Bicliques, Cliques, Neighborhoods y la Propiedad de Helly

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Cidade maravilhosa,
meu ***Rio de Janeiro***

Agradecimientos

Tenía 17 años, me gustaba ir al cine, bailar, escuchar música, y,...matemática. Tambén me gustaba el arte y la creatividad, y por eso, me anoté para estudiar publicidad con la ilusión adolescente de convertirme en una exitosa publicista. Por otro lado, para satisfacer mi (oculto) lado nerd, me anoté en la carrera de matemática, como quien se anota en un curso de guitarra. ”Como hobbie” solía decir ...

El primer año de carrera, dos semanas antes de rendir mi primer parcial, ya como una estudiante universitaria, me plantee dejar la carrera. ”Mi cabeza no da para esto”, pensaba ...

Mientras viajaba en el 160, un entonces docente, me preguntó por qué no me anotaba en el concurso de ayudante de segunda, que se estaba llamando en ese entonces. ”Yo, ayudante?!!!”, me preguntaba ...

Se habían terminado las materias a cursar, y los finales pendientes y la tesis de licenciatura, constituían un lejano y oscuro horizonte, el que de ninguna manera pensaba alcanzar. ”Largo la carrera, para qué quiero el título”, repetía muchas veces...

Se abrió un concurso exclusiva en computación, me anoté. ”Siempre supimos que este era mi destino”, solíamos decir entre mis amigos de computación.

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Resumen

Un grafo es biclique-Helly cuando el conjunto de bicliques verifica la propiedad de Helly. En esta tesis caracterizamos a la familia de grafos biclique-Helly, y presentamos dos algoritmos polinomiales para el problema de reconocimiento. Por otro lado, relacionamos las clases de grafos biclique-Helly, clique-Helly, discos-Helly y vecindad-Helly.

Es natural preguntarse si la propiedad de Helly es hereditaria para subgrafos inducidos. En este caso, nos referimos a los grafos clique-Helly hereditarios, discos-Helly hereditarios, biclique-Helly hereditarios y vecindad-Helly hereditarios, respectivamente. Las primeras dos clases fueron estudiadas en la literatura. En esta tesis, estudiamos las dos clases restantes. Presentamos caracterizaciones que se basan en subgrafos prohibidos. Ya que esta familia de subgrafos prohibidos tiene tamaño fijo, las caracterizaciones mencionadas dan lugar a algoritmos polinomiales de reconocimiento de las clases.

Dado un grafo G, la matriz biclique de G es una matriz con valores en el conjunto $\{0, 1, -1\}$, donde las columnas y las filas representan los vértices y las bicliques de G, respectivamente, y los valores 1,-1 en una fila correponden a dos vértices adyacentes de una biclique. Es esta tesis, describimos una caracterización de las matrices bicliques, en forma similar a la empleada en la caracterización de las matrices biclique. En esta caracterización, empleamos el concepto de hypergrafos bipartitos-conformal. Por otra parte, consideramos el caso particular de matrices bicliques de grafos bipartidos.

Dada una familia de subconjuntos $\mathcal{F}$, el grafo de intersección de $\mathcal{F}$ es un grafo cuyos vértices se corresponden con los conjuntos de $\mathcal{F}$, donde dos vértices son adyacentes si los correspondientes conjuntos se intersecan. En esta tesis definimos el grafo biclique de G, $KB(G)$, como el grafo de intersección de la familia de bicliques de un grafo. Un grafo G es grafo biclique si $KB(H) = G$, para algún grafo H. En esta tesis presentamos una caracterización de los grafos biclique.

Dado G, definimos $N_c(G)$ como el grafo de intersección de las vecindades cerradas de G. En esta tesis estudiamos el grafo $N_c(G)$ en relación con la propiedad de Helly.

Los grafos perfectos son importantes desde el punto de vista algorítmico. En este trabajo estudiamos los grafos cuyo grafo biclique es perfecto, es decir, grafos KB-perfectos. Damos una caracterización de los grafos KB-perfectos tales que no continenen al grafo P_5 como subgrafo inducido.

Abstract

A graph is biclique-Helly when its family of (maximal) bicliques is a Helly family. We describe characterizations for biclique-Helly graphs, leading to polynomial time recognition algorithms. In addition, we relate biclique-Helly graphs to the classes of clique-Helly, disk-Helly and neighborhood-Helly graphs. A natural question is to determine for which graphs the corresponding Helly property holds for every induced subgraph. This leads to the classes of hereditary clique-Helly, hereditary disk-Helly, hereditary biclique-Helly and hereditary neighborhood-Helly graphs, respectively. The first two of them have already been characterized. In this thesis, we describe characterizations for the remaining ones, by families of forbidden subgraphs. The forbidden subgraphs are all of fixed size, implying polynomial time recognition for these classes.

Given a graph G, the biclique matrix of G is a $\{0, 1, -1\}$ matrix having one row for each biclique and one column for each vertex of G, and such that a pair of 1,-1 entries in a same row corresponds exactly to adjacent vertices in the corresponding biclique. We describe a characterization for biclique matrices, in similar terms as those employed in the characterization of clique matrices. In the characterizations, we employ the concept of bipartite-conformal hypergraphs. The special case of biclique matrices of bipartite graphs is also considered.

Given a family of subsets of some set $\mathcal{F}$, the *intersection graph* of $\mathcal{F}$ is a graph having one vertex for each set of $\mathcal{F}$, and two vertices are adjacent whenever their corresponding sets intersect. sIn this thesis we define the biclique graph of G, $KB(G)$, as the intersection graph of the family of all bicliques of G. A graph G is a biclique graph if there exists a graph H such that $KB(H) = G$. We present a characterization for biclique graph. The special case of biclique graphs of bipartite graphs is also considered.

The closed neighborhood graph is the intersection graph of the closed neighborhoods of G. We study closed neighborhood graphs in relation to the Helly property.

Perfect graphs are very interesting from an algorithmic point of view. We study the graphs for which the biclique graph is perfect, the KB-perfect graphs. We give a characterization of the of KB-perfect graphs with no induced P_5.

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Introduction

The Helly property plays a very important role in the study of families of subsets of a set. A family of subsets is Helly if every pairwise intersecting subfamily has a common element.

In the scope of graph theory, the study of the Helly property has motivated the introduction of some classes of graphs, as clique-Helly graphs [12, 23, 32, 36, 49], disk-Helly graphs [5, 6], and neighborhood-Helly graphs [10] (i.e., the families of cliques, disks and neighborhoods of a graphs are Helly, respectively.)

Berge ([10]) described a polynomial algorithm to verify if a given family has is Helly. However that the problem of recognizing clique-Helly graphs, can not be solved by using the algorithm proposed by Berge, since the number of cliques of a graph can be exponential. In [52] there is a characterization of clique-Helly graphs that leads to a polynomial time recognition algorithm.

Disk-Helly graphs were studied by Bandelt and Pesch and others [5, 7, 22]. They have described an algorithm with complexity $O(mn^2)$ for recognizing disk-Helly graphs.

In this thesis we study some other families of subsets of a set. In particular, we consider the set of vertices of a graph G. We focus on the family of bicliques of a graph. A biclique of a graph is a subset of vertices that induces a maximal complete bipartite subgraph of G.

We consider in this thesis the graphs whose bicliques form a Helly family, the biclique-Helly graphs. We remark that a graph would have an exponential number of bicliques ([48]). Therefore, as in the case of clique-Helly graphs, the algorithm proposed by Berge would not be efficient for recognizing biclique-Helly graphs. In this thesis, we study this problem and give a characterization of biclique-Helly graphs that leads to a polynomial time algorithm for the recognition problem.

Besides the interest of examining bicliques in the scope of the Helly property, we mention that bicliques have been considered in some different contexts, e.g. [33, 43, 45, 47, 53].

None of the mentioned classes are closed under induced subgraphs. So, a question would be to characterize the graphs for which the Helly property is preserved for every induced subgraph. It leads to the hereditary classes of clique-Helly, biclique-Helly, neighborhood-Helly and disk-Helly graphs. Hereditary clique-Helly graphs have been characterized in [46], while [22] (c.f. [16]) contains a characterization of hereditary disk-Helly graphs. In this work, we describe forbidden subgraph characterizations for the classes of hereditary biclique-Helly and hereditary neighborhood-Helly graphs. All graphs in these forbidden families are of fixed size. In fact they have at most 8 vertices. Consequently, the characterizations imply polynomial time recognition for hereditary biclique-Helly, hereditary open neighbourhood-Helly and hereditary closed neighbourhood-Helly graphs.

Clique matrices of a graph have been characterized by Gilmore in 1960, and have been employed in different contexts. For example, in the characterizations of interval graphs [28], Helly circular-arc graphs [26] and self-clique graphs [12, 36], as well as in different covering problems involving cliques.

Motivated by the above concept, in this thesis we consider biclique matrices of a graph. We define the biclique matrix of a graph G as the $\{0, 1, -1\}$-matrix whose rows are the incident vectors of the bicliques of G. We give a characterization for such matrices, in similar terms as those used in the characterization of clique matrices. Biclique matrices can be employed, for instance in covering problems involving bicliques. Such problems have been considered, among others, by [3, 53].

Given a family of subsets of some set $\mathcal{F}$, the *intersection graph* of $\mathcal{F}$ is a graph having a vertex for each set of $\mathcal{F}$, and two vertices are adjacent whenever their corresponding sets intersect. On the other hand, given a graph G, it is an *intersection graph* when there exists a family $\mathcal{F}$ of subsets of some set such that G is its intersection graph. It is easy to prove that all graphs are intersection graphs (Marczewski, [44]). Intersection graphs were studied in several contexts (See [39]).

One of the most important problem concerning intersection graphs is the recognition problem. It consists on deciding whether for a given graph G and a family $\mathcal{F}$ of subsets of a set, G is the intersection graph of $\mathcal{F}$.

Many classes of intersection graphs have been defined, by fixing a suitable family of subsets. For example, clique graphs, interval graphs, chordal graphs, and line graphs (See [15, 16, 23, 24, 27, 38]) .

Clique graphs, i.e., the intersection graph of the family of bicliques of a graph has played an important role in the intersection graph theory. Roberts and Spencer, in [49] give the first characterization for clique graphs. A different one is shown in [2]. However, none of them appeared to lead to a polynomial time algorithm for the recognition problem. Moreover, it was an open problem determining the computational complexity of the clique graph recognition problem. Recently, in [1] it is proved that the mentioned problem is NP-complete.

Motivated by the concept of clique graphs, we define the biclique graph of G as the intersection graph of the family of all bicliques of G. We give a characterization of biclique graphs and show some families of graphs that are not biclique graphs. However, we leave as open to determine the computational complexity of the recognition problem.

The closed neighborhood graphs (or square graphs) are the intersection graphs of the closed neighborhoods of a graphs. It is proved that recognizing square graphs is NP-complete in the general case, while square graphs of bipartite graphs can be recognized in polynomial time (See [22, 37, 40, 41, 54]). We concentrate on the case of closed neighborhood graphs of open neighborhood-Helly bipartite graphs and study its clique graph and its relation to the biclique graphs.

Berge [8] defined a graph G to be perfect whenever the chromatic number coincides with the cardinality of a maximum clique, for every induced subgraph H of G. Perfect graphs are very interesting from an algorithmic point of view. While determining the clique number and the chromatic number of a graph are NP-complete problems, they are solvable in polynomial time for perfect graphs [30]. Besides, it has been proved recently that perfect graphs can be recognized in polynomial time [17]. For more background information on perfect graphs see [29].

We study the classes of graphs such that their biclique graphs are perfect. For that purpose we define the b-biclique-perfect and b-coordinated graphs, in a similar way as the classes of c-clique-perfect and c-coordinated ([13, 14]) were defined. We present some conditions in which b-biclique-perfect and b-coordinated graphs have a perfect biclique graph.

1.1 Results of the thesis

We study some properties of bicliques of graphs. We study the Helly property associated to bicliques, for bipartite graphs and in the general case. We give a characterization of biclique-Helly graphs that leads to an algorithm of complexity $O(|V(G)|^3|E(G)|)$ for the recognition problem. We employ a similar concept to extended triangle, which has been used to characterize clique-Helly graphs.

On the other hand, we define the bichromatic-Helly graphs. We give a characterization and an $O(|V(G)|^3|E(G)|)$ time algorithm for recognizing this class of graphs. Also we relate bichromatic-Helly graphs to biclique-Helly graphs, obtaining a second algorithm for recognizing biclique-Helly graphs with the same computational complexity.

Also, the Helly property is studied in the context of neighborhoods, both open and closed neighborhoods are considered. We study the relations between biclique-Helly, clique-Helly and open and closed neighborhood-Helly graphs.

Since the Helly property is not hereditary under induced subgraphs, we study the corresponding hereditary versions separately. We characterize hereditary biclique-Helly graphs by forbidden induced subgraph of finite size, which leads to a polynomial time algorithm for recognizing this class.

Analogously, we characterize hereditary open neighborhood-Helly graphs and closed neighborhood-Helly graphs with no triangles. Finally, we compare these three classes with hereditary clique-Helly graphs.

We define the biclique matrix as the matrix whose rows are the incident vectors of the bicliques of G. We give a characterization of biclique matrices. For that purpose we define the concept of bipartite-conformal hypergraphs, in a similar way as it was defined the concept of conformal hypergraphs, employed to characterize clique matrices (Gilmore, c.f [10]). We prove that recognizing bipartite-conformal hypergraphs with a compatible bicoloring and biclique matrices can be done in $O(m^2n + mn^3)$ steps.

We define the biclique graph, as the intersection graph of the family of bicliques of a graph. We study the recognition problem and give a characterization for biclique graphs and the particular case of biclique graphs of bipartite graphs, using the characterization of biclique matrices. For that purpose, we define the bipartite-Helly property for a labeled family of subsets. We give an algorithm of complexity $O(m^3n + m^3n)$ for recognizing bipartite-Helly labeled families. This result proves that the recognition problem for biclique graphs is in NP. We study the neighborhood graph of bipartite graphs. We give a relation between bicliques of a bipartite graph G and the cliques of the closed neighborhood graph. We use that relation to prove that, for the case that G is open neighborhood-Helly, the biclique graph of G and the clique graph of the closed neighborhood graph of G coincide.

This result is a useful tool for studying properties of biclique graphs, based on properties of the clique graphs. We employ this tool when we study the class of graphs for which their biclique graph is perfect. We prove that when we restrict to the subclass of graphs without induced P_5 (bicliqual), that class coincides with the classes of b-coordinated graphs, c-coordinated graphs, c-clique-perfect and b-biclique-perfect graphs. Finally, as an appendix of this thesis, we study a generalization of the concept of cliques, the p-cliques. We characterize the (p,q)-Helly property from a matricial point of view.

1.2 Definitions

Let G be a finite undirected graph, $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. An *independent set* in a graph G is a subset of pairwise non-adjacent vertices of G. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set of G. Say that a graph is a *complete graph* when every two vertices are adjacent. A *clique* of G is a complete subgraph maximal under inclusion. The clique number of G, denoted by $\omega(G)$, is the cardinality of the maximum clique of G. A *biclique* is a maximal subset $B \subseteq V(G)$ inducing a complete bipartite subgraph in G. Write $B = X \cup Y$ for the corresponding bipartition, restricting to $X, Y \neq \emptyset$. See an example in Figure 1.1. B_1, B_2, and B_3 are bicliques of G.

A spanning subgraph of G is a graph H such as $V(H) = V(G)$ and

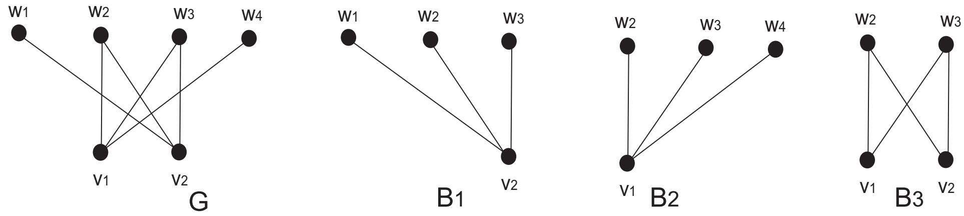


Figure 1.1: Example of bicliques if a graph G

$E(H) \subseteq E(G)$. A vertex $v \in V(G)$ is *universal* when it is adjacent to every other vertex of G.

A sequence $v_1, ..., v_k$ of distinct vertices $v_1, \ldots, v_k$ is a *path* of length k when $v_i v_{i+1}$ is an edge of G, for each i, $1 \leq i \leq k-1$. On the other hand, when $v_1 v_k$ is also an edge, say that $v_1, ..., v_k$ is a *cycle*. Denote by P_k the induced path with k vertices. Write C_k for a cycle having k vertices. By $\overline{G}$ denote the complement of G. The *open neighborhood* of a vertex v, $N(v)$ is the set of vertices adjacent to v, while the *closed neighborhood* of v, denoted by $N[v]$ is $N(v) \cup \{v\}$. A vertex with empty an open neighborhood is an *isolated* vertex. The *disk* $D_l(v)$ of $v \in V(G)$ is the subset of vertices of G whose distance to v is less or equal to l.

A graph is *chordal* when every cycle of length greater than 3 has a chord.

For $S \subseteq V(G)$, denote by $G[S]$ the subgraph induced in G by S. Say that vertex u dominates vertex v if $N(v) \subseteq N(u)$. We say that v is a *dominated* vertex. Two vertices are *twins* if they have the same open neighborhood. A dominated vertex is strictly dominated if it is dominated by a vertex that is not its twin.

Let $\mathcal{B}$ be some family of bicliques of G. The graph $G_{\mathcal{B}}$, formed exactly by the vertices and edges involved in the bicliques of $\mathcal{B}$ is called the *biclique subgraph* of G, relative to $\mathcal{B}$. When every biclique of $G_{\mathcal{B}}$ is also a biclique of G, say that it is a *special biclique subgraph.* Given a family $\mathcal{C}$ of cliques of a graph G, denote by $G_{\mathcal{C}}$ its corresponding clique subgraph, that is, the subgraph formed by the vertices and edges of the cliques of $\mathcal{C}$. A clique subgraph $G_{\mathcal{C}}$ is *special* when every clique of $G_{\mathcal{C}}$ is a clique in G.

A property P is hereditary if for every graph G that verifies P, it holds for every induced subgraph. A graph is dismantable if there is an ordering on the vertices $v_1, ..., v_n$ such that v_i is a dominated vertex in the graph $G - \{v_1, ..v_{i-1}\}$. Let $\mathcal{F}$ be a family of subsets of some set. Say that $\mathcal{F}$ is *intersecting* when the subsets of $\mathcal{F}$ pairwise intersect. On the other hand, when every intersecting subfamily of $\mathcal{F}$ has a common element then $\mathcal{F}$ is a *Helly* family. A graph G is *biclique-Helly* (clique-Helly, neighborhood-Helly, disk-Helly) when its family of bicliques (cliques, neighborhoods, disks) is Helly. In Figure 1.2, graph G is not biclique-Helly, since the family of bicliques of G is an intersecting family, but there is no a common intersection. On the other hand, graph G' is biclique-Helly.

A graph G is *hereditary biclique-Helly* (clique-Helly, neighborhood-Helly, disk-Helly) if every induced H is biclique-Helly (cliques-Helly, neighborhood-Helly, disk-Helly).

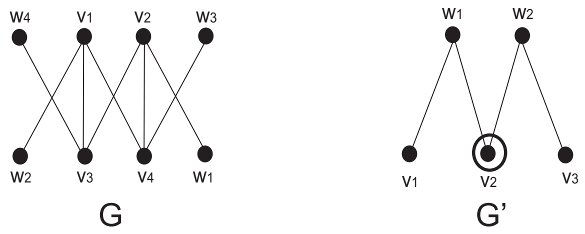


Figure 1.2: G is not biclique-Helly, G' is biclique-Helly.

If G has c cliques $\{C_1, \ldots, C_c\}$ then the *clique matrix* of G is the $c \times n$ $\{0,1\}$-matrix A, defined as $a_{ki} = 1$ if and only if $v_i \in C_k$. Finally, when G has d bicliques $B_1, \ldots, B_d \subseteq V(G)$, the *biclique matrix* of G is the $d \times n$ $\{0, 1, -1\}$-matrix A, where $a_{ki} = -a_{kj} \neq 0$, precisely when $v_i, v_j \in B_k$ and v_i, v_j are adjacent, for all $1 \leq k \leq n$ and $1 \leq i \neq j \leq n$.

Given a family of subsets of some set $\mathcal{F}$, the *intersection graph* of $\mathcal{F}$ is a graph having vertex for each set of $\mathcal{F}$, and two vertices are adjacent whenever their corresponding sets intersect. The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. Denote by $K^2(G)$ the clique graph of $K(G)$. A graph G is *self-clique* if $K(G)$ is isomorphic to G. Analogously, define the *biclique graph* $KB(G)$ as the intersection graph of the bicliques of G. The closed neighborhood graph of G, $N_c(G)$, is the intersection graphs of closed neighborhoods of G.

A graph is *weakly 2-colorable* when there is a bipartition U, W of vertices of G such that every clique of G has vertices of both parts.

Given a set S of elements, a family $\mathcal{F}$ of subsets of S is a *split* of S if for every pair of elements $x, y \in S$, there exist a set in $\mathcal{F}$ containing x, and not containing y.

Given a graph G, say that a family of subgraphs $\mathcal{C}$ *covers* the edges of G, if every edge belongs to some subgraph of the family.

A *clique cover* of a graph G is a subset of cliques covering all the vertices of G. The *clique-covering number* $\theta(G)$ is the cardinality of a minimum clique cover of G.

The *chromatic number* of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of G, the *clique number* of G, denoted by $\omega(G)$.

A graph G is *perfect* when $\theta(H) = \alpha(H)$ for every induced subgraph H of G (or equivalently, when $\chi(H) = \omega(H)$ for every induced subgraph H). A graph G is *K-perfect* if $K(G)$ is perfect. Analogously, a graph is *KB-perfect* if its biclique graph is perfect.

A matrix $A \in R^{k\times n}$ is *perfect* if the polyhedron $P(A) = \{x \in R^n / Ax \leq \mathbf{1}, x \geq 0\}$ has only integer extreme points.

Given a set $S = \{x_1, ..., x_n\}$ and a family $\mathcal{F} = \{F_1, ..., F_k\}$ of subsets of S, the incidence matrix of $\mathcal{F}$ is a $\{0, 1\} - matrix$ with n columns and k rows such that $a_{ij} = 1$ if x_j belongs to F_i and $a_{ij} = 0$ otherwise.

A hypergraph $\mathcal{H}$ is defined by a set of vertices $V(\mathcal{H}) = \{v_1, ..., v_s\}$ and a set of hyperedges $E(\mathcal{H}) = \{E_1, ..., E_k\}$, where E_i is a subset of $V(\mathcal{H})$, $\mid H_i \mid \geq 2$. The dual hypergraph of $\mathcal{H}$ is a hypergraph $\mathcal{H}^*$, with vertex set $V(\mathcal{H}^*) = \{E_1, ..., E_k\}$, and hyperedges $E(\mathcal{H}^*) = \{E_1^*, ..., E_s^*\}$, where $E_i^* = \{E_j,\ such\ that\ v_i \in E_j\}$, for $1 \leq i \leq s$.

1.3 Different problems on bicliques and related concepts

Bicliques were study in several contexts [3, 18, 25, 34, 43, 45, 47, 48, 55].

In [48], Prisner determined an upper bound for the cardinality of the family of bicliques of a bipartite graph and general graphs. Also, he presented a family of graph, the Cocktail party graphs, for which this bound is reached. Moreover, he proved that no other family verifies that property.

The *Cocktail party graph*, of order j, denoted by $CP(j)$, is a bipartite graph, $\mid V_1 \mid = \mid V_2 \mid = j$, where $v_i \in V_1$ is adjacent to $w_j \in V_2$ for every $i \neq j$.

Theorem 1.1 *[48] Let G be a bipartite graph and let $\mathcal{B}(G)$ be the family of bicliques of G. Then, $\mid \mathcal{B}(G) \mid \leq 2^{\frac{n}{2}}$. The cardinality of the family of cliques of the graph $CP(j)$ is exactly $\mid \mathcal{B}(G) \mid = 2^j$ and no other bipartite graph of $2j$ vertices has this property.*

Observe that the family of bicliques of $CP(j)$ is $\{(a_i, b_j), i \in I,\ j \in \{1...j\} \backslash I, \forall I \subseteq \{1...j\}\}$. It follows that its cardinality is 2^j=$2^{\frac{n}{2}}$ (Figure 1.3).

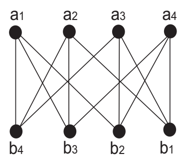


Figure 1.3: Cocktail party of order 4, $CP(4)$

Theorem 1.2 *[48] Any graph has at most $n^{\frac{5}{2}}(1,618034^n + o(1))$ bicliques.*

It is worth mention that in the proof of this Theorem, appears the gold ratio.

Theorem 1.3 *[48] Let G be a bipartite graph with bipartition V_1, V_2, $CP(j)$-free, for some j. Then, G has at most $(\mid V_1 \mid\mid V_2 \mid)^{j-1}$ bicliques*

As a Corollary, he proved the following:

Corollary 1.1 *Let* Γ *be a family of bipartite graphs closed under induced subgraphs. Then there exists a polynomial function* $f(n)$ *such that every member of the class has at most* $f(\mid V \mid)$ *bicliques if and only if* $CP(j) \notin \Gamma$ *for some* j*.*

Several concepts related to bicliques were studied. It is worth to mention that in the literature, some of them are also call bicliques.

A *c-biclique* is a complete subgraph (not necessary induced or maximal). A *balanced biclique* is a biclique were the bipartition V_1, V_2 has the same cardinality.

Some decision problems related to bicliques, c-bicliques and balanced bicliques were studied. As examples, we mention the *Maximun vertex biclique problem* and the *Maximun edge biclique problem (MBP).* The first one consists of deciding if a given graph contains a biclique of at least k vertices $(\mid V_1 \mid + \mid V_2 \mid \geq k)$. The second problem consists of deciding if G contains a biclique with at least k edges, $(\mid A \mid \cdot \mid B \mid \geq k)$. Those problems have been studied in [25, 45, 55].

1.4 How the thesis is organized

The thesis is organized as follows.

In Chapter 2, we study biclique-Helly graphs, and give a characterization for this family. We also study the Helly property related to open and closed neighborhoods. We also define the bichromatic-Helly graphs and characterize them.

In Chapter 3, we study the classes of hereditary biclique-Helly, hereditary neighborhood-Helly gaphs (both open and closed are considered). Characterizations of these classes are given and the relations between them and the hereditary clique-Helly graphs.

In Chapter 4, we define and characterize biclique matrices, for the general case and for the particular case of bipartite graphs. Also, the concept of bipartite-conformal is defined. Finally, a polynomial algorithm for identifying bipartite-conformal hypergraphs is given and two polynomial algorithms are proposed for recognizing biclique matrices.

In Chapter 5 we define and characterize biclique graphs. We present some families of graphs which are not biclique graphs and study the classes of biclique graphs of some classes of graphs. Also we define the bipartite-Helly property and give a polynomial algorithm for recognizing bipartite-Helly labeled families.

Chapter 6 contains a characterization of the closed neighborhood graph of a biclique graph. Relations between the cliques of the neighborhood graph of a bipartite graph G and bicliques of G are given. The Helly property is also studied in this context.

In Chapter 7 we study those graphs for which the biclique graphs is perfect, i.e., the KB-perfect graphs. We define the b-coordinated graphs and the b-biclique-perfect graphs.

Chapter 8 contains the conclusions of the thesis.

1.4.1 Notation and basic graphs

We remark that along the thesis, we will say that a vertex or an edge belongs to a graph, meaning that it belongs to the vertex set of the graph or the edge set of the graph, respectively. On the same way, we will consider a subgraph of a graph as a subset of vertices or/and edges.

Notation: .

- $G - \{v\}$ is a graph G obtained by removing a vertex v and its incident edges.
- K_n, is the complete graph of n vertices.
- C_n, is a cycle of n vertices
- $F - free$, a graph which does not contain the graph F as an induced subgraph.
- $n - fan$, is an induced path of $n+1$ vertices, and an additional universal vertex. See Figure 1.4
- Hajós graph, with center T: See Figure 1.4
- k-extended Hajós graph with center T, a graph obtained from the Hajós graph by adding k edges between vertices not in the center, $k = 1, 2, 3$. See Figure 1.4

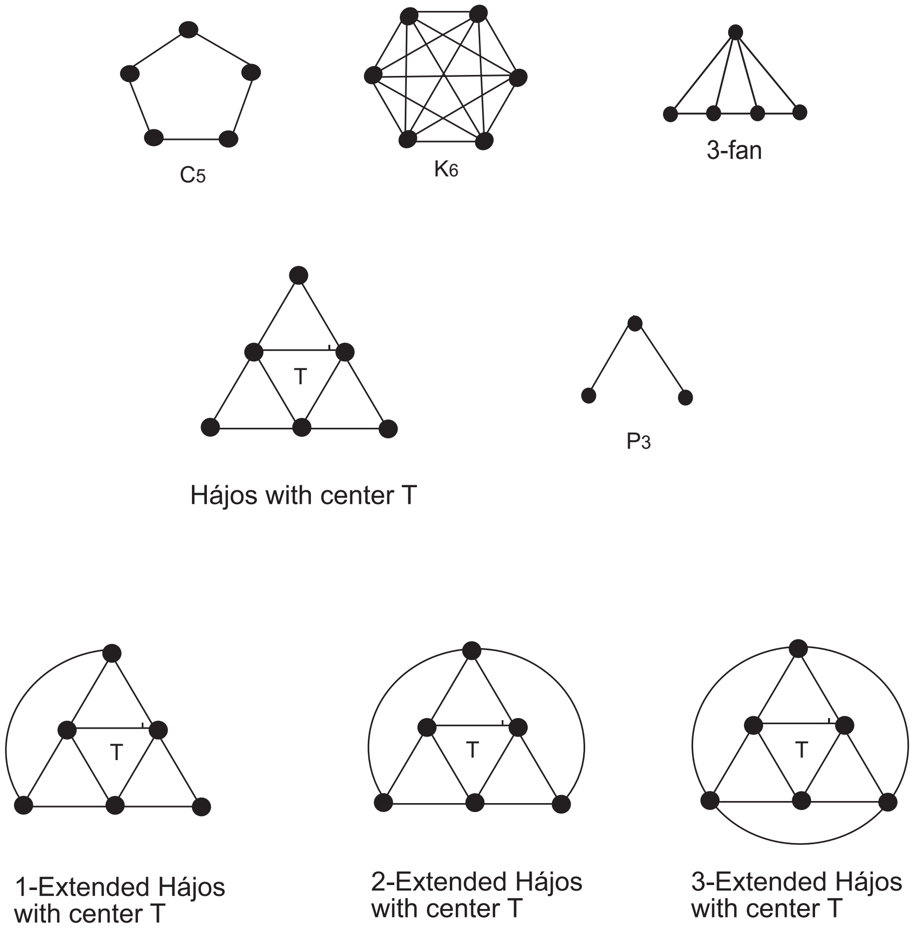


Figure 1.4: Basic graphs

Chapter 2

Bicliques and the Helly property

2.1 Introduction

In this Chapter we study the Helly property in the context of bicliques and neighborhoods of a graph. We define the biclique-Helly graphs and give two characterizations for this class, along with polynomial time algorithms for recognizing biclique-Helly graphs. Also we study the classes of open and closed neighborhood-Helly graphs, and relate them to clique-Helly and biclique-Helly graphs.

Helly families of subsets have been studied in different contexts. Berge proposed a polynomial time algorithm ($O(n^3)$) to decide whether a given family is Helly or not. It is based on the following Theorem.

Theorem 2.1 *[10] A family $\mathcal{F}$ of subsets of a set S is Helly if and only if for every 3-subset S' of S, the subfamily of subsets in $\mathcal{F}$ containing at least two elements S', have a common intersection.*

In the scope of graph theory, the study of the Helly property has motivated the introduction of some classes of graphs, as clique-Helly graphs [23, 32, 49], disk-Helly graphs [5, 6], and neighborhood-Helly graphs [10]. These classes correspond to the cases where the families subject to the Helly property are cliques, disks and neighborhoods, respectively.

For recognizing clique-Helly graphs the algorithm proposed by Berge is not efficient since the number of cliques can be exponential. Clique-Helly graphs have been characterized in [52]. It is based on the concept of extended triangles and it leads to a polynomial time recognition algorithm. Given a triangle T in G, the *extended triangle* of T, denoted by $E(T)$ is the subgraph of G induced by the vertices which form a triangle with at least one edge of T. The characterization is given by the following Theorem and it leads to a polynomial time recognition algorithm.

Theorem 2.2 *[52] A graph is clique-Helly if and only if every extended triangle has a universal vertex.*

In [14] there is another characterization of clique-Helly graphs, which relates properties of the graph with properties of a polyhedron generated by the clique matrix of G. However, note that $K(G)$ might be of exponential size, implying that it does not lead directly to a polynomial time recognition algorithm of clique-Helly graphs.

Theorem 2.3 *[14] Let G be a graph with n vertices and m cliques, and let A_G be a clique matrix of G and $K(G)$ its clique graph. Then it is equivalent:*

1. *G is clique-Helly.*
2. *The clique matrix of $K(G)$ is A_G^T without including rows.*
3. *The polyhedrom $\{x \in \Re^m / A^T \cdot x \leq 1, x \geq 0\}$ is the same as $\{x \in \Re^m / A_{K(G)} \cdot x \leq 1, x \geq 0\}$.*

Disk-Helly graphs were studied by Bandelt and Pesch and others [5, 7, 22]. They give an algorithm with complexity $O(m.n^2)$ for recognizing disk-Helly graphs:

Theorem 2.4 *A graph G is disk-Helly if and only if it is clique-Helly and dismantlable.*

In this Chapter, we consider the graphs whose bicliques form a Helly family, the biclique-Helly graphs. Besides the interest of examining bicliques in the scope of the Helly property, these graphs could be of interest in the study of retracts [33]. We mention that bicliques have been considered in some different contexts, e.g. [43, 45, 47, 53].

Recall that a graph is biclique-Helly when its family of bicliques is a Helly family. In addition, we relate biclique-Helly graphs to the classes of clique-Helly, disk-Helly and neighborhood-Helly graphs.

The Chapter is organized as follows.

In Section 2 we describe a characterizations for biclique-Helly graphs. It generalizes the notion of extended triangles.

In Section 3 a weaker notion of clique-Helly graphs is defined, the bichromatic-Helly graphs. We give a characterization for bichromatic-Helly graphs which leads to an algorithm for recognizing bichromatic-Helly graphs in in polynomial-time complexity.

In Section 4, a second characterization for biclique-Helly graphs is given. It relates biclique-Helly graphs to bichromatic-Helly graphs. Both characterizations given in this Chapter lead to algorithms for recognizing biclique-Helly graphs, in polynomial-time complexity. Recall that a graph might have an exponential number of bicliques [48]. Therefore, the algorithm by Berge (Theorem 2.1, see [11], [10] [49]) for recognizing Helly families would not recognize biclique-Helly graphs within polynomial time.

In Section 5 we give a characterization for open neighborhood-Helly graphs, a partial characterization for closed neighborhood-Helly graphs and finally, we relate the class of biclique-Helly graphs to those of clique-Helly graphs, disk-Helly graphs and neighborhood-Helly graphs.

2.2 Biclique-Helly Graphs

In this Section, we describe a characterization for biclique-Helly graphs. The characterization employs a generalization of the idea of extended triangle [22, 52]. We employ the following generalization of the concept of extended triangles.

Say that a vertex v *dominates* an edge e, when one of the extremes of e either coincides or is adjacent to v. When v dominates every edge of G then v is an *edge dominator* of G. Clearly, a universal vertex is an edge dominator. For $S \subseteq V(G)$, denote by $G[S]$ the subgraph induced in G by S.

Let $S \subseteq V(G)$, $|S| = 3$. Denote by $\mathcal{B}_S$ the family of bicliques of G, each of them containing at least two vertices of S. Let S^* be the set of vertices of the biclique subgraph $G_{\mathcal{B}_S}$. The induced subgraph $G[S^*]$ is called the *extension* of S. Clearly, $G_{\mathcal{B}_S}$ is a spanning subgraph of $G[S^*]$. See some examples in Figure 2.1.

Before we describe the characterization, we prove two useful Lemmas.

Lemma 2.1 *Let G be a graph. Then G has neither triangles nor C_5's if and only if each of the extensions in G is a bipartite graph.*

Proof: Let G be a graph with no triangles nor induced $C_5's$ and $S = \{v_1, v_2, v_3\} \subseteq V(G)$. If $G[S^*]$ is empty there is nothing to prove. Otherwise $G[S^*]$ has at least two vertices in S. First, examine the alternative when there are only two vertices v_1, v_2 of S in $G[S^*]$. Suppose v_1 and v_2 are adjacent. Let X be the subset of vertices of $G[S^*]$ non adjacent to v_1 and $Y = S^* \setminus X$. By definition of $G[S^*]$, any vertex of X must be adjacent to v_2, and any of Y is adjacent to v_1. Because G has no triangles, any two vertices of X are not adjacent. Similarly, no two vertices of Y are adjacent. Then $G[S^*]$ is bipartite. The second alternative is to assume that v_1 and v_2 are not adjacent. Let X be the subset of vertices of $G[S^*]$, simultaneously non adjacent to v_1 and v_2, and $Y = S^* \setminus X$. By a similar argument as above, we conclude that $X \cup Y$ is a bipartition of $G[S^*]$, as required.

We examine the situation when all vertices of S are in $G[S^*]$. The alternatives for S are to be a triangle, or a P_3, or their complements. However, S can not be a triangle, by hypothesis. We discuss below the other three possibilities.

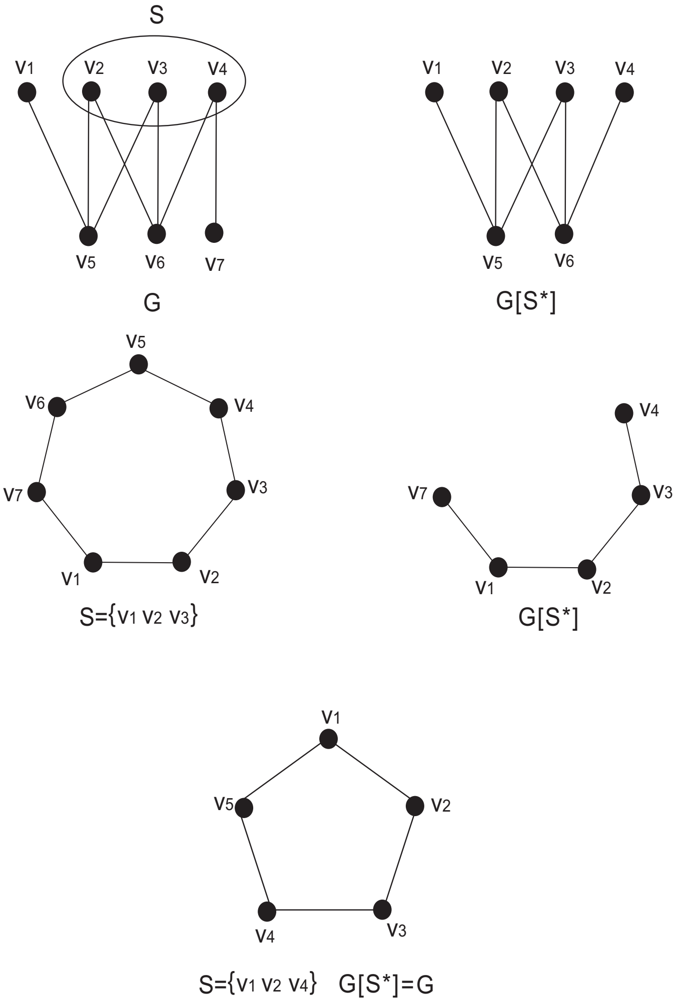


Figure 2.1: Examples of extensions of S

Suppose S is a P_3, with v_1v_3 non adjacent. Denote by $X \subseteq S^*$ the subset consisting of those vertices simultaneously non adjacent to v_1 and v_3, and $Y = S^* \setminus X$. We show that $X \cup Y$ is a bipartition of $G[S^*]$. Let $v_i, v_j \in X$, $v_i \neq v_j$. Suppose that v_i and v_j are adjacent. Since v_i is in $G[S^*]$, there exists v'_i adjacent to v_1, v_3 and v_i. Similarly, there exists vertex v'_j adjacent to v_1, v_3 and v_j. Clearly, $v'_i, v'_j \in Y$. Because G has no triangles, $v'_i \neq v'_j$ and v'_j is neither adjacent to v_i nor v'_i. However, in the latter situation, $v_i, v_j, v'_j, v_3, v'_i$ form a C_5, a contradiction. Consequently, v_i and v_j are not adjacent, as required.

Next, let $v_k, v_l \in Y$, $v_k \neq v_l$. Assume that v_k and v_l are adjacent. By definition of $G[S^*]$, both v_k and v_l are adjacent to at least one of v_1, v_3. Then v_k, v_l are both non adjacent to v_2, otherwise G has a triangle. Without loss of generality, we may assume that v_1 and v_k are adjacent. Then v_1, v_l can not be adjacent, implying that v_3, v_l are adjacent, implying that v_3, v_k are not. In this case, v_1, v_k, v_l, v_3, v_2 induce a C_5, contrary to the hypothesis. Therefore

v_k and v_l are not adjacent, meaning that $G[S^*]$ is bipartite. Consequently, whenever S is a P_3, $G[S^*]$ is indeed bipartite.(Figure 2.2)

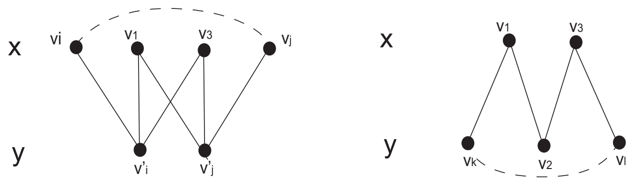


Figure 2.2: Extension of P_3

In the next alternative, assume that S is an independent set. Let X be the subset of vertices of S^* which are non adjacent simultaneously to v_1, v_2, v_3, while $Y = S^* \setminus X$. Let $v_i, v_j \in X$, $v_i \neq v_j$. Assume that v_i, v_j are adjacent. By definition of $G[S^*]$, there exist vertices $v'_i, v'_j \in Y$, both of them adjacent to at least two of the three vertices of S. Consequently, one of the vertices of S, say v_1, is adjacent to both v'_i, v'_j. Because G has no triangles, $v'_i \neq v'_j$, with v'_i, v_j not adjacent, v_i, v'_j also not adjacent. Then $v_1, v'_j, v_j, v_i, v'_i$ form a C_5, which is impossible. Hence v_i, v_j can not be adjacent.(Figure 2.3)

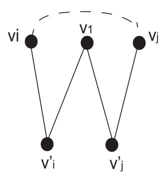


Figure 2.3: Extension of an independent set

In the sequel, examine Y. First, we show that any vertex $v_k \in Y$ must be adjacent to at least two vertices of S. By contrary, suppose that v_k is adjacent solely to one vertex of S, say v_3. By definition of $G[S^*]$, there exists some vertex v', simultaneously adjacent to v_1, v_2 and v_k, because no biclique of G contains v_3, v_k, v_1 nor v_3, v_k, v_2. Again, by definition of $G[S^*]$, there exists some vertex v'' adjacent to v_3, and to v_2 or v_1. Without loss of generality, let v'' be adjacent to v_1. Because G has no triangles, v_3 and v' are not adjacent, and v'' is also not adjacent to both v_k and v'. However, in the latter situation, v_1, v'', v_3, v_k, v' form a C_5, contrary to the hypothesis. Consequently, $v_k \in Y$ indeed implies that v_k is adjacent to at least two vertices of S. Finally, let $v_l, v_t \in Y$, $v_l \neq v_t$. Then one of the vertices of S, say v_1, is adjacent to both v_l and v_t. Since G has no triangles, v_l and v_t can not be adjacent. Therefore $G[S^*]$ is bipartite, with bipartition $X \cup Y$.

In the last alternative, S induces a $\overline{P_3}$. Let v_1 and v_2 be the adjacent vertices in S. By definition of $G[S^*]$, there exists a vertex v'_1 adjacent to both v_1 and v_3. Similarly, there exists v'_2 adjacent to v_2 and v_3. It follows that either a triangle or a C_5 exist among the vertices $v_1, v_2, v_3, v'_1, v'_2$. Consequently. $G[S^*]$ can not contain simultaneously v_1, v_2 and v_3, meaning that this case does not occur (see Figure 2.4). The proof is complete.

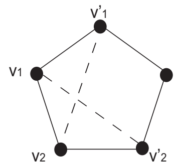


Figure 2.4: Extension of $\overline{P_3}$

Conversely, suppose G has an induced triangle, $T = \{v_1, v_2, v_3\}$. It is clear that T is included in the extension of T. Suppose now that G contains a C_5., $v_1, ..., v_5$. The extension of $S = \{v_1, v_3, v_5\}$ contains the induced C_5. In both cases, the extensions are not bipartite graphs. ∎

Lemma 2.2 *Let G be a biclique-Helly graph. Then G contains neither triangles nor induced C_5's.*

Proof: By hypothesis, G is biclique-Helly. First, we show that G has no triangles. By contrary, assume that vertices v_1, v_2, v_3 form a triangle and denote by e_i the edge $v_i v_{i+1(mod\ 3)}$, $1 \leq i \leq 3$. There exists some biclique B_i containing e_i, for each i. Then $\{B_1, B_2, B_3\}$ constitutes a family of distinct intersecting bicliques. Because G is biclique-Helly, there exists a vertex v common to B_1, B_2, B_3. Clearly, $v \neq v_1, v_2, v_3$, as $v_i \notin B_j$ if and only if $j = i + 1(mod\ 3)$. Because $v \in B_1$, v is adjacent to exactly one between v_1 and v_2, say adjacent to v_1 and not v_2. The latter implies v to be adjacent to v_3, because $v \in B_2$. Consequently, v, v_1, v_3 form a triangle, meaning that $v \notin B_3$. That is, G contains no vertex common to B_1, B_2, B_3, contradicting G to be biclique-Helly.

Next, we show that G has no C_5's. By contrary assume that G does contain such a cycle. Extend each triple of successive vertices of the cycle to a biclique. This family is intersecting, so it has a common vertex v. Since G has no triangles, there is a pair of successive vertices in the cycle that are not adjacent to v. However, they lie with v in a biclique, a contradiction. Consequently, G does not contain C_5's. ∎

The main Theorem is the following. It gives a characterization of biclique-Helly graphs.

Theorem 2.5 *A graph G is biclique-Helly if and only if G has no triangles and each of its extensions has an edge dominator.*

Proof: By hypothesis, G is biclique-Helly, then by Lemma 2.2 it follows that G has no triangles. Our aim is to prove that each non empty extension $G[S^*]$ contains an edge dominator, for $S \subseteq V(G)$, $|S| = 3$. Denote by $\mathcal{B}_S$ the family of bicliques of G, each of them containing at least two vertices of S. Because G is biclique-Helly, there exists some vertex v common to all bicliques of $\mathcal{B}_S$.

On the other hand, since by Lemma 2.2, G has neither triangles nor C_5's, we can apply Lemma 2.1 and conclude that $G[S^*]$ is bipartite. Let $X \cup Y$ be a bipartition of it. Because $G_{\mathcal{B}_S}$ is a spanning subgraph of $G[S^*]$, we know that $G_{\mathcal{B}_S}$ is bipartite and $X \cup Y$ a bipartition of it. Without loss of generality, let $v \in X$. We show that v is adjacent to all vertices of Y. Otherwise, if $w \in Y$ is not adjacent to v, because $G_{\mathcal{B}_S}$ has no isolated vertices, there is an edge e in $G_{\mathcal{B}_S}$ incident to w. Clearly, any biclique which contains the extreme vertices of e does not contain v. The latter implies that v is not common to all bicliques of $\mathcal{B}_S$, a contradiction. Consequently, v is adjacent to all vertices of Y, meaning that v is an edge dominator of both $G_{\mathcal{B}_S}$ and $G[S^*]$. The proof of necessity is complete.

Conversely, let G be a graph with no triangles and whose extensions contain edge dominators. We prove that G is biclique-Helly. Let $S \subseteq V(G)$, $|S| = 3$, and $\mathcal{B}_S$ the collection of bicliques of G, containing at least two of the the three vertices of S. Let $G_{\mathcal{B}_S}$ be the biclique subgraph of G, relative to $\mathcal{B}_S$, and $G[S^*]$ the extension of S. By hypothesis, $G[S^*]$ has an edge dominator v. It is clear that if $G[S^*]$ is bipartite, then v belongs to every biclique of $G_{\mathcal{B}_S}$. We show that $G[S^*]$ is bipartite. Suppose it has an induced odd cycle C_{2k+1}. As v is an edge dominator and G has no triangles, v is alternatively adjacent to vertices v_1, v_3,..., v_{2k+1}, forming a triangle, a contradiction. The case $v_i = v$ is similar.

As $G[S^*]$ is bipartite, every biclique of $G_{\mathcal{B}_S}$ is also a biclique of $G[S^*]$. Consequently, v is also contained in every biclique of $G_{\mathcal{B}_S}$. Finally, every biclique of $\mathcal{B}_S$ is also a biclique of $G_{\mathcal{B}_S}$. The latter implies that v is common to all bicliques of $\mathcal{B}_S$.

The conclusion is that, for every subset S of three vertices, the family of bicliques of G having at least two vertices of S contains a common vertex. By [11], G is biclique-Helly. ∎

An algorithm for recognizing whether or not a given graph is biclique-Helly follows directly from Theorem 2.5. Let G be a graph

Algorithm 2.1 ***Biclique-Helly graphs (I)*** *First verify if G has triangles. If positive, answer NO. Otherwise, for each $S \subseteq V(G)$, $|S| = 3$, construct S^* and $G[S^*]$, and check if $G[S^*]$ has an edge dominator. If the answer is negative for some S, answer NO; otherwise answer YES.*

Algorithm. Input: A graph G

- For every $S \subseteq V(G)$, S different from the complement of a P_3, $S = \{x, y, z\}$
  - $S^\star = \emptyset$
  - If S is a triangle, answer NO. Complexity $O(|V(G)|^3)$
  - Otherwise, if $y, z \in N(x)$
    - For every $k \in N(y)$, if $k \in N(z)$

      · $S^* = S^* \cup N[k]$
    * $S^* = S^* \cup N[x] \cup N[z] \cup N[y]$. Complexity $O(|E(G)|)$
  – Otherwise, if $y, z \notin N(x)$ and z, y are not adjacent
    * For $k \in N(x)$
      · If y or $z \in N(k)$, $S^* = S^* \cup N[k]$
    * For $k \in N(y)$
      · If $z \in N(k)$, $S^* = S^* \cup N[k]$. Complexity $O(|E(G)|)$
  – If $S^\star \neq \emptyset$
    * If S^* has not an edge dominator, answer NO. Complexity $O(|E(G)|)$

- Answer YES

Checking the existence of a triangle has a cost of $O(n^3)$ time. Constructing $G[S^*]$ and checking the existence of an edge dominator can be done in $O(| \ E(G) \ |)$ time. Then, the Algorithm 2.1 terminates within $O(|V(G)|^3|E(G)|)$ steps.

In the next section we study a variation of clique-Helly graphs, the bichromatic-Helly graphs.

2.3 Bichromatic-Helly Graphs

In this section, we define a weaker notion of clique-Helly graphs, the bichromatic-Helly graphs.

Let G be a graph and $U \cup W$ an arbitrary bipartition of its vertices. A clique C of G is called *bichromatic* when it contains at least one vertex of each of the parts U and W. Say that G is *bichromatic-Helly* (relative to U, W) when its bichromatic cliques form a Helly family. Observe that the bichromatic-Helly property depends strongly on the bipartition $U \cup W$. In Figure 2.5 we can see an example: The graph G is bichromatic-Helly relative to the bipartition $U' \cup W'$, while G is not bichromatic-Helly relative to the bipartition $U \cup W$.

Let T be a triangle of G. The *extended triangle* of T, denoted by $E(T)$ is the subgraph of G induced by the vertices which form a triangle with at least one edge of T. Let G be a graph and $U \cup W$ a bipartition of its vertices. A vertex $v \in E(T)$ is a *relevant* vertex of T if it belongs to some bichromatic clique of G which contains at least one edge of T. The extended relevant triangle of T, $ER(T)$, is the subgraph of $E(T)$ induced by the relevant vertices of T. In Figure 2.5.d we can see an example of the extended relevant triangle of v_1, v_2, v_3 relative to U', V' for the graph G. In 2.5.d, we can see the extended relevant triangle of v_1, v_2, v_3 relative to U, V for the graph G.

Let G be a graph and let $\mathcal{C}$ be the set of cliques of G. For each subset $\mathcal{C}' \subseteq \mathcal{C}$, define the clique subgraph $G_{\mathcal{C}'}$ as the subgraph of G formed by the vertices and edges of the cliques of the subset $\mathcal{C}'$ (See an example of $G_{\{v_1,v_2,v_2\}}$ in Figure 2.5.f).

We formulate a characterization for bichromatic-Helly graphs, which is employed in that for biclique-Helly graphs.

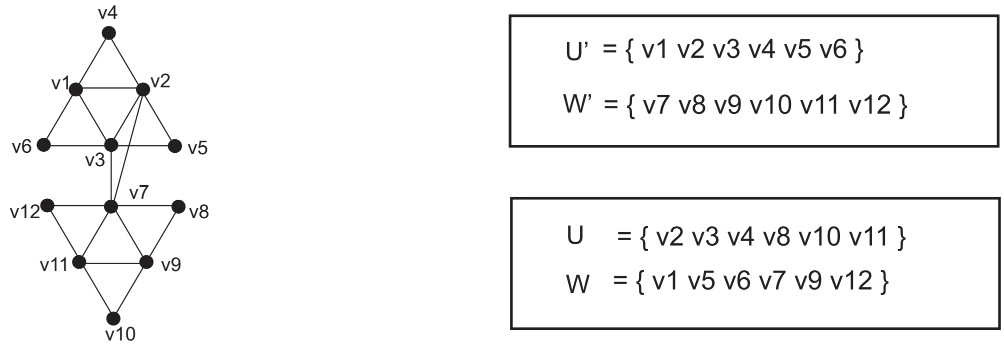
The following Lemma is direct, but it is useful in our proof of the next Theorem.

Lemma 2.3 *Let G be a graph and $U \cup W$ a bipartition of its vertices. Let T be a triangle of G and let $\mathcal{C}$ be the set of bichromatic cliques of G having at least one edge of T. Then, the clique subgraph $G_{\mathcal{C}}$ is a spanning subgraph of the extended relevant triangle $ER(T)$.*

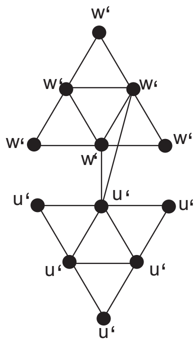
Proof: Let T be a triangle of G. Let $\mathcal{C}$ be the subfamily of bichromatic cliques of G having at least one edge of T and let $ER(T)$ be the extended relevant triangle of $E(T)$. First we prove that the set of vertices of $G_{\mathcal{C}}$ is the same as the set of vertices of $ER(T)$. Suppose v is a vertex of the clique subgraph $G_{\mathcal{C}}$. Then v belongs to some bichromatic clique of C and therefore v is a relevant vertex of the extended triangle $E(T)$. Conversely, let w be a vertex of $ER(T)$.

It follows that w belongs to some bichromatic clique that contains at least one edge of T. Then, $w \in V(G_{\mathcal{C}})$.

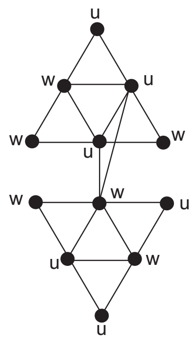
Now, let vw be an edge of $E(G_C)$. Then, v and w are vertices of $ER(T)$. As $ER(T)$ is an induced subgraph, vw is an edge of $ER(T)$. ∎



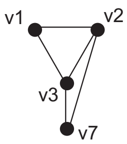
(a) Graph G. Bipartitions U', W' and U,V



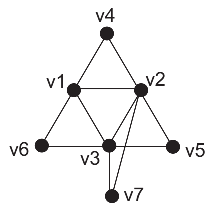
(b) G is bichromatic Helly relative to bipartition U', W'



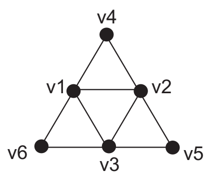
(c) G is not bichromatic Helly relative to bipartition U, W



(d) The extended relevant triangle of {v1,v2,v3} relative to bipartition U', W'



(e) The extended relevant triangle of {v1,v2,v3} relative to bipartition U, W



(f) Clique subgraph $G_{\{v1,v2,v3\}}$

Figure 2.5: Examples

The following Theorem characterizes bichromatic-Helly graphs for general bipartitions.

Theorem 2.6 *Let G be a graph and $U \cup W$ a bipartition of its vertices. Then G is bichromatic-Helly if and only if for every triangle T, $ER(T)$ has a universal vertex.*

Proof: Let G be a bichromatic-Helly graph and $U \cup W$ a bipartition of its vertices. Let T be a triangle of G and $ER(T)$ its extended relevant triangle. We will show that $ER(T)$ has a universal vertex. Let $\mathcal{C}$ be the set of bichromatic cliques of G having at least one edge of T. Observe that the cliques of C pairwise intersect. By hypothesis, G is bichromatic-Helly. Then, there exists a vertex v belonging to every clique of $\mathcal{C}$. It follows that v is a universal vertex of the clique subgraph $G_{\mathcal{C}}$. By Lemma 2.3, $G_{\mathcal{C}}$ is a spanning subgraph of $ER(T)$, then $ER(T)$ has a universal vertex.

Conversely, suppose that there exists a subfamily $\mathcal{C}$ of bichromatic cliques which does not verify the Helly property. Consider $\mathcal{C}' = \{C_1, C_2, \cdots, C_k\} \subseteq \mathcal{C}$ a minimal not Helly subfamily of $\mathcal{C}$. Clearly $k \geq 3$. By the minimality of $\mathcal{C}'$, there is a vertex w_i which belongs to every bichromatic clique of the subfamily $\mathcal{C}'_i = \mathcal{C}' \setminus \{C_i\}$. Consider vertices w_1, w_2 and w_3. They induce a triangle T in G.

Let $\mathcal{C}''$ be the subfamily of bichromatic cliques of G that contains at least one edge of T. Consider the extended relevant triangle $ER(T)$. By hypothesis, $ER(T)$ has a universal vertex v. That means that v belongs to every bichromatic clique that contains at least one edge of T, i.e., v belongs to every bichromatic clique of $\mathcal{C}''$.

As $\mathcal{C}' \subseteq \mathcal{C}''$, v belongs to every bichromatic clique of $\mathcal{C}'$. ∎

An algorithm for recognizing whether or not a given graph is bichromatic-Helly follows directly from Theorem 2.6. Let G be a graph with bipartition $U \cup W$.

Algorithm 2.2 ***Bichromatic-Helly graphs.*** *For every triangle T, construct $ER(T)$ and check if it has a universal vertex. If the answer is negative for some T, answer NO; otherwise answer YES.*

Construct $ER(T)$ and checking if it has a universal vertex can be done in $O(|\ (E(G)\ |)$ time. The Algorithm 2.2 terminates within $O(|V(G)|^3|E(G)|)$ steps.

We study the relation between bichromatic-Helly graphs and clique-Helly graphs. If G is bichromatic-Helly relative to some weak 2-coloring, it is clear that G is clique-Helly. On the other hand, a clique-Helly graph G is bichromatic-Helly for every bipartition $U \cup W$. The converse is given by the following Theorem.

Theorem 2.7 *G is clique Helly if and only if G is bichromatic-Helly for every bipartition $U \cup W$.*

Proof: It is clear that if G is clique-Helly, then G is bichromatic-Helly for every bipartition $U \cup W$.

Conversely, we prove that every extended triangle of G has a universal vertex. Let v_1, v_2, v_3 be a triangle and let $E(T)$ be its extended triangle. Consider any bipartition $U \cup W$ of G such

1. $v_1, v_2 \in U$, $v_3 \in W$
2. every vertex $w \in E(T)$ which is not adjacent to v_3, belongs to W
3. every vertex that is not adjacent to v_1, belongs to U

Consider $ER(T)$ the extended relevant triangle of T, relative to bipartition $U \cup W$. It is clear that $E(T) \subseteq ER(T)$. Since, by hypothesis, G is bichromatic-Helly relative to U, W, $ER(T)$ has a universal vertex. Then, it follows that $E(T)$ has a universal vertex and G is clique-Helly. ∎

2.4 Bichromatic-Helly graphs and biclique-Helly graphs

In this section we formulate another characterization for biclique-Helly graphs, based on bichromatic-Helly graphs. We want to relate bichromatic cliques to bicliques. For that purpose, we make the following construction.

Given a graph G, with vertices $v_i \in V(G)$, denote by $H(G)$ the graph obtained from G, by the following construction. For each $v_i \in V(G)$, there is a pair of distinct vertices u_i and w_i in $H(G)$. The edges of $H(G)$ are as follows: u_i, u_j and w_i, w_j are adjacent precisely when v_i, v_j are not, while u_i, w_j are adjacent when v_i, v_j are also adjacent. Denote $U = \cup u_i$ and $W = \cup w_i$. The bipartition $U \cup W$ is called the *canonical bipartition* of $H(G)$. See an example in Figure 2.6.

The relation between bicliques of a graph G and bichromatic cliques of $H(G)$ is given by the following Lemma.

Lemma 2.4 *Let G be a graph. Then there is a 1-2 correspondence between the bicliques of G and the bichromatic cliques of $H(G)$. Moreover, the two cliques of $H(G)$ that correspond to a biclique of G are disjoint.*

Proof: Let $B = \{z_{i_1}, z_{i_2}, ..., z_{i_k}\} \cup \{z_{i'_1}, z_{i'_2}, ..., z_{i'_{k'}}\}$ be a biclique of G. Consider the complete subgraph of H induced by $\{v_{i_1}, v_{i_2}, ..., v_{i_k}, w_{i'_1}, w_{i'_2}, ..., w_{i'_{k'}}\}$. We are going to prove that it is a clique. Suppose there is a vertex u of H adjacent to every vertex of the complete subgraph. Without loss of generality, we can assume that $u \in W$. Let z be the corresponding vertex of u

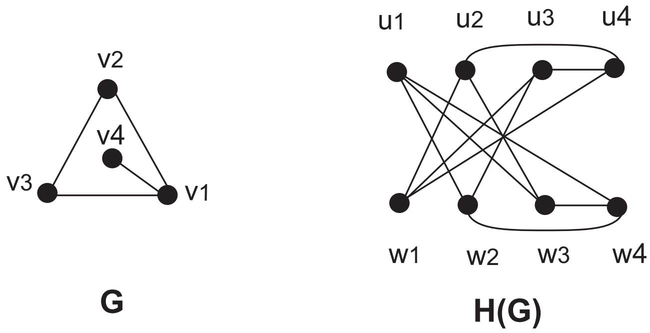


Figure 2.6: Example of $H(G)$

in G. It follows that z is adjacent to every vertex of the set $\{z_{i_1}, z_{i_2}, ..., z_{i_k}\}$, and not adjacent to every vertex of the set $\{z_{i'_1}, z_{i'_2}, ..., z_{i'_{k'}}\}$. Hence, B is not maximal, which is an absurd. Analogously, it can be proved that the complete subgraph induced by $\{w_{i_1}, w_{i_2}, ..., w_{i_k}, v_{i'_1}, v_{i'_2}, ..., v_{i'_{k'}}\}$ is a clique of H. It is clear that both cliques are disjoint.

Finally, let $C = \{v_{i_1}, v_{i_2}, ..., v_{i_k}, w_{i'_1}, w_{i'_2}, ..., w_{i'_{k'}}\}$ be a bichromatic clique of H. Observe that $i_j \neq i'_s$ for all j, s because w_k is not adjacent to v_k. Then, the subgraph induced by vertices $\{z_{i_1}, z_{i_2}, ..., z_{i_k}\} \cup \{z_{i'_1}, z_{i'_2}, ..., z_{i'_{k'}}\}$ is a complete bipartite graph. With similar arguments that before, it can be proved that it is a biclique. ■

Based on the relation between bicliques and bichromatic cliques we present the second characterization for biclique-Helly graphs.

Theorem 2.8 *A graph G is biclique-Helly if and only if*

1. *G contains neither triangles nor induced C_5's.*

2. *The graph $H(G)$ is bichromatic-Helly, relative to its canonical partition.*

Proof: Let G be a graph, and $U \cup W$ the canonical bipartition of $H(G)$. Suppose G is biclique-Helly. Then Condition 1 holds because of Lemma 2.2. The following observation is useful for proving Condition 2.

Remark 2.1 *Let $u_i \in U$ and $w_i \in W$ be the pair of vertices of $H(G)$, corresponding to $v_i \in V(G)$. Then no vertex of $H(G)$ is adjacent to both u_i and w_i.*

The reason why the above assertion is true is simple. Suppose $u_j \in U$ is adjacent to $u_i \in U$ in $H(G)$. Then v_i, v_j are not adjacent in G. The latter implies w_i, u_j not adjacent in $H(G)$. Similarly, $w_j \in W$ adjacent to u_i implies w_j not adjacent to w_i. Consequently, Observation 2.1 is true.

Denote by $\mathcal{C}$ a family of intersecting bichromatic cliques of $H(G)$. Let $\mathcal{B}$ be the family of bicliques of G, corresponding to $\mathcal{C}$. Observe that this is a 1-1 correspondence, because the pair of cliques of $H(G)$ which correspond to the same biclique of G are disjoint. Since G is biclique-Helly, the bicliques of G contain a common vertex $v_i \in V(G)$. Let B_j be the biclique of $\mathcal{B}$ corresponding to the bichromatic clique $C_j \in \mathcal{C}$. Because $v_i \in B_1$, it follows $u_i \in C_1$ or $w_i \in C_1$. Without loss of generality, assume $u_i \in C_1$. We show that $u_i \in C_j$, also for clique $C_j \in \mathcal{C}$, $j \neq 1$. Suppose the contrary and let $u_i \notin C_j$. Because $v_i \in B_j$, it follows $w_i \in C_j$. If $w_i \in C_1$ then u_i and w_i must be adjacent, which can not occur. Then $u_i \in C_1 \setminus C_j$ and $w_i \in C_j \setminus C_1$. Because C_1 and C_j intersect, there must be some vertex of $H(G)$ belonging to $C_1 \cap C_j$. Such a vertex would have to be simultaneously adjacent to u_i and w_i, which contradicts Observation 2.1. Consequently, C_j contains u_i, implying that $H(G)$ is indeed bichromatic-Helly.

Conversely, by hypothesis Conditions 1 and 2 are true for some graph G. We show that G is biclique-Helly. Let $\mathcal{B}$ be an intersecting family of bicliques of G. We construct a family $\mathcal{C}$ of bichromatic cliques of $H(G)$. Each $B_i \in \mathcal{B}$ corresponds in $H(G)$ to a pair of disjoint bichromatic cliques C_i' and C_i''. We choose $C_i \in \mathcal{C}$ as to be C_i' or C_i'', according to the following rule. Arbitrarily, choose $C_1 = C_1'$. Note that since B_1 and B_i intersect, C_1 must intersect C_i' or C_i''. Then, for $i > 1$, if C_1 and C_i' intersect then choose $C_i = C_i'$, and otherwise $C_i = C_i''$. First, we show that $\mathcal{C}$, as above obtained, is an intersecting family. Let $C_i, C_j \in \mathcal{C}$. If $i = 1$ then C_1 intersects any C_j, by construction. Let $i, j \neq 1$. Since B_i and B_j intersect there exists $v_s \in B_i \cap B_j$. Then C_i contains u_s or w_s. Without loss of generality, let $u_s \in C_i$. We will show that $u_s \in C_j$, meaning that C_i and C_j intersect. Suppose the contrary, $u_s \notin C_j$. Then $w_s \in C_j$. Consider the following alternatives for locating $C_1 \cap C_i$ and $C_1 \cap C_j$.

Case 1: $C_1 \cap C_i \cap U \neq \emptyset$ and $C_1 \cap C_j \cap U \neq \emptyset$

Let $u_i \in C_1 \cap C_i \cap U$ and $u_j \in C_1 \cap C_j \cap U$. If $u_i = u_s$ then C_1 also contains u_s, implying that u_j is adjacent to u_s, because $u_s, u_j \in C_1$. Since $w_s \in C_j$, u_j is also adjacent to w_s, contradicting Observation 2.1. Consequently, $u_i \neq u_s$. Similarly, $u_j \neq u_s$. Then u_i, u_j, u_s are distinct. Since $u_s, u_i \in C_i$, v_s and v_i are not adjacent in G. On the other hand, because $w_s, u_j \in C_j$, v_s and v_j are adjacent. Finally, v_i and v_j are not adjacent, because $u_i, u_j \in C_1$. Since $v_i, v_j \in B_1$, there exists $v_p \in B_1$ adjacent to both v_i, v_j. If v_p, v_s are adjacent then v_p, v_j, v_s form a triangle, which contradicts Condition 1. Then v_p, v_s are not adjacent. Because $v_i, v_s \in B_i$, there exists a vertex $v_q \in B_i$, simultaneously adjacent to v_i and v_s. We conclude that v_q is not adjacent neither to v_p nor v_j, otherwise G would contain a triangle. However in this situation, the vertices v_i, v_j, v_s, v_p, v_q induce a C_5 in G, not possible.

Case 2: $C_1 \cap C_i \cap U \neq \emptyset$ and $C_1 \cap C_j \cap W \neq \emptyset$

Let $u_i \in C_1 \cap C_i \cap U$ and $w_j \in C_1 \cap C_j \cap W$. Similarly as in Case 1, we know that v_i and v_s are not adjacent. Because $w_s, w_j \in C_j$, v_s and v_j are also not adjacent, and $u_i, w_j \in C_1$ implies v_i, v_j to be adjacent. Since $v_s, v_i \in B_i$, there exists some vertex $v_p \in B_i$ adjacent to v_s and v_i. We know that v_p, v_j are not adjacent, otherwise G would contain a triangle. Also, because $v_s, v_j \in B_j$, there exists $v_q \in B_j$, adjacent to v_s, v_j. If v_p, v_q or v_q, v_i

are adjacent then G contains a triangle, otherwise the vertices v_i, v_j, v_s, v_p, v_q induce a C_5, which is forbidden by Condition 1. Therefore Case 2 can also not occur.

Case 3: $C_1 \cap C_i \cap W \neq \emptyset$ and $C_1 \cap C_j \cap U \neq \emptyset$

With the similar arguments as in Case 1, we conclude that Case 3 can not occur.

The alternative $C_1 \cap C_i \cap W \neq \emptyset$ and $C_1 \cap C_j \cap W \neq \emptyset$ is Case 1. The conclusion is that C_i and C_j intersect, meaning that $\mathcal{C}$ is an intersecting family. By Condition 2, $H(G)$ is bichromatic-Helly. Then $H(G)$ contains a vertex common to all cliques of $\mathcal{C}$. Such a vertex corresponds in G to a vertex common to all bicliques of $\mathcal{B}$. Therefore G is biclique-Helly. ∎

An algorithm for recognizing whether or not a given graph is biclique-Helly follows directly from Theorem 2.8. Let G be a graph.

Algorithm 2.3 ***Biclique-Helly graphs (II)*** *First verify if G has triangles or $C_5's$. Then construct the graph $H(G)$. Run the Algorithm 2.2 for recognizing bichromatic-Helly graphs applied to the graph $H(G)$.*

Constructing the graph $H(G)$ takes $O(|V(G)|^2)$ steps. The Algorithm 2.3 terminates within $O(|V(G)|^5)$ steps.

2.5 Neighborhood-Helly graphs and other related classes

In this section we relate biclique-Helly graphs to the classes of clique-Helly, neighborhood-Helly, both open and closed neighborhoods are considered, disk-Helly and clique-Helly.

Recall that a graph is neighborhood-Helly (disk-Helly, clique-Helly) when its family of neighborhoods (disks, cliques, respectively) is Helly. We remark that the classes of open and closed neighborhood-Helly overlap. Neighborhood Helly graphs appears in the context of retracts, for example [4]. Consider the class of open neighborhood-Helly graphs. We describe a characterization of this class of graphs, in terms of extensions. It will be useful for our purpose of relating it to biclique-Helly graphs. Start with the following lemma.

Lemma 2.5 *Let G be a open neighborhood-Helly graph. Then G has no triangles.*

Proof: Let G be a graph and T be a triangle of G, with vertices v_1, v_2 and v_3, respectively. Consider the family of neighborhoods $W = \{N(v_1), N(v_2), N(v_3)\}$. Since this family is intersecting, there exists a vertex v_4 in G adjacent to v_1, v_2 and v_3. Now consider the intersecting family $W_4 = W \cup \{N(v_4)\}$. As $v_j \notin N(v_j)$, there exists a vertex $v_5 \neq v_j$, $j = \{1, 2, 3, 4\}$, adjacent to v_1, v_2, v_3 and v_4. Following this procedure, we can construct the family $W_i = W \cup \{N(v_i)\}$ of intersecting neighborhoods and assure the existence of a vertex $v_{i+1} \neq v_j$, adjacent to v_j for $1 \leq j \leq i$. However, G is finite, leading to a contradiction. Then G can not have triangles. ■

Let S be an independent set of G, $|S| = 3$, and $G[S^*]$ its extension. Denote by $S_2^* \subseteq S^*$ the subset of vertices which are adjacent to at least two vertices of S.

The following Theorem gives a characterization of open neighborhood-Helly graphs.

Theorem 2.9 *A graph G is open neighborhood-Helly if and only if G has no triangles and for every independent set S, $|S| = 3$, S^* contains a vertex adjacent to all vertices of S_2^*.*

Proof: Let $S = \{v_1, v_2, v_3\}$ be a set of non adjacent vertices and $G[S^*]$ its extension. Consider the vertices of S_2^*. It is clear that the family of open neighborhoods of vertices of S_2^* intersect. By hypothesis, there is a vertex w which belongs to every neighborhood of the family. Then, w is adjacent to all vertices of S_2^*.

Conversely, let G be a graph satisfying the hypothesis. By contrary, assume that G is not open neighborhood-Helly. For $v_i \in V(G)$, denote by N_i the open neighborhood of v_i. Let $N = \{N_1, N_2, ..., N_l\}$, $l \geq 3$, be a minimal subfamily of the neighborhoods of G which is not Helly. Then $N - N_i$ is a Helly family, meaning that it contains a common element $w_i \in N_j$, $j \neq i$. We claim that $v_i \neq w_j$, $1 \leq i, j \leq l$. Clearly, if $i \neq j$ the claim holds because $w_i \in N_j$ and $v_j \notin N_j$. Examine the alternative $v_i = w_i$. It implies that v_i is adjacent to v_k for any $k \neq i$. Since N_i and N_k intersect, there exists a vertex w which forms a triangle with v_i and v_k, a contradiction. Consequently, $w_i \neq v_j$. Furthermore, w_i, w_j must be distinct, for $i \neq j$. Finally, we assert that w_i, w_j are non adjacent. Otherwise, if w_i, w_j are adjacent, consider any v_k, $k \neq i, j$. By the above claim, $v_k \neq w_i, w_j$. Since $w_i, w_j \in N_k$, it follows that w_i, w_j, v_k form a triangle, a contradiction. Then $S = \{w_1, w_2, w_3\}$ is an independent set. Consider the extension $G[S^*]$, where $S = \{w_1, w_2, w_3\}$. Observe that as every N_i contains at least two of the vertices of S, v_i is adjacent to at least two vertices of S and then belongs to S_2^*. By hypothesis, there is a vertex w adjacent to every vertex of S_2^*. Then, w belongs to N_i for $1 \leq i \leq l$, meaning that G is open neighborhood-Helly. ■

The characterization given by Theorem 2.9 leads to a possibly faster recognition algorithm than the one resulting by the application of the general test for Helly families ([10]). Furthermore, the characterization of Theorem 2.9 is also useful for proving relations to other classes of Helly families.

The following Proposition gives conditions for a graph with induced C_6 to be open neighborhood-Helly. It will be useful in the characterization of Hereditary neighborhood-Helly graphs given in Chapter 3.

Proposition 2.1 *Let G be an open neighborhood-Helly graph. If G has C_6 as an induced subgraph, then it contains the graph J of Figure 2.7 as induced subgraph.*

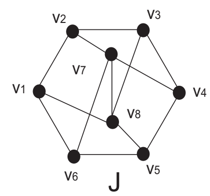


Figure 2.7: Graph J

Proof: Suppose G has the graph C_6 as an induced subgraph. Let v_1, v_2, v_3, v_4, v_5 and v_6 be the vertices of C_6. Consider the neighborhoods $N(v_1)$, $N(v_3)$, $N(v_5)$. As they have pair intersection, there exists a vertex $v_8 \neq v_i$, v_8 adjacent to v_i, for $i = 1, 3, 5$. Consider now the neighborhoods $N(v_2)$, $N(v_4)$, $N(v_6)$ and $N(v_8)$. As they have pair intersection, there exists a vertex $v_7 \neq v_i$, v_7 adjacent to v_i, for $i = 2, 4, 6, 8$. By Lemma 2.5, G is has no triangles, then vertices v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 induce the graph J. ∎

A characterization of closed neighborhood-Helly graphs, restricted to the triangle-free class, is given in the following Theorem.

Theorem 2.10 *Let G be a triangle-free graph. Then G is $\{C_4, C_5, C_6\}-free$ if and only if it is closed neighborhood-Helly.*

Proof: Let $N' = \{N_{i_1}, N_{i_2}, ..., N_{i_s}\}$ be a minimal subfamily of closed neighborhoods which does not verify the Helly property. Consider the vertex w_1 which belongs to every N_{i_j}, $j \neq 1$ and vertices w_2 and w_3, $w_2 \in N_{i_j}$, $j \neq 2$, and $w_3 \in N_{i_j}$, $j \neq 3$. Is is clear that $w_{i_j} \neq v_{i_j}$, $j = 1, 2, 3$.

Suppose it is the case that $w_j = v_{i_s}$ for some j, s. Without loss of generality, suppose $w_1 = v_{i_2}$. It means that v_{i_2} is not adjacent to v_{i1}. Then, $v_{i1} \neq w_3$. Also, v_{i_2} is adjacent to v_{i_3} and w_2 is not adjacent to v_{i_2}. Then, $w_1, w_2 \neq v_{i_3}$. As G is $C_4 - free$, w_2 and w_3 are not adjacent. Then $v_{i_1} \neq w_2$. If v_{i_1} is adjacent to v_{i_3}, it forms a C_4, otherwise, they induce a C_5. Then this case can not occur.

Now, suppose $w_j \neq v_{i_s}$ for all j, s. As G is $C_6 - free$, the the sets $\{w_j\}$ and $\{v_{i_s}\}$ can not be both independent sets. Suppose w_2 is adjacent to w_3. As G is $\{C_5, C_4\} - free$, w_1 is adjacent to w_2 and w_3. Then, these vertices induce the Hajós graph, contradicting the hypothesis. If v_{i_1} is adjacent to

v_{i_2}, with the same argument we conclude that these six vertices induce the Hajós graph. It follows that G is hereditary closed neighbourhood Helly.

Conversely, suppose G has C_4 as induced subgraph. Consider $N[v_1]$, $N[v_2]$, $N[v_3]$ and $N[v_4]$. They have a pairwise intersection. Then there is a $v \in N[v_i]$, i=1,...,4., such that among vertices v_1, v_2, v_3, v_4, v there is an induced triangle. Absurd.

Analogously, suppose G has C_5 as induced subgraph. Consider $N[v_1]$, $N[v_2]$, $N[v_3]$, $N[v_4]$ and $N[v_5]$. They have a pairwise intersection. Then, there is a vertex $v \in N[v_i]$, i=1,...,5. Then, among $v_1, v_2, v_3, v_4, v_5, v$ there is an induced triangle. For the graph C_6, consider neighborhoods $N[v_1]$, $N[v_3]$ and $N[v_5]$. They form an intersecting family. Then, there is a vertex $v \in N[v_i]$, i=1,3,5. Then, among vertices v_1, v_3, v_5, v there is an induced triangle. ■

We study some relations between biclique- Helly graphs and open neighborhood-Helly graphs. The following results relate both classes of graphs. For both proofs, we use the characterizations we have presented of the mentioned classes.

Theorem 2.11 *Let A, B, C and D be the graphs of Figure 6.3. Let G be a open neighborhood-Helly graph, with no C_5's, and such that every induced subgraph A of it extends to one of the induced subgraphs B, C or D . Then G is biclique-Helly.*

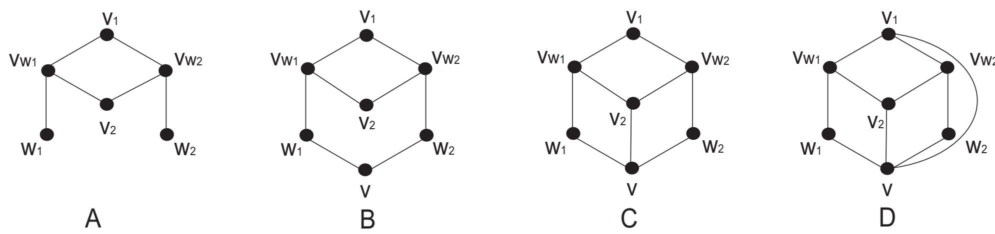


Figure 2.8: Graphs A, B, C and D.

Proof: By Lemma 2.5, G has no triangles. We prove that each non empty extension has an edge dominator. Let $S \subseteq V(G)$, $|S| = 3$, and $G[S^*]$ the extension of S. By Lemma 2.1, $G[S^*]$ is a bipartite graph. In the case where there are only two vertices of S in S^*, clearly the theorem follows.

Discuss the case where $S \subseteq S^*$. First, suppose that S is an independent set. By Theorem 2.9 there exists a vertex v in S^* adjacent to every vertex of S_2^*. As $G[S^*]$ is a bipartite graph, it follows that v is an edge dominator of $G[S^*]$, as required.

Consider the case when the subset $S = \{v_1, v_2, v_3\}$ induces a P_3, with v_1, v_2 non adjacent. Let X and Y be the bipartiton of $G[S^*]$, v_1, $v_2 \in X$. In the case $|X| = 2$, it is clear that v_3 is an edge dominator of $G[S^*]$. We analyze the case $|X| \geq 3$. First we prove that the family $\{N(v), v \in X\}$ is intersecting. Suppose the contrary. Let w_1, w_2 be two vertices of X with no common neighbor. If w_1 or w_2 coincides with v_1 or v_2, it follows that there is a vertex adjacent to v_1,v_2,w_1,w_2, implying that w_1 and w_2 have indeed a common neighbor. Assume $w_1, w_2 \neq v_1, v_2$.

By definition of $G[S^*]$, there exist vertices $v_{w_1}, v_{w_2} \in Y$, $v_{w_1} \neq v_{w_2}$ adjacent to w_1 and w_2 respectively, and both adjacent to v_1 and v_2. As w_1 and w_2 have no common neighbor, vertices $w_1, v_{w_1}, v_1, v_{w_2}, v_2$ induce the graph A which does not extend to any of the graphs B, C or D, what leads to a contradiction. Then $\{N(v), v \in X\}$ is an intersecting family. As G is open neighborhood-Helly, there is a vertex v adjacent to every vertex $w \in X$, $v \in S^*$. Then, v is an edge dominator in $G[S^*]$. By Theorem 2.5, G is biclique-Helly.

Examine the case where S induces the complement of a P_3 in G. Similarly as in Lemma 1, by applying the definition of $G[S^*]$, we conclude that G contains a triangle or a C_5, contradicting the hypothesis. Finally, the case where S induces a triangle also does not occur, completing the proof. ■

The converse for a restricted family of graphs is given by the following Theorem.

Theorem 2.12 *Let G be a $cube-free$ graph. If G is biclique-Helly then it is open neighborhood-Helly.*

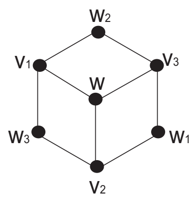


Figure 2.9: The Cube graph

Proof: We prove that for every independent set of three vertices $S = \{v_1, v_2, v_3\}$, its extension $G[S^*]$ has a vertex v adjacent to S_2^*. Recall that as G is biclique Helly, by Lemma 1, $G[S^*]$ is a bipartite graph. Let X, Y be a bipartition of vertices of $G[S^*]$, $S \subseteq X$. Suppose there is no edge dominator in X. By Theorem 2.5, since G is biclique-Helly, $G[S^*]$ has an edge dominator w. Then $w \in Y$. As none of the vertices $v_i, i = 1, 2, 3$ are edge dominators, there exist distinct vertices $w_1, w_2, w_3 \in Y$ with v_i adjacent to w_j precisely when $i \neq j$, for $i, j = 1, 2, 3$. Then, vertices $v_1, w_3, v_2, w_1, v_3, w_2, w$ induce the *cube* graph, absurd. Consequently, $G[S^*]$ has an edge dominator in X. By Theorem 2.9, G is open neighborhood-Helly. ■

The relation between closed neighborhood-Helly graphs and biclique-Helly graphs is given in Chapter 3. We prove that closed neighborhood-Helly graphs with no triangles are also biclique-Helly.

Next we relate biclique-Helly graphs, clique-Helly and disk-Helly graphs, and open and closed neighborhood-Helly graphs.

Proposition 2.2 *If G is biclique-Helly or open neighborhood-Helly then G is clique-Helly*

Proof: Biclique-Helly and open neighborhood-Helly graphs do not contain triangles and so they are clique-Helly. ∎

Proposition 2.3 *Let G be a graph with no triangles. Then if G is closed neighborhood-Helly, then it is open neighborhood-Helly.*

Proof: Let $W = \{N(v_1), N(v_2), N(v_3), ..., N(v_k)\}$ be a pair intersecting family of open neighborhoods. Consider the family $W' = \{N[v_1], N[v_2], ..., N[v_k]\}$. By hypothesis there is a vertex v which belongs to $N[v_i]$, for $1 \leq i \leq k$. Suppose $v = v_j$ for some j, $1 \leq j \leq k$. Consider a vertex v_i, $i \neq j$. As $N(v_i) \cap N(v_j) \neq \emptyset$, there exists a vertex v' adjacent to v_i and v_j. As $v_j \in N(v_i)$, the vertices v_i, v_j and v' induce the a triangle which is an absurd. Then, $v \neq v_j$ for $1 \leq j \leq k$. It follows that v belongs to $N(v_i)$ for $1 \leq i \leq k$. ∎

Finally, consider disk-Helly graphs.

Proposition 2.4 *Let G be a disk-Helly graph with no triangles. Then G is biclique-Helly.*

Proof: First, we will prove that if G is disk-Helly and has no triangles, then it is $C_5 - free$. By contrary assume that the vertices $v_1, \ldots, v_5$ form a C_5. Consider the disks $D_1(v_2)$, $D_1(v_3)$ and $D_1(v_5)$. Since these disks are a Helly family, there is a vertex v adjacent to v_2, v_3, v_5. Then v forms a triangle with v_2 and v_3, contradicting the hypothesis. Now we will prove that the family of bicliques of G is Helly. Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be an intersecting family of bicliques. For each vertex v of each biclique B_j of $\mathcal{B}$, consider the disk $D_2(v)$. Observe that if $v \in B_j$, the vertices of B_j are included in the set of vertices of $D_2(v)$. Consequently, the family of disks $\{D_2(v)/v \in \cup_1^k B_j\}$ is also intersecting. As G is disk-Helly, there is a vertex z which belongs to every $D_2(v)$. We are going to prove that z belongs to every biclique B_i of $\mathcal{B}$. Let X_i, Y_i be the bipartition of the vertices of B_i into independent sets. Suppose z does not belong to B_i. As G has no triangles, z can not be adjacent simultaneously to a vertex of X_i and a vertex of Y_i. Then, we can assume that $X_i \cup \{z\}$ is an independent set. Then there is a vertex $w \in Y_i$ which is not adjacent to z. As $z \in D_2(w)$, there exists a vertex z' adjacent to z and w. Let v be a vertex of X_i. If z' is adjacent to v then z', v, w forms a triangle contradicting the hypothesis. As $z \in D_2(v)$, there is a vertex z'' adjacent to z and v. It follows that vertices v, w, z', z, z'' induce

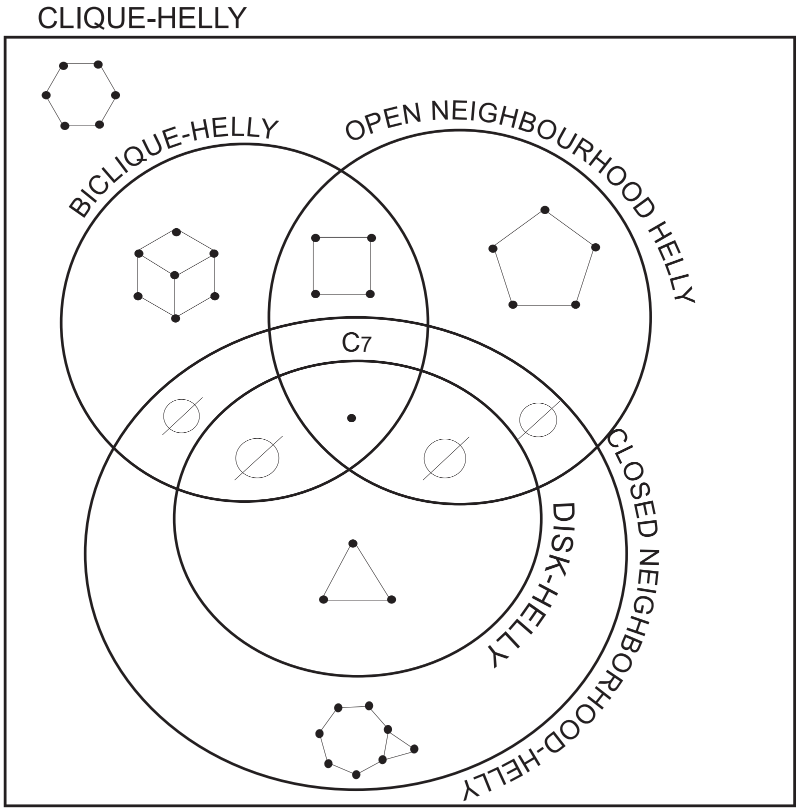


Figure 2.10: Intersections between clique-Helly, biclique-Helly, open neghborhood-Helly, closed neighborhood-Helly and disk-Helly graphs.

a C_5 or contain a triangle as an induced subgraph which is an absurd. Then, z belongs to B_i for $1 \leq j \leq k$. ■

It is clear that disk-Helly graphs are closed neighborhood-Helly graphs.

The result of Theorem 2.4 can be obtained as a corollary of Theorem 3.5 of Chapter 3, which relates closed neighborhood-Helly graphs, triangle-free, with biclique-Helly graphs.

In Figure 2.10 the relations between the classes we have studied in this Chapter are shown. The empty intersections follows directly from the results we have proved, considering that open neighborhood-Helly and a biclique-Helly graphs have no triangles.

Chapter 3

Helly hereditary classes

3.1 Introduction

In Chapter 2 we have studied Helly families of a graph. A question would be to characterize graphs for which the Helly property is preserved for every induced subgraph. Considering the graphs of Chapter 2, we refer to hereditary classes of clique-Helly, biclique-Helly, neighborhood-Helly and disk-Helly graphs. Hereditary clique-Helly graphs have been characterized in [46], while [22] (c.f. [16]) contains a characterization of hereditary disk-Helly graphs. In this Chapter, we describe forbidden subgraph characterizations for the classes of hereditary biclique-Helly and hereditary neighborhood-Helly graphs. Both open and closed neighborhoods are considered. All graphs in these forbidden families are of fixed size. In fact they have at most 8 vertices. Consequently, the characterizations imply polynomial time recognition for hereditary biclique-Helly, hereditary open neighborhood-Helly and hereditary closed neighborhood-Helly graphs.

A graph G is *hereditary clique-Helly* when every of its induced subgraphs is clique-Helly. Similarly, define *hereditary biclique-Helly*, *hereditary open neighborhood-Helly* and *hereditary closed neighborhood Helly.* See examples in Figure 3.1.

Figure 3.1: Examples of Hereditary biclique-Helly, and (open and closed) neighborhood-Helly graphs

Hereditary clique-Helly graphs have been characterized as follows.

Theorem 3.1 *[46]: A graph G is hereditary clique-Helly if and only if it does not contain as induced subgraphs neither the Hajós graph, neither any k-extended Hajós graph, k=1,2,3.*

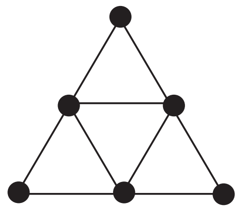


Figure 3.2: The Hajós graph

Proof: Suppose H is not a clique-Helly subgraph of G. Let $\mathcal{C} = \{C_1, ..., C_k\}$ be a minimal family of intersecting cliques of H without common intersection in H. Observe that $k \geq 3$. Consider $\mathcal{C} \setminus \{C_1\}$. There is a vertex v_1 in every clique C_i, for $i \neq 1$. Analogously, there are vertices v_2, and v_3 such that $v_j \in V_i$ for every $i \neq j$, $j = 2, 3$.

It is clear that vertices v_1, v_2, v_3 induce a triangle of H. As v_1 does not belong to C_1, there is a vertex w_1 in C_1 not adjacent to v_1 but adjacent to v_2 and v_3. Analogously, there exist vertices w_2, w_3, w_2 not adjacent to v_2 and adjacent to v_1 and v_3, and w_3 not adjacent to v_3, adjacent to v_1 and v_2. It follows that among vertices $v_1, v_2, v_3, w_1, w_2, w_3$, there is an induced graph of Figure 3.2.

For the converse, it is clear that none of the forbidden subgraphs are clique Helly. ∎

The following is a characterization for hereditary disk-Helly graphs.

Theorem 3.2 *[22]: A graph G is hereditary disk-Helly if and only if it is chordal and does not contain the Hajós graph as induced subgraph.*

Proof: If G is disk-Helly, by Theorem 2.4 it is clique Helly. Then if G is hereditary disk-Helly, it is hereditary clique-Helly and, by Theorem 3.1 G does not contains the Hajós graph as an induced subgraph. It is easy to prove that C_k is not disk-Helly, for $k \geq 4$. Consider the family of disks $D1 = \{D_{|(k-1)|/2}\}$ if k is odd and $D2 = \{D_{|(k-2)|/2}\}$ if k is even. Both are intersecting families without a common vertex. Then G is chordal.

Conversely, let H be a minimal subgraph that is not disk-Helly. If it is not disk-Helly, as G has neither C_4 nor the Hajos graph as induced subgraphs, by Theorem 3.1, H is clique Helly. By Theorem 2.4, it follows that H is not dismantable. This means that H has not dominated vertices which implies that H has no simplicial vertices. Since chordal graphs are have simplicial vertices, H can not be cordal, leading to an absurd. [20]. ∎

In Section 2, we describe a characterization for hereditary biclique-Helly graphs while the proposed characterizations for hereditary open and closed neighborhood-Helly graphs are in Section 3. Relations among these classes are formulated in Section 4.

3.2 Hereditary biclique-Helly graphs

In this Section, we describe a characterization by a finite family of forbidden subgraphs for the class of hereditary biclique-Helly graphs. Recall the following definitions.

Let $S \subseteq V(G)$, $|S| = 3$. Denote by $\mathcal{B}_S$ the family of bicliques of G, each of them containing at least two vertices of S. Consider the graph $G_{\mathcal{B}_S}$ and denote its vertex set by $S^* \subseteq V(G)$. The induced subgraph $G[S^*]$ is extension of S. Clearly, $G_{\mathcal{B}_S}$ is a spanning subgraph of $G[S^*]$. The lemma below is useful. Recall that we proved in Chapter 2 that if G is a graph with neither triangles nor C_5's, then each of its extensions is a bipartite graph (Lemma 2.1).

The characterization of hereditary biclique-Helly graphs is next formulated in the following Theorem.

Theorem 3.3 *A graph G is hereditary biclique-Helly if an only if it does not contain any of the graphs of Figure 3.3, as induced subgraphs.*

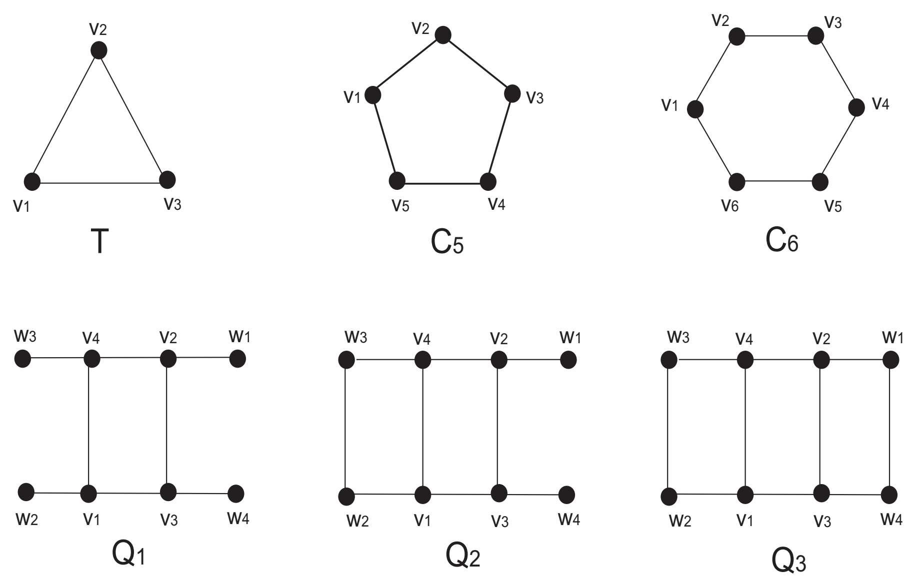


Figure 3.3: Graphs T, C_5, C_6, Q_1, Q_2 and Q_3

Proof: To prove that the graphs of Figure 3.3 are not biclique-Helly, we show a pairwise intersecting family $\mathcal{B}$ of bicliques with no common vertex, in each case. For the triangle, $\mathcal{B} = \{\{v_1, v_2\}, \{v_1, v_3\},\{v_2, v_3\}\}$ and $\mathcal{B} = \{\{v_1, v_2, v_3\},$

$\{v_3, v_4, v_5\}$, $\{v_1, v_2, v_5\}\}$ for the C_5. For the C_6, the family is $\mathcal{B} = \{\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_1, v_5, v_6\}\}$. Finally, for the graphs Q_1, Q_2 and Q_3, $\mathcal{B} = \{\{v_1,$ w_2, v_3, $v_4\}$, $\{v_2, v_3, v_4, w_1\}$,$\{v_3, w_4, v_1, v_2\}$, $\{v_1, v_2, v_4, w_3\}\}$.

Conversely, let G be a graph which does not contain any of the graphs of Figure 3.3, as an induced subgraph. Suppose it is not hereditary biclique-Helly. Let H be an induced subgraph which is not biclique-Helly, and $\mathcal{B}$ a non Helly family of bicliques of H. We can choose $\mathcal{B} = \{B_1, \ldots, B_k\}$ as a minimal such family. Clearly, $k \geq 3$. As for every i, $\mathcal{B} \setminus B_i$ is a Helly family, there exists a vertex v_i which belongs to B_j and not to B_i, for all $j \neq i$. Let $S = \{v_1, \ldots, v_k\}$ be the collection of such vertices. Write $\alpha_S = \alpha(H[S])$.

First, we show that $\alpha_S \leq 2$. By contrary, suppose that $\{v_1, v_2, v_3\} \subseteq S$ is an independent set of H. Since H has no triangles and $v_i \notin B_i$, there exists a vertex $w_i \in B_i$ adjacent to v_j and not to v_i, for $j \neq i$ and $1 \leq i \leq 3$. Then, $\{w_1, w_2, w_3\}$ is also an independent set. In the latter situation, $\{v_1, w_2, v_3, w_1, v_2, w_3\}$ induces a C_6, which is forbidden. Consequently, indeed $\alpha_S \leq 2$. See Figure 3.4.

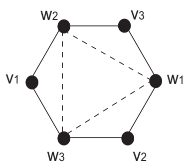


Figure 3.4: C_6 induced by $\{v_1, w_2, v_3, w_1, v_2, w_3\}$

In the sequel, we discuss the possible values k can assume.

Let $k \geq 5$. Denote $S' = \{v_1, v_2, v_3, v_4\}$. Clearly, $S' \subseteq B_5$. For this reason and considering that $\alpha_S \leq 2$, we know that S' induces a C_4 in H. Let v_1v_3 and v_2v_4 be the non adjacent pairs in S'. Again, because $\alpha_S \leq 2$, v_5 must be adjacent to at least one vertex of S', say adjacent to v_1. Because H has no triangles, v_5 can not be adjacent to v_2, nor to v_4. However, in this situation, $\{v_2, v_4, v_5\}$ is an independent set of size 3, a contradiction. Consequently, $k < 5$.

Next, discuss the case $k = 4$. Let $S' = \{v_1, v_2, v_3\} \subseteq S$. Since $S' \subseteq B_4$ and $\alpha_S \leq 2$, S' induces a P_3 in H. Let v_1 and v_2 be the non adjacent vertices in S'. As $v_1, v_2, v_4 \in B_3$ and $\alpha_S \leq 2$, it follows that S must induce a C_4 in H. On the other hand, because $v_i \notin B_i$ and $v_j \in B_i$ for $j \neq i$, each B_i has an additional vertex $w_i \in B_i$, $1 \leq i \leq 4$ with the following properties: because $v_1 \notin B_1$ and H has no triangles, w_1 is adjacent to v_2 and not adjacent to v_1, v_3, v_4. Similarly, w_2 is adjacent to v_1, and not to v_2, v_3, v_4, and w_3 is adjacent to v_4 and not v_1, v_2, v_3, while w_4 is adjacent to v_3 and not v_1, v_2, v_4. See Figure 3.5.

Examine the possible adjacencies among the w_i's. If w_1w_2 or w_3w_4 are adjacent then H contains a C_5, forbidden. So, assume these pairs are not adjacent. Let $W = \{w_1w_3, w_1w_4, w_2w_3, w_2w_4\}$ be the set of the other possible pairs of w_i's and denote $W' = \{v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4\}$. If none of the

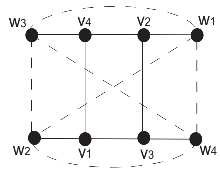


Figure 3.5: Case $k = 4$

pairs of W is adjacent then W' induces the graph Q_1. When exactly one of the pairs of W is adjacent then W' forms the graph Q_2. When precisely the pairs w_1w_3 and w_2w_4, or w_1w_4 and w_2w_3 are adjacent then W' induces Q_3. Finally, when at least w_1w_3 and w_1w_4, or w_2w_3 and w_2w_4 are adjacent then a C_6 exists. Consequently, $k = 4$ is not possible.

Next, consider the case $k = 3$, and let $S = \{v_1, v_2, v_3\}$. Denote by $\mathcal{B}_S$ the subset of bicliques of H containing at least two vertices of S. Then $\{B_1, B_2, B_3\} \subseteq \mathcal{B}_S$. Discuss the possibilities for S. Clearly, S is not a triangle. Since $\alpha_S \leq 2$, S is also not an independent set. Suppose S induces a $\overline{P_3}$, with v_1, v_2, adjacent. As $v_2, v_3 \in B_1$, there is a vertex $w_1 \in B_1$ adjacent to v_2 and v_3. Similarly, there exists $w_2 \in B_2$, with w_2 adjacent to v_1 and v_3. In this case, either a triangle is formed or $\{v_1, v_2, w_1, v_3, w_2\}$ induces a C_5, which is forbidden.

Examine the remaining alternative, that is S induces a P_3. By Lemma 1, $H[S^*]$ is bipartite. Let $X \cup Y$ be a bipartition of it. Let v_1, v_3 be the non adjacent pair of vertices of S. Let $v_1, v_3 \in X$ and $v_2 \in Y$. Because $v_i \notin B_i$ and $v_j \in B_i$, for $j \neq i$, there is a vertex $w_i \in B_i$, $w_i \neq v_j$, $i, j = 1, 3$, such that $w_1 \in Y$ is adjacent to v_3 and not adjacent to v_1 and v_2, while the vertex $w_3 \in Y$ is adjacent to v_1 and not to v_3 and v_2. Also, as B_2 is a biclique, there exists some vertex $w_2 \in Y$ adjacent to v_1 and v_3. Because $v_2 \notin B_2$, there is also a vertex $w_2' \in X \cap B_2$ which is adjacent to w_2 and not to v_2. Since $\{B_1, B_2, B_3\}$ is not a Helly family, w_2 is not common to these three bicliques. Without loss of generality, assume $w_2 \notin B_3$. Then there exists a vertex $w_3' \in X$ which belongs to B_3, w_3' adjacent to v_2 and w_3, and not to w_2. See Figure 3.6.

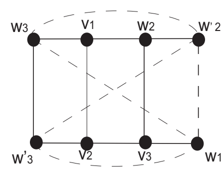


Figure 3.6: Case $k = 3$

Let $W' = \{v_1, v_2, v_3, w_1, w_2, w_2', w_3, w_3'\}$ and $W = \{w_1w_2', w_1w_3', w_3w_2'\}$. If none of the pairs of vertices of W are adjacent then W' induces the graph Q_2. Otherwise, if w_1w_2' is the only adjacent pair of W then W' induces Q_3.

In the remaining alternatives, a C_6 exists among the vertices of W'. All cases lead to forbidden subgraphs.

Consequently, $k = 3$ is also not possible. However, $k \geq 3$. This contradiction completes the proof. ∎

The characterization given in Theorem 3.3 leads to a polynomial time recognition algorithm which consist of checking if the graph G has any of the forbidden subgraphs.

We remark that, as bipartite hereditary biclique-Helly graphs does not contain the graph C_6 as induced subgraph, (i.e., the the cocktail party CP_3 of order 3), the number of bicliques is at most $(\mid V_1 \mid\mid V_2 \mid)^2$, according to Theorem 1.3.

3.3 Hereditary open and closed neighborhood Helly graphs

In this Section we study the Helly property relative to the family of open and closed neighborhoods in every subgraph. We describe below characterizations for hereditary open and closed neighborhood-Helly graphs.

Theorem 3.4 *Let G be a graph. Then G is hereditary open neighbourhood-Helly if and only if G does not contain C_6 nor triangles as induced subgraphs.*

Proof: Let G be a hereditary open neighborhood-Helly graph. Clearly, any subset $\{v_1, v_2, v_3\} \subseteq V(G)$ does not induce a triangle in G, otherwise $N(v_1)$, $N(v_2)$, $N(v_3)$ pairwise intersect, but there is no common vertex. Suppose G contains a C_6. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the ordering of the vertices in this cycle. Since $N(v_1), N(v_3), N(v_5)$ also pairwise intersect with no common vertex, we conclude that G neither contains a C_6.

Conversely, assume the theorem is not true. Then G contains an induced subgraph H which is not open neighborhood-Helly. Consider a minimal family $N = \{N(v_1), N(v_2), ..., N(v_l)\}$ of intersecting neighborhoods with no common vertex in H. Then $l \geq 3$. Because of the minimality, there is a vertex $w_i \in N(v_j)$ precisely when $i \neq j$, for all $1 \leq i \leq l$. Since $v_i \notin N(v_i)$, we conclude that $w_i \neq v_j$, for all $i \neq j$. Moreover, we show that $w_i \neq v_i$ for all i. By contrary, let $w_i = v_i$. Then, v_i and v_j are adjacent. On the other hand, since $N(v_i)$ and $N(v_j)$ intersect, there is a vertex w which forms a triangle with v_i and v_j, which is forbidden by hypothesis. Consequently, $w_i \neq v_j$, for $1 \leq i, j \leq l$. We claim that $w_1, ..., w_l$ form an independent set in G. The latter is true because, if w_i, w_j are adjacent, the fact that $w_i, w_j \in N(v_k)$, for $k \neq i, j$, implies that w_i, w_j, v_k form a triangle of G, contradicting the hypothesis. Similarly, $v_1, ..., v_l$ also form an independent set, otherwise $v_i \in N(v_j)$ implies that v_i, v_j, w_k, $k \neq i, j$ form a triangle. In this situation, $w_i, w_j, w_k, v_i, v_j, v_k$ induce a C_6 in G, impossible. The proof is complete. ∎

As a corollary of the Theorem 3.4, we can relate open neighborhood-Helly graphs with hereditary open neighborhood-Helly.

Next we consider the closed neighborhood-Helly class. We give a characterization of hereditary closed neighborhood-Helly graphs in terms of forbidden subgraphs.

Theorem 3.5 *A graph G is hereditary closed neighborhood-Helly if and only if it does not contain C_4, C_5, C_6 nor the Hajós graph as induced subgraphs.*

Proof: Suppose G contains the Hajós graph H. The family of the closed neighborhoods of the three vertices with degree two in H is intersecting and has no common vertex. Consequently, G can not contain the Hajós graph. Suppose G contains a C_4. The family of closed neighborhoods of its four vertices is intersecting and has no common vertex in its induced subgraph. Therefore, no $C_4's$ can exist. Similarly, the families of the closed neighborhoods of three vertices in a C_5, two of them non adjacent, and the closed neighborhoods of the three mutually non adjacent vertices in a C_6, are both intersecting and have no common vertex. Consequently, G does not contain neither C_5 nor C_6.

Conversely, by hypothesis G contains neither the Hajós graph, nor any of C_4, C_5,C_6 as induced subgraphs. By contrary, assume that G is not hereditary closed neighborhood-Helly. Then G contains an induced subgraph H which is not closed neighborhood-Helly. Let $N = \{N[v_1], N[v_2], ..., N[v_l]\}$ be a minimal such intersecting family of H with no common vertex. Clearly, $l \geq 3$. By the minimality of l, there exist vertices w_i, such that $w_i \in N[v_j]$, exactly for $i \neq j$. Compare w_i and v_j. It is clear, $v_i \neq w_i$. Suppose $w_i = v_j$, for some i, j. Without loss of generality, let $w_1 = v_2$. Then, v_1, v_2 are not adjacent, which implies $v_1 \neq w_3$. The latter means that $w_1, w_2 \neq v_3$. In this situation, if v_1, v_3 are adjacent, the vertices w_1, w_3, v_1, v_3 induce a C_4, forbidden. Consequently, v_1, v_3 are not adjacent. Examine vertex w_2. It follows that when w_2, w_3 are adjacent, the vertices w_1, w_2, w_3, v_3 induce a C_4, and otherwise w_1, w_2, w_3, v_1, v_3 induce a C_5 in G. Hence, this assumption cannot occur. Finally, assume $w_i \neq v_j$, for all i, j. Since G does not contain a C_6, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ can not be both independent sets. Suppose w_2, w_3 are adjacent. Since G does not contain neither C_4 nor C_5, we conclude that w_1 must be adjacent to both w_2, w_3. In this situation, if $\{v_1, v_2, v_3\}$ is an independent set, $w_1, w_2, w_3, v_1, v_2, v_3$ induce the Hajós graph, otherwise, they induce a C_4. In any alternative, a forbidden subgraph arises. The alternative v_2, v_3 to be adjacent instead of w_2, w_3, is similar, terminating the proof. ∎

3.4 Relations among the hereditary classes

In this Section we relate the classes of hereditary biclique-Helly graphs, neighborhood-Helly graphs, both open and closed, hereditary disk-Helly and hereditary clique-Helly graphs.

We obtain two corollaries of the characterizations formulated in the previous Section.

The corollary below relates hereditary open neighborhood-Helly graphs to the open neighborhood-Helly graphs. It is a consequence of Theorem 3.4 and Lemma 2.5.

Corollary 3.1 *A graph is hereditary open neighborhood-Helly if and only if it is open neighborhood-Helly and has no induced subgraph isomorphic to the graph J of Figure 2.7.*

Proof: By Lemma 2.5 and Proposition 2.1, G is $C_6 - free$ and has no triangles. Then, by Theorem 3.4 G is hereditary open neighborhood-Helly. Conversely, if G is hereditary open neighborhood-Helly, it contains no C_6 and hence, no subgraph isomorphic to J. ∎

The next Corollary relates hereditary closed neighborhood-Helly graphs to closed neighborhood-Helly graphs. It is a direct consequence of Theorem 3.2.

Corollary 3.2 *Let G be a closed neighborhood-Helly graph, with no triangles. Then G is hereditary closed neighborhood-Helly.*

Next, we relate open neighborhood-Helly graphs to hereditary closed neighborhood-Helly graphs.

Proposition 3.1 *Let G be a graph with no triangles, $C_4's$, nor $C_5's$. Then if G is open neighborhood-Helly, it is hereditary closed neighborhood-Helly.*

Proof: As G is $C_4 - free$ and has no triangles, it is $J - free$. By Theorem 2.1, G is $C_6 - free$. By Theorem 3.5 it follows that is hereditary closed neighbourhood-Helly. ∎

The following Theorem relates hereditary closed neighborhood-Helly graph to hereditary biclique-Helly graphs.

Theorem 3.6 *Let G be a hereditary closed neighborhood-Helly graph with no triangles. Then G is hereditary biclique-Helly.*

Proof: By Theorem 3.5, if G is hereditary closed neighborhood-Helly and has no triangles, then it is $\{C_4, C_5, C_6\} - free$. Then, by Theorem 3.3 it is hereditary biclique-Helly. Observe that in the proof of this Theorem, every biclique is in fact a closed neighborhood. ∎

The converse is not true. For example, the graph C_4 is hereditary biclique-Helly but it is not closed neighborhood-Helly. (See Figure 3.7)

Corollary 3.3 *Let G be a $C_4 - free$ graph. Then G is hereditary closed neighborhood-Helly and triangle-free if and only if it is hereditary biclique-Helly.*

Proof: If G is $C_4 - free$, has no triangles and hereditary closed neighborhood-Helly, by Theorem 3.5 it is also $\{C_5, C_6\}$-free. By Theorem 3.3 we conclude G is hereditary biclique-Helly.

Conversely, if G is $C_4 - free$ and hereditary biclique-Helly, according to Theorem 3.3 it is $\{C_5, C_6\}$-free and Hajós-free. From Theorem 3.5 it follows that G is hereditary closed neighborhood-Helly. ∎

Corollary 3.4 *Let G be a open neighborhood-Helly graph with no triangles, $C_4's$, nor $C_5's$. Then:*

1. *G is hereditary open neighborhood-Helly.*
2. *G is hereditary closed neighborhood-Helly.*
3. *G is hereditary biclique-Helly.*

Proof: 1) As G has no $C_4's$, it is $J - free$. By Corollary 3.1 and Theorem 3.4, G is hereditary open neighborhood-Helly.

2) It follows from Proposition 3.1.

3) It follows from Theorem 3.6 ∎

As a consequence of the results presented, we obtain the following relations between closed neighborhood-Helly graphs and hereditary biclique-Helly, hereditary open neighbourhood-helly and hereditary clique-Helly graphs graphs.

Corollary 3.5 *If G is closed neighborhood-Helly with no triangles, then it is hereditary biclique-Helly, hereditary open neighborhood-Helly, hereditary closed neighbourhood-Helly and hereditary clique-Helly.*

Finally we can also conclude:

Corollary 3.6 *Let G be a bipartite $C_4 - free$ graph, then it is equivalent:*

1. *G is closed neighborhood-Helly*
2. *G is open neighborhood-Helly*
3. *G is hereditary open neighborhood-Helly.*
4. *G is hereditary closed neighborhood-Helly.*

Proof: 1) $\Longrightarrow$ 2) As G is a bipartite graph, it has no triangles. By Proposition2.3, G is open neighborhood-Helly.

2) $\Longrightarrow$ 3) If G is $C_4 - free$, it is $J - free$ (Figure 2.7), then, by Corollary 3.1, it is hereditary open neighborhood-Helly.

3) $\Longrightarrow$ 4) It follows directly from Proposition3.1.

4) $\Longrightarrow$ 1) It follows directly. ∎

Corollary 3.7 *Let G be a graph with girth at least 7. Then G is hereditary biclique-Helly, hereditary open neighborhood-Helly and hereditary closed neighborhood-Helly.*

The following two results about hereditary disk-Helly graphs are direct.

Corollary 3.8 *If G hereditary disk-Helly, then it is hereditary clique-Helly and hereditary closed neighborhood-Helly.*

Corollary 3.9 *If G is hereditary disk-Helly, with no triangles, then it is hereditary biclique-Helly and consequently, hereditary open neighborhood-Helly.*

Figure 3.7 shows the relations between the mentioned hereditary classes.

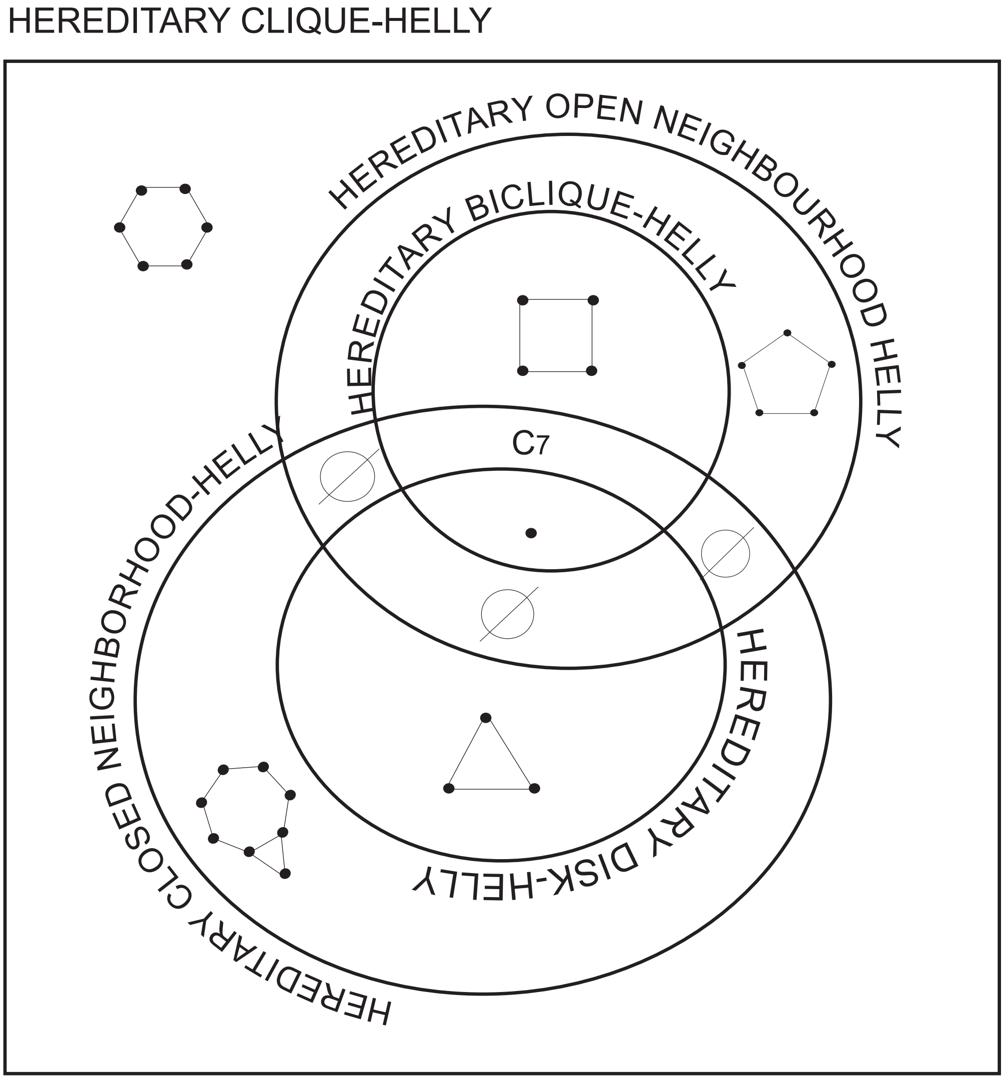


Figure 3.7: Relations between hereditary classes

Chapter 4

Biclique matrices and bipartite-conformal hypergraphs

4.1 Introduction

In this Chapter we give a characterization for biclique matrices, in similar terms as those employed in the characterization of clique matrices. The special case of biclique matrices of bipartite graphs is also considered. In the characterizations, we employ the concept of bipartite-conformal.

Clique matrices of a graph have been characterized by Gilmore in 1960, and have been employed in different contexts. For example, in the characterizations of interval graphs [28], Helly circular-arc graphs [26] and self-clique graphs [12, 36], as well as in different covering problems involving cliques.

Motivated by the above concept, we consider biclique matrices of a graph. We describe a characterization for such matrices, in similar terms as those used in the characterization of clique matrices. Biclique matrices can be employed, for instance in covering problems involving bicliques. Such problems have been considered, among others, by [3, 53]. We recall that we use biclique matrices for characterizing biclique graphs.

Denote by $\mathcal{H}$ a hypergraph, with vertex set $V(\mathcal{H})$ and hyperedge set $E(\mathcal{H})$. Write $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ and $E(\mathcal{H}) = \{E_1, \ldots, E_m\}$. The *2-section* of a hypergraph $\mathcal{H}$ is a graph G_2, where $V(G_2) = V(\mathcal{H})$ and such that there is an edge $v_iv_j \in E(G_2)$ precisely when there exists some hyperedge $E_k \supseteq \{v_i, v_j\}$, for all $1 \leq i \neq j \leq n$. Say that $\mathcal{H}$ is *conformal* when each clique of G_2 is contained in some hyperedge of $\mathcal{H}$. Say that $\mathcal{H}$ is *Helly* when every subfamily of intersecting hyperedges contains a common vertex. Given a $\{0,1\}$-matrix A, with n columns and m rows, the *associated hypergraph* H of A is a hypergraph with n vertices and m hyperedges such that vertex $v_i \in V(H)$ belongs to hyperedge E_j if and only if $a_{ij} = 1$.

The characterization given by Gilmore (c.f. Berge [10]).of clique matrices can be formulated in terms of the above concepts, applied to $\{0,1\}$-matrices.

Theorem 4.1 *([28]): Let A be a $\{0,1\}$-matrix and $\mathcal{H}$ its associated hypergraph. Then A is a clique matrix of some graph if and only if*

(i) each row of A has at least one 1,

(ii) A has no included rows, and

(iii) $\mathcal{H}$ is conformal.

We recall some concepts we use during this Chapter. A bicoloring of G is a bipartition of the vertices of G into subsets V_1, V_2. A clique of G is bichromatic relative to a bicoloring V_1, V_2 if it contains at least a vertex of V_1 and a vertex of V_2. A weak 2-coloring of G is a bicoloring V_1, V_2 such that every clique of G is bichromatic, relative to V_1, V_2.

Recall that, if G has c cliques $\{C_1, \ldots, C_c\}$ then the clique matrix of G is the $c \times n$ $\{0,1\}$-matrix A, where $a_{ki} = 1$ if and only if $v_i \in C_k$. Finally, when G has d bicliques $B_1, \ldots, B_d \subseteq V(G)$, the biclique matrix of G is the $d \times n$ $\{0,1,-1\}$-matrix A, where $a_{ki} = -a_{kj} \neq 0$, precisely when $v_i, v_j \in B_k$ and v_i, v_j are adjacent, for all $1 \leq k \leq n$ and $1 \leq i \neq j \leq n$.

In Section 2, we describe the proposed characterization for biclique matrices. The special case of biclique matrices of bipartite graphs is considered in Section 3. In Section 4 we characterize the class of bipartite-conformal hypergraphs whith compatible bicoloring, which is useful for our purposes. Section 5 contains algorithms for recognizing biclique matrices and bipartite-conformal hypergraphs.

4.2 Characterization of a biclique matrix

In this section, we give a characterization of biclique matrices of a graph. We employ the following concepts.

Let $\mathcal{H}$ be a hypergraph in which there is a bicoloring $\mathcal{C}$ of the occurrances of each vertex in the hyperedges of $\mathcal{H}$, using the colors *white* and *black*. That is, if vertex v belongs to hyperedges $E_1, ..., E_k$, then v is assigned a color in each of these hyperedges, and these colors are independent. Define a bicoloring of the edges of the 2-section G_2 of $\mathcal{H}$ as follows. Each $v_iv_j \in E(G_2)$ is *black* when there exists some edge $E_k \supseteq \{v_i, v_j\}$, where v_i and v_j have different colors in E_k; otherwise v_iv_j is *white*. Define the *black section* of $\mathcal{H}$, as the subgraph G_b of G_2, containing exactly the black edges of G_2. Say that $\mathcal{H}$ is *bipartite-conformal*, relative to $\mathcal{C}$, when each biclique B of G_b is contained in some hyperedge of $\mathcal{H}$. That is, there is a hyperedge E_k such that v_iv_j is an edge of B precisely when v_i, v_j have different colors in E_k. When every two vertices contained in a hyperedge of $\mathcal{H}$ with the same color are not adjacent in G_b , we say that $\mathcal{C}$ is a *compatible bicoloring.*

(a)

$$A_1 = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

(b)

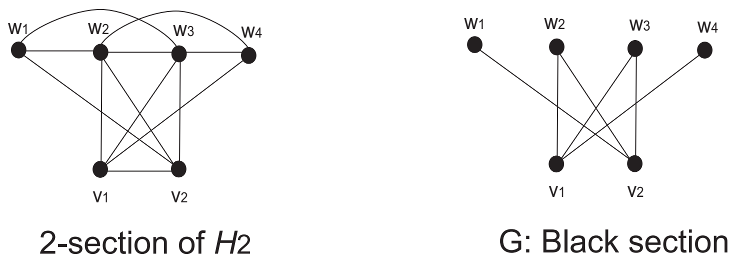


Figure 4.1: $\{0, 1, -1\}$-matrices and the graph G

Given a $m \times n$ $\{0, 1, -1\}$-matrix A, the *associated hypergraph* $\mathcal{H}$ of A is the hypergraph having one vertex v_i for each column i and one hyperedge E_k for each row k of A, such that $v_i \in E_k$ precisely when $a_{ki} \neq 0$. Define a special bicoloring of occurrances of each vertex in the hyperedges of $\mathcal{H}$ as follows: vertex $v_i \in V(\mathcal{H})$ is *white in* E_k when $a_{ki} = 1$ and v_i is *black in* E_k when $a_{ki} = -1$. When $v_i \notin E_k$ then v_i is uncolored for E_k. Such a bicoloring is called the *canonical bicoloring of* $\mathcal{H}$.

Let A, A' be $\{0, 1, -1\}$-matrices. Denote by A_k the vector consisting of row k of A. Say that row k is *included* in row l, when $a_{ki} = 1$ implies $a'_{li} = 1$ and $a_{ki} = -1$ implies $a_{li} = -1$, for all $1 \leq i \leq n$, where $A'_l = A_l$ or $A'_l = -A_l$. In general, say that A, A' are *row-similar* when $A_k = A'_k$ or $A_k = -A'_k$, for all $1 \leq k \leq m$.

Figure 4.1(a) illustrates an example of a $\{0, 1, -1\}$-matrix with included rows. The last row of A_1 is included in the first row. The hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$, associated to the matrices A_1 and A_2 respectively, have as vertex sets $V(\mathcal{H}_1) = V(\mathcal{H}_2) = \{v_1, v_2, w_1, w_2, w_3, w_4\}$, and hyperedges $\mathcal{H}_1 = \{E_1, E_2, E_3\}$, $\mathcal{H}_2 = \{E_1, E_2, E'_3\}$, where $E_1 = \{v_1, w_2, w_3, w_4\}$, $E_2 = \{v_2, w_1, w_2, w_3\}$, $E_3 = \{v_1, v_2, w_2, w_3\}$ and $E'_3 = \{v_1, w_2, w_4\}$. Finally, in Figure 4.1(b) we show that the black section G of both hypergrahs $\mathcal{H}_1$ and $\mathcal{H}_2$ coincides. Observe that $\mathcal{H}_1$ is not bipartite-conformal, while A_2 is a biclique matrix of G.

Remark that whenever A, A' are two row-similar matrices then the 2-sections G_2, G'_2 of their corresponding associated hypergraphs are isomorphic.

Moreover, if $e \in E(G_2)$ and $e' \in E(G_2')$ are two corresponding edges in the isomorphism $G_2 \cong G_2'$ then they have identical colors in the respective canonical bicolorings.

The following Theorem is the main result of this Chapter. It characterizes biclique matrices of graphs.

Theorem 4.2 : *Let A be a $m \times n$ $\{0, 1, -1\}$-matrix, and $\mathcal{H}$ its associated hypergraph. Then A is a biclique matrix of some graph if and only if*

(i) each row of A has at least one 1 and at least one -1,

(ii) A has no included rows,

(iii) A does not contain as a submatrix, neither M_1, nor M_2, nor any matrix row-similar to M_1 or M_2, and

$$M1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad M2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$(a)\ Matrix\ M_1 \qquad (b)\ Matrix\ M_2$

Figure 4.2: Matrices M_1, M_2

(iv) $\mathcal{H}$ is bipartite-conformal, relative to its canonical bicoloring.

Proof: By hypothesis, A is a biclique matrix of some graph G. Let $V(G) = \{v_1, \ldots, v_n\}$, and denote its bicliques by $B_1, \ldots, B_m \subseteq V(G)$. We know that $a_{ki} = -a_{kj} \neq 0$, precisely when v_i, v_j are adjacent and belong to B_k. By definition, there is at least one edge v_iv_j in biclique B_k. In this case, $a_{ki} = -a_{kj} \neq 0$, meaning that row k has at least one 1 and one -1. Then (i) holds. Next, observe that A is a biclique matrix of some graph if and only if any of the matrices row-similar to A are so. Consequently, row k can not be included in any other row, otherwise B_k would not be maximal. Hence (ii) holds.

For (iii), assume that A contains M_1 as a submatrix. Let k, l and i, j be the pairs of rows and columns of A, respectively, which contain M_1. Then row k implies that v_i, v_j are not adjacent, while row l implies that they are adjacent, impossible. The cases of the remaining forbidden matrices are similar.

Next, examine (iv). Let B_k be a biclique of G, with bipartition $V_1 \cup V_2 = B_k$. Then row k of A has entries

$$a_{ki} = \begin{cases} 0, & \text{if } v_i \neq B_k \\ 1, & \text{if } v_i \in V_1 \\ -1, & \text{if } v_i \in V_2, \end{cases}$$

for all $1 \leq i \leq n$, where the choice of V_1, V_2 is arbitrary. By the construction of the associated hypergraph $\mathcal{H}$, the hyperedge $E_k \in E(\mathcal{H})$ contains all vertices v_i, such that $a_{ki} \neq 0$. Then $E_k \supseteq B_k$. Let G_2 be the 2-section of $\mathcal{H}$ and G_b its black section. We show that $G = G_b$. Clearly, $V(G_b) = V(G) = \{v_1, \ldots, v_n\}$. Let $v_i, v_j \in \{v_1, \ldots v_n\}$, $i \neq j$. First, suppose $v_i v_j \in E(G_b)$. Then $v_i v_j$ is a black edge of G_2, meaning that v_i, v_j are assigned different colors in some edge $E_l \in E(\mathcal{H})$. That is, $a_{li} = -a_{lj} \neq 0$. However, A is a biclique matrix of G. Then row l implies that v_i, v_j are adjacent also in G. Consequently, $E(G_b) \subseteq E(G)$. Finally, consider $v_i v_j \in E(G)$. Then $v_i v_j$ belong to some biclique B_r of G. That is, there is a row r of A, such that $a_{ri} = -a_{rj} \neq 0$. The latter implies that $v_i, v_j \in E_r \in E(\mathcal{H})$, meaning that $v_i v_j$ is a black edge of G_2, i.e. $v_i v_j \in E(G_b)$. Consequently $E(G) \subseteq E(G_b)$. That is, $G = G_b$. Then B_k is an arbitrary biclique of G_b. Since $E_k \supseteq B_k$, it follows that $\mathcal{H}$ is bipartite-conformal.

Conversely, by hypothesis A satisfies (i)-(iv). We show that A is a biclique matrix. In fact, we show that A is a biclique matrix of the black section G_b of $\mathcal{H}$, relative to the canonical bicoloring.

To start, we show that every biclique B of G_b corresponds to a row of A. Let $V_1 \cup V_2 = B$ be the bipartition of B, $V_1, V_2 \neq \emptyset$. From (iv), we conclude that B is contained in some hyperedge E_k of $\mathcal{H}$. Let $v_i, v_j \in B$ and examine the possible alternatives. In the first alternative, suppose $v_i \in V_1$ and $v_j \in V_2$. Then $v_i v_j \in E(G_b)$. By definition, $v_i v_j$ is a black edge of G_2. Consequently, v_i, v_j have distinct colors in some hyperedge of $\mathcal{H}$. In addition, we know that v_i, v_j must have distinct colors in any hyperedge of $\mathcal{H}$ that contain both of these vertices. Otherwise A would contain as a submatrix the matrix M_1, or M_2, or any of matrices row-similar to M_1 or M_2, contradicting (iii). Consequently, the row k of A, corresponding to E_k, is such that $a_{ki} = -a_{kj} \neq 0$. In the next alternative, let $v_i, v_j \in V_1$. Since $v_i, v_j \in E_k$, each of these vertices has a color in E_k. Because $v_i v_j$ is not a black edge of G_2, it follows that both vertices v_i, v_j must have identical colors in E_k. That is, $a_{ki} = a_{kj} \neq 0$. Finally, when $v_i \notin B_k$ it easily follows that $a_{ki} = 0$. The alternative $v_i, v_j \in V_2$, is similar. Consequently, B corresponds to E_k, hence to row k of A.

In the sequel, we show that every row k of A corresponds to some biclique of G_b. Let $V_1 \subseteq V(\mathcal{H})$ be the set of vertices of $\mathcal{H}$ corresponding to the 1 entries of row k of A, and $V_2 \subset V(\mathcal{H})$, those corresponding to the -1 entries. From (i), it follows that $V_1, V_2 \neq \emptyset$. First, let $v_i \in V_1$ and $v_j \in V_2$. Then v_i, v_j are assigned distinct colors in the hyperedge $E_k \in E(\mathcal{H})$. Consequently, $v_i v_j$ is a black edge of G_2, hence $v_i v_j \in E(G_b)$. Next, let $v_i, v_j \in V_1$. Then v_i, v_j are both white in E_k. Again, we know that whenever v_i, v_j are both contained in some hyperedge $E_l \in E(\mathcal{H})$, then v_i, v_j have identical colors in E_l, otherwise A would contain a forbiden submarix of (iii). Consequently, $v_i v_j$ is a white edge of G_2, meaning that $v_i v_j \notin E(G_b)$. The situation where $v_i, v_j \in V_2$ is similar. Consequently, $V_1 \cup V_2$ is a complete bipartite set of G_b included in a biclique B. Let l be the row corresponding to B. Because of (ii), row k is not included in any other row. Consequently, $l = k$ and $V_1 \cup V_2$ is indeed the biclique B of G_b, completing the proof. ∎

The following property is a consequence of Theorem 4.2

Corollary 4.1 *A matrix is a biclique matrix of some graph if and only if it is a biclique matrix of the black section of its associated hypergraph.*

4.3 Bipartite graphs

In this section we examine biclique matrices of bipartite graphs.

The following concept is useful. A $\{0, 1, -1\}$-matrix A is *bipartite* when it admits a matrix row-similar to A', such that no column of A' has both entries 1 and -1. It is clear that a graph is bipartite if and only if its biclique matrix is bipartite. We observe that bipartite matrices can be recognized in polynomial-time.

As a direct corollary of the Theorem 4.2, follows a characterization for biclique matrices of bipartite graphs.

Corollary 4.2 *: Let A be a $m \times n$ $\{0, 1, -1\}$-matrix, and $\mathcal{H}$ its associated hypergraph. Then A is a biclique matrix of some bipartite graph if and only if*

(i) each row of A has at least one 1 and at least one -1,

(ii) A has no included rows,

(iii) $\mathcal{H}$ is bipartite-conformal, relative to its canonical bicoloring,

(iv) A is bipartite.

Given a graph G with d bicliques $B_1, \ldots, B_d \subseteq V(G)$, a *positive biclique* matrix A of G is a $d \times n$ $\{0, 1\}$-matrix such that $a_{ij} = 1$ if vertex v_j belongs to biclique B_i and $a_{ij} = 0$ otherwise. Clearly, a biclique matrix corresponds to a positive matrix by replacing each -1 by 1.

Theorem 4.3 *Let G be a graph, A be a clique matrix of G and $\mathcal{H}$ the associated hypergraph of A. Then, the following statements are equivalent:*

(1) A is a positive biclique matrix of a bipartite graph H.

(2) A is a positive biclique matrix of a neighborhood-Helly bipartite graph H.

(3) G admits a weak 2-coloring $V_1, V_2 \subseteq V(G)$ and $\mathcal{H}$ is bipartite-conformal, relative to the bicoloring V_1, V_2.

Proof: (1) $\Rightarrow$ (2) : Suppose A is a positive matrix of a bipartite graph H, with bipartition V_1, V_2. We prove that H is neighborhood-Helly. Let V' be a set of vertices of H whose neighborhoods pairwise intersect. Without loss of generality, $V' \subseteq V_1$. The columns of A corresponding to V' pairwise intersect, since any two vertices in V' belong to a same biclique. Then, columns of A corresponding to V pairwise intersect and V' is a complete subset of G, contained in some clique C. Finally, the columns of A corresponding to V' intersect at the row which corresponds to C in A.

(2) $\Rightarrow$ (3) : Let V_1, V_2 be the bipartition of H. Considering V_1, V_2 as a bicoloring of vertices of G. Since A is a positive biclique matrix of H, every clique of G is bichromatic, following that V_1, V_2 is a weak 2-coloring of G. It is clear that $\mathcal{H}$ is bipartite conformal, by Corollary 4.2.

(3) $\Rightarrow$ (1) : Let V_1, V_2 be the bicoloring of G. Define the bipartite matrix B as follows: for every i, $b_{ij} = a_{ij}$ if $j \in V_1$ and $b_{ij} = -a_{ij}$ for $j \in V_2$. Since V_1, V_2 is a weak 2-coloring of G, every row of B has at least a 1 and a -1. Since A is a clique matrix, B has not included rows. Finally, by hypothesis, $\mathcal{H}$ is bipartite conformal. Corollary 4.2 says that B is a biclique matrix of a bipartite graph H, ie. A is a positive biclique matrix of H. ∎

4.4 Bipartite-conformal hypergraphs

In this section we characterize bipartite-conformal hypergraphs, having compatible bicoloring. Let $\mathcal{H}$ be a hypergraph, $E(\mathcal{H}) = \{E_1, ..E_k\}$ and let $\mathcal{C}$ be a bicoloring of the ocurrances of each vertex in the hyeperedges of $\mathcal{H}$. Let G_b be its black section. For every subfamily $\mathcal{E}' = \{E_i, E_j, E_k\}$ of three hyperedges of $\mathcal{H}$, consider every triple l_i, l_j, l_k, $1 \leq i, j, k \leq m$, $l = 1, -1$ (white and black, respectively). Let $\mathcal{V}^1_{\{l_i,l_j,l_k\}}$ be the subfamily of vertices of $\mathcal{H}$ which belong to at least two hyperedges $E_s \in \mathcal{E}'$, $E_r \in \mathcal{E}'$, with color l_s, l_r, respectively. Similarly, let $\mathcal{V}^2_{\{l_i,l_j,l_k\}}$ be the subfamily of vertices of $\mathcal{H}$ which belong to at least two hyperedges $E_s \in \mathcal{E}'$, $E_r \in \mathcal{E}'$, with color $-l_s$, $-l_r$, respectively.

Theorem 4.4 *Let $\mathcal{H}$ be a hypergraph, and let $\mathcal{C}$ be a compatible bicoloring. Then $\mathcal{H}$ is bipartite-conformal if and only if every induced P_3 of G_b is contained in an hyperedge of $\mathcal{H}$ and every subfamily $\mathcal{V}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{V}^2_{\{l_i,l_j,l_k\}}$ is contained in an hyperedge of $\mathcal{H}$.*

Proof: Assume that is bipartite-conformal $\mathcal{H}$. It is clear that every P_3 is contained in an hyperedge of $\mathcal{H}$. We prove that $\mathcal{V}^1_{\{l_i,l_j,l_k\}}$ and $\mathcal{V}^2_{\{l_i,l_j,l_k\}}$ induce independent sets in G_b. Let $v_r, v_s \in \mathcal{V}^1_{\{l_i,l_j,l_k\}}$. Then there is an hyperedge, suppose E_i, which contains both vertices with color l_i. As $\mathcal{C}$ is a compatible bicoloring, v_r, v_s are not adjacent in G_b. Analogously, $\mathcal{V}^2_{\{l_i,l_j,l_k\}}$ is an independent set in G_b. Finally, let $v_r \in \mathcal{V}^1_{\{l_i,l_j,l_k\}}$, $v_s \in \mathcal{V}^2_{\{l_i,l_j,l_k\}}$. There is an hyperedge in $\mathcal{H}$ that contains v_r, v_s with different colors, what means that in G_b they are adjacent. Then, the complete bipartite subgraph $\mathcal{V}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{V}^2_{\{l_i,l_j,l_k\}}$ must be contained in an hyperedge of $\mathcal{H}$.

Conversely, let B be a biclique with bipartition V_1, V_2. Suppose it is not contained in an hyperedge. Let B' be a minimal bipartite subgraph of B with bipartitions $V'_1 \subseteq V_1$, $V'_2 \subseteq V_2$ ($V'_1, V'_2 \neq \emptyset$) which is not contained in an hyperedge. Then, $\mid V_1 + V_2 \mid \geq 4$. Consider the case $\mid V_1 \mid = \mid V_2 \mid = 2$. Let E_1 be the hyperedge containing $V'_1 \setminus \{v_{i_1}\} \cup V'_2$. Let l_1 be the color of vertices of V'_2 in E_1 (recall that since V'_2 is an independent set, every vertex has the same color in E_1). Analogously, let E_2 be the hyperedge containing $V'_1 \setminus \{v_{i_2}\} \cup V'_2$. Let l_2 be the color of the vertices of V'_2 in E_2. Finally, let E_3 be the hyperedge containing $V'_1 \cup V'_2 \setminus \{v_{j_1}\}$. Let l_3 be the color of vertices of V'_2 in E_3. Consider $\mathcal{V}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{V}^2_{\{l_1,l_2,l_3\}}$. We prove that B' is included in $\mathcal{V}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{V}^2_{\{l_1,l_2,l_3\}}$. Vertex v_{i_1} is contained in E_2 and E_3. Since $v_{j_2} \in V'_2$ is contained in E_1, E_2 and E_3, the colors of v_{j_2} in E_1, E_2 and E_3 are l_1, l_2, l_3 respectively. Then, as v_{j_2} is adjacent to v_{i_1} and both are contained in E_2 and E_3, colors of v_{i_1} in E_2 and E_3 are $-l_2$, $-l_3$ respectively. Analogously, v_{i_2} is contained in E_1, E_3 with colors $-l_1$, $-l_3$, respectively. Finally, v_{j_1} is contained in E_1, E_2 with colors l_1, l_2 respectively. It follows that $V'_1 \subseteq \mathcal{V}^1_{\{l_1,l_2,l_3\}}$, $V'_2 \subseteq \mathcal{V}^2_{\{l_1,l_2,l_3\}}$. The case where $V'_1 \geq 3$ is similar. We consider $V'_1 \setminus \{v_{i_1}\} \cup V'_2$, $V'_1 \setminus \{v_{i_2}\} \cup V'_2$ and $V'_1 \setminus \{v_{i_3}\} \cup V'_2$ and conclude there are hyperedges E_1, E_2, E_3, such that $v_{i_j} \notin E_j$. Finally, consider l_1, l_2, l_3 the colors corresponding to vertices of V'_2 in hyperedges E_1, E_2, E_3 respectively. In any case, it follows that B' is contained in a hyperedge, what is an absurd. ■

4.5 Algorithms for recognizing biclique matrices and bipartite-conformal hypergraphs with compatible bicoloring

In this Section, we describe an algorithm for deciding if a given hypergraph with a compatible bicoloring $\mathcal{C}$ is bipartite-conformal, and also algorithms for recognizing biclique matrices.

For recognizing biclique matrices, we describe two algorithms. The first is based on Theorem 4.2. The second follows from Corollary 4.1 and employs an algorithm for generating the bicliques of a graph.

The following algorithm for recognizing bipartite-conformal hypergraphs having a compatible bicoloring follows from Theorem 4.4. Let $\mathcal{H}$ be an hypergraph and let $\mathcal{C}$ be its compatible bicoloring.

Algorithm 4.1 ***Bipartite-conformal hyperhraphs.*** *Check if every induced P_3 of G_b is contained in an hyperedge of $\mathcal{H}$. If negative, answer NO and stop. For every l_i, l_j, l_k, $1 \leq i, j, k \leq m$, $l = 1, -1$, consider $\mathcal{V}^1_{\{l_i,l_j,l_k\}}$, $\mathcal{V}^2_{\{l_i,l_j,l_k\}}$. Check if there is an hyperedge E_t that contains $\mathcal{V}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{V}^2_{\{l_i,l_j,l_k\}}$. If negative answer NO. Otherwise, answer YES.*

The Algorithm 4.1 requires $O(m^2n + mn^3)$ steps.

The algorithm for recognizing biclique matrices follows directly from Theorem 4.2. Let A be a $m \times n$ $\{0, 1, -1\}$-matrix and let $\mathcal{H}$ be the associated hypergraph of A. We remark that A does not contain as a submatrix, neither M_1, nor M_2, nor any matrix row-similar to M_1 or M_2 (Figure 4.2), if and only if the canonical bicoloring of $\mathcal{H}$ is compatible.

Algorithm 4.2 ***Biclique matrices (I)***. *First check the conditions* $(i), (ii), (iii)$ *of Theorem 4.2. If any of these conditions is not satisfied by A, answer NO and stop. Otherwise, check condition* (iv) *applying the Algorithm 4.1.*

The Algorithm 4.2 requires $O(m^2n + mn^3)$ steps.

Alternatively, we can also recognize biclique matrices by employing the following algorithm.

Algorithm 4.3 ***Biclique matrices (II).*** *First, construct the associated hypergraph* $\mathcal{H}$ *and the black section* G_b*, relative to the canonical bicoloring of A. Let* $V(G_b) = \{v_1, \ldots, v_n\}$*. Generate the bicliques of* G_b *employing algorithm [19]. For every biclique B of* G_b*, construct its corresponding row entry in a biclique matrix of* G_b*. That is, construct an n-vector* B'*, where* $b'_i = -b'_j \neq 0$ *if and only if* $v_i, v_j \in B$ *and* $v_iv_j \in E(G_b)$*, making arbitrary choices of 1 and -1, whenever possible. Then find one row l of A, such that* $A_l = B'$ *or* $A_l = -B'$*. If this condition is satisfied for some row l then remove it from A. Otherwise, answer NO and stop. Perform this procedure m times. If all rows of A have been removed and all bicliques of* G_b *have been generated, then answer YES, otherwise answer NO.*

The Algorithm 4.3 requires $O(m^2n + mn^3)$ time and $O(nm)$ space.

We remark that similar techniques as above, can be applied to obtain a polynomial time algorithm for recognizing general bipartite-conformal hypergraphs, i.e. not necessarily compatible. For such algorithm, we would again employ the algorithm for biclique generation [19].

Finally, we consider the bipartite case. A polynomial time algorithm for recognizing biclique matrices of bipartite graphs follows directly from Corollary 4.2, noting that checking if a $\{0, 1, -1\}$-matrix is bipartite can be done in $O(mn)$ time.

We remark that if A is a bipartite matrix, we can apply the following algorithm instead of Algorithm 4.1 to check if the associated hypergraph $\mathcal{H}$ is bipartite-conformal.

Let V_1, V_2 the columns with entries 1 and -1, respectively. Construct A' by adding to A one row containing 1's in the columns of V_1 and another row containing -1's in the columns of V_2. Run the algorithm proposed by Berge [10] to identify if the columns of A' are Helly. It follows that columns of A' are Helly if and only if $\mathcal{H}$ is bipartite-conformal

Chapter 5

Biclique graphs

5.1 Introduction

In this Chapter, we consider some special classes of intersection graphs. In particular we consider biclique graphs. We present a characterization for biclique graph. The special case of biclique graphs of bipartite graphs is also considered.

Given a family of subsets of some set $\mathcal{F}$, the *intersection graph* of $\mathcal{F}$ is a graph having a vertex for each set of $\mathcal{F}$, and two vertices are adjacent whenever their corresponding sets intersect. On the other hand, given a graph G, it is an *intersection graph* when there exists a family $\mathcal{F}$ of subsets of some set such that G is its intersection graph. Intersection graphs were studied in several contexts (See [39]).

All graphs are intersection graphs (Marczewski, [44]). In fact, given a graph G, we can always construct a family $\mathcal{F}$ of subsets $F_i \subseteq E(G)$, as follows: For each $v_i \in V(G)$, let F_i be the set of edges incident to v_i. It it is not difficult to prove that G is the intersection graph of $\mathcal{F}$.

There are many classes of intersection graphs which can be defined, by finding suitable families of subsets. For example, clique graphs, interval graphs, chordal graphs, and line graphs (See [15, 16, 23, 24, 27, 38]) .

Two problems arise when leading with intersection graphs. The recognition problem which consists on deciding whether a given graph G and a family $\mathcal{F}$ of subsets of a set, G is the intersection graph of $\mathcal{F}$. The second a goal is to study the computational complexity of the mentioned problem.

Recall that the clique graph $K(G)$ of a graph G is the intersection graph of the family of all cliques of G. Two characterizations for clique graphs were given. In [49], Robert and Spencer give the first characterization for clique graphs. A different one is shown in [2]. However, none of them appeared to lead to a polynomial time algorithm for the recognition problem. Moreover, it was an open problem determining the computational complexity of the clique graph recognition problem. Recently, in [1] it is proved that the mentioned problem is NP-complete.

Motivated by the concept of clique graphs, we define the biclique graph of G, $KB(G)$, as the intersection graph of the family of all bicliques of G. A graph G is a biclique graph if there exists a graph H such that $KB(H) = G$. (Figure 5.1)

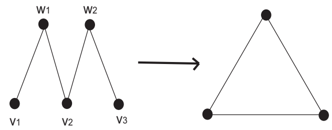


Figure 5.1: Graph P_5 and its biclique graph

Recall that given a set S of elements, a family $\mathcal{F}$ of subsets of S is a *split* of S if for every pair of elements $x, y \in S$, there exist a set in $\mathcal{F}$ containing x, and not containing y. A covering of a family set S is a family $\mathcal{F}$ of subset of S such that each element of S belongs to some set of $\mathcal{F}$.

Let S be a set of elements, and let L be a set of labels. A subset of F is a *L-labeled subset* when every element of it has a label of the set L. A family of labeled subsets is a *labeled family*. We distiguish two labeled sets by its elements considering its labels.

A labeled family of sets $\mathcal{F}$ is *well labeled* if every element belonging to a set of $\mathcal{F}$ have the same label in all sets of $\mathcal{F}$.

A labeled family of sets $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is *bipartite-intersecting* , if $\mathcal{F}_1, \mathcal{F}_2$ are well labeled and every set of $\mathcal{F}_1$ intersects every set of $\mathcal{F}_2$ in at least one element with different label.

A bipartite-intersecting labeled family $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ has a *good intersection* if every element that belong to some set of $\mathcal{F}_1$ and some set of $\mathcal{F}_2$, have different labels in each subfamily $\mathcal{F}_1, \mathcal{F}_2$.

When every bipartite-intersecting subfamily of $\mathcal{F}$ has a good intersection and contains a common element, then $\mathcal{F}$ is *bipartite-Helly*.

Given a set S of elements and a set L of labels, $\mid L \mid= 2$, an L-labeled family $\mathcal{F}$ is a *labeled bicovering* of S if for every element $x \in S$, there exist two sets in $\mathcal{F}$ containing x, each containing x with different label.

In Figure 5.2 there is an example of a $\{1, -1\}$-labeled family, where v_a means that element v has label a. The family $\mathcal{F}$ is a bicovering of S. The subset $\mathcal{F}_2$ of S is an example of a well labeled family whereas the subset $\mathcal{F}_1$ is not well labeled. The subfamily $\mathcal{F}'_1 \cup \mathcal{F}'_2$ of $\mathcal{F}$ is bipartite-intersecting, but it has not a good intersection. The subfamily $\mathcal{F}''_1 \cup \mathcal{F}''_2$ is a bipartite-intersecting subfamily of $\mathcal{F}$ with a good intersection, but it has not a common element. Finally, $\mathcal{F}'''_1 \cup \mathcal{F}'''_2$ is a bipartite-intersecting subfamily of $\mathcal{F}$ with a good intersection and contains a common element.

Let A be a $\{0, 1, -1\}$-matrix and let $\mathcal{F}$ be the family of columns. We consider $\mathcal{F}$ as a labeled family, where the row i is an element of the column j with label a_{ij} if and only if $a_{ij} \neq 0$. We say that two columns i, j intersect if $a_{ki} \neq 0$ and $a_{kj} \neq 0$ for some row k.

- $S = \{v, w, z, r, s\}$, $L = \{1, -1\}$
- Subsets of S: $F_1 = \{v_1, w_{-1}\}$, $F_2 = \{v_{-1}, w_{-1}\}$, $F_3 = \{s_{-1}, w_{-1}\}$, $F_4 = \{v_1, z_1, r_{-1}, s_1\}$, $F_5 = \{v_{-1}, r_1, w_1\}$, $F_6 = \{v_1, w_1\}$, $F_7 = \{v_{-1}, w_1\}$
- $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$
- $\mathcal{F}_1 = \{F_1, F_2\}$, $\mathcal{F}_2 = \{F_1, F_3\}$
- $\mathcal{F}'_1 = \{F_1\}$, $\mathcal{F}'_2 = \{F_2\}$
- $\mathcal{F}''_1 = \{F_3\}$, $\mathcal{F}''_2 = \{F_4, F_6\}$
- $\mathcal{F}'''_1 = \{F_1, F_4\}$, $\mathcal{F}'''_2 = \{F_5, F_7\}$

Figure 5.2: Example of a labeled family

On the other hand, given a set $S = \{x_1, ...x_n\}$ and an L-labeled family $\mathcal{F} = \{F_1, ..., F_k\}$, the *labeled incidence matrix* of $\mathcal{F}$ is a L-matrix A with n columns and k rows, where $a_{ij} = l$ if element x_j belongs to F_i with label l, and $a_{ij} = 0$ otherwise.

Given a $\{0, 1, -1\}$-vector v, the *symmetric of* v is the vector $v' = -v$. We say that vector v *is bi-included* in vector w if for every k such that $v_k = 1$, then $w_k = 1$ and for k such that $v_k = -1$, then $w_k = -1$. Let A be a $\{0, 1, -1\}$-matrix. We say that a row A_i *is included* in row A_j if A_i is bi-included in A_j or it holds for the symmetric of A_j. The induced graph of a $\{0, 1\}$-matrix $A \in R^{m\times n}$ is a graph of n vertices where v_i is adjacent to v_j if there exists a row k in A such that $a_{ki} = a_{kj} = 1$. Given a $\{0, 1, -1\}$-matrix $A \in R^{m\times n}$, the graph *bi-induced* by A is a graph of n vertices v_1,v_2,...,v_n where v_i is adjacent to v_j if there exists a row k in A such that $a_{ki} = 1, a_{kj} = -1$.

Recall that a $\{0, 1, -1\}$-matrix A is *bipartite* when it admits a row-symmetric matrix A', such that no column of A' has both entries 1 and -1. It is clear that a graph is bipartite if and only if its biclique matrix is bipartite.

This Chapter is organized as follows. In Section 2 we study clique graphs. We give a different proof for the characterization of clique graphs given in [49]. In Section 3, we characterize biclique graphs and describe families of graphs which are not biclique graphs. In Section 4 we characterize bipartite-Helly families. This characterization leads to a polynomial time algorithm for recognizing bipartite-Helly families. In Section 5 we characterize classes of biclique graphs of some families of graphs. In Section 6, other class of intersection graphs is considered, the E-biclique graphs.

5.2 Clique Graphs

In this Section we give a different proof for the Theorem of Robert and Spencer, which characterize clique graphs. It is a simpler proof and it employs techniques we use for the characterization of biclique graphs.

Lemma 5.1 *[9] Let $\mathcal{H}$ be an hypergraph and let $\mathcal{H}$ be its dual hypergraph. Then, $\mathcal{H}$ is conformal if and only if the family of hyperedges $\mathcal{H}$ is Helly.*

Theorem 5.1 *[49] A graph G is a clique graph if and only if there exists a family of complete subgraphs $\mathcal{C}$ verifying:*

1. *$\mathcal{C}$ covers the edges of G*
2. *$\mathcal{C}$ is a Helly family*

Proof: Suppose G is a clique graph, i.e., $G = K(H)$ for some graph H. Let A be a clique matrix of H. Observe that the graph induced by A^T is exactly G.

Let $\mathcal{C}$ be the family of complete subgraphs induced by the rows of A^T. We prove that it verifies the conditions 1) and 2) of the Theorem. As A is a clique matrix, by Theorem 4.1 and considering the reformulation in Theorem 5.2, its columns are Helly and its rows are not included. Consequently, $\mathcal{C}$ is Helly. Finally, since G is the graph induced by A^T, every edge of G is contained in some complete subgraph of $\mathcal{C}$.

Conversely. Let $\mathcal{C}$ be a family of k complete subgraphs of G, $C_1, ..., C_k$ verifying conditions 1) and 2). Consider the family $\mathcal{C}' = \mathcal{C} \cup \{\{v_1\}, ..., \{v_n\}\}$. Consider $B \in \{0,1\}^{(k+n)\times n}$, the incidence matrix of $\mathcal{C}'$, i.e. $b_{ij} = 1$ if vertex j belongs to the complete C'_i and $b_{ij} = 0$ otherwise.

Since $b_{(k+j)i} = 1$ if and only if $j = i$, $\mathcal{C}'$ has no included columns. By hypothesis, $\mathcal{C}$ is a Helly family and therefore, so are the rows of B.

Consider B^T. It has no included rows and its columns are Helly, i.e., it is a clique matrix, according to Theorem 5.2. Let H be the graph induced by B^T, as mentioned before, $(B^T)^T$ induces the clique graph of H. Hence, B induces $K(H)$. On the other hand, since $\mathcal{C}$ covers edges of G, B induces the graph G. We conclude that $G = K(H)$. ∎

The following Corollary is direct.

Corollary 5.1 *[49] Let $\mathcal{C}$ be the family of cliques of a graph G. Then, $\mathcal{C}$ verifies properties 1) and 2) of Theorem 5.1 if and only if G is a clique graph of a clique Helly graph.*

5.3 Biclique Graphs

In this Section we give a characterization for bicliques graphs. Also, we prove that complete graphs are biclique graphs and we prove that graphs without induced diamonds, are not biclique graphs.

For our purpose we will need the following Lemmas and Corollaries.

Lemma 5.2 *Let $\mathcal{F} = \{F_1, F_2, ...F_k\}$ be a $\{1, -1\}$-labeled family. Then, every bipartite-intersecting subfamily has a good intersection if and only if every bipartite-intersecting subfamily $\mathcal{F}' = \{F_i\} \cup \{F_j\}$ has a good intersection.*

Proof: It is clear that if every bipartite-intersecting subfamily has a good intersection, also does the bipartite-intersecting subfamily $\mathcal{F}' = \{F_i\} \cup \{F_j\}$.

Conversely, suppose there is a bipartite-intersecting subfamily $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2$ which has not a good intersection. Then, there are $F_i \in \mathcal{F}_1, F_j \in \mathcal{F}_2$ such that intersect in an element with same label. As $\mathcal{F}'$ is a bipartite-intersecting family, so it is the subfamily $\{F_i\} \cup \{F_j\}$. It follows that $\{F_i\} \cup \{F_j\}$ is a bipartite-intersecting family which has not a good intersection, contradicting the hypothesis. ∎

As a corollary, we obtain the following Lemma.

Lemma 5.3 *Let A be a $\{0, 1, -1\}$-matrix. Then, every bipartite-intersecting subfamily of columns has a good intersection if and only of A does not contain as a submatrix, neither M_1, nor M_2 of Figure 5.3, nor any row-symmetric matrix of M_1 or M_2.*

$$M1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad M2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Figure 5.3: Matrices M_1, M_2

Corollary 5.2 *Let A be a $\{0, 1, -1\} - matrix$, whose columns verify the bipartite-Helly property . Then, A does not contain as a submatrix, neither M_1, nor M_2, nor any row-symmetric matrix of M_1 or M_2, of Figure 5.3.*

Lemma 5.1 relates the Helly property to the conformal condition for hypergraphs. Similarly, we are going to relate the bipartite-Helly property to the bipartite-conformal condition. We need the following definitions.

We say that a *hypergraph is labeled* when the family of hyperedges is an L-labeled family. Given a labeled hypergraph $\mathcal{H}$, we label the family of edges of the dual hypergraph $\mathcal{H}^*$ as follows: vertex E_i has label j in hyperedge E_k^* of $\mathcal{H}^*$ if and only if vertex v_k has label j in hyperedge E_i in $\mathcal{H}$.

Observe that when every pair of hyperedges of $\mathcal{H}$ has a good intersection, the labeling $\mathcal{H}$ induce a compatible bicoloring of $\mathcal{H}$. In that case, we say that $\mathcal{H}$ has a *compatible labeling.*

Lemma 5.4 *Let $L = \{black, white\}$ be a set of labels, $\mathcal{H}$ an L-labeled hypergraph and $\mathcal{H}^*$ its dual L-labeled hypergraph. Then the labeling of $\mathcal{H}$ is compatible and $\mathcal{H}$ is bipartite-conformal if and only if the hyperedges of $\mathcal{H}^*$ are bipartite-Helly.*

Proof: Observe that every pair of bipartite-intersecting hyperedges of $\mathcal{H}$ has a good intersection if and only if every pair of bipartite-intersecting hyperedges of $\mathcal{H}^*$ has a good intersection. We need to prove that $\mathcal{H}$ is bipartite-conformal if and only every bipartite-intersecting family of hypedeges of $\mathcal{H}^*$ has common vertex.

Suppose $\mathcal{H}$ is bipartite-conformal. Let G be its black section. Consider $\mathcal{E}_1 \cup \mathcal{E}_2$ a bipartite-intersecting family of hyperedges of $\mathcal{H}^*$, where $\mathcal{E}_1 = \{E^*_{i_1}, ...E^*_{i_k}\}$, $\mathcal{E}_2 = \{E^*_{i_{k+1}}, ...E^*_{i_s}\}$.

Since $\mathcal{E}_1, \mathcal{E}_2$ are well labeled, $V_1 = \{v_{i_1}, ...v_{i_k}\}$, $V_2 = \{v_{i_{k+1}}...v_{i_s}\}$ are both independent sets in G. On the other hand, since every $E^*_i \in \mathcal{E}_1, E^*_j \in \mathcal{E}_2$ intersect in a different label, vertices $v_i \in V_1, v_j \in V_2$ are adjacent in G. It follows that V_1, V_2 induce a bipartite complete subgraph in G. Since $\mathcal{H}$ is bipartite-conformal, there is an hyperedge E_t which contains the vertices of $V_1 \cup V_2$. It follows that E_t in $\mathcal{H}^*$ is a common vertex of $\mathcal{E}_1 \cup \mathcal{E}_2$.

Conversely, let B be a biclique of G with bipartition $V_1 = \{v_{i_1}, ..., v_{i_s}\}$, $V_2 = \{v_{i_{s+1}}, ..., v_{i_t}\}$. Consider $\mathcal{E}_1 = \{E^*_{i_1}, ...E^*_{i_s}\}$, $\mathcal{E}_1 = \{E^*_{i_{s+1}}, ...E^*_{it}\}$, hyperedges of $\mathcal{H}^*$. Since V_1, V_2 are independent sets, $\mathcal{E}_1, \mathcal{E}_2$ are well labeled. Since every vertex of V_1 intersects every vertex of V_2, $\mathcal{E}_1 \cup \mathcal{E}_2$ is a bipartite-intersecting family in $\mathcal{H}^*$. By hypothesis there is a vertex E_t common to $\mathcal{E}_1, \mathcal{E}_2$. Then, edge E_t of $\mathcal{H}$ contains the vertices of B. ∎

Remark 5.1 *Let A be a $\{0, 1, -1\}$-matrix which does not contain any of the matrices M_1, M_2, of Figure 5.3, nor any of its symmetric matrices, as a submatrix. Let $\mathcal{H}$ be its associated labeled hypergraph. Then $\mathcal{H}$ is bipartite-conformal if and only if the columns of A are bipartite-Helly.*

The following Theorems are reformulations of Theorems 4.1 4.2 and Corollary 4.2, considering the Helly and the bipartite-Helly property. They will be useful in the proof of the characterization of biclique graphs.

Theorem 5.2 *: Let A be a $\{0, 1\}$-matrix and $\mathcal{H}$ its associated hypergraph. Then A is a clique matrix of some graph if and only if*

1. *Each row of A has at least one 1,*
2. *A has no included rows, and*
3. *The family of columns of A is Helly.*

Theorem 5.3 *Let A be a $\{0, 1, -1\}$-matrix. Then A is a biclique matrix if and only if:*

1. *A does not contain included rows and has at least a 1 and a -1 in each row.*

2. *The family of columns of A is bipartite-Helly*

As a Corollary of Theorem 5.3 it follows a characterization for biclique matrices of bipartite graphs.

Theorem 5.4 *Let A be a $\{0, 1, -1\}$-matrix. Then A is a biclique matrix of a bipartite graph if and only if*

1. *A has not included rows*

2. *A is bipartite with partition V_1, V_2.*

3. *The family of columns of A is bipartite-Helly*

The following is the main Theorem of the Chapter. It characterizes biclique graphs for general graphs. The proof of this Theorem employs the characterization given in Theorem 5.3 for bicliques matrices.

Theorem 5.5 *A graph G is a biclique graph if and only if there exists a $\{1, -1\}$-labeled family $\mathcal{C}$ of subsets of $V(G)$, such that:*

1. *Each subset induces a complete graph in G and $\mathcal{C}$ covers the edges of G*

2. *C is a labeled bicovering of $V(G)$*

3. *C is bipartite-Helly*

4. *C is a split of $V(G)$*

Proof: Suppose G is the biclique graph of a graph H and let A be a biclique matrix of H. Let A^+ be the matrix of values $\{\mid a_{ij} \mid\}$ and consider its transpose $(A^+)^T$. The graph G induced by $(A^+)^T$ is the intersection graph of the columns of $(A^+)^T$, i.e., $KB(H) = G$. Let $\mathcal{C}$ be the labeled family of complete subgraphs in G induced by the rows of A^T, where vertex w_i has label a_{ij} if it belongs to the complete C_j.

As A is a biclique matrix, by Theorem 5.3, A has at least a 1 and a -1 in each row and its columns verify the bipartite-Helly property. It follows that $\mathcal{C}$ is a bicovering of G and $\mathcal{C}$ is bipartite-Helly. As G is induced by $(A^+)^T$, every edge of G belongs to a complete of $\mathcal{C}$. We prove that $\mathcal{C}$ is a split. Suppose the contrary, i.e., there exist two vertices v_i, v_j such that the subfamily of complete subgraphs containing v_i is included in the sufamily of complete subgraphs containing vertex v_j. In other words, suppose row i of A^+ is included in row j. Since by Theorem 5.3 A does not have included rows, without loss of generality we can assume that there is a column k such that a_{ki}=1, a_{kj}=-1. As neither the symmetric of row i is bi-included in row j, either there exists a column k' such that either $a_{k'i}$=1, $a_{k'j} = 1$ or $a_{k'i} = -1$, $a_{k'j} = -1$. In both cases, $a_{k'i}$, $a_{k'j}$ $a_{k'i}$, $a_{k'j}$ form one of the matrices of Figure 5.3, what leads to a contradiction, according to Lemma 5.2 and Theorem 5.3. Then, $\mathcal{C}$ verifies the conditions of the Theorem.

Conversely, let $\mathcal{C}$ be the labeled family of k complete subgraphs of G given by hypothesis. Consider the matrix $B \in \{0, 1, -1\}^{k\times n}$ where $b_{ij} = 1$ (respectively, -1) if vertex v_j belongs to the complete C_i with label 1 (respectively -1) and $b_{ij} = 0$ otherwise. Consider B^T. Since $\mathcal{C}$ is a bicovering, B^T contains at least a 1 and a -1 in each row. As $\mathcal{C}$ is a split of G and is bipartite-Helly, B^T has no included rows and its columns are bipartite-Helly. According to Theorem 5.3, B^T is a biclique matrix of a graph H. As observed below, the induced graph of $((B^T)^+)^T$ is the biclique graph of H, i.e. $((B^T)^+)^T = B^+$ induces $KB(H)$. By hypothesis, every edge of G belongs to a complete of $\mathcal{C}$, implying that the graph induced by B^+ is G. We conclude that $G = KB(H)$. ■

An example of the proof of Theorem 5.5 is given in Figure 5.4. Consider the triangle. The family of complete subgraphs $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_{-1}$ verifies the hypothesis of Theorem 5.5. The subfamily $\mathcal{C}_1$ has all its vertices are labeled with 1, and $\mathcal{C}_{-1}$ uses only labels -1. The matrix B is the labeled incidence $\{0, 1, -1\}$-matrix of the family of complete subgraphs $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_{-1}$. The graph G is the graph bi-induced by B^T. It follows that the triangle is the biclique graph of G.

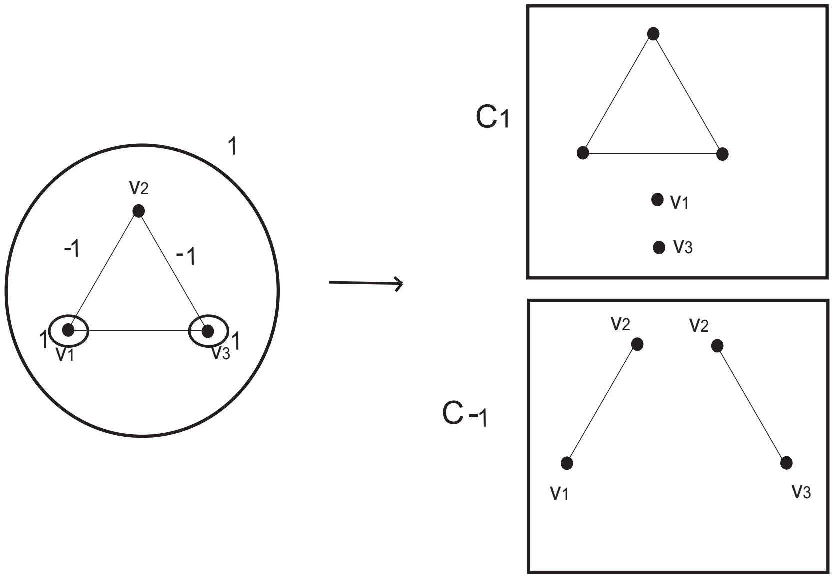
The following Remark is direct from the proof of Theorem 5.5

Remark 5.2 *Let G be a graph and let $\mathcal{C}$ be a $\{1, -1\}$-labeled family of subsets of $V(G)$. Let B be the labeled incidence $\{0, 1, -1\}$-matrix of $\mathcal{C}$. Then, $\mathcal{C}$ verifies the conditions of Theorem 5.5 if and only if B^T is the biclique matrix of a graph H. Furtheremore, $KB(H) = G$.*

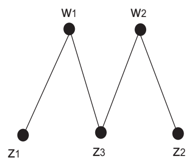
As a corollary of Theorem 5.5, we prove that the recognition problem for bicliques graphs is in NP.

Theorem 5.6 *Let G be a graph with n vertices. The problem of determining if G is a biclique graph is NP.*

Proof: A certificate for G being a biclique graph is a $\{1, -1\}$labeled family $\mathcal{C}$ of m completes subgraphs of G, satisfying the conditions of Theorem 5.5. First, we show that we can restrict to families $\mathcal{C}$ of size $O(n^2)$, where $V(G) = n$. For every vertex v_i, consider C^{i_1}, $C^{i_{-1}}$ the subsets in $\mathcal{C}$ containing vertex v_1 with labels 1, -1, respectively. For every edge v_iv_j, consider the subgraph $C^{ij} \in \mathcal{C}$ such that $v_iv_j \in C^{ij}$. Finally, for every pair of vertices v_i, v_j, consider the subgraphs $C^{i,j}$, $C^{j,i}$ such that $v_i \in C^{i,j}$, $v_j \notin C^{i,j}$ and $v_j \in C^{j,i}$, $v_i \notin C^{j,i}$. The subfamily $\mathcal{C}' = \{C^{i_1}, C^{i_{-1}}, C^{i,j}, C^{j,i}, C^{ij}\}_{i,j=1\ldots n}$ verifies conditions 1), 2) 3) of Theorem 5.5 and contains $O(n^2)$ subsets. Remark 5.2 completes the proof of the Theorem. ■



$$B = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \end{array} \quad B^T = \begin{array}{c} \begin{matrix} z_1 & z_2 & z_3 & w_1 & w_2 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix} \end{array}$$



G

Figure 5.4: The triangle is the biclique graph of G

The next Theorem proves that graphs of more than 3 vertices without induced diamonds are not biclique graphs.

Theorem 5.7 *Let G be a connected diamond-free graph with at least four vertices. If G is not a complete graph, then G is not a biclique graph.*

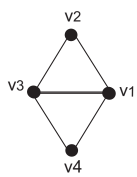
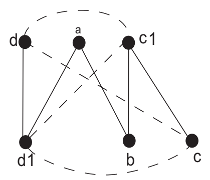


Figure 5.5: Diamond

Proof: Let v_1, v_3, v_2 be the vertices that induce a P_3, and let $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$. Suppose G is the biclique graph of the graph H. Let B_1, B_2 and B_3 be the corresponding bicliques of v_1, v_2, v_3 in H. Let a be the vertex in the intersection of B_1 and B_2, and let b be a vertex in the intersection of B_2 and B_3. Suppose it is the case, ab is an edge of H. If a forms a triangle with some vertex c of B_3 and b, then ac is contained in some biclique B which intersects B_1, B_2 and B_3. Observe that c is not in B_1 because B_1 and B_3 are disjoint bicliques and c is not in B_2 because $b \in B_2$, $B \neq B_1, B_2, B_3$. It follows that the corresponding vertex v in G along with v_1, v_2 and v_3 induce a diamond, which leads to a contradiction. As $a \notin B_3$ there exists some vertex $c \in B_3, c \notin B_1$, c not adjacent to b such that ac is not an edge of H. Similarly, we can conclude that there is a vertex $d \in B_1, d \notin B_3$, d not adjacent to a such that bd is not an edge of H. Take a vertex $c_1 \in B_3$ adjacent to c and $d_1 \in B_1$ adjacent to d. We have already proved that a can not be adjacent to c_1, neither b is adjacent to d_1. If c_1,d_1 are not adjacent vertices, then either the complete bipartite subgraph induced by $\{d_1, a, b\}$ or $\{a, b, c_1\}$ is included in some biclique different from B_i, $i = 1, 2, 3$. Then, the corresponding vertex in G forms a diamond with v_1, v_2 and v_3 what is not possible. (Figure 5.6)



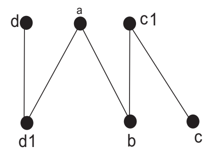


Figure 5.6: Alternatives for vertices a, b, c, c_1, d_1

Then, c_1d_1 is an edge of H. If d is not adjacent to c_1, consider the bipartite complete subgraph $\{d_1, d, a, c_1\}$. It is included in some biclique B. As $d \notin B_3$, it is clear that $B \neq B_3$. As $c_1 \in B_3$, and B_1, B_3 are disjoint bicliques, it is clear that $B \neq B_1$. Finally, as $b \in B_2$ and b is not adjacent to d it follows $B \neq B_2$. As B intersects B_i, i=1,2,3, there is a vertex in G which forms a diamond with v_1, v_2, and v_3 which leads to a contradiction.

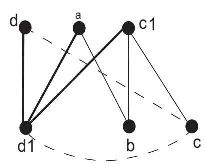


Figure 5.7: Case c_1d is not an edge

It follows that c_1 and d are adjacent. Consider the complete bipartite subgraph $\{d, c_1, b\}$. It is included in some biclique B. As $d \in B_1$, and B_1 and B_3 are disjoint, $B \neq B_3$. As $b \notin B_1$, $B \neq B_1$. Finally, $B \neq B_2$ because a is in B_2, and it is not adjacent to d nor to c_1. (Figure 5.8)

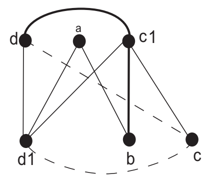


Figure 5.8: Bipartite subgraph induced by vertices d, c_1, b

Then, the corresponding vertex in G forms a triangle with vertices v_1,v_2,v_3, what contradicts the hypothesis.

Therefore, the case where ab is an edge cannot occur.

We examine the situation where ab is not an edge of H. Let c be a vertex in B_2 adjacent to a and b. We have just proved that c can not be in B_1 nor in B_3. If a is adjacent to two adjacent vertices d, d_1 of B_3 as in Figure 5.9 , then the biclique B which contains the edge ad_1 intersecs B_1, B_2, B_3. Since it contains a, $B \neq B_3$, because $b \in B_2$ and b is not adjacent to a nor to d_1, and $B \neq B_1$ while $B \neq B_2$ because $d_1 \notin B_1$. It follows that the corresponding vertex of the biclique B in G forms a diamond with v_1, v_2, v_3, what is not possible.

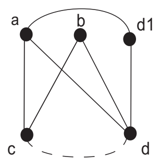


Figure 5.9: Alternatives for vertices a, b, c, d, d_1

As $a \notin B_3$ and does not form a triangle with two vertices of B_3, there is a vertex $d \in B_3$ not adjacent to a and adjacent to b. Suppose c is adjacent to d. Consider the biclique B containing $\{d, c, a\}$, because $c \notin B_1, B_3$ and $b \in B_2$, $B \neq B_1, B_2, B_3$. The corresponding vertex of B in G together with v_1, v_2, v_3 induce a diamond, what can not occur.

Analogously, we can affirm that there is a vertex f in B_1 adjacent to a, not adjacent to b. As we already have proved, c can not be adjacent to f. Consider the two different bicliques B_i and B_j containing the vertices $\{f, a, c\}$ and $\{b, c, d\}$ respectively. It is clear that $B_i, B_j \neq B_2$ because, as a and b are in B_2, neither f or d are in B_2. It follows that there is an induced diamond among the vertices v_i, v_j v_1, v_2 and v_3 (Figure 5.10). ■

As a direct consequence of the proof of Theorem 5.7 we obtain the following Corollary.

Corollary 5.3 *If G is a biclique graph, every edge which belongs to an induced P_3 is contained in an induced diamond.*

The following result about trees is direct from Theorem 5.7.

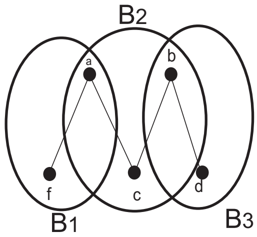


Figure 5.10: Bicliques B_1, B_2, B_2

Corollary 5.4 *Trees with more than 2 vertices are not biclique graphs.*

The special case of biclique graphs of bipartite graphs is considered in the next Theorem.

As a Corollary of Theorem 5.5 we obtain the following characterization of biclique graphs of bipartite graphs.

Theorem 5.8 *A graph G is a biclique graph of a bipartite graph if and only if there exists a $\{1,-1\}$-labeled family $\mathcal{C}$ of subsets of $V(G)$, such that:*

1. *$\mathcal{C}$ can be divided into two subfamilies, $\mathcal{C}_1, \mathcal{C}_2$ such that $\mathcal{C}_1$ is {1}-labeled and $\mathcal{C}_2$ is a {-1}-labeled subfamily.*
2. *Each subset induces a complete graph in G and $\mathcal{C}$ covers the edges of G*
3. *$\mathcal{C}$ is a bicovering of $V(G)$*
4. *$\mathcal{C}$ is bipartite-Helly*
5. *$\mathcal{C}$ is a split of $V(G)$*

Proof: Let G be a biclique graph of a bipartite graph H with bipartition V_1, V_2, and let A be a bipartite biclique matrix of H. Let G be the graph induced by $(A^+)^T$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be the labeled family of complete subgraphs of G induced by the rows of A^T, relative to the bipartition V_1, V_2 respectively. As we have proved in Theorem 5.5, the family $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ verifies properties $2, 3, 4, 5$. As A is bipartite, it is easy to see that $\mathcal{C}_1$, $\mathcal{C}_2$ also verifies property 1.

The converse follows directly from Theorem 5.5 . ∎

We prove that complete graphs are biclique graphs of bipartite graphs.

Proposition 5.1 *Complete graphs are biclique graphs of bipartite graphs.*

Proof: We will use an inductive argument. First, observe that $KB(P_4) = K_3$. Assume that $KB(G) = K_n$, G being a bipartite graph. We construct inductively a bipartite graph G' such that $K_B(G') = K_{n+1}$. Let V_1 and V_2 be the bipartition of G. Add to G vertices v_1' and v_2' and the set of edges

$\{(v_1', w),\ w \in V_2\} \cup \{(v_1', v_2')\}$. The resulting bipartite graph G' verifies that $K(G') = K_{n+1}$. ■

Using the given characterization for biclique graphs of bipartite graphs, we obtain a different proof for Theorem 5.7, restricted to the bipartite graphs of bipartite graphs. First, we prove that cycles greater that 4 are not biclique graphs of bipartite graphs. Although this is a direct consequence of Theorem 5.7, we formulate a separate proof of it, using similar techniques, as those we employed in Theorem 5.9.

Proposition 5.2 *Graphs C_k, $k \geq 4$ are not biclique graphs of bipartite graphs.*

Proof: Let $G = C_k$ be an induced cycle of more than 3 vertices. Suppose it is a biclique graph of a bipartite graph. Hence, by Theorem 5.8 there exist in G a $\{1, -1\}$-labeled bicovering with properties of Theorem 5.1. Let $\{v_1, v_2, \cdots v_k\}$ be vertices of G. As $\mathcal{C}$ covers edges of G, every edge $l_i = (v_i, v_{i+1})$, is a subset of $\mathcal{C}$. Observe that $\mathcal{C}_1$ (analogously, $\mathcal{C}_2$), can not contain an edge, not having a consecutive edge in $\mathcal{C}_1$. Otherwise, suppose $l_i = (v_i, v_{i+1}) \in \mathcal{C}_1$, $l_{i+1} = (v_{i+1}, v_{i+2}) \in \mathcal{C}_2$ and $l_{i+2} = (v_{i+2}, v_{i+3}) \in \mathcal{C}_1$. It contradicts the hypothesis of $\mathcal{C}$ being bipartite Helly. Consider the edge (v_1, v_2) and suppose it belongs to $\mathcal{C}_1$. If it is the case that no edge of G belongs to $\mathcal{C}_2$, since $\mathcal{C}$ is a split, every vertex belongs to $\mathcal{C}_2$. In particular v_1 and v_2 belong to $\mathcal{C}_2$. Since both complete subgraphs, $\{v_1\}$, $\{v_2\}$ intersect the complete (v_1, v_2) of $\mathcal{C}_1$, it contradicts the hypothesis of $\mathcal{C}$ being bipartite-Helly. If it is the case that every edge also belongs to $\mathcal{C}_2$, as we have proved before, every three consecutive edges contradict the bipartite-Helly property. Then, there is an edge, suppose l_1, which belongs to $\mathcal{C}_1$ and not to $\mathcal{C}_2$ and an edge l' belonging to $\mathcal{C}_2$ and not to $\mathcal{C}_1$. Let s be the minimum edge (v_s, v_{s+1}) belonging to $\mathcal{C}_2$. As we observed before, there exist an ordering for vertices of G such that $s \geq 3$. Since s is minimum, $l_{s-1}, l_{s-2} \notin \mathcal{C}_2$. As $\mathcal{C}$ is a split, vertex v_{s-1} must belong to a complete of $\mathcal{C}_1$ and $\mathcal{C}_2$, so the complete $\{v_{s-1}\}$ belongs to $\mathcal{C}_2$. It follows that $\{v_{s-1}\} \in \mathcal{C}_2$, $l_s \in \mathcal{C}_2$, and $l_{s-1} \in \mathcal{C}_1$ are a bipartite intersecting family of $\mathcal{C}$ with no common intersection. Absurd. ■

We can see by Figure 5.11 that P_3 is not a biclique graph of a bipartite graph, since all possible labels are considered for its edges and all of them lead to a contradiction.

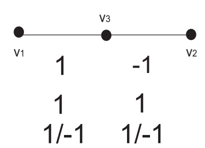


Figure 5.11: Graph P_3 and all possible labels of its edges.

Remark 5.3 *The graph C_3 is the biclique graph of P_5*

Theorem 5.9 is a direct consequence of Theorem 5.7. However, we give another proof from a new point of view based on the characterization of biclique graphs of bipartite graphs.

Theorem 5.9 *If G is not a complete graph, diamond-free graph, with more that 3 vertices, then G is not a biclique graph of a bipartite graph.*

Proof: Let v_1, v_3, v_2 vertices that induce a P_3, and let $l_1 = (v_1, v_3)$, $l_2 = (v_3, v_2)$. Suppose G is a biclique graph. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be the family of complete graphs given by Theorem 5.8.

As $\mathcal{C}$ is a bicovering and every edge of G belongs to a complete subgraph of $\mathcal{C}$, there exists some complete subgraph C_1 containing l_1.

Case a): Suppose every complete subgraph that contains l_1 is in the subfamily $\mathcal{C}_1$ (analogously $\mathcal{C}_2$). As v_1 must be labeled by both labels, there must be a complete subgraph C_2 in $\mathcal{C}_2$ such that $v_3 \notin C_2$, (Figure, 5.12).

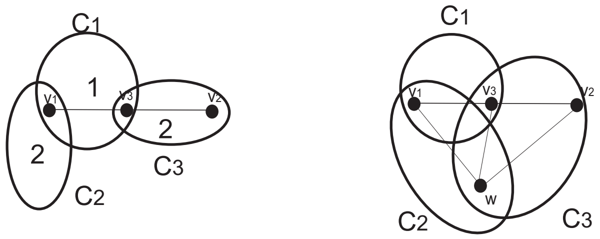


Figure 5.12: Complete subgraphs C_1, C_2, C_3

If l_2 is contained in some complete subgraph C_3 of $\mathcal{C}_2$, consider the bipartite intersecting subfamily $\{C_1\} \cup \{C_2, C_3\}$. By hypothesis, $\mathcal{C}$ is bipartite-Helly, then it must have a common intersection, i.e. there exist a vertex w which forms a diamond with vertices v_1, v_2, v_3, which is an absurd. We conclude that every complete subgraph containing l_2 is in $\mathcal{C}_1$.

As v_3 is contained in $\mathcal{C}_2$, there exists a C_4 in $\mathcal{C}_2$ which contains v_3 such that $v_2, v_1 \notin C_4$. Then the subfamily $\{C_1\} \cup \{C_2, C_4\}$ has a common vertex w'. Observe that w' is not adjacent to v_2, otherwise vertices v_1,v_2, v_3, w' would induce a diamond, (Figure 5.13).

Let C_6 be a complete subgraph containing l_2 in $\mathcal{C}_1$. Clearly, $v_1, w' \notin C_6$. As v_3 is covered by $\mathcal{C}_2$ and l_2 is not covered by $\mathcal{C}_2$, there exists a complete subgraph C_5 in $\mathcal{C}_2$ such that $v_3, v_1, w' \notin C_5$. Again, the subfamily $\{C_6\} \cup \{C_4, C_5\}$ has a common vertex w''. Then, v_2, v_3, w' and w'' induce a diamond contradicting the hypothesis, (Figure 5.14).

Case b). Edges l_1 and l_2 are both covered by $\mathcal{C}_1$ and $\mathcal{C}_2$. As $\mathcal{C}$ is a split of the vertices of G, there exists a complete subgraph C_3 which contains v_1, and not v_3. Without loss of generality, we can suppose $C_3 \in \mathcal{C}_1$. Let $C_1 \in \mathcal{C}_2$

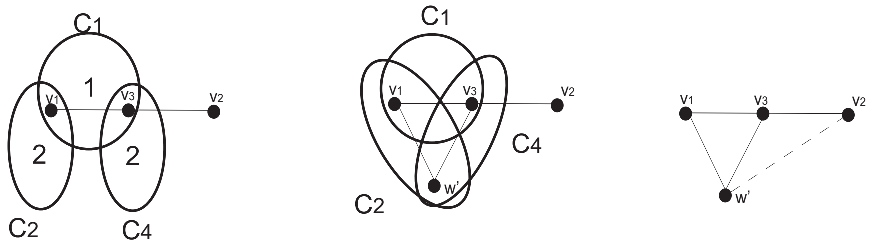


Figure 5.13: Complete subgraphs C_1, C_2, C_4 and the induced diamond v_1, v_2, v_3, w'

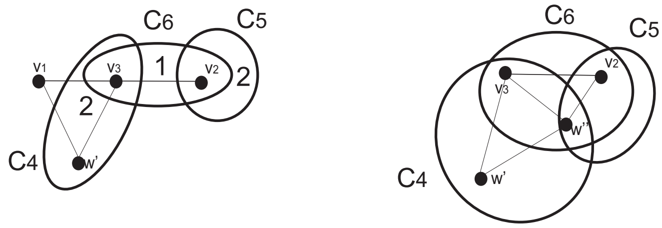


Figure 5.14: Complete subgraphs C_4, C_5, C_6 and the induced diamond v_2, v_3, w', w''

and $C_2 \in \mathcal{C}_1$ the complete subgraphs covering l_1 and l_2 respectively. Then, the subfamily $\{C_2, C_3\} \cup \{C_1\}$ has a common vertex w. Then, v_1 v_2, v_3, and w induce a diamond, (Figure 5.15). ■

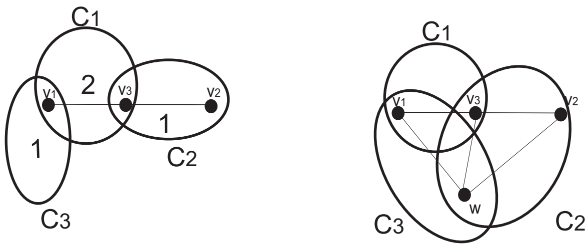


Figure 5.15: Complete subgraphs C_1, C_2, C_3 and the induced diamond v_1, v_2, v_3, w

5.4 Bipartite-Helly families

In this section we study bipartite-Helly labeled-families. We give a characterization for bipartite-Helly labeled-families that leads to a polynomial time algorithm for the recognition problem.

Recall the following definitions that we have introduced in Chapter 4. Let S a set of n elements, $L = \{1, -1\}$ a set of labels and let $\mathcal{F}$ be an L-labelled family of m subsets of S. For every subset $S' = \{x_i, x_j, x_k\}$ of three elements of S, consider every triple l_i, l_j, l_k, $1 \leq i, j, k \leq m$, $l = 1, -1$. Let $\mathcal{F}^1\{l_i, l_j, l_k\}$ be the subfamily of $\mathcal{F}$ of subsets which contains at least two elements $x_s \in S'$, $x_r \in S'$, with label l_s, l_r, respectively. Similarly, let $\mathcal{F}^2_{\{l_i,l_j,l_k\}}$ be the subfamily of $\mathcal{F}$ of subsets which contains at least two elements $x_s \in S'$, $x_r \in S'$, with label $-l_s$, $-l_r$, respectively.

The following Theorem characterizes $\{1, -1\}$-labeled bipartite-Helly families.

Theorem 5.10 *A $\{1, -1\}$-labeled family $\mathcal{F}$ is bipartite-Helly if and only if*

1. *Every bipartite-intersecting subfamily $\mathcal{F}' = \{F_i\} \cup \{F_j\}$ of $\mathcal{F}$ has a good intersection*

2. *Every bipartite-intersecting subfamily $\mathcal{F} = \{F_i\} \cup \{F_j, F_k\}$ has a common element*

3. *Every subfamily $\mathcal{F}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{F}^2_{\{l_i,l_j,l_k\}}$ has a common intersection.*

Proof: We prove that $\mathcal{F}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{F}^2_{\{l_i,l_j,l_k\}}$ is a bipartite intersecting family. First, we prove that $\mathcal{F}^1_{\{l_i,l_j,l_k\}}$ and $\mathcal{F}^2_{\{l_i,l_j,l_k\}}$ are well labeled. Let $F_r, F_s \in \mathcal{F}^1_{\{l_i,l_j,l_k\}}$. Then there is an element, suppose x_i, which belongs to both subsets with label l_i. If F_r, F_s intersect in an element with different label, $\{F_r\} \cup \{F_s\}$ is a bipartite-intersecting family with not a good intersection, absurd. Analogously, $\mathcal{F}^2_{\{l_i,l_j,l_k\}}$ is a well labeled family. Finally, let $F_r \in \mathcal{F}^1_{\{l_i,l_j,l_k\}}$, $F_s \in \mathcal{F}^2_{\{l_i,l_j,l_k\}}$. There is an a common element, suppose x_j, that belongs to F_r, F_s with different label, since x_j has label l_j in F_r, and $-l_j$ in F_s. We conclude that $\mathcal{F}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{F}^2_{\{l_i,l_j,l_k,l_k\}}$ is a bipartite-intersecting family and therefore, it has a common element.

Conversely. Let $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2$ be minimal bipartite-intersecting subfamily with no common element. Then, $\mid \mathcal{F}_1 + \mathcal{F}_2 \mid \geq 4$. Consider the case $\mid \mathcal{F}_1 \mid = \mid \mathcal{F}_2 \mid = 2$. Let x_1 be the common element to $\mathcal{F}_1 \setminus \{F_{i_1}\} \cup \mathcal{F}_2$. Let l_1 be the label of x_1 in $\mathcal{F}_2$ (recall that since $\mathcal{F}_1$ is well labeled, every element has the same label in $\mathcal{F}_2$). Analogously, let x_2 be the common element to $\mathcal{F}_1 \setminus \{F_{i_2}\} \cup \mathcal{F}_1$. Let l_2 be the label of x_2 in $\mathcal{F}_2$. Finally, let x_3 be the element belonging to $\mathcal{F}_1 \cup \mathcal{F}_2 \setminus \{F_{j_1}\}$. Let l_3 be the label of x_3 $\mathcal{F}_2$. Consider $\mathcal{F}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{F}^2_{\{l_1,l_2,l_3\}}$. We prove that $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2$ is included in $\mathcal{F}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{F}^2_{\{l_1,l_2,l_3\}}$. Subset F_{i_1} contains x_2 and x_3. Since $F_{j_2} \in \mathcal{F}_2$ contains x_1, x_2 and x_3, the labels of x_{j_2} in F_1, F_2 and F_3 are l_1, l_2, l_3 respectively. Then, as F_{j_2} intersect F_{i_1} and both contain x_2 and x_3, theirs labels in F_{i_1} are $-l_2$, $-l_3$ respectively. Analogously, F_{i_2} is contains x_1, x_3 with labels $-l_1$, $-l_3$, respectively. Finally, F_{j_1} contains in x_1, x_2 with labels l_1, l_2 respectively. It follows that $\mathcal{F}_1 \subseteq \mathcal{F}^1_{\{l_1,l_2,l_3\}}$, $\mathcal{F}_2 \subseteq \mathcal{F}^2_{\{l_1,l_2,l_3\}}$.

The case where $\mathcal{F}_1 \geq 3$ is similar. We consider $\mathcal{F}_1 \setminus \{F_{i_1}\} \cup \mathcal{F}_2$, $\mathcal{F}_1 \setminus \{F_{i_2}\} \cup \mathcal{F}_2$ and $\mathcal{F}_1 \setminus \{F_{i_3}\} \cup \mathcal{F}_2$ and conclude there are elements x_1, x_2, x_3, such that $x_j \notin F_{i_j}$. Finally, consider l_1, l_2, l_3 the labels of the $x_1, x_2, x_3 \in \mathcal{F}_1$, respectively. In any case, it follows that $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2$ has a common element, what is an absurd.

Finally, by Lemma 5.3, every bipartite-intersecting subfamily has a good intersection. We conclude that $\mathcal{F}$ is bipartite-Helly. ∎

We remark that Theorem 5.10 implies that bipartite-Helly $\{1, -1\}$-labeled families can be recognized in polynomial time.

The algorithm is described below. For a given labeled family $\mathcal{F}$, it answers YES or NO, depending on whether $\mathcal{F}$ is a bipartite-Helly. Let S a set of n elements, $L = \{1, -1\}$ a set of labels, and $\mathcal{F}$ a family of labeled m subsets of S.

Algorithm 5.1 ***Bipartite-Helly.*** *Check if every bipartite-intersecting subfamily $\{F_1\} \cup \{F_j\}$ has a good intersection. Consider every bipartite-intersecting subfamily $\{F_i\} \cup \{F_j, F_k\}$. Check if it has a common intersection. If not, answer NO ad stop. For every v_i, v_j, v_k, $1 \leq i, j, k \leq n$, $l = 1, -1$, consider $\mathcal{F}^1_{\{l_i,l_j,l_k\}}$, $\mathcal{F}^2_{\{l_i,l_j,l_k\}}$. Check if $\mathcal{F}^1_{\{l_i,l_j,l_k\}}$, $\mathcal{F}^2_{\{l_i,l_j,l_k\}}$ has a common intersection. If this condition is not satisfied for any l_i, l_j, l_k , answer No and stop. Otherwise answer Yes.*

The Algorithm 5.1 requires $O(m^3 n + m^3 n)$ time and $O(nm)$ space.

5.5 Classes of biclique graphs

Is this section we study some subclasses of biclique graphs.

We need the following definitions. Given a family $\mathcal{C}$ of subgraphs of a graph G, a $\mathcal{C}$-bicovering of the family $\mathcal{C}$ is a pair of subfamilies $\mathcal{C}_1$, $\mathcal{C}_2$, such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and each vertex of G is covered by $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e. each vertex belongs to at least one subgraph of $\mathcal{C}_1$ and one subgraph of $\mathcal{C}_2$. A family $\mathcal{C}$ of subgraphs is Helly-bicovered if there exists a $\mathcal{C}$-bicovering, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ that verifies the bipartite-Helly property, considering $\mathcal{C}_1$ as a $\{1\}$-labeled family, and $\mathcal{C}_2$ as a $\{-1\}$-labeled family. When $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, we say that the $\mathcal{C}_1, \mathcal{C}_2$ is an *independent bicovering.*

Let G be a a bipartite graph, with bipartition $V_1 \cup V_2$. Say that G has no *dominated bipartition* if both parts V_1, V_2 have vertices which are not dominated by some vertex of the other bipartition.

Let $\mathcal{CHB}$ be the class of graphs such that the family of cliques is Helly-bicovered.

Let $\mathcal{CHBD} \subseteq \mathcal{CHB}$ be the class of graphs such that the family of cliques is Helly-bicovered and has no dominated vertices.

Let $\mathcal{CHBDI}$ be the class of graphs such that the family of cliques is independent Helly-bicovered and has no dominated vertices.

Let $BHED$ be the class of bipartite biclique-Helly graphs without strictly dominated vertices.

Let $\mathcal{BHD}$ be the class of bipartite biclique-Helly graphs with no dominated vertices.

Let $\mathcal{BHDB}$ be the class of bipartite biclique-Helly graphs with no dominated bipartition.

We study the classes of $KB(\mathcal{BHDB})$, i.e. biclique graphs of bipartite biclique-Helly graph with no dominated bipartition.

Theorem 5.11 *Let $H \in \mathcal{BHDB}$, and let $G = KB(H)$, then:*

1. *There exists a bicoloring V_1, V_2 of the vertices of the clique graph of G such that non empty family of bichromatic cliques of $K(G)$ is Helly and the family of columns of the clique matrix of $K(G)$ is bipartite-Helly, relative to the bipartition $V_1 \cup V_2$.*

2. *$K(G)$ can be constructed by removing dominated vertices from the induced subgraph of the closed neighborhood graph of a bipartite biclique-Helly graph.*

Proof: By hypothesis, $G = KB(H)$, where H is a bipartite biclique-Helly graph with bipartition $V_1 \cup V_2$, and no dominated bipartition. Consider H', the bipartite biclique-Helly graph obtained by removing the dominated vertices of H. Since H has no dominated bipartition, H' has at least one edge. Let $A^B_{H'}$ be the biclique matrix of H'. By Theorem 6.5, $(A^{B+}_{H'})^T$ is the clique matrix of G.

It is clear that the transpose of the clique matrix of G induces $K(G)$, i.e. $A^{B+}_{H'}$ induces $K(G)$. On the other hand, $A^{B+}_{H'}$ induces the subgraph of the closed neighborhood graph $N_c(H)$, obtained by eliminating the dominated vertices, since the set of dominated vertices of H is the same that of $N_c(H)$. We have already proved in Proposition 6.1 that every row of A^{B+}_{H} induces a clique in $N_c(G)$. Then, by removing dominated vertices of H, every not included row of $A^B_{H'}$ is still a biclique in H and then, a clique in $N_c(H)$ and so in $K(G)$. As H is biclique-Helly, so are the remaining bicliques after removing dominated vertices. The bicoloring V_1, V_2 of vertices of $N_c(H)$ gives a bicoloring to $K(G)$, where every clique induced by rows of A^{B+}_{H} is bichromatic. Moreover, we already proved in Proposition 6.1 that those cliques are the only bichromatic cliques of $N_c(H)$. Then, $K(H)$ is bichromatic-Helly. Finally, as the clique matrix of $K(G)$ consists of the matrix $A^{B+}_{H'}$, with additional rows corresponding to monochromatic cliques. It follows that the columns of the clique matrix of $K(G)$ is bipartite-Helly for the bipartition V_1, V_2 and every two vertices that belong to a monochomatic clique, also belong to some bichromatic clique. ∎

Theorem 5.12 *The following statements hold.*

1. *$KB(BHD) = CHBDI$*

2. $CHBD \subseteq KB(BHED)$.

Proof: First we prove that $KB(BHD) \subseteq CHBDI$. Let G be a bipartite biclique-Helly graph with no dominated vertices. Let A_B be a biclique matrix of G where the first k columns correspond to the vertices of the bipartition V_1 and the next $n-k+1$ correspond to vertices of V_2. As G has no dominated vertices and is biclique-Helly, by Theorem 6.5, $(A^{B+})^T$ is the clique matrix of $KB(G)$. Let $\mathcal{C}_1$ be the set of cliques that correspond to the first k rows of A_B^T and let $\mathcal{C}_2$ be the set of cliques that correspond to the next $n-k+1$ rows. It is clear that $\mathcal{C}_1$ and $\mathcal{C}_2$ are not empty and verify the bipartite-Helly property.

As A_B is a biclique matrix, by Theorem 5.4 $(A^{B+})^T$ does not have included rows and every row has an entry equal to 1 in the first k columns and another in the last $n-k+1$ columns. Then, every vertex of $KB(G)$ belongs to some clique of $\mathcal{C}_1$ and some clique of $\mathcal{C}_2$, where $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ and $KB(G)$ has no dominated vertices. It follows that the cliques of $KB(G)$ are independent Helly-bicovered by $\mathcal{C}_1$ and $\mathcal{C}_2$.

Before we prove that $CHBDI \subseteq KB(BHD)$, first we prove that $CHBD \subseteq KB(BHED)$. Let H be a graph of n vertices in $CHBD$. Let $\mathcal{C}_1 \cup \mathcal{C}_2$ the $\mathcal{C}$-bicovering of the cliques of H, $\mid \mathcal{C}_1 \mid = k_1$, $\mid \mathcal{C}_2 \mid = k_2$.

Let A be a $\{0, 1, -1\}$ matrix with n columns and $k_1 + k_2$ rows, where the columns represent the vertices of H and the rows represent the cliques of $\mathcal{C}_1$ and $\mathcal{C}_2$ in the order such that $a_{ij} = 1$ if vertex j belongs the clique C_i in V_1, $a_{(i+k_1)j} = -1$ if vertex j belongs to the ith clique of $\mathcal{C}_2$. Consider A^T. Its columns verify the bipartite-Helly property for the bipartition $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1$ is a $\{1\}$-labeled family and $\mathcal{C}_2$ is a $\{-1\}$-labeled family. Each row has a non zero entry for some column of $\mathcal{C}_1$ and some column of $\mathcal{C}_2$ and it has not included rows, since H has no dominated vertices. By Theorem 5.4 it is a biclique matrix of some graph G. As the transpose of $(A^+)^T$ induces the biclique graph of G, then $H = KB(G)$. Observe that the clique matrix of H is the matrix obtained by eliminating the twin rows of A. As the columns of A^T represent cliques of H, it has no strictly included columns, so there are no strictly dominated vertices in G. As the columns of the clique matrix of H are Helly, by Theorem 4.1, and Theorem 2.1, so are the columns of A. Then, the rows of A^T verify the Helly property which implies that the family of bicliques of G is Helly.

To prove that $KB(BHD) = CHBDI$, observe that if the subfamilies C_1 and C_2 has no common clique, then A is the clique matrix of H and A^T has no included columns. ∎

Theorem 5.13 *Let G be a biclique-Helly bipartite graph, with no dominated bipartition V_1, V_2. Let $\mathcal{C}$ be the cliques of the biclique graph $KB(G)$. Then, there exists a Helly-bicovering $\mathcal{C}_1 \cup \mathcal{C}_2$ of $\mathcal{C}$.*

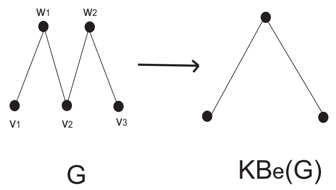


Figure 5.16: The graph P_3 is the E-biclique graph of the graph P_5

Proof: Let A^B be the biclique matrix of G. By Theorem 6.5, $(A^{B+})^T$ without the included rows is the clique matrix of $KB(G)$, $A_{KB(G)}$. Consider $\mathcal{C}_1$ and $\mathcal{C}_2$ the subfamily of cliques induced by the rows of $A_{KB(G)}$ corresponding to columns of V_1 and V_2 in A^B, respectively. By hypothesis, G has no dominated parts V_1, V_2, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are non empty subfamilies. By Theorem 5.4, C_1, C_2 is a Helly-bicovering of $\mathcal{C}$.

5.6 E-biclique graphs

In this section we define the E-biclique graphs and give necessary conditions for a graph to be an E-biclique graph. We also prove that E-biclique graphs are clique graphs.

The biclique graph was defined based on vertex intersection of bicliques. For the E-biclique graphs, we consider the edge intersection of bicliques. Say that two bicliques *edge intersect* if they contain a common edge. The *E-biclique graph* of a graph, denoted by $KB_e(G)$, is the edge intersection graph of the bicliques of G. See an example in Figure 5.16.

The following Theorem gives necessary conditions for a graph to be an E-biclique graph of a bipartite graph.

Theorem 5.14 *Let G a bipartite graph. There exist in $KB_e(G)$ a family of complete subgraphs which verify the Helly property and covers the edges of $KB_e(G)$.*

Proof: Let G be a bipartite graph, $E(G) = \{e_1, ..., e_k\}$ the edges of G and $B_1, ...B_q$ the bicliques of G. Let A be the incidence matrix of the family of bicliques, considering as columns the edges of G, i.e., $a_{ij} = 1$ if edge e_j belongs to biclique B_i. Consider A^T. It induces the graph $KB_e(G)$. Consider the family of complete subgraphs induced by the rows of A^T, i.e., the complete subgraph of $V(KB_e(G))$ L_{e_j}, is the subgraph induced by $\{ w_s \in V(KB_e(G))$ such that $e_j \in B_s\}$. We prove that this family is Helly.

Let $L_{e_{i_1}}, ..., L_{e_{i_l}}$ be a pair intersecting subfamily. Clearly, $L_{e_{i_h}} \cap L_{e_{i_s}} \neq \emptyset$ implies taht there is a biclique in G that contains e_{i_h} and e_{i_s}. Then, its endpoints induce a bipartite subgraph in G. Since $L_{e_{i_1}}, ..., L_{e_{i_l}}$ pairwise intersect and G is bipartite, $e_{i_1}, ..., e_{i_l}$ induce a bipartite subgraph in G. Therefore, there is a biclique B in G containing $e_{i_1}, ..., e_{i_l}$. Consequently,

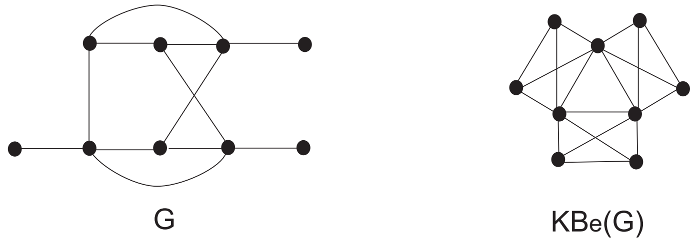


Figure 5.17: Graph G and $K_e(G)$.

the corresponding vertex w_B in $KB_e(G)$ belongs to $L_{e_{i_s}}$ for every s. Finally, to prove that the family covers the edges of G, observe that there is an edge between two vertices w_s, w_t in $KB_e(G)$ if bicliques B_s, B_t in G have a common edge e. Then, vertices w_s, w_t belong to L_e. ∎

Corollary 5.5 *If G is an E-biclique graph of a bipartite graph, then it is a clique graph.*

Proof: It is direct from Theorems 5.1 and 5.14.

Remark 5.4 *If G is not a bipartite graph, the family L_e, $e \in E(G)$, is not necessary a Helly family. In Figure 5.17 we give an example of a graph G, such that there exists no family that verifies the conditions of Theorem 5.14 in its E-biclique graph, $KB_e(G)$.*

Chapter 6

Closed neighborhood graphs and the Helly property

6.1 Introduction

In this Chapter we study the closed neighborhood graph and relate it to the concepts we have studied in the previous chapters.

In Section 2 we give a proof of a characterization of closed neighborhood graphs, using the same techniques we have used to characterized biclique graphs. We consider the general case and the subclass of closed neighborhood graphs of bipartite graphs.

In Section 3 we relate closed neighborhood graphs to biclique graphs. We give a relation between bicliques of a bipartite graph and cliques of its closed neighborhood graph. As a consequence, we obtain conditions for a biclique graph of a bipartite graph to be the clique graph of the closed neighborhood graph. Also we relate the Helly property to properties of the biclique matrix and the mentioned classes.

In Section 4 we define the class of bicliqual graphs and give a characterization. We obtain some conditions for a closed neghborhood graph of a bicliqual graph to be clique-Helly.

The technique we have employed in the characterization of biclique graphs and we use for the characterization of closed neighborhood graphs, is a variation of the *Krausz type* (See also [31, 35, 39, 50, 51]). It is based on properties of matrices: Suppose we want to characterize the intersection graph H of a family $\mathcal{F}$ of subgraphs of G.

1. First, construct the incidence matrix A of the family $\mathcal{F}$.

2. Characterize the incidence matrix based on properties P of the matrix.

3. Consider every column of the matrix as a complete graph in the intersection graph. Ask conditions for the family of complete subgraphs, according to the properties P of A.

4. Finally, conclude that H is an intersection graph of the family $\mathcal{F}$ if and only if there exists a family of complete subgraphs $\mathcal{C}$ in H such that verifies those conditions.

5. The proof follows considering the fact that A^T induces the intersection graph of $\mathcal{F}$.

6.2 The closed neighborhood graph

In this section we give a different proof for the characterization of closed neighborhood graphs using the ideas mentioned above. Also we study the closed neighborhood graphs of a restricted class, the bipartite graphs.

Let G be a graph, the closed neighborhood graph of G, $N_c(G)$, is the intersection graphs of closed neighborhoods of G. If G is bipartite, and $V_1 \cup V_2$ its bipartition, then we say that a vertex of $N_c(G)$ belongs to a part V_i if it corresponds to a neighborhood of a vertex of V_i. Observe that two vertices of V_1 and V_2 are adjacent if and only if the vertices in G of the corresponding neighborhoods are adjacent.(Figure 6.1)

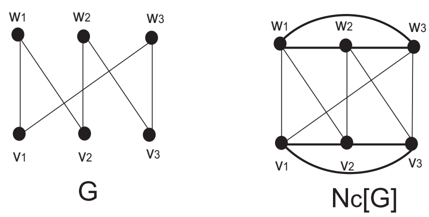


Figure 6.1: Graph G and its closed neighborhood graph

Closed neighborhood graphs are useful in the study of biclique graphs. We remark that closed neighborhood graphs are also known in the literature as *s*quare graphs (See [22, 37, 40, 41, 54]).

The following Theorem gives a characterization of closed neighborhood graphs.

Theorem 6.1 *[41] [42] Let G be a graph and $v_1, ..., v_n$ its vertices. Then G is a closed neighborhood graph if and only if there exists in G a family $\mathcal{F} = \{F_1, F_2, ...F_n\}$ of n complete subgraphs such that*

1. *For every i, $v_i \in F_i$*

2. *Vertex $v_j \in F_k$ if and only if $v_k \in F_j$.*

3. *$\mathcal{F}$ covers the edges of G*

Proof: Suppose G is the closed neighborhood graph of H. Let A be the incidence matrix of the family of closed neighborhoods of H, with n rows and n columns, i.e. $a_{ij} = 1$ if and only if v_iv_j is and edge in G or $i = j$. It is clear that the induced graph of A is G (in fact, matrix A^T induces G, but we use the fact that $A^T = A$). For $i = 1, ..., n$, consider F_i as the complete subgraph of G induced by each row i of A. We prove that $F_1, ..., F_n$ verifies conditions of the Theorem. Since $a_{ii} = 1$ and $a_{ij} = a_{ji}$, conditions $1), 2)$ are satisfied. Finally, since A induces G, the family satisfies condition 3).

Conversely. Let B be the incidence matrix of the family $\mathcal{F}$, with n columns and n rows, i.e., $a_{ij} = 1$ if and only if vertex v_j belongs to the complete subgraph F_i. Construct the graph H of n vertices as follows: v_i is adjacent to v_j if and only if $v_i \in F_j$, i.e. consider row j of B as the incidence vector of the closed neighborhood of v_j. Conditions $1), 2)$ imply that H is well defined, since $a_{ij} = a_{ji}$ and $a_{ii} = 1$. It is clear that the graph induced by B is $N_c(H)$. By condition condition 3), $G = N_c(H)$ ■

Next, a characterization for closed neighborhood graphs of bipartite graphs.

Theorem 6.2 *Let G be a graph with vertices $\{v_1, v_2, ..., v_n\}$. The following statements are equivalent:*

1. *G is a closed neighborhood graph of a bipartite graph*
2. *There exists in G a family $\mathcal{F} = \{F_1, F_2, ...F_n\}$ of n complete subgraphs such that*
   - *(a) $v_i \in F_i$*
   - *(b) If $v_j \in F_i$, then $v_i \in F_j$*
   - *(c) $\mathcal{F}$ covers the edges of G*
   - *(d) There is k such that*
     - *If $j < k$, $v_j \notin F_t$ for every $t < k$, $s \neq t$*
     - *If $j \geq k$, $v_j \notin F_t$ for every $t \geq k$, $s \neq t$.*
3. *There is bipartition $V_1 \cup V_2$ of the vertices of G such that $v_i, v_j \in V_1$ (analogously, V_2) are adjacent if and only if $N(v_i) \cap N(v_j) \cap V_2 \neq \emptyset$ (analogously $N(v_i) \cap N(v_j) \cap V_1 \neq \emptyset$).*
4. *There exists a bipartition $V_1 \cup V_2$ of vertices of G such that*
   - *(a) For every triangle v_1, v_2, v_3 contained in V_1 (analogously, V_2), either there is a vertex in V_2 (V_1) adjacent to all v_1, v_2, v_3 or there exist $w_1, w_2, w_3 \in V_2$ such that $v_1, v_2, v_3, w_1, w_2, w_3$ induce the 3-extended Hajós graph with center v_1, v_2, v_3.(Figure 1.4).*
   - *(b) For every induced graph P_3, with vertices v_1, v_2, v_3, included in V_1, there exist vertices $w_1, w_2 \in V_2$ such that v_1, w_1, w_2, v_3, v_2 induce a 3-fan.*
   - *(c) For every induced P_3, v_1, v_2, v_3, such that $v_1, v_2 \in V_1$ (analogously, V_2), and $v_3 \in V_2$ (V_1), there is a vertex $w \in V_2$ (V_1) such that v_1, w, v_3, v_2 induce a diamond.*

(d) *There is no induced P_3, v_1, v_2, v_3, such that $v_1, v_3 \in V_1$ (analogously, V_2), $v_2 \in V_2$ (V_1).*

(e) *Every edge v_1v_2 which is not contained in a triangle, nor in a P_3, has its endpoints in different parts.*

Proof:

1) $\Longrightarrow$ 2): By Theorem Theorem 6.1, we construct a family $\mathcal{F} = \{F_1, ..., F_n\}$, where $v_i \in F_i$, and $v_i \in F_j$ if and only if $v_j \in F_i$. It is clear that if G is bipartite, with bipartition $\{v_1, ..., v_{s-1}\} \cup \{v_s, ..., v_n\}$, then $\mathcal{F}$ verifies the required conditions, for $k = s$.

2) $\Longrightarrow$ 3): Let $\mathcal{F} = \{F_1, ..., F_n\}$ be a family verifying the conditions of 2. Let $V_1 = \{v_1, ..., v_{k-1}\}$, and $V_2 = \{v_k, ...v_n\}$. We prove that the bipartition $V_1 \cup V_2$ verifies the requirements of 3. Let $v_i, v_j \in V_1$ be two adjacent vertices $i, j < k$. There is $t \geq k$ such that $v_iv_j \in F_t$, since $\mathcal{F}$ covers the edges of G. Then, $v_t \in N(v_i) \cap N(v_j) \cap V_2$. Conversely, if $v_t \in N(v_i) \cap N(v_j) \cap V_2$, for $v_i, v_j \in V_1$, since $\mathcal{F}$ covers all edges, it follows that $v_iv_t \in F_l$. Then $l = i$ or $l = t$, since $v_i \notin F_m$ for $m < k$, $j \neq m$ and $v_t \notin F_m$ for every $m \geq k$, $m \neq t$. Analogously, $v_jv_t \in F_{l'}$, $l' = j$ or $l' = t$. In any case, it follows that $v_i, v_j \in F_t$ concluding that v_i, v_j are adjacent.

3) $\Longrightarrow$ 4) We prove that the bipartition given by the hypothesis verifies the requirements. Let v_1, v_2, v_3 be a triangle T. Suppose T is contained in V_1 (analogously , V_2). By hypothesis, there is at least a vertex $w \in V_2$ adjacent to v_1, v_2. If some vertex $w \in V_2$ is adjacent to v_1, v_2 and also v_3, the proof is finished. Otherwise, there exist a vertex $w' \in V_2$ adjacent to v_1, v_3, $w \neq w'$. Analogously, there is a vertex $w'' \in V_2$ adjacent to v_2, v_3. Since $N(w), N(w'), N(w'')$ have pairwise intersection in V_1, by hypothesis, they are adjacent. We conclude that the vertices $v_1, v_2, v_3, w, w', w''$ induce the 3-extended Hajós graph with center in v_1, v_2, v_3.

If v_1, v_2, v_3 induce a P_3, then it follows that either it is contained in V_1 (V_2) or v_1, v_2 V_1, and $v_3 \in V_2$ (analogously, V_1, V_2 respectively). If v_1, v_2, v_3 is in V_1, there exists a vertex $w_1 \in V_2$ (V_1) adjacent to v_2. Analogously, there is a vertex $w_2 \in V_2$ (V_1) adjacent to v_2, v_3. Since v_1 and v_3 are not adjacent, $w_2 \neq w_1$. It follows that v_1, v_2, v_3, w_1, w_2 induce a 3-fan.

If it is the case that $v_1, v_2 \in V_1$, and $v_3 \in V_2$ (analogously, $v_1, v_2 \in V_2$, $v_3 \in V_1$), there exists a vertex $w \in V_2$ (V_1) adjacent to v_2. As $N(w) \cap N(v_3)$, it follows that v_1, v_2, v_3, w induce a diamond.

Finally, it is clear that any edge v_iv_j, where $v_i, v_j \in V_1$ (V_2) is contained in a triangle.

4) $\Longrightarrow$ 1) Define the bipartite graph H as follows: For every v_i, there is a vertex $z_i \in V(H)$, and z_iz_j is an edge of H if and only if $v_i \in V_1, v_2 \in V_2$ and v_iv_j is an edge of G. We prove that G is isomorphic to the closed neighborhood graph of H, under the isomorphism which relates v_i to $N[z_i]$.

First, we prove that v_iv_j is an edge of G if and only if $N[z_i]$, $N[z_j]$ intersect. Let v_iv_j be an edge of G. The case where z_i and z_j belong to different parts, is clear. Consider the case $v_i, v_j \in V_1$. By hypothesis, if v_iv_j is contained in a triangle or a P_3. In either cases there is a vertex $w \in V_2$ belonging to v_iv_j. The proof is complete. ∎

We remark that closed neighborhood graphs of bipartite graphs can be recognized in $O(n^4)$ [37] , whereas the recognition problem is NP-complete in the general case [40].

6.3 Closed neighborhood graphs and biclique graphs

Along this section, when we focus on the class of closed neighborhood graphs of bipartite graph.

Closed neighborhood graphs may be a useful tool in the study of biclique graphs. Moreover, bicliques of a graph G can be related to cliques of its closed neighborhood graph. Consequently, their corresponding biclique graph and clique graph, respectively, are related. Properties of the biclique graph of a bipartite graph G can be obtained by looking at properties of the clique graph of the closed neighborhood graph of G.

In this section, we study relations between bicliques of a bipartite graph G and cliques of its closed neighborhood graph, and between the biclique graph of G and the clique graph of G and $N_c(G)$, respectively.

Start with a definition. Let G be a bipartite graph and let $N_c(G)$ be the closed neighborhood graph of G. Let B be a biclique of G. Define $B^\star$ as the complete subgraph of $N_c(G)$ induced by vertices of G corresponding to the closed neighborhoods of the vertices of B. This notation is employed through Chapters 6 and 7.

A relation between bicliques of G and cliques of $N_c(G)$ is given by the following Proposition.

Proposition 6.1 *Let G be a bipartite graph with bipartition $V_1 \cup V_2$ and let $N_c(G)$ be its closed neighborhood graph. Let $\mathcal{B}(G)$ be the set of bicliques of G and let $\mathcal{C}(N_c(G))$ be the set of cliques of $N_c(H)$. Then,*

1. *For every biclique B of G, $B^\star$ is a clique of $N_c(G)$*
2. *If B_1, B_2 are two different bicliques of G, so are the cliques $B_1^\star$ and $B_2^\star$ of $N_c(G)$*
3. *Biclique B_1 intersects biclique B_2 if and only if $B_1^\star$ and $B_2^\star$ intersects.*
4. *$\mathcal{C}(N_c(G))$ can be divided into two disjoint subfamilies as follows: $\mathcal{C}(N_c(G)) = \{B^\star :\ B \in \mathcal{B}(G)\} \cup \{\ C$ such that $V(C) \subseteq V_1$ or $V(C) \subseteq V_2\}$.*

Proof: Let B be a biclique and let $B^\star$ the complete subgraph in $N_c(G)$. First observe that if B_1, B_2 are different bicliques or intersect, also $B_1^\star$ and

$B_2^\star$ are. This follows from the fact that B_1 and B_2 intersect in G if and only if they have a common vertex v, if and only if $B_1^\star$ and $B_2^\star$ contain vertex of $N[v]$. Next, we prove that $B^\star$ is a clique of $N_c(G)$. Suppose it is not a clique. It means there is a vertex v adjacent to every vertex of $B^\star$ in $N_c(G)$. Suppose v corresponds to $N[w]$, $w \in V_1$. Then, v is adjacent to all vertices of $B_1^\star \cap V_2$. As G is bipartite, w is adjacent to every vertex of $B \cap V_2$. We conclude that $B \cup v$ is a complete bipartite subgraph of G, absurd. It follows that $B^\star$ is a clique of H. Finally, it remains to prove that every clique in $N_c(G)$ containing vertices from V_1 and V_2 corresponds to a clique of the form $B^\star$, since it is clear that that every $B^\star$ contains vertices from V_1 and V_2. Let C be a clique of $N_c(G)$ and consider the vertices of G corresponding to the neighborhoods of vertices of $N_c(G)$. Clearly, it is a complete bipartite subgraph B' of G. If it is not a biclique of G, it is included in a biclique B. Then, it follows that C in included in $B^\star$. Consequently, $C = B^\star$ and $B' = B$. ■

By the close correspondence between bicliques of G and cliques of $N_c(G)$, we obtain the following Proposition, relating clique graphs of closed neighborhood graphs and biclique graphs. It is a corollary of the Proposition 6.1.

Proposition 6.2 *Given a graph G, the biclique graph $KB(G)$, is an induced subgraph of $K(N_c(G))$.*

6.3.1 The Helly property and intersection graphs

In Proposition 6.2 we have related graph $KB(G)$, with $K(N_c(G))$. Our next goal is to know when both graphs are indeed, isomorphic. We prove that $KB(G)$ coincides with $K(N_c(G))$ just when the graph G is open neighborhood-Helly.

The next Lemma is clear, but it is useful for the Theorems that follow.

Lemma 6.1 *Let G be a bipartite graphs with bipartition $V_1 \cup V_2$ and let A be its biclique matrix. Then, it is equivalent:*

1. *Each of the subfamilies of the columns of A corresponding to the parts V_1 and V_2 is Helly*
2. *G is open neighborhood-Helly*

Recall that, given a bicoloring $V_1 \cup V_2$ of a graph G, a bichromatic clique is a clique which contains at least a vertex of each parts V_1 and V_2. Recall that a graph is *2-weak colorable* when there is a bicoloring of vertices of G such that every clique of G is bichromatic.

Theorem 6.3 *Let G be a bipartite graph without isolated vertices, $N_c(G)$ its closed neighborhood graph, A_G^B the biclique matrix and $A_{N_c(G)}$ the clique matrix of $N_c(G)$. Finally, let $\mathcal{C}(N_c(G))$ be the set of cliques of $N_c(G)$. Then, it is equivalent:*

1. *G is open neighborhood-Helly*
2. *$\mathcal{C}(N_c(G)) = \{B^\star,\ B$ biclique of G $\}$*
3. *$A_G^{B+} = A_{N_c(G)}$.*

Proof: We prove that 1) and 2) are equivalent. Let G be a open neighborhood-Helly bipartite graph. Suppose there is a clique C in $N_c(G)$ containing vertices only from V_1. Let $v_1, ..., v_k$ be the vertices of C. By definition, vertices $v_i, v_j \in V_1$ are adjacent if and only if the neighborhoods $N[z_i]$, $N[z_j]$ intersect. Then $N[z_1], ..., N[z_k]$ is an intersecting subfamily of neighborhoods. As two vertices of the same part are not adjacent, the subfamily of open neighborhoods $N(z_1)..., N(z_k)$ is intersecting. By hypothesis, there is a vertex $z_t \in V_2$ in G adjacent to $z_1, ..., z_k$. It follows that vertex v_t together with C induce a complete subgraph of $N_c(G)$, absurd. Then, $\mathcal{C}(N_c(G)) = \{B^\star$, B biclique of G $\}$.

Conversely. By contrary, suppose G is not neighborhood-Helly. Let $N(z_1)...N(z_k)$ be an intersecting subfamily of open neighborhoods. Clearly, without loss of generality we can assume that $z_i \in V_1$, $i = 1, ..., k$. Then vertices $v_1, ..., v_k \in N_c(G)$ induce a complete subgraph F. Let $B^\star$ be the clique containing F. It follows that the biclique B which contains vertices $z_1, ..., z_k$ also contains a vertex $w \in V_2$. Clearly, $w \in N(z_i)$ for every i, $i = 1...k$.

Finally, we prove 2) $\iff$ 3). It follows directly from the fact that A_G^{B+} induces the closed neighborhood graph $N_c(G)$ of G, and every row induces $B^\star$. ∎

As corollaries, we obtain the following direct results.

Corollary 6.1 *Given a graph H, the following statements are equivalent:*

1. *H is a closed neighborhood graph of an open neighborhood-Helly bipartite graph*
2. *H is a closed neighborhood graph of a bipartite graph G, and the bipartition of vertices of G induces a 2-weak coloring of H.*
3. *H is 2-weak colorable by the bicoloring induced by a bipartition $V_1 \cup V_2$, where two vertices of V_1 (analogously, V_2) are adjacent if and only if they induce a triangle with some vertex of V_2 (analogously, V_1)*
4. *The columns of the clique matrix of H can be partitioned into V_1, V_2 such that it verifies the bipartite-Helly property, where columns of V_1 have label 1, and columns of V_2, label -1.*

Proof: It is a direct consequence of Theorem 6.3

The next Corollary characterizes the graph G such that $KB(G) = K(N_c(G))$.

Corollary 6.2 *Let G be a bipartite graph. Then, G is open neighborhood-Helly if and only if $KB(G) = K(N_c(G))$.*

Proof: It is a direct consequence of Theorem 6.3. ■

We study the cliques of $N_c(G)$ when the bipartite graph G is not neighborhood-Helly. In other words, we examine the cliques which are contained in one of the parts.

Proposition 6.3 *Let G be a bipartite graph and let $V_1 \cup V_2$ be its bipartition. The cliques of $N_c(G)$ which are contained in one of the parts of the bipartition of G are in correspondence with the maximal intersecting families of open neighborhoods without common vertex.*

Proof: Let C be a clique of $N_c(G)$ included in V_1, (analogously, V_2). Consider the subfamily of the corresponding neighborhoods of G. Clearly, it is a maximal intersecting subfamily. Suppose $w \in V_2$ is a common vertex, then the vertex $N(w)$ of $N_c(G)$ belongs to C, which is a contradiction.

Conversely, it is clear that such an intersecting family $\mathcal{F}$ of neighborhoods contains only neighborhoods of vertices of one part (suppose V_1) and induces a complete subgraph H' of $N_c(G)$. Then, H' is included in a clique C of $N_c(G)$.

Suppose $C = B^\star$, where B is a biclique of G. Let $w \in V_2$ be a vertex of B. It follows that w is a common vertex of the family $\mathcal{F}$, absurd. Then, C is included in V_1. If H' is not C, there is a vertex $N(z_k) \in V_1$ adjacent to every vertex of H'. Then, $\{N(z_k)\} \cup \mathcal{F}$ is an intersecting family, which is an absurd since $\mathcal{F}$ is maximal. ■

As a consequence, we can organize the clique matrix of a closed neighborhood graph of bipartite graphs as follows.

Remark 6.1 *Let G be a bipartite graph with k bicliques. The clique matrix of $N_c(G)$ can be organized as follows.*

-The first k rows are the rows of the biclique matrix of G while the columns are divided into V_1 and V_2. .

-The remaining rows are divided into two groups, first those which have zero entries in columns of V_1 and the other, which have zero entries in columns of V_2.

In Figure 6.2 we can observe the matrices of the graphs of Figure 6.1.

Next we will relate the Helly property applied to the family of bicliques of G, to cliques of $N_c(G)$ and cliques in $KB(G)$. Some of the following properties are a direct consequence of results already presented in the thesis.

$$A_G^B = \begin{pmatrix} v_1 & v_2 & v_3 & | & w_1 & w_2 & w_3 \\ 1 & 0 & 0 & | & -1 & 0 & -1 \\ 0 & 1 & 0 & | & -1 & -1 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & -1 \\ 1 & 1 & 0 & | & -1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & -1 & 0 \\ 1 & 0 & 1 & | & 0 & 0 & -1 \end{pmatrix}$$

$$A_{N_c(G)} = \begin{pmatrix} v_1 & v_2 & v_3 & & w_1 & w_2 & w_3 \\ 1 & 0 & 0 & & 1 & 0 & 1 \\ 0 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 1 \\ 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 0 & 1 & | & 0 & 0 & 1 \\ - & - & - & | & - & - & - \\ 1 & 1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 1 & 1 & 1 \end{pmatrix}$$

Figure 6.2: The Biclique matrix of G and $N_c(G)$

Proposition 6.4 *Let G be a bipartite graph with bipartition $V_1 \cup V_2$. Then:*

1. *If $N_c(G)$ is clique-Helly, then G is biclique-Helly.*

2. *If G is open neighborhood-Helly, then G is biclique-Helly if and only if $N_c(G)$ is clique Helly.*

Proof:

1. By Proposition 6.1, the set $\mathcal{B}^\star = \{B^\star, B \textit{ biclique of } \mathrm{G}\}$ is a subfamily of cliques of $N_c(G)$. Recalling that two cliques of $\mathcal{B}$ intersect if and only if their corresponding bicliques in G intersect, it follows that G is biclique-Helly.

2. Is a direct consequence of Theorem 6.3.

We can use the closed neighborhood graph to prove properties of biclique graphs. We study graphs for which their biclique graph is biclique-Helly. We will use the following result on clique graphs.

Theorem 6.4 *[49] The clique graph of a clique-Helly graph is clique-Helly*

Proof: It a direct consequence from the fact that the transpose of the clique matrix without including rows is the clique matrix of the clique graph (Theorem 2.3). ∎

Proposition 6.5 *If G is open neighborhood-Helly and biclique-Helly, then $KB(G)$ is clique Helly.*

Proof: As G is open neighborhood-Helly and biclique-Helly, $N_c(G)$ is clique-Helly. By Theorem 6.3, $K(N_c(G)) = KB(G)$. By Theorem 6.4 the clique graph of a clique-Helly graph is clique-Helly. ∎

The Helly property related to bicliques in a graphs has a close relation with the biclique graph, by looking at its clique matrix. It is exposed in the following Theorem.

Let A_G^B be the biclique matrix of a graph G. Let $A(G)$ be the matrix obtained from $(A_G^{B+})^T$ by removing the included rows.

Theorem 6.5 *Let A_G^B be a biclique matrix of a bipartite graph G. Let H be the graph induced by $(A_G^{B+})^T$. Then, the following statements are equivalent:*

1. *G is biclique-Helly.*
2. *$A(G)$ is the clique matrix of $KB(G)$.*

Proof: Suppose G is a bipartite biclique-Helly graph. It is clear that $(A_G^{B+})^T$ induces the biclique graph of G, $KB(G)$. As G is biclique-Helly, columns of $(A_G^{B+})^T$ are Helly. Then, $A(G)$ is a clique matrix, according to Theorem 5.2.

Conversely, suppose $A(G)$ is the clique matrix of $KB(G)$. Then, rows of A_G^B without the columns corresponding to the dominated vertices are Helly. We prove that by adding the included columns, the Helly property is preserved. Let $\mathcal{F}$ be a subfamily of intersecting rows. Suppose two rows intersect at an added column. Clearly, the rows also intersect another column, in other words, every intersecting family of columns of A_G^B is still an intersecting family after removing included columns. Then, rows of A_G^B are Helly, i.e. G is biclique Helly. ∎

Remark 6.2 *Proposition 6.5 also can be proved by using the fact that columns of A_G^B are Helly (if G is open neighborhood-Helly) together with Theorem 6.5.*

As a Corollary, we obtain the following result:

Corollary 6.3 *Let A_G^B be a biclique matrix of a bipartite graph G and $A_{N_c(G)}$ be the clique matrix of $N_c(G)$. Let H be the graph induced by $(A_G^{B+})^T$. Then it is equivalent:*

1. *G is open neighborhood-Helly and biclique-Helly.*
2. *$A(G)$ is the clique matrix of $K(N_c(G))$.*

Proof: Suppose G is open neighborhood-Helly and biclique-Helly.By Theorem 6.5, $A(G)$ is the clique matrix of $KB(G)$. If G is open neighborhood-Helly, by Corollary 6.2, $KB(G) = K(N_c(G))$.

Conversely, suppose $A(G)$ is the clique matrix of $K(N_c(G))$. Observe that $A(G)$ induces $KB(G)$. Then, $K(N_c(G)) = KB(G)$ and, according to Corollary 6.2, G is open neighborhood-Helly. It follows that $A_G^{B+} = A_{N_c(G)}$. Then, $A^T_{N_c(G)}$ is the clique matrix of $K(N_c(G))$. By Theorem 6.5, G is biclique-Helly. ■

Theorem 6.6 *A clique graph $K(G)$ is a closed neighborhood graph of a biclique-Helly bipartite graph if and only if there exists a bipartition of vertices of the clique graph of $K(G)$ such that:*

1. *The bichromatic cliques of $K(G)$ are Helly*
2. *Every pair of vertices of a monochromatic clique belongs to a bichromatic clique.*

Proof: First we show that $K(G)$ is a closed neighborhood graph. Let $V_1 \cup V_2$ be the bipartition given by the hypothesis. Since every pair of adjacent vertices belongs to a bichromatic clique, it follows that two vertices of the same bipartition are adjacent if and only if they have a common neighbor in the other bipartition. By Theorem 6.2 $K(G)$ is a closed neighborhood graph of a bipartite graph H, with bipartition V_1, V_2. Since bichromatic cliques of $N_c(H) = K(G)$ correspond to bicliques of H, it follows that H is biclique-Helly. Conversely. Considering bipartitions V_1, V_2 given by the bipartite graph, the result follows directly from Proposition 6.1 and Theorem 6.2 . ■

6.4 About closed neighborhood graphs and open neighborhood Helly graphs

We analyze the possible induced diamonds in a closed neighborhood graph $N_c(G)$ of a bipartite graph G. Also, we study the alternatives for the extended triangles of $N_c(G)$.

Let G be a bipartite graph, and let $V_1 \cup V_2$ be its bipartition. Let v_1, v_2, v_3, v_4 be an induced diamond in $N_c(G)$, v_1, v_2 of degree 3. We consider only the cases where v_1, v_2, v_3, v_4 is not contained in one part of the bipartition:

Case 1: v_1, $v_2 \in V_1$. It follows that $v_3 \in V_2$, or $v_4 \in V_2$. Without loss of generality, suppose $v_3 \in V_2$. Then, $v_4 \in V_1$, since v_3 and v_4 are not adjacent. The case where v_1, $v_2 \in V_2$ is analogous. We conclude that if v_1, v_2 belong to one part, suppose V_1, also does v_4.

Case 2: $v_1 \in V_1$, $v_2 \in V_2$. As v_3 and v_4 are not adjacent, the can not belong to the same part of the bipartition. We conclude that if v_1 and v_2 belong to different parts, also do vertices v_3 and v_4.

Observe that in both cases, given a triangle v_1, v_2, v_3 in $N_c(G)$, every vertex of its extended triangle adjacent to v_i, v_j belong to the same part V_1 or V_2, determined by the vertices v_i, v_j.

In the following Proposition, we study the bipartite graphs for which their closed neighborhood does not have induced diamonds included in one part of the bipartition. It is a similar to the result given by the Theorem 2.11

Proposition 6.6 *Let $G = V_1 \cup V_2$ be an open neigbourhood-Helly, bipartite graph. Let Q be the graph of Figure 6.3. Then G is $Q - free$, where $N(w_1) \cap N(w_1) = \emptyset$ if and only if $G^\star$ does not have and induced diamond $\{w_1, w_2, v_3, v_4\} \in V_1$ (analogously, V_2)*

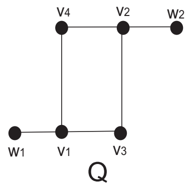


Figure 6.3: Graph Q

Proof: Suppose $G^\star$ has a diamond which vertices $\{w_1, w_2, v_3, v_4\}$ belong to V_1. As $N(w_1) \cap N(v_i)$, $N(w_2) \cap N(v_i)$ and $N(v_3) \cap N(v_4)$ are non empty sets and G is open Neigbourhood-Helly, there exist vertices $v_1, v_2 \in V_2$ adjacent to w_1, v_3, v_4, and w_1, v_3, v_4 respectively. As w_1 and w_2 are non adjacent in $N_c(G)$, $N(w_1) \cap N(w_1) = \emptyset$.

Conversely, suppose G has the graph Q as induced subgraph, and $N(w_1) \cap N(w_1) = \emptyset$. Then, in $N_c(G)$, w_1 is not adjacent to w_2 and both vertices are adjacent to v_4 and v_3 (See Figure 6.4). ■

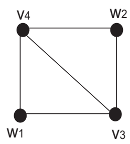


Figure 6.4:

6.5 Bicliqual graphs

In this section we introduce the bicliqual graphs. We give a characterization of bicliqual graphs by forbidden subgraphs. We use bicliqual graphs to obtain conditions for a closed neighborhood graph to be hereditary clique-Helly.

In [13] are defined the cliqual graphs as the graphs for which every clique subgraph G_C has C as the family of cliques. In [13] it is also proved that the class of cliqual graphs is equivalent to the class of hereditary clique-Helly grahs. It was useful in the characterization of k-perfect graphs ([13, 14]).

Theorem 6.7 *[13] A graph is hereditary clique-Helly if and only if it is cliqual.*

Recall that, given a graph G, a clique subgraph $G_{\mathcal{C}}$ of G is the subgraph formed by vertices and edges of the subfamily $\mathcal{C}$ of cliques of G. Similarly, the biclique graph $G_{\mathcal{B}}$ is the subgraph formed by the vertices and edges of the subfamily of bicliques $\mathcal{B}$. A *special clique subgraph* is a clique subgraph such that its family of cliques is included in the family of cliques of G. Analogously, define the *special biclique subgraph*, as the biclique subgraph which has its family of bicliques included in the family of bicliques of G. Given a family of bicliques $\mathcal{B}$, say that it is *bicliqual* if for every subfamily $\mathcal{B}'$ of $\mathcal{B}$, the biclique subgraph $G_{\mathcal{B}'}$ has $\mathcal{B}'$ as the family of bicliques. A graph is called *bicliqual* if its bicliques form a bicliqual family. It is clear that if the graph is bicliqual, every biclique subgraph is special.

Next, follows a characterization of bicliqual graphs for bipartite graphs, by a forbidden subgraph.

Theorem 6.8 *Let G be a bipartite graph. Then G is bicliqual if and only if it does not contain P_5 as induced subgraph. (Figure 6.5)*

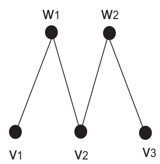


Figure 6.5: P_5

Proof: Let $V_1 \cup V_2$ be the bipartition of vertices of G and suppose vertices v_1, w_1, v_2, w_2, v_3, where $v_i \in V_1$, $w_j \in V_2$, induce a P_5 in G. Consider the bicliques B_1, B_2 containing v_1, w_1, v_2 and v_2, w_2, v_3 respectively. Let G_{B_1,B_2} the biclique subgraph generated by B_1, B_2. It is clear that there is a biclique $B_3 \neq B_1, B_2$ of G_{B_1,B_2} which contains vertices w_1, v_2, w_2. It follows that G is is not a bicliqual graph.

Coversely, let G be a bipartite graph with no P_5 as an induced subgraph. Suppose there is a subfamily $H = B_1, ..., B_k$ of bicliques of G which is not bicliqual. Let G_H be the biclique subgraph of G generated by H and let B be a biclique of G_H, $B \neq B_i$, $i = 1, ..., k$. Consider the following set:

$A = \{j : \forall\ U \subseteq B, \mid U \mid= j,$ *there exists a biclique* B_s *containing* $U\}$. Suppose B has m vertices. Observe that no element of A is greater that $m-1$, since there is no biclique in H containing B. Let $r = max\{j \in A\}$. It is clear that $j \geq 2$, since every edge of B belongs to a biclique of H. Consider a subset R of $r+1$ vertices of B. As $r+1 \geq 3$, let v_1, v_2, v_3 be three vertices of R. Suppose it is the case that they belong to V_1. Consider the subset $R \setminus \{v_1\}$. As it has r elements, there is a biclique B_s in H which does not contain v_1.

Hence, there is a vertex w_1 adjacent to v_2, v_3, and not adjacent to v_1. Analogously, consider the subset $H \setminus \{v_3\}$. With the same argument, we prove that there is a vertex w_3 adjacent to v_2, v_1, and not adjacent to v_3. We conclude that v_1, w_3, v_2, w_1 v_3 induce a P_5. Absurd. Without loss of generality, consider the case $v_1, v_3 \in V_1$, $v_3 \in V_2$. It follows that v_1, v_2, v_3 induce the graph P_3. Consider $H \setminus \{v_1\}$. By the same argument as we used before, we conclude that there is a vertex w_1 adjacent to v_3, not adjacent to v_1. Analogously, there is a vertex w_3 adjacent to v_1, not adjacent to v_3. It follows that w_1, v_3, v_2, v_1, w_3 induce the graph P_5, which is an absurd.

■

The following corollary, relates bicliqual graphs to hereditary biclique-Helly graphs.

Corollary 6.4 *Let* G *be a* $P_5 - free$*, bipartite graph. Then,* G *is hereditary biclique-Helly.*

Proof: Suppose the contrary. Then, by Theorem 3.3, G contains some of the graphs of Figure 3.3 as induced subgraph. Hence, G contains a P_5, absurd. ■

Remark 6.3 *The converse of Corollary 6.4 does not holds. The graph* P_5 *is not bicliqual but it is hereditary biclique-Helly.*

As an application of the previous results, we can relate hereditary clique-Helly, closed neighborhood graphs to biclique-Helly graphs.

Theorem 6.9 *If the closed neighborhood graph* $N_c(G)$ *of a bipartite graph* G *is hereditary clique-Helly, then* G *is open neighborhood-Helly and biclique-Helly and the columns of the clique matrix of* $N_c(G)$ *are Helly-bicovered.*

Proof: Suppose $N_c(G)$ is hereditary clique-Helly. Observe that $N_c(G) = G_{\mathcal{B}^\star}$, i.e. $N_c(G)$ is the clique subgraph generated by the family of cliques $\{B^\star,\ B\ is\ a\ biclique\ of\ G\}$. As $N_c(G)$ is hereditary clique-Helly, by Theorem 6.7, $\mathcal{B}^\star$ is the family of cliques of $N_c(G)$. Hence, $\mid A_{N_c(G)} \mid= A_G^B$. By Theorem 6.3, G is open neighborhood-Helly. On the other hand, as $A_{N_c(G)}$ is a biclique matrix of a bipartite graph, columns of $A_{N_c(G)}$ are bipartite-Helly for the bipartition $V_1 \cup V_2$, according to Theorem 5.4. ■

Remark 6.4 *The converse does not hold. In Figure 6.6 there is an example of a graph G that is open neighborhood-Helly and biclique-Helly, and $N_c(G)$ is not hereditary clique-Helly.*

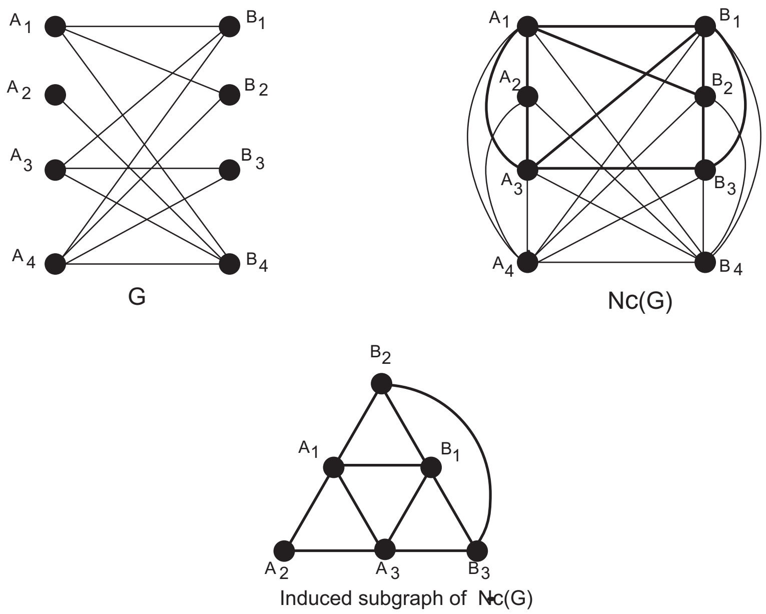


Figure 6.6: Graph G and $N_c(G)$

The following Corollary explains when the closed neighborhood graph of a bipartite graph is hereditary clique-Helly.

Corollary 6.5 *Let G be a $P_5 - free$, bipartite graph. Then $N_c(G)$ is hereditary clique-Helly and G is biclique-Helly.*

Proof: If G is P_5-free. it is a direct consequence of Theorem 3.4 that G is open neighborhood-Helly. Then, by Theorem 6.3, $\mathcal{C} = \{B^\star,\ B\ is\ a\ biclique\ of\ G\}$ is the family of cliques of $N_c(G)$. On the other hand, by Theorem 6.8 G is bicliqual.

Using Theorem 3.1, we prove that $N_c(G)$ is cliqual. Consider the clique subgraph $G_{\{B_1^\star ... B_k^\star\}}$ of $N_c(G)$ generated by $\{B_1^\star ... B_k^\star\}$. Suppose $\{B_1^\star ... B_k^\star\}$ is not cliqual, meaning that there is a clique $C \notin \{B_1^\star ... B_k^\star\}$ of $G_{\{B_1^\star ... B_k^\star\}}$. Consider the case that C is contained in one part, say V_1. Let C' be a minimal subgraph of C such that no clique of the family $\{B_1^\star ... B_k^\star\}$ contains C'. It is clear that C' has at least 3 vertices. By minimality of C', there is a clique $B_1^\star$ containing $C' \setminus \{v_{i_1}\}$. As v_{i_1} is not in $B_1^\star$, there is a vertex $w_1 \in V_2$ not adjacent to v_{i_1}, adjacent to every vertex of $C' \setminus \{v_{i_1}\}$. Analogously, there are vertices $w_2 \in V_2$ such that w_2 is not adjacent to v_{i_2}, and it is adjacent to every vertex of $C' \setminus \{v_{i_2}\}$. Let v_{i_3} be a vertex of C', $v_{i_1} \neq v_{i_1}, v_{i_2}$. It follows that in G, vertices $v_{i_1}, w_2, v_{i_3}, w_1, v_{i_2}$ induce a P_5, absurd. Then, C is not included in a part. Then, consider $B^\star$ the clique that contains the complete subgraph C in $N_c(G)$. In G, construct the biclique subgraph $G_{B_1 \ldots B_k}$. The corresponding vertices of C in G induce a complete bipartite subgraph contained in a biclique of $G_{\{B_1 \ldots B_k\}}$. By hypothesis, since G is bicliqual, vertices of C are included in a biclique B_i. It follows that C is included in $B_i^{\star}$. Absurd. ∎

Remark 6.5 *The converse of Corollary 3.1 does not holds. i.e., the closed neighborhood graph of P_5, $N_c(P_5)$ is cliqual. (Figure 6.7)*

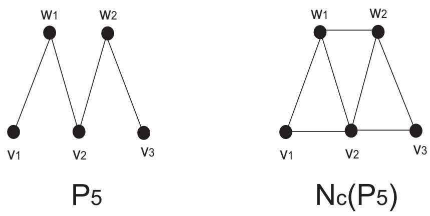


Figure 6.7: Graph P_5 and its closed neigborhood graph

Chapter 7

Clique-perfectness and biclique-perfectness

7.1 Introduction

In this Chapter we study some classes related to perfect graphs. We define the class of b-coordinated and b-biclique-perfect graphs. We relate them to the class of graphs for which the biclique graph is perfect, i.e., KB-perfect graphs.

We also study the relation between the mentioned classes and the c-clique-perfect, c-coordinated and K-perfect graphs.

Recall that K-perfect graphs are those for which the clique graph is perfect. Analogously, say that a graph is *KB-perfect* when its biclique graph is perfect.

Denote by $M(G)$, the maximum number of cliques that has a common vertex. Given a graph G and a vertex v of G, denote by $m_b(v)$ the number of bicliques that contains v. Define $MB(G)$ as the maximum number of bicliques that has a common vertex, i.e., $MB(G) = max_v\{m_b(v)\}$. On the other hand, a clique coloring is a function from a set of colors to the set of cliques of a graph, in such a way that two intersecting cliques have different colors, i.e., a coloring for $K(G)$. Analogously, define the *biclique coloring* as an assignment of colors to bicliques of a graph, in such a way that two intersecting cliques have different colors, i.e., a coloring of $KB(G)$. Denote by $\chi(K(G))$ the minimum clique coloring, and $\chi(KB(G))$, the minimum biclique coloring.

Remark that $MB(G) \leq M(N_c(G))$ and $\chi(KB(G)) \leq \chi(K(N_c(G)))$. If G is open neighborhood-Helly, then $MB(G) = M(N_c(G))$ and $\chi(KB(G)) = \chi(K(N_c(G)))$. In [13], it is defined the class of coordinated graphs as the graphs for which $M(H)$ and $\chi(K(H))$ coincide for every induced subgraph H. Also, the c-coordinated are the graphs for which the equality holds for every special clique subgraph H. On the other hand we define the $b-$*coordinated* graphs as the graphs for which $MB(H) = \chi(KB(H))$ for every special biclique subgraph of H.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *biclique-transversal* of a graph G is a subset of vertices that meets all the bicliques of G. A *biclique-independent set* is a collection of pairwise vertex-disjoint bicliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_c(G)$ and $\alpha_c(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. Analogously, denote by $\tau_b(G)$ and $\alpha_b(G)$, the sizes of a minimum biclique-transversal and a maximum biclique-independent set of G, respectively.

It is easy to see that $\tau_c(G) \geq \alpha_c(G)$ and $\tau_b(G) \geq \alpha_b(G)$ for any graph G. For G bipartite, observe that $\tau_b(G) \leq \tau_c(N_c(G))$ and $\alpha_b(G) \leq \alpha_c(N_c(G))$. Indeed, when G is open neighborhood-Helly, the equality holds, i.e. $\tau_b(G) = \tau_c(N_c(G))$, $\alpha_b(G) = \alpha_c(N_c(G))$.

A graph G is *clique-perfect* if $\tau_c(H) = \alpha_c(H)$ for every induced subgraph H of G. The *c-clique-perfect graphs*, defined in [14] are graphs such that for every special clique subgraph H, $\tau_c(H) = \alpha_c(H)$. On the other hand, define the $b-biclique-perfect\ graphs$ as the family of graphs such that $\tau_b(H) = \alpha_b(H)$ for every special biclique subgraph H.

The c-coordinated and c-clique-perfect graphs are related to K-perfect graphs. In [13, 14] hereditary clique-Helly, K-perfect graphs were characterized as follows.

Theorem 7.1 *[13] Let G be a clique-Helly K-perfect graph. Then G is c–coordinated and c-clique-perfect.*

Theorem 7.2 *Let G be an hereditary clique-Helly graph. Then the following statements are equivalent:*

1. *G is K-perfect.*
2. *G is c-coordinated.*
3. $|C(H)| \leq \alpha_C(H)M(H)$ *for every clique subgraph H of G.*
4. *G is c-clique-perfect.*

7.2 B-coordinated graphs, b-biclique-perfect graphs and KB-perfects graphs

We study the b-coordinated and b-biclique-perfect graphs in relation to KB-perfect graphs, i.e., graphs whose biclique graphs are perfect. Since bicliques of G are related to cliques of $N_c(G)$, it is intuitive to think of a relation between b-coordinated graphs and the c-coordination of its neighborhood graph. The following result summarizes some of these relations, for the class of P_5-free graphs.

Theorem 7.3 *Let G be a $P_5 - free$, bipartite, graph. The following statements are equivalent:*

1. *G is $b - coordinated$*
2. *$N_c(G)$ is $c - coordinated$*
3. *G is b-biclique-perfect*
4. *$N_c(G)$ is $c - clique - perfect$*
5. *$KB(G)$ is perfect*
6. *$K(N_c(G))$ is perfect*
7. *matrix $(A_{N_c(G)})^T$ is perfect*
8. *matrix $(A_G^B)^T$ is perfect*
9. *$|C(H)| \leq \alpha_c(H)M(H)$ for every clique subgraph H of $N_c(G)$.*
10. *$|B(H)| \leq \alpha_b(H)MB(H)$ for every biclique subgraph H of G.*

Proof: $1) \iff 2), 3) \iff 4)$ As G has no induced P_5, G is open neighborhood-Helly. Therefore, by Theorem 6.3, $\{B^\star,\ B\ biclique\ of\ G\}$ is the family of cliques of $N_c(G)$. Let H be the clique subgraph $G_{\{B_1^\star \ldots B_k^\star\}}$ of $N_c(G)$. By Corollary 6.5, $N_c(G)$ is hereditary clique-Helly and according to Theorem 6.7, the cliques of $G_{\{B_1^\star \ldots B_k^\star\}}$ are exactly $\{B_1^\star \ldots B_k^\star\}$.

Consider the biclique subgraph $G_{\{B_1 \ldots B_k\}}$. By Theorem 6.8, $B_1 \ldots B_k$ are all the bicliques of $G_{\{B_1 \ldots B_k\}}$. Then, it is clear that $N_c(G_{\{B_1 \ldots B_k\}}) = G_{B_1^\star \ldots B_k^\star}$. It follows that, if $MB(G_{\{B_1 \ldots B_k \ldots B_s\}}) = \chi(KB(G_{B_1 \ldots B_k \ldots B_s}))$, then $M(H) = \chi(K(H))$, meaning that if G is b-coordinated, $N_c(G)$ is c-coordinated and, if G is b-biclique-perfect, $N_c(G)$ is c-clique-perfect.

Conversely, let $H = G_{\{B_1 \ldots B_k\}}$ be a biclique subgraph of G. Since G is $P_5 - free$, the family of bicliques of H are $\{B_1 \ldots B_k\}$. Let $N_c(H)$ be the closed neighborhood graph of H. To conclude our proof, observe that $N_c(H) = G_{\{B_1^\star \ldots B_k^\star\}}$. Then, $N_c(H)$ is a clique subgraph of $N_c(G)$ and since $M(N_c(H)) = F(N_c(H))$, and $\tau_c(N_c(H)) = \alpha_c(N_c(H))$ then $MB(H) = \chi(KB(H))$ and $\tau_b(H) = \alpha_b(H)$. We conclude that G is b-coordinated and b-biclique-perfect.

The equivalence between 2), 4) , 6) and 7) is direct from Theorem 7.2

Next, prove that $2), 5)$ and 6) are equivalent: Since G is $P_5 - free$, by Corollary 6.5, $N_c(G)$ is hereditary clique Helly. By Theorem 7.2, $N_c(G)$ is $c - coordinated$ if and only if $K(N_c(G))$ is perfect. By Corollary 6.2, G is open neighborhood-Helly, and $KB(G) = K(N_c(G))$. It follows that $KB(G)$ is a perfect graph if and only if $K(N_c(G))$ is perfect.

We prove that 6) and 7) are equivalent. By Theorem 2.3, since $N_c(G)$ is hereditary clique-Helly, the polyhedrom $\{x \in \Re^m / (A_G^\star)^T \cdot x \leq 1, x \geq 0\}$ is the same as $\{x \in \Re^m / A_K(G^\star) \cdot x \leq 1, x \geq 0\}$. We conclude that $K(N_c(G))$ is perfect if and only if $A_K(N_c(G))$ is perfect if and only if $(A_{N_c(G)})^T$ is perfect.

The equivalence between 7) and 8) is a direct consequence from the Theorem 6.3.

Finally, we prove the equivalence of items 9) and 10). Given a clique subgraph H formed by cliques $\{B_1^\star, ...B_k^\star\}$, the biclique subgraph $G_{\{B_1,...,B_k\}}$ of G has $\{B_1, ..., B_k\}$ as its set of bicliques. Analogously, given a biclique subgraph $G_{\{B_1,...,B_k\}}$ of G, it is true that $N_c(G_{\{B_1,...,B_k\}}) = \{B_1^\star, ...B_k^\star\}$, since G is bicliqual (P_5-free) and $N_c(G)$ is cliqual (hereditary clique Helly, by Corollary 6.5 and Theorems 6.7,6.8. It follows that for every clique subgraph of H of $N_c(G)$, there is a biclique subgraph H' in G such that $|C(H)| = |B(H')|$, $M(H) = MB(H')$ and $\alpha_c(H) = \alpha_b(H')$, and conversely, for every biclique subgraph $G_{\mathcal{B}}$, $N_c(G_{\mathcal{B}})$ is a clique subgraph of $N_c(G)$, verifying that $|C(N_c(G_{\mathcal{B}})| = |B(G_{\mathcal{B}})|$, $M(N_c(G_{\mathcal{B}})) = MB(G_{\mathcal{B}})$ and $\alpha_c(N_c(G)) = \alpha_b(G)$.

Theorem 7.2 completes the proof. ∎

To study the case when G has indeed a P_5, we need to analyze properties of $KB(G)$ disregarding the closed neighborhood graph.

The following Proposition gives some relations between the graph G and its biclique graph.

Proposition 7.1 *Let G be a graph. Then:*

1. $MB(G) \leq \omega(KB(G))$.
2. *If G is biclique-Helly then* $M^B(G) = \omega(KB(G))$.
3. $\alpha_b(G) = \alpha(KB(G))$.
4. $\tau_b(G) \geq \theta(KB(G))$.
5. *If G is biclique-Helly then* $\tau_b(G) = \theta(KB(G))$.

Proof:

1. Observe that $m_b(v) \leq \omega(KB(G))$, $\forall v \in V(G)$, since all the vertices that correspond to the $m(v)$ bicliques containing v induce a complete subgraph in $KB(G)$. In particular, $MB(G) \leq \omega(K(G))$.
2. We only need to prove that if G is biclique-Helly, then $\omega(KB(G)) \leq MB(G)$. Let L be a maximum biclique of $KB(G)$ and $B_1, \ldots, B_r$ be the bicliques of G that correspond to the vertices of L. Since G is a biclique-Helly graph, there is at least one vertex v_L in G which belongs to the intersection of all the r bicliques. So, it is easy to see that $M^B(G) \geq m^b(v_L) = \omega(KB(G))$.
3. It follows from the fact that disjoint bicliques of G correspond to non adjacent vertices in $KB(G)$, and conversely.

4. Let $v_1, \ldots, v_{\tau_b(G)}$ be a biclique-transversal set of G. For each i, analyze the $m_b(v_i)$ vertices in $KB(G)$ corresponding to the bicliques in G that contain the vertex v_i. They form a complete set of $KB(G)$. This complete set must be included in some clique L_i of $KB(G)$. Observe that these cliques L_i ($i = 1, \ldots, \tau_b(G)$) are not all necessarily different. We prove that there is a minimum clique cover of size at most $\tau_b(G)$. Let w be a vertex of $KB(G)$. Then, w corresponds to some biclique B_w of G. As the set $v_1, \ldots, v_{\tau_b(G)}$ intersects all the bicliques of G, there is some vertex v_j that belongs to B_w. This means that the corresponding vertex of B_w in $KB(G)$ belongs to the clique L_j, i.e., $w \in L_j$. Then, the size of the minimum clique cover of $KB(G)$ is at most the size of this biclique cover which is at most $\tau_b(G)$.

5. All we need to prove is that if G is biclique-Helly, then $\tau_b(G) \leq \theta(KB(G))$. Recall that, as G is biclique-Helly, then by Theorem 6.5 every clique C of $KB(G)$ has a correspondence with a vertex of G, v_C satisfying that vertices of C correspond exactly to the family of bicliques of G containing v_C. Now, let $L_1, \ldots, L_{\theta(KB(G))}$ be a clique cover of $KB(G)$. Let $v_{L_1}, \ldots, v_{L_{\theta(KB(G))}}$ be the corresponding vertices in G. We need to prove that they are a biclique-transversal set of G. Let B be a biclique of G and w_B its corresponding vertex in $KB(G)$. Then there is an index j such that w_B belongs to the clique L_j in $KB(G)$. It follows that the associated vertex v_{L_j} belongs to B in G. ■

The next Theorem relates KB-perfect graphs to b-coordinated and b-biclique-perfect graphs.

Theorem 7.4 *If G is a biclique-Helly, KB-perfect graph, then G is b-coordinated and b-biclique-perfect.*

Proof: Consider a special biclique subgraph $G_{\mathcal{B}}$ of G and let $\{B_1, ..., B_s\}$ be its family of bicliques. As they are bicliques in G, let H be the subgraph of $KB(G)$ induced by the vertices corresponding to bicliques $B_1, ..., B_s$ in $KB(G)$. Then, by Proposition 7.1, $\alpha_b(G) = \alpha(KB(G))$. On the other hand, it is clear that since G is biclique-Helly, so are the special biclique subgraphs. Then, $MB(G_{B_1,\ldots,B_k}) = \omega(H)$ and $\tau_b(G) = \theta(KB(G))$. Finally, as $KB(G)$ is perfect, $\chi(H) = \omega(H)$ and $\alpha(KB(G)) = \theta(KB(G))$. We conclude that $\chi(KB((G_{B_1,\ldots,B_k})) = MB(G_{B_1,\ldots,B_k})$ and $\alpha_b(G) = \tau_b(G)$. ■

In the next Corollary, we study the graphs for which the biclque graph is c–coordinated and c-clique-perfect. The following Lemma is usefull.

Lemma 7.1 *[23] Let G be a graph. Denote by $K^2(G)$ the clique graph of $K(G)$. Then, $K^2(G)$, is an induced subgraph of G.*

Corollary 7.1 *Let G be an open neigborhood-Helly and biclique-Helly graph such that $N_c(G)$ is perfect. Then $KB(G)$ is c–coordinated and c-clique-perfect.*

Proof: Recall that if G is biclique-Helly and open neighborhood Helly, $N_c(G)$ and $KB(G)$ are clique-Helly and $KB(G) = K(N_c(G)$, according to Propositions 6.4, 6.5 and Corollary 6.2. By Lemma 7.1 $K^2(N_c(G))$ is an induced subgraph of $N_c(G)$. Consequently, if $N_c(G)$ is perfect, so is $K^2(N_c(G))$. Then $K(N_c(G))$ is K-perfect and clique-Helly, and by Theorem 7.1, $K(G)$ is c-coordinated and c-clique-perfect, and so is $KB(G)$. ∎

Chapter 8

Conclusions

A biclique of G is a maximal complete bipartite subgraph of G. A graph is biclique-Helly when its family of bicliques is a Helly family.

We have described characterizations for biclique-Helly graphs, leading to two polynomial time recognition algorithms. We also have considered open and closed neighborhood-Helly graphs.

We have defined the bichromatic-Helly graphs and characterized them, giving a polynomial time algorithm for recognizing this class.

We have described characterizations for hereditary biclique-Helly, and hereditary open and closed neighborhood-Helly graphs, by families of forbidden subgraphs. The forbidden subgraphs are all of fixed size, implying polynomial time recognition for these classes.

We have considered the biclique matrix of a graph and described a characterization of it. We have formulated two polynomial time algorithms for recognizing biclique matrices. On the other hand we have also characterized bipartite-conformal hypergraphs with compatible bicolorings and gave a polynomial time algorithm for the recognition problem.

A biclique graph is the intersection graph of the bicliques of G. We have given a characterization of biclique graphs. We also have studied the classes of biclique graphs of same classes.

We have introduced the concept of bipartite-Helly hypergraphs and proposed a polynomial time algoritmm for recognizing bipartite-Helly labeled families.

The neighborhood graphs have been considered. We have studied the clique graph of this class and relate it to biclique graphs.

We have considered the KB-perfect graphs, i.e., graphs having a perfect biclique graph. We have characterized this class under the restriction of not having induced P_5's.

8.1 Open problems and future work

We leave the following as open questions and future work:

1. Characterize positive biclique matrices of a graph.
2. Determine the computational complexity of recognizing biclique graphs.
3. Find families of graphs for which the problem of recognizing their biclique graphs can be solved in polynomial time.
4. Study the iterated biclique operator, that is, $KB^i(G) = KB(KB^{i-1}(G))$.
5. Characterize E-biclique graphs
6. Define a $b-coloring$ as a coloring of the vertices of a graph, where two vertices have different colors if they belong to a same biclique. Define the $b - perfect\ graphs$ as graphs for which the maximum biclique coincides with the minimum b-coloring. Study b-perfect graphs.
7. Define a *b-weak coloring* as the minimum number of colors that are necessary to color vertices of G in such a way that every biclique of G contains at least two colors. Characterize graphs that are b-weakly 2-colorable, that is, graphs graphs with b-weak coloring number equal to 2.
8. Define the *edge-biclique graph* H of a graph G as the graph such that $V(H) = E(G)$, and two vertices of H are adjacent when their corresponding edges in G belong to a same biclique. Characterize edge-biclique graphs.

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Appendix A

p-Completes Sets and p-Cliques

In this appendix, we consider a generalization of cliques of a graph, the p-cliques. Also, we relates the to a generalization of the concept of the Helly property. We give a characterization of (p,q)-Helly graphs in terms of the the clique matrix of an certain graph.

We remark that (p,q)-Helly graphs were consider in [21]

Let $\mathcal{F} = \{M_1...M_r\}$ be a family of sets. We say that it is a *p-intersecting* family if for every subfamily $\mathcal{F}' = \{M_{i_1}...M_{i_s}\}$, $s \leq p$, $\cap_1^p M_i \neq \emptyset$.

Let $\mathcal{F} = \{M_1...M_r\}$ be a family of sets. We say that it is a *(p,q)-intersecting family* if for every subfamily $\mathcal{F}' = \{M_{i_1}...M_{i_p}\}$, the cardinality of $\cap_1^p M_i \neq \emptyset$ is greater than q-1. We say that a family is p-intersecting precisely when it is (p,1)-intersecting.

Let $\mathcal{F} = \{M_1...M_r\}$ be a family. We say that it verifies the *p-Helly* property if for every p-intersecting subfamily $F' = \{M_{i_1}...M_{i_s}\}$, $\cap_1^s M_i \neq \emptyset$. A graph G is p-clique-Helly if the family of cliques of G satisfies the p-Helly property. We say that a graph is *(p,q)-clique-Helly* if every (p,q)-intersecting subfamily of cliques have common intersection in more than q-1 vertices. For the special case q=1, we say that the family is p-clique-Helly.

A *p-complete* subgraph in a graph $G = K(H)$ is a complete subgraph $\{v_1...v_n\}$ such that the family of cliques of H corresponding to the vertices $v_1...v_n$ form a p-intersecting family. A p-complete subgraph is a *p-clique* if it is maximal under inclusion. A *p-clique matrix* of a graph G is the incident matrix of the p-cliques of G.

The following Theorem gives some properties of the p-clique matrix of a graph.

Theorem A.1 *Let $G = K(H)$, and let A_G^P be a p-clique matrix of G. Then,*

1. *A_G^P does not have dominated rows.*
2. *A_G^P does not contain zero columns.*
3. *The family of columns of A_G^P satisfy the p-Helly property.*

Proof: Let A_G^P be a p-clique matrix of a graph G, and suppose it has some row dominated by another row. Since every row represents a p-clique in the graph G, this means there is a p-clique included in another p-clique in G, which is an absurd.

As every vertex is contained in some p-clique, A_G^P does not contain zero columns.

Let $j_1, \ldots, j_r$ be a p-intersecting family of columns. Take p of those columns and its corresponding cliques in H. As they have common intersection, there is a p-clique in G containing them. This implies that its corresponding cliques in H belong to a p-intersecting family. Then, these p cliques of H have common intersection. Therefore, the cliques of H corresponding to the vertices $j_1, \ldots, j_r$ of G form a p-intersecting family and so $j_1, \ldots, j_r$ must be included in some p-clique of G. Therefore, columns $j_1, \ldots, j_r$ have common intersection. ∎

We study the p-clque-Helly graphs in relation to the p-clique matrix of the clique graph. Start with the following Lemma.

Lemma A.1 *Let G be a p-clique-Helly graph and let $K(G)$ be its clique graph. Then, each p-clique L of $K(G)$ has an associated vertex v_L in G such that the vertices of L in $K(G)$ are exactly those corresponding to the cliques of $C(v_L)$ in G.*

Proof: Let L be a p-clique of $K(G)$. Let $w_1, w_2, \ldots, w_r$ be the vertices of $K(G)$ that form the p-clique L, and let $M_1, M_2, \ldots, M_r$ be the cliques of G that correspond to those vertices. As L is a p-clique of $K(G)$, $M_1, M_2, \ldots, M_r$ must be a p-intersecting family. As G is a p-clique-Helly graph, there is at least one vertex in G which belongs to the intersection of all the r cliques. This will be the the associated vertex v_L of L. Suppose that there is a clique $M \in C(v_L) \setminus \{M_1, M_2, \ldots, M_r\}$, and let w be its corresponding vertex in $K(G)$. Then $M \cup \{M_1, M_2, \ldots, M_r\}$, is a p-intersecting family and $w, w_1, w_2, \ldots, w_r$ induce a complete subgraph of $K(G)$, which contradicts the maximality of L. ∎

Theorem A.2 *Let G be a graph, let A_G be a clique matrix of G and let $A_{K(G)}^P$ be a p-clique matrix of $K(G)$ with the vertices in the same order as their corresponding p-cliques in A_G. Then, the following statements are equivalent:*

1. *G is p-clique-Helly.*
2. *The matrix A_G^t without the dominated rows is a p-clique matrix of $K(G)$.*

3. *The polyhedron* $\{x \in R^k / A^t_G x \leq \mathbf{1}, x \geq 0\}$
   is the same as the polyhedron $\{x \in R^k / A^P_{K(G)} x \leq \mathbf{1}, x \geq 0\}$.

Proof: $(i) \Rightarrow (ii)$ Let G be a p-clique-Helly graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$ the vertices of G. Note that $C(v_1), \ldots, C(v_n)$ are identified by the columns of A_G, so by Lemma A.1, every p-clique of $K(G)$ is identified by a column of A_G. On the other hand, every column of a vertex of G in A_G represents a p-complete subgraph $K(G)$ so it must be included in some p-clique, which is represented in A^t_G by a row of a vertex of G. This means that this vertex dominates the other. Then the submatrix of A^t_G obtained by removing the dominated rows is a clique matrix of $K(G)$.

$(ii) \Rightarrow (iii)$ As the variables are nonnegative, the dominated rows of the matrix A^t_G can be removed without loosing restrictions.

$(iii) \Rightarrow (i)$ Suppose that G is not p-clique-Helly. Let $M_1, M_2, \ldots, M_r$, $r \geq 2$, be a p-intersecting family of cliques in G without common intersection. Without loss of generality, we can assume that those cliques correspond to the first r rows of A_G. Then, for each vertex v_j there exist a clique M_{i_j} not containing it. If we look at the clique matrix of G this means that for every column j, there is some $i_j \leq r$ such that $a_{i_j} = 0$. Let $x = (x_i)$ be the vector: $x_i = \frac{1}{r-1}$ for $1 \leq i \leq r$, and $x_i = 0$ for $r+1 \leq i \leq k$ and compute $(A^t_G x)_j = \sum_{i=1}^r a_{ij} x_i + \sum_{i=r+1}^k a_{ij} 0$. As for each j there is at least one $i_j \leq r$ such that $a_{i_j j} = 0$, then $(A^t_G x)_j \leq \frac{r-1}{r-1} = 1$ Then, the vector x belongs to the polyhedron $\{x \in R^k / A^t_G x \leq \mathbf{1}, x \geq 0\}$. Now, let $A^P_{K(G)} = \{b_{ij}\}$ be the p-clique matrix of $K(G)$. As $M_1, M_2, \ldots, M_r$ form a p-intersecting family of cliques in G, there must be a p-clique in $K(G)$ containing their corresponding vertices in the clique graph. Therefore, there is a row i in $A^P_{K(G)}$ such that $b_{ij} = 1$ for $j \leq r$. Then $(A^P_{K(G)} x)_i = \frac{r}{r-1} > 1$ and so x does not belong to the polyhedron $\{x \in R^k / A_{K(G)} x \leq \mathbf{1}, x \geq 0\}$, which is a contradiction. ∎

We define the intersection graphs of a family of completes of q elements. Let G be a graph, the graph $\phi_q(G)$ is defined in the following way: the vertices of $\phi_q(G)$ correspond to completes of G of q vertices, two vertices being adjacent in $\phi_q(G)$ if the corresponding completes are contained in a clique.

The Helly property applied to cliques of ϕ_q is related to the (2,q)-clique-Helly property. The relation is given by the following Lemma.

Lemma A.2 *G is (2,q)-clique-Helly if and only if* ϕ_q *is clique-Helly.*

Theorem A.3 *[21]* ϕ_q *is p-clique-Helly if and only if G is (p,q)-clique-Helly.*

Proof: $\Longrightarrow$) Let $C_1, C_2, ..., C_k$ be a (p,q)-intersecting family of cliques. Then, each clique has more than q vertices. Let $\overline{C_1}, \overline{C_2}, ..., \overline{C_k}$ be the corresponding cliques in ϕ_q. As every subfamily of $C_1, C_2, ..., C_k$ intersects in at least q vertices, $\overline{C_1}, \overline{C_2}, ..., \overline{C_k}$ is a p-intersecting family of cliques in ϕ_q. As ϕ_q is p-clique-Helly, there is a vertex w belonging to $\cap_1^k \overline{C_i}$. This vertex corresponds to a subgraph of G of q elements which belongs to $C_1, C_2, ..., C_k$.

$\Longleftarrow$) Suppose G is (p,q)-clique-Helly. Let $\overline{C_1}, \overline{C_2}, ..., \overline{C_k}$ be a p-intersecting family of cliques of ϕ_q. Take the corresponding cliques in G, $C_1, C_2, ..., C_k$. Then they form a (p,q)-intersecting family. As G is is (p,q)-clique-Helly, there exist a complete subgraph of q vertices which belong to $C_1, C_2, ..., C_k$. Therefore, the corresponding vertex in ϕ_q which belong to $\overline{C_1}, \overline{C_2}, ..., \overline{C_k}$. ∎

We give a characterization of (p,q)-clique-Helly graphs.

Theorem A.4 *Let G be a graph. Let $A^T_{(\theta_q)}$ be the traspose of the clique matrix of the graph θ_q. Then $A^P_{K(\theta_q)}$=$A^T_{(\theta_q)}$ if and only if G is (p,q)-clique-Helly.*

Proof: By Theorem A.2, $A^P_{K(\theta_q)}$=$A^T_{\theta_q)}$ if and only if θ_q is p-clique-Helly. By Theorem A.3, this is equivalent to the fact that the graph G is (p,q)-Clqiue-Helly. ∎

A.1 Maximum p-cliques

In this section, we extended the concept of $\omega(G)$, considering p-cliques. We relate this concept to $M(G)$.

Let G be a graph and $K(G)$ its clique graph. Denote by $\omega_p(K(G))$ the cardinality of the maximum p-clique of the graph $K(G)$.

Theorem A.5 *Let G be a graph. Then $M(G) \leq \omega_p(K(G))$ and the equality holds whenever G is a p-clique-Helly graph.*

Proof: First, observe that $m(v) \leq \omega_p(K(G))$ $\forall v \in V(G)$, since all the vertices that correspond to the $m(v)$ cliques containing v induce a p-complete subgraph in $K(G)$. In particular, $M(G) \leq \omega_p(K(G))$. To prove the equality, we need to prove that if G is p-clique-Helly, then $\omega(K(G)) \leq M(G)$. Let L be a maximum p-clique of $K(G)$ and let v_L be the associated vertex in G given by Lemma A.1. Then, $M(G) \geq m(v_L) = \omega(K(G))$. ∎

Theorem A.6 *Let G be a graph. If $M(G) = F(G)$, then $M(G) = \omega_p(K(G)) = \omega(K(G))$*

Proof: By Theorem A.5, $M(G) \leq \omega_p(K(G))$. As every p-clique is contained in some clique, it follows that $\omega_p(K(G)) \leq \omega(K(G))$. Then, $M(G) \leq \omega_p(K(G)) \leq \omega(K(G)) \leq \chi(K(G)) = F(G)$. ∎