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Espacios Invariantes por Traslaciones con Generador Refinable

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Resumen. Analizamos la estrucutura de espacios invariantes por traslaciones con generador refinable y de soporte compacto. Primero estudiamos el caso unidimensional con dilatación 2. Demostramos que existe una nueva representación de estos espacios en término de funciones con un cierto tipo de homogeneidad. En particular, esta clase de funciones incluye a todos los polinomios homogéneos que son reproducibles por el generador, lo cual relaciona esta representación con el grado de precisión o "accuracy" del espacio. Mostramos que estas funciones se pueden construir a partir de vectores asociados al espectro de la matriz de escala del generador. Caracterizamos completamente la clase de todas las funciones homogéneas y demostramos que reproducen al generador. Esto lo generalizamos a espacios invariantes por traslaciones en $\mathbb{R}^d$, cuyo generador cumple una ecuación de refinabilidad con factor de dilatación matricial. Estos resultados son potencialmente útiles en aplicaciones de teoría de aproximación, teoría de wavelets y teoría de muestreo. Finalmente, consideramos el problema del muestreo o "sampling" en espacios invariantes por traslaciones de $L^2(\mathbb{R})$ generados por funciones cuyas traslaciones enteras son un marco para el espacio. En particular estudiamos los espacios de muestreo ([SZ04], [SZ99]). Caracterizamos las funciones que pertenecen a espacios de muestreo y obtuvimos descomposiciones atómicas de estos espacios en subespacios de muestreo.

Palabras claves. Funciones homogéneas, espacios invariantes por traslaciones, grado de precisión, funciones refinables, espacios de muestreo, marcos.

Refinable Shift Invariant Spaces

Abstract. We analyze the structure of refinable shift invariant spaces with a compactly supported generator. First we study the one-dimensional case with dilation 2. We provide a new representation of these spaces in terms of functions with a special property of homogeneity. In particular, this class of functions includes all the homogeneous polynomials that are reproducible by the generator, which links this representation to the accuracy of the space. We show that these functions can be constructed from vectors associated to the spectrum of the scale matrix of the generator. We completely characterize the class of all homogeneous functions and show that they reproduce the generator. We generalize these results to shift invariant spaces in $\mathbb{R}^d$ with a generator that satisfies a refinement equation which dilation factor is an expansive matrix. These results are potentially useful in applications to approximation theory, wavelet theory and sampling. Finally, we consider the sampling problem in shift invariant spaces of $L^2(\mathbb{R})$, generated by functions which integer translates are a frame for the space. In particular we study sampling spaces ([SZ04], [SZ99]). We characterize the functions that belong to sampling spaces and we obtain atomic decompositions of these spaces in sampling subspaces.

Key words. Homogeneous functions, shift invariant spaces, accuracy, refinable functions, sampling spaces, frames.

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Introducción

Una función $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ de soporte compacto se llama *refinable* si satisface la ecuación

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x - k), \tag{1}$$

donde $c_0, ..., c_N$ son escalares complejos. Los escalares c_k se denominan la *máscara* de la función.

El *espacio invariante por traslaciones* (EIT) generado por φ se define como

$$\mathcal{S}(\varphi) = \{f : \mathbb{R} \longrightarrow \mathbb{C} : f(x) = \sum_{k \in \mathbb{Z}} y_k \varphi(x + k), y_k \in \mathbb{C}\}.$$

Espacios invariantes por traslaciones con generador refinable suscitan especial interés, dado que aparecen naturalmente en el estudio de la teoría de wavelets, la teoría de aproximación y la teoría de sampling. El objetivo general de esta tesis es proporcionar información sobre la estructura de estos espacios.

En varios casos, no se conoce la fórmula explícita de φ, sin embargo, muchas propiedades de φ pueden obtenerse a partir de la máscara. Una cuestión fundamental es cuándo el espacio $\mathcal{S}(\varphi)$ contiene polinomios y de qué grado. El *grado de precisión* de φ es el mayor entero κ, tal que todos los polinomios de grado menor o igual que $\kappa - 1$ pertenecen a $\mathcal{S}(\varphi)$.

El grado de precisión de φ está relacionado con el orden de aproximación de $\mathcal{S}(\varphi)$, ([Jia95a],[dB90]), y con los momentos nulos y la suavidad de la wavelet asociada, en el caso en que φ genera un análisis de multiresolución ([Mey92]). Existen varias equivalencias conocidas para el grado de precisión. En particular, bajo la hipótesis de independencia lineal de sus traslaciones enteras, φ tiene grado de precisión κ, si y sólo si $\{1, 2^{-1}, ..., 2^{-(\kappa-1)}\}$ son autovalores de la matriz $T \in \mathbb{C}^{(N+1)\times(N+1)}$ definida por $T = \{c_{2i-j}\}_{i,j=0,...,N}$ (la *matriz de escala*), y además existen polinomios $p_0, ..., p_{\kappa-1}$ con grado$(p_i) = i$, tal que cada uno de los vectores $v_i = \{p_i(k)\}_{k=0,...,N}$ es un autovector a izquierda de T correspondiente al autovalor 2^{-i} ([Dau88], [CHM98]). Además, si φ tiene grado de precisión κ, entonces para $s = 0, 1, ..., \kappa - 1$ se tiene que $x^s = \sum_{k\in\mathbb{Z}} p_s(k)\varphi(x-k)$, donde p_s es el polinomio que provee el autovector para el autovalor 2^{-s} (ver [CHM98]).

Dado que φ tiene soporte compacto, $\mathcal{S}(\varphi)$ tiene "localmente" un número finito de generadores. Llamaremos *base local* de $\mathcal{S}(\varphi)$ a un conjunto de funciones en $\mathcal{S}(\varphi)$, cuyas restricciones a $[0, 1]$ forman una base de todas las funciones en $\mathcal{S}(\varphi)$ restringidas a $[0, 1]$.

Por lo observado, si φ tiene grado de precisión κ, conocemos κ funciones linealmente independientes en $[0, 1]$ (i.e. los monomios mónicos de grado menor que κ), cada una asociada a un autovalor de T. Sea φ el spline cardinal de grado $n-1$, es decir la convolución n-veces de la función característica del intervalo $[0, 1]$. Entonces todos los polinomios de grado menor o igual que $n-1$ pertenecen a $\mathcal{S}(\varphi)$. Es más, el conjunto $\{1, x, x^2, \ldots, x^{n-1}\}$ es una base local para $\mathcal{S}(\varphi)$, y el espectro de T es exactamente el conjunto $\{1, 2^{-1}, \ldots, 2^{-(n-1)}\}$.

Ahora, si φ no es un spline cardinal, entonces T podría tener algún autovalor λ distinto a una potencia de $1/2$.

Si las potencias de $1/2$ están asociadas a polinomios homogéneos, ¿qué funciones en $\mathcal{S}(\varphi)$ estarán asociadas a autovalores arbitrarios λ? Si uno considera todos los autovalores y sus funciones asociadas, ¿se logrará obtener una base local de $\mathcal{S}(\varphi)$?

Blu y Unser [BU02] estudiaron las que ellos llaman funciones de base radiales autosimilares y mostraron la relación que hay entre estas funciones y generadores de análisis de multiresolución. Por otro lado, Zhou [Zho02] descubrió la existencia en $\mathcal{S}(\varphi)$ de funciones que satisfacen $h(x) = \lambda h(2x)$. A cada autovalor simple λ de T se le puede asociar una función que satisface una ecuación de este tipo. Sin embargo, estas funciones no alcanzan para obtener una representación completa del espacio. Por este motivo, nosotros consideramos todo el espectro de T para una función refinable y de soporte compacto general. A partir de esto, logramos reconstruir el generador φ en término de funciones asociadas al espectro de T, obteniendo así una nueva representación de $\mathcal{S}(\varphi)$. Probamos que estas funciones proveen una base local de $\mathcal{S}(\varphi)$. La ventaja de esta base es que incluye a todos los monomios x^s que pertenecen a $\mathcal{S}(\varphi)$, y aquellas funciones de la base que no son polinomios homogéneos aún preservan cierto tipo de homogeneidad. Además, estas funciones se pueden obtener a partir de los vectores que pertenecen a la base de Jordan de T. También mostramos que T necesariamente es inversible, si se supone que las trasladadas enteras de φ son linealmente independientes.

Luego pasamos a analizar el caso multidimensional con un generador que cumple una ecuación de refinamiento con factor de dilatación arbitraria. Más precisamente, consideramos una función $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ de soporte compacto que satisface

$$\varphi(x) = \sum_{k \in \Lambda} c_k \varphi(Ax - k), \quad c_k \in \mathbb{C}, \tag{2}$$

donde $\Gamma \subset \mathbb{R}^d$ es un reticulado arbitrario, Λ es un subconjunto finito de Γ, y A es tal que $A(\Gamma) \subset \Gamma$ y todos los autovalores de A satisfacen $|\lambda| > 1$. El EIT $\mathcal{S}(\varphi)$ ahora es el espacio de funciones que se escriben como combinación lineal infinita de las Γ-trasladadas de φ. Sea $Q \subset \mathbb{R}^d$ un mosaico o "tile" para Γ, i.e. las trasladadas $\{Q+k\}_{k\in\Gamma}$ cubren $\mathbb{R}^d$ con intersección de medida de Lebesgue cero. Una base local de $\mathcal{S}(\varphi)$ ahora es un conjunto de funciones en $\mathcal{S}(\varphi)$ cuyas restricciones a Q constituyen una base de las funciones de $\mathcal{S}(\varphi)$ restringidas a Q. Se obtuvieron resultados análogos al caso unidimensional, en particular se mostró que se puede construir una base local de $\mathcal{S}(\varphi)$ que consiste solamente de funciones que satisfacen una ecuación de homogeneidad.

En la última parte de este trabajo estudiamos el problema del muestreo (sampling) en EIT. Sea F un espacio de funciones definidas en $\mathbb{R}$, y $X \subset \mathbb{R}$ un subconjunto discreto. La teoría de muestreo estudia cuándo una función puede ser reconstruida a partir de sus valores $\{f(x_k)\}_{k\in\mathbb{Z}}$. El resultado básico es el llamado "Teorema del Sampling Clásico", que resuelve el problema en el espacio de Paley-Wiener de funciones de banda limitada (funciones en $L^2(\mathbb{R})$ con transformada de Fourier soportada en $[-\frac{1}{2}, \frac{1}{2}]$). Toda función en este espacio puede ser representada por la fórmula

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\frac{\sin(\pi(x-k))}{\pi(x-k)},$$

donde la convergencia es uniforme y en $L^2(\mathbb{R})$. El espacio de Paley-Wiener es un EIT refinable con función generadora $sinc(x) = \frac{\sin \pi x}{\pi x}$. En muchas aplicaciones no corresponde a la situación real asumir que una función es de banda limitada. Además, la función $sinc(x)$ tiene decaimiento lento, lo cual se traslada en una reconstrucción pobre. Por ello se ha comenzado recientemente a estudiar el problema del muestreo en otros espacios de funciones. En [AG00], Aldroubi y Gröchenig lo estudiaron en espacios generados por splines, y en [Wal92] Walter lo estudió en subespacios de wavelet. Las trasladadas de la función $sinc(x)$ constituyen una base ortonormal del espacio de Paley-Wiener, en particular un marco. Esto lleva a considerar el muestreo en ciertos subespacios de $L^2(\mathbb{R})$ generados por funciones cuyas trasladadas forman un marco, llamados espacios de muestreo ([SZ04], [SZ99]). Logramos caracterizar las funciones que pertenecen a estos espacios y obtener una descomposición atómica de ellos.

El trabajo está organizado de la siguiente manera: En el capítulo 1 presentamos un resumen de las propiedades básicas de EIT definidos en $\mathbb{R}$, en particular también de los EIT con generador refinable (con factor de dilatación 2). En el capítulo 2 mostramos la nueva representación de estos últimos en término de funciones que llamaremos $(2, \lambda, r)$-homogéneas y proveemos algunos ejemplos. En el capítulo 3 introducimos los conceptos y las herramientas correspondientes al caso multidimensional con factor de dilatación matricial, y en el capítulo 4 extendemos los resultados del capítulo 2 a este nuevo caso. En el capítulo 5 presentamos algunos resultados obtenidos sobre espacios de muestreo.

No están incluidas las demostraciones de resultados ya conocidos.

Descripción de los resultados originales

Los resultados originales de esta tesis se concentran principalmente en los capítulos 2, 4 y 5. Se basan en los artículos [CHM06], [CHM05] y [BCH05].

Sea $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ de soporte compacto que satisface (1). Sea $\ell(\mathbb{Z})$ el espacio de todas las sucesiones definidas en $\mathbb{Z}$. El *operador de subdivisión* asociado a la máscara c_k es el operador $S_c : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ definido por $S_c(\alpha)_j = \sum_{i\in\mathbb{Z}} \alpha_i c_{2i-j}$. (Asumimos $c_t = 0$ si $t \neq 0, \dots, N$). Sea T la matriz de escala definida anteriormente. Mostramos la conexión que hay entre las propiedades espectrales de S_c y T. En particular, probamos que cada vector de la base de Jordan de T se puede extender a una sucesión en $\ell(\mathbb{Z})$ que satisface la misma relación pero para S_c. Luego mostramos, usando la teoría de ecuaciones en diferencias, que las dimensiones de los núcleos de T y de S_c son iguales. Además, a cada vector no nulo en el núcleo de T, le corresponde una combinación lineal no trivial de las trasladadas enteras de φ que produce la función cero.

Sea $\lambda \in \mathbb{C}, \lambda \neq 0$, y sea $r \geq 1$ un entero. Decimos que una función h es $(2, \lambda, r)$-*homogénea* si satisface la siguiente ecuación:

$$\sum_{k=0}^{r} \binom{r}{k} (-\lambda)^{r-k} h(2^{-k}x) = 0 \ \text{ p.c.t.p.},$$

donde r se denomina el orden de homogeneidad, y λ el grado. En particular, para $r = 1$, tenemos $h(x) = \lambda h(2x)$.

Pudimos caracterizar completamente todas las funciones $(2, \lambda, r)$-homogéneas en $\mathcal{S}(\varphi)$. Sea $\mathcal{H} \subset \mathcal{S}(\varphi)$ el subespacio generado por todas las funciones $(2, \lambda, r)$-homogéneas en $\mathcal{S}(\varphi)$ con $\lambda \in \mathbb{C}$ cualquiera y $r \in \mathbb{N}$. Suponiendo que las trasladadas enteras de φ son linealmente independientes, mostramos que $\dim(\mathcal{H}) = N+1$ y que hay una base de $\mathcal{H}$ relacionada con el espectro de T. Más precisamente dada una base $\mathcal{B} = \{v_0, \dots, v_N\}$ de $\mathbb{C}^{N+1}$ que produce la forma de Jordan de T, asociamos a cada vector $v \in \mathcal{B}$ una única función $(2, \lambda, r)$-homogénea $\mathcal{S}(\varphi)$, donde λ y r satisfacen $v(T - \lambda I)^r = 0$.

Las primeras N de estas funciones son una base local de $\mathcal{S}(\varphi)$. Esto permite reconstruir al generador φ a partir de las funciones homogéneas y da una nueva representación de las funciones en $\mathcal{S}(\varphi)$.

Luego consideramos una función $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ de soporte compacto que satisface (2) y el EIT generado por ella. Al pasar al caso multidimensional general, la situación se tornó más compleja, por lo que tuvimos que utilizar técnicas distintas. Una de las dificultades que aparecieron fue encontrar la matriz apropiada T, dado que hallar el soporte exacto de φ puede resultar difícil en dimensiones mayores. Tuvimos que utilizar propiedades geométricas de atractores, mosaicos y conjuntos admisibles, relacionadas con el soporte de φ, para poder definir esta matriz. Esta fue fundamental para el análisis de la clase de funciones (A, λ, r)-homogéneas, i.e. funciones con dominio en $\mathbb{R}^d$ tal que $(\mathcal{D}_A - \lambda I)^r h = 0$, donde $\mathcal{D}_A$ es el operador de dilatación dado por $\mathcal{D}_A f(x) = f(A^{-1}x)$. Nuevamente pudimos asociar a cada vector de la base de Jordan de T una función (A, λ, r)-homogénea y obtener así una base local de $\mathcal{S}(\varphi)$. Además, demostramos que si φ tiene grado de presición κ, el espacio de todas las funciones (A, λ, r)-homogéneas en $\mathcal{S}(\varphi)$ contiene $\alpha_\kappa = \sum_{s=0}^{\kappa-1} d_s$ polinomios linealmente independientes, donde d_s es el número de monomios de grado s linealmente independientes.

En el último capítulo obtuvimos condiciones necesarias y suficientes para que una función en $L^2(\mathbb{R})$ pertenezca a un espacio de muestreo. Para ello, probamos primero que si una función f pertenece a un espacio de muestreo, entonces el EIT generado por f también es un espacio de muestreo. Luego aplicamos los resultados obtenidos al problema de los conjuntos determinantes. Básicamente, un conjunto determinante es un conjunto de generadores para un EIT ([ACH$^+$04]). Mostramos que dado un conjunto determinante de un espacio de muestreo, el espacio puede descomponerse como suma de subespacios de muestreo, cada uno generado por un elemento del conjunto determinante.

Introduction

A compactly supported function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ is called *refinable* if it satisfies the equation

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x - k), \tag{1}$$

where $c_0, ..., c_N$ are complex scalars. The scalars c_k are the *mask* of the function.

The *shift invariant space* (SIS) generated by φ is defined by

$$\mathcal{S}(\varphi) = \{f : \mathbb{R} \longrightarrow \mathbb{C} : f(x) = \sum_{k \in \mathbb{Z}} y_k \varphi(x + k), y_k \in \mathbb{C}\}.$$

Shift invariant spaces with refinable generator arise particular attention because they appear in wavelet theory, approximation theory and sampling theory. The main goal of this thesis is to provide information about the structure of these spaces.

In many cases we don't know the explicit formula of φ, nevertheless many properties of φ can be obtained from the mask. One fundamental question is when $\mathcal{S}(\varphi)$ contains polynomials and of which degree. The *accuracy* of φ is the maximum integer κ such that all polynomials of degree less or equal $\kappa - 1$ are contained in $\mathcal{S}(\varphi)$.

The accuracy is related to the approximation order of $\mathcal{S}(\varphi)$ ([Jia95a], [dB90] and the references therein), and with the zero moments and the smoothness of the associated wavelet when φ generates a multiresolution analysis [Mey92]. There are many well known equivalent conditions for accuracy. In particular, under the hypothesis of linear independence of its integer translates, φ has accuracy κ, if and only if $\{1, 2^{-1}, ..., 2^{-(\kappa-1)}\}$ are eigenvalues of the $(N + 1) \times (N + 1)$ matrix T defined by $T = \{c_{2i-j}\}_{i,j=0,\dots,N}$ (the *scale matrix*), and there exist polynomials $p_0, \dots, p_{\kappa-1}$ of degree$(p_i) = i$ such that each of the vectors $v_i = \{p_i(k)\}_{k=0,\dots,N}$ is a left eigenvector of T corresponding to the eigenvalue 2^{-i} ([Dau88], [CHM98]). Furthermore, if φ has accuracy κ, then for $s = 0, 1, \dots, \kappa - 1$ it is true that $x^s = \sum_{k\in\mathbb{Z}} p_s(k)\varphi(x-k)$, where p_s is the polynomial that provides the eigenvector for 2^{-s}.

Since φ is compactly supported, $\mathcal{S}(\varphi)$ has "locally" a finite number of generators. A set of functions in $\mathcal{S}(\varphi)$ which restrictions to $[0, 1]$ is a basis for all the functions in $\mathcal{S}(\varphi)$ restricted to $[0, 1]$, is called a *local basis*.

If φ has accuracy κ, by the preceding observations we know κ linearly independent functions in $[0, 1]$ (i.e. the monomials x^s with $s < \kappa$), each of them associated to an eigenvalue of T. Let φ be the B-spline of degree $n - 1$, i.e. the convolution n-times of the characteristic function of the interval $[0, 1]$. Then all the polynomials with degree less or equal $n-1$ belong to $\mathcal{S}(\varphi)$, in fact, $\{1, x, x^2, \dots, x^{n-1}\}$ is a local basis of $\mathcal{S}(\varphi)$, and the set $\{1, 2^{-1}, \dots, 2^{-(n-1)}\}$ is the spectrum of T.

Now, if φ is not a B-spline, T could have an eigenvalue which is not a power of 1/2.

If powers of 1/2 are associated to homogeneous polynomials, what type of functions in $\mathcal{S}(\varphi)$ are associated to an arbitrary eigenvalue λ? If we consider the whole spectrum of T, is it possible to obtain a local basis of $\mathcal{S}(\varphi)$?

Blu and Unser [BU02] studied what they call self-similar radial basis functions, and showed the relations between these functions and generators of multiresolution analysis. On the other side, Zhou [Zho02] recently showed the existence of functions in $\mathcal{S}(\varphi)$ that satisfy $h(x) = \lambda h(2x)$. Each simple eigenvalue λ of T can be associated to a function that satisfies this equation. However, since we wanted a complete representation of $\mathcal{S}(\varphi)$, we considered the whole spectrum of T. From here, we could reconstruct the generator φ in terms of functions associated to the spectrum of T, and obtained a new representation of $\mathcal{S}(\varphi)$. We showed that these functions provide a local basis of $\mathcal{S}(\varphi)$. The advantage of this basis is that it includes all the monomials x^s in $\mathcal{S}(\varphi)$, and the functions of the basis which are not homogeneous polynomials still preserve certain type of homogeneity. Furthermore, these functions can be obtained from vectors of the Jordan basis of T. We also showed that T is necessarily invertible if we assume that the integer translates of φ are linearly independent.

Then we analyzed the multidimensional case, with a generator that satisfies a refinement equation with arbitrary dilation factor. More precisely, we considered a compactly supported function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ such that

$$\varphi(x) = \sum_{k\in\Lambda} c_k \varphi(Ax - k), \quad c_k \in \mathbb{C}, \tag{2}$$

where $\Gamma \subset \mathbb{R}^d$ is an arbitrary lattice, Λ is a finite subset of Γ, and A is such that $A(\Gamma) \subset \Gamma$ and all the eigenvalues of A satisfy $|\lambda| > 1$. The SIS $\mathcal{S}(\varphi)$ is the space of functions that can be written as an infinite linear combination of the Γ-translates of φ. Let $Q \subset \mathbb{R}^d$ be a tile for Γ, i.e. the translates $\{Q + k\}_{k\in\Gamma}$ cover $\mathbb{R}^d$ with overlaps of Lebesgue measure zero. A local basis of $\mathcal{S}(\varphi)$ is a set of functions in $\mathcal{S}(\varphi)$ which restrictions to Q are a basis of the functions of $\mathcal{S}(\varphi)$ restricted to Q. We obtained analogous results to the one-dimensional case, in particular we showed that a local basis of $\mathcal{S}(\varphi)$ can be constructed, which consists solely of functions which satisfy an equation of homogeneity.

In the last part of this work we considered the sampling problem in SIS. Let F be a space of functions defined in $\mathbb{R}$, and $X \subset \mathbb{R}$ a discrete subset. The sampling theory studies when a function can be reconstructed from its values $\{f(x_k)\}_{k\in\mathbb{Z}}$. The basic result is the "Classical Sampling Theorem" which resolves the problem in the Paley-Wiener space of band-limited functions (functions in $L^2(\mathbb{R})$ with Fourier Transform supported in $[-\frac{1}{2}, \frac{1}{2}]$). Every function of this space can be represented by the formula

$$f(x) = \sum_{k\in\mathbb{Z}} f(k) \frac{\sin(\pi(x-k))}{\pi(x-k)},$$

where the convergence is uniform and in $L^2(\mathbb{R})$. The Paley-Wiener space is a refinable SIS with generating function $sinc(x) = \frac{\sin \pi x}{\pi x}$. In many applications the real situation is that the functions are not band-limited. Besides, the sinc function has slow decay. This is why recently the sampling theory in other spaces received more attention. In [AG00], Aldroubi and Gröchenig studied the sampling problem in spaces generated by

B-splines and in [Wal92] Walter studied it in wavelet subspaces. We know that for the Paley-Wiener space the integer translates of the function $sinc(x)$ are an orthonormal basis, in particular a frame. Therefore it is natural to consider sampling in certain subspaces of $L^2(\mathbb{R})$, called sampling spaces, which are generated by functions which integer shifts are a frame ([SZ04], [SZ99]). We characterized the functions that belong to sampling spaces and obtained atomic decompositions of sampling spaces.

This work is organized as follows: In chapter 1 we present a review of the basic properties of SIS defined on the real line, in particular also properties of refinable SIS (with dilation factor 2). In chapter 2 we show the new representation of refinable SIS in terms of what we will call $(2, \lambda, r)$-homogeneous functions. In chapter 3 we introduce the concepts and tools used for the multidimensional case with arbitrary dilation, and in chapter 4 we extend the results obtained in chapter 2 to this new case. In Chapter 5 we present some results about sampling spaces.

Proofs of known results are not included.

Description of original results

The original results are mainly concentrated in chapters 2, 4 and 5. They are based on the articles [CHM06], [CHM05] and [BCH05].

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a compactly supported function that satisfies (1). Let $\ell(\mathbb{Z})$ be the space of all the sequences defined in $\mathbb{Z}$. The *subdivision operator* associated to the mask c_k is the operator $S_c : \ell(\mathbb{Z}) \rightarrow \ell(\mathbb{Z})$ defined by $S_c(\alpha)_j = \sum_{i\in\mathbb{Z}} \alpha_i c_{2i-j}$. (We assume $c_t = 0$ if $t \neq 0, \dots, N$). Let T be the scale matrix defined before. We showed the relation between the spectral properties of S_c and T. In particular, we proved that every vector of the Jordan can be extended to a sequence in $\ell(\mathbb{Z})$ that satisfies the same relation but for S_c. Using the theory of difference equations, we then showed that the dimension of the kernel of T is the same as the one of S_c. Furthermore, we showed that to each non-zero vector in the kernel of T, there corresponds a non-trivial linear combination of the integer translates of φ yielding the zero function.

Let $\lambda \in \mathbb{C}, \lambda \neq 0$, and $r \geq 1$ an integer. We say that a function h is $(2, \lambda, r)$-*homogeneous* if it satisfies the following equation:

$$\sum_{k=0}^{r} \binom{r}{k} (-\lambda)^{r-k} h(2^{-k}x) = 0 \text{ p.c.t.p.},$$

where r is called the order of homogeneity and λ the degree. In particular, for $r = 1$, we have $h(x) = \lambda h(2x)$.

We characterized completely all $(2, \lambda, r)$-homogeneous functions in $\mathcal{S}(\varphi)$. Let $\mathcal{H} \subset \mathcal{S}(\varphi)$ the span of all the $(2, \lambda, r)$-homogenoeus functions in $\mathcal{S}(\varphi)$, for any $\lambda \in \mathbb{C}$ and $r \in \mathbb{N}$. We showed that under the hypothesis of linear independence of the translates of φ, $\dim(\mathcal{H}) = N + 1$, and that there is a basis of $\mathcal{H}$, corresponding to the spectrum of T. More precisely, given a basis $\mathcal{B} = \{v_0, \ldots, v_N\}$ of $\mathbb{C}^{N+1}$ that yields the Jordan form of T we associated to each vector $v \in \mathcal{B}$ a unique $(2, \lambda, r)$-homogeneous function in $S(\varphi)$, where λ and r satisfy $v(T - \lambda I)^r = 0$.

The first N of these functions are a local basis of functions in $S(\varphi)$ restricted to $[0, 1]$. This allows to reconstruct the generator φ from the homogeneous functions, and gives a new representation for the functions in $S(\varphi)$.

We then considered a compactly supported function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ which satisfies (2), and the generated shift invariant space. When moving to higher dimensions, the situations turned out to be complex. This is why we had to use other techniques. One difficulty that appeared was to find the appropriate matrix ,T since the determination of the exact support of φ can be difficult. We had to use geometric properties of attractors, tiles and admissible sets, related to the support of φ, in order to define this matrix. This matrix was fundamental for the analysis of the class of (A, λ, r)-homogeneous functions, i.e. functions defined in $\mathbb{R}^d$ such that $(\mathcal{D}_A - \lambda I)^r h = 0$. Here $\mathcal{D}_A$ is the dilation operator given by $\mathcal{D}_A f(x) = f(A^{-1}x)$.

Again we could associate to each vector of the Jordan basis of T a (A, λ, r)-homogeneous function and from here obtain a local basis of $\mathcal{S}(\varphi)$. Furthermore, we proved that if φ has accuracy κ, the space of all (A, λ, r)-homogeneous functions contains $\alpha_\kappa = \sum_{s=0}^{\kappa-1} d_s$ linearly independent polynomials, where d_s is the number of linearly independent monomials of degree s.

In the last chapter we obtained necessary and sufficient conditions in order that a function in $L^2(\mathbb{R})$ belongs to a sampling space. To see that, we first proved that if a function belongs to a sampling space, then the SIS generated by f also is a sampling space.Then we applied these results to the problem of determinig sets. Basically, a determining set is a generating set for a SIS in $L^2(\mathbb{R})$ ([ACH$^+$04]). We showed that given a determining set of a sampling space, the space can be decomposed as a sum of sampling subspaces, each of them generated by an element of the determining set.

Chapter 1

Refinable Shift Invariant Spaces in $\mathbb{R}$

1.1 Introduction

The *theory of shift invariant spaces* is of great importance in the study of wavelets, splines, approximation theory, sampling theory, Gabor systems and other areas. We will focus in this chapter on shift invariant spaces defined on the real line.

Let Γ be a lattice, i.e. the image of $\mathbb{Z}$ under any nonsingular linear translation.

Definition 1. A *shift invariant space* (SIS) S is a space of functions on $\mathbb{R}$ that is invariant under lattice translates, i.e.,

$$f \in S \;\Rightarrow\; f(\cdot - k) \in S, \quad \text{for every } k \in \Gamma. \tag{1.1.1}$$

In this and the following chapter we will assume that Γ is the integer lattice $\Gamma = \mathbb{Z}$. Let $\Phi : \mathbb{R} \longrightarrow \mathbb{C}^r$ defined by $\Phi(x) = (\varphi_1(x), \dots, \varphi_r(x))^t$. We define the *SIS generated by* Φ as

$$\mathcal{S}(\Phi) = \{f : \mathbb{R} \longrightarrow \mathbb{C}^r : f(x) = \sum_{k\in\mathbb{Z}} \sum_{i=1}^{r} y_{k,i}\varphi_i(x+k),\ y_{k,i} \in \mathbb{C}\} =$$

$$= \{\sum_{k\in\mathbb{Z}} y_k\Phi(x+k),\ y_k \in \mathbb{C}^{1\times r}\}.$$

We will assume that the functions φ_i are compactly supported. If this is the case, the previous expression is well defined.

If $r > 1$, then $\mathcal{S}(\Phi)$ is called a *finitely generated shift invariant space* and $\varphi_1, \dots, \varphi_r$ are the generators of $\mathcal{S}(\Phi)$. In this work we will mostly deal with SIS generated by a single function φ (i.e. $r = 1$), which are called *principal shift invariant spaces* (PSI). We will then write $\mathcal{S}(\varphi)$ instead of $\mathcal{S}(\Phi)$.

A simple example is the PSI generated by $\varphi_0 = \chi_{[0,1]}$. It is the space of all piecewise constants with possible discontinuities at $\mathbb{Z}$. In wavelet theory the space $V_0 = \mathcal{S}(\varphi) \cap L^2(\mathbb{R})$ is a SIS that plays an important role. Under appropiate conditions the nested spaces $V_j = \{f(2^j \cdot) : f \in V_0\}$ form a *multiresolution analysis* from which a wavelet basis for $L^2(\mathbb{R})$ can be constructed. In *uniform sampling* typically band-limited SIS are employed, i.e. they are generated by a function whose Fourier transform is compactly supported. The most known example in this context is the Paley-Wiener space generated by the function $sinc(x) = \frac{\sin \pi x}{\pi x}$, where we have the classical representation of a signal in terms of the Whittaker cardinal series. In Gabor Analysis, the Weyl-Heisenberg system of SIS is constructed from one SIS W, which is generated by a (or several) *window function*. The other SIS of this system are *modulations* of W, i.e. they are obtained by multiplying W by an exponential function.

SIS that are generated by refinable functions are of particular interest, since they appear naturally in the study of wavelets, splines and subdivision schemes. The space $V_0 = \mathcal{S}(\varphi) \cap L^2(\mathbb{R})$ generated by the *scaling function* φ of the multiresolution analysis mentioned before, is a refinable SIS, as it is any SIS generated by splines.

A crucial property of a SIS is if it contains polynomials and up to which degree, since this is closely related to the approximation properties of the space. This is why the *accuracy* of the generating function plays an important role.

In this chapter we will present a review of the basic properties of SIS and refinable SIS. The references are [Ron01], [Jia95a], [Dau92], [Woj97], [HW96] and [Mey92].

1.2 Properties of Shift Invariant Spaces

An important operator that came up in the theory of SIS is the *synthesis operator*. We denote $\ell(\mathbb{Z})$ the space of all the sequences defined on $\mathbb{Z}$ and $\ell_0(\mathbb{Z})$ the space of all the finitely supported sequences on $\mathbb{Z}$.

Definition 2. Let $\Phi : \mathbb{R} \longrightarrow \mathbb{C}^r$ such that φ_i is compactly supported for every $1 \leq i \leq r$. The linear mapping defined on $(\ell(\mathbb{Z}))^r$ by

$$T_\Phi(y) = \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}} y_{k,i}\varphi_i(x-k) \tag{1.2.1}$$

is called the *synthesis operator* of $\mathcal{S}(\Phi)$.

We will concentrate now on the PSI case. The synthesis operator of $\mathcal{S}(\varphi)$ is then

$$T_\varphi(y) = \sum_{k\in\mathbb{Z}} y_k\varphi(x-k).$$

Definition 3. We say that the integer translates $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ of $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ are *globally linearly independent* or *linearly independent*, if

$$\sum_{k\in\mathbb{Z}} \alpha_k\varphi(\cdot - k) \equiv 0, \quad \Longrightarrow \quad \alpha_k = 0, \forall k$$

for any sequence $\alpha \in \ell(\mathbb{Z})$.

Observe that this definition is stronger than the classical linear independence, where the sequences α are only the finitely supported ones. The linear independence of the integer translates of φ, or equivalently the injectivity of T_φ, is one of the most important properties of a SIS. Note that the restriction of T_φ to $\ell^2(\mathbb{Z})$ is always injective.

The following necessary and sufficient conditions for linear independence (which hold also in $\mathbb{R}^d$), are due to A. Ron.

Theorem 1. *([Ron89]) The integer translates of a compactly supported integrable function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ are linearly independent if and only if $\hat{\varphi}$ does not have any* 1*-periodic zero (in* $\mathbb{C}$*).*

The next result is valid if the spatial dimension is 1 but not in the multidimensional case (the space generated by the shifts of the characteristic function of the squares with vertices $(0,0),(1,1),(0,2),(-1,1)$ is a counterexample). It can be found in [Ron90] and more general for the FSI case in [Jia97].

Theorem 2. *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a compactly supported integrable function. Then $\mathcal{S}(\varphi)$ contains a compactly supported generator θ whose integer translates are linearly independent.*

Even though $S(\varphi)$ is an infinite dimensional space, the fact that φ is compactly supported implies that $S(\varphi)$ is "locally" finite dimensional. This motivates the following definition.

Definition 4. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a compactly supported function and $E \subset \mathbb{R}$ such that $E^0 \neq \emptyset$. We will say that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ are *locally linearly independent* on E if

$$\sum_{k\in\mathbb{Z}} \alpha_k \varphi(\cdot - k) \equiv 0 \quad \text{on } E \implies \quad \alpha_k = 0 \;\; \forall k \in K_\varphi(E),$$

where

$$K_\varphi(E) = \{k \in \mathbb{Z} : |(\mathrm{Supp}(\varphi) + k) \cap E| > 0\}.$$

The integer translates of φ are called *strongly locally linearly independent* if they are locally linearly independent on *any* set E with $E^0 \neq \emptyset$.

Obviously local linear independence implies global linear independence.

A basic notion, which is important in many areas like wavelet theory and sampling theory is the *stability* of the integer translates of the generators of SIS.

Definition 5. Let $\varphi \in L^p(\mathbb{R})$. We will say that φ has *stable integer translates* if there exist positive constants c_1 and c_2 such that

$$c_1\|a\|_p \leq \left\| \sum_{k\in\mathbb{Z}} a_k\varphi_i(\cdot - k) \right\|_p \leq c_2\|a\|_p \tag{1.2.2}$$

for all $a \in \ell_0(\mathbb{Z})$.

If $p = 2$ it is said that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a *Riesz sequence*. In this case, condition (1.2.2) is often stated in the Fourier domain. That is, there exist $A, B > 0$ such that

$$A \leq \sum_{k\in\mathbb{Z}} |\hat{\varphi}(w + k)|^2 \leq B \quad \text{a.e. } w \in \mathbb{R}. \tag{1.2.3}$$

A Riesz sequence $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is called a *Riesz basis* for a closed subspace $V \subseteq L^2(\mathbb{R})$ if $\overline{\text{span}}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}} = V$. For $\varphi \in L^2(\mathbb{R})$ the 1-periodic function

$$G_\varphi(w) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(w + k)|^2 \tag{1.2.4}$$

is called the *grammian* of φ. If in (1.2.3) $A = B = 1$, i.e. $G_\varphi \equiv 1$, the system $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is orthonormal.

Constructions of wavelets from a multiresolution analysis depend on the stability of the integer translates of the underlying *scaling function*. In sampling theory, the stability is important for the reconstruction of a function from its discrete sampled values in a stable way, i.e., small perturbations of the sampled values of a function produce a function "close" to the original one.

The next result which is very useful in wavelet theory can be found in [Dau92].

Proposition 1. *Suppose that $\varphi \in L^2(\mathbb{R})$ is a function such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz sequence. Then there exists a function $\phi \in \overline{\text{span}}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ such that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal system. For each such ϕ, the system $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for the space $\overline{\text{span}}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$.*

Remark. The most natural choice for a function ϕ as in Proposition 1 that gives the orthonormal basis for $\overline{\text{span}}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$, is given by

$$\hat{\phi} = \frac{\hat{\varphi}(w)}{G_{\varphi}(w)^{\frac{1}{2}}}.$$

1.3 Refinability

A function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ with compact support is called *refinable* if there exists a complex sequence $\{c_k\}_{k\in\mathbb{Z}}$ such that φ satisfies the equation

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(2x - k), \tag{1.3.1}$$

The scalars $\{c_k\}_{k\in\mathbb{Z}}$ are the *mask* of the refinable function.

Note. Equation (1.3.1) is often called the *scaling equation*.

A key goal is the determination of properties of a refinable φ based on the mask. Given a refinement mask, it is a matter of interest to determine the existence, uniqueness and smoothness of the solutions of the refinement equation (1.3.1). See [CDM91], [CH94], [DL91], [DL92], [DGL91], [Eir92], [Rio92], [Wan95], among others. It is known that if the sum of the coefficients of the mask is 2, there always exists a solution to (1.3.1), but this may be a solution in the distributional sense.

Theorem 3. *If the mask $\{c_k\}_{k\in\mathbb{Z}}$ satisfies $\sum_{k\in\mathbb{Z}} c_k = 2$, the equation (1.3.1) has a unique compactly supported disributional solution φ satisfying $\hat{\varphi}(0) = 1$*

The solution φ with $\hat{\varphi}(0) = 1$ is called the *normalized solution* of (1.3.1).

Equation (1.3.1) is also fundamental in the context of subdivision schemes. Computations of solutions to (1.3.1) can be done using the subdivision scheme [CDM91].

Examples The function $\varphi_0 = \chi_{[0,1]}$ is refinable since

$$\varphi_0(x) = \varphi_0(2x) + \varphi_0(2x - 1).$$

The so called *hat function*

$$\varphi_1(x) = \chi_{[0,1]} * \chi_{[0,1]} = \begin{cases} x & \text{for } 0 \le x < 1 \\ 2 - x & \text{for } 1 \le x \le 2 \\ 0 & \text{otherwise,} \end{cases}$$

satisfies

$$\varphi_1(x) = \frac{1}{2}\varphi_1(2x) + \varphi_1(2x-1) + \frac{1}{2}\varphi_1(2x-2).$$

For

$$\varphi_2(x) = \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]} = \begin{cases} \frac{1}{2}x^2 & \text{for } 0 \le x < 1 \\ -(x-\frac{3}{2})^2 + \frac{3}{4} & \text{for } 1 \le x \le 2 \\ \frac{1}{2}(x-3)^2 & \text{for } 2 \le x \le 3 \\ 0 & \text{otherwise,} \end{cases}$$

holds

$$\varphi_2(x) = \frac{1}{4}\varphi_2(2x) + \frac{3}{4}\varphi_2(2x-1) + \frac{3}{4}\varphi_2(2x-2) + \frac{1}{4}\varphi_2(2x-3).$$

In fact, for every $n \in \mathbb{N}$ the *B-spline of degree* n, which is obtained by convoluting $n+1$ times the function $\chi_{[0,1]}$ by itself, is a refinable function, which satisfies

$$\varphi_n(x) = \sum_{k=0}^{n+1} 2^{-n} \binom{n+1}{k} \varphi_n(2x-k).$$

It is also positive and supported on the interval $[0, n+1]$.

The graph of the refinable function given by

$$f(x) = \frac{1}{3}f(2x) + \frac{2}{3}f(2x-1) + \frac{2}{3}f(2x-2) + \frac{1}{3}f(2x-3) \tag{1.3.2}$$

can be seen in Figure 1.1.

Observe that if φ is refinable, we obtain in (1.3.1)

$$\varphi(\frac{\beta}{2^m}) = \sum_{k\in\mathbb{Z}} c_k \varphi(\frac{\beta}{2^m} - k), \quad \text{for every } \beta \in \mathbb{Z}. \tag{1.3.3}$$

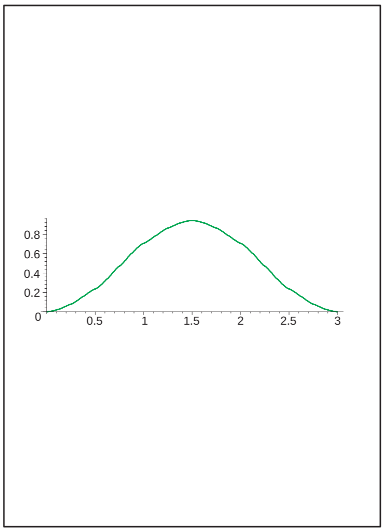

Figure 1.1: Refinable function for coefficients $1/3, 2/3, 2/3, 1/3$

Hence φ can be determined for all dyadics, thus everywhere in the case that φ is continuous.

We can assume that only finitely many c_k are nonzero, since φ is compactly supported. So the refinement equation (1.3.1) can be written (translating if necessary)

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x - k). \tag{1.3.4}$$

We will consider the shift invariant space generated by φ

$$\mathcal{S}(\varphi) = \{f : \mathbb{R} \longrightarrow \mathbb{C} : f(x) = \sum_{k \in \mathbb{Z}} y_k \varphi(x + k),\ y_k \in \mathbb{C}\}.$$

A refinable SIS is a SIS with a refinable generator. Refinable SIS and refinable generators have been studied extensively, since they are very important in wavelet theory and approximation theory.

1.3.1 Refinability in Wavelet Theory

The development of wavelets as a tool for a new representation of functions in L^2 began in 1980. The main goal of wavelet theory is the study of orthonormal bases in L^2

generated by dilations and translations of certain functions. Wavelets arise particular attention because of their multiple applications in signal representation and image compression. They constitute an alternative to traditional signal processing methods which are based on Fourier analysis. The advantage of wavelet analysis is that it is well localized in time and frequency, and hence produces better approximation of data that have discontinuities and pronounced local variations.

Definition 6. A *wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that the family

$$\left\{\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) : j, k \in \mathbb{Z}\right\}, \tag{1.3.5}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

The family (1.3.5) is called a *wavelet basis* and the function ψ the *mother wavelet*. The series

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}$$

turned out to converge in most of the function spaces. The use of these bases in signal treatment is based on the fact that for sufficiently smooth signals they produce a great number of scalar products $\langle f, \psi_{j,k}\rangle$ near to zero. This is a desirable property in many applications, in particular in image compression. The first example of a wavelet basis that has been constructed is the Haar basis, generated by the mother wavelet

$$\psi(x) = \begin{cases} x & \text{for } 0 \leq x < \frac{1}{2} \\ 2 - x & \text{for } \frac{1}{2} \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Its main limitation is that it is discontinue. In 1986 a general method for constructing smooth wavelets was developed by S. Mallat and Y. Meyer. It is based on the notion of *multiresolution analysis*, which plays a crucial role in signal theory and image processing.

Multiresolution Analysis

Most of the known "good" wavelets arise from a multiresolution analysis. It is the starting point of a pyramidal algorithm formulated by S. Mallat for the construction of the wavelet coefficients of a signal. A signal is decomposed in this way in subbands. Multiresolution analysis allows to study images at different scales.

Definition 7. A *multiresolution analysis* (MRA) is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ that satisfy:

(1) $V_j \subset V_{j+1}$;

(2) $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$ for every $j \in \mathbb{Z}$;

(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(4) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;

(5) there exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

The function φ in (5) is called the *scaling function* of the multiresolution analysis.

Remark. By Proposition 1 we can can relax condition (5) replacing "orthonormal basis" by "Riesz basis", in a lot of literature you can find it stated that way.

Condition (4) tells us that every function in $L^2(\mathbb{R})$ can be approximated as closely as we want by its projection on V_j for a sufficiently big j.

The Scaling Function

From Definition 7 we can see that the multiresolution analysis is completely determined by the scaling function φ. So a method to obtain a MRA is to start with a φ

and set $V_0 = \overline{\text{span}}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$. The other spaces V_j will be determined by condition (2), so we will have to check conditions (1), (3), (4) and (5).

Examples of MRA The function $\chi_{[0,1]}$ generates the MRA $\{V_j\}_{j\in\mathbb{Z}}$, where V_j is

$$V_j = \left\{f \in L^2(\mathbb{R}) : f \text{ is constant on } \left[2^{-j}k, 2^{-j}(k+1)\right] \ \forall k \in \mathbb{Z}\right\}.$$

The sequence of spaces $\{V_j\}_{j\in\mathbb{Z}}$, where

$$V_j = \left\{f \in L^2(\mathbb{R}) : f \text{ is continuous and } f \text{ is linear on } \left[2^{-j}k, 2^{-j}(k+1)\right] \ \forall k \in \mathbb{Z}\right\},$$

is a MRA.

More general, for

$$V_j = \left\{f \in L^2(\mathbb{R}) : f \in C^{n-1} \text{ is polynomial of degree } n \text{ on } \left[2^{-j}k, 2^{-j}(k+1)\right] \ \forall k \in \mathbb{Z}\right\}$$

we have that $\{V_j\}_{j\in\mathbb{Z}}$ is a MRA.

The MRA associated to the function $sinc(x) = \frac{\sin \pi x}{\pi x}$ is given by

$$V_j = \left\{f \in L^2(\mathbb{R}) : \text{Supp}(\hat{\varphi}) \subset \left[-2^{j-1}, 2^{j-1}\right]\right\}. \tag{1.3.6}$$

Since $\varphi \in V_0 \subset V_1$, by condition (2) we have that $\varphi(\frac{\cdot}{2}) \in V_0$. Since $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a basis of V_0,

$$\varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} c_k \varphi(x-k), \tag{1.3.7}$$

which is equivalent to

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(2x-k),$$

and hence φ is a refinable function. So we have that every scaling function of a MRA is necessarily refinable.

Note. If $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is orthonormal, the coefficients are given by

$$c_k = \langle \varphi(\frac{\cdot}{2}), \varphi(\cdot - k)\rangle.$$

We will now describe how to construct a wavelet basis given a multiresolution $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ with scaling function φ. Let W_0 be the orthogonal complement of V_0 in V_1, i.e.,

$$V_1 = V_0 \bigoplus W_0 \tag{1.3.8}$$

Dilating the elements of W_0 by 2^j, we get a closed subspace W_j of V_{j+1} such that

$$V_{j+1} = V_j \bigoplus W_j \qquad \text{for every } j \in \mathbb{Z}. \tag{1.3.9}$$

As $V_j \longrightarrow \{0\}$ if $j \longrightarrow -\infty$, it follows that

$$V_{j+1} = V_j \bigoplus W_j = \bigoplus_{l=-\infty}^{j} W_l. \tag{1.3.10}$$

As $V_j \longrightarrow L^2(\mathbb{R})$ if $j \longrightarrow \infty$, we have

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j. \tag{1.3.11}$$

Hence, in order to have a wavelet, we need a function $\psi \in W_0$ such that $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_0. Any such a ψ satisfies, by condition (2) and the definition of W_j, that

$$\left\{2^{\frac{j}{2}}\psi(2^j \cdot -k) : k \in \mathbb{Z}\right\}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

Observe that the space W_j contains the "details" that are added when we pass from approximation at the scale 2^j to the approximation at the finer scale 2^{j+1}.

An explicit formula of a wavelet ψ in terms of the scaling function is given in the following theorem.

Theorem 4. *([Mal89]) Let $\varphi \in L^2(\mathbb{R})$ be a scaling function with orthonormal shifts associated to a MRA. If φ satisfies the refinement equation $\varphi(x) = \sum_{k\in\mathbb{Z}} c_k\varphi(2x-k)$, then with*

$$\psi := \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}}\varphi(2 \cdot -k) \tag{1.3.12}$$

the collection $\left\{2^{\frac{j}{2}}\psi(2^j\cdot -k) : j,k\in\mathbb{Z}\right\}$ *forms an orthonormal basis of wavelets for* $L^2(\mathbb{R})$.

Thus the problem of finding a wavelet with "good" properties such as compact support, smoothnes and high approximation order of the spaces V_j, is reduced to find the appropiate scaling function. A decisive step in this direction was made by Daubechies in [Dau88]. She defines the scaling function implicitly as a solution to a refinement equation (1.3.1) with a mask chosen in such a way that the solution has the desired properties. In particular $\{c_k\}_{k\in\mathbb{Z}}$ should decrease rapidly.

Usually there is not an explicit mathematical formula for the scaling function φ nor the wavelet ψ. Nevertheless there are algorithms used to approximate it. The *cascade algorithm* designed by Daubechies and Lagarias [DL91] for the construction of the scaling function of compactly supported orthonormal wavelets associated to a MRA is a particular case of a *subdivision scheme*.

1.3.2 Refinability and Subdivision Schemes

Subdivision methods are recursive schemes used in computer graphics to approximate curves and surfaces interpolating discrete data. The iteration of certain algorithm produces at every stage a denser data set. The curve (or surface) is then approximated by the polygon (polyhedral surface) that interpolates the data.

The following is an example of a subdivision method proposed first by de Rham. Let θ be defined by

$$\theta(x) = \begin{cases} 1+x & \text{for } -1\le x<0 \\ 1-x & \text{for } 0\le x\le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.13}$$

For a given refinement mask $\{c_k\}_{k\in\mathbb{Z}}$, begin with

$$\varphi_0(x) = \theta(x).$$

Then replace φ_0 by the refined polygon

$$\varphi_1(x) = \sum_{k \in \mathbb{Z}} c_k \varphi(2x - k).$$

The iteration scheme

$$\varphi_{n+1}(x) = \sum_{k \in \mathbb{Z}} c_k \varphi_n(2x - k), \quad n \in \mathbb{N} \tag{1.3.14}$$

is called the *subdivision scheme* or *refinement scheme* associated with the mask $\{c_k\}_{k \in \mathbb{Z}}$.

We say that the subdivision scheme converges in L^p if there exists a function $\varphi \in L^p(\mathbb{R})$ such that

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_p = 0. \tag{1.3.15}$$

Cavaretta, Dahmen and Michelli proved that if the subdivision scheme is convergent, then the limit function φ is the normalized solution of the refinement equation (1.3.1). They also proved the following result about convergence of the subdivision scheme for $p = \infty$ ([CDM91]). Jia extended it to the the case $1 \le p \le \infty$.

Theorem 5. *([Jia95b]) Let φ be the normalized solution to the refinement equation (1.3.1), where $\sum_{k \in \mathbb{Z}} c_k = 2$. If $\varphi \in L^p(\mathbb{R})$ $(1 \le p < \infty)$ or φ is a continuous function in the case $p = \infty$, and if the integer translates of φ are stable, then the subdivision scheme associated with mask $\{c_k\}_{k \in \mathbb{Z}}$ converges to φ in L^p. If the subdivision scheme associated with mask $\{c_k\}_{k \in \mathbb{Z}}$ converges in L^p $(1 \le p \le \infty)$, then*

$$\sum_{i \in \mathbb{Z}} c_{j-2i} = 1, \quad \text{for every } j \in \mathbb{Z}. \tag{1.3.16}$$

The *subdivision operator* associated to the mask $\{c_k\}_{k \in \mathbb{Z}}$ is the operator

$$S_c : \ell(\mathbb{Z}) \to \ell(\mathbb{Z}) \tag{1.3.17}$$

defined by

$$S_c(\alpha)_j = \sum_{i \in \mathbb{Z}} \alpha_i c_{2i-j}.$$

Note. The subdivision operator is sometimes defined in a different but equivalent way as

$$\tilde{S}_c(\alpha)_j = \sum_{i\in\mathbb{Z}} \alpha_i c_{j-2i}. \tag{1.3.18}$$

If $h : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ is the operator $h(\alpha)_k = \alpha_{-k}$, then $h\tilde{S}_c h = S_c$ and therefore S_c and $\tilde{S}_c$ share most of the properties. For a nice account of properties of the subdivision operator see, [BJ02].

1.4 Accuracy

Definition 8. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a compactly supported function. Then a function f is *reproducible by integer translates of* φ, (or shorter *reproducible by* φ) if there exist complex scalars $\{y_k\}_{k\in\mathbb{Z}}$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} y_k \varphi(x+k), \tag{1.4.1}$$

i.e., $f \in \mathcal{S}(\varphi)$.

Given a φ, it is important to know what conditions must satisfy φ if we want all polynomials up to a certain degree to be reproducible from integer translates of φ. Let $\kappa \in \mathbb{N}$. Denote Π_κ the space of all polynomials of degree less or equal than κ, i.e.,

$$\Pi_\kappa = \left\{ p : p(x) = \sum_{i=0}^{\kappa} a_i x^i : a_i \in \mathbb{C} \right\}.$$

It is known that if a polynomial of degree m can be reproduced by a compactly supported φ, then every polynomial of degree less than m can also be reproduced by translates of φ, i.e. $\Pi_m \subset \mathcal{S}(\varphi)$.

So it is natural to ask if there is a maximum integer κ such that $\Pi_\kappa \subset \mathcal{S}(\varphi)$. It is known that this is always true for a compactly supported φ (see for [CHM99] and references therein). These observations lead to the following definition.

Definition 9. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a compactly supported function. Then φ has *accuracy* κ if κ is the maximum integer such that $\Pi_{\kappa-1} \subset \mathcal{S}(\varphi)$.

The next proposition presents a property of the scalars used in (1.4.1) for the case that f is a polynomial.

Proposition 2. *([CHM99]). Assume that φ is compactly supported and its integer translates are linearly independent. Let $p \in \mathcal{S}(\varphi)$ be a polynomial of degree m, and write $p(x) = \sum_{k\in\mathbb{Z}} y_k \varphi(x+k)$. Then there exists a polynomial $u_p(x)$ of degree m such that $y_k = u_p(k)$.*

Corollary 1. *([CHM99]). Assume that φ is compactly supported and its integer translates are linearly independent. If φ has accuracy κ, then the map $\mu : \Pi_{\kappa-1} \longrightarrow \Pi_{\kappa-1}$ defined by $\mu(p) = u_p$ is a linear bijection of $\Pi_{\kappa-1}$ onto itself.*

As a consequence, if φ is as in Corollary 1, for each polynomial $u \in \Pi_{\kappa-1}$ the function $q(x) = \sum_{k\in\mathbb{Z}} u(k)\varphi(x+k))$ is itself a polynomial with $deg(q) = deg(u)$.

Let K be a compact set and $\delta(K)$ its diameter. Denote [a] the integer part of a real number a, and Π the set of all polynomials with complex coefficients. The following result shows that κ is related to the diameter of the support of φ.

Proposition 3. *([CHM99]). Let φ be a compactly supported function. Then the set of polynomials reproducible by φ is a finite-dimensional subspace of Π, and*

$$\dim(\Pi \cap \mathcal{S}(\varphi)) \leq [\delta(\mathrm{Supp}(\varphi))] + 1.$$

1.4.1 Accuracy and Approximation Power

The main goal of approximation theory is to approximate arbitrary functions by simpler ones, i.e. polynomials, trigonometric sums, etc., which are easier to compute. If we want a more precise approximation usually we have to increase the complexity of the approximants. As we will see, the approximation power of function is tied to its accuracy.

We consider approximation in $L^p(\mathbb{R}),\ 1 \leq p \leq \infty$. For a subset G of $L^p(\mathbb{R})$, the distance of a function $f \in L^p(\mathbb{R})$ to G is given by

$$dist_p(f, G) := \inf_{g \in G} \|f - g\|_p.$$

Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$.

Define $\mathcal{S} = \mathcal{S}(\varphi) \cap L^p(\mathbb{R})$. For $h > 0$, let $\mathcal{D}_h$ be the operator defined by $\mathcal{D}_h f = f(\cdot/h)$, where f is an arbitrary function on $\mathbb{R}$. We denote

$$\mathcal{S}^h = \{\mathcal{D}_h(f) : f \in \mathcal{S}\} = \mathcal{D}_h(S).$$

The Sobolev space

$$W_m^p(\mathbb{R}) = \{f \in L^p(\mathbb{R}) : \forall \alpha \leq m,\ \exists u_\alpha \in L^p(\mathbb{R}) \text{ such that}$$
$$\int_{-\infty}^{\infty} f \nu^{(\alpha)} = (-1)^\alpha \int_{-\infty}^{\infty} u_\alpha \nu \ \forall\, \nu \in C_c^\infty(\mathbb{R})\}$$

is the space of all functions whose weak derivatives up to order m lie in $L^p(\mathbb{R})$.

Definition 10. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a compactly supported function in $L^p(\mathbb{R})$. Given a real number $m \in \mathbb{N}$ we say that $\mathcal{S}(\varphi)$ provides L^p- *approximation order* m if, for each $f \in W_m^p(\mathbb{R})$ there exists a positive constant C_f such that

$$dist_p(f, S^h) \leq C_f h^m \tag{1.4.2}$$

for every $h > 0$. (C_f is independent of h).

Definition 11. We say that $\mathcal{S}(\varphi)$ provides L^p- *density order* m if for each $f \in W_m^p(\mathbb{R})$,

$$\lim_{h\to 0^+} \frac{dist_p(f, S^h)}{h^r} = 0. \tag{1.4.3}$$

The following theorem shows the close relation between the approximation order provided by $\mathcal{S}(\varphi)$ and polynomial reproducibility.

Theorem 6. *([Jia97]) Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a compactly supported function in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Let $\kappa \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *$\mathcal{S}(\varphi)$ provides L^p- approximation order κ.*

(2) *$\mathcal{S}(\varphi)$ provides L^p- density order $\kappa - 1$.*

(3) *$\mathcal{S}(\varphi)$ has accuracy κ.*

(4) *$\mathcal{S}(\varphi)$ contains a compactly supported function η such that*

$$\sum_{j\in\mathbb{Z}} p(j)\eta(\cdot - j) = p \quad \text{for every } p \in \Pi_{\kappa-1}.$$

This result is no longer true for FSI (with more than one generator) in $\mathbb{R}^d$, $d > 1$. In [dBH83] a counterexample for the case $d = 2$ is presented.

Accuracy and order of approximation can also be characterized in terms of the Fourier transform. The classical *Strang-Fix* conditions appeared first in [Sch46] and [SF73].

Definition 12. A compactly supported function $\varphi \in L^2(\mathbb{R})$ satisfies the *Strang-Fix conditions of order* m if

$$\hat{\varphi}(0) \neq 0 \text{ and } \hat{\varphi}^{(s)}(l) = 0, \ \forall l \in \mathbb{Z} - \{0\}, \ 0 \leq s \leq m - 1. \tag{1.4.4}$$

Actually there exists the following equivalence between accuracy and the Strang-Fix conditions.

Theorem 7. *[SF73] Assume that $\varphi \in C^1(\mathbb{R})$ is compactly supported and has linearly independent integer translates. Then the following statements are equivalent:*

(1) *φ satisfies the Strang-Fix conditions of order κ.*

(2) *φ has accuracy κ.*

Now let us consider a compactly supported function in $L^2(\mathbb{R})$ that is refinable. Taking Fourier transform on both sides of the equation

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x-k).$$

yields

$$\hat{\varphi}(2w) = \frac{1}{2}\left(\sum_{k=0}^{N} c_k e^{-2\pi i w k}\right)\hat{\varphi}(w).$$

The trigonometric polynomial

$$m_0(w) = \frac{1}{2}\sum_{k=0}^{N} c_k e^{-2\pi i w k}$$

is called the *symbol* of φ. If we assume that the integer shifts of φ are orthonormal, we have

$$\begin{aligned}
1 &= \sum_k |\hat{\varphi}(2w+k)|^2 \\
&= \sum_k |\hat{\varphi}(w+\frac{k}{2})|^2 |m_0(w+\frac{k}{2})|^2 \\
&= \sum_k |\hat{\varphi}(w+k)|^2 m_0(w)^2 \\
&+ \sum_k |\hat{\varphi}(w+\frac{1}{2}+k)|^2 m_0(w+\frac{1}{2})^2,
\end{aligned}$$

and so

$$|m_0(w)|^2 + |m_0(w + \frac{1}{2})|^2 = 1.$$

Many properties of a refinable function can also be obtained in terms of properties of the symbol m_0, in particular the Strang-Fix conditions can be characterized in the following way:

Theorem 8. *Assume that $\varphi \in L^2(\mathbb{R})$ is a refinable compactly supported function with linearly independent integer translates. Then the following statements are equivalent:*

(1) *φ satisfies the Strang-Fix conditions of order κ*

(2) *$\frac{1}{2}$ is a zero of order κ of m_0, i.e., $m_0^{(s)}(\frac{1}{2}) = 0$ for $0 \leq s \leq \kappa - 1$.*

As a consequence we have that the symbol can be factorized as

$$m_0 = (1 + e^{2\pi i w})^{\kappa - 1} R(w).$$

For a refinable function the Strang-Fix conditions can also be obtained from the well known "sum-rules". Assuming proper hypothesis, a function φ that satisfies (1.3.4) has accuracy κ if and only if

$$\sum_{k=0}^{N} c_k = 2 \text{ and } \sum_{k=0}^{N} (-1)^k k^j c_k = 0 \text{ for } j = 0, \dots, \kappa - 1. \tag{1.4.5}$$

1.4.2 Accuracy and Wavelets

For a wavelet that stems form a MRA, we want the associated scaling function to have high accuracy. It leads to good approximation properties of the spaces V_0 which define the multiresolution analysis. The accuracy of the scaling function is closely related to the vanishing moments of the wavelet as we can see from the following proposition.

Proposition 4. *Assume that $\varphi \in L^2(\mathbb{R})$ is a compactly supported scaling function of a MRA and has orthonormal shifts. Let ψ be the associated wavelet, defined by (1.3.12). Then the following are equivalent.*

(1) $\int_{-\infty}^{+\infty} x^s \psi(x)\, dx = 0$ *for* $s = 0, \dots, \kappa - 1$.

(2) φ *has accuracy* κ.

Thus, if the accuracy κ of the scaling function φ is high, as a consequence of Taylor's theorem, the wavelet coefficients of the smooth part of a signal will be small. This yields good signal compression for the applications. As we can deduce from (1.3.12), the scaling function and the wavelet have the same smoothness. It is also true that accuracy is necessary for a compactly supported scaling function to be smooth, as we can deduce from the following theorem and Proposition 4.

Theorem 9. *([Mey92]) Assume that $\varphi \in L^2(\mathbb{R})$ is a compactly supported scaling function of a MRA and has orthonormal shifts, and let ψ be the corresponding wavelet. If $\psi \in C^k$ then $\int_{-\infty}^{+\infty} x^s \psi(x)\, dx = 0$ for $s = 0, \dots, k$.*

Chapter 2

Local Bases for Refinable Shift Invariant Spaces on the Real Line

2.1 Introduction

In this chapter we will consider a compactly supported function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ that satisfies the refinement equation (1.3.4), i.e.

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x - k).$$

It can be shown that φ must be supported in the interval $[0, N]$ (see for [LRM91]).

Let T be the $(N+1) \times (N+1)$ matrix, *the scale matrix*, defined by

$$T = \{c_{2i-j}\}_{i,j=0,\dots,N}. \tag{2.1.1}$$

Here, and always throughout this work, we assume $c_t = 0$, if $t \neq 0, \dots, N$.

In Chapter 1 we already presented equivalent conditions for accuracy. The next proposition gives a characterization of accuracy which is of particular importance for what we are concerned with in this thesis.

Proposition 5. *([Dau92], [CHM98]) Let φ be a compactly supported function satisfying* (1.3.4)*. Assume that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ are linearly independent. Then the following statements are equivalent*

(1) *The function φ has accuracy κ.*

(2) *The numbers $\{1, 2^{-1}, ..., 2^{-(\kappa-1)}\}$ are eigenvalues of the matrix T defined above and there exist polynomials $p_0, ..., p_{\kappa-1}$ with degree$(p_i) = i$ such that the vector $v_i = \{p_i(k)\}_{k=0,...,N}$ is a left eigenvector of T corresponding to the eigenvalue 2^{-i}, $i = 0, ..., \kappa - 1$.*

One interesting property is that if φ has accuracy κ, then for $s = 1, ..., \kappa - 1$ it is true that the scalars used to reproduce x^s from φ are precisely the components of the eigenvector v_s, i.e.

$$x^s = \sum_{k\in\mathbb{Z}} p_s(k)\varphi(x-k). \tag{2.1.2}$$

where p_s is the polynomial that provides the eigenvector for 2^{-s} (see for [CHM98]).

Hence there exists a relation between the spectrum of T and certain functions in $S(\varphi)$ (more precisely between the eigenvalues 2^{-s} and the monomials x^s). This motivates the study of the spectral properties of T to further investigate its relationship with functions in $S(\varphi)$. This is our main purpose.

Consider the space

$$I(\varphi) = \{f/_{[0,1]} : [0,1] \longrightarrow \mathbb{C} : f/_{[0,1]}(x) = f(x)\ \forall x \in [0,1], f \in \mathcal{S}(\varphi)\}. \tag{2.1.3}$$

The space $I(\varphi)$ is finite dimensional. A canonical set of generators is the set

$$\{\varphi(x-k)/_{[0,1]} : [0,1] \longrightarrow \mathbb{C};\ k \text{ such that } |(\mathrm{Supp}(\varphi)+k) \cap [0,1]| > 0\}.$$

We will call the algebraic dimension of the vector space $I(\varphi)$ the *local dimension* of $S(\varphi)$ and we will note it $dim_{[0,1]}\mathcal{S}(\varphi)$. A basis of $I(\varphi)$ is named a *local basis* for $S(\varphi)$. It is easy to see that

$$dim_{[0,1]}\mathcal{S}(\varphi) \leq length(\mathrm{Supp}(\varphi)) = N.$$

The following is a remarkable property of refinable compactly supported univariate functions.

Theorem 10. *([LRM91]) Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ a refinable compactly supported function such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ are globally linearly independent. Then $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ are strongly locally linearly independent.*

This result implies that for a univariate function that satisfies (1.3.4), actually we have $dim_{[0,1]}\mathcal{S}(\varphi) = N$.

By Proposition 3 we know that if φ has accuracy κ, then $\kappa \leq N$.

When φ is the B-spline of degree $N-1$, then the accuracy is N, i.e. it is maximum. Moreover, the set $\{1, x, x^2, \ldots, x^{N-1}\}$ is a local basis for $\mathcal{S}(\varphi)$, and the spectrum of T consists exactly of $\{1, 2^{-1}, \ldots, 2^{-(N-1)}\}$.

Let φ be a univariate refinable compactly supported function with accuracy κ which is not a B-spline. We saw that powers of $1/2$ are eigenvalues associated to monomials. What can we say about the other eigenvalues and eigenvectors if $\kappa < N$? Which kind of functions in $\mathcal{S}(\varphi)$ are associated to an arbitrary eigenvalue λ? Since $dim_{[0,1]}\mathcal{S}(\varphi) = N$, we know that we can complete the set of functions $\{1, x, x^2, \ldots, x^{\kappa-1}\}$ to a local basis of $\mathcal{S}(\varphi)$. Could this be done with functions associated to the other eigenvalues?

In the case that λ is a simple eigenvalue, Blu and Unser [BU02] and later Zhou [Zho02] showed that λ is associated to what they call a 2-scale λ-homogeneous function, that is, a function in the SIS that satisfies the relation $h(x) = \lambda h(2x)$. In particular, the monomial x^k satisfies this property for $\lambda = 2^{-k}$. However, to obtain a complete representation of the space it is necessary to consider the whole spectrum of T. This is what we will do in this chapter. We will show that there exists a local basis which contains the monomials $\{1, x, x^2, \ldots, x^{\kappa-1}\}$, and certain functions, which can also be obtained from the spectrum of T, which are not homogeneous polynomials but still satisfy a property of homogeneity. Hence one obtains a different way of writing the functions of $\mathcal{S}(\varphi)$. We also show that if $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ are linearly independent, then T is invertible.

2.2 Notation

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a function supported in $[0, N]$ satisfying (1.3.4).

For each $x \in \mathbb{R}$ let $\phi(x)$ be the infinite column vector associated to φ, namely

$$\phi(x) = \{\varphi(x+k)\}_{k\in\mathbb{Z}} = \begin{pmatrix} \vdots \\ \varphi(x-1) \\ \varphi(x) \\ \varphi(x+1) \\ \vdots \end{pmatrix}. \tag{2.2.1}$$

Define the double bi-infinite matrix $L = L_\varphi$ by

$$L = [c_{2i-j}]_{i,j\in\mathbb{Z}}\,. \tag{2.2.2}$$

Observe that each row of L is a double shift of the preceding one. Hence L is a "downsampled Toeplitz matrix" or a "two-slanted matrix."

If φ satisfies the refinement equation, then for $k \in \mathbb{Z}$

$$\begin{aligned} [L\phi(2x)]_k &= \sum_{j\in\mathbb{Z}} c_{2k-j}\varphi(2x+j) \\ &= \sum_{l\in\mathbb{Z}} c_l\varphi(2x+2k-l) \\ &= \varphi(x+k) = [\phi(x)]_k. \end{aligned}$$

The converse also holds, hence the refinement equation can be written

$$\phi(x) = L\phi(2x).$$

Using the matrix L, the subdivision operator (1.3.17) can be recast as

$$S_c\alpha = \alpha L \quad \alpha \in \ell(\mathbb{Z})$$

were α on the right-hand side of the equation is thought as an infinite row vector. Note that the scaling matrix T defined in (2.1.1) is a finite submatrix of L. We will consider in our analysis the matrices M, T_0, T_1, that are submatrices of T and are defined as $M = [c_{2i-j}]_{i,j=1,\ldots,N-1}$, $T_0 = [c_{2i-j}]_{i,j=0,\ldots,N-1}$, $T_1 = [c_{2i-j}]_{i,j=1,\ldots,N}$. That is,

$$T = \begin{bmatrix} c_0 & 0 & \ldots\ldots\ldots\ldots & & & 0 \\ c_2 & c_1 & c_0 & 0 & \ldots & 0 \\ \vdots & c_3 & c_2 & c_1 & \ldots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots & c_{N-2} \\ 0 & \ldots\ldots\ldots & & & 0 & c_N \end{bmatrix} = \left[\begin{array}{c|c|c} c_0 & 0 & 0 \\ \hline \vdots & M & \vdots \\ \hline 0 & 0 & c_N \end{array}\right]. \tag{2.2.3}$$

Note that c_0 and c_N must be non-zero, since $\mathrm{Supp}(\varphi) = [0, N]$.

Now, if $Y \in \ell(\mathbb{Z})$, define Y^0 and Y^M as the restriction of Y to the indexes $\{0, \ldots, N\}$, and $\{1, \ldots, N-1\}$, respectively, i.e., $Y^0 = (Y_0, \ldots, Y_N)$,
$Y^M = (Y_1, \ldots, Y_{N-1})$.

Note. Throughout this work, $(L - \lambda I)$ is considered as an operator on $\ell(\mathbb{Z})$, defined by left-multiplication (i.e. $Y \longmapsto Y(L - \lambda I)$, where Y is a double infinite row vector). I is the identity operator acting on $\ell(\mathbb{Z})$. By an abuse of notation, we will use the notation I for all identity operators, without distinguishing the space they are acting on. Note also that properties of the matrix L translate directly into properties of the subdivision operator S_c.

2.3 Spectral properties of L

The following proposition will show how the spectral properties of L are related to those of T. The case $r = 1$ has been studied earlier by [CHM00], [JRZ98], [Zho01, Zho02].

Proposition 6. *Let* $\lambda \in \mathbb{C}$

(1) *Let* $Y \in \ell(\mathbb{Z})$ *and* $r \in \mathbb{N}$, $r \geq 1$. *If* $Y \in \operatorname{Ker}(L - \lambda I)^r$, *then* $Y^0 \in \operatorname{Ker}(T - \lambda I)^r$.

Moreover, if $\lambda \neq 0$, $Y \neq 0$ *and* $Y \in \operatorname{Ker}(L - \lambda I)^r$, *then* $Y^0 \neq 0$.

(2) *If* $v \in \operatorname{Ker}(T - \lambda I)^r$ *and* $\lambda \neq 0$, *then there exists an extension* $Y_v \in \ell(\mathbb{Z})$ *of* v *(i.e.* $Y_v^0 = v$*) such that* $Y_v \in \operatorname{Ker}(L - \lambda I)^r$.

Proof. The matrix L can be decomposed in blocks as

$$L = \left[\begin{array}{c|c|c} R & 0 & 0 \\ \hline P & T & Q \\ \hline 0 & 0 & S \end{array}\right] \tag{2.3.1}$$

where we decompose $\mathbb{Z}$ as

$$\mathbb{Z} = A^- \cup \quad A^0 \cup \quad A^+ \tag{2.3.2}$$

with $A^- = \mathbb{Z} \cap (-\infty, -1]$, $A^0 = \mathbb{Z} \cap [0, N]$ and $A^+ = \mathbb{Z} \cap [N+1, +\infty)$, and

$$R = L|_{A^- \times A^-} \quad P = L|_{A^0 \times A^-} \quad T = L|_{A^0 \times A^0} \quad Q = L|_{A^0 \times A^+} \quad S = L|_{A^+ \times A^+}.$$

This block form of the matrix, is closed under multiplication. So if $r \geq 1, r \in \mathbb{N}$,

$$L^r = \left[\begin{array}{c|c|c} R^r & 0 & 0 \\ \hline P_r & T^r & Q_r \\ \hline 0 & 0 & S^r \end{array}\right]$$

where

$$P_r = \sum_{k=0}^{r-1} T^k P R^{r-k-1} \quad \text{and} \quad Q_r = \sum_{k=0}^{r-1} T^k Q S^{r-k-1}. \tag{2.3.3}$$

Since

$$(L-\lambda I)=\left[\begin{array}{c|c|c} R-\lambda I & 0 & 0 \\ \hline P & T-\lambda I & Q \\ \hline 0 & 0 & S-\lambda I \end{array}\right],$$

we have

$$(L-\lambda I)^r=\left[\begin{array}{c|c|c} (R-\lambda I)^r & 0 & 0 \\ \hline P_r^\lambda & (T-\lambda I)^r & Q_r^\lambda \\ \hline 0 & 0 & (S-\lambda I)^r \end{array}\right],$$

where P_r^λ and Q_r^λ are as in (2.3.3) with the obvious changes.

Note that the matrix S is upper triangular, with diagonal $(0,0,0,\dots)$, and hence $(S-\lambda I)^r$ is upper triangular, with diagonal $((-\lambda)^r,(-\lambda)^r,(-\lambda^r),\dots)$.

Analogously, we observe that R is lower triangular with zeroes in the main diagonal, so $(R-\lambda I)^r$ is lower triangular with diagonal $((-\lambda)^r,(-\lambda)^r,(-\lambda)^r,\dots)$.

If $Y=(Y^-,Y^0,Y^+)$, then $Y(L-\lambda I)^r$ can be written as

$$(Y^-(R-\lambda I)^r+Y^0P_r^\lambda,Y^0(T-\lambda I)^r,Y^0Q_r^\lambda+Y^+(S-\lambda I)^r). \tag{2.3.4}$$

So if $Y\in \mathrm{Ker}(L-\lambda I)^r$, then $Y^0\in \mathrm{Ker}(T-\lambda I)^r$.

We now want to show that if $Y\in \mathrm{Ker}(L-\lambda I)^r$, $\lambda\neq 0$, $Y\neq 0$, then $Y^0\neq 0$.

For this, let $k_0\in\mathbb{Z}$ be such that $Y_{k_0}\neq 0$. If $0\le k_0\le N$, we are done. Assume that $k_0>N$. Then, since $Y(L-\lambda I)^r=0$, in particular, the k_0 element of this product is 0. But since $\lambda\neq 0$, $(S-\lambda I)^r$ is upper triangular with $(-\lambda)^r$ in the diagonal, therefore the only non-zero elements of column k_0 of $(L-\lambda I)^r$ are between 0 and k_0. Hence there has to be a $k_1, 0\le k_1<k_0$, such that $Y_{k_1}\neq 0$. Again, if $0\le k_1\le N$ we are done, otherwise we repeat the argument until k_j is in the desired interval.

If $k_0 < 0$, the argument works in the same way, reversing the role of $(S - \lambda I)^r$ and $(R - \lambda I)^r$.

For the proof of part (2), assume that $v \in \mathbb{C}^{N+1}$, $v \in \mathrm{Ker}(T - \lambda I)^r$. We want to find an infinite vector $Y \in \ell(\mathbb{Z})$, such that $Y^0 = v$ and $Y \in \mathrm{Ker}(L - \lambda I)^r$. From equation (2.3.4) we know that if $Y \in \ell(\mathbb{Z})$,

$$[Y(L - \lambda I)^r]^+ = Y^0 Q_r^\lambda + Y^+(S - \lambda I)^r, \quad \text{and} \quad [Y(L - \lambda I)^r]^- = Y^0 P_r^\lambda + Y^-(R - \lambda I)^r.$$

Therefore, if $Y \in \mathrm{Ker}(L - \lambda I)^r$, and $Y^0 = v$, then Y^+ and Y^- have to satisfy

$$Y^+(S - \lambda I)^r = -vQ_r^\lambda \quad \text{and} \quad Y^-(R - \lambda I)^r = -vP_r^\lambda. \tag{2.3.5}$$

Again using the fact that $(S - \lambda I)^r$ and $(R - \lambda I)^r$ are triangular, if $\lambda \neq 0$, there are unique solutions for Y^+ and Y^- and they can be obtained recursively.

$\square$

The last proposition tells us that the elements of the spectrum of T are intimately related to those of L. But by the special form of T (see equation (2.2.3)), we can actually use the $(N-1) \times (N-1)$ matrix M to obtain the spectrum of T, as the following proposition shows:

Proposition 7. *Let $\lambda \neq 0 \in \mathbb{C}$.*

(1) *Let $v^0 = (v_0, \ldots, v_N) \in \mathbb{C}^{N+1}$ and $r \in \mathbb{N}$, $r \geq 1$. If $v^0 \in \mathrm{Ker}(T - \lambda I)^r$ then $v^M = (v_1, \ldots, v_{N-1}) \in \mathrm{Ker}(M - \lambda I)^r$. Moreover, if $\lambda \neq c_0$, $\lambda \neq c_N$, $v^0 \in \mathrm{Ker}(T - \lambda I)^r$, and $v^0 \neq 0$, then $v^M \neq 0$.*

(2) *If $v^M = (v_1, \ldots, v_{N-1}) \in \mathrm{Ker}(M - \lambda I)^r$ and $\lambda \neq c_0$ and $\lambda \neq c_N$, then there exists an extension $v^0 \in \mathbb{C}^{N+1}$ of v, such that $v^0 \in \mathrm{Ker}(T - \lambda I)^r$.*

Proof. The proof is immediate by noting the special block-form of T given in equation (2.2.3). □

2.4 The kernel of L

The case $\lambda = 0$ could not be handled with the methods of Proposition 6, since the matrices R and S in (2.3.1) have zeros in the main diagonal. Instead, we need some results from the theory of difference equations which we present below [Hen62].

2.4.1 Difference Equations

Consider the linear difference equation with constant coefficients of order r

$$u_0 y_n + u_1 y_{n+1} + \cdots + u_r y_{n+r} = 0 \quad y = \{y_n\}_{n \in \mathbb{Z}}, \tag{2.4.1}$$

where $u_k \in \mathbb{C}, u_0 \neq 0, u_r \neq 0$ with characteristic polynomial $P(x) = \sum_{k=0}^{r} u_k x^k$.

A *solution* to the equation (2.4.1) is a sequence Y in $\ell(\mathbb{Z})$ that satisfies (2.4.1) for all $k \in \mathbb{Z}$. A vector $y = (y_0, \ldots, y_m)$ with $m \geq r+1$ is a *finite solution* of (2.4.1) if it satisfies (2.4.1) for $n = 0$ to $n = m - r - 1$.

The *space of solutions* $S \subset \ell(\mathbb{Z})$ has dimension r, and a basis of this space (the fundamental basis) can be written in the following way:

Let $h \geq 1$ be an integer, and let $d_1, \ldots, d_h$ be arbitrary non-zero complex numbers with $d_i \neq d_j$ if $i \neq j$. Let $r_1, \ldots, r_h$ be positive integers. To each pair (d_i, r_i), $i = 1, \ldots, h$, we will associate a sequence $a_i = \{a_{ik}\}_{k \in \mathbb{Z}}$ defined as follows: Set $r = r_1 + \cdots + r_h$ and $r_0 = 0$. Let $0 \leq i \leq r - 1$ and $s = s(i), j = j(i)$ be the unique integers that satisfy $r_0 + \cdots + r_{s-1} \leq i < r_1 + \cdots + r_s$, $j(i) = i - \sum_{k=0}^{s(i)-1} r_k$. Define

$$a_{ik} = \begin{cases} \operatorname{sg}(k) \frac{|k|!}{(|k|-j(i))!} \, d_{s(i)}^k & \text{for } |k| \geq j(i) \\ 0 & |k| < j(i). \end{cases} \qquad i = 0, \ldots, r-1, \quad k \in \mathbb{Z}, \tag{2.4.2}$$

where $\operatorname{sg}(k)$ is the sign of k.

So, if P is the characteristic polynomial associated to (2.4.1), consider the pairs $\{(d_i, r_i)$: where d_i is a root of P and r_i its multiplicity$\}$. The sequences $\{a_{ik}\}_{k\in\mathbb{Z}}$, $i = 0, \ldots, r-1$, form a basis of S, the subspace of $\ell(\mathbb{Z})$ of the solutions to (2.4.1).

It is also known from the theory of difference equations that every solution is determined unequivocally by any r consecutive elements of it. Hence, if y is a solution such that r consecutive elements are 0, then y is the zero solution.

We will now associate to the pairs $\{(d_i, r_i) : i = 1, \ldots, h\}$ the $r \times r$ matrix $\mathcal{A} = [a_{ij}]_{i,j=0,\ldots,r-1}$. Then (cf. Henrici [Hen62], p. 214)

$$\det(\mathcal{A}) = \prod_{1\le l<s\le h} (d_l - d_s)^{r_l + r_s} \prod_{i=1}^{h} (r_i - 1)!!, \tag{2.4.3}$$

where $0!! = 1$ and $k!! = k!(k-1)! \ldots 1!$. Since $d_i \neq d_j$ for $i \neq j$, $\det(\mathcal{A}) \neq 0$ and $\mathcal{A}$ is invertible.

Let us now consider a system of k linear difference equations with constant coefficients of order r :

$$u_{i0} y_n + \cdots + u_{ir} y_{n+r} = 0, \quad i = 1, \ldots, k, \quad n \in \mathbb{Z}, \tag{2.4.4}$$

and let P_i be the characteristic polynomial of equation i, $P_i(x) = \sum_{j=0}^{r} u_{ij} x^j$. Define $\mathcal{D} = \bigcup_{i=1}^{k} \{d : P_i(d) = 0\} = \{d_1, \ldots, d_s\}$, and for each $d \in \mathcal{D}$ define $r_d = max\{r_i : r_i \text{ is the multiplicity of } d \text{ in } P_i\}$. Note that $r_d \geq 1$ $\forall d \in \mathcal{D}$. We then have the pairs $(d_i, r_{d_i}) = (d_i, r_i)$. Define the index of the system to be $t = \sum_{d\in\mathcal{D}} r_d \leq kr$. Let ℓ be the degree of the maximum common divisor p, of $\{P_a, \ldots, P_k\}$. Hence, $P_i(x) = p(x)\tilde{P}_i(x)$, with degree $\tilde{P}_i = r - \ell$. (Note that ℓ could be 0.) With the above notation, we have the following proposition:

Proposition 8. *The space S_k of solutions to the system* (2.4.4) *has dimension ℓ, where ℓ is the degree of the maximum common divisor of the characteristic polynomials. Moreover, if t is the index of the system* (2.4.4)*, and z is a vector of length t*

that satisfies (2.4.4)*, then it can be extended to a sequence* $y_z = \{y_j\}_{j\in\mathbb{Z}}$ *solution of* (2.4.4) *and such that* $y_j = z_j,\ j = 1,\dots,t$.

Proof. Let p be the maximum common divisor of $P_1,\dots,P_k$, and let $\ell = \deg(p)$. It is clear that $\dim(S_k) \geq \ell$.

For the other inequality, consider the $t \times t$ matrix $\mathcal{A} = [a_{ij}]_{i,j=0,\dots,t-1}$, with a_{ij} defined in (2.4.2) for the pairs $\{(d_i, r_i)\}$ defined above and t being the index of the system (2.4.4). Since $d_i \neq d_j$ for $i \neq j$, $\det(\mathcal{A}) \neq 0$ by (2.4.3), and $\mathcal{A}$ is invertible.

Assume now that $y \in S_k$. Then y is a solution to all k difference equations of (2.4.4), hence there exist $\alpha^1,\dots,\alpha^k$ vectors of length r, such that

$$\mathcal{A}_i\alpha^i = [y_0,\dots,y_{t-1}]^t \quad 1 \leq i \leq k, \tag{2.4.5}$$

where $\mathcal{A}_i$ is an $t \times r$ matrix whose columns are a fundamental system for equation i. Note that $\mathcal{A}_i$ is a sub-matrix of $\mathcal{A}$, whose columns correspond to some columns $\{i_1,\dots,i_r\}$ of $\mathcal{A}$.

Now let $\tilde{\alpha}^i$ be vectors of length t, such that $\tilde{\alpha}^i_h = 0$ whenever $h \notin \{i_1,\dots,i_r\}$ and $\tilde{\alpha}^i_{i_s} = \alpha^i_s$, $s = 1,\dots,r$. Then we have for $i, j = 1,\dots,k$

$$\mathcal{A}\tilde{\alpha}^i = \mathcal{A}_i\alpha^i = [y_0,\dots,y_{t-1}]^t = \mathcal{A}_j\alpha^j = \mathcal{A}\tilde{\alpha}^j, \tag{2.4.6}$$

and hence $\mathcal{A}(\tilde{\alpha}^i - \tilde{\alpha}^j) = 0$, for all $i \neq j$ and therefore, by the invertibility of $\mathcal{A}$, $\tilde{\alpha}^i = \tilde{\alpha}^j$, for all $i \neq j$. Therefore the only non-zero elements of α^i can be those corresponding to the columns associated to the roots of p. Hence y is a linear combination of ℓ columns, and therefore $\dim(S_k) \leq \ell$. $\square$

By noting that for the previous proof we only used the first t coordinates of the infinite sequences, we have the following immediate corollary.

Corollary 2. *If z is a vector of length t that satisfies* (2.4.4)*, then it can be extended to a sequence $y_z = \{y_j\}_{j\in\mathbb{Z}}$ solution of* (2.4.4) *and such that $y_j = z_j,\ j = 1, \ldots, t$.*

2.4.2 The $\mathrm{Ker}(L)$

We can now return to our double infinite matrix L and look at the special case $\lambda = 0$. As it turns out, the kernel of L is characterized by the vectors in the kernel of M. Since c_0 and c_N are non-zero, the matrices T and M have kernels of the same dimension. Moreover, we have the following proposition:

Proposition 9. *Consider the polynomials p_e and p_o of degree $q = \frac{N-1}{2}$ (we assume N to be odd) $p_e(x) = c_0 + c_2x + \cdots + c_{2q}x^q,\ p_o(x) = c_1 + c_3x + \cdots + c_{2q+1}x^q$. Then $\dim(\mathrm{Ker}(L)) = \dim(\mathrm{Ker}(M)) =$ degree(p), where p is the maximum common divisor of the polynomials p_e and p_o. In particular, if $\dim(\mathrm{Ker}(M)) > 0$, p_e and p_o have a common root. Furthermore:*

(1) *For every $Y \in \mathrm{Ker}(L)$, $Y \neq 0$, we have $Y^M \neq 0$ and $Y^M \in \mathrm{Ker}(M)$.*

(2) *Conversely, for each $v \in \mathrm{Ker}(M), v \neq 0$, we have $Y_v \neq 0$ and $Y_v \in \mathrm{Ker}(L)$.*

Proof. Let us first observe that $Y \in \ell(\mathbb{Z})$ is in the kernel of L, if and only if Y satisfies the system of difference equations

$$\begin{cases} c_0v_n + c_2v_{n+1} + \cdots + c_{2q}v_{n+q} & = 0, \\ c_1v_n + c_3v_{n+1} + \cdots + c_{2q+1}v_{n+q} & = 0. \end{cases} \tag{2.4.7}$$

Therefore, by Proposition 8, $\mathrm{Ker}(L)$ is the subspace generated by the fundamental solutions associated to the roots of p, the maximum common divisor of p_o and p_e. This shows that $\dim(\mathrm{Ker}(L)) = \mathrm{degree}(p)$.

On the other side, if $Y \in \mathrm{Ker}(L)$, since $(YL)^M = Y^M M$ we conclude that $Y^M \in \mathrm{Ker}(M)$, and if Y^M is the zero vector, then the solution Y of (2.4.7) has $N-1$ consecutive zeros, so $Y = 0$. Hence, if $Y \neq 0$, then $Y^M \neq 0$, which proves (1).

To see that if $v = (v_1, \ldots, v_{N-1})$ satisfies $vM = 0$, then v can be extended, just note that the sequence $v_1, \ldots, v_{N-1}$ must satisfy the difference equations system of order $q = \frac{N-1}{2}$ (we assumed N to be odd) given by (2.4.7). Since the index t of the system (2.4.7) satisfies $t \leq 2q = N-1$, and v is a non-trivial common solution of length $N-1$, by Corollary 2 this solution can be extended in such a way that the extension satisfies both difference equations. This proves (2).

From (1) and (2) it is immediate that $\dim(\mathrm{Ker}(L)) = \dim(\mathrm{Ker}(M))$. $\square$

Note. The fact that $\dim(\mathrm{Ker}(M)) > 0$ implies that p_e and p_o have a common root was proved under some minor technical conditions by Meyer [Mey91]. Related results can also be found in [JW93].

2.4.3 Invertibility of L

Propositions 6 and 7 relate the spectral properties of the matrix M to the ones of the operator L. The next proposition shows a necessary condition for the independence of the integer translates of the function φ, in terms of the matrix M.

Proposition 10. *With the above notation, consider the following properties:*

(1) $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ *are linearly independent.*

(2) *The operator* $L : \ell(\mathbb{Z}) \longrightarrow \ell(\mathbb{Z}), Y \longmapsto YL$ *is one-to-one.*

(3) *The matrix* M *is invertible.*

Then (2) $\iff$ (3) and (1) $\implies$ (2).

Proof. **(1) $\implies$ (2)** Assume $YL = 0$. Define $F(x) = Y\phi(x)$. Then we have $F(x) = Y\phi(x) = YL\phi(2x) = 0$. Now, $Y\phi(x) = 0 \implies Y = 0$, therefore $\text{Ker}(L) = \{0\}$.

(2) $\iff$ (3) is a consequence of Proposition 9. □

Note: We do not know if either (2) or (3) implies (1).

2.5 Homogeneous functions

Assume now that $Y \in \text{Ker}(L - \lambda I)^r$, and define $h(x) = Y\phi(x)$. So, h satisfies

$$\begin{aligned} 0 = Y(L-\lambda I)^r\phi(x) &= Y\left(\sum_{k=0}^{r}\begin{pmatrix} r \\ k \end{pmatrix}(-\lambda)^k L^{r-k}\right)\phi(x) \\ &= Y\left(\sum_{k=0}^{r}\begin{pmatrix} r \\ k \end{pmatrix}(-\lambda)^k \phi(\frac{x}{2^{r-k}})\right), \\ &= \sum_{k=0}^{r}\begin{pmatrix} r \\ k \end{pmatrix}(-\lambda)^k h(\frac{x}{2^{r-k}}). \end{aligned}$$

Equivalently, if $\mathcal{D}_a$ is the operator defined by $(\mathcal{D}_a f)(x) = f(a^{-1}x)$, we have that h satisfies

$$(\mathcal{D}_2 - \lambda I)^r h = 0. \tag{2.5.1}$$

We will say that a function h is $(2, \lambda, r)$ homogeneous, if h satisfies (2.5.1) (r is the order of homogeneity, and λ is the degree), and we will denote by $\mathcal{H}(2, \lambda, r)$ the space of all $(2, \lambda, r)$- homogeneous functions.

Remark (1). A $(2, \lambda, r)$-homogeneous function is a particular case of a poly-scale refinable distribution. The concept of poly-scale refinable distribution is a generalization of refinability. See for instance [DD02, Sun05a].

Remark (2). Note that if

$$h \in \mathcal{H}(2, \lambda, r), \quad \text{then} \quad h \in \mathcal{H}(2, \lambda, s) \quad \text{for every} \quad s \geq r.$$

Therefore the "order of homogeneity" will be defined by

$$\min\{s : h \in \mathcal{H}(2, \lambda, s)\}.$$

Remark (3). If h is homogeneous (of any order), then $h(0) = 0$. The values of any homogeneous function of order r in $(0, +\infty)$ are completely determined by its values on any interval of the type $\left[\frac{1}{2^{k+r}}, \frac{1}{2^k}\right)$, $k \in \mathbb{Z}$. (Analogously, the values on $(-\infty, 0)$, are obtained from the values in any interval of the type $\left(-\frac{1}{2^k}, -\frac{1}{2^{k+r}}\right]$). To see this, note that by choosing $x = 2^r t$ in (2.5.1), we obtain

$$\sum_{k=0}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-\lambda)^k h(2^k x) = 0,$$

and therefore

$$h(x) = -\sum_{j=1}^{r} \begin{pmatrix} r \\ j \end{pmatrix} (-\lambda)^j h(2^j x) \quad \text{and} \tag{2.5.2}$$

$$h(2^r x) = -\sum_{j=0}^{r-1} \begin{pmatrix} r \\ j \end{pmatrix} (-\lambda)^{j-r} h(2^j x) \quad \text{which is equivalent to}$$

$$h(x) = -\sum_{j=1}^{r} \begin{pmatrix} r \\ j \end{pmatrix} (-\lambda)^{-j} h(2^{-j} x). \tag{2.5.3}$$

So, if $x \in \left[\frac{1}{2^{k+r+1}}, \frac{1}{2^{k+r}}\right)$, then for $j = 1, \ldots, r$, $\;2^j x \in \left[\frac{1}{2^{k+r-j+1}}, \frac{1}{2^{k+r-j}}\right) \subset \left[\frac{1}{2^{k+r}}, \frac{1}{2^k}\right)$, and using (2.5.2), we determine $h(x)$. Iterating this procedure, we see that all values in the interval $\left(0, \frac{1}{2^{k+r}}\right)$ can be determined.

On the other hand, for $x \in \left[\frac{1}{2^k}, +\infty\right)$, we use (2.5.3), and observe that if $x \in \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right)$, for $j = 1, \ldots, r$, $\quad 2^{-j}x \in \left[\frac{1}{2^{k+j}}, \frac{1}{2^{k+j-1}}\right) \subset \left[\frac{1}{2^{k+r}}, \frac{1}{2^k}\right)$.

Remark (4). In the case of order of homogeneity 1, (e.g. $r = 1$), h is a 2-scale homogeneous function as described by Zhou [Zho02].

Proposition 11. *Assume $\{\varphi(\cdot - k)\}$ are linearly independent. Let ϕ be as in* (2.2.1)*. If $g_1, \ldots, g_n \in \mathcal{S}(\varphi)$, $g_i = Y^i\phi$, then $\{g_1, \ldots, g_n\}$ are linearly independent functions if and only if $\{Y^1, \ldots, Y^n\}$ are linearly independent in $\ell(\mathbb{Z})$.*

Proof. We observe that

$$\sum_{i=1}^{n} \alpha_i g_i = \sum_{i=1}^{n} \alpha_i \left(Y^i\phi\right) = \left(\sum_{i=1}^{n} \alpha_i Y^i\right) \phi.$$

This equation, together with the linear independence of the translates of φ, tells us that $\sum_i \alpha_i g_i \equiv 0$ if and only if $(\sum_i \alpha_i Y^i) = 0$, which proves the desired result. $\square$

Theorem 11. *Assume that $\{\varphi(\cdot - k)\}$ are linearly independent. If $h = Y\phi$ ($h \in \mathcal{S}(\varphi)$), and $h \in \mathcal{H}(2, \lambda, r)$, then $v_h = Y^0 \in \mathrm{Ker}(T - \lambda I)^r$. Reciprocally, if $v \in \mathrm{Ker}(T - \lambda I)^r$, then the function $h = Y_v\phi$ is in $\mathcal{H}(2, \lambda, r)$. (Here Y_v is the unique extension of v to a vector in $\mathrm{Ker}(L - \lambda I)^r$ by Proposition 6.)*

Proof. For the first claim, note that

$$\begin{aligned}
0 &= \sum_{k=0}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-\lambda)^{r-k} h(2^{-k}x) \\
&= \sum_{k=0}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-\lambda)^{r-k} Y L^k \phi(x) \\
&= Y \left(L - \lambda I\right)^r \phi(x).
\end{aligned}$$

Then, $Y\,(L-\lambda I)^r = 0$ and by Proposition 6, $v_h \in \mathrm{Ker}(T-\lambda I)^r$.

For the converse first observe that if $v = 0$ the result is trivial. Assume $v \neq 0$ and $v \in \mathrm{Ker}(T-\lambda I)^r$; then by Proposition 10 $\lambda \neq 0$. Hence (by Proposition 6) there is a unique extension $Y_v \in \mathrm{Ker}(L-\lambda I)^r$, so $h = Y_v \phi$ is in $\mathcal{H}(2, \lambda, r)$. $\square$

2.5.1 Local basis of homogeneous functions

Theorem 12. *Assume that $\{\varphi(\cdot - k)\}$ are linearly independent. Let Λ be the set of eigenvalues of T, and let $\mathcal{B} = \{v_0, \ldots, v_N\}$ be a basis of $\mathbb{C}^{N+1}$ that gives the Jordan form of T. Let $\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda \subset \mathcal{S}(\varphi)$, where $\mathcal{H}_\lambda(\varphi) = \{h \in \mathcal{S}(\varphi) : h \in \mathcal{H}(2, \lambda, k),$ for some $k \geq 1\}, \lambda \in \Lambda$. Then we have that* $\dim(\mathcal{H}) = N + 1$.

Remark. Note that we can choose both $v_0 = (1, 0, \ldots, 0)$ and $v_N = (0, \ldots, 0, 1)$ to be in the basis $\mathcal{B}$, corresponding to the eigenvalues c_0 and c_N, respectively.

Proof. If $v_i \in \mathcal{B}\,(0 \leq i \leq N)$, then $v_i \in \mathrm{Ker}(T-\lambda I)^k$ and $v_i \notin \mathrm{Ker}(T-\lambda I)^{k-1}$, for some $\lambda \in \Lambda$, and $k \geq 1$. So to each $v_i \in \mathcal{B}$, we can associate a unique pair (λ, k). Let us denote such $v_i = v(\lambda, k)$. (Note that by the previous observation, $v_0 = v(c_0, 1)$ and $v_N = v(c_N, 1)$.)

After Theorem 11, we can associate to each $v(\lambda, k)$ a function $h_{v(\lambda,k)}$ in $\mathcal{H}(2, \lambda, k) \cap \mathcal{S}(\varphi)$. Furthermore, the functions $\{h_{v(\lambda,k)}\}_{v \in \mathcal{B}}$, are linearly independent.

For this, observe that since the vectors in $\mathcal{B}$ are linearly independent, its extensions $\{Y_v\}$ are linearly independent in $\ell(\mathbb{Z})$, and therefore the functions $\{h_{v(\lambda,k)}\}_{v \in \mathcal{B}}$ are linearly independent.

One can see that if a finite number of functions are homogeneous for the same λ, then a linear combination of them is also homogeneous for the same λ. More precisely,

$$\sum_{i=0}^{n} \alpha_i h_i(\lambda, k_i) = h(\lambda, k), \quad \text{where} \quad k = \max_i (k_i),$$

for

$$(\mathcal{D}_2 - \lambda I)^k h = (\mathcal{D}_2 - \lambda I)^k \sum_{i=0}^{n} \alpha_i h_i = \sum_{i=0}^{n} \alpha_i (\mathcal{D}_2 - \lambda I)^k h_i = 0.$$

Hence, if p_T, the characteristic polynomial of T, is factorized as

$$p_T(x) = \prod_{\lambda \in \Lambda} (x - \lambda)^{r_\lambda},$$

then $\dim(\mathcal{H}_\lambda) = r_\lambda$, and a basis of $\mathcal{H}_\lambda$ is the set of $(2, \lambda, k)$-homogeneous functions associated to the vectors $v \in \mathcal{B}$, such that $v = v(\lambda, k)$, for some $k \geq 1$.

Now consider

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda \subset \mathcal{S}(\varphi).$$

With the notation above we have: The correspondence

$$v(\lambda, k) \in \mathcal{B} \longmapsto h_{v(\lambda,k)} \in \mathcal{H}, \tag{2.5.4}$$

extends linearly to a linear isomorphism $\tau : \mathbb{C}^{N+1} \longrightarrow \mathcal{H}$. $\square$

Corollary 3. *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathbb{C}^{N+1} & \longleftrightarrow & \mathcal{H} \\ \updownarrow & \nearrow & \\ W & & \end{array}$$

where $W = \mathrm{span}\{Y_v : v \in \mathcal{B}\}$.

Proof. The extension

$$v(\lambda, k) \longmapsto Y_{v(\lambda,k)},$$

defined earlier, can also be extended to an isomorphism $\tilde{\tau} : \mathbb{C}^{N+1} \longrightarrow W \subset \ell(\mathbb{Z})$. $\square$

Theorem 13. *Assume that φ has linearly independent integer translates. Let $\mathcal{B} = \{v_0, \ldots, v_N\}$ be as before, a Jordan basis for T, and let B be the $(N+1) \times (N+1)$ matrix that has the vectors v_i as rows. Let*

$$\phi^0(x) = \begin{bmatrix} \varphi(x) \\ \varphi(x+1) \\ \vdots \\ \varphi(x+N) \end{bmatrix} \quad \text{and} \quad h(x) = \begin{bmatrix} h_0(x) \\ h_1(x) \\ \vdots \\ h_N(x) \end{bmatrix},$$

where h_i is the homogeneous function associated to the vector v_i. Then we have

(i) $h(x) = B\phi^0(x)$ $x \in [-1, 1]$.

(ii) $\phi^0(x) = T\phi^0(2x)$ *and* $h(x) = BTB^{-1}h(2x)$ $\quad x \in [-1/2, 1/2]$, *where* BTB^{-1} *is in Jordan form.*

(iii) *There exists a local basis of $\mathcal{S}(\varphi)$ consisting of homogeneous functions. Moreover, if φ has accuracy κ, this basis can be chosen to contain the polynomials $\{1, x, \cdots, x^{\kappa-1}\}$.*

Remark. Note that (ii) is a statement about the refinability of both ϕ^0 and h, where the scaling matrix of h is the Jordan form of the scaling matrix of ϕ^0.

Proof. Since the support of φ is $[0, N]$, $\varphi(x+k) = 0$ if $k \notin \{0, 1, \ldots, N\}$ for $x \in [-1, 1]$. Then

$$h_i(x) = Y^i\Phi(x) = v_i\Phi^0(x) \;\; x \in [-1, 1].$$

So we have

$$h(x) = B\phi^0(x)\ x \in [-1,1]. \tag{2.5.5}$$

This shows that for every interval $I \subset [-1,1]$ the functions $\{\varphi(x), \varphi(x+1), \cdots, \varphi(x+N)\}$ span the same space than the functions $\{h_0, \cdots, h_N\}$ when restricted to I.

Since $\{\varphi(x), \varphi(x+1), \cdots, \varphi(x+N-1)\}$ is a local basis of $\mathcal{S}(\varphi)$, if v_N has been chosen to be $v_N = (0, \cdots, 0, 1)$, using equation (2.5.5) $\{h_0, \cdots, h_{N-1}\}$ are a local basis for $\mathcal{S}(\varphi)$. Moreover, if φ has accuracy $\kappa \leq N$, then we can choose $v_0, \cdots, v_{\kappa-1}$ to be the eigenvectors associated to the eigenvalues $\{1, \cdots, 2^{-\kappa+1}\}$, and hence $\{h_0, \cdots, h_{N-1}\} = \{1, x, \cdots, x^{\kappa-1}, h_\kappa, \cdots, h_{N-1}\}$.

This proves (i) and (iii).

For (ii), again using the fact that the support of φ is $[0, N]$, it is easily seen that if $x \in [-\frac{1}{2}, \frac{1}{2}]$ then $\phi(x) = T\phi(2x)$. Then we have $B\Phi^0(x) = BTB^{-1}B\Phi^0(2x)$, $x \in [-\frac{1}{2}, \frac{1}{2}]$. $\square$

2.5.2 The $\mathcal{D}_2$ operator

Let $\mathcal{D}_2$ be the operator defined earlier $(\mathcal{D}_2 g)(x) = g(\frac{x}{2})$. Then $\mathcal{D}_2 : \mathcal{S}(\varphi) \longrightarrow \mathcal{S}(\varphi)$ is well defined. To see this, consider $g \in \mathcal{S}(\varphi)$, $g(x) = Y\phi(x)$ with $Y \in \ell(\mathbb{Z})$.

$$t(x) \equiv \mathcal{D}_2 g(x) = g(\frac{x}{2}) = Y\phi(\frac{x}{2}) = YL\phi(x).$$

Since $YL \in \ell(\mathbb{Z})$ we have that $t \in \mathcal{S}(\varphi)$.

Assume that the translates of φ are linearly independent. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}(\varphi) & \xrightarrow{\mathcal{D}_2} & \mathcal{S}(\varphi) \\ \mathcal{P}\downarrow & & \downarrow\mathcal{P} \\ \ell(\mathbb{Z}) & \xrightarrow{L} & \ell(\mathbb{Z}) \end{array}$$

where $\mathcal{P}$ is the function that associates to each element of $\mathcal{S}(\varphi)$ its coordinates in $\{\varphi(x+k)\}$. $\mathcal{P}$ is a linear isomorphism.

Let now $\pi : \ell(Z) \longrightarrow \mathbb{C}^{N+1}$ be the projection to the $N+1$ first coordinates, $\pi(Y) = Y^0 = (Y_0, \ldots, Y_N)$. We then have

$$\begin{array}{ccc} \mathcal{S}(\varphi) & \xrightarrow{\mathcal{D}_2} & \mathcal{S}(\varphi) \\ \mathcal{P}\Big\downarrow & & \Big\downarrow\mathcal{P} \\ \ell(\mathbb{Z}) & \xrightarrow{L} & \ell(\mathbb{Z}) \\ \pi\Big\downarrow & & \Big\downarrow\pi \\ \mathbb{C}^{N+1} & \xrightarrow{T} & \mathbb{C}^{N+1} \end{array}$$

where the second diagram also commutes by (2.3.4).

In addition, we have the inclusion

$$\mathbb{C}^{N+1} \overset{\mathcal{E}}{\hookrightarrow} \ell(\mathbb{Z}) \overset{\mathcal{P}^{-1}}{\longrightarrow} \mathcal{S}(\varphi),$$

where $\mathcal{E}(v) = Y_v$, which makes the inclusion $\tau : v \mapsto h_v$ defined in (2.5.4) also one to one.

Let $\mathcal{H} = \tau(\mathbb{C}^{N+1})$. Then $\mathcal{H} \subset \mathcal{S}(\varphi)$ is a subspace of $\mathcal{S}(\varphi)$ that is $\mathcal{D}_2$-invariant.

2.5.3 Generalizations

Throughout this work "function" has meant "measurable function". However, with the obvious modifications, the results of this chapter hold for the case that φ is a generalized function or distribution. It is interesting to consider the generalization of the results to arbitrary dilations $m \geq 1$ in $\mathbb{R}$, and in higher dimensions with arbitrary dilation matrices. We will see later that most of the results are still true in these cases.

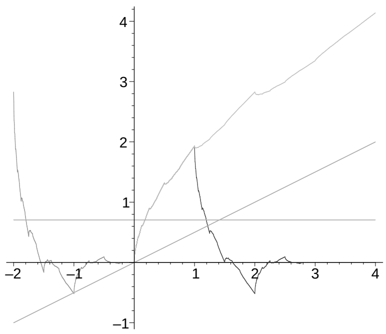

Figure 2.1: Daubechies D4 with the homogeneous functions.

2.6 Examples for $N = 3$

B-spline. The B-spline of degree 2 is the refinable function that satisfies

$$b(x) = \frac{1}{4}b(2x) + \frac{3}{4}b(2x-1) + \frac{3}{4}b(2x-2) + \frac{1}{4}b(2x-3). \tag{2.6.1}$$

The B-splines are those functions for which the accuracy is maximum and so coincides with the dimension of the matrix T_0. Therefore, in this case, the eigenvalues of T_0 are 1 (for the constant functions), $\frac{1}{2}$ (for the linear functions), and $\frac{1}{4}$ (for the quadratic functions).

Daubechies $\mathbf{D}_4$. Daubechies wavelets are those refinable functions of N coefficients that are orthogonal and provide the highest order of accuracy possible. (Note that the splines do not form an orthonormal basis.) The scaling function D_4 satisfies:

$$D_4(x) = \frac{1+\sqrt{3}}{4}D_4(2x) + \frac{3+\sqrt{3}}{4}D_4(2x-1) + \frac{3-\sqrt{3}}{4}D_4(2x-2) + \frac{1-\sqrt{3}}{4}D_4(2x-3).$$

D_4 has accuracy 2 (it reproduces the constant and the linear functions). In this case the matrix T_0 has eigenvalues 1, $\frac{1}{2}$ and $c_0 = \frac{1+\sqrt{3}}{4}$. So a basis for

$$\operatorname{span}\{D_4(x), D_4(x+1), D_4(x+2)\}_{x\in[0,1]}$$

is also given by $\operatorname{span}\{1, x, h_{c_0}(x)\}_{x\in[0,1]}$, where h_{c_0} is the homogeneous function associated to c_0 (see Figure 2.1).

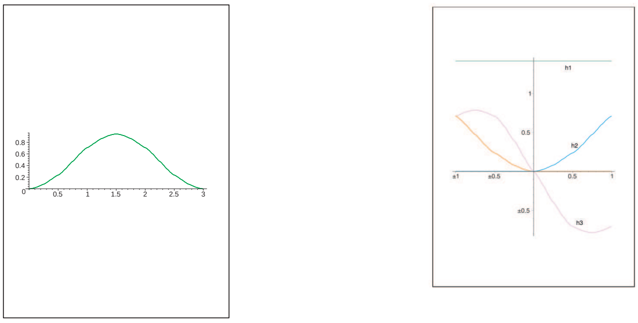

Figure 2.2: Scaling function for coefficients $1/3, 2/3, 2/3, 1/3$ with Homogeneous functions - h_1, h_2, h_3. h_3 is a 2-homogeneous function

$(\lambda, 1)$**-Homogeneous functions are not enough.** In the two previous examples, we could always obtain a basis of $\text{span}\{f(x), f(x+1), f(x+2)\}_{x\in[0,1]}$ just by using 1-homogeneous functions. The following example is to illustrate that even in the simple case of only 4 coefficients it may be necessary to use homogeneous functions of order bigger than 1. Consider the function

$$f(x) = \frac{1}{3}f(2x) + \frac{2}{3}f(2x-1) + \frac{2}{3}f(2x-2) + \frac{1}{3}f(2x-3). \tag{2.6.2}$$

It can be shown that f has accuracy 1, and the eigenvalues of T are $\{1, \frac{1}{3}\}$. So in this case, $\text{span}\{f(x), f(x+1), f(x+2)\}_{x\in[0,1]} = \text{span}\{1, h_{\{1/3,1\}}(x), h_{\{1/3,2\}}(x)\}_{x\in[0,1]}$, where $h_{\{1/3,1\}}$ is a 1-homogeneous function corresponding to the eigenvalue $1/3$, and $h_{\{1/3,2\}}$ is a 2-homogeneous function corresponding to the eigenvalue $1/3$ (see Figure 2.2).

Chapter 3

The Multidimensional Setting

3.1 Introduction

In this chapter we will introduce the notation and concepts that arise if we want to extend the theory developed so far to the multivariate case with arbitrary dilation. The 2-refinable or 2-scaling functions we studied before are a special case of what are called self-similar functions.

Let Γ be a lattice in $\mathbb{Z}^d$. A *dilation matrix* associated to Γ is $d \times d$ matrix A such that

(1) $A(\Gamma) \subset \Gamma$ and

(2) A is expansive, i.e. every eigenvalue λ of A satisfies $|\lambda| > 1$.

We will say that a compactly supported function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is *refinable* with respect to A and Γ, if it satisfies the *dilation equation*

$$\varphi(x) = \sum_{k \in \Lambda} c_k \varphi(Ax - k), \quad x \in \mathbb{R}^d, \tag{3.1.1}$$

for some finite subset $\Lambda \subset \Gamma$, and coefficients $c_k \in \mathbb{C}$.

The *shift invariant space* (SIS) generated by φ is the space

$$\mathcal{S}(\varphi) = \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{C} : f(x) = \sum_{k \in \Gamma} y_k \varphi(x + k), \quad y_k \in \mathbb{C}, k \in \Gamma \right\}.$$

Note again that since φ is compactly supported, the right hand side of the previous equation is well defined.

Definition 13. We say that the Γ translates $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are *linearly independent* or *globally linearly independent*, if for any sequence $\{\alpha_k\}_{k\in\Gamma}$ in $\ell(\Gamma)$,

$$\sum_{k\in\Gamma} \alpha_k \varphi(\cdot - k) \equiv 0 \quad \text{implies} \quad \alpha_k = 0.$$

As in the one-dimensional case, a function that satisfies (3.1.1) and has orthonormal lattice translates is the starting point for the construction of orthogonal wavelet bases associated with a multiresolution analysis [GM92],[KV92],[Mey92],[CD93]. The generalized concept of multivariate wavelets is the following.

Definition 14. A *collection of wavelets* associated with a dilation matrix A is a finite set of functions $\psi^1, \ldots, \psi^l$ in $L^2(\mathbb{R}^d)$ such that the system

$$\left\{\psi^i_{j,k}(x) = |detA|^{\frac{j}{2}}\psi(A^j x - k) : j, k \in \Gamma,\, 1 \le i \le l\right\},$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$.

A multiresolution analysis in the general setting is defined as follows.

Definition 15. A *multiresolution analysis* (MRA) associated with a dilation matrix A is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ which satisfy:

(1) $V_j \subset V_{j+1}$;

(2) $f \in V_j$ if and only if $f(A\cdot) \in V_{j+1}$ for every $j \in \mathbb{Z}$;

(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(4) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;

(5) there exist a function $\varphi \in V_0$ called the scaling function such that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ is an orthonormal basis for V_0.

We again have that condition (1) implies that $\varphi \in V_0 \subset V_1$, and so by conditions (2) and (5) we obtain that φ is refinable with respect to A and Γ. Analogously to the one-variable case we define W_0 as the orthogonal complement of V_0 in V_1, and

$$W_j = \left\{ f(A^j \cdot) : f \in W_0 \right\}.$$

So in order to have a *collection of wavelets* associated with the dilation matrix A we need $\psi^1, \dots, \psi^l \in W_0$ such that

$$\left\{ \psi^i(\cdot - k) : k \in \Gamma,\, 1 \leq i \leq l \right\}$$

is an orthonormal basis for W_0.

There exist relatively simple examples on the real line if we take $A = m$ integer. Given V_0 and the scaling function of a MRA with dilation 2, we define $V_j = \{ f(m^j \cdot) : f \in V_0 \}$.

Actually if $d > 1$, finding wavelets associated to a MRA can be a difficult task. Gröchenig and Madych [GM92] showed that there exists Haar-type multiresolution analysis in $L^2(\mathbb{R}^d)$.

3.2 Attractors, Tiles and Admissible Sets

Let Γ be a lattice and A a dilation matrix associated to Γ. Since $\Gamma = U(\mathbb{Z}^d)$, where U is an invertible matrix, the matrix $U^{-1}AU$ maps $\mathbb{Z}^d$ into itself. Hence $U^{-1}AU$ is an integer matrix which has integer determinant, and consequently A has also integer determinant. The group $\Gamma/A(\Gamma)$ has order $|\det(A)|$ (see for example [Woj97]). Set

$$m = |\det(A)|,$$

and let $D = \{d_1, \dots, d_m\}$ be a set of representatives of the group $\Gamma/A(\Gamma)$ of order m, i.e. Γ is partitioned into the disjoint cosets

$$\Gamma_i = A(\Gamma) - d_i = \{Ak - d_i : k \in \Gamma\}.$$

We call D a *full set of digits*, or *digit set.* For example, a full set of digits of the group $\mathbb{Z}/m(\mathbb{Z})$ (i.e. $d = 1$, $\Gamma = \mathbb{Z}$, and $A = m$) is the set $\mathbb{Z}_m = \{0, \dots, m-1\}$.

Assume that $\gamma_1, \dots, \gamma_d$ is a set of generators for Γ, that is, $\gamma_1, \dots, \gamma_d$ are linearly independent vectors in $\mathbb{R}^d$ and

$$\Gamma = \{l_1\gamma_1 + \ldots + l_d\gamma_d : l_i \in \mathbb{Z}\}.$$

The set

$$P = \{x_1\gamma_1 + \ldots + x_d\gamma_d : 0 \leq x_i < 1\}$$

is a *fundamental domain* for the group $\mathbb{R}^d/\Gamma$, i.e. it is a full set of representatives of $\mathbb{R}^d/\Gamma$. This means that $\mathbb{R}^d$ is partitioned into the disjoint sets $\{P + k\}_{k\in\Gamma}$. Observe that there exists am isomorphism between $\mathbb{R}^d/\Gamma$ and the d-dimensional torus, since $[0, 1)^d = \{U^{-1}x : x \in P\}$.

3.2.1 Attractors

For each $k \in \Gamma$, we define $w_k : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ by

$$w_k(x) = A^{-1}(x + k).$$

Since A is a dilation matrix, A^{-1} is contractive for some appropriate norm in $\mathbb{R}^d$, so each w_k is a contractive mapping on $\mathbb{R}^d$ for that norm.

The space

$$\mathcal{H}(\mathbb{R}^d) = \{K \subset \mathbb{R}^d : K \neq \emptyset \text{ and } K \text{ is compact}\},$$

is a complete metric space under the Hausdorff metric d defined by

$$d(B, C) = \inf\{\varepsilon > 0 : B \subset C_\varepsilon \text{ and } C \subset B_\varepsilon\},$$

where

$$B_\varepsilon = \left\{x \in \mathbb{R}^d : \text{ dist}(x, B) < \varepsilon\right\}.$$

For each finite subset $H \subset \Gamma$, we define

$$w_H(B) = \bigcup_{k \in H} w_k(B) = A^{-1}(B + H)$$

It can be shown that w_H is a contractive map in $\mathcal{H}(\mathbb{R}^d)$ (using that each w_k is a contractive mapping on $\mathbb{R}^d$). Consequently, by the Contraction Mapping Theorem, there exists a unique nonempty compact set $K_H \subset \mathbb{R}^d$ such that

$$w_H(K_H) = K_H, \quad i.e. \quad K_H = A^{-1}(K_H + H).$$

In fact, we can write

$$K_H = \sum_{j=1}^{\infty} A^{-j}(H) = \left\{ \sum_{j=1}^{\infty} A^{-j} h_j : h_j \in H \right\}. \tag{3.2.1}$$

The set K_H is called the *attractor* of the iterated function system (IFS)generated by $\{w_k\}_{k \in H}$. [Hut81].

3.2.2 Tiles

Given the digit set $D = \{d_1, \ldots, d_m\}$, we consider the attractor

$$Q = K_D = \sum_{j=1}^{\infty} A^{-j}(D) = \left\{ \sum_{j=1}^{\infty} A^{-j} \varepsilon_j : \varepsilon_j \in D \right\} \tag{3.2.2}$$

of the iterated system generated by $\{w_d\}_{d \in D}$. We have that, for $\gamma \in \Gamma$

$$K_{D+\gamma} = \sum_{j=1}^{\infty} A^{-j}(D + \gamma) = \sum_{j=1}^{\infty} A^{-j} D + (A - I)^{-1}\gamma = K_D + (A - I)^{-1}\gamma. \tag{3.2.3}$$

Therefore we can assume without loss of generality that $0 \in D$, and hence, by equation (3.2.2), we have $0 \in Q$.

The set Q satisfies the following properties (see [Ban91] and [GM92]):

a) $\bigcup_{k \in \Gamma} Q + k = \mathbb{R}^d$.

b) $Q^0 \neq \emptyset, Q = \overline{Q^0}$, *and* $|\partial Q| = 0$.

c) $|Q \cap (Q+k)| = 0$ *for every* $k \in \Gamma - \{0\}$ *if and only if* $|Q| = |P|$, *where* P *is a fundamental domain for* $\mathbb{R}^d/\Gamma$. *In this case* $Q \cap (Q+k) \subset \partial Q$ *for all* $k \in \Gamma - \{0\}$.

A longstanding problem was the question of whether for each dilation matrix A there exists a full set of digits D such that the corresponding attractor Q is a *tile*, that is, the Γ- translates $\{Q+k\}_{k\in\Gamma}$ cover $\mathbb{R}^d$ with overlaps of measure zero (hence $|Q| = |P|$). Lagarias and Wang showed that this happens if $d = 1, 2, 3$ or if $|det(A)| > d$ (see for [LW95],[LW96], [LW97]). A counterexample was found recently. In [Pot97] it is shown that for $d = 4$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & -1 & 1 \end{pmatrix} \tag{3.2.4}$$

there does not exist a set of digits D such that $Q = K_D$ is a tile.

In [GM92] it is proved that if Q is a tile then the function χ_Q generates a multiresolution analysis of $L^2(\mathbb{R}^d)$ (the Haar-type MRA mentioned before).

We will assume in this work that Q is a tile.

Examples An example of a tile in $\mathbb{R}^2$ is the set known as the "twin dragon" obtained if we choose the dilation matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.2.5}$$

(which maps the lattice $\Gamma = \mathbb{Z}^2$ into itself via an expansion by $\sqrt{2}$ and a rotation by $\frac{\pi}{4}$), and the digit set $D = \{(0,0), (1,0)\}$. This tile has a fractal boundary. Its characteristic function is the solution to the refinement equation with $\Lambda = \{d_1, d_2\}$ and $c_{d_1} = c_{d_2} = 1$ (see Figure 3.1).

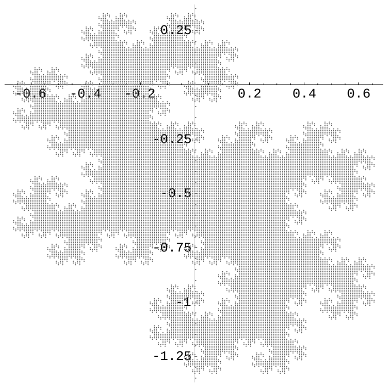

Figure 3.1: Twin Dragon Attractor

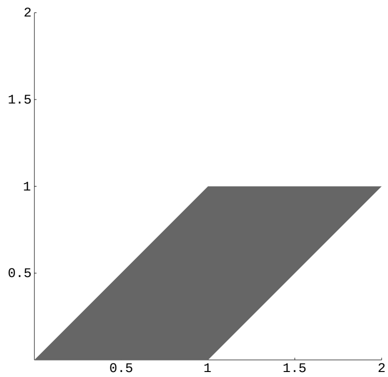

Figure 3.2: Parallelogram Attractor

For the dilation matrix

$$A' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.2.6}$$

and $D = \{(0,0),(1,0)\}$, the tile is the parallelogram with vertices

$$\{(0,0),(1,0),(2,1),(1,1)\}$$

(see Figure 3.2).

The sublattices $A(\mathbb{Z}^2)$ and $A'(\mathbb{Z}^2)$ coincide. These two matrices are known as the *quincunx dilation matrices* and the sublattice as the *quincunx lattice* of $\mathbb{Z}^2$.

3.2.3 Admissible sets

Let H be a fixed finite subset of Γ

Definition 16. We say that a set $\Omega \subset \Gamma$ is *H-admissible* if

$$A^{-1}(\Omega + H) \cap \Gamma \subset \Omega, \tag{3.2.7}$$

which is equivalent to say that $w_H(\Omega) \cap \Gamma \subset \Omega$.

Remark. If $H \subset H'$ and Ω is H'-admissible, then Ω is H-admissible.

We immediately have the following Proposition:

Proposition 12. *If Ω_H is defined as $\Omega_H = K_H \cap \Gamma$, then Ω_H is an H-admissible set.*

Proof. Since $\Omega_H \subset K_H$, we have

$$w_H(\Omega_H) \cap \Gamma \subset w_H(K_H) \cap \Gamma = \Omega_H,$$

which shows the desired property. □

Let $\ell(\Gamma)$ be the space of all sequences defined in Γ, and let L be the infinite matrix associated to the refinement equation (3.1.1)

$$L_{ij} = \begin{cases} c_{Ai-j} & \text{for } Ai - j \in \Lambda \\ 0 & \text{otherwise .} \end{cases} \tag{3.2.8}$$

In this work we will mainly be interested in Λ-admissible sets. The reason for that is that if $\Omega \subset \Gamma$ is Λ-admissible, then the space $\ell(\Omega) = \{Y \in \ell(\Gamma) : y_k = 0, k \notin \Omega\}$ is right invariant under L.

We will need to "extend" finite vectors to infinite ones with certain prescribed properties, and such that they coincide with the finite one if restricted to a finite subset of the lattice. Therefore, the following result which is due to [CHM00] (not exactly as we state it here), will be very useful.

Proposition 13. *For each finite $H \subset \Gamma$, there exists a strictly increasing sequence $\{\Omega_n\}_{n\geq 0}$ of H-admissible sets whose union is Γ, such that $\Omega_0 = \Omega_H$ and*

$$w_H(\Omega_{n+1}) \cap \Gamma \subset \Omega_n \tag{3.2.9}$$

for all $n \geq 0$.

Proof. Let $\|\cdot\|$ be any norm in $\mathbb{R}^d$ such that $\|A^{-1}\| < 1$ and fix $\varepsilon > 0$, such that $H \subset B(\varepsilon)$, where $B(\varepsilon) = \{x \in \mathbb{R}^d : \|x\| \leq \varepsilon\}$, the closed ball with radius ε centered at the origin. Now set

$$\delta_0 = \frac{\varepsilon}{\|A^{-1}\|^{-1} - 1}. \tag{3.2.10}$$

Choose $\delta > \delta_0$ in such a way that $\Omega_H \subset B(\delta)$ and set $F_0 = B(\delta)$. Since $\delta > \delta_0$, we have $\|A^{-1}\|(\delta + \varepsilon) < \delta$. Hence,

$$w_H(F_0) = A^{-1}(B(\delta) + H) \subset A^{-1}(B(\delta + \varepsilon)) \subset B(\|A^{-1}\|(\delta + \varepsilon)) \subset B(\delta) = F_0.$$

We define recursively $F_{j+1} = w_H(F_j)$ for $j \geq 0$. It is easy to see, by induction, that $F_{j+1} \subset F_j$ for every j. Since F_0 is compact, the Contraction Mapping Theorem tells us that $\bigcap F_j = K_H$. It follows that $F_j \cap \Gamma = \Omega_H$ for every j large enough, and consequently $\{F_j \cap \Gamma\}$ is a finite collection of sets. Let

$$\Omega_H = \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_N = F_0 \cap \Gamma$$

be the distinct elements of this collection and fix $0 \leq n < N$. Since there exists a $j \in \mathbb{N}$ such that

$$\Omega_n = F_j \cap \Gamma \subsetneq F_{j-1} \cap \Gamma = \Omega_{n+1},$$

we have

$$w_H(\Omega_{n+1}) \cap \Gamma \subset w_H(F_{j-1}) \cap \Gamma = F_j \cap \Gamma = \Omega_n. \tag{3.2.11}$$

So inclusion (3.2.9) holds for $n = 0, 1, \ldots, N-1$.

Now we set $\delta_N = \delta$ and define recursively, $\delta_{n+1} = \frac{\delta_n}{\|A^{-1}\|} - \varepsilon$ for $n \geq N$. The sequence of numbers $\delta_N < \delta_{N+1} < \cdots$ is increasing. Define $\Omega_n = B(\delta_n) \cap \Gamma$ for $n > N$. If $\Omega_{n+1} = \Omega_n$, we skip that one and continue until $\Omega_{n+k} \neq \Omega_n$. In this way we obtain a strictly increasing sequence of sets $\{\Omega_k\}_{k \geq N}$. Combining with the sets $\Omega_0, \ldots, \Omega_N$ constructed previously, we have a strictly increasing sequence $\{\Omega_n\}_{n \geq 0}$. The inclusion

$$w_H(\Omega_{n+1}) \cap \Gamma \subset \Omega_n \tag{3.2.12}$$

holds for every $n \in \mathbb{N}_0$, since for $n \geq N$, again there exist a $j \in \mathbb{N}$ such that

$$\Omega_n = B(\delta_j) \cap \Gamma \subsetneq B(\delta_{j+1}) \cap \Gamma = \Omega_{n+1},$$

and then

$$w_H(\Omega_{n+1}) = A^{-1}(\Omega_{n+1} + H) \subset B(\|A^{-1}\|(\delta_{j+1} + \varepsilon)) = B(\delta_j). \tag{3.2.13}$$

We already showed that $\Omega_0 = \Omega_H$ is H-admissible. Since $\Omega_n \subset \Omega_{n+1}$, it follows from (3.2.12) that Ω_{n+1} is H-admissible for every $n \in \mathbb{N}_0$, which completes the proof. $\square$

Corollary 4. *If $H \subsetneq H' \subset \Gamma$, then there exists $n_0 \geq 1$ and a strictly increasing sequence $\{\Omega_n\}_{n \geq 0}$ of H-admissible sets whose union is Γ, such that*

$$\Omega_0 = \Omega_H, \quad \Omega_{n_0} = \Omega_{H'}, \quad \textit{and} \quad w_H(\Omega_{n+1}) \cap \Gamma \subset \Omega_n \textit{ for all } n \geq 0.$$

Proof. First construct the sequence $\{\Omega'_n\}$ associated to H' using the previous proposition. Note that by Remark 3.2.3 the sets Ω'_n are also H-admissible.

Using the notation of the previous proof, let j_0 be such that $F_{j_0} \cap \Gamma = \Omega_{H'}$. Now consider the sequence $\{G_j\}_{j \geq j_0}$, where $G_{j_0} = F_{j_0}$ and $G_{j+1} = w_H(G_j)$, for $j \geq j_0$. Since

$$G_{j_0+1} = w_H(G_{j_0}) = A^{-1}(F_{j_0} + H) \subseteq A^{-1}(F_{j_0} + H') = w_{H'}(F_{j_0}) \subseteq F_{j_0} = G_{j_0},$$

then $G_{j+1} \subseteq G_j$ and therefore $\cap_{j \geq j_0} G_j = K_H$ and hence, $\{G_j \cap \Gamma\}$ is again a finite collection of sets of say $n_0 + 1$ elements. Consider now the distinct elements

$$\Omega_H = \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_{n_0} = F_{j_0} \cap \Gamma = \Omega_{H'},$$

and let

$$\Omega_{n_0+k} = \Omega'_k.$$

This new sequence satisfies all the desired properties. □

For a more complete treatment of admissible sets see [CHM04] and also Jia [Jia98].

3.3 Accuracy in higher dimensions

We will use the notation of [CHM98].

Let $x = (x_1, \dots, x_d)^t \in \mathbb{R}^d$. With the standard multi-index notation we write $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ with each α_i a nonnegative integer. Denote by

$$|\alpha| = \alpha_1 + \cdots + \alpha_d$$

the degree of α. The number of multi-indices α of degree s is

$$d_s = \binom{s+d-1}{d-1}.$$

We write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for $1 \leq i \leq d$, and we set

$$\binom{\alpha}{\beta} = \begin{cases} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} & \text{if } \beta \leq \alpha \\ 0 & \text{if } \beta_i > \alpha_i \text{ for some } i. \end{cases}$$

Definition 17. The *accuracy* of φ is the highest degree κ such that all multivariate polynomials q with degree $(q) < \kappa$ are in $\mathcal{S}(\varphi)$.

It can be shown that if φ is a compactly supported solution of the refinement equation (3.1.1) then $\text{Supp}(\varphi) \subset K_\Lambda$, where the set K_Λ is the particular case taking $H = \Lambda$ in (3.2.1). As a consequence, any such φ has necessarily finite accuracy (see for [CHM04]).

For each integer $s \geq 0$, we define the vector valued function $X_{[s]} : \mathbb{R}^d \longrightarrow \mathbb{C}^{d_s}$ by

$$X_{[s]}(x) = [x^\alpha]_{|\alpha|=s}, \quad x \in \mathbb{R}^d.$$

The ordering of the multi-indices α of degree s is not important as long as the same ordering is used throughout.

We will now look at the behavior of $X_{[s]}(x)$ under the multiplication by an arbitrary $d \times d$ matrix Z with scalar entries $z_{i,j}$. If $|\alpha| = s$, then $(Zx)^\alpha$ is not in general a monomial, except for the special case $A = cI_d$. Instead, it is a new polynomial of degree s, that is still homogeneous, but possibly involves all terms x^β with $|\beta| = s$.

Let $Z_{[s]} = [z^s_{\alpha,\beta}]_{|\alpha|=s,|\beta|=s}$ be the $d_s \times d_s$ matrix whose scalar entries $z^s_{\alpha,\beta}$ are defined by the equation

$$\sum_{|\beta|=s} z^s_{\alpha,\beta} x^\beta = (Zx)^\alpha = \prod_{i=1}^{d} (z_{i,1}x_1 + \cdots + z_{i,d}x_d)^{\alpha_i}.$$

The matrices $Z_{[s]}$ and their properties have been intensively studied in [CHM98], [CHM99].

Examples If I_d denotes the identity matrix in $\mathbb{R}^d$, we have that

$$(I_d)_{[s]} = I_{d_s}.$$

For $d = 1$ (that is, Z is a scalar) holds

$$Z_{[s]} = Z^s, \ \ X_{[s]}(x) = x^s.$$

For $d = 2$, let A' be the quincunx matrix defined before

$$A' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and consider the monomial $x^{(1,2)} = x_1 x_2^2$. We have

$$A'x = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix},$$

so

$$\begin{aligned} (A'x)^{(1,2)} &= \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}^{(1,2)} \\ &= (x_1 + x_2)(x_1 - x_2)^2 \\ &= x_1^3 - x_1^2 x_2 - x_1 x_2^2 + x_2^3 \\ &= x^{(3,0)} - x^{(2,1)} - x^{(1,2)} + x^{(0,3)}. \end{aligned}$$

As we can see $(A'x)^{(1,2)}$ is not a monomial anymore. The vector of all monomials of degree 3 is

$$X_{[3]} = \begin{pmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{pmatrix}.$$

Dilation by A' gives

$$
\begin{aligned}
X_{[3]}(A'x) &= \begin{pmatrix} (x_1+x_2)^3 \\ (x_1+x_2)^2(x_1-x_2) \\ (x_1+x_2)(x_1-x_2)^2 \\ (x_1-x_2)^3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{pmatrix} \\
&= A'_{[3]} X_{[3]}(x).
\end{aligned}
$$

This is analogous to the one-dimensional equation $(2x)^3 = 2^3 x^3$, except that now the factor $A'_{[3]}$ is not a scalar but a matrix.

In general, dilation of $X_{[s]}(x)$ by a matrix Z obeys the rule

$$X_{[s]}(Zx) = Z_{[s]} X_{[s]}(x), \tag{3.3.1}$$

since

$$
\begin{aligned}
X_{[s]}(Zx) &= [(Zx)^{\alpha}]_{|\alpha|=s} \\
&= [z^{s}_{\alpha,\beta}]_{|\alpha|=s,|\beta|=s} [x^{\alpha}]_{|\alpha|=s} \\
&= Z_{[s]} X_{[s]}(x).
\end{aligned}
$$

Hence, for $s = 1$ and any matrix Z we have

$$
\begin{aligned}
Z_{[1]} X_{[1]}(x) = X_{[1]}(Zx) &= \begin{pmatrix} (Zx)_1 \\ \vdots \\ (Zx)_d \end{pmatrix} \\
&= Z \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = Z X_{[1]},
\end{aligned}
$$

and thus $Z_{[1]} = Z$.

If Z and W are two matrices,

$$(ZW)_{[s]}X_{[s]}(x) = X_{[s]}(ZWx) = Z_{[s]}X_{[s]}(Wx) = Z_{[s]}W_{[s]}X_{[s]}(x).$$

Hence

$$(ZW)_{[s]} = Z_{[s]}W_{[s]}$$

and consequently, if Z is invertible,

$$(Z^{-1})_{[s]} = (Z_{[s]})^{-1}.$$

The following generalization of Proposition 2 states that if a compactly supported φ with independent translates has accuracy κ, then the coefficients used to reconstruct polynomials from translates of φ are also polynomials evaluated at lattice points.

Proposition 14. *([CHM98]) Assume that $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is compactly supported and that the translates of φ along Γ are linearly independent. If φ has accuracy κ, and q is any polynomial with deg$(q) < \kappa$, then there exists a unique polynomial $u_q : \mathbb{R}^d \longrightarrow \mathbb{C}$ with deg$(u_q) =$ deg(q) such that*

$$q(x) = \sum_{k\in\Gamma} u_q(k)\varphi(x+k).$$

The next result for the multivariate case is analogous to Proposition 5. It provides sufficient conditions for a refinable function to have accuracy κ, which are also necessary in the case that φ has independent translates.

Theorem 14. *([CHM98]) Assume that $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfies the refinement equation (3.1.1), and that φ is integrable and compactly supported. Consider the following statements.*

(I) *φ has accuracy κ*

(II) *There exists a collection of scalars* $\{v_\alpha : 0 \le |\alpha| < \kappa\}$ *such that*

(i) $v_0 \hat{\varphi}(0) \neq 0$, *and*

(ii) $Y_{[s]} = A_{[s]} Y_{[s]} L$ *for* $0 \le s < \kappa$,

where $Y_{[s]} = (y_{[s]}(k))_{k \in \Gamma}$ *is the row vector of evaluations at lattice points of the column vector of polynomials* $y_{[s]} : \mathbb{R}^d \longrightarrow \mathbb{C}^{d_s \times 1}$

$$y_{[s]}(x) = \sum_{0 \le \beta \le \alpha} (-1)^{|\alpha| - |\beta|} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} v_\beta x^{\alpha - \beta}.$$

Then we have the following.

(a) If the translates of φ *along* Γ *are linearly independent, then statement (I) implies statement (II).*

(b) Statement (II) implies statement (I).

Chapter 4

Homogeneous Functions in $\mathbb{R}^d$

4.1 Introduction

In chapter 2 we considered a function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ supported in $[0, N]$ that satisfies the 2-scale refinement equation (1.3.4). We showed that to each vector from a basis that gives the Jordan of the matrix T it is possible to associate a function (λ, r)-homogeneous in $\mathcal{S}(\varphi)$ that satisfies

$$(\mathcal{D}_2 - \lambda I)^r h = 0.$$

These functions are linearly independent and provide a local basis which contains all the monomials x^k within the accuracy. The generator φ can be completely obtained from this local basis.

In this chapter we extend this study to $\mathbb{R}^d$, with a general dilation matrix A and an arbitrary full rank lattice $\Gamma \subset \mathbb{R}^d$, i.e. we consider a compactly supported function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ which satisfies the refinement equation (3.1.1)

$$\varphi(x) = \sum_{k \in \Lambda} c_k \varphi(Ax - k), \quad c_k \in \mathbb{C}.$$

Analogously to the one-dimensional case, we will consider functions that satisfy the relation $\lambda h(Ax) = h(x)$ (see section 4.3). Here $h : \mathbb{R}^d \to \mathbb{C}$, A is a $d \times d$ invertible matrix and $\lambda \in \mathbb{C}$. More in general, we will consider functions satisfying that $(\mathcal{D}_A - \lambda I)^r h = 0$. To avoid any ambiguity, we will say that functions satisfying this equation are in the class $\mathcal{H}(A, \lambda, r)$, in place to use the word homogeneous, since we will also be dealing with polynomials that are homogeneous in the standard way, (i.e. a polynomial p of degree s is homogeneous, if $p(ax) = a^s p(x), x \in \mathbb{R}^d\, \forall a \in \mathbb{R}$). Note however, that with this definition, for $d = 2$ the monomial $h(x_1, x_2) = x_1 x_2$ will be in $\mathcal{H}(A, \lambda, 1)$ only if A is diagonal and $\lambda = \frac{1}{A_{11}A_{22}}$.

Let $Q \subset \mathbb{R}^d$ be a tile for Γ. The dimension of the space

$$Q(\varphi) = \{f/_Q : Q \longrightarrow \mathbb{C} : f/_Q(x) = f(x)\ \forall x \in Q, f \in \mathcal{S}(\varphi)\} \tag{4.1.1}$$

will be called the local dimension of $\mathcal{S}(\varphi)$ and a basis of $Q(\varphi)$ a local basis of $\mathcal{S}(\varphi)$.

In one dimension, the accuracy of φ was immediately related to spectral properties of the finite submatrix T of $L = [c_{Ai-j}]_{i,j\in\Gamma}$. In higher dimension the situation is much more complex (see [CHM98], [CHM99], [CHM00]). The difficulty here is to find the appropriate matrix T. Since we are in $\mathbb{R}^d$, the indexes vary along a d-dimensional lattice, so to write L as a matrix, one has to order the points. Which order is not important, as long as it is always the same. In the one dimensional case, it was straightforward to look at a submatrix of L that was intimately related to the support of φ. In the higher dimensional setting, it may be a difficult problem to determine the support exactly. This is one of the problems one has to overcome to solve the question raised here.

However, analogous results to the one-dimensional case could be obtained. We show that that local basis of $\mathcal{S}(\varphi)$ can be obtained using solely functions from $\mathcal{H}(A, \lambda, r)$, where λ is an eigenvalue for a finite submatrix of L. Hence each function in $\mathcal{S}(\varphi)$ can be written locally as a linear combination of the translates of functions in $\mathcal{H}(A, \lambda, r)$.

We further show that if φ has accuracy κ, the space of all the functions in the class $\mathcal{H}(A, \lambda, r)$ in $\mathcal{S}(\varphi)$ contains $\alpha_k = \sum_{s=0}^{\kappa-1} d_s$ linearly independent polynomials.

4.2 The spectrum of L

Let us now return to the refinement equation (3.1.1), $\varphi(x) = \sum_{k\in\Lambda} c_k\varphi(Ax-k)$. If we consider the infinite column vector

$$\Phi(x) = \{\varphi(x+k)\}_{k\in\Gamma}, \tag{4.2.1}$$

this equation becomes, analogously to the one-dimensional case

$$\Phi(x) = L\Phi(Ax), \tag{4.2.2}$$

since, for $i \in \Gamma$

$$\begin{aligned}[L\Phi(Ax)]_k &= \sum_{j\in\Gamma} c_{Ak-j}\varphi(Ax+j)\\ &= \sum_{l\in\Gamma} c_l\varphi(Ax+Ak-l)\\ &= \varphi(x+k) = [\Phi(x)]_k.\end{aligned}$$

As already mentioned, the set K_Λ, which is the particular case taking $H = \Lambda$ in (3.2.1), satisfies that if φ is a compactly supported solution of the refinement equation (3.1.1), then $\mathrm{Supp}(\varphi) \subset K_\Lambda$ (see[CHM04]). Also, by Proposition 12, the set $\Omega_\Lambda = K_\Lambda \cap \Gamma$ is Λ-admissible. However, it is not necessarily true that $\mathrm{Supp}\,\varphi \subset \cup_{\lambda\in\Omega_\Lambda} Q + \lambda$.

We will therefore consider the bigger set

$$\Omega' = K_{\Lambda'} \cap \Gamma,$$

where

$$\Lambda' = \Lambda - D \supset \Lambda.$$

In [CHM04] it was shown that the translations of Q using all elements of Ω' cover the support of the compactly solution to (3.1.1), more precisely,

$$K_\Lambda \subset Q + \Omega'.$$

Moreover, Ω' is Λ' admissible, and hence also Λ-admissible. As noted earlier, the Λ-admissibility of Ω' guarantees that the space $\ell(\Omega') = \{Y \in \ell(\Gamma) : y_k = 0, k \notin \Omega'\}$ is right invariant under L.

Let now $\{\Omega_n\}_{n\geq 0}$ be a sequence of subsets of Γ that satisfies:

- $\Omega_0 = \Omega_\Lambda$
- For $i \geq 0$, $\Omega_i \subsetneq \Omega_{i+1}$, and $\cup_i \Omega_i = \Gamma$
- For $i \geq 0$ Ω_i are Λ-admissible and $w_\Lambda(\Omega_{i+1}) \cap \Gamma \subseteq \Omega_i$.
- $\Omega_{n_0} = \Omega'$
- For $i \geq n_0$, Ω_i are Λ'-admissible and $w_{\Lambda'}(\Omega_{i+1}) \cap \Gamma \subseteq \Omega_i$.

These sets exist by Proposition 13 and its Corollary 4.

We denote by $\{T_n\}_{n\geq 0}$ the finite submatrices of L

$$T_n = [c_{Ai-j}]_{i,j\in\Omega_n}. \tag{4.2.3}$$

Since $\Omega_n \subset \Omega_{n+1}$, if the order in Γ is appropriately chosen, actually T_n is a submatrix of T_{n+1}, for each n.

Let $Y = \{y_k\}_{k\in\Gamma} \in \ell(\Gamma)$ be an infinite row vector, and $P_n : \ell(\Gamma) \longrightarrow \mathbb{C}^{1\times\Omega_n}, n \geq 0$ be the restriction mappings defined by

$$P_n Y = \{y_k\}_{k\in\Omega_n}. \tag{4.2.4}$$

We consider $L - \lambda I : \ell(\Gamma) \longrightarrow \ell(\Gamma)$ the left-multiplication operator who maps $Y \longrightarrow Y(L - \lambda I)$ (where I is the identity operator acting on $\ell(\Gamma)$). By abuse of notation, I will be any identity operator, no matter on which space it is acting on.

Note. In what follows we will use powers of the matrix $(L - \lambda I)$. Note that these powers are point-wise well defined, since the rows of the matrix L have a finite number of non-zero elements.

The next proposition shows the relation between the spectrum of L and T_n:

Proposition 15. *Consider $\lambda \in \mathbb{C}$, $r \in \mathbb{N}$ and $n \geq 0$.*

(1) *Let* $Y \in \ell(\Gamma)$. *We have*

$$Y \in \mathrm{Ker}(L - \lambda I)^r \quad \textit{implies} \quad P_n Y \in \mathrm{Ker}(T_n - \lambda I)^r. \tag{4.2.5}$$

Conversely,

(2) *If* $v \in \mathrm{Ker}(T_n - \lambda I)^r$ *and* $\lambda \neq 0$, *then we can extend* v *to an infinite row vector* Y_v *(i.e.* $Y_v \in \ell(\Gamma)$ *and* $P_n Y_v = v$*), so that* $Y_v \in \mathrm{Ker}(L - \lambda I)^r$.

(3) *If* $\lambda \neq 0, Y \neq 0$ *and* $Y \in \mathrm{Ker}(L - \lambda I)^r$, *then* $P_n Y \neq 0$. *In particular the extension in* (2) *of* v *to* Y_v *is unique.*

Proof.

(1) First note that $j \in \Omega_n$ and $Ai - j \in \Lambda$ implies $i \in \Omega_n$. For, in this case, $Ai \in \Omega_n + \Lambda$ and since Ω_n is a Λ-admissible set it follows that $i \in A^{-1}(\Omega_n + \Lambda) \cap \Gamma \subset \Omega_n$. Hence

$$j \in \Omega_n, \; i \notin \Omega_n \implies [L - \lambda I]_{ij} = 0. \tag{4.2.6}$$

Moreover, we will show by induction on r that,

$$\text{if } j \in \Omega_n \text{ and } i \notin \Omega_n, \text{then } [(L - \lambda I)^r]_{ij} = 0. \tag{4.2.7}$$

ii

(a) The case $r = 1$ is simply (4.2.6), since we assume that $c_k = 0$ if $k \notin \Lambda$.

(b) Suppose now that (4.2.7) holds for some fixed $r \geq 1$. Using (a), for $j \in \Omega_n$ we have

$$\begin{aligned} [(L - \lambda I)^{r+1}]_{ij} &= \sum_{k \in \Gamma} [(L - \lambda I)^r]_{ik} [L - \lambda I]_{kj} \\ &= \sum_{k \in \Omega_n} [(L - \lambda I)^r]_{ik} [L - \lambda I]_{kj}. \end{aligned}$$

Now, if $i \notin \Omega_n$, the inductive hypothesis yields that the last sum is zero.

Therefore, the statement is true for all $r \in \mathbb{N}$.

To prove the first part of the Proposition, let $Y \in \ell(\Gamma)$ and $Y \in \mathrm{Ker}(L - \lambda I)^r$.

Applying the preceding equality, we obtain for each $j \in \Omega_n$

$$\begin{aligned}
[(P_n Y)(T_n - \lambda I)^r]_j &= \sum_{i \in \Omega_n} y_i [(L - \lambda I)^r]_{ij} \\
&= \sum_{i \in \Gamma} y_i [(L - \lambda I)^r]_{ij} \\
&= [Y(L - \lambda I)^r]_j = 0.
\end{aligned}$$

This completes the proof of (1).

(2) Assume that $v \in \mathbb{C}^{1 \times \Omega_n}, \lambda \neq 0$ and $v \in \mathrm{Ker}(T_n - \lambda I)^r$. We want to construct a vector $Y_v \in \ell(\Gamma)$ such that $P_n Y_v = v$ and $Y_v \in \mathrm{Ker}(L - \lambda I)^r$.

We now prove by induction on r that,

$$\text{for } j \in \Omega_{k+1},\ i \notin \Omega_k, k \geq 0,\ [(L - \lambda I)^r]_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ (-\lambda)^r & \text{for } i = j. \end{cases} \tag{4.2.8}$$

(a) Case $r = 1$. If i were such that $Ai - j \in \Lambda$, then $i \in A^{-1}(\Omega_{k+1} + \Lambda) \cap \Gamma \subset \Omega_k$. Hence, if $i \notin \Omega_k$ then $Ai - j \notin \Lambda$ and so $L_{ij} = 0$, and therefore $[L - \lambda I]_{ij} = 0$ for $i \neq j$, and $[(L - \lambda I)]_{jj} = -\lambda$.

(b) Assume that (4.2.8) holds for $r \geq 1$. Then for $j \in \Omega_{k+1}$ and $i \notin \Omega_k, k \geq 0$, we have

$$\begin{aligned}
[(L - \lambda I)^{r+1}]_{ij} &= \sum_{\ell \in \Gamma} [(L - \lambda I)^r]_{i\ell} [L - \lambda I]_{\ell j} \\
&= \sum_{\ell \notin \Omega_k} [(L - \lambda I)^r]_{i\ell} [L - \lambda I]_{\ell j},
\end{aligned}$$

since by the inductive hypothesis $[(L-\lambda I)^r]_{i\ell}=0$ if $\ell\in\Omega_k\subset\Omega_{k+1}$ and $i\notin\Omega_k$. It follows using the case $r=1$ that

$$[(L-\lambda I)^{r+1}]_{ij}=\begin{cases}0 & \text{for } i\neq j\\ (-\lambda)^{r+1} & \text{for } i=j,\end{cases} \tag{4.2.9}$$

which proves (4.2.8), for every $r\in\mathbb{N}$.

Define now $y_j=v_j$ for $j\in\Omega_n$, and define recursively, for $j\notin\Omega_n$,

$$y_j=\frac{-1}{(-\lambda)^r}\sum_{i\in\Omega_k}y_i[(L-\lambda I)^r]_{ij},\quad j\in\Omega_{k+1}\backslash\Omega_k, k\geq n. \tag{4.2.10}$$

The vector $Y_v=\{y_j\}_{j\in\Gamma}$ is an extension of v. To see that $Y_v\in\operatorname{Ker}(L-\lambda I)^r$, since $(Y(L-\lambda I)^r)_j=\sum_{i\in\Gamma}y_i[(L-\lambda I)^r]_{ij}$, we have:

- If $j\in\Omega_n$, then by (4.2.6)

$$\begin{aligned}\sum_{i\in\Gamma}y_i[(L-\lambda I)^r]_{ij}&=\sum_{i\in\Omega_n}y_i[(T_n-\lambda I)^r]_{ij}\\&=\sum_{i\in\Omega_n}v_i[(T_n-\lambda I)^r]_{ij}=0.\end{aligned} \tag{4.2.11}$$

- If $j\notin\Omega_n$, then there exists $k\in\mathbb{N}_0, k\geq n$ such that $j\in\Omega_{k+1}\backslash\Omega_k$. Therefore,

$$\begin{aligned}\sum_{i\in\Gamma}y_i[(L-\lambda I)^r]_{ij}&=\sum_{i\in\Omega_k}y_i[(L-\lambda I)^r]_{ij}+\sum_{i\notin\Omega_k}y_i[(L-\lambda I)^r]_{ij}\\&=\sum_{i\in\Omega_k}y_i[(L-\lambda I)^r]_{ij}+y_j(-\lambda)^r)\\&=0.\ (\text{by (4.2.10)})\end{aligned}$$

(3) For the last part of the Proposition, assume that $\lambda\neq 0, Y\neq 0$, and $Y\in\operatorname{Ker}(L-\lambda I)^r$. To show that $P_nY\neq 0$, take $k_0\in\Gamma$ such that $y_{k_0}\neq 0$. If $k_0\in\Omega_n$, there is

nothing to prove. Otherwise, let

$$t_0 = \min\{k \in \mathbb{N} : k_0 \in \Omega_k\}. \tag{4.2.12}$$

Since $Y(L - \lambda I)^r = 0$,

$$\begin{aligned}\sum_{i\in\Gamma} y_i[(L-\lambda I)^r]_{ik_0} &= \sum_{i\in\Omega_{t_0-1}} y_i[(L-\lambda I)^r]_{ik_0} + \sum_{i\notin\Omega_{t_0-1}} y_i[(L-\lambda I)^r]_{ik_0} \\ &= \sum_{i\in\Omega_{t_0-1}} y_i[(L-\lambda I)^r]_{ik_0} + y_{k_0}(-\lambda)^r = 0.\end{aligned}$$

So, there exist $k_1 \in \Omega_k, 0 < k < t_0$, such that $y_{k_1} \neq 0$. If $k_1 \in \Omega_n$, we can stop here. If not, we repeat the procedure until $k_j \in \Omega_n$.

$\square$

Remark.

- Since the previous Proposition is true for **any** set of the sequence Ω_n, in fact the smallest matrix T_0 already has all the spectral information of L.
- The extension of the vectors of $\mathrm{Ker}(T_0 - \lambda I)^r$ to vectors of $\mathrm{Ker}(L - \lambda I)^r$ will produce *intermediate* vectors of $\mathrm{Ker}(T_n - \lambda I)^r$, by the construction of the sets Ω_n produced in Corollary 4.

For the special case $\lambda = 0$, under some mild assumptions, we have an additional property.

Lemma 1. *If $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent, then the operator $L : \ell(\Gamma) \longrightarrow \ell(\Gamma), Y \longmapsto YL$ is one to one.*

Proof. Let $YL = 0$. Then

$$Y\Phi(x) = YL\Phi(Ax) = 0. \tag{4.2.13}$$

Since $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent, $Y\Phi = 0$ implies $Y = 0$, so $\mathrm{Ker}(L) = \{0\}$.

□

4.3 The class $\mathcal{H}(A, \lambda, r)$

Assume $Y \in \mathrm{Ker}(L - \lambda I)^r$. If we define $h(x) = Y\Phi(x)$, we have

$$\begin{aligned} 0 = Y(L-\lambda I)^r\Phi(x) &= Y\left(\sum_{k=0}^{r}\begin{pmatrix} r \\ k \end{pmatrix}(-\lambda)^{r-k}L^k\right)\Phi(x) \\ &= Y\left(\sum_{k=0}^{r}\begin{pmatrix} r \\ k \end{pmatrix}(-\lambda)^{r-k}\Phi(A^{-k}x)\right), \\ &= \sum_{k=0}^{r}\begin{pmatrix} r \\ k \end{pmatrix}(-\lambda)^{k}h(A^{k-r}x). \end{aligned}$$

This leads to the following definition:

Definition 18. A function h is in the class $\mathcal{H}(A, \lambda, r)$, if it satisfies

$$\sum_{k=0}^{r}\begin{pmatrix} r \\ k \end{pmatrix}(-\lambda)^{k}h(A^{k-r}x) = 0 \quad \text{for every } x \in \mathbb{R}^d. \tag{4.3.1}$$

If we define the operator $\mathcal{D}_A$ by $\mathcal{D}_A(f)(x) = f(A^{-1}x)$, then h is in $\mathcal{H}(A, \lambda, r)$ if and only if

$$(\mathcal{D}_A - \lambda I)^r h = 0.$$

A function in $\mathcal{H}(A, \lambda, r)$ will also be said to be of class $\mathcal{H}(A, \lambda, r)$.

Note that if $h \in \mathcal{H}(A, \lambda, r)$, then $h \in \mathcal{H}(A, \lambda, s)$ for every $s \geq r$.

Proposition 16. *Let $V \subset \mathbb{R}^d$ be a bounded set such that $0 \in V^0$ and $V \subset AV$. Set $C = AV \setminus V$ and $a \in \mathbb{Z}$. Let h be a function of class $\mathcal{H}(A, \lambda, r)$. Then the values of h in $\mathbb{R}^d \setminus \{0\}$ can be determined from its values in any set of the type:*

$$\tilde{C} = \bigcup_{k=a}^{a+r} A^k C.$$

Furthermore, if $\lambda \neq 1$ then $h(0) = 0$.

Proof. Since $h \in \mathcal{H}(A, \lambda, r)$ we get that

$$h(x) = -\sum_{k=1}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-\lambda)^k h(A^k x) \quad \text{and} \tag{4.3.2}$$

$$h(x) = -\sum_{k=1}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-\lambda)^{-k} h(A^{-k} x). \tag{4.3.3}$$

On the other side, it has been proved in [ACM04] that the set C satisfies:

a) $\bigcup_{j \in \mathbb{Z}} A^j C = \mathbb{R}^d \setminus \{0\}$

b) The sets $\{A^j C\}_{j \in \mathbb{Z}}$ are pairwise disjoint.

So, from (4.3.2) we deduce that the values of h in $A^{a-1}C$ can be obtained from the values in $\tilde{C}$, and analogously, from (4.3.3) the values of h in $A^{a+r+1}C$ can be obtained from the values in $\tilde{C}$.

Then we proceed inductively to obtain all the values in $\mathbb{R}^d \setminus \{0\}$. Finally, it is immediate from the definition, that $h(0) = 0$ when $\lambda \neq 1$. $\square$

Proposition 17. *Suppose $\{\varphi(\cdot - k)\}_{k \in \Gamma}$ are linearly independent. Let $f_1, \ldots, f_l \in \mathcal{S}(\varphi)$, $f_i = Y^i \Phi$, where $Y^i \in \ell(\Gamma)$. Then $f_1, \ldots, f_l$ are linearly independent functions if and only if $Y^1, \ldots, Y^l$ are linearly independent in $\ell(\Gamma)$.*

Proof. Since

$$\sum_{i=1}^{l} \alpha_i f_i = \sum_{i=1}^{l} \alpha_i (Y^i \Phi) = \left(\sum_{i=1}^{l} \alpha_i Y^i \right) \Phi, \tag{4.3.4}$$

and the translates of φ along the lattice Γ are linearly independent, we conclude that $\sum_{i=1}^{l} \alpha_i f_i \equiv 0$ if and only if $(\sum_{i=1}^{l} \alpha_i Y^i) = 0$, which leads to the desired result. □

Remark. Let $E : \mathcal{S}(\varphi) \longrightarrow \ell(\Gamma)$ be the function that associates to each element of $\mathcal{S}(\varphi)$, its coordinates in $\{\varphi(x+k)\}$. Proposition 17 shows that E is an isomorphism.

Proposition 18. *Assume that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent.*

(1) *If $h \in \mathcal{S}(\varphi), h = Y\Phi$ and $h \in \mathcal{H}(A, \lambda, r)$, then $Y \in \operatorname{Ker}(L - \lambda I)^r$ and $P_n Y \in \operatorname{Ker}(T_n - \lambda I)^r$.*

Conversely

(2) *Assume that $\lambda \neq 0, v \in \operatorname{Ker}(T_n - \lambda I)^r$ and that Y_v is the unique extension of v such that $Y_v \in \operatorname{Ker}(L - \lambda I)^r$ (see Proposition 15). Then the function $h = Y_v\Phi$ belongs to $\mathcal{H}(A, \lambda, r)$.*

Proof. If $h = Y\Phi$ is of class $\mathcal{H}(A, \lambda, r)$, then we have

$$\begin{aligned} 0 &= \sum_{k=0}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-\lambda)^k h(A^{k-r}x) \\ &= \sum_{k=0}^{r} \begin{pmatrix} r \\ k \end{pmatrix} (-\lambda)^k Y L^{r-k} \Phi(x) \\ &= Y \left(L - \lambda I\right)^r \Phi(x). \end{aligned}$$

Since the Γ translates of φ are linearly independent, it follows that $Y(L - \lambda I)^r = 0$, and consequently, by Proposition 15, $P_n Y \in \operatorname{Ker}(T_n - \lambda I)^r$.

To prove the second part, note that if $v = 0$, the statement is trivially true. If $\lambda \neq 0$ and $v \in \mathrm{Ker}(T_n - \lambda I)^r$ and $v \neq 0$, then, by Proposition 15 we can extend v to a vector $Y_v \in \mathrm{Ker}(L - \lambda I)^r$, and so the function $h = Y_v \Phi$ is of class $\mathcal{H}(A, \lambda, r)$. □

4.3.1 Jordan decomposition of T_n

Let $m_n = \#\Omega_n$. Consider the set Δ_n of eigenvalues of T_n and the associated Jordan basis $\mathcal{B}_n = \{v_1, \ldots, v_{m_n}\}$ of $\mathbb{C}^{m_n}$. For each $v_i \in \mathcal{B}_n$ we have that $v_i \in \mathrm{Ker}(T_n - \lambda I)^k$ and $v_i \notin \mathrm{Ker}(T_n - \lambda I)^{k-1}$ for some $\lambda \in \Delta_n$, and for some $k \geq 1$. So to each v_i there corresponds a unique pair (λ, k). Note however that to two different v_is in the basis there could correspond the same pair. For each vector of $\mathcal{B}_n$, set $v_i = v_i(\lambda, k)$. If $\lambda \neq 0$, by Proposition 18 we can associate to each $v_i(\lambda, k)$ a function $h_{v_i(\lambda,k)}$ in $\mathcal{H}(A, \lambda, k) \cap \mathcal{S}(\varphi)$. Since the vectors $v_i(\lambda, k)$ are linearly independent, its extensions $\{Y_{v_i}\}$ are linearly independent in $\ell(\Gamma)$, so the functions $\{h_{v_i(\lambda,k)}\}_{v_i \in \mathcal{B}_n, \lambda \neq 0}$ are linearly independent.

If $h_1, \ldots, h_l$ are of class $\mathcal{H}(A, \lambda, k_i)$, for some k_i, $i = 1, \ldots, l$, then a linear combination of them is of class $\mathcal{H}(A, \lambda, k)$, with $k = \max_{1 \leq i \leq l}(k_i)$, for if $h = \sum_{i=0}^{l} \alpha_i h_i(\lambda, k_i)$, then

$$(\mathcal{D}_A - \lambda I)^k h = (\mathcal{D}_A - \lambda I)^k \sum_{i=0}^{l} \alpha_i h_i = \sum_{i=0}^{l} \alpha_i (\mathcal{D}_A - \lambda I)^k h_i = 0,$$

and consequently, $h \in \mathcal{H}(A, \lambda, k)$.

Let

$$\chi_{T_n}(x) = \prod_{\lambda \in \Delta_n} (x - \lambda)^{r_\lambda} \tag{4.3.5}$$

be the characteristic polynomial of T_n, and set

$$\mathcal{H}_\lambda(\varphi) = \{h \in \mathcal{S}(\varphi) : h \in \mathcal{H}(A, \lambda, k), \text{ for some } k \geq 1\ \}, \quad \lambda \in \Delta_n.$$

Then, if $\lambda \neq 0$, using Proposition 18, $\dim(\mathcal{H}_\lambda(\varphi)) = r_\lambda$ and a basis for $\mathcal{H}_\lambda(\varphi)$ are the functions of class $\mathcal{H}(A, \lambda, k)$ corresponding to the vectors $v_i \in \mathcal{B}_n$, such that

$v_i = v_i(\lambda, k)$, for some $k \geq 1$. So, if we denote

$$\mathcal{H} = \bigoplus_{\lambda \in \Delta_n, \lambda \neq 0} \mathcal{H}_\lambda(\varphi) \subset \mathcal{S}(\varphi),$$

then, $\dim(\mathcal{H}) = m_n - r_0$, where r_0 is the dimension of the subspace generated by the vectors of the Jordan basis associated to $\lambda = 0$. (Note that $m_n = \sum_{\lambda \in \Delta_n, \lambda \neq 0} r_\lambda$.)

In order to be able to include the case $\lambda = 0$ in our analysis, we need to consider the case in which $\operatorname{Supp} \varphi \subset \cup_{\omega \in \Omega_n} Q + \omega$. This will guarantee, that (except for a possible set of measure zero), $\varphi(x + k) = 0$ if $k \notin \Omega_n$.

4.3.2 The case in which Ω_n contains $\operatorname{Supp}(\varphi)$

If we recall the choice of the sequence $\{\Omega_n\}$ at the beginning of section 4.2, it is clear, that for $n \geq n_0$, we have that $\operatorname{Supp} \varphi \subseteq \cup_{\omega \in \Omega_n} Q + \omega$, and hence, the local dimension of $\mathcal{S}(\varphi)$ is $\dim \operatorname{span}\{\varphi(x + k)\}_{k \in \Omega_n}$.

In that case for $x \in Q^\circ$, $\lambda \neq 0$, $h_{v_i(\lambda,k)}(x) = \sum_{l \in \Omega_n} [v_i(\lambda, k)]_l \varphi(x+l)$ since $\varphi(x+j) = 0$ if $j \notin \Omega_n$.

Moreover for $\lambda = 0$ we have the following Proposition:

Proposition 19. *Let $n \geq n_0$, and let r_0 be the power of x in χ_{T_n} (c.f. (4.3.5)). Consider $v_i = v_i(0, r)$, with $r \leq r_0$. Define $h(x) = \sum_{k \in \Omega_n} [v_i]_k \varphi(x + k)$. Then $h \equiv 0$ a.e. on Q.*

We postpone the proof to remark that with this Proposition, if $n \geq n_0$, and $\mathcal{B}_n$ is (as before) the matrix whose rows are the vectors of the Jordan basis for T_n, then

$$\begin{bmatrix} h_{v_1(\lambda,k)}(x) \\ \vdots \\ h_{v_{m_n}(\lambda,k)}(x) \end{bmatrix} = \mathcal{B}_n P_n \Phi(x) \quad \text{a.e. } x \in Q. \tag{4.3.6}$$

Hence, since the matrix $\mathcal{B}_n$ is invertible the local dimension of $\mathcal{S}(\varphi)$ coincides with $\dim \operatorname{span}\{h_{v_i(\lambda,k)}(x), x \in Q, v_i \in \mathcal{B}_n\}$, which is equal to the dimension of $\mathcal{H}$.

So the local dimension of $\mathcal{S}(\varphi)$ can be found by finding the Jordan form of any of the finite matrices T_n as long as $n \geq n_0$.

Moreover, any function of the shift-invariant space $\mathcal{S}(\varphi)$ can be written as a linear combination of the lattice translates of the homogeneous functions. Namely, let $f \in \mathcal{S}(\varphi)$, then

$$f(x) = \sum_{\gamma\in\Gamma} \alpha_\gamma \varphi(x+\gamma) \quad x \in \mathbb{R}^d, \alpha_\gamma \in \mathbb{C}. \tag{4.3.7}$$

If we call $\mathbf{g} : \mathbb{R}^n \longrightarrow \mathbb{C}^{m_n}$ and $\mathbf{h} : \mathbb{R}^n \longrightarrow \mathbb{C}^{m_n}$ the functions

$$\mathbf{g}(x) = P_n \Phi(x) \chi_Q(x) \quad \text{and} \quad \mathbf{h}(x) = \begin{bmatrix} h_{v_1(\lambda,k)}(x) \\ \vdots \\ h_{v_{m_n}(\lambda,k)}(x) \end{bmatrix} \chi_Q(x), \tag{4.3.8}$$

and for $\gamma \in \Gamma$ we denote by $\overline{\alpha}_\gamma = (\alpha_{i_1+\gamma}, \ldots, \alpha_{i_{m_n}+\gamma})$ the vector of length m_n whose indices are in $\Omega_n + \gamma$ (here $\Omega_n = \{i_1, \ldots, i_{m_n}\}$), then (4.3.7) becomes

$$f(x) = \sum_{\gamma\in\Gamma} \overline{\alpha}_\gamma \mathbf{g}(x+\gamma), \tag{4.3.9}$$

and using (4.3.6) we obtain

$$f(x) = \sum_{\gamma\in\Gamma} \beta_\gamma \mathbf{h}(x+\gamma) \quad \text{where} \quad \beta_\gamma = \overline{\alpha}_\gamma \mathcal{B}_n^{-1}. \tag{4.3.10}$$

We will now prove Proposition 19. For this, let r_0 be the power of x in χ_{T_n} (c.f. (4.3.5)). Choose $m \geq n$ large enough such that

$$\Omega_m \supset \Omega_{n_0} - \left(D + AD + \cdots + A^{r_0-1}D\right). \tag{4.3.11}$$

Define the matrices $[(T_k)_d]_{ij} = c_{Ai-j+d}, i, j \in \Omega_k$, for any $k \in \mathbb{N}$, and $d \in D$. It is shown in [CHM04], that if $x \in Q$, for any $r \geq 1$ there exists $y_r \in Q$, $\gamma_r \in \Gamma$, such that

$$x = A^{-r}(y_r + \gamma_r) \quad \text{with} \quad \gamma_r = d_r + Ad_{r-1} + \cdots + A^{r-1}d_1 \tag{4.3.12}$$

where $d_i \in D, 1 \leq i \leq r$. Therefore, if $k \geq n_0$ and P_k is as in (4.2.4)

$$P_k\Phi(x) = (T_k)_{d_1} \ldots (T_k)_{d_r} P_k \Phi(A^r x - \gamma_r) \quad x \in Q. \tag{4.3.13}$$

For convenience, we will call $\Omega = \Omega_m$.

Lemma 2. *With the previous notation, if $r \leq r_0$, then for $k \in \Omega$ and $j \in \Omega_{n_0}$, we have*

$$[T_m^r]_{k(j-\gamma_r)} = [(T_m)_{d_1} \ldots (T_m)_{d_r}]_{kj} \, ,$$

where $\gamma_r \in D + AD + \cdots + A^{r-1}D$.

Remark. Note that the preceding equation does not state that both matrices are equal.

Proof. We will prove the Lemma by induction on r. Let γ_r be as in (4.3.12).

- The case $r = 1$ is trivial by the definition of $(T_m)_{d_1}$.
- $r - 1 \Longrightarrow r$ Observe first that by the choice of Ω,

  $$[T_m]_{u(j-\gamma_r)} = [(T_m)_{d_r}]_{u(j-A\gamma_{r-1})} \, , \quad u \in \Omega, j \in \Omega_{n_0}.$$

  Now

  $$\begin{aligned} [T_m^r]_{k(j-\gamma_r)} &= \sum_{u\in\Omega} \left[T_m^{r-1}\right]_{ku} [T_m]_{u(j-\gamma_r)} \\ &= \sum_{u\in\Omega} \left[T_m^{r-1}\right]_{ku} [(T_m)_{d_r}]_{u(j-A\gamma_{r-1})} \\ &= \sum_{u\in\Omega_{n_0}-\gamma_{r-1}} \left[T_m^{r-1}\right]_{ku} [(T_m)_{d_r}]_{u(j-A\gamma_{r-1})} \, , \end{aligned} \tag{4.3.14}$$

  where the last equality follows from the Λ'-admissibility of Ω_{n_0}. But

  $$[(T_m)_{d_r}]_{u(j-A\gamma_{r-1})} = [(T_m)_{d_r}]_{(u+\gamma_{r-1})j} \, , u \in \Omega_{n_0} - \gamma_{r-1}, j \in \Omega_{n_0},$$

and therefore, using induction and the Λ'-admissibility of Ω_{n_0}, (4.3.14) becomes

$$
\begin{aligned}
[T_m^r]_{k(j-\gamma_r)} &= \sum_{u\in\Omega_{n_0}-\gamma_{r-1}} \left[T_m^{r-1}\right]_{ku} [(T_m)_{d_r}]_{(u+\gamma_{r-1})j} \\
&= \sum_{\ell\in\Omega_{n_0}} \left[T_m^{r-1}\right]_{k(\ell-\gamma_{r-1})} [(T_m)_{d_r}]_{\ell j} \\
&= \sum_{\ell\in\Omega_{n_0}} \left[(T_m)_{d_1}\dots(T_m)_{d_{r-1}}\right]_{k\ell} [(T_m)_{d_r}]_{\ell j} \\
&= \sum_{\ell\in\Omega} \left[(T_m)_{d_1}\dots(T_m)_{d_{r-1}}\right]_{k\ell} [(T_m)_{d_r}]_{\ell j}
\end{aligned}
$$

which completes the inductive step.

$\square$

We can now prove Proposition 19.

Proof. Let $\Omega = \Omega_m$ be as before, and let $x \in Q \setminus (\partial Q \cup \bigcup_{i=1}^r A^{-i}\partial Q + A^{-i}D + \cdots + A^{-1}D)$. Note that with this choice of x, $A^r x - \gamma_r \in Q^\circ$, and therefore, if $u \notin \Omega_{n_0}$, then $\varphi(A^r x - \gamma_r + u) = 0$.

Using this, together with the previous lemma, and equation (4.3.13) for $k = m$, we have

$$
\begin{aligned}
[T_m^r P_m \Phi(A^r x)]_k &= \sum_{j\in\Omega} [T_m^r]_{kj}\, \varphi(A^r x + j) \\
&= \sum_{j\in\Omega} [T_m^r]_{k(j-\gamma_r)}\, \varphi(A^r x - \gamma_r + j) \\
&= \sum_{j\in\Omega} [(T_m)_{d_1}\dots(T_m)_{d_r}]_{kj}\, \varphi(A^r x - \gamma_r + j) \\
&= \varphi(x+k).
\end{aligned}
$$

Therefore, for $x \in Q \setminus (\partial Q \cup \bigcup_{i=1}^{r} A^{-i}\partial Q + A^{-i}D + \cdots + A^{-1}D)$

$$\begin{aligned} h(x) &= \sum_{k\in\Omega_n} [v_i]_k \varphi(x+k) \\ &= \sum_{k\in\Omega_n} [v_i]_k \sum_{j\in\Omega} [T_m^r]_{kj}\, \varphi(A^r x + j) \\ &= \sum_{j\in\Omega} \left(\sum_{k\in\Omega_n} [v_i]_k\, [T_m^r]_{kj} \right) \varphi(A^r x + j) = 0 \end{aligned}$$

□

4.4 Accuracy and homogeneous polynomials

In this section we will relate the previously obtained results, to the accuracy of a refinable function. If A is the dilation matrix corresponding to the refinement equation (3.1.1), by (3.3.1)

$$X_{[s]}(A^{-1}x) = A^{-1}{}_{[s]} X_{[s]}(x).$$

If J_s is the Jordan form of $A^{-1}{}_{[s]}$, then there exists an invertible $d_s \times d_s$ matrix Q_s such that $Q_s A^{-1}{}_{[s]} Q_s^{-1} = J_s$. So we have that

$$Q_s X_{[s]}(A^{-1}x) = (Q_s A^{-1}{}_{[s]} Q_s^{-1}) Q_s X_{[s]}(x).$$

Denote by $\widetilde{Q}_s(x) = Q_s X_{[s]}(x)$. Observe that $\widetilde{Q}_s(x) = \left(\widetilde{Q}_s^1(x), \ldots, \widetilde{Q}_s^{d_s}(x)\right)^t$ is a column vector of polynomials of degree s that are homogeneous. By the previous equation, we have that

$$\widetilde{Q}_s(A^{-1}x) = J_s \widetilde{Q}_s(x).$$

Let β be an eigenvalue of ${A^{-1}}_{[s]}$ and B the Jordan block of order ℓ associated to β, i.e.,

$$B = \begin{pmatrix} \beta & 0 & \dots & 0 & 0 \\ 1 & \beta & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \beta & 0 \\ 0 & 0 & \dots & 1 & \beta \end{pmatrix} \in \mathbb{C}^{\ell\times\ell}.$$

We write $\widetilde{Q}_B(x)$ for the vector that is the restriction of $\widetilde{Q}_s(x)$ to the coordinates that correspond to the block B, i.e, if $j, j+1, \dots, j+\ell-1$ are the columns of B in J_s then $\widetilde{Q}_B(x) = \left(\widetilde{Q}_s^j(x), \dots, \widetilde{Q}_s^{j+\ell-1}(x)\right)^t$. Since

$$\widetilde{Q}_s(A^{-1}x) = J_s\widetilde{Q}_s(x),$$

we have

$$\widetilde{Q}_B(A^{-1}x) = B\widetilde{Q}_B(x). \tag{4.4.1}$$

This relation will enable us to show how, under the hypothesis of accuracy, we can relate the Jordan form of ${A^{-1}}_{[s]}$ to the one of T_n. This relation also gives a necessary condition for φ to have accuracy κ.

Proposition 20. *Assume that φ has accuracy κ and that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent. Let $s < \kappa$. If β is an eigenvalue of ${A^{-1}}_{[s]}$ and B is a Jordan block of ${A^{-1}}_{[s]}$ associated to β of order ℓ, then T_n has a Jordan block associated to β of order ℓ' with $\ell' \geq \ell$.*

Proof. Consider $\widetilde{Q}_B(x) = (\widetilde{Q}_B^1(x), \dots, \widetilde{Q}_B^\ell(x))$. It follows from (4.4.1) that

$$\begin{array}{rcl} \widetilde{Q}_B^1(A^{-1}x) & = & \beta\widetilde{Q}_B^1(x) \\ \widetilde{Q}_B^2(A^{-1}x) & = & \widetilde{Q}_B^1(x) + \beta\widetilde{Q}_B^2(x) \\ \vdots & & \vdots \\ \widetilde{Q}_B^\ell(A^{-1}x) & = & \widetilde{Q}_B^{\ell-1}(x) + \beta\widetilde{Q}_B^\ell(x). \end{array} \tag{4.4.2}$$

Since $\widetilde{Q}^i_B(x) \in \mathcal{S}(\varphi)$ for $1 \leq i \leq \ell$, we can write

$$\widetilde{Q}^i_B(x) = Y^i \Phi(x),$$

for some infinite column vector Y^i. From (4.4.2) we have for $2 \leq i \leq \ell$,

$$Y^i \Phi(A^{-1}x) = Y^{i-1}\Phi(x) + \beta Y^i \Phi(x),$$

which implies

$$Y^i L\Phi(x) - \beta Y^i \Phi(x) = Y^{i-1}\Phi(x).$$

So, the linear independence of $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ yields

$$Y^i(L - \beta I) = Y^{i-1}. \tag{4.4.3}$$

Since $\widetilde{Q}^\ell_B(x) \in \mathcal{H}(A, \beta, \ell)$, by Proposition 18 we have that $v = P_n Y^\ell \in \mathrm{Ker}(T_n - \beta I)^\ell$. Consider the vectors $v_1 = P_n Y^\ell, v_2 = (P_n Y^\ell)(T_n - \beta I), \ldots, v_\ell = (P_n Y^\ell)(T_n - \beta I)^{\ell-1}$. Let us show that $v_1, \ldots, v_\ell$ are linearly independent: Assume that

$$\sum_{i=1}^{\ell} \alpha_i v_i = 0. \tag{4.4.4}$$

Since

$$\begin{aligned} \left(\sum_{i=1}^{\ell} \alpha_i v_i\right)(T_n - \beta I)^{\ell-1} &= \left(\sum_{i=1}^{l} \alpha_i v(T_n - \beta I)^{i-1}\right)(T_n - \beta I)^{\ell-1} \\ &= (\sum_{i=1}^{\ell} \alpha_i v(T_n - \beta I)^{\ell+i-2} \\ &= \alpha_1 v(T_n - \beta I)^{\ell-1}, \end{aligned}$$

it follows from (4.4.4) that

$$\alpha_1 v(T_n - \beta I)^{\ell-1} = 0.$$

Since for every $Y \in \ell(\Gamma), r \in \mathbb{N}$ we have that

$$(P_n Y)(T_n - \beta I)^r = P_n(Y(L - \beta I)^r),$$

part 3 of Proposition 15 tells us that $v(T_n - \beta I)^{\ell-1} \neq 0$. Hence $\alpha_1 = 0$. If we multiply each side of (4.4.4) by $(T_n - \beta I)^{\ell-2}$ we see that $\alpha_2 = 0$. Analogously $\alpha_3 = \ldots = \alpha_\ell = 0$ and therefore $v_1, \ldots, v_\ell$ are linearly independent. This implies that we have a Jordan block of T_n associated to β of order at least ℓ. We can repeat this procedure for every Jordan block $B_1, \ldots, B_k$ of $A^{-1}{}_{[s]}$ associated to β of respective orders $l_1 \geq l_2 \geq \ldots \geq l_k$. Let, for $1 \leq j \leq k$

$$\widetilde{Q}^{l_j}_{B_j}(x) = Y_j^{l_j}\Phi(x).$$

All we have to prove now is that

$$\begin{array}{cccc} P_n Y_1^{l_1}, & (P_n Y_1^{l_1})(T_n - \beta I), & \ldots, & (P_n Y_1^{l_1})(T_n - \beta I)^{l_1 - 1}, \\ & & \vdots & \\ P_n Y_k^{l_k}, & (P_n Y_k^{l_k})(T_n - \beta I), & \ldots, & (P_n Y_k^{l_k})(T_n - \beta I)^{l_k - 1} \end{array}$$

are linearly independent. Let

$$\begin{array}{cccc} \alpha_1^1 P_n Y_1^{l_1} + & \alpha_1^2 (P_n Y_1^{l_1})(T_n - \beta I) + & \ldots + & \alpha_1^{l_1}(P_n Y_1^{l_1})(T_n - \beta I)^{l_1 - 1} + \\ & & \vdots & \\ \alpha_k^1 P_n Y_k^{l_k} + & \alpha_k^2 (P_n Y_k^{l_k})(T_n - \beta I) + & \ldots + & \alpha_k^{l_k}(P_n Y_k^{l_k})(T_n - \beta I)^{l_k - 1} = 0 \end{array} \tag{4.4.5}$$

Let $B_1, \ldots, B_t$ the Jordan blocks of order l_1. If we multiply each side of the previous equation by $(T_n - \beta I)^{l_1 - 1}$, we obtain

$$\sum_{i=1}^{t} \alpha_i^1 (P_n Y_i^{l_1})(T_n - \beta I)^{l_1 - 1} = 0, \text{ i.e.}$$

$$P_n\left(\sum_{i=1}^{t}\alpha_i^1 Y_i^{l_1}(L-\beta I)^{l_1-1}\right)=0.$$

Since $\sum_{i=1}^{t}\alpha_i^1 Y_i^{l_1}(L-\beta I)^{l_1-1}\in \mathrm{Ker}(L-\beta I)$, part 3 of Proposition 15 implies that

$$\sum_{i=1}^{t}\alpha_i^1 Y_i^{l_1}(L-\beta I)^{l_1-1}=0.$$

So, since by (4.4.3) and Proposition 17, $Y_1^{l_1}(L-\beta I)^{l_1-1},\ldots,Y_t^{l_1}(L-\beta I)^{l_1-1}$ are linearly independent, it follows that $\alpha_1^1=\ldots=\alpha_t^1=0$. Repeating a similar argument for every $l_j, 2\le j\le k$ we can see that every scalar of (4.4.5) is equal to zero. This completes the proof. $\square$

Let us now recall (4.4.1), and notice that

$$\widetilde{Q}_B(A^{-1}x)-\beta\widetilde{Q}_B(x)=(B-\beta I)\widetilde{Q}_B(x).$$

Equivalently, if we recall the definition of $\mathcal{D}_A$ of the previous section, $\mathcal{D}_A(f)(x)=f(A^{-1}x)$, we have

$$(\mathcal{D}_A-\beta I)\widetilde{Q}_B(x)=(B-\beta I)\widetilde{Q}_B(x),$$

where the product on the left side is understood coordinatewise. Moreover, for $k\in\mathbb{N}$,

$$\begin{aligned}(B-\beta I)^k\widetilde{Q}_B(x)&=\sum_{i=0}^{k}\binom{k}{i}(-\beta)^{k-i}B^i\widetilde{Q}_B(x)\\&=\sum_{i=0}^{k}\binom{k}{i}(-\beta)^{k-i}\mathcal{D}_A^i\widetilde{Q}_B(x)\\&=(\mathcal{D}_A-\beta I)^k\widetilde{Q}_B(x).\end{aligned}$$

In particular, since $(B-\beta I)$ is nilpotent of order ℓ, we have

$$(\mathcal{D}_A-\beta I)^\ell\widetilde{Q}_B(x)=(B-\beta I)^\ell\widetilde{Q}_B(x)=0.$$

Hence, all entries of $\widetilde{Q}_B(x)$ belong to $\mathcal{H}(A,\beta,\ell)$. We can repeat this argument for every Jordan block associated to β and every eigenvalue β of ${A^{-1}}_{[s]}$. It follows that each component of $\widetilde{Q}_s(x)$ belongs to $\mathcal{H}(A, \lambda, r)$ for some eigenvalue λ of $A^{-1}{}_{[s]}$, and some $r \in \mathbb{N}$. Since Q_s is an invertible matrix and the monomials x^α with $|\alpha| = s$ are linearly independent, it follows that $\widetilde{Q}_s^1(x), \dots, \widetilde{Q}_s^{d_s}(x)$ are linearly independent, and all homogeneous polynomials $q(x) = q(x_1, \dots, x_n)$ with $deg(q) = s$, are a linear combination of $\widetilde{Q}_s^1(x), \dots, \widetilde{Q}_s^{d_s}(x)$.

We can now state the next theorem:

Theorem 15. *Assume that φ has accuracy κ and that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent. If q is a homogeneous polynomial in $\mathbb{R}^d$ with $deg(q) < \kappa$, then $q \in \mathcal{H} = \bigoplus_{\lambda\in\Delta_n} \mathcal{H}_\lambda(\varphi)$, where Δ_n is the set of eigenvalues of T_n.*

Proof. Let $s < \kappa$, and let $\widetilde{Q}_s$ and $\widetilde{Q}_B$ be as before. Since φ has accuracy κ, and $s < \kappa$, all components of $\widetilde{Q}_B$ (in fact all components of $\widetilde{Q}_s$) are in $\mathcal{S}(\varphi)$, and satisfy

$$\widetilde{Q}_B(A^{-1}x) = \begin{pmatrix} \beta & 0 & \dots & 0 & 0 \\ 1 & \beta & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \beta & 0 \\ 0 & 0 & \dots & 1 & \beta \end{pmatrix} \widetilde{Q}_B(x). \tag{4.4.6}$$

If we denote by $\widetilde{Q}_B^1(x)$ the first coordinate of $\widetilde{Q}_B(x)$ we see that $\widetilde{Q}_B^1(x)$ is actually of class $\mathcal{H}(A, \beta, 1)$. Hence, by Proposition 18, $\widetilde{Q}_B^1(x) = Y\Phi$, where $P_nY \in \mathrm{Ker}(T_n - \beta I)$. This means that β is also an eigenvalue of T_n and the theorem follows. $\square$

The following corollary imposes conditions on the eigenvalues of T_n, under the hypothesis of accuracy.

Corollary 5. *Assume that φ has accuracy κ and that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of A (counted with multiplicity). If $\eta = (\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_d})$, then $[\eta^\alpha]_{|\alpha|=s}$ are eigenvalues of T_n, for $s = 0, 1, \dots, \kappa - 1$.*

Proof. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A. By [CHM98], $[\lambda^\alpha]_{|\alpha|=s}$ are the eigenvalues of $A_{[s]}$. Also, recall that since A is invertible, $A_{[s]}$ is also invertible and $(A_{[s]})^{-1} = {A^{-1}}_{[s]}$. So the eigenvalues of ${A^{-1}}_{[s]}$ are $[\eta^\alpha]_{|\alpha|=s}$. We have already proved that if φ has accuracy κ and $s < \kappa$, then every eigenvalue of ${A^{-1}}_{[s]}$ is also an eigenvalue of T_n. So the result follows. $\square$

Chapter 5

Sampling in Shift Invariant Spaces

5.1 Introduction

Given a function f which is defined on the real line. What information of f can be obtained if we only know its values on a discrete set? This question only has sense if we restrict ourselves to an adequate class of functions.

Let F be a space of functions defined on $\mathbb{R}$, and $X \subset \mathbb{R}$ a discrete subset. The main goal of the *sampling theory* is to recover a function from its values $\{f(x_k)\}_{k\in\mathbb{Z}}$. If the elements of X form a regular grid (i.e. they are equidistant), the sampling is *uniform*.

Let $\sigma > 0$. The space of σ band-limited functions is defined by

$$P_\sigma = \left\{ f \in L^2(\mathbb{R}) : \ \mathrm{Supp}(\hat{f}) \subset [-\sigma, \sigma] \right\}.$$

The space $P_{1/2}$ is called the Paley-Wiener space.

The classical result in sampling theory is the *Whittaker-Shannon-Theorem* which states that every function $f \in P_\sigma$ can be recovered from its samples $f(\frac{k}{2\sigma})_{k\in\mathbb{Z}}$ by the following formula

$$f(x) = \sum_{k\in\mathbb{Z}} f(\frac{k}{2\sigma}) \frac{2\sigma \sin(\pi(2\sigma x - k))}{\pi(2\sigma x - k)},$$

where the series on the right hand side converges uniformly and in $L^2(\mathbb{R})$.

The sampling theory developed into different directions (see for [BF00]). Lately there have been discovered deep connections with wavelet theory, frames and reproductive Hilbert spaces.

The Paley-Wiener space mentioned before is a refinable SIS space generated by the function $sinc(x) = \frac{\sin \pi x}{\pi x}$, which is the scaling function associated to the MRA given in (1.3.6). Sampling theory in spaces other than the space of band-limited functions recently aroused considerable interest. This is in part because the band-limitedness assumption is not very realistic in many applications. Besides, the sinc function has slow decay which translates in poor reconstruction.

Aldroubi and Gröchenig studied sampling in spline-type spaces which are refinable SIS generated by the spline of degree n ([AG00]).

In [Wal92], Walter extended the classical sampling theorem to wavelet subspaces generated by scaling functions which satisfy certain conditions.

Let us recall that a SIS in $L^2(\mathbb{R})$ is a subspace of $L^2(\mathbb{R})$, such that it is invariant under integer translations.

Given functions $f_1, f_2, \ldots, f_n \in L^2(\mathbb{R})$, we will denote by $S(f_1, \ldots, f_n)$, the SIS generated by the integer translates of these functions, i.e. the $L^2(\mathbb{R})$- closure of the span of the set $\{f_i(\cdot - k) : i = 1, ..., n, k \in \mathbb{Z}\}$.

Definition 19. A sequence $\{\phi_k\}_{k\in\mathbb{Z}}$ is a *frame* for a separable Hilbert space $\mathcal{H}$ if there exist positive constants A and B that satisfy

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, \phi_k\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

Note. If $A = B$, we say that the frame is *tight*, and if $\{\phi_k\}_{k\in\mathbb{Z}}$ satisfies the right inequality in the above formula, it is called a *Bessel sequence*.

When the set $\{f_i(\cdot - k) : i = 1, ..., n, k \in \mathbb{Z}\}$, forms a frame of $S(f_1, \ldots, f_n)$, we will write sometimes $V(f_1, \ldots, f_n)$ instead of $S(f_1, \ldots, f_n)$, to stress this fact. It is known that every space $S(f_1, \ldots, f_n)$ contains functions $g_1, \ldots, g_l$, with $l \leq n$, such that $S(f_1, \ldots, f_n) = V(g_1, \ldots, g_l)$.

The integer translates of the function $sinc(x) = \frac{\sin \pi x}{\pi x}$ are an orthonormal base of $P_{1/2}$, in particular a frame. This leads to consider sampling in the more general setting of spaces generated by functions which integer translates form a frame.

A sampling space is a space $V(\varphi)$ where the generator has special properties which we will state below.

Our main goal in this chapter is to find characterizations of functions of $L^2(\mathbb{R})$ that belong to a general sampling space $V(\varphi)$ and to study the structure of $V(\varphi)$.

5.2 Definitions and Preliminaries

We will consider the following definition of sampling spaces, that appears in [SZ04], [SZ99].

Definition 20. A closed subspace $V(\varphi)$ of $L^2(\mathbb{R})$ is called a *sampling space* if there exists a function s such that:

(1) The translates $\{s(\cdot - k)\}_{k\in\mathbb{Z}}$ are a frame for the space $V(\varphi)$.

(2) For every sequence $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ the series $\sum_{k\in\mathbb{Z}} c_k s(\cdot - k)$ converges pointwise to a continuous function.

(3) For every $f \in V(\varphi)$,

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)s(x-k), \tag{5.2.1}$$

where the convergence is in $L^2(\mathbb{R})$ and uniform in $\mathbb{R}$.

The function s is called the *sampling function* of $V(\varphi)$.

Recall that the grammian of a function $\varphi \in L^2(\mathbb{R})$ is the function $G_\varphi(\omega) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega + k)|^2$. We will denote by E_φ the set $E_\varphi = \{w \in \mathbb{R} : G_\varphi(\omega) > 0\}$. The set E_φ is periodic i.e. $E_\varphi = E_\varphi + k$ for every integer k.

For a shift invariant space $S(\phi)$ it is known ([dBVR94a]) that the integer translates of the function φ defined by

$$\hat{\varphi}(\omega) = \begin{cases} \frac{\hat{\phi}(\omega)}{G_\phi(\omega)^{\frac{1}{2}}} & \text{for } \omega \in E_\phi \\ 0 & \text{otherwise,} \end{cases} \tag{5.2.2}$$

form a tight frame of $S(\phi)$, in particular $S(\phi) = V(\varphi)$.

A basic tool in the analysis of sampling and signal processing is the *Zak transform.*

Definition 21. For $f \in L^2(\mathbb{R})$ the Zak transform of f is the function on $\mathbb{R}^2$:

$$Z_f(x, \omega) = \sum_{k \in \mathbb{Z}} f(x + k) e^{-2\pi i k \omega}.$$

For properties of the Zak transform see [Jan88].

Sun and Zhou gave the following characterization of the sampling spaces defined above:

Proposition 21. *([SZ99]) Let $V(\varphi)$ be a shift invariant space. Then the following two assertions are equivalent:*

(i) The space $V(\varphi)$ is a sampling space

(ii) The function φ is continuous, $\sum_{k \in \mathbb{Z}} |\varphi(x - k)|^2$ is bounded on $\mathbb{R}$ and

$$A \chi_{E_\varphi}(\omega) \leq |Z_\varphi(0, \omega)| \leq B \chi_{E_\varphi}(\omega) \quad a.e.\ \omega$$

for some constants $A, B > 0$.

The next result can be found in [BL98].

Proposition 22. *The sequence* $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ *is a frame for the closure in* $L^2(\mathbb{R})$ *of the space it spans if and only if there exist positive constants* A *and* B *that satisfy*

$$A \leq G_\phi(\omega) \leq B \qquad a.e.\ \omega \in E_\phi. \tag{5.2.3}$$

Note. Recall that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for the closure in $L^2(\mathbb{R})$ of the space it spans if and only if equation (5.2.3) holds a.e. $\omega \in \mathbb{R}$.

It is known that if the sequence $\{\phi_k\}_{k\in\mathbb{Z}}$ is a frame for the Hilbert space $\mathcal{H}$, then $f \in \mathcal{H}$ if and only if $f = \sum_{k\in\mathbb{Z}} c_k\phi_k$, for some $c_k \in \ell^2(\mathbb{Z})$ with convergence in $L^2(\mathbb{R})$.

The following is an important characterization for a PSI of $L^2(\mathbb{R})$ in terms of the Fourier transform of the generator.

Theorem 16. *([dBVR94a]) Let* $S(\phi)$ *be a shift invariant space. A function* f *is in* $S(\phi)$ *if and only if* $\hat{f} = r\hat{\phi}$ *for some periodic function* r *of period one, with* $r\hat{\phi} \in L^2(\mathbb{R})$.

The analogous result for FSI has been proved in [dBVR94b].

Notation. Let $f, g \in L^2(\mathbb{R})$. We denote

$$[f, g](x) = \sum_{k\in\mathbb{Z}} f(x+k)\overline{g(x+k)}.$$

Observe that $G_f(\omega) = [\hat{f}, \hat{f}](\omega)$.

Since the Fourier transform preserves the scalar product, if V is a closed subspace of $L^2(\mathbb{R})$ and P_V is its orthogonal projection, we have that $\widehat{P_V(f)} = P_{\hat{V}}(\hat{f})$, where $\hat{V} = \{\hat{f} : f \in V\}$.

In [dBVR94a] this formula for the orthogonal projection was obtained:

$$P_{\widehat{S(\varphi)}}(\hat{f})(\omega) = r(\omega)\hat{\varphi}(\omega),$$

where

$$r(\omega) = \begin{cases} \frac{[\hat{f},\hat{\varphi}](\omega)}{[\hat{\varphi},\hat{\varphi}](\omega)} & \text{for } \omega \in E_\varphi \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.4}$$

5.3 Functions in Sampling Spaces

In this section we will first show that if a function f belongs to a sampling space, then $S(f)$ is a sampling space itself. We will see that the sampling function of $S(f)$ is the orthogonal projection, onto $S(f)$, of the sampling function of the original space. From here, using the characterization of sampling spaces given by Sun and Zhou in [SZ99], we obtain necessary and sufficient conditions for a function f to belong to a sampling space.

We will need the following results.

Proposition 23. *([SZ04]) Let $V(\varphi)$ be a sampling space with sampling function s. Then there exists a sampling space $V(\tilde{\varphi})$ with sampling function $\tilde{s}$ such that:*

(1) *The space $V(\varphi)$ is a subspace of $V(\tilde{\varphi})$*

(2) *The sequence $\{\tilde{s}(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for $V(\tilde{\varphi})$.*

Lemma 3. *Assume $V(\varphi) \subset L^2(\mathbb{R})$ is a shift invariant space and φ is a continuous function. Let $\phi \in V(\varphi)$ such that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Bessel sequence. If $\sum_{k\in\mathbb{Z}} |\varphi(x+k)|^2 < L < +\infty \ \ \forall x \in \mathbb{R}$, then $\sum_{k\in\mathbb{Z}} |\phi(x+k)|^2 < L' < +\infty \ \ \forall x \in \mathbb{R}$.*

The proof of this lemma is in [SZ99] for the case that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a frame of $V(\varphi)$, but the same proof works if it is only a Bessel sequence.

Lemma 4. *([Sun05b]) If $\varphi \in L^2(\mathbb{R})$ is continuous and $\sum_{k\in\mathbb{Z}} |\varphi(x+k)|^2 < L < +\infty$, then $Z_\varphi(x,\omega) = 0 \ \ \forall x \in \mathbb{R}, \quad a.e.\ \omega \in \mathbb{R} \setminus E_\varphi$.*

Remark. Let $V(\varphi)$ be a sampling space and s its sampling function. For $f \in V(\varphi)$,

$$\hat{f}(\omega) = Z_f(0,\omega)\hat{s}(\omega) \quad \text{a.e.}\ \omega \in \mathbb{R}.$$

Therefore we obtain

$$G_f(\omega) = |Z_f(0,\omega)|^2 G_s(\omega) \quad \text{a.e.}\ \omega \in \mathbb{R}.$$

To see this, observe that

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)s(x-k),$$

with uniform convergence and in $L^2(\mathbb{R})$, so

$$\hat{f}(\omega) = (\sum_{k\in\mathbb{Z}} f(k)e^{-2\pi ik\omega})\hat{s}(\omega) = Z_f(0,\omega)\hat{s}(\omega) \quad \text{a.e.}\, \omega \in \mathbb{R},$$

and then

$$\sum_k |\hat{f}(\omega+k)|^2 = |Z_f(0,\omega)|^2 \sum_k |\hat{s}(\omega+k)|^2 \quad \text{a.e.}\, \omega \in \mathbb{R},$$

hence

$$G_f(\omega) = |Z_f(0,\omega)|^2 G_s(\omega) \quad \text{a.e.}\, \omega \in \mathbb{R}.$$

We obtained the following properties:

Lemma 5. *Let $V(\varphi)$ be a sampling space and s its sampling function. Then we have:*

i) If $\phi_1, \phi_2 \in L^2(\mathbb{R})$ and $S(\phi_1) = S(\phi_2)$, then $E_{\phi_1} = E_{\phi_2}$ (up to a set of measure zero). In particular if $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a frame for $V(\varphi)$ then $E_\varphi = E_\phi$.

ii) The sampling function satisfies $Z_s(0,\omega) = \chi_{E_s}(\omega) \quad a.e.\, \omega \in \mathbb{R}$.

iii) For $f \in V(\varphi)$, $E_f = \{\omega \in \mathbb{R} : Z_f(0,\omega) \neq 0\}$ (up to a set of measure zero).

iv) The sampling function s is unique, up to a set of measure zero, and satisfies $\hat{s}(\omega) = \frac{\hat{\phi}(\omega)}{Z_\phi(0,\omega)}\chi_{E_\varphi}(\omega)$ for each generator ϕ whose translates form a frame of $V(\varphi)$.

Proof. *i)* If $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a frame for $V(\varphi)$ then there exists $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k \phi(x-k).$$

So $G_\varphi(\omega) = |(\sum_{k\in\mathbb{Z}} c_k e^{-2\pi ik\omega})|^2 G_\phi(\omega)$ a.e. $\omega \in \mathbb{R}$, which yields $E_\varphi \subseteq E_\phi$ (up to a set of measure zero). Similarly we obtain the other inclusion. Now, if ϕ_1, ϕ_2 generate the same SIS, we can modify these generators as in (5.2.2) to obtain tight frames, and the result follows.

ii) Since we have $\hat{s}(\omega+k) = Z_s(0,\omega)\hat{s}(\omega+k)$ a.e. $\omega \in \mathbb{R}$, it follows that $Z_s(0,\omega) = 1$ for almost every $\omega \in E_s$. On the other side, by Proposition 21, $Z_s(0,\omega) = 0$ for almost every $\omega \notin E_s$.

iii) We have

$$G_f(\omega) = |Z_f(0,\omega)|^2 G_s(\omega) \quad \text{a.e.}\, \omega \in \mathbb{R}.$$

Using Proposition 23 we can assume that $\{s(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of the sampling space, so $G_s(\omega) \neq 0$ a.e. $\omega \in \mathbb{R}$, which implies *iii*).

iv) Let ϕ be a generator whose translates form a frame of $V(\varphi)$. We have $\hat{\phi}(\omega) = Z_\phi(0,\omega)\hat{s}(\omega)$ a.e. $\omega \in \mathbb{R}$. By *i*), $E_\varphi = E_\phi$, so using *iii*),

$$\hat{s}(\omega) = \frac{\hat{\phi}(\omega)}{Z_\phi(0,\omega)} \quad \text{a.e. } \omega \in E_\varphi$$

and the result follows. □

Now we are ready to proof the following theorem.

Theorem 17. *Let $V(\varphi)$ be a sampling space with sampling function s and $f \in L^2(\mathbb{R})$. If $f \in V(\varphi)$ then $S(f)$ is a sampling space with sampling function $s_f = P_{S(f)}(s)$, where $P_{S(f)}(s)$ is the orthogonal projection of s onto $S(f)$. In this case $\hat{s_f} = \hat{s}\chi_{E_f}$.*

Proof. For $f \in V(\varphi)$, we define

$$\hat{h}(\omega) = \begin{cases} \frac{\hat{f}(\omega)}{Z_f(0,\omega)} & \text{for } \omega \in E_f \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.1}$$

Observe that by Lemma 5 *iv)* the function h is well defined. Using Theorem 16, it is easy to see that $S(f) = S(h)$. To show that $\{h(\cdot - k)\}_{k\in\mathbb{Z}}$ is a frame sequence, first observe that $E_h = E_f$, so for almost every $\omega \in E_h$,

$$G_h(\omega) = \frac{G_f(\omega)}{|Z_f(0,\omega)|^2} = G_s(\omega),$$

which is uniformly bounded below and above.

Now we will see that $S(f) = V(h)$ is a sampling space:

Since $h \in S(f) \subseteq V(\varphi)$, h is continuous. By Proposition 21 and Lemma 3, $\sum_{k\in\mathbb{Z}} |h(x+k)|^2$ is uniformly bounded in $\mathbb{R}$. So by Lemma 4, $Z_h(0,\omega) = 0 \quad \text{a.e.}\, \omega \in \mathbb{R} \setminus E_h$. Furthermore, for almost every $\omega \in E_h$,

$$G_h(\omega) = |Z_h(0,\omega)|^2 G_s(\omega).$$

Since $E_h \subseteq E_s$ (up to a set of measure zero), we can write for almost all $\omega \in E_h$

$$\frac{G_h(\omega)}{G_s(\omega)} = |Z_h(0,\omega)|^2,$$

and using that $G_h(\omega)$ and $G_s(\omega)$ are both bounded above and below in E_h, we have that $|Z_h(0,\omega)|$ is also bounded above and below. Hence, using Proposition 21, we can conclude that $S(f)$ is a sampling space. Moreover, h is its sampling function. To see this, it suffices to prove that for every $g \in S(f)$, $\hat{g}(\omega) = Z_g(0,\omega)\hat{h}(\omega)$. But for $g \in S(f) \subseteq V(\varphi)$,

$$\hat{g}(\omega) = Z_g(0,\omega)\hat{s}(\omega),$$

and since $\hat{h}(\omega) = \hat{s}(\omega)$ for $\omega \in E_\varphi$, we have that $\hat{g}(\omega) = Z_g(0,\omega)\hat{h}(\omega)$ for almost every $\omega \in E_f$. On the other hand, since $g \in S(f)$, there exists a 1-periodic function r, such that $\hat{g}(\omega) = r(\omega)\hat{f}(\omega)$, and therefore the equality also holds for almost every $\omega \in \mathbb{R} \setminus E_f$.

Hence

$$\hat{s_f}(\omega) = \hat{h}(\omega) = \hat{s}(\omega)\chi_{E_f}(\omega) \quad \text{a.e.} \quad \omega \in \mathbb{R}.$$

Finally we prove that $\hat{h}$ is actually the projection of s onto $S(f)$, i.e. $\hat{h} = \widehat{P_{S(f)}(s)}$. For almost every $\omega \in E_f$,

$$\widehat{P_{S(f)}(s)}(\omega) = \frac{[\hat{s}, \hat{f}](\omega)}{[\hat{f}, \hat{f}](\omega)}\hat{f}(\omega) =$$

$$\frac{[\hat{s}, Z_f(0,\cdot)\hat{s}](\omega)}{[\hat{f}, \hat{f}](\omega)}Z_f(0,\omega)\hat{h}(\omega) = |Z_f(0,\omega)|^2\frac{[\hat{s}, \hat{s}](\omega)}{[\hat{f}, \hat{f}](\omega)}\hat{h}(\omega) = \hat{h}(\omega),$$

and for almost every $\omega \in \mathbb{R} \setminus E_f$, we have

$$\widehat{P_{S(f)}(s)}(\omega) = \hat{h}(\omega) = 0.$$

$\square$

As a consequence, by Proposition 21, we have the following necessary and sufficient conditions for a function to belong to a sampling space.

Theorem 18. *Assume $f \in L^2(\mathbb{R})$. Then f belongs to a sampling space if and only if the function h defined by $\hat{h} = \frac{\hat{f}}{G_f^{\frac{1}{2}}}$ in E_f and zero otherwise, satisfies:*

(1) *h is continuous.*

(2) *The function $\sum_{k\in\mathbb{Z}} |h(x-k)|^2$ is bounded on $\mathbb{R}$.*

(3) *There exist constants $A, B > 0$ such that*

$$A\chi_{E_f}(\omega) \leq |Z_h(0,\omega)| \leq B\chi_{E_f}(\omega) \quad \textit{a.e. } \omega.$$

5.4 Decompositions of Sampling Spaces

The results of the preceding section can now be applied to the problem of *determining sets*.

Given a set $\mathcal{F} = \{f_1, \dots, f_m\} \subset L^2(\mathbb{R})$ where f_i are functions that belong to an unknown shift invariant space $V(\varphi)$, it is an important matter to be able to decide whether this set $\mathcal{F}$ is sufficient to determine $V(\varphi)$. This leads to the concept of *determining sets* of shift invariant spaces. In [ACH$^+$04], the problem is solved if $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of $V(\varphi)$.

We study the problem for the case that $V(\varphi)$ is a sampling space. We give necessary and sufficient conditions on $\mathcal{F}$, needed for determining the unknown sampling space $V(\varphi)$. Moreover we decompose the sampling space $V(\varphi)$ as the sum of the sampling spaces $S(f_i)$. In particular the sampling function s of $V(\varphi)$ can be recovered from the functions f_i.

Definition 22. Let $V(\varphi)$ be a shift invariant space. The set $\mathcal{F} = \{f_1, \dots, f_m\} \subset V(\varphi)$ is a *determining set* for $V(\varphi)$ if for any $g \in V(\varphi)$ there exist $\alpha_1, \dots, \alpha_m$ 1-periodic measurable functions such that:

$$\hat{g} = \alpha_1 \hat{f}_1 + \cdots + \alpha_m \hat{f}_m.$$

Theorem 19. *Assume $V(\varphi)$ is a sampling space. Then $\mathcal{F} = \{f_1, \dots, f_m\} \subset V(\varphi)$ is a determining set for $V(\varphi)$ if and only if the set*

$$Z = \left(\bigcup_{i=1}^{m} E_{f_i}\right) \triangle E_\varphi$$

has Lebesgue measure zero (where $\triangle$ denotes the symmetric difference of sets[1]).

Moreover, if $\mathcal{F}$ is a determining set for $V(\varphi)$, then

$$V(\varphi) = S(f_1) + \cdots + S(f_m).$$

[1] $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Proof. Let s be the sampling function of $V(\varphi)$.

Assume that $\mathcal{F} = \{f_1, \dots, f_m\} \subset V(\varphi)$ is a determining set for $V(\varphi)$. Recall that by Lemma 5, $E_\varphi = E_s$ (up to a set of measure zero). Since

$$G_{f_i}(\omega) = |Z_{f_i}(0, \omega)|^2 G_s(\omega) \quad \text{a.e. } \omega \in \mathbb{R},$$

for almost every $\omega \in E_{f_i}$, ω belongs to E_φ, so the set $\cup_{i=1}^m E_{f_i} \setminus E_\varphi$ has Lebesgue measure zero.

On the other side there exist $\alpha_1, \dots, \alpha_m$ 1-periodic measurable functions such that

$$\hat{\varphi}(\omega) = \sum_{i=1}^m \alpha_i(\omega) \hat{f}_i(\omega),$$

so

$$G_\varphi(\omega) = \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\omega + k)|^2 \leq \sum_{k \in \mathbb{Z}} \left(\sum_{i=1}^m |\alpha_i(\omega) \hat{f}_i(\omega + k)| \right)^2.$$

Hence the set $E_\varphi \setminus \cup_{i=1}^m E_{f_i}$ has Lebesgue measure zero.

To prove the reciprocal it suffices to show that there exist $\alpha_1, \dots, \alpha_m$ 1-periodic measurable functions such that

$$\hat{s}(\omega) = \sum_{i=1}^m \alpha_i(\omega) \hat{f}_i(\omega). \tag{5.4.1}$$

Define the sets B_i inductively by $B_1 = E_{f_1}$, and for $2 \leq i \leq m$, $B_i = E_{f_i} \setminus \bigcup_{j=1}^{i-1} B_j$. For $1 \leq i \leq m$ set

$$\alpha_i(\omega) := \begin{cases} \frac{1}{Z_{f_i}(0,\omega)} & \text{for } \omega \in B_i \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.2}$$

Since $\hat{s}(\omega) = \alpha_i(\omega) \hat{f}_i(\omega)$ a.e. $\omega \in B_i$, equation (5.4.1) holds.

Finally we will see that $V(\varphi) = S(f_1) + \dots + S(f_m)$. Since

$$\int_{\mathbb{R}} |\alpha_i(\omega)|^2 |\hat{f}_i(\omega)|^2 d\omega = \int_{B_i} |\hat{s}(\omega)|^2 d\omega < +\infty,$$

it follows that $\alpha_i \hat{f}_i \in S(f_i)$ (see Theorem 16), so $V(\varphi) \subseteq S(f_1) + \cdots + S(f_m)$.

To show the other inclusion, let $g \in S(f_1) + \cdots + S(f_m)$. Then there exist $\beta_1, \ldots, \beta_m$ 1-periodic measurable functions such that $\beta_i \hat{f}_i \in L^2(\mathbb{R})$, $1 \leq i \leq m$, and $\hat{g} = \beta_1 \hat{f}_1 + \cdots + \beta_m \hat{f}_m$. So, using that $\hat{f}_i(\omega) = Z_{f_i}(0, \omega)\hat{s}(\omega)$, we have that $\hat{g}(\omega) = (\beta_1(\omega) Z_{f_1}(0,\omega) + \cdots + \beta_m(\omega) Z_{f_m}(0,\omega))\hat{s}(\omega)$. Since $\hat{g} \in L^2(\mathbb{R})$ and $\beta_1 Z_{f_1}(0,\omega) + \cdots + \beta_m Z_{f_m}(0,\omega)$ is a 1-periodic function, applying again Theorem 16, g belongs to $V(\varphi)$. $\square$

From Theorem 17 we can also obtain the following decomposition of a sampling space.

Proposition 24. *Let $V(\varphi)$ be a sampling space and let $\{E_j\}_{j \in \mathbb{N}}$ be a partition of E_φ in periodic measurable sets (i.e. $E_\varphi = \bigcup_j E_j$, $|E_j \cap E_l| = 0$ for all $j \neq l$, and $E_j + k = E_j$ for every integer k).*

Define for $j \in \mathbb{N}$, φ_j by $\hat{\varphi}_j = \hat{\varphi}\chi_{E_j}$. Then we have:

$$V(\varphi) = \bigoplus_{j \in \mathbb{N}} V(\varphi_j),$$

where $V(\varphi_j)$ is a sampling space for each j. Furthermore, if s is the sampling function of $V(\varphi)$ then the Fourier transform of the sampling function s_j of $V(\varphi_j)$ is $\hat{s}_j = \hat{s}\chi_{E_j}$.

Proof. Since φ satisfies that there exist $A, B \geq 0$ such that

$$A \leq G_\varphi(\omega) \leq B \quad \text{a.e. } \omega \in E_\varphi,$$

then each φ_j has the same property in E_j. We conclude that $\{\varphi_j(\cdot - k)\}_{k \in \mathbb{Z}}$ is a frame sequence. Furthermore, note that $\varphi_j \in V(\varphi)$ since χ_{E_j} is 1-periodic and $\chi_{E_j}\hat{\varphi} \in L^2(\mathbb{R})$. Hence, by by Theorem 17, $V(\varphi_j)$ is a sampling space.

Let us show now that $V(\varphi) = \bigoplus_j V(\varphi_j)$. Assume that $f \in V(\varphi)$. Then $\hat{f} = m_f \hat{\varphi}$ with $m_f \in L^2[0,1)$.

So, $\hat{f} = \sum_j (m_f \chi_{E_j})(\chi_{E_j} \hat{\varphi}) = \sum_j m_j \hat{\varphi}_j$, where $m_j = m_f \chi_{E_j} \in L^2[0,1)$. Then $\hat{f}_j := m_j \hat{\varphi}_j \in V(\varphi_j)$. That is $f = \sum_j f_j, f_j \in V(\varphi_j)$.

On the other side, assume that $g_j \in V(\varphi_j)$ and $\sum_j g_j = 0$. Write $\hat{g}_j = \theta_j \hat{\varphi}_j$ with $\theta_j \in L^2[0,1)$.

Suppose that for some $r \in \mathbb{N}$, the set $M := \{\omega : \hat{g}_r(\omega) \neq 0\}$ has positive Lebesgue measure. Since $M \subset E_r$ we have that for almost all $\omega \in M$,

$$0 = \sum_j \hat{g}_j(\omega) = \hat{g}_r(\omega) = \theta_r(\omega)\hat{\varphi}_r(\omega).$$

The fact that $\omega \in E_r$ implies that for some integer k, $\hat{\varphi}(\omega + k) \neq 0$.

Then we can write $0 = \theta_r(\omega + k)\hat{\varphi}_r(\omega + k) = \theta_r(\omega)\hat{\varphi}_r(\omega + k)$.

So, $\theta_r(\omega) = 0$. That is $\theta_r \equiv 0 \quad$ a.e. in M, which is a contradiction. We conclude that $g_j \equiv 0$ for all j.

Since $\varphi_j \in V(\varphi)$, by Theorem 17, $\hat{s_j} = \hat{s}\chi_{E_{\varphi_j}} = \hat{s}\chi_{E_j}$ and this completes the proof.

□

Example We will give an example of the decomposition mentioned in Proposition 24.

The sampling space generated by $\varphi(x) = \frac{\sin \pi x}{\pi x}$ is the Paley Wiener Space $P_{1/2}$. In this case $E_\varphi = \mathbb{R}$. Let $\{a_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_{>0}$ a strictly decreasing sequence such that $a_1 = \frac{1}{2}$ and $\lim_{j \to +\infty} a_j = 0$.
Set $E_j = ([-a_j, -a_{j+1}] \cup [a_{j+1}, a_j]) + \mathbb{Z}$. Following Proposition 24, it is clear that $\{E_j\}_{j \in \mathbb{N}}$ is a partition of E_φ in periodic measurable sets and

$$V(\varphi) = \bigoplus_{j \in \mathbb{N}} V(\varphi_j),$$

where $\hat{\varphi}_j(\omega) = \chi_{[-a_j,-a_{j+1}]\cup[a_{j+1},a_j]}(\omega)$. In this way we produce a decomposition in sampling spaces corresponding to different frequency bands. Notice that when $a_j = \frac{1}{2^j}$, we obtain the wavelet subspaces for the Shannon wavelet.

Remark. We observe that it is natural to consider the sampling problem on a general lattice $a\mathbb{Z} + b$, with $a \in \mathbb{R}_{>0}, b \in \mathbb{R}$. For this, let $\mathcal{T} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be an unitary operator and $V = V(\varphi)$ a sampling space. Set $W = \mathcal{T}(V)$.

Consider $g \in W$ and call $f = \mathcal{T}^{-1}g$. Then $f = \sum_k f(k)t_k\varphi$, where t_k is the translation operator $t_kh(x) = h(x-k)$. So, $g = \mathcal{T}f = \sum_k f(k)(\mathcal{T} \circ t_k)\varphi = \sum_k \mathcal{T}^{-1}(\mathcal{T}f)(k)(\mathcal{T} \circ t_k)\varphi$. That is

$$g = \sum_k (\mathcal{T}^{-1}g)(k)(\mathcal{T} \circ t_k)\varphi, \quad \forall g \in W.$$

Let us now define the unitary dilation operator D_a by $D_af(x) = \sqrt{a}f(ax)$ and $\mathcal{T}$ by $\mathcal{T} = D_a \circ t_b$. Denote $\phi = \mathcal{T}\varphi$. Then, due to the commutation relation $D_at_k = t_{\frac{k}{a}}D_a$, the following sampling formula holds:

$$g = \sum_k g(\frac{k+b}{a})\phi(x - \frac{k}{a}), \qquad \forall g \in V_{a,b} = \overline{span}(\{\phi(\cdot - \frac{k}{a}), k \in \mathbb{Z}\})).$$

Index of Symbols

$B(\varepsilon)$	$= \{x \in \mathbb{R}^d : \|x\| \leq \varepsilon\}$
E^0	interior of $E \subset \mathbb{R}^d$
∂E	boundary of $E \subset \mathbb{R}^d$
$\overline{E}$	closure of $E \subset \mathbb{R}^d$
$\lvert E\rvert$	Lebesgue measure of $E \subset \mathbb{R}^d$
$\hat{f}$	Fourier transform of f given by the formula
	$\hat{f}(w) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x w}\, dx$
$\langle f, g\rangle$	inner product in $L^2(\mathbb{R})$ given by $\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\, dx$
A	dilation matrix
$A_{[s]}$	matrix related to dilation of $X_{[s]}$ by A
c_k	coefficients in the refinement equation
D	$= \{d_1, \dots , d_m\}$, digit set associated with A
d_s	number of multi-indices of degree s
$\mathcal{H}(A, \lambda, r)$	space of (A, λ, r)-homogeneous functions
K_H	attractor of IFS $\{w_k\}_{k\in H}$
L	$= [c_{Ai-j}]_{i,j\in\Gamma}$
$L^p(E)$	$= \left\{f : E \longrightarrow \mathbb{C} \text{ such that } \int_E \lvert f(x)\rvert^p\, dx < \infty\right\}$
$\\|f\\|_p$	$\left(\int_E \lvert f(x)\rvert^p\, dx\right)^{\frac{1}{p}} \qquad (1 \leq p < +\infty)$
$\\|f\\|_\infty$	essential supremum of f on E
m	$= \lvert det(A)\rvert$
$m_0(w)$	symbol of the refinement equation

d	dimension of domain of the refinable function
Q	$= K_D$, tile associated with A and D
$\mathrm{Supp}(f)$	support of f
V_j	subspaces in multiresolution analysis
w_k	affine map $w_k(x) = A^{-1}(x+k)$
w_H	$w_H(B) = \bigcup_{k \in H} w_k(B)$
$X_{[s]}$	vector of all polynomials of degree s
$\delta_{j,k}$	Kronecker delta
Γ	lattice in $\mathbb{R}^d$ invariant under A
κ	accuracy
Λ	support of coefficients in refinement equation
Λ'	$= D - \Lambda$
χ_E	characteristic function of a set E
$\ell(\Gamma)$	the space of all the sequences defined on Γ
$\ell_0(\Gamma)$	the space of all the finitely supported sequences on Γ

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