rho(x) = \begin{cases} x - \rho^2 \phi\left(\dfrac{|x - x_1|}{\rho}\right) \nu_{u_\varepsilon}(x_1) & \text{for } x \in B_\rho(x_1), \\ x & \text{elsewhere,} \end{cases}$$

where $\phi \in C_0^\infty(-1, 1)$ with $\phi'(0) = 0$.

Now choose ρ such that

$$\delta_r = \rho^2 \int_{B_\rho(x_1) \cap \partial\{u_\varepsilon > 0\}} \phi\left(\frac{|x - x_1|}{\rho}\right) d\mathcal{H}^{N-1}.$$

Take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$ and

$$v = \begin{cases} v_0 & \text{in } B_r(x_0) \\ v_\rho & \text{in } B_\rho(x_1) \\ u_\varepsilon & \text{elsewhere.} \end{cases}$$

Thus, we have that

$$|\{v > 0\}| = |\{u_\varepsilon > 0\}|. \tag{3.2.14}$$

On the other hand as in Lemma 3.1.4, we have

$$\begin{aligned} &\int_{B_\rho(x_1)} (G(|\nabla v_\rho|) - G(|\nabla u_\varepsilon|))\; dy \\ &\qquad = \int_{\tau_\rho(B_\rho(x_1)) \cap \{v_\rho > 0\}} G(|\nabla v_\rho|)\; dy - \int_{B_\rho(x_1)} G(|\nabla u|)\; dx \\ &\qquad = \int_{B_\rho(x_1) \cap \{u > 0\}} \rho(G(|\nabla u_\varepsilon|) \operatorname{div} \eta - \; F(|\nabla u_\varepsilon|) \nabla u_\varepsilon D\eta \nabla u_\varepsilon) + o(\rho)\; dx \end{aligned}$$

where $\eta(y) = -\phi(|y|)\nu(x_1)$. Using the fact that $|\nabla \eta|$ is bounded from above by a constant independent of ρ and ε, and that $|\nabla u_\varepsilon| = \lambda_\varepsilon + O(\rho^\alpha)$ in $B_\rho(x_1)$ we have

$$\int_{B_\rho(x_1)} (G(|\nabla v_\rho|) - G(|\nabla u_\varepsilon|))\, dy \leq kG(\lambda_\varepsilon)\rho^{N+1} + o(\rho^{N+1})$$

but, δ_r has the same order of ρ^{N+1} then

$$\int_{B_\rho(x_1)} (G(|\nabla v_\rho|) - G(|\nabla u_\varepsilon|))\, dy \leq KG(\lambda_\varepsilon)\delta_r + o(\delta_r). \tag{3.2.15}$$

Therefore by (3.2.13), (3.2.15) and (3.2.14) we have

$$0 \leq \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) \leq -c\delta_r + KG(\lambda_\varepsilon)\delta_r + o(\delta_r)$$

and then $\lambda_\varepsilon \ge c > 0$. $\square$

LEMMA 3.2.16. *For every $\varepsilon > 0$ there exists a neighborhood of A in Ω such that $u_\varepsilon > 0$ in this neighborhood.*

PROOF. Let $y_0 \in A$ and let $B_\delta(z_0)$ be an exterior tangent ball to $\partial\Omega$ at y_0 such that $\overline{\Omega} \cap \overline{B} = \{y_0\}$. Let us take δ small enough so that $B_{2\delta}(z_0) \cap \partial\Omega \subset\subset A$. Let w_ε be a minimizer of

$$\widetilde{J}_\varepsilon(w) := \int_{\mathcal{R}} G(|\nabla w|)\,dx + \frac{1}{\varepsilon}|\{w > 0\} \cap \mathcal{R}| \tag{3.2.17}$$

in $\{w \in W^{1,G}(\mathcal{R})\,,\ w = 0 \text{ on } \partial B_{2\delta}(z_0),\ w = c_0 \text{ on } \partial B_\delta(z_0)\}$. Here $\mathcal{R} = B_{2\delta}(z_0) \setminus \overline{B}_\delta(z_0)$.

Every minimizer of (3.2.17) is radially symmetric and radially decreasing with respect to z_0. This is seen by using Schwartz symmetrization after extending w_ε to $B_\delta(z_0)$ as the constant function c_0 (see Chapter 1, section 7). This symmetrization preserves the distribution function and strictly decreases the $\int_B G(|\nabla u|)\,dx$ unless the function is already radially symmetric and radially decreasing (see Proposition 1.7.16). Moreover, these minimizers are ordered and their supports are nested. Let us take as w_ε the smallest minimizer.

By the properties of w_ε there holds that w_ε is strictly positive in a ring around $B_\delta(z_0)$. Also w_ε is continuous in $\mathcal{R}$. Recall that u_ε is continuous in Ω. Let us see that $u_\varepsilon \ge w_\varepsilon$ in $\mathcal{R} \cap \Omega$. This will prove the statement.

Assume instead that $\{w_\varepsilon > u_\varepsilon\} \neq \emptyset$.

Let us first consider the function $v = \min\{u_\varepsilon, w_\varepsilon\}$ in $\mathcal{R} \cap \Omega$. Since $u_\varepsilon \ge c_0 \ge w_\varepsilon$ on $\partial\Omega \cap \mathcal{R}$ and $u_\varepsilon \ge 0 = w_\varepsilon$ on $\Omega \cap \partial\mathcal{R}$ there holds that $v = w_\varepsilon$ on $\partial(\mathcal{R} \cap \Omega)$. Therefore, the function $\underline{v} = v$ in $\mathcal{R} \cap \Omega$, $\underline{v} = w_\varepsilon$ in $\mathcal{R} \setminus \Omega$ is an admissible function for the minimization problem (3.2.17). Since w_ε is the smallest minimizer and, by assumption $\underline{v} \le w_\varepsilon$ and $\underline{v} \neq w_\varepsilon$, there holds that $\widetilde{J}_\varepsilon(\underline{v}) > \widetilde{J}_\varepsilon(w_\varepsilon)$. Since $\underline{v} = w_\varepsilon$ in $\mathcal{R} \setminus \Omega$ and in $\mathcal{R} \cap \Omega \cap \{w_\varepsilon \le u_\varepsilon\}$ and equal to u_ε outside those sets there holds that (with $\mathcal{D} = \mathcal{R} \cap \Omega \cap \{w_\varepsilon > u_\varepsilon\}$),

$$\int_{\mathcal{D}} G(|\nabla u_\varepsilon|)\,dx + \frac{1}{\varepsilon}|\{u_\varepsilon > 0\} \cap \mathcal{D}| > \int_{\mathcal{D}} G(|\nabla w_\varepsilon|)\,dx + \frac{1}{\varepsilon}|\{w_\varepsilon > 0\} \cap \mathcal{D}|. \tag{3.2.18}$$

Let now $\bar{v} = \max\{u_\varepsilon, w_\varepsilon\}$ in $\mathcal{R} \cap \Omega$, $\bar{v} = u_\varepsilon$ in $\Omega \setminus \mathcal{R}$. This function is admissible for (P_ε) so that

$$\int_\Omega G(|\nabla \bar{v}|)\,dx + F_\varepsilon\big(|\{\bar{v} > 0\}|\big) \ge \int_\Omega G(|\nabla u_\varepsilon|)\,dx + F_\varepsilon\big(|\{u_\varepsilon > 0\}|\big).$$

Since $\bar{v} = w_\varepsilon$ in $\mathcal{D}$ and $\bar{v} = u_\varepsilon$ in $\Omega \setminus \mathcal{D}$,

$$\begin{aligned} &\int_{\mathcal{D}} G(|\nabla w_\varepsilon|)\,dx + F_\varepsilon\big(|\{u_\varepsilon > 0\}| + |\{w_\varepsilon > 0\} \cap \mathcal{D}| - |\{u_\varepsilon > 0\} \cap \mathcal{D}|\big) \\ &\qquad \ge \int_{\mathcal{D}} G(|\nabla u_\varepsilon|)\,dx + F_\varepsilon\big(|\{u_\varepsilon > 0\}|\big). \end{aligned} \tag{3.2.19}$$

By (3.2.18) and (3.2.19) (with $C_w = |\{w_\varepsilon > 0\} \cap \mathcal{D}|$ and $C_u = |\{u_\varepsilon > 0\} \cap \mathcal{D}|$) we have,

$$\begin{aligned}\int_{\mathcal{D}} G(|\nabla u_\varepsilon|)\,dx &> \int_{\mathcal{D}} G(|\nabla w_\varepsilon|)\,dx + \frac{1}{\varepsilon}(C_w - C_u)\\ &\geq \int_{\mathcal{D}} G(|\nabla u_\varepsilon|)\,dx + F_\varepsilon(|\{u_\varepsilon > 0\}|) - F_\varepsilon(|\{u_\varepsilon > 0\}| + C_w - C_u)\\ &\quad + \frac{1}{\varepsilon}(C_w - C_u).\end{aligned}$$

Thus,

$$F_\varepsilon(|\{u_\varepsilon > 0\}| + C_w - C_u) - F_\varepsilon(|\{u_\varepsilon > 0\}) > \frac{1}{\varepsilon}(C_w - C_u)$$

which is a contradiction since $F_\varepsilon(A) - F_\varepsilon(B) \leq \frac{1}{\varepsilon}(A - B)$ if $A \geq B$ and $C_w \geq C_u$ by the definition of D.

Therefore, $u_\varepsilon \geq w_\varepsilon$ in $\mathcal{R} \cap \Omega$ and the lemma is proved. $\square$

With these uniform bounds on λ_ε, we can prove that $|\{u_\varepsilon > 0\} \cap \Omega| = \alpha$.

Theorem 3.2.20. *Under the same hypotheses of Lemma 3.2.10, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $|\{u_\varepsilon > 0\}| = \alpha$. Therefore, u_ε is a minimizer of $\mathcal{J}$ in $\mathcal{K}_\alpha$.*

Proof. Arguing by contradiction, assume first that $|\{u_\varepsilon > 0\}| > \alpha$. Let $x_1 \in \partial\{u_\varepsilon > 0\} \cap \Omega$ be a regular point. We will proceed as in the proof of Lemma 3.2.10. Given $\delta > 0$, we perturb the domain $\{u_\varepsilon > 0\}$ in a neighborhood of x_1, decreasing its measure by δ. We choose δ small so that the measure of the perturbed set is still larger than α. Take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$, and let

$$v = \begin{cases} v_\rho & \text{in } B_\rho(x_1)\\ u_\varepsilon & \text{elsewhere,}\end{cases}$$

where τ_ρ is the function that we have considered in the previous lemma.

We have

$$\begin{aligned}0 \leq \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) &= \int_\Omega G(|\nabla v|)\,dx - \int_\Omega G(|\nabla u_\varepsilon|)\,dx + F_\varepsilon(|\{v > 0\}|)\\ &\quad - \int_\Omega F_\varepsilon(|\{u_\varepsilon > 0\}|)\\ &\leq kG(\lambda_\varepsilon)\delta + o_\varepsilon(\delta) - \frac{1}{\varepsilon}\delta\\ &\leq \left(kG(C) - \frac{1}{\varepsilon}\right)\delta + o_\varepsilon(\delta) < 0,\end{aligned}$$

if $\varepsilon < \varepsilon_0$ and then $\delta < \delta_0(\varepsilon)$. A contradiction.

Now assume that $|\{u_\varepsilon > 0\}| < \alpha$. This case, is a little bit different from the other. First, we proceed as in the previous case but this time we perturb in a neighborhood of x_1 the set $\{u_\varepsilon > 0\}$ increasing its measure. That is, take

$$\tau_\rho(x) = \begin{cases} x + \rho^2 \phi\left(\dfrac{|x-x_1|}{\rho}\right)\nu_{u_\varepsilon}(x_1) & \text{for } x \in B_\rho(x_1), \\ x & \text{elsewhere,} \end{cases}$$

where $\phi \in C_0^\infty$ supported in the unit interval, take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$ and

$$v = \begin{cases} v_\rho & \text{in } B_\rho(x_1) \\ u_\varepsilon & \text{elsewhere.} \end{cases}$$

For ρ small enough we have $|\{v > 0\}| < \alpha$ and

$$|\{v > 0\}| - |\{u_\varepsilon > 0\}| = C\rho^{N+1} + o(\rho^{N+1}),$$

therefore

$$F_\varepsilon(|\{v > 0\}|) - F_\varepsilon(|\{u_\varepsilon > 0\}|) \le C\varepsilon\rho^{N+1} + o_\varepsilon(\rho^{N+1}). \tag{3.2.21}$$

In order to estimate the other term, we will make use of a blow up argument as in Lemma 3.1.4. In fact, we take $u_\rho(y) = \frac{1}{\rho}u(x_1 + \rho y)$ and we change variables to obtain,

$$\begin{aligned} \rho^{-N} &\int_{B_\rho(x_1)} (G(|\nabla v_\rho|) - G(|\nabla u_\varepsilon|))\, dx \\ &= \int_{B_1(0)\cap\{u_\rho>0\}} \rho[G(|\nabla u_\rho|)\mathrm{div}(\eta) - F(|\nabla u_\rho|)(\nabla u_\rho)^t D\eta \nabla u_\rho] + o(\rho)\, dy \end{aligned}$$

where $\eta(y) = \phi(|y|)\nu(x_1)$. Now, as in Lemma 3.1.4 we get,

$$\begin{aligned} \rho^{-N-1} &\int_{B_\rho(x_1)} (G(|\nabla v_\rho|) - G(|\nabla u_\varepsilon|))\, dx \to \\ &\qquad - \Phi(\lambda_\varepsilon) \int_{B_1(0)\cap\{y\cdot\nu=0\}} \phi(|y|)\, d\mathcal{H}^{N-1}(y). \end{aligned}$$

Therefore

$$\int_{B_\rho(x_1)} (G(|\nabla v_\rho|) - G(|\nabla u_\varepsilon|))\, dx = -C\rho^{N+1}\Phi(\lambda_\varepsilon) + o(\rho^{N+1}). \tag{3.2.22}$$

Finally, combining (3.2.21) and (3.2.22) we have

$$\begin{aligned} 0 \le \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) &= \int_\Omega G(|\nabla v|)\, dx - \int_\Omega G(|\nabla u_\varepsilon|)\, dx \\ &\quad + F_\varepsilon(|\{v > 0\}|) - F_\varepsilon(|\{u_\varepsilon > 0\}|) \\ &\le -\Phi(\lambda_\varepsilon)\rho^{N+1} + o_\varepsilon(\rho^{N+1}) + C\varepsilon\rho^{N+1} \\ &\le (-\Phi(c) + C\varepsilon)\rho^{N+1} + o_\varepsilon(\rho^{N+1}) < 0, \end{aligned}$$

if $\varepsilon < \varepsilon_1$ and then $\rho < \rho_0(\varepsilon)$. Again a contradiction that ends the proof.

$\square$

As a corollary, we have the desired result for our problem

COROLLARY 3.2.23. *Suppose that g satisfies* (0.0.2). *Then any minimizer u of $\mathcal{J}$ in $\mathcal{K}_\alpha$ is a locally Lipschitz continuous function, $\partial_{red}\{u > 0\}$ is a $C^{1,\beta}$ surface locally in Ω and the remainder of the free boundary has $\mathcal{H}^{N-1}-$measure zero. Moreover if $N = 2$ and g satisfies* (2.7.57) *then $\partial\{u > 0\}$ is a $C^{1,\beta}$ surface locally in Ω.*

PROOF. If u is minimizer of $\mathcal{J}$ in $\mathcal{K}_\alpha$, by Theorem 3.2.20 we have that for small ε there exists a solution u_ε to (P_ε) such that $|\{u_\varepsilon > 0\}| = \alpha$ then u is a solution to (P_ε). Therefore, the result follows. $\square$

CHAPTER 4

The singular perturbation problem

In this chapter we study the following problem. For any $\varepsilon > 0$, take u^ε a solution of,

$$(P_\varepsilon) \qquad \mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon), \quad u^\varepsilon \geq 0.$$

A solution to (P_ε) is a function $u^\varepsilon \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega g(|\nabla u^\varepsilon|)\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\nabla\varphi\,dx = -\int_\Omega \varphi\beta_\varepsilon(u^\varepsilon)\,dx \tag{4.0.1}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Where $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$, with $\beta \in \mathrm{Lip}(\mathbb{R})$, supported in $[0,1]$, and such that satisfies, $\int_0^1 \beta(s)\,ds = M$ for a constant M.

We are interested in what happens with the limiting problem, when $\varepsilon \to 0$.

In this chapter we will assume all the time, that g satisfies condition (0.0.2).

1. Uniform bound of the gradient

We begin by proving that solutions of the perturbation problem are locally uniformly Lipschitz. That is, the u^ε's are locally Lipschitz, and the Lipschitz constant is independent of ε. To prove this result, we will first need to prove a couple of lemmas.

Lemma 4.1.1. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } B_{r_0}(x_0)$$

such that $u^\varepsilon(x_0) \leq 2\varepsilon$. Then, there exists $C = C(N, r_0, \delta, g_0, \|\beta\|_\infty, g(1))$ such that, if $\varepsilon \leq 1$,

$$|\nabla u^\varepsilon(x_0)| \leq C.$$

Proof. Let $v(x) = \frac{1}{\varepsilon}u^\varepsilon(x_0+\varepsilon x)$. Then if $\varepsilon \leq 1$, $\mathcal{L}v = \beta(v)$ in B_{r_0} and $v(0) \leq 2$. By Harnack inequality (see Remark 1.2.21) we have that $0 \leq v(x) \leq C_1$ in $B_{r_0/2}$ with $C_1 = C_1(N, g_0, \delta, \|\beta\|_\infty)$ therefore, again by Remark 1.2.21 we have that

$$|\nabla v(0)| \leq C$$

with $C = C(N, \delta, g_0, \|\beta\|_\infty, r_0, g(1))$. □

LEMMA 4.1.2. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } B_1,$$

and $0 \in \partial\{u^\varepsilon > \varepsilon\}$, then for $x \in B_{1/4} \cap \{u^\varepsilon > \varepsilon\}$,

$$u^\varepsilon(x) \le \varepsilon + C\,dist(x, \{u^\varepsilon \le \varepsilon\}),$$

with $C = C(N, \delta, g_0, \|\beta\|_\infty, g(1))$.

PROOF. For $x_0 \in B_{1/4} \cap \{u^\varepsilon > \varepsilon\}$ take, $m_0 = u^\varepsilon(x_0) - \varepsilon$ and $\delta_0 = \operatorname{dist}(x_0, \{u^\varepsilon \le \varepsilon\} \cap B_1)$. Since $0 \in \partial\{u^\varepsilon > \varepsilon\} \cap B_1$, $\delta_0 \le 1/4$. We want to prove that, $m_0 \le C(N, \delta, g_0, \|\beta\|_\infty, g(1))\delta_0$.

Since, $B_{\delta_0}(x_0) \subset \{u^\varepsilon > \varepsilon\} \cap B_1$ we have that, $u^\varepsilon - \varepsilon \ge 0$ in $B_{\delta_0}(x_0)$ and $\mathcal{L}(u^\varepsilon - \varepsilon) = 0$. By Harnack inequality there exists $c_1 = c_1(N, g_0, \delta)$ such that

$$\min_{B_{\delta_0/2}(x_0)} (u^\varepsilon - \varepsilon) \ge c_1 m_0.$$

Let as take $\varphi = e^{-\mu|x|^2} - e^{-\mu\delta_0^2}$ with $\mu = \frac{2K}{\delta\delta_0^2}$, and where K is the constant defined in Lemma 1.2.47 that depends only on N and g_0. Then, we have that $\mathcal{L}\varphi > 0$ in $B_{\delta_0} \setminus B_{\delta_0/2}$ (see the proof of Lemma 1.2.47).

Let now $\psi(x) = c_2 m_0 \varphi(x - x_0)$ for $x \in \overline{B}_{\delta_0}(x_0) \setminus \overline{B}_{\delta_0/2}(x_0)$. Then, again, by Lemma 1.2.47 we have that, if we choose c_2 conveniently depending on N, δ, g_0

$$\begin{cases} \mathcal{L}\psi(x) > 0 & \text{in } B_{\delta_0}(x_0) \setminus \overline{B}_{\delta_0/2}(x_0) \\ \psi = 0 & \text{on } \partial B_{\delta_0}(x_0) \\ \psi = c_1 m_0 & \text{on } \partial B_{\delta_0/2}(x_0). \end{cases}$$

By the comparison principle (see Lemma 1.2.35) we have,

$$\psi(x) \le u^\varepsilon(x) - \varepsilon \quad \text{in } \overline{B}_{\delta_0}(x_0) \setminus \overline{B}_{\delta_0/2}(x_0). \tag{4.1.3}$$

Take $y_0 \in \partial B_{\delta_0}(x_0) \cap \partial\{u^\varepsilon > \varepsilon\}$. Then, $y_0 \in \overline{B}_{1/2}$ and

$$\psi(y_0) = u^\varepsilon(y_0) - \varepsilon = 0. \tag{4.1.4}$$

Let $v^\varepsilon = \frac{u^\varepsilon(y_0 + \varepsilon x)}{\varepsilon}$, then if $\varepsilon < 1$ we have that $\mathcal{L}v^\varepsilon = \beta(v^\varepsilon)$ in $B_{1/2}$ and $v^\varepsilon(0) = 1$. Therefore, by Harnack inequality (see Remark 1.2.21) we have that $\max_{\overline{B}_{1/4}} v^\varepsilon \le \widetilde{c}$ and

$$|\nabla u^\varepsilon(y_0)| = |\nabla v^\varepsilon(0)| \le C \max_{\overline{B}_{1/4}} v^\varepsilon \le c_3. \tag{4.1.5}$$

Finally, by (4.1.3), (4.1.4) and (4.1.5) we have that, $|\nabla\psi(y_0)| \le |\nabla u^\varepsilon(y_0)| \le c_3$. Observe that $|\nabla\psi(y_0)| = c_2 m_0 e^{-\mu\delta_0^2} 2\mu\delta_0 \le c_3$, therefore

$$m_0 \le \frac{c_3 e^{\mu\delta_0^2}}{c_2 2\mu\delta_0} = \frac{c_3 \delta e^{2K/\delta}}{c_2 4K}\delta_0$$

and the result follows.

□

Now, we can prove the main result of this section,

PROPOSITION 4.1.6. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } B_1,$$

$0 \in \partial\{u^\varepsilon > \varepsilon\}$, *then we have for $x \in B_{1/8}$,*

$$|\nabla u^\varepsilon(x)| \le C$$

with $C = C(N, \delta, g_0, \|\beta\|_\infty, g(1))$.

PROOF. By Lemma 4.1.1 we know that if $x_0 \in \{u^\varepsilon \le 2\varepsilon\} \cap B_{3/4}$ then,

$$|\nabla u^\varepsilon(x_0)| \le C_0$$

with $C_0 = C_0(N, \delta, g_0, \|\beta\|_\infty, g(1))$.

Let $x_0 \in B_{1/8} \cap \{u^\varepsilon > \varepsilon\}$ and $\delta_0 = \text{dist}(x_0, \{u^\varepsilon \le \varepsilon\})$.

As $0 \in \partial\{u^\varepsilon > \varepsilon\}$ we have that $\delta_0 \le 1/8$. Therefore, $B_{\delta_0}(x_0) \subset \{u^\varepsilon > \varepsilon\} \cap B_{1/4}$ and then $\mathcal{L}u^\varepsilon = 0$ in $B_{\delta_0}(x_0)$ and by Lemma 4.1.2

$$u^\varepsilon(x) \le \varepsilon + C_1 \text{dist}(x, \{u^\varepsilon \le \varepsilon\}) \quad \text{in } B_{\delta_0}(x_0). \tag{4.1.7}$$

1. Suppose that $\varepsilon < \bar{c}\delta_0$ with $\bar{c}$ to be determined. Then, if $x \in B_{\delta_0}(x_0)$ there holds that $\text{dist}(x, \partial\{u^\varepsilon > \varepsilon\} < 2\delta_0$. Let $v(x) = \frac{u^\varepsilon(x_0+\delta_0 x)}{\delta_0}$ then $\mathcal{L}v = \beta_\varepsilon(u^\varepsilon(x_0+\delta_0 x)) = 0$ in B_1. Therefore, by Lemma 1.2.18
$$|\nabla v(0)| \le C \sup_{B_1} v,$$
with $\widetilde{C} = \widetilde{C}(N, g_0, \delta, g(1))$. We obtain,
$$|\nabla u^\varepsilon(x_0)| \le \frac{\widetilde{C}}{\delta_0} \sup_{B_{\delta_0}(x_0)} u^\varepsilon \le \frac{\widetilde{C}}{\delta_0}(\varepsilon + C\delta_0) \le \widetilde{C}(\bar{c} + C).$$
2. Suppose that $\varepsilon \ge \bar{c}\delta_0$. By (4.1.7) we have,
$$u^\varepsilon(x_0) \le \varepsilon + C_1\delta_0 \le \Big(1 + \frac{C_1}{\bar{c}}\Big)\varepsilon < 2\varepsilon,$$
if we choose $\bar{c}$ big enough. By Lemma 4.1.1, we have $|\nabla u^\varepsilon(x_0)| \le C$, with $C = C(N, g_0, \delta, \|\beta\|_\infty, g(1))$.

The result follows. □

With this lemmas we obtain the following,

COROLLARY 4.1.8. *Let u^ε be a solution of*

$$\mathcal{L}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } \Omega,$$

with $\|u^\varepsilon\|_{L^\infty(\Omega)} \le L$. Then, we have for $\Omega' \subset\subset \Omega$, that there exists $\varepsilon_0(\Omega, \Omega')$ such that if $\varepsilon \le \varepsilon_0(\Omega, \Omega')$,

$$|\nabla u^\varepsilon(x)| \le C \quad \text{in } \Omega'$$

with $C = C(N, \delta, g_0, L, \|\beta\|_\infty, g(1), dist(\Omega', \partial\Omega))$.

Proof. Let $\tau > 0$ such that $\forall x \in \Omega'$, $\overline{B_\tau(x)} \subset \Omega$ and $\varepsilon \leq \tau$. Let $x_0 \in \Omega'$.

1. If $\delta_0 = \operatorname{dist}(x_0, \partial\{u^\varepsilon > \varepsilon\}) \leq \tau/8$, let $y_0 \in \partial\{u^\varepsilon > \varepsilon\}$ such that $|x_0 - y_0| = \delta_0$. Let $v(x) = \frac{u^\varepsilon(y_0+\tau x)}{\tau}$, and $\bar{x} = \frac{x_0-y_0}{\tau}$, then $|\bar{x}| < 1/8$. As $0 \in \partial\{v > \varepsilon/\tau\}$ and $\mathcal{L}v = \beta_{\varepsilon/\tau}(v)$ in B_1, we have by Proposition 4.1.6
$$|\nabla u^\varepsilon(x_0)| = |\nabla v(\bar{x})| \leq C.$$
2. If $\delta_0 = \operatorname{dist}(x_0, \partial\{u^\varepsilon > \varepsilon\}) \geq \tau/8$, there holds that
   a) $B_{\tau/8}(x_0) \subset \{u^\varepsilon > \varepsilon\}$, or
   b) $B_{\tau/8}(x_0) \subset \{u^\varepsilon \leq \varepsilon\}$,
   In the first case, $\mathcal{L}u^\varepsilon = 0$ in $B_{\tau/8}(x_0)$. Therefore,
$$|\nabla u^\varepsilon(x_0)| \leq C(N, g_0, \delta, \tau, g(1), L).$$
   In the second case, we can apply Lemma 4.1.1 and we have,
$$|\nabla u^\varepsilon(x_0)| \leq C(N, g_0, \delta, \tau, g(1), 2\|\beta\|_\infty).$$
   The result is proved.

□

2. Passage to the limit

Since we have that $|\nabla u^\varepsilon|$ is bounded by a constant independent of ε, we have that there exists a function $u \in Lip(\Omega)$ such that, for a subsequence $\varepsilon_j \to 0$, $u^{\varepsilon_j} \to u$. In this section we will prove some properties of the function u.

Proposition 4.2.1. *Let $\{u^\varepsilon\}$ be a uniformly bounded family of solutions of (P_ε). Then for any sequence $\varepsilon_j \to 0$ there exists a subsequence $\varepsilon'_j \to 0$ and $u \in Lip_{loc}(\Omega)$ such that,*

1. *$u^{\varepsilon'_j} \to u$ uniformly in compact subsets of Ω,*
2. *$\mathcal{L}u = 0$ in $\Omega \setminus \partial\{u > 0\}$*
3. *There exists a locally finite measure μ such that $\beta_{\varepsilon'_j}(u^{\varepsilon'_j}) \rightharpoonup \mu$ as measures in Ω', for every $\Omega' \subset\subset \Omega$,*
4. *$\nabla u^{\varepsilon'_j} \to \nabla u$ in $L^{g_0+1}_{loc}(\Omega)$,*
5. $$\int_\Omega F(|\nabla u|)\nabla u \nabla\varphi = -\int_\Omega \varphi \, d\mu$$
   for all $\varphi \in C_0^\infty(\Omega)$. Moreover μ is supported on $\Omega \cap \partial\{u > 0\}$.

Proof. (1) follows by Corollary 4.1.8.

To prove (2), take $E \subset\subset \{u > 0\}$ then $u \geq c > 0$ in E. Therefore, $u^{\varepsilon'_j} > c/2$ in E for ε'_j small. If we take $\varepsilon'_j < c/2$ as $\mathcal{L}u^{\varepsilon'_j} = 0$ in $\{u^{\varepsilon'_j} > \varepsilon'_j\}$, we have that $\mathcal{L}u^{\varepsilon'_j} = 0$ in E, then $\|u^{\varepsilon'_j}\|_{C^{1,\alpha}(E)} \leq C$. For a subsequence we have,
$$\nabla u^{\varepsilon'_j} \to \nabla u \quad \text{uniformly in } E$$
therefore, $\mathcal{L}u = 0$.

To prove (3), let us take $\Omega' \subset\subset \Omega$, and $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ in Ω'. As in Ω' $\|\nabla u^{\varepsilon'_j}\| \leq C$, we have that

$$C(\varphi) \geq \int_\Omega \beta_{\varepsilon'_j}(u^{\varepsilon'_j})\varphi\, dx \geq \int_{\Omega'} \beta_{\varepsilon'_j}(u^{\varepsilon'_j})\, dx.$$

Therefore, $\beta_{\varepsilon'_j}(u^{\varepsilon'_j})$ is bounded in $L^1_{loc}(\Omega)$, so that, there exists a locally finite measure μ such that

$$\beta_{\varepsilon'_j}(u^{\varepsilon'_j}) \rightharpoonup \mu \quad \text{as measures}$$

that is, for every $\varphi \in C_0(\Omega)$,

$$\int_\Omega \beta_{\varepsilon'_j}(u^{\varepsilon'_j})\varphi\, dx \to \int_\Omega \varphi\, d\mu$$

We divide the proof of (4) in several steps.

Step 1. Let $\Omega' \subset\subset \Omega$, then by Corollary 4.1.8, $|\nabla u^{\varepsilon_j}| \leq C$ in Ω'. Therefore for a subsequence ε'_j we have that there exists $\xi \in (L^\infty(\Omega'))^N$ such that,

$$\begin{aligned} &\nabla u^{\varepsilon'_j} \rightharpoonup \nabla u && * - \text{weakly in } (L^\infty(\Omega'))^N \\ &A(\nabla u^{\varepsilon'_j}) \rightharpoonup \xi && * - \text{weakly in } (L^\infty(\Omega'))^N \\ &u^{\varepsilon'_j} \to u && \text{uniformly in } \Omega' \end{aligned} \tag{4.2.2}$$

where $A(p) = F(|p|)p$. For simplicity we call $\varepsilon'_j = \varepsilon$.

We want to prove that, for any $v \in C_0^\infty(\Omega')$

$$\int_{\Omega'} A(\nabla u^\varepsilon)\nabla v\, dx \to \int_{\Omega'} A(\nabla u)\nabla v\, dx. \tag{4.2.3}$$

First, as A is monotone (i.e $\langle A(\eta) - A(\xi), \eta - \xi\rangle \geq 0\ \forall\ \eta, \xi \in \mathbb{R}^N$) we have that, for any $w \in W^{1,G}(\Omega')$,

$$I = \int_{\Omega'} (A(\nabla u^\varepsilon) - A(\nabla w))(\nabla u^\varepsilon - \nabla w)\, dx \geq 0. \tag{4.2.4}$$

Therefore,

(4.2.5)

$$
\begin{aligned}
&-\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla w\,dx - \int_{\Omega'} A(\nabla w)(\nabla u^\varepsilon - \nabla w)\,dx\\
&= -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx + I\\
&= -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\,dx - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)\psi\,dx - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)(1-\psi)\,dx\\
&\quad - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx + I\\
&\geq -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\,dx + \int_{\Omega'} A(\nabla u^\varepsilon)\nabla(u^\varepsilon - u)\psi\,dx + \int_{\Omega'} A(\nabla u^\varepsilon)(u^\varepsilon - u)\nabla\psi\,dx\\
&\quad - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)(1-\psi)\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx,
\end{aligned}
$$

where in the last inequality we are using (4.2.4) and (0.3.2).

Now, take $\psi = \psi_j \to \chi_{\Omega'}$. Then if Ω' is regular we have $\int_{\Omega'} |\nabla\psi_j|\,dx \to \operatorname{Per}\Omega'$. We have that,

$$
\Big|\int_{\Omega'} A(\nabla u^\varepsilon)(u^\varepsilon - u)\nabla\psi_j\,dx\Big| \leq C\|u^\varepsilon - u\|_{L^\infty(\Omega')}\int_{\Omega'} |\nabla\psi_j|\,dx \leq C\|u^\varepsilon - u\|_{L^\infty(\Omega')}
$$

taking $\psi_j \to \chi_{\Omega'}$ in (4.2.5) we obtain,

$$
\begin{aligned}
&-\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla w\,dx - \int_{\Omega'} A(\nabla w)(\nabla u^\varepsilon - \nabla w)\,dx\\
&\geq -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\,dx + \int_{\Omega'} A(\nabla u^\varepsilon)\nabla(u^\varepsilon - u)\,dx - C\|u^\varepsilon - u\|_{L^\infty(\Omega')}\\
&\quad - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx\\
&= -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\,dx - \int_{\Omega'} A(\nabla u^\varepsilon)\nabla u\,dx - C\|u^\varepsilon - u\|_{L^\infty(\Omega')}
\end{aligned}
$$

therefore, taking $\varepsilon \to 0$ we get using (4.2.2) and (3) that,

$$
-\int_{\Omega'} u\,d\mu - \int_{\Omega'} \xi\nabla w\,dx - \int_{\Omega'} A(\nabla w)(\nabla u - \nabla w)\,dx \geq -\int_{\Omega'} u\,d\mu - \int_{\Omega'} \xi\nabla u\,dx
$$

and then,

$$
\int_{\Omega'} (\xi - A(\nabla w))(\nabla u - \nabla w)\,dx \geq 0. \tag{4.2.6}
$$

Take now $w = u + \lambda v$ with $v \in C_0^\infty(\Omega')$. Dividing by λ and taking $\lambda \to 0^+$ in (4.2.6), we obtain,

$$
\int_{\Omega'} (\xi - A(\nabla u))\nabla v\,dx \geq 0,
$$

chaining v by $-v$ we obtain the desired result.

Step 2 Therefore passing to the limit in the equation

$$0=\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla\phi+\int_{\Omega'}\beta_{\varepsilon}(u^{\varepsilon})\phi\,dx, \tag{4.2.7}$$

we have by Step 1, that for every $\phi\in C_0^{\infty}(\Omega')$,

$$0=\int_{\Omega'} A(\nabla u)\nabla\phi+\int_{\Omega'}\phi\,d\mu. \tag{4.2.8}$$

Taking $\phi=u^{\varepsilon}\psi$ in (4.2.7) with $\psi\in C_0^{\infty}(\Omega')$ we have that

$$0=\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla u^{\varepsilon}\psi\,dx+\int_{\Omega'} A(\nabla u^{\varepsilon})u^{\varepsilon}\nabla\psi\,dx+\int_{\Omega'}\beta_{\varepsilon}(u^{\varepsilon})u^{\varepsilon}\psi\,dx.$$

Using that,

$$\int_{\Omega'} A(\nabla u^{\varepsilon})u^{\varepsilon}\nabla\psi\,dx\to\int_{\Omega'} A(\nabla u)u\nabla\psi\,dx$$
$$\int_{\Omega'}\beta_{\varepsilon}(u^{\varepsilon})u^{\varepsilon}\psi\,dx\to\int_{\Omega'} u\psi d\mu$$

we obtain

$$0=\lim_{\varepsilon\to0}\Big(\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla u^{\varepsilon}\psi\,dx\Big)+\int_{\Omega'} A(\nabla u)u\nabla\psi\,dx+\int_{\Omega'} u\psi d\mu.$$

Taking now, $\phi=u\psi$ in (4.2.8) we have,

$$0=\int_{\Omega'} A(\nabla u)\nabla u\psi\,dx+\int_{\Omega'} A(\nabla u)u\nabla\psi\,dx+\int_{\Omega'} u\psi\,d\mu.$$

Therefore,

$$\lim_{\varepsilon\to0}\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla u^{\varepsilon}\psi\,dx=\int_{\Omega'} A(\nabla u)\nabla u\psi\,dx.$$

Then,

$$\begin{aligned}
&\left|\int_{\Omega'}(A(\nabla u^{\varepsilon})\nabla u^{\varepsilon}-A(\nabla u)\nabla u)\,dx\right|\\
&\le\left|\int_{\Omega'}(A(\nabla u^{\varepsilon})\nabla u^{\varepsilon}-A(\nabla u)\nabla u)\psi\,dx\right|+\left|\int_{\Omega'}(A(\nabla u^{\varepsilon})\nabla u^{\varepsilon})(1-\psi)\,dx\right|\\
&\quad+\left|\int_{\Omega'} A(\nabla u)\nabla u(1-\psi)\,dx\right|\\
&\le\left|\int_{\Omega'}(A(\nabla u^{\varepsilon})\nabla u^{\varepsilon}-A(\nabla u)\nabla u)\psi\,dx\right|+C\int_{\Omega'}|1-\psi|\,dx.
\end{aligned}$$

So that, taking $\varepsilon\to0$ and then $\psi\to1$ a.e with $0\le\psi\le1$ we obtain,

$$\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla u^{\varepsilon}dx\to\int_{\Omega'} A(\nabla u)\nabla u\,dx. \tag{4.2.9}$$

As

$$\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla u\psi\,dx \to \int_{\Omega'} A(\nabla u)\nabla u\psi\,dx$$

we also have, doing the same calculation, that

$$\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla u\,dx \to \int_{\Omega'} A(\nabla u)\nabla u\,dx. \tag{4.2.10}$$

Step 3. By the monotonicity of A we have,

$$\begin{aligned}\int_{\Omega'} G(|\nabla u^{\varepsilon}|)\,dx - \int_{\Omega'} G(|\nabla u|)\,dx &= \int_{\Omega'}\int_0^1 A(\nabla u + t(\nabla u^{\varepsilon} - \nabla u))\nabla(u^{\varepsilon} - u)\,dx\\ &\geq \int_{\Omega'} A(\nabla u)\nabla(u^{\varepsilon} - u)\,dx.\end{aligned}$$

Therefore, by step 3 we have,

$$\liminf_{\varepsilon\to 0}\int_{\Omega'} G(|\nabla u^{\varepsilon}|)\,dx - \int_{\Omega'} G(|\nabla u|)\,dx \geq 0.$$

Step 4. By Step 2 we have,

$$\begin{aligned}\int_{\Omega'} G(|\nabla u^{\varepsilon}|)\,dx - \int_{\Omega'} G(|\nabla u|)\,dx &= \int_{\Omega'}\int_0^1 A(\nabla u + t(\nabla u^{\varepsilon} - \nabla u))\nabla(u^{\varepsilon} - u)\,dx\\ &\leq \int_{\Omega'} A(\nabla u^{\varepsilon})\nabla(u^{\varepsilon} - u)\,dx \to 0.\end{aligned}$$

Then, Step 3 implies,

$$\int_{\Omega'} G(|\nabla u^{\varepsilon}|)\,dx \to \int_{\Omega'} G(|\nabla u|)\,dx. \tag{4.2.11}$$

Step 5. Let $u^s = su + (1-s)u^{\varepsilon}$. Then,

(4.2.12)

$$\begin{aligned}\int_{\Omega'} G(|\nabla u|)\,dx - \int_{\Omega'} G(|\nabla u^{\varepsilon}|)\,dx &= \int_{\Omega'}\int_0^1 A(\nabla u^s)\nabla(u - u^{\varepsilon})\,ds\,dx\\ = \int_{\Omega'}\int_0^1 (A(\nabla u^s) - A(\nabla u^{\varepsilon}))\nabla(u^s - u^{\varepsilon})\,ds\,dx &+ \frac{1}{2}\int_{\Omega'} A(\nabla u^{\varepsilon})\nabla(u - u^{\varepsilon})\,dx.\end{aligned}$$

As in the proof of Theorem 1.2.38, we have that

$$\begin{aligned}&\int_{\Omega'}\int_0^1 (A(\nabla u^s) - A(\nabla u^{\varepsilon}))\nabla(u^s - u^{\varepsilon})\,ds\,dx\\ &\qquad\geq C\Big(\int_{A_2} G(|\nabla u - \nabla u^{\varepsilon}|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla u^{\varepsilon}|^2\,dx\Big),\end{aligned}$$

where A_1 and A_2 were define in Theorem 1.2.38. Therefore, by (4.2.9), (4.2.10), (4.2.11) and (4.2.12) we have,

$$\Big(\int_{A_2} G(|\nabla u - \nabla u^{\varepsilon}|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla u^{\varepsilon}|^2\,dx\Big) \to 0.$$

Then, if we prove that

$$\Big(\int_{A_2} G(|\nabla u-\nabla u^\varepsilon|)\,dx+\int_{A_1} F(|\nabla u|)|\nabla u-\nabla u^\varepsilon|^2\,dx\Big)\geq C\int_{\Omega'}|\nabla u-\nabla u^\varepsilon|^{g_0+1}\,dx$$

the result follows.

In fact, we can suppose that, $g_0\geq 1$. If $t\leq C_0$ then $g(t)\geq C_1t^{g_0}$. As $|\nabla u|\leq C_0$ and $|\nabla u-\nabla u^\varepsilon|\leq C_0$, for some constant C_0 we have

$$\begin{aligned}&G(|\nabla u_\varepsilon-\nabla u|)\geq C|\nabla u_\varepsilon-\nabla u|^{g_0+1}\\&F(|\nabla u_\varepsilon|)\geq C_1|\nabla u_\varepsilon|^{g_0-1}\geq C|\nabla u_\varepsilon-\nabla u|^{g_0-1}\qquad\text{in } A_1.\end{aligned}$$

and the claim follows.

Finally (5) holds by (4), (3) and (2). $\square$

LEMMA 4.2.13. *Let $\{u^{\varepsilon_j}\}$ be a uniformly bounded family of solutions of (P_{ε_j}) in Ω such that $u^{\varepsilon_j}\to u$ uniformly on compact subsets of Ω and $\varepsilon_j\to 0$. Let $x_0,\ x_n\in\Omega\cap\partial\{u>0\}$ be such that $x_n\to x_0$ as $n\to\infty$. Let $\lambda_n\to 0$, $u_{\lambda_n}(x)=\frac{1}{\lambda_n}u(x_n+\lambda_nx)$ and $(u^{\varepsilon_j})_{\lambda_n}(x)=\frac{1}{\lambda_n}u^{\varepsilon_j}(x_n+\lambda_nx)$. Suppose that $u_{\lambda_n}\to U$ as $n\to\infty$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exists, $j(n)\to\infty$ such that for every $j_n\geq j(n)$ there holds that $\varepsilon_{j_n}/\lambda_n\to 0$ and*

1. *$(u^{\varepsilon_{j_n}})_{\lambda_n}\to U$ uniformly in compact subsets of $\mathbb{R}^N$.*
2. *$\nabla(u^{\varepsilon_{j_n}})_{\lambda_n}\to\nabla U$ in $L^{g_0+1}_{loc}(\mathbb{R}^N)$,*
3. *$\nabla u_{\lambda_n}\to\nabla U$ in $L^{g_0+1}_{loc}(\mathbb{R}^N)$.*

PROOF. By simplicity we assume $x_n=x_0$. Then

$$|u^{\varepsilon_j}_{\lambda_n}(x)-U(x)|\leq|\frac{1}{\lambda_n}u^{\varepsilon_j}(x_0+x\lambda_n)-\frac{1}{\lambda_n}u(x_0+x\lambda_n)|+|u_{\lambda_n}-U(x)|=I+II.$$

Fix $k>0$ and $\delta>0$. Then, if $n\geq n(k,\delta)$ there holds that $II<\delta$ in $B_k(0)$.

For the other bound, let $r>0$ such that $B_r(x_0)\subset\Omega'\subset\subset\Omega$. Observe that, for each n there exists $j(n)$ such that if $j\geq j(n)$

$$|u^{\varepsilon_j}(x)-u(x)|\leq\frac{\lambda_n}{n}\qquad\text{for } x\in B_r(x_0).$$

Therefore, if $j\geq j(n)$ with n large such that $\lambda_n\leq r/k$ we have $I\leq\frac{1}{n}$ for $x\in B_k(0)$. So, for $j\geq j(n)$ and n large

$$|(u^{\varepsilon_j})_{\lambda_n}(x)-U(x)|<\delta+\frac{1}{n}\qquad\text{for } x\in B_k(0).$$

Then, if $j_n\geq j(n)$ $(u^{\varepsilon_j})_{\lambda_n}\to U$ uniformly in $B_k(0)$. We may assume, without loss of generality that $\varepsilon_j/\lambda_n<1/n$ for $j\geq j(n)$. So (1) is proved.

It is easy to see that $(u^{\varepsilon_j})_{\lambda_n}$ are solutions to $(P_{\varepsilon_j/\lambda_n})$ in $B_{2k}(0)$ for n large.

By Proposition 4.2.1 there exists a subsequence j'_n such that $\nabla(u^{\varepsilon_{j'_n}})_{\lambda'_n}\to\nabla U$ in $L^{g_0+1}(B_k)$. Then also (2) holds.

Let now, $\delta > 0$ and consider,

$$\|\nabla u_{\lambda_n} - \nabla U\| \leq \|\nabla u_{\lambda_n} - \nabla (u^{\varepsilon_{j_n}})_{\lambda_n}\| + \|\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} - \nabla U\| = I + II,$$

where all the norms are in $L^{g_0+1}(B_k)$. By (2) we know that $II < \delta$ if $j \geq j_n$ and n large. Moreover, by Proposition 4.2.1 we have,

$$\begin{aligned} I^{g_0+1} &= \int_{B_k} |\nabla u - \nabla u^{\varepsilon_j}|^{g_0+1}(x_0 + \lambda_n x)\, dx \\ &= \frac{1}{{\lambda_n}^N} \int_{B_{\lambda_n k}(x_0)} |\nabla u - \nabla u^{\varepsilon_j}|^{g_0+1}(x)\, dx < \delta^{g_0+1} \end{aligned}$$

if n is sufficiently large so that $\lambda_n k \leq r$ and then, j large enough. This proves (3).

□

Now we prove a technical lemma that is the basis of our main results.

Lemma 4.2.14. *Let u^ε be solutions to*

$$\mathcal{L} u^\varepsilon = \beta_\varepsilon(u^\varepsilon)$$

in Ω. Then, for any $\psi \in C_0^\infty(\Omega)$ we have,

$$-\int_\Omega G(|\nabla u^\varepsilon|)\psi_{x_1}\, dx + \int_\Omega F(|\nabla u^\varepsilon|)\nabla u^\varepsilon \nabla \psi\, u^\varepsilon_{x_1}\, dx = \int_\Omega B_\varepsilon(u^\varepsilon)\psi_{x_1}, \tag{4.2.15}$$

where $B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau)\, d\tau$.

Proof. For simplicity, since ε will be fixed throughout the proof, we will denote $u^\varepsilon = u$. We know that $|\nabla u| \leq C_0$. Take $g_n(t) = g(t) + \frac{t}{n}$, then

$$\min\{1, \delta\} \leq \frac{g_n'(t)t}{g_n(t)} \leq \max\{1, g_0\}. \tag{4.2.16}$$

Take $A_n(p) = \frac{g_n(|p|)}{|p|}p$, and $\mathcal{L}_n(v) = \operatorname{div}(A_n(\nabla v))$. Then, if $\Omega' \subset\subset \Omega$ and we take u_n the solution of

$$\begin{cases} \mathcal{L}_n u_n = \beta_\varepsilon(u) & \text{in } \Omega' \\ u_n = u & \text{on } \partial\Omega' \end{cases} \tag{4.2.17}$$

we have by (4.2.16) that all the $g_n's$ belong to the same class and then, by Theorem 1.2.19, $\|u_n\|_{C^{1,\alpha}(\Omega')} \leq C$ with C independent of n. Therefore, there exists u_0 such that, for a subsequence

$$\begin{aligned} &u_n \to u_0 \quad \text{uniformly in } \Omega' \\ &\nabla u_n \to \nabla u_0 \quad \text{uniformly in } \Omega'. \end{aligned}$$

On the other hand, $A_n(p) \to A(p)$ uniformly in compact sets of $\mathbb{R}^N$. Then, $\mathcal{L}u_0 = \beta_\varepsilon(u^\varepsilon)$, and, as $u_0 = u$ on $\partial\Omega$ and $\mathcal{L}u^\varepsilon = \beta_\varepsilon(u)$, there holds that $u_0 = u^\varepsilon$. (Observe that in the proof of the Comparison Principle, in Lemma 1.2.35 we can change the equation $\mathcal{L}u = 0$, by $\mathcal{L}u = f(x)$ with $f \in L^\infty(\Omega)$ to prove uniqueness of the Dirichlet problem).

Now let us prove that the following equality holds,

$$-\int_\Omega G_n(|\nabla u_n|)\psi_{x_1}\,dx + \int_\Omega F_n(|\nabla u_n|)\nabla u_n \nabla\psi\, u_{nx_1}\,dx = -\int_\Omega \beta_\varepsilon(u) u_{nx_1}\psi.$$

In fact, for n fixed we have that $F_n(t) = g_n(t)/t \geq 1/n$ and then by Theorem 1.2.34, $u_n \in W^{2,2}(\Omega)$. As u_n is a weak solution of (4.2.17) and as $u_n \in W^{2,2}(\Omega)$, taking as test function in the weak formulation ψu_{nx_1} we have that

$$\int_\Omega F_n(|\nabla u_n|)\nabla u_n \nabla(\psi u_{nx_1})\,dx = -\int_\Omega \beta_\varepsilon(u) u_{nx_1}\psi\,dx.$$

As $(G_n(|\nabla u_n|))_{x_1} = g_n(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|}(\nabla u_n)_{x_1} = F(|\nabla u_n|)\nabla u_n(\nabla u_n)_{x_1}$ we have that

$$-\int_\Omega G_n(|\nabla u_n|)\psi_{x_1}\,dx + \int_\Omega F_n(|\nabla u_n|)\nabla u_n \nabla\psi u_{nx_1}\,dx = -\int_\Omega \beta_\varepsilon(u) u_{nx_1}\psi\,dx,$$

Passing to the limit as $n \to 0$ and integrating by parts, we get,

$$-\int_\Omega G(|\nabla u|)\psi_{x_1}\,dx + \int_\Omega F(|\nabla u|)\nabla u \nabla\psi\, u_{x_1}\,dx = \int_\Omega B_\varepsilon(u)\psi_{x_1}\,dx.$$

□

Proposition 4.2.18. *Let $x_0 \in \Omega$ and let u^{ε_k} be solutions to*

$$\mathcal{L}u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k})$$

in Ω. If u^{ε_k} converge to $\alpha(x-x_0)_1^+$ uniformly in compact subsets of Ω, with $\varepsilon_k \to 0$ as $k \to \infty$ and $\alpha \in \mathbb{R}$, then

$$\alpha = 0 \quad or \quad \alpha = \Phi^{-1}(M).$$

Where $\Phi(t) = g(t)t - G(t)$.

Proof. Assume that $x_0 = 0$. Since $u^{\varepsilon_k} \geq 0$, we have that $\alpha \geq 0$. If $\alpha = 0$ there is nothing to prove. So let us assume that $\alpha > 0$. Let $\psi \in C_0^\infty(\Omega)$. By Lemma 4.2.14 we have,

$$-\int_\Omega G(|\nabla u^{\varepsilon_k}|)\psi_{x_1}\,dx + \int_\Omega F(|\nabla u^{\varepsilon_k}|)\nabla u^{\varepsilon_k}\nabla\psi\, u^{\varepsilon_k}_{x_1}\,dx = \int_\Omega B_{\varepsilon_k}(u^{\varepsilon_k})\psi_{x_1}. \tag{4.2.19}$$

Since $0 \leq B_{\varepsilon_k}(s) \leq M$, there exists $M(x) \in L^\infty(\Omega)$, $0 \leq M(x) \leq M$, such that $B_{\varepsilon_k} \to M(x)$ $*$- weakly in $L^\infty(\Omega)$. If $y \in \Omega \cap \{x_1 > 0\}$, then $u^{\varepsilon_k} \geq \frac{\alpha y_1}{2}$ in a neighborhood of y for k large. Thus, $u^{\varepsilon_k} \geq \varepsilon_k$ and we have

$$B_{\varepsilon_k}(u^{\varepsilon_k})(x) = \int_0^{u^{\varepsilon_k}/\varepsilon_k}\beta(s)\,ds = M.$$

Using Proposition 4.2.1 we have that

$$\nabla B_{\varepsilon_k}(u_{\varepsilon_k}) = \beta_{\varepsilon_k}(u_{\varepsilon_k})\nabla u^{\varepsilon_k} \to 0 \text{ in } L^1_{loc}(\Omega \cap \{x_1 < 0\}).$$

Hence, $M(x) = \overline{M} \in [0, M]$ in $\Omega \cap \{x_1 < 0\}$. Passing to the limit in (4.2.19), using the strong convergence result in Proposition 4.2.1 we have

$$-\int_{\{x_1>0\}} G(\alpha)\psi_{x_1}\,dx + \int_{\{x_1>0\}} F(\alpha)\alpha^2\psi_{x_1}\,dx = M\int_{\{x_1>0\}} \psi_{x_1} + \overline{M}\int_{\{x_1<0\}} \psi_{x_1}.$$

Then,

$$(-G(\alpha) + g(\alpha)\alpha)\int_{\{x_1>0\}} \psi_{x_1}\,dx = M\int_{\{x_1>0\}} \psi_{x_1}\,dx + \overline{M}\int_{\{x_1<0\}} \psi_{x_1}\,dx.$$

And, integrating by parts, we obtain

$$(-G(\alpha) + g(\alpha)\alpha)\int_{\{x_1=0\}} \psi\,dx' = M\int_{\{x_1=0\}} \psi\,dx' - \overline{M}\int_{\{x_1=0\}} \psi_{x_1}\,dx'.$$

Thus, $(-G(\alpha) + g(\alpha)\alpha) = M - \overline{M}$. Let as see that $\alpha = \Phi^{-1}(M)$, we want to show that $\overline{M} = 0$.

Let $K \subset\subset \{x_1 < 0\}$. Then for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that,

$$\begin{aligned}|K \cap \{\varepsilon < B_{\varepsilon_j}(u^{\varepsilon_j}) < M - \varepsilon\}| &\le |K \cap \{\delta < u^{\varepsilon_j}/\varepsilon_j < 1 - \delta\}| \\ &\le |K \cap \{\beta_{\varepsilon_j}(u^{\varepsilon_j}) \ge a/\varepsilon_j\}| \to 0\end{aligned}$$

as $j \to 0$, where $a = \inf_{[\delta,1-\delta]}\beta > 0$, and we are using that $\beta(u^{\varepsilon_j}) \to 0$ in $L^1(K)$ by Proposition 4.2.1. And as $B(u^{\varepsilon_j}) \to \overline{M}$ in $L^1(K)$ we conclude that,

$$|K \cap \{\varepsilon < \overline{M} < M - \varepsilon\}| = 0$$

for every $\varepsilon > 0$, then $\overline{M} = 0$ or $\overline{M} = M$, since $\alpha > 0$ we must have $\overline{M} = 0$. □

PROPOSITION 4.2.20. *Let $x_0 \in \Omega$, and let u^{ε_k} be a solution to $\mathcal{L}u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k})$ in Ω, where g' satisfies* (4.2.24). *If u^{ε_k} converges to $\alpha(x - x_0)_1^+ + \gamma(x - x_0)_1^-$ uniformly in compact subsets of Ω, with $\alpha, \gamma > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$, then*

$$\alpha = \gamma \le \Phi^{-1}(M).$$

PROOF. We can assume that $x_0 = 0$. Since u^{ε_k} satisfies (4.2.15) and $\alpha, \gamma > 0$ we have, if $y \in \Omega \cap \{x_1 > 0\}$, then $u^{\varepsilon_k} \ge \frac{\alpha y_1}{2}$ in a neighborhood of y for k large and then, $u^{\varepsilon_k} \ge \varepsilon_k$. So that

$$B_{\varepsilon_k}(u^{\varepsilon_k})(x) = \int_0^{u^{\varepsilon_k}/\varepsilon_k} \beta(s)\,ds = M.$$

The same happens if $y_1 < 0$. Then $B_{\varepsilon_j}(u^{\varepsilon_j}) \to M$ in $L^1_{loc}(\Omega)$. Passing to the limit in (4.2.15) we deduce that, for ant $\Psi \in C_0^\infty(\Omega)$ we have,

$$-\int_{\{x_1>0\}} \Phi(\alpha)\psi_{x_1}\,dx - \int_{\{x_1<0\}} \Phi(\gamma)\psi_{x_1}\,dx = \int_\Omega M\psi_{x_1},$$

Integrating by parts we obtain,

$$\int_{\{x_1=0\}} \Phi(\alpha)\psi\,dx' - \int_{\{x_1=0\}} \Phi(\gamma)\psi\,dx' = 0$$

and then $\alpha = \gamma$.

Now assume that $\alpha > \Phi^{-1}(M)$. We will prove that this is a contradiction.

Step 1. Let $\mathcal{R}_2 = \{x = (x_1, x') \in \mathbb{R}^N : |x_1| < 2, |x'| < 2\}$. We can assume that $\mathcal{R}_2 \subset \Omega$.

We will construct a family $\{v^{\varepsilon_j}\}$ of solutions of (P_{ε_j}) in $\mathcal{R}_2$ satisfying $v^{\varepsilon_j}(x_1, x') = v^{\varepsilon_j}(-x_1, x')$ in $\mathcal{R}_2$, and such that $v^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{R}_2$, where $u(x) = \alpha|x_1|$. To this end, we take $b_{\varepsilon_j} = \sup_{\mathcal{R}_2} |u^{\varepsilon_j} - u|$ and v^{ε_j} the minimal solution (the minimum of all supersolutions) to (P_ε) in $\mathcal{R}_2$ with boundary values $v^{\varepsilon_j} = u - b_{\varepsilon_j}$ on $\partial\mathcal{R}_2$. By Proposition 4.2.1, there exists $v \in Lip_{loc}(\mathcal{R}_2)$ such that, for a subsequence that we still denote v^{ε_j}, $v^{\varepsilon_j} \to v$ uniformly on compact subsets of $\mathcal{R}_2$. From the minimality of v^{ε_j} we have that $u \geq v$.

To prove the other inequality, let $w \in C^{1,\beta}(\mathbb{R})$, satisfying

$$(F(|w'|)w')' = \beta(w) \quad \text{in } \mathbb{R}, \quad w(0) = 1, \quad w'(0) = \alpha.$$

Observe that, when $w'(s) > 0$ the equation is locally uniformly elliptic and then $w \in C^2(\mathbb{R})$. Suppose that there exists an $s \in \mathbb{R}$ such that $w'(s) = 0$, take s_1 the first positive time such that this happens. Then, if we multiply by w' in the equation and we integrate, we obtain

$$-\Phi(\alpha) = \Phi(w'(s_1)) - \Phi(\alpha) = \int_{w'(0)}^{w'(s_1)} g'(s)s\,ds = B(w(s_1)) - M \geq -M$$

which means that $\Phi(\alpha) \leq M$, a contradiction. Analogously if $w'(s) = 0$ somewhere in $\{s < 0\}$. Then, $w' > 0$ everywhere. By the same calculation as before, we obtain that for any $s \in \mathbb{R}$ we have,

$$\Phi(w'(s)) - \Phi(\alpha) = B(w(s)) - M \leq 0,$$

and

$$\Phi(w'(s)) = B(w(s)) + \Phi(\alpha) - M \geq \Phi(\alpha) - M = \Phi(\bar{\alpha}), \tag{4.2.21}$$

for some $\alpha > \bar{\alpha} > 0$. Then, $\bar{\alpha} \leq w'(s) \leq \alpha$. As w is increasing, we obtain that $w'(s) = \alpha$ for $s \geq 0$, and there exists $\bar{s} < 0$ such that $w(\bar{s}) = 0$, which means by (4.2.21) that $w'(\bar{s}) = \bar{\alpha}$, and then $w'(s) = \bar{\alpha}$ for all $s \leq \bar{s}$, therefore

$$w(s) = \begin{cases} 1 + \alpha s & s > 0 \\ \bar{\alpha}(s - \bar{s}) & s \leq \bar{s}. \end{cases}$$

Let $w^{\varepsilon_j}(x_1) = \varepsilon_j w\Big(\frac{x_1}{\varepsilon_j} - \frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} + \bar{s}\Big)$, then

$$w^{\varepsilon_j}(0) = \varepsilon_j w\Big(-\frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} + \bar{s}\Big) = \varepsilon_j \bar{\alpha}\Big(\bar{s} - \frac{b_{\varepsilon_j}}{\bar{\alpha}\varepsilon_j} - \bar{s}\Big) = -b_{\varepsilon_j}$$

and $w^{\varepsilon_j}{}'(s) \leq \alpha$. Therefore, $w^{\varepsilon_j} \leq u - b_{\varepsilon_j}$ in $\mathbb{R}$, so that, $w^{\varepsilon_j} \leq v^{\varepsilon_j}$ on $\partial\mathcal{R}_2$ and then by the comparison principle below (Lemma 4.2.25) we have that $w^{\varepsilon_j} \leq v^{\varepsilon_j}$ in $\mathcal{R}_2$. Take $x_1 > 0$, then for j large $x_1 - \frac{b_{\varepsilon_j}}{\bar{\alpha}} > \frac{x_1}{2}$ then $\frac{1}{\varepsilon_j}(x_1 - \frac{b_{\varepsilon_j}}{\bar{\alpha}}) + \bar{s} > \frac{x_1}{2\varepsilon_j} + \bar{s} > 0$, and therefore $w^{\varepsilon_j} = \varepsilon_j + \alpha x_1 - \frac{\alpha}{\bar{\alpha}} b_{\varepsilon_j} + \alpha\varepsilon_j \bar{s}$, then in any compact set of $\{x_1 > 0\}$, $w^{\varepsilon_j} \to u$ uniformly. Passing to the limit, we get that $u \leq v$ in $\mathcal{R}_2 \cap \{x_1 > 0\}$. Observe that, by the uniqueness of the minimal solution, we have that $v^{\varepsilon_j}(x_1, x') = v^{\varepsilon_j}(-x_1, x')$. Thus, we obtain that $u \le v$ in $\mathcal{R}_2$.

Step 2. Let $\mathcal{R}^+ = \{x : 0 < x_1 < 1, |x'| < 1\}$. Define,

$$F_j = \int_{\partial\mathcal{R}^+\cap\{x_1=1\}} F(|\nabla v^{\varepsilon_j}|)(v^{\varepsilon_j}_{x_1})^2\,dx' + \int_{\partial\mathcal{R}^+\cap\{|x'|=1\}} F(|\nabla v^{\varepsilon_j}|)v^{\varepsilon_j}_n v^{\varepsilon_j}_{x_1}\,dS,$$

where ${v_n}^{\varepsilon_j}$ is the exterior normal of v^{ε_j} on $\partial\mathcal{R}^+ \cap \{|x'| = 1\}$. We first want to prove that,

$$F_j \le \int_{\partial\mathcal{R}^+\cap\{x_1=1\}} \Big(G(|\nabla v^{\varepsilon_j}|) + B_{\varepsilon_j}(v^{\varepsilon_j})\Big)\,dx'.$$

To prove this, we proceed as in the proof of Lemma 4.2.14. This is, we can suppose that $F(s) \ge c$, by using an approximation argument. Therefore, we can suppose that $v^{\varepsilon_j} \in W^{2,2}(\mathcal{R}_2)$. Using the weak formulation of (P_ε) in $\mathcal{R}^+$ we have,

$$\begin{aligned} E_j &:= \int\int_{\mathcal{R}^+} \frac{\partial}{\partial x_1}\Big(G(|\nabla v^{\varepsilon_j}|)\Big)\,dx = \int\int_{\mathcal{R}^+} F(|\nabla v^{\varepsilon_j}|)\nabla v^{\varepsilon_j}\nabla v^{\varepsilon_j}_{x_1}\,dx \\ &= \int\int_{\mathcal{R}^+} \operatorname{div}(F(|\nabla v^{\varepsilon_j}|)\nabla v^{\varepsilon_j} v^{\varepsilon_j}_{x_1})\,dx - \int\int_{\mathcal{R}^+} \beta_{\varepsilon_j}(v^{\varepsilon_j})v^{\varepsilon_j}_{x_1} =: H_j - G_j. \end{aligned}$$

Using the divergence theorem and the fact that $v^{\varepsilon_j}_{x_1}(0, x') = 0$ (by the symmetry in the x_1 variable) we find that, $H_j = F_j$.

From the convergence of $v^{\varepsilon_j} \to u = \alpha|x_1|$ in $\mathcal{R}_2$ and Proposition 4.2.1 we have that

$$\nabla v^{\varepsilon_j}_{x_1} \to \alpha e_1 \quad \text{a.e in } \mathcal{R}_2^+ = \mathcal{R}_2 \cap \{x_1 > 0\}.$$

Since $|\nabla v^{\varepsilon_j}|$ are uniformly bounded, from the dominate convergence theorem we deduce that,

$$\lim_{j\to\infty} F_j = \int_{\partial\mathcal{R}^+\cap\{x_1=1\}} g(\alpha)\alpha\,dx' \tag{4.2.22}$$

and

$$\begin{aligned} F_j = E_j + G_j &= \int\int_{\mathcal{R}^+} \frac{\partial}{\partial x_1}\Big(G(|\nabla v^{\varepsilon_j}| + B_{\varepsilon_j}(v^{\varepsilon_j}))\Big)\,dx \\ &= \int_{\partial\mathcal{R}^+\cap\{x_1=0\}} -\Big(G(|\nabla v^{\varepsilon_j}| + B_{\varepsilon_j}(v^{\varepsilon_j}))\Big)\,dx' \\ &\quad + \int_{\partial\mathcal{R}^+\cap\{x_1=1\}} \Big(G(|\nabla v^{\varepsilon_j}| + B_{\varepsilon_j}(v^{\varepsilon_j}))\Big)\,dx' \\ &\le \int_{\partial\mathcal{R}^+\cap\{x_1=1\}} \Big(G(|\nabla v^{\varepsilon_j}| + B_{\varepsilon_j}(v^{\varepsilon_j}))\Big)\,dx'. \end{aligned}$$

Using again that $v^{\varepsilon_j} \to u = \alpha|x_1|$ uniformly on compact subsets of $\mathcal{R}_2$, we have that $|\nabla v^{\varepsilon_j}| \to \alpha$ uniformly and $B_{\varepsilon_j}(v^{\varepsilon_j}) = M$ on $\partial\mathcal{R}^+ \cap \{x_1 = 1\}$, and therefore

$$\limsup_{j\to\infty} F_j \le \int_{\partial\mathcal{R}^+\cap\{x_1=1\}} (G(\alpha) + M)\, dx'. \tag{4.2.23}$$

Therefore, by (4.2.22) and (4.2.23) we obtain $\Phi(\alpha) \le M$ which is a contradiction.

□

Now we prove the comparison principle needed in the proof of the lemma above. This is the step where we need an additional hypothesis: There exist $-1 < \alpha_1 \le \alpha_2$ such that for any $s, t \ge 0$ we have,

$$g'(ts) \ge \min\{s^{\alpha_1}, s^{\alpha_2}\} g'(t) \tag{4.2.24}$$

LEMMA 4.2.25. *Let $w^\varepsilon(x_1)$ be a strictly increasing solution of $\mathcal{L}(w^\varepsilon) = \beta_\varepsilon(w^\varepsilon)$ on $\mathbb{R}$ such that $\lim_{s\to-\infty} w^\varepsilon(s) < 0$ and $v^\varepsilon(x) \ge 0$ a solution of $\mathcal{L}v^\varepsilon = \beta_\varepsilon(v^\varepsilon)$ in $\mathcal{R} = \{x = (x_1, x') : a < x_1 < b, |x'| < r\}$, continuous up to $\partial\mathcal{R}$. Then, if g' satisfies condition (4.2.24) the following comparison principle holds: if $v^\varepsilon(x) \ge w^\varepsilon(x_1)$ for all $x \in \partial\mathcal{R}$ then $v^\varepsilon(x) \ge w^\varepsilon(x_1)$ for all $x \in \mathcal{R}$.*

PROOF. Since $\lim_{s\to-\infty} w^\varepsilon(s) < 0$ and $v^\varepsilon(x) \ge 0$, we can find τ such that,

$$w^\varepsilon(x_1 - \tau) < v^\varepsilon(x) \quad \text{on } \bar{\mathcal{R}}.$$

For $\eta > 0$ sufficiently small define,

$$w^{\varepsilon,\eta}(x_1) := w^\varepsilon(\varphi_\eta(x_1 - c_\eta)),$$

where $\varphi_\eta(s) = s + \eta s^2$ and $c_\eta > 0$ is the smallest constant such that $\varphi_\eta(s - c_\eta) \le s$ on $[-2\tau, 2\tau]$ (observe that $c_\eta \to 0$ when $\eta \to 0$). If $c_\eta - \frac{1}{\eta} \le -2\tau$ then $\varphi_\eta(s - c_\eta) \le 0$ for $s \le c_\eta$. Observe that, in $[-2\tau, 2\tau]$ $w^{\varepsilon,\eta} \le w^\varepsilon$ and as $\eta \to 0$ $w^{\varepsilon,\eta} \to w^\varepsilon$ uniformly.

If we call $\widetilde{\varphi}_\eta(s) = \varphi_\eta(s - c_\eta)$ computing we have,

$$\mathcal{L}(w^{\varepsilon,\eta}) = g'(w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\widetilde{\varphi}'_\eta) w^{\varepsilon\prime\prime}(\widetilde{\varphi}_\eta)(\widetilde{\varphi}'_\eta)^2 + g'(w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\widetilde{\varphi}'_\eta) w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)\widetilde{\varphi}''_\eta.$$

If we define

$$\gamma_1(s) = \begin{cases} \alpha_1 & \text{if } \widetilde{\varphi}'_\eta(s) > 1 \\ \alpha_2 & \text{if } \widetilde{\varphi}'_\eta(s) \le 1 \end{cases} \qquad \gamma_2(s) = \begin{cases} \alpha_1 & \text{if } w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)(s) > 1 \\ \alpha_2 & \text{if } w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)(s) \le 1. \end{cases}$$

Using condition (4.2.24), that $\mathcal{L}\varphi_\eta > 0$ and $w^{\varepsilon\prime} > 0$ we have,

$$\begin{aligned} \mathcal{L}(w^{\varepsilon,\eta}) &\ge g'(w^{\varepsilon\prime}(\widetilde{\varphi}_\eta)) w^{\varepsilon\prime\prime}(\widetilde{\varphi}_\eta)(\widetilde{\varphi}'_\eta)^{\gamma_1+2} + g'(\widetilde{\varphi}'_\eta)\widetilde{\varphi}''_\eta (w^{\varepsilon\prime}(\widetilde{\varphi}_\eta))^{\gamma_2+1} \\ &= \mathcal{L}w^\varepsilon(\widetilde{\varphi}_\eta)(\widetilde{\varphi}'_\eta)^{\gamma_1+2} + \mathcal{L}\widetilde{\varphi}_\eta (w^{\varepsilon\prime}(\widetilde{\varphi}_\eta))^{\gamma_2+1} > \mathcal{L}w^\varepsilon(\widetilde{\varphi}_\eta)(\widetilde{\varphi}'_\eta)^{\gamma_1+2} \\ &= \beta_\varepsilon(w^{\varepsilon,\eta})(\widetilde{\varphi}'_\eta)^{\gamma_1+2}. \end{aligned}$$

Since, $\beta_\varepsilon(w^{\varepsilon,\eta}) = 0$ when $x_1 \le c_\eta$ and $\widetilde{\varphi}'_\eta \ge 1$ when $x_1 \ge c_\eta$, we have that $\mathcal{L}(w^{\varepsilon,\eta}) > \beta_\varepsilon(w^{\varepsilon,\eta})$. Summarizing,

$$\mathcal{L}w^{\varepsilon,\eta} > \beta_\varepsilon(w^{\varepsilon,\eta}), \quad w^{\varepsilon,\eta} \to w^\varepsilon \text{ as } \eta \to 0 \quad \text{and } w^{\varepsilon,\eta} \le w^\varepsilon.$$

Let now $\tau^* \leq 0$ the smallest constant such that

$$w^{\varepsilon,\eta}(x_1 - \tau^*) \leq v^{\varepsilon}(x) \quad \text{on } \overline{\mathcal{R}}.$$

We want to prove that $\tau^* = 0$. By the minimality of τ^*, there exists a point $x^* \in \overline{\mathcal{R}}$ such that $w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^{\varepsilon}(x^*)$. If $\tau^* > 0$, then $w^{\varepsilon,\eta}(x_1 - \tau^*) < w^{\varepsilon,\eta}(x_1) \leq w^{\varepsilon}(x_1) \leq v^{\varepsilon}(x)$ on $\partial\mathcal{R}$, and hence, x^* is an interior point of $\mathcal{R}$. At this point observe that the gradient of $w^{\varepsilon,\eta}(x_1 - \tau^*)$ is non-degenerate. Since $\mathcal{L}w^{\varepsilon,\eta}(x_1^* - \tau^*) > \beta_{\varepsilon}(w^{\varepsilon,\eta}(x^* - \tau^*)) = \beta_{\varepsilon}(v^{\varepsilon}(x^*)) = \mathcal{L}v^{\varepsilon}(x^*)$, we have that this happens in an open set. We also have $w^{\varepsilon,\eta}(x_1 - \tau^*) \leq v^{\varepsilon}(x)$ in $\mathcal{R}$ and $w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^{\varepsilon}(x^*)$.

As $w^{\varepsilon,\eta}(x_1^* - \tau^*) \leq v^{\varepsilon}(x^*)$ in $\mathcal{R}$ and $w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^{\varepsilon}(x^*)$ we have that $\nabla w^{\varepsilon,\eta}(x_1^* - \tau^*) = \nabla v^{\varepsilon}(x^*)$. Let,

$$Lv = a_{ij}(\nabla w^{\varepsilon,\eta}(x_1 - \tau^*))v_{x_i x_j},$$

where a_{ij} was define in (1.2.23). Since $\nabla w^{\varepsilon,\eta}$ is nondegenerate, L is uniformly elliptic near the point x^*, and since $\nabla w^{\varepsilon,\eta}(x_1^* - \tau^*) = \nabla v^{\varepsilon}(x^*)$, we have that $Lw^{\varepsilon,\eta}(x_1^* - \tau^*) > Lv^{\varepsilon}(x^*)$ and $Lv^{\varepsilon}(x^*) = \beta_{\varepsilon}(v^{\varepsilon})$ therefore v^{ε} is C^2 near that point and, we have, for some $\delta > 0$

$$\begin{cases} Lw^{\varepsilon,\eta}(x_1 - \tau^*) > Lv^{\varepsilon}(x) & \text{in } B_{\delta}(x^*) \\ w^{\varepsilon,\eta}(x_1^* - \tau^*) = v^{\varepsilon}(x^*) & \\ w^{\varepsilon,\eta}(x_1 - \tau^*) \leq v^{\varepsilon}(x) & \text{in } \overline{\mathcal{R}}, \end{cases}$$

but since L is uniformly elliptic near x^*, these three statements contradict the strong comparison principle. Therefore $\tau^* = 0$ and then $w^{\varepsilon,\eta} \leq v^{\varepsilon}$ on $\overline{\mathcal{R}}$. Letting $\eta \to 0$ we obtain the desired result.

□

3. Asymptotic behavior of limit solutions

Now we want to prove for g satisfying conditions 2.7.57 and 4.2.24 the asymptotic development of the limiting function u. We will obtain this result, under suitable assumptions on the function u. First we give the following,

DEFINITION 4.3.1. *Let v be a continuous nonnegative function in a domain $\Omega \in \mathcal{R}^N$. We say that v is non-degenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist c, $r_0 > 0$ such that*

$$\frac{1}{r^N}\int_{B_r(x_0)} v\,dx \geq cr \quad \text{for } 0 < r \leq r_0. \tag{4.3.2}$$

We say that u is uniformly non-degenerate on a set $A \subset \{u = 0\}$ if (4.3.2) holds for every $x_0 \in A$ with the same constants c and r_0.

We have the following,

THEOREM 4.3.3. *Suppose that g' satisfies condition (4.2.24). Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has an inward unit* *normal η in the measure theoretic sense at x_0, and suppose that u is non-degenerate at x_0. Under these assumptions, we have*

$$u(x) = \Phi^{-1}(M)\langle x - x_0, \eta\rangle^+ + o(|x - x_0|).$$

The proof of this theorem is based on the following result,

THEOREM 4.3.4. *Suppose that g' satisfies condition* (4.2.24). *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω as $\varepsilon_j \to 0$. Then,*

$$\limsup_{x\to x_0} |\nabla u(x)| \leq \Phi^{-1}(M).$$

PROOF. Let

$$\alpha := \limsup_{x\to x_0} |\nabla u(x)|.$$

Since $u \in Lip_{loc}(\Omega)$ $\alpha < \infty$. If, $\alpha = 0$ we are done. So, suppose that $\alpha > 0$. Then there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0, \qquad |\nabla u(z_k)| \to \alpha.$$

Let y_k be the nearest point from z_k to $\Omega \cap \partial\{u > 0\}$ and let $d_k = |z_k - y_k|$. Consider the blow up sequence u_{d_k} with respect to $B_{d_k}(y_k)$. Since u is Lipschitz, and $u_{d_k}(0) = 0$ for every k, there exists $u_0 \in Lip(\mathbb{R}^N)$, such that $u_{d_k} \to u_0$ uniformly in compact sets of $\mathbb{R}^N$. And we also have that u_0 is an $\mathcal{L}-$ solution in $\{u_0 > 0\}$ (see proof of (6) in Lemma 1.6.13).

Now, set $\bar{z}_k = (z_k - y_k)/d_k \in \partial B_1$. We may assume that $\bar{z}_k \to \bar{z} \in \partial B_1$. Take,

$$\nu_k := \frac{\nabla u_{d_k}(\bar{z}_k)}{|\nabla u_{d_k}(\bar{z}_k)|} = \frac{\nabla u(z_k)}{|\nabla u(z_k)|}.$$

Passing to a subsequence, we can assume, that $\nu_k \to e_1$. Observe that $B_{2/3}(\bar{z}) \subset B_1(\bar{z}_k)$ for k large, and therefore u_0 is an $\mathcal{L}-$ solution there. By interior C^α gradient estimates, we have $\nabla u_{d_k} \to \nabla u_0$ uniformly in $B_{1/3}(\bar{z})$, and therefore $|\nabla u(z_k)| \to \partial_{x_1} u_0(\bar{x})$. Thus, $\partial_{x_1} u_0(\bar{x}) = \alpha$.

Next, we claim that $|\nabla u_0| \leq \alpha$ in $\mathbb{R}^N$. In fact, let $R > 1$ and $\delta > 0$. Then, there exists, $\tau_0 > 0$ such that $|\nabla u(x)| \leq \alpha + \delta$ for any $x \in B_{\tau_0 R}(x_0)$. For $|z_k - x_0| < \tau_0 R/2$ and $d_k < \tau_0/2$ we have, $B_{d_k R}(z_k) \subset B_{\tau_0 R}(x_0)$ and therefore $|\nabla u_{d_k}(x)| \leq \alpha + \delta$ in B_R for k large. Passing to the limit, we obtain $|\nabla u_0| \leq \alpha + \delta$ in B_R, and since δ and R were arbitrary, the claim holds.

Lemma 1.2.28 says that if $w = \frac{\partial u_0}{\partial x_1}$ then

$$\partial_{x_j}(a_{ij}(\nabla u_0) D_j w) = 0 \quad \text{ in } B_1(\bar{x}).$$

Since this is a uniformly elliptic equation near $\bar{x}$, $w \leq \alpha$ in $B_1(\bar{x})$ and $w(\bar{x}) = \alpha$, the strong maximum principle implies that $w = \alpha$ in $B_r(\bar{x})$. Thus $|\nabla u_0| = \alpha$ in $B_r(\bar{x})$ for some $r > 0$. Now, by a continuation argument, we have $w = \alpha$ in $B_1(\bar{x})$. As $\partial_{x_1} u_0 = \alpha$ then, for some $y \in \mathbb{R}^N$ we have $u_0(x) = \alpha(x_1 - y_1)$ in $B_1(\bar{x})$. Since $\mathcal{L}u_0 = 0$ in $\{u_0 > 0\}$ by continuation we have, $u_0(x) = \alpha(x_1 - y_1)$ in $\{x_1 \geq y_1\}$.

As $u_0 \geq 0$ and $\mathcal{L}u_0 = 0$ in $\{u_0 > 0\}$ and $u_0 = 0$ in $x_1 = y_1$ we have by Lemma 1.5.10 that

$$u_0 = \gamma(y_1 - x_1) + o(|x - y|) \quad \text{ in } x_1 - y_1 < 0$$

for some $\gamma \geq 0$.

Now, define for $\lambda > 0$ $(u_0)_\lambda(x) = \frac{1}{\lambda}u_0(\lambda x + y)$. There exists a sequence $\lambda_n \to 0$ and $u_{00} \in Lip(\mathbb{R}^N)$ such that $(u_0)\lambda_n \to u_{00}$ uniformly in compact subsets of $\mathbb{R}^N$. We have $u_{00}(x) = \alpha x_1^+ + \gamma x_1^-$. By Lemma 4.2.13 there exists a sequence $\varepsilon_j' \to 0$ such that $u^{\varepsilon_j'}$ is a solution to $(P_{\varepsilon_j'})$ and $u^{\varepsilon_j'} \to u_0$ uniformly on compact subsets of $\mathbb{R}^N$. Applying a second time Lemma 4.2.13 we find a sequence $\varepsilon_j'' \to 0$ and a solution $u^{\varepsilon_j''}$ to $(P_{\varepsilon_j''})$ converging uniformly in compact subsets of $\mathbb{R}^N$ to u_{00}. Now we can apply Proposition 4.2.18 in the case that $\gamma = 0$ or Proposition 4.2.20 in the case that $\gamma > 0$, and we conclude that $\alpha \leq \Phi^{-1}(M)$.

□

PROOF OF THEOREM 4.3.3. Assume that $x_0 = 0$, and $\eta = e_1$. Take $u_\lambda(x) = \frac{1}{\lambda}u(\lambda x)$. Let $\rho > 0$ such that $B_\rho \subset\subset \Omega$, since $u_\lambda \in Lip(B_{\rho/\lambda})$ uniformly in λ, $u_\lambda(0) = 0$, there exists $\lambda_j \to 0$ and $U \in Lip(\mathbb{R}^N)$ such that $u_{\lambda_j} \to U$ uniformly on compact subsets of $\mathbb{R}^N$. From Proposition 4.2.1 and Lemma 4.2.13, $\mathcal{L}u_\lambda = 0$ in $\{u_\lambda > 0\}$. Using that we are at a point where we have an inward normal in the measure theoretic sense, we have, for fixed k,

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap B_k| \to 0 \quad \text{ as } \lambda \to 0.$$

Hence, U is non negative in $\{x_1 > 0\}$, $\mathcal{L}U = 0$ in $\{U > 0\}$ and U vanishes in $\{x_1 \leq 0\}$. Then, by Lemma 1.5.10 we have that, there exists $\alpha \geq 0$ such that,

$$U(x) = \alpha x_1^+ + o(|x|) \quad \text{ in } \{x_1 > 0\}.$$

By Lemma 4.2.13 we can find a sequence $\varepsilon_j' \to 0$ and solutions $u^{\varepsilon_j'}$ to $(P_{\varepsilon_j'})$ such that $u_{\varepsilon_j'} \to U$ uniformly on compact subsets of $\mathbb{R}^N$ as $j \to \infty$. Define $U_\lambda(x) = \frac{1}{\lambda}U(\lambda x)$, then $U_\lambda \to \alpha x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$. Applying again Lemma 4.2.13 we find a second sequence $\sigma_j \to 0$ and u^{σ_j} solution to (P_{σ_j}) such that $u^{\sigma_j} \to \alpha x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$ and,

$$\nabla u^{\sigma_j} \to \alpha \chi_{\{x_1 > 0\}} e_1 \quad \text{ in } L^{g_0+1}_{loc}(\mathbb{R}^N).$$

Now we proceed as in the proof of Proposition 4.2.18. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ and choose $u^{\sigma_j}_{x_1}\psi$ as test function in the weak formulation of $\mathcal{L}u^{\sigma_j} = \beta_{\sigma_j}(u^{\sigma_j})$. Then,

$$B_{\sigma_j}(u^{\sigma_j}) \to M\chi_{\{x_1 > 0\}} + \overline{M}\chi_{\{x_1 < 0\}} \quad * \text{ weakly in } L^\infty$$

with $\overline{M} = 0$ or $\overline{M} = M$. Moreover $\Phi(\alpha) = M - \overline{M}$.

By the non degeneracy assumption on u we have,

$$\frac{1}{r^N}\int_{B_r} u_{\lambda_j}\, dx \geq cr$$

and then,

$$\frac{1}{r^N}\int_{B_r} U_{\lambda_j}\,dx \geq cr.$$

Therefore $\alpha > 0$. So that we have that $\overline{M} = 0$. Then, $\alpha = \Phi^{-1}(M)$.

We have shown that,

$$U(x) = \begin{cases} \Phi^{-1}(M)x_1 + o(|x|) & x_1 > 0 \\ 0 & x_1 \leq 0. \end{cases}$$

By Theorem 4.3.4, $|\nabla U| \leq \Phi^{-1}(M)$ in $\mathbb{R}^N$. As $U = 0$ on $\{x_1 = 0\}$ we have, $U \leq \Phi^{-1}(M)x_1$ in $\{x_1 > 0\}$.

Since $|\nabla U(0)| = \Phi^{-1}(M) > 0$, near zero U satisfies a linear uniformly elliptic equation (defined in (1.2.26)) and the same equation is satisfied by $w = U - \Phi^{-1}(M)x_1$ in $\{x_1 > 0\} \cap B_r(0)$ for some $r > 0$. We also have $w \leq 0$ so that by Hopf's boundary principle we have that $w = 0$ in $\{x_1 > 0\}$. And the proof is completed. □

THEOREM 4.3.5. *Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω as $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and suppose that u is non-degenerate at x_0. Assume there is a ball B contained in $\{u = 0\}$ touching x_0 then,*

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} \frac{u(x)}{dist(x, B)} = \Phi^{-1}(M). \tag{4.3.6}$$

PROOF. Let l be the finite limit in the left of (4.3.6), and $y_k \to x_0$ with $u(y_k) > 0$ and

$$\frac{u(y_k)}{d_k} \to l, \quad d_k = \operatorname{dist}(y_k, B).$$

Consider the blow up sequence u_k with respect to $B_{d_k}(x_k)$, where $x_k \in \partial B$ are points with $|x_k - y_k| = d_k$, and choose a subsequence with blow up limit u_0, such that

$$e := \lim_{k\to\infty} \frac{x_k - y_k}{d_k}$$

exists. Then, by construction, $u_0(-e) = l$, $u_0(x) \leq -l\langle x, e\rangle$ for $\langle x, e\rangle \leq 0$, $u_0(x) = 0$ for $\langle x, e\rangle \geq 0$. By the non-degeneracy assumption, we have that $l > 0$. Both, u_0 and $l\langle x \cdot e\rangle^+$ are $\mathcal{L}$ solutions in $\{u_0 > 0\}$, and from the maximum principle we have that (since $l > 0$) they must coincide in a neighborhood of $-e$, by continuation we have that $u_0 = l(x \cdot e)^+$. Then, we have by, Proposition 4.2.18, that $l = \Phi^{-1}(M)$.

□

4. Regularity of the free boundary

Now, we can prove a regularity result for the free boundary of limits of solution of problem (P_ε), THEOREM 4.4.7. *Suppose that g satisfies* (0.0.2) *and moreover g' satisfies condition* (4.2.24)*. Let u^{ε_j} be a solution to (P_{ε_j}) in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly in compact subsets of Ω as $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$, be such that there is an inward unit normal η in the measure theoretic sense at x_0. Suppose that u is uniformly non-degenerate at the free boundary in a neighborhood of x_0 (see Definition 4.3.1). Then, there exists $r > 0$ so that $B_r(x_0) \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface.*

PROOF. By Corollary 4.1.8, Theorem 4.3.3, Theorem 4.3.5 and the non-degeneracy assumption we have that u is a weak solution in the sense of Definition 2.6.2. Therefore Theorem 2.7.56 applies, and the result follows.

□

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