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Convexidad Geodésica, Espacios Simétricos y Operadores de Hilbert-Schmidt

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Título: Convexidad Geodésica, Espacios Simétricos y Operadores de Hilbert-Schmidt

Resumen: En un conjunto de operadores inversibles y positivos (concretamente en el grupo de operadores Hilbert-Schmidt con unidad adjunta) se introduce una estructura Riemanniana natural que convierte este espacio en una variedad simétrica de curvatura seccional no positiva. Este espacio puede describirse como un cociente mediante la acción de automorfismos interiores. Estudiamos las subvariedades Riemannianas geodésicamente convexas, que resultan ser caracterizables por una propiedad algebraica de su tangente; en particular estudiamos el grupo de isometrías de estas subvariedades. Mostramos cómo cualquier espacio simétrico del tipo no compacto puede ser isométricamente identificado con una de estas subvariedades mencionadas. Para cualquier subvariedad convexa y cerrada, construimos una proyección ortogonal que permite factorizar cualquier operador de la variedad mediante un factor en la subvariedad y un factor ortogonal a la misma. Esta factorización es única (y depende analíticamente de los parámetros). Incluimos una sección dedicada al estudio de la geometría de las órbitas unitarias de un operador fijo, donde calculamos las geodésicas de estas órbitas para las distintas métricas que pueden introducirse.

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Palabras clave: operador de Hilbert-Schmidt, operador positivo, geodésica, convexidad, factorización, espacio simétrico, grupo unitario, espacio homogéneo

Title: GEODESIC CONVEXITY, SYMMETRIC SPACES AND HILBERT-SCHMIDT OPERATORS

ABSTRACT: A natural Riemannian structure is given to the set of positive invertible (unitized) Hilbert-Schmidt operators; this metric makes this set a nonpositively curved, infinite dimensional Hilbert manifold. We give an intrinsic (algebraic) characterization of such submanifolds, and we study their group of isometries. We show that any symmetric space of the noncompact type can be isometrically embedded in this manifold. For any convex, closed submanifold we construct an orthogonal projection by means of the Riemannian exponential, a projection which provides a unique factorization for any operator in the manifold; the factors being an operator in the submanifold and the exponential of an operator orthogonal to the submanifold. We include a final section devoted to the study of the unitary orbits of a fixed operator and the diverse geometries that arise from endowing this orbit with different Riemannian metrics.

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Keywords and phrases: Hilbert-Schmidt class, positive operator, geodesic, convexity, factorization, symmetric space, unitary group, homogeneous space

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Para mi padre,
Angel Rafael Larotonda
1939-2005

Allá al fondo está la muerte, pero no tenga miedo. Sujete el reloj con una mano, tome con dos dedos la llave de la cuerda, remóntela suavemente. Ahora se abre otro plazo, los árboles despliegan sus hojas, las barcas corren regatas, el tiempo como un abanico se va llenando de sí mismo y de él brotan el aire, las brisas de la tierra, la sombra de una mujer, el perfume del pan.

¿Qué más quiere, qué más quiere? Átelo pronto a su muñeca, déjelo latir en libertad, imítelo anhelante. El miedo herrumbra las áncoras, cada cosa que pudo alcanzarse y fue olvidada va corroyendo las venas del reloj, gangrenando la fría sangre de sus rubíes. Y allá en el fondo está la muerte si no corremos y llegamos antes y comprendemos que ya no importa.

Julio Cortázar, "Instrucciones para dar cuerda al reloj"

Introducción

Una variedad de Hadamard es una variedad diferenciable, Riemanniana, conexa, simplemente conexa y de curvatura seccional no positiva. Desde el punto de vista topológico, es un objeto extremadamente simple.

Sin embargo, [Eb96] para cualquier variedad M de curvatura no positiva, los grupos de homotopía $\pi_k(M)$, $k \geq 2$ son nulos, y M puede ser expresada como un cociente de una variedad de Hadamard (el revestimiento universal de M) por un grupo de isometrias del revestimiento (el grupo de isometrias en cuestión es isomorfo a $\pi_1(M)$).

La geometría de los espacios de curvatura no positiva es ciertamente rica y tiene aplicaciones en muchas otras ramas de la matemática, como las funciones armónicas ([Cor92], [GS92], [KS93], [MSY93]), las variedades de dimensión 3 y los grupos de Klein ([MS84], [Gab92], [Can93], [CJ94], [Min94], [McM96], [Otal96], [Gab97], [Otal98], [Min99], [Kap01], [GMT03]), teoría de rango y rigidez ([Ball85], [BBE85], [BBS85], [BS97], [EH90], [BB95], [Lee97]), topología de altas dimensiones ([FH81], [FJ93] and [CGM90]), grupos hiperbólicos y geometría cuasi-conforme ([Gro87], [Pan89], [BM91], [RS94], [Sela95], [Bow98a], [Bow98b], [BP99], [BP00], [HK98]), teoría de grupos geométrica y combinatoria ([Gro87], [DJ91], [Sch95], [CD95], [BM97], [KL97a], [KL97b], [Esk98]) y dinámica ([Cro90], [Otal90], [BCS95], [BFK98]).

Los tratados clásicos [Hel62] de Sigurdur Helgason y [BGS85] de Wallman *et al.*, la introducción a la geometría de los espacios de tipo no compacto de Patrick Eberlein [Eb96], o el artículo de difusión por el mismo autor [Eb89] contienen muchos (sino todos) los resultados relevantes concernientes a la geometría de espacios de curvatura no positiva, como la Ley de Cosenos, proyecciones ortogonales, convexidad de la función distancia, la construcción del espacio de frontera y los teoremas sobre rigidez y rango.

Concentrémonos brevemente en seis resultados que son válidos [Hel62] en cualquier variedad de Hadamard M de dimensión finita:

1. La función exponencial $\mathrm{Exp}_p : T_pM \to M$ es un difeomorfismo para cada punto $p \in M$.

2. Para cada par de puntos $p, q \in M$ existe una única geodésica minimizante normal (*i.e.* de velocidad unitaria) que une p con q.

3. Para cualquier triángulo geodésico en M (cuyos lados son las geodésicas de longitudes a, b y c) se tiene la Ley de Cosenos Hiperbólica, que dice:

$$c^2 \geq a^2 + b^2 - 2ab\cos(\theta), \text{ donde } \theta \text{ es el ángulo opuesto a } c$$

4. La suma de los ángulos internos de cualquier triángulo geodésico es a lo sumo π.

5. Para cualquier par de geodésicas α, β en M, la función

$$f(t) = \mathrm{dist}(\alpha(t), \beta(t))$$

es una función real convexa.

6. Supongamos que C es un conjunto convexo cerrado de M. Entonces para cada $p \in M$ existe un único punto $\Pi_C(p) \in C$ tal que

$$\operatorname{dist}(p, \Pi_C(p)) \leq \operatorname{dist}(p, q) \text{ para cualquier } q \in C$$

En el contexto Riemanniano, el punto $\Pi_C(p)$ se denomina *pie de la perpendicular* de C a p.

Las nociones de completitud como espacio métrico y de completitud en el sentido geodésico están íntimamente ligadas por el teorema de Hopf-Rinow. Como este teorema es falso en dimensión infinita [Atkin75] [Atkin97], la compacidad de los entornos de M parece ser relevante para que los resultados mencionados más arriba sean ciertos. Sin embargo, esto no es así ya que todos estos resultados son válidos en el contexto de los espacios de curvatura no positiva (que son espacios métricos donde alguna desigualdad de comparación de triángulos es válida). En particular, la prueba de la existencia de un único punto que realice la distancia a un conjunto convexo cerrado (sin suponer la compacidad de los entornos) puede encontrarse en [Jost97].

Nosotros vamos a ir en una dirección distinta, y lo que haremos será extender estos resultados a una variedad diferenciable Σ_∞ que es localmente isomorfa a un espacio de Hilbert de dimensión infinita (en realidad, a la parte real de una cierta álgebra de Banach $\mathcal{B}$). La variedad Σ_∞ es simplemente conexa, completa en el sentido geodésico (y en el métrico), y tiene curvatura seccional no positiva: es más, $\Sigma_\infty = GL^+(\mathcal{B})$ resulta ser un espacio simétrico en el sentido usual (Riemanniano) de la palabra. Todas las herramientas de la geometría Riemanniana estarán a mano y podremos explorar relaciones entre el álgebra de Banach y la geometría de la variedad.

Por ejemplo, probaremos que la única geodésica minimizante que realiza la distancia entre un punto y un conjunto convexo y cerrado debe ser ortogonal al conjunto, obteniendo de esta manera un teorema de factorización para operadores, con muchas aplicaciones inmediatas.

El primer resultado de la lista será obvio a partir de la definición de Σ_∞; para probar los otros cinco, vamos a tener que revisar y poner en contexto algunos resultados de la literatura existente sobre geometría en espacios de operadores.

El espacio Σ_∞ es simétrico y tiene curvatura no positiva, y es universal en esta categoría, en el sentido siguiente: cualquier espacio simétrico y de curvatura no positiva puede identificarse isométricamente con alguna subvariedad cerrada y convexa de Σ_∞.

Aunque no vamos a hacer uso de ella en este manuscrito, hacemos notar al lector que la teoría de clasificación de L^*-álgebras (ver [Sch60] [Sch61] por J.R. Schue, [CGM90] por Mira, Martin and González, o [Neh93] por E. Neher) provee un ambiente más abstracto (más general si se quiere) de trabajo para esta variedad y sus subvariedades: la parte real de cualquier L^*-álgebra puede ser naturalmente identificada con una subvariedad cerrada y convexa de Σ_∞.
A lo largo de este manuscrito, vamos a usar Exp_p para denotar la exponencial Riemannian de nuestra variedad en el punto p, y usaremos asimismo exp en vez de Exp_1, que es la exponencial usual de operadores.

Comentamos brevemente la organización y los resultados más relevantes de este manuscrito. Los resultados previos están mencionados como tales y los resultados nuevos son los indicados a continuación como **Teorema 1**, **Teorema 2**, ... hasta el **Teorema 14**:

En la sección II, introducimos la notación y los preliminares necesarios para la construcción de una variedad de Hilbert de dimensión infinita que resulta ser completa, simplemente conexa y tiene curvatura seccional no positiva.
El espacio ambiente para casi todos los cálculos es el espacio de Banach con producto interno dado por la traza $\mathcal{H}_\mathbb{R} = \{\lambda + a\}$, donde λ es un número real y a es un operador autoadjunto Hilbert-Schmidt que actúa en un espacio de Hilbert separable H. Como conjunto, $\Sigma_\infty := \exp(\mathcal{H}_\mathbb{R})$. Como la exponencial es analítica es fácil ver que Σ_∞ es abierto en $\mathcal{H}_\mathbb{R}$ (Proposición II.3).
La métrica que introducimos en Σ_∞ (II.4) es similar a la métrica que hace de las matrices inversibles y positivas un espacio simétrico:

$$\langle X, Y\rangle_p = \left\langle Yp^{-1}, p^{-1}X\right\rangle_2 \qquad \text{para } p \in \Sigma_\infty \text{ y } X, Y \in \mathcal{H}_\mathbb{R},$$

donde $< \alpha + a, \beta + b >_2 = \alpha\overline{\beta} + 2tr(b^* a)$. Con esta métrica la variedad Σ_∞ tiene una derivada covariante (II.5) dada por

$$\nabla_X Y = X(Y) - \frac{1}{2}\left(Xp^{-1}Y + Yp^{-1}X\right)$$

donde $X(Y)$ denota derivación del campo Y en la dirección de X (derivación hecha en el espacio lineal $\mathcal{H}_\mathbb{R}$); la curvatura seccional (4) está dada por la fórmula

$$\mathcal{R}_p(X,Y)Z = -\frac{1}{4}\, p\left[\left[p^{-1}X, p^{-1}Y\right], p^{-1}Z\right]$$

donde $[x,y] = xy - yx$ denota el conmutador usual de operadores en $L(H)$

En (IV.9) probamos existencia y unicidad de curvas minimizantes:

Teorema 1 *Pongamos* $\|X\|_p = \|p^{-\frac{1}{2}}Xp^{-\frac{1}{2}}\|_2$ *y* $L(\alpha) = \int_0^1 \|\alpha(t)\|_{\dot{\alpha}(t)}\, dt$. *Si*

$$\operatorname{dist}(p,q) = \inf\{L(\alpha) : \alpha \subset \Sigma_\infty,\ \alpha\, es\, suave,\ \alpha(0) = p, \alpha(1) = q\}$$

entonces la curva $\gamma_{pq}(t) = p^{\frac{1}{2}}\left(p^{-\frac{1}{2}}qp^{-\frac{1}{2}}\right)^t p^{\frac{1}{2}}$ *es el camino mas corto en* Σ_∞ *que une* p *con* q*; es mas,*

$$\operatorname{dist}(p,q) = L(\gamma_{pq}) = \|\ln(p^{\frac{1}{2}}q^{-1}p^{\frac{1}{2}})\|_2 \equiv \|\dot{\gamma_{pq}}(t)\|_{\gamma_{pq}(t)}$$

Probamos (III.5) que los campos de Jacobi a lo largo de geodésicas γ son convexos (en el sentido siguiente: $t \mapsto \|J(t)\|_{\gamma(t)} = \langle J(t), J(t)\rangle_{\gamma(t)}$ es una función convexa), y como un corolario (III.6), obtenemos

Teorema 2 *La función real* $t \mapsto \operatorname{dist}(\gamma(t), \delta(t))$ *es convexa para cualquier par de geodésicas* $\gamma, \delta \in \Sigma_\infty$.

Como conocemos la expresión de las geodésicas, conocemos la expresión para la exponencial Riemanniana $\mathrm{Exp}_p : T_p\Sigma_\infty \to \Sigma_\infty$, que está dada por

$$\mathrm{Exp}_p(v) = p^{\frac{1}{2}} \exp\left(p^{-\frac{1}{2}}vp^{-\frac{1}{2}}\right) p^{\frac{1}{2}} = pe^{p^{-1}v}$$

Esta función es un C^ω-difeomorfismo (analítico) sobreyectivo para cada p (IV.6), y lo mismo se aplica para la restricción de Exp_p al fibrado tangente de cualquier subvariedad cerrada y geodésicamente convexa $M \subset \Sigma_\infty$.

También probamos (IV.11) que la suma de los ángulos internos de cualquier triángulo geodésico en Σ_∞ es menor o igual que π (que es una condición de no positividad para la curvatura seccional); probamos explícitamente que la curvatura seccional es no positiva en la Proposición III.3.

Como un corolario de todas estas desigualdades, obtenemos

Teorema 3 *La variedad Σ_∞ con la distancia geodésica es un espacio métrico completo*

En la sección V, recordamos algunas definiciones y una serie de resultados sobre conjuntos cerrados y geodésicamente convexos, que son la categoría de subvariedades para los cuales el teorema de proyección (Teorema 5) se aplica. En particular, se tiene el siguiente resultado:

Resultado *Supongamos que $\mathfrak{m}$ es un subespacio cerrado del tangente tal que*

$$[X,[X,Y]] \in \mathfrak{m} \textit{ siempre que } X, Y \in \mathfrak{m}$$

Entonces $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ con la métrica inducida es una subvariedad cerrada y geodésicamente convexa.

Este resultado se debe principalmente a Mostow [Mos55] (aunque Pierre de la Harpe sugiere que su demostración se extiende trivialmente a operadores Hilbert-Schmidt en [Har72]). Es debido a este resultado (que por otra parte caracteriza todas las subvariedades que pasan por 1 con esta propiedad de convexidad) que uno está en condiciones de afirmar que estos conjuntos convexos existen en abundancia (ver el Corolario V.11).
En particular, cualquier subálgebra (cerrada) de los operadores Hilbert-Schmidt da lugar a un ejemplo de subvariedad convexa. Otros ejemplos se obtienen considerando el conjunto de operadores que actúan en un subespacio determinado de H. En la sección V.2.1 damos una lista extensa (pero por supuesto no completa) de conjuntos convexos.

En la sección V.3 adoptamos el punto de vista de Élie Cartan, y estudiamos las subvariedades convexas de M como espacios simétricos homogéneos para la acción de un grupo de operadores inversibles G_M. Este grupo es el grupo de Lie más pequeño que contiene a M (dentro del grupo de los operadores inversibles de la forma $\lambda + a$ con a Hilbert-Schmidt y λ un escalar). El resultado principal de esta sección es el Teorema IV de más abajo (V.29). A lo largo de todo el manuscrito, usamos $GL(\mathcal{B})$ para denotar el grupo de elementos inversibles de un álgebra de Banach $\mathcal{B}$; asimismo escribimos $\mathcal{U}(\mathcal{B})$ para denotar el gupor de elementos unitarios. La notación $I_0(M)$ se utilizará para referirnos a la componente arcoconexa de la identidad del grupo de isometrías de M.

Teorema 4 *Si $M = \exp(\mathfrak{m})$ es convexa y cerrada, y $G_M \subset GL(\mathcal{H}_\mathbb{C})$ es el subgrupo de Lie con álgebra de Lie $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m}, \mathfrak{m}]}$, entonces*

(a) $P(G_M) = M$, *con lo cual M es un espacio homogéneo para G_M.*

(b) Para cada $g = \mid g \mid u_g$ (su descomposición polar de Cauchy) en G_M, se tiene $\mid g \mid = \sqrt{gg^*} \in M \subset G_M$, y además $u_g \in K \subset G_M$ donde K es el subgrupo de Lie de isotropía $K = \{g \in G_M : gg^* = 1\}$ con álgebra de Lie $\mathfrak{k} = \overline{[\mathfrak{m}, \mathfrak{m}]}$. En particular, G_M tiene una descomposición polar

$$G_M \simeq M \times K = P(G_M) \times U(G_M)$$

(c) $M = P(G_M) \simeq G_M / K$

(d) M tiene curvatura seccional no positiva.

(e) Para $g \in G_M$, consideremos $I_g(r) = grg^*$. Entonces $I : G_M \to I_0(M)$.

(f) Tomemos $p, q \in M$, y definamos $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}} q p^{-\frac{1}{2}})^{\frac{1}{2}} p^{-\frac{1}{2}} \in G_M$. Entonces I_g es una isometría en $I_0(M)$ que mapea p en q, es decir G_M actúa transitivamente e isométricamente en M.

En la sección VI enunciamos y demostramos el teorema principal sobre existencia y unicidad de la geodésica minimizante entre un punto y un convexo (VI.9):

Teorema 5 *Sea M una subvariedad cerrada y geodésicamente convexa de Σ_∞. Entonces para cada punto $p \in \Sigma_\infty$, existe una única geodésica normal γ_p que une p con M tal que*

$$\operatorname{Long}(\gamma) = \operatorname{dist}(p, M)$$

Es más, esta geodésica es ortogonal a M*, y si* $\Pi_M : \Sigma_\infty \to M$ *es la función que asigna a* p *el otro extremo de* γ_p *en* M*, entonces* Π_M *es una función contractiva para la distancia geodésica.*

Como un corolario directo (VI.13), obtenemos una descomposición polar para elementos inversibles relativa a una subvariedad convexa dada. Esta descomposición se asemeja fuertemente a la descomposición de Iwasawa (ver [Hel62]) para grupos de Lie:

Teorema 6 *Supongamos que* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *es una subvariedad cerrada y convexa. Entonces para todo* $g \in GL(\mathcal{H}_\mathbb{C})$ *existe una factorización única de la forma*

$$g = pe^{\nu}u, \text{ donde } p \in M, \nu \in \mathfrak{m}^\perp, u \in \mathcal{U}(\mathcal{H}_\mathbb{C}) \text{ es un operador unitario.}$$

La función $g \mapsto (p, e^{\nu}, u)$ *es una biyección analítica que da un isomorfismo*

$$GL(\mathcal{H}_\mathbb{C}) \simeq M \times \exp(\mathfrak{m}^\perp) \times \mathcal{U}(\mathcal{H}_\mathbb{C})$$

La sección VII trata algunas aplicaciones del teorema de factorización. Cuando éste se aplica a la variedad de operadores diagonales (VII.2), se obtiene una descomposición de los operadores positivos como un producto de un operador diagonal positivo y la exponencial de un operador autoadjunto codiagonal:

Teorema 7 *Tomemos un operador* A *(Hilbert-Schmidt y autoadjunto) tal que* $1 + A > 0$*. Entonces existen: un operador* D *estrictamente positivo y diagonal (perturbación de un múltiplo de la identidad por un operador de Hilbert-Schmidt) y un operador autoadjunto Hilbert-Schmidt* V *(de diagonal nula) tales que vale la siguiente factorización:*

$$1 + A = D\, e^V D$$

Es más, D *y* V *son los únicos con las propiedades mencionadas que hacen válida esta factorización, y la función que asigna* $1 + A \mapsto (D, V)$ *es analítica real.*

Una aplicación directa (VII.4) de este último teorema nos da una demostración alternativa de una ya conocida factorización para matrices (Teorema 3 del artículo [Mos55] por G.D. Mostow, ver también el Teorema 1 del artículo [CPR91] por G. Corach, H. Porta y L. Recht)

Resultado *Tomemos una matriz positiva e inversible* $A \in M_n^+$. *Entonces existen* únicas *matrices* $D, V \in M_n$ *tales que* D *es diagonal y estrictamente positiva,* V *es autoadjunta y tiene diagonal nula, y vale la siguiente fórmula:*

$$A = D\, e^V D$$

Las funciones $A \mapsto D$ *y* $A \mapsto V$ *son analíticas reales.*

Un corolario particularmente agradable (VII.3) de los Teoremas 6 y 7 es el siguiente. Esta descomposición es comparable a la descomposición de Iwasawa para grupos de Lie de dimensión finita, ver [Hel62]:

Teorema 8 *Para todo* $g \in GL(\mathcal{H}_{\mathbb{C}})$, *existe una única factorización*

$$g = de^w u,$$

donde d *es un operador diagonal, positivo e inversible de* $\mathcal{H}_{\mathbb{C}}$, w *es un operador autodjunto con diagonal nula de* $\mathcal{H}_{\mathbb{C}}$, *y* u *es un operador unitario de* $\mathcal{H}_{\mathbb{C}}$.

En la sección VIII discutimos una foliación de codimensión uno del espacio total dada por hojas cerradas y totalmente geodésicas. El espacio tangente de cada hoja es el conjunto de todos los operadores Hilbert-Schmidt autoadjuntos (conjunto que de aquí en más abreviaremos HS^h). Las hojas también resultan paralelas en el sentido siguiente: la distancia entre dos hojas es constante y está dada por la longitud de cualquier geodésica que sea simultáneamente ortogonal a ambas (VIII.4).

Probamos que la curvatura seccional es trivial para 2-planos verticales con respecto a la foliación (Proposición VIII.5), y también (VIII.6) que Σ_∞ es isométrica al producto directo de dos subvariedades completas y totalmente geodésicas:

$\Sigma_1 = \exp(\mathrm{HS}^h)$ y Λ (la subvarieda de escalares positivos). Es decir, hay un isomorfismo Riemanniano

$$\Sigma_\infty \simeq \Sigma_1 \times \Lambda$$

La hoja Σ_1 contiene a la identidad, y su espacio tangente es el conjunto de operadores Hilbert-Schmidt autoadjuntos, así que toda vez que sea posible trabajamos dentro de Σ_1 para evitar la manipulación innecesaria de escalares.

La versión intrínseca del teorema de factorización toma una forma más simple en Σ_1; basándonos en resultados de la sección V, se lee:

Teorema 9 *Supongamos que $\mathfrak{m} \subset \mathrm{HS}^h$ es un subespacio cerrado tal que*

$$[x,[x,y]] \in \mathfrak{m} \qquad \textit{para todo } \; x, y \in \mathfrak{m}$$

Entonces para cualquier $a \in \mathrm{HS}^h$ existe una única descomposición de la forma

$$e^a = e^x e^v e^x$$

donde $x \in \mathfrak{m}$, y $v \in \mathrm{HS}^h$ verifica $\operatorname{tr}(vz) = 0$ para todo $z \in \mathfrak{m}$. El operador x es el único minimizante en $\mathfrak{m}$ de la aplicación

$$y \mapsto \operatorname{tr}\left(\ln^2(e^{a/2}e^{-y}e^{a/2})\right)$$

No podemos dejar de señalar que este resultado es un análogo en dimensión infinita de un teorema de G.D. Mostow para matrices [Mos55].

En la sección VIII.2, construimos una inclusión topológica del espacio M_n^+ de matrices positivas e inversibles (de $n \times n$) en Σ_1 (esta inclusión también puede hallarse -aunque con otro formalismo- en [AV03]). La inclusión resulta cerrada y geodésicamente convexa; en (VIII.10) sólo consideramos elementos $p \in \Sigma_1$ y mostramos otra aplicación del teorema de factorización:

Teorema 10 *Si identificamos M_n^+ con el primer bloque de la representación matricial de los operadores Hilbert-Schmidt (en cualquier base ortonormal prefijada), entonces para todo operador positivo e inversible e^b (b es Hilbert-*

Schmidt y autoadjunto) existe una única factorización de la forma

$$e^{b} = \begin{pmatrix} e^{A} & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} e^{-A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{O}_{n\times n} & Y^{*} \\ Y & X \end{pmatrix} \right\}$$

En la sección IX, bosquejamos la demostración de la inclusión de las variedades simétricas del tipo no compacto en M_n^+ (este resultado se debe a Patrick Eberlein, ver [Eb85]). Esta inclusión junto con la inclusión isométrica de M_n^+ en Σ_1 (sección VIII.2) nos da el siguiente resultado:

Teorema 11 *Para cualquier variedad simétrica M del tipo no compacto existe una inclusión de M en Σ_1 que es un difeomorfismo entre M y una variedad cerrada y geodésicamente convexa de Σ_1. Esta inclusión preserva el tensor métrico en el siguiente sentido: el pull-back en M del producto interno de Σ_1 resulta ser un múltiplo constante y positivo del producto interno de M, en cada componente irreducible de de Rham. Identificando M con su imagen, M factoriza Σ_∞* via *la aplicación contractiva Π_M.*

En la sección X, para un operador $e^{a} \in \Sigma_1$ fijo, consideramos la acción del grupo unitario de $L(H)$ mediante la conjugación $g \mapsto g e^{a} g^{*}$; también consideramos la acción (mediante la misma conjugación) del grupo de operadores unitarios que son perturbaciones escalares de operadores Hilbert-Schmidt. La órbita para los dos grupos no es necesariamente el mismo conjunto (Ejemplo X.5). Recordemos que utilizamos $\mathcal{U}(\mathcal{B})$ para denotar el grupo de unitarios del álgebra de Banach involutiva $\mathcal{B}$.

Discutimos condiciones necesarias y suficientes para que la órbita Ω tenga una estructura analítica de subvariedad (aquí Ω denota la órbita para alguno de los dos grupos mencionados). Una respuesta parcial al problema está dada por el Teorema 12 (X.6) y el Teorema 13 (X.3), que afirman:

Teorema 12 *Si la C^{*}-álgebra generada por a y 1 es de dimensión finita, entonces la órbita de e^{a} para la acción del grupo de unitarios Hilbert-Schmidt admite una estructura analítica de subvariedad de Σ_∞.*

Teorema 13 *La órbita de* e^a *para la acción del grupo de unitarios de* $L(H)$ *admite una estructura analítica de subvariedad de* Σ_∞ *si y sólo si la* C^*-*álgebra generada por* a *y por* 1 *es de dimensión finita.*

Los resultados de la sección X.2 están vinculados con el estudio de las geodésicas de Ω para las diferentes métricas Riemannianas que este conjunto admite. En la sección X.2.1 miramos la órbita como subespacio del espacio Euclídeo de los operadores Hilbert-Schmidt; mostramos que para cualquier h autoadjunto, la curva

$$\gamma(t) = e^{ith} e^a e^{-ith}$$

es una geodésica de Ω siempre que $e^a - 1$ es un proyector ortogonal y h es codiagonal en la representación asociada a este proyector (X.11).
También demostramos que, para cualquier e^a, estas curvas son las geodésicas usuales de Σ_1 sólo en el caso trivial, es decir, cuando se reducen a un punto (esta es la Proposición X.9); en particular, cuando la órbita es considerada como subvariedad del espacio Euclídeo de los operadores Hilbert-Schmidt, se deduce que la misma no es geodésica en ninguno de sus puntos (en el caso en que $e^a - 1$ es un proyector).

En la sección X.2.2 consideramos la órbita de un operador e^a como subvariedad Riemanniana $\Omega \subset \Sigma_1$; el resultado principal es el Teorema XII que enunciamos a continuación (X.14). A lo largo de este manuscrito, usaremos $[\ ,\ ]$ para denotar el conmutador usual de operadores. Estos resultados son válidos para la órbita por la acción de cualquiera de los dos grupos $\mathcal{U}(L(H))$ o $\mathcal{U}(\mathcal{H}_\mathbb{C})$, ya que las dos acciones inducen la misma subvariedad de Σ_1 cuando $e^a - 1$ es un proyector (esto se demuestra en el Lema X.7):

Teorema 14 *Supongamos que* $e^a = 1 + A$ *con* A *un proyector ortogonal, y* $\Omega \subset \Sigma_1$ *es la órbita unitaria de* e^a*. Entonces*

1. Ω *es una subvariedad Riemannian de* Σ_1

2. $T_p\Omega = \{i[x,p] : x \in HS^h\}$ *y* $T_p\Omega^\perp = \{x \in HS^h : [x,p] = 0\}$

3. *La acción del grupo unitario es isométrica en* Ω*, es decir*

$$\mathrm{dist}^{\Omega}(upu^*, uqu^*) = \mathrm{dist}^{\Omega}(p, q)$$

para cualquier operador unitario $u \in L(H)$.

4. *Para cualquier* $v = i[x, p] \in T_p\Omega$*, la exponencial de la variedad está dada por*

$$\mathrm{Exp}_p^{\Omega}(v) = e^{ighg^*} p\, e^{-ighg^*}$$

donde $p = ge^a g^*$ *y* h *es la parte codiagonal de* g^*xg *(en la representación matricial asociada al proyector* A*, ver la Proposición X.11). En particular, la exponencial está definida en todo el tangente de la órbita.*

5. *Si* $p = ge^a g^*$, $q = we^a w^*$*, y* h *es un operador codiagonal, autoadjunto tal que* $w^* ge^{ih}$ *conmuta con* e^a*, entonces la curva*

$$\gamma(t) = e^{itghg^*} p e^{-itghg^*}$$

es una geodésica de $\Omega \subset \Sigma_1$*, que une* p *con* q.

6. *Si tomamos* $h \in HS^h$*, entonces* $L(\gamma) = \frac{\sqrt{2}}{2} \|h\|_2$

7. *La exponencial* $\mathrm{Exp}_p^{\Omega} : T_p\Omega \to \Omega$ *es sobreyectiva.*

En la última sección concluimos el manuscrito con algunas preguntas pendientes y comentarios.

Precedentes

- En su tesis doctoral de 1955, "Infinite Dimensional Manifolds and Morse Theory" [McA65], J. McAlpin estableció los fundamentos de la geometría Riemanniana en dimensión infinita. Entre otros resultados relevantes, probó que una variedad de Hilbert de curvatura seccional no positiva tiene una exponencial Riemanniana que es un isomorfismo entre el tangente y la variedad para cualquier punto $p \in M$. También probó que la exponencial tiene diferencial expansiva en cualquier punto: este resultado está íntimamente conectado con la convexidad de los campos de Jacobi y de la distancia geodésica, dos hechos que juegan un papel central en las construcciones de este manuscrito.

- La convexidad de la distancia geodésica y los campos de Jacobi en variedades modeladas por álgebras de operadores es objeto de estudio en varios trabajos de G. Corach, H. Porta y L. Recht [CPR92], [CPR94]. La convexidad de la distancia en el contexto de operadores acotados positivos puede pensarse como una reinterpretación de la clásica desigualdad de Segal para operadores en $L(H)$: $\|e^{x+y}\| \le \|e^{x/2}e^{y}e^{x/2}\|$.

- Basándose en la construcción clásica de una estructura Riemanniana para el conjunto M_n^+ de matrices positivas e inversibles (la primera publicación sobre el particular parece ser el artículo [Mos55] de G.D. Mostow), E. Andruchow y A. Varela muestran en un artículo reciente [AV03] como los operadores Hilbert-Schmidt con el producto interno dado por la traza proveen un marco conveniente para la construcción de una variedad Hilbertiana Σ_∞ que resulta ser una variedad de Hadamard en el sentido clásico (Riemanniano). Este manuscrito está basado en la mencionada construcción. Ver tambien [Har72] por P. de la Harpe.

- El teorema de factorización de este manuscrito tiene precedentes obvios en la descomposición polar de Cauchy para operadores, pero también cabe mencionar el artículo [CPR91] por Corach *et al.* (ver también [Mos55]). En un artículo de H. Porta y L. Recht [PR94] se demuestra un resultado de descomposición similar pero en el contexto de C^*-álgebras y esperanzas condicionales.

- En [Eb85], Patrick Eberlein muestra como cualquier espacio simétrico M del tipo no compacto puede ser incluido topológicamente en $P(\mathfrak{g})$ (los operadores positivos inversibles que actúan en el álgebra del Lie del grupo de isometrías de M). Esta inclusión da un conjunto cerrado y geodésicamente convexo, y resulta una isometría en el siguiente sentido: si g^* es el pull-back de la métrica de $P(\mathfrak{g})$, entonces g^* es un múltiplo constante de la métrica de M en cada componente irreducible de de Rham de M.

- La conexión entre el espectro de un operador, y la existencia de una estructura homogénea reductiva para la órbita del operador en cuestión ha sido objeto de estudio a través de los años para diversos autores, incluyendo Andruchow, Deckard, Fialkow, Raeburn y Stojanoff en [DF79], [AFHS90], [Rae77], [AS89], [AS91], [Fial79] and [AS94]. En particular, [DF79] parece ser el primer estudio sistemático del tema.

- La geometría de los espacios homogéneos reductivos que aparecen naturalmente en álgebras de Banach ha sido extensamente estudiada, y mencionaremos sólo algunos artículos: Corach, Porta y Recht estudian el espacio de idempotentes en ([PR87a], [PR87b], [CPR93b], [CPR90b]), el conjunto de operadores positivos inversibles es tratado en [CPR92], [CPR93a], [AV03], y el espacio de elementos relativamente regulares en [CPR90a].

Banderas generalizadas (grassmanianas, etc) son también estudiadas por Andruchow, Durán, Mata-Lorenzo, Recht, Stojanoff, y Wilkins en [ARS92], [DMR00], [DMR04a], [DMR04b], [Wilk90]. Las isometrías parciales se estudian en [AC04], la esfera de un módulo de Hilbert se trata en [ACS99], y los pesos en álgebras de von Neumann algebras han sido estudiados por Andruchow y Varela en [AV99].

Geodesic Convexity

Symmetric Spaces

and

Hilbert-Schmidt Operators

Gabriel Larotonda

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ABSTRACT: A natural Riemannian structure is introduced on the set of positive invertible (unitized) Hilbert-Schmidt operators, in order to obtain several decomposition theorems by means of geodesically convex submanifolds. We also give an intrinsic (algebraic) characterization of such submanifolds, and we study the group of isometries. We show that any symmetric space of the noncompact type can be isometrically embedded in this manifold. We include a final section devoted to the study of the unitary orbits of a fixed operator and the diverse geometries that arise from endowing this orbit with different Riemannian metrics.

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To my father,
Angel Rafael Larotonda
1939-2005

Allá al fondo está la muerte, pero no tenga miedo. Sujete el reloj con una mano, tome con dos dedos la llave de la cuerda, remóntela suavemente. Ahora se abre otro plazo, los árboles despliegan sus hojas, las barcas corren regatas, el tiempo como un abanico se va llenando de sí mismo y de él brotan el aire, las brisas de la tierra, la sombra de una mujer, el perfume del pan.

¿Qué más quiere, qué más quiere? Átelo pronto a su muñeca, déjelo latir en libertad, imítelo anhelante. El miedo herrumbra las áncoras, cada cosa que pudo alcanzarse y fue olvidada va corroyendo las venas del reloj, gangrenando la fría sangre de sus rubíes. Y allá en el fondo está la muerte si no corremos y llegamos antes y comprendemos que ya no importa.

Julio Cortázar, "Instrucciones para dar cuerda al reloj"

I Introduction

I.1 Hadamard manifolds

A Hadamard manifold is a Riemannian manifold which is simply connected, complete, and has nonpositive sectional curvature. From the topological viewpoint, it is a very simple object.

However, (see [Eb96]) for any manifold M of nonpositive sectional curvature, the higher homotopy groups ($\pi_k(M)$, $k \geq 2$) vanish, and M can be expressed as a quotient space of a Hadamard manifold (the universal covering of M) and a suitable deckgroup of isometries of the covering which is isomorphic to $\pi_1(M)$.

The geometry of nonpositevely curved spaces is indeed rich and has applications in many other branches of mathematics, such as harmonic maps ([Cor92], [GS92], [KS93], [MSY93]), 3-manifolds and Kleinian groups ([MS84], [Gab92], [Can93], [CJ94], [Min94], [McM96], [Otal96], [Gab97], [Otal98], [Min99], [Kap01], [GMT03]), structure theory and rigidity ([Ball85], [BBE85], [BBS85], [BS97], [EH90], [BB95], [Lee97]), high dimensional topology ([FH81], [FJ93], [CGM90]), hyperbolic groups and quasi conformal geometry ([Gro87], [Pan89], [BM91], [RS94], [Sela95], [Bow98a], [Bow98b], [BP99], [BP00] and [HK98]), geometric and combinatorial group theory ([Gro87], [DJ91], [Sch95], [CD95], [BM97], [KL97a], [KL97b] and [Esk98]) and dynamics ([Cro90], [Otal90], [BCS95], [BFK98]).

The classical treatises [Hel62] by Sigurdur Helgason and [BGS85] by Wallman *et al.*, the introduction to the geometry of spaces of the noncompact type by Patrick Eberlein [Eb96], or the expository survey by the same author [Eb89] collect many of the relevant facts concerning the geometry of these objects, such as the Law of Cosines, orthogonal projections, convexity of the distance function, the construction of the boundary space, and rank rigidity theorems.

Let's focus briefly on six basic results which are valid (see [Hel62]) in any Hadamard manifold M of finite dimension:

1. The exponential map $\mathrm{Exp}_p : T_pM \to M$ is a diffeomorphism for each $p \in M$.
2. For each pair $p, q \in M$ there exists a unique normal (*i.e.* unit speed), minimizing geodesic from p to q.
3. For any geodesic triangle in M whose sides are geodesics of length a, b and c, we have the Hyperbolic Law of Cosines, which states:
$$c^2 \geq a^2 + b^2 - 2ab\cos(\theta), \text{ where } \theta \text{ is the angle opposite to } c$$
4. The sum of the interior angles of any such triangle is at most π.
5. For any pair of geodesics α, β in M, the function
$$f(t) = \mathrm{dist}(\alpha(t), \beta(t))$$
is a real convex function.
6. Let C be a convex closed subset of M. Then for each $p \in M$ there exists a unique point $\Pi_C(p) \in C$ such that
$$\mathrm{dist}\,(p, \Pi_C(p)) \leq \mathrm{dist}(p, q) \text{ for any } q \in C$$

In the Riemannian context, the point $\Pi_C(p)$ is called the *foot of the perpendicular* from p to C.

The notions of completeness as metric space and completeness in the geodesic sense are intimately related by Hopf-Rinow's theorem. Since this theorem is false in infinite dimensions (see [Atkin75], [Atkin97]), compactness of neighbourhoods of M seems to be relevant for these results to hold true. However, statements 1 through 6 are known to be valid in the setting of nonpositively curved spaces (which are metric spaces where some geodesic triangle comparison inequality is valid). In particular, the proof of existence of a unique distance-realizing point for any closed convex set (without assuming local compactness of neighbourhoods) can be found in [Jost97].

We will go in an alternate direction, in order to extend these results to a manifold Σ_∞ which is locally isomorphic to an infinite dimensional Hilbert space (in fact, the real part of a Banach algebra $\mathcal{B}$). The manifold Σ_∞ is simply connected, complete, and has nonpositive sectional curvature; moreover, $\Sigma_\infty = GL^+(\mathcal{B})$ is a symmetric space in the usual Riemannian sense. All the tools of the Riemannian geometry will be at hand, and we will be able to explore relationships between the Banach algebra and the geometry of the manifold.

For instance, we will prove that the unique minimizing geodesic that realices distance between a point and a convex set must be orthogonal to that set, obtaining in this way a decomposition theorem for operators, with many immediate applications.

The first result of the list will be apparent from the definition of Σ_∞; to prove the second, the third, the fourth and the fifth we will have to collect some facts from the existing literature of geometry on spaces of operators.

The space Σ_∞ is symmetric and nonpositively curved, and universal in this category in the sense that every symmetric space of the noncompact type can be (almost) isometrically embedded as a geodesically convex, closed submanifold.

Though we will not need it along this manuscript, it should be noted that the general classification theory of L^*-algebras (see [Sch60] and [Sch61] by J.R.Schue, [CGM90] by Mira, Martin and González, or [Neh93] by E. Neher) provides a general abstract framework for this manifold and its convex submanifods: the real part of any L^*-algebra can be naturally embedded as a convex closed submanifold of Σ_∞.

I.2 The main results

A few words about notation: we will use greek characters $\alpha, \beta, \delta, \ldots$ to denote real and complex numbers, and capital characters $\Sigma, \Lambda, \Delta, \Omega, \ldots$ to denote manifolds. The first characters of the alphabet $a, b, c, d, \cdots$ will be reserved for Hilbert-Schmidt operators and as usual, $p, q, r, s, \ldots$ will be used for points (in Σ_∞); sometimes we will use capital letters $A, B, C, D, \ldots$ to stress the fact that this points are positive invertible operators in the unitized Banach algebra of Hilbert-Schmidt operators. The capital letters $X, Y, Z, W, \ldots$ will be used sometimes to denote selfadjoint operators (tangent vectors) in the mentioned Banach algebra. German characters $\mathfrak{a}, \mathfrak{k}, \mathfrak{m}, \mathfrak{p}, \ldots$ will be used as customary in Lie group theory to denote Lie algebras (or to denote certain subspaces of Lie algebras). Throughout, Exp_p will denote the exponential of the Riemannian manifold at the point p, and we will use exp instead of Exp_1, which is the usual exponential of operators.

Now we outline the organization and main results of this work (previous results are mentioned as such, and new results are **Theorem 1**, **Theorem 2**, ... trough **Theorem 14**):

In section II, we introduce some notation and recall a few results we will need for the construction of a Hilbert manifold of infinite dimension Σ_∞, which is complete, simply connected and has nonpositive sectional curvature.
The ambient space for most of the computations is the Banach space with trace inner product $\mathcal{H}_\mathbb{R} = \{\lambda + a\}$, where λ is a real number and a is a selfadjoint Hilbert-Schmidt operator acting on a separable Hilbert space H. As a set, $\Sigma_\infty := \exp(\mathcal{H}_\mathbb{R})$. The exponential is an open mapping so Σ_∞ is open in $\mathcal{H}_\mathbb{R}$ (Proposition II.3).

The metric we introduce (II.4) resembles the metric of the positive invertible matrices when they are regarded as symmetric space:

$$\langle X, Y \rangle_p = \left\langle Yp^{-1}, p^{-1}X \right\rangle_2 \qquad \text{for } p \in \Sigma_\infty \text{ and } X, Y \in \mathcal{H}_\mathbb{R},$$

where $<\alpha+a,\beta+b>_2=\alpha\overline{\beta}+tr(b^*a)$. With this metric the manifold Σ_∞ has covariant derivative (II.5) given by

$$\nabla_X Y = X(Y) - \frac{1}{2}\left(Xp^{-1}Y + Yp^{-1}X\right)$$

where $X(Y)$ denotes derivation of the vector field Y in the direction of X (performed in the ambient space $\mathcal{H}_{\mathbb{R}}$); the sectional curvature (4) is given by

$$\mathcal{R}_p(X,Y)Z = -\frac{1}{4}\,p\left[\left[p^{-1}X, p^{-1}Y\right], p^{-1}Z\right]$$

where $[x,y]=xy-yx$ denotes the standard commutator of operators in $L(H)$

In Theorem I (IV.9) we prove that the unique geodesic for the connection introduced is given by an explicit formula which involves only the starting and endpoints of the curve:

Theorem 1 *Set* $\|X\|_p = \|p^{-\frac{1}{2}}Xp^{-\frac{1}{2}}\|_2$, *and* $L(\alpha)=\int_0^1 \|\alpha(t)\|_{\dot{\alpha}(t)}\,dt$. *If*

$$\mathrm{dist}(p,q) = \inf\{L(\alpha) : \alpha \subset \Sigma_\infty,\ \alpha \textit{ is smooth },\ \alpha(0)=p, \alpha(1)=q\}$$

then the curve $\gamma_{pq}(t) = p^{\frac{1}{2}}\left(p^{-\frac{1}{2}}qp^{-\frac{1}{2}}\right)^t p^{\frac{1}{2}}$ *is the shortest path joining* p *to* q *in* Σ_∞*; moreover,*

$$\mathrm{dist}(p,q) = L(\gamma_{pq}) = \|\ln(p^{\frac{1}{2}}q^{-1}p^{\frac{1}{2}})\|_2 \equiv \|\dot{\gamma_{pq}}(t)\|_{\gamma_{pq}(t)}$$

We prove (III.5) that Jacobi fields along geodesics γ are convex (in the sense that the real map $t\mapsto \|J(t)\|_{\gamma(t)} = \langle J(t), J(t)\rangle_{\gamma(t)}$ is convex), and as a corollary (III.6), we get

Theorem 2 *The real map* $t \mapsto \mathrm{dist}(\gamma(t),\delta(t))$ *is convex for any pair of geodesics* $\gamma,\delta\in\Sigma_\infty$.

Since we know the formula for the geodesics, we also know that the Riemannian exponential $\mathrm{Exp}_p : T_p\Sigma_\infty \to \Sigma_\infty$ is given by

$$\mathrm{Exp}_p(v) = p^{\frac{1}{2}}\exp\left(p^{-\frac{1}{2}}vp^{-\frac{1}{2}}\right)p^{\frac{1}{2}} = pe^{p^{-1}v}$$

This map is a C^{ω} diffeomorphism onto Σ_∞ for each p (IV.6), and the same is true for the restriction of Exp_p to the tangent bundle of any geodesically convex, closed submanifold M of Σ_∞.

A corollary for all these inequalities is
Theorem 3 *The manifold Σ_∞ with the geodesic distance is a complete metric space*

We also prove (IV.11) that the sum of the inner angles of any geodesic triangle in Σ_∞ is less or equal than π, which is nonpositive constrain on sectional curvature; we prove explicitly that sectional curvature is nonpositive in Proposition III.3.

In section V, we recall some definitions and facts about closed, geodesically convex subsets, which are the submanifolds where the projection theorem (Theorem 5) applies. In particular, we have the following result

Result *Assume $\mathfrak{m}$ is a closed subspace such that $[X,[X,Y]] \in \mathfrak{m}$ whenever $X, Y \in \mathfrak{m}$. Then $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ with the induced metric is a closed, geodesically convex submanifold.*

This result is mainly due to Mostow [Mos55] (though Pierre de la Harpe sketches the proof for Hilbert-Schmidt operators in [Har72]), and it shows that there are plenty of this sets (see Corollary V.11).

In particular, any closed abelian subalgebra of Hilbert-Schmidt operators provides an example of a convex submanifold. Other examples are provided by operators acting on fixed subspaces of H. In section V.2.1 we give a list of convex sets; this list is exhaustive but by no means complete.

In section V.3 we take Élie Cartan's viewpoint, and study convex submanifolds M as homogeneous symmetric spaces for the action of a convenient group G_M. This group is the smaller Lie group -inside the invertible operators of the Banach algebra- containing M. The main result is Theorem IV below (V.29). Throughout, $GL(\mathcal{B})$ stands for the group of invertible elements in the Banach algebra $\mathcal{B}$ and $I_0(M)$ for the connected component of the identity of the group of isometries of M :

Theorem 4 *If* $M = \exp(\mathfrak{m})$ *is convex and closed, and* $G_M \subset GL(\mathcal{H}_{\mathbb{C}})$ *is the Lie subgroup with Lie algebra* $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m}, \mathfrak{m}]}$, *then*

(a) $P(G_M) = M$, *so* M *is a homogeneous space for* G_M.

(b) For any $g = |\, g \,|\, u_g$ *(Cauchy polar decomposition) in* G_M, *we have* $|\, g \,| = \sqrt{gg^*} \in M \subset G_M$, *and also* $u_g \in K \subset G_M$ *where* K *is the isotropy Lie subgroup* $K = \{g \in G_M : gg^* = 1\}$ *with Lie algebra* $\mathfrak{k} = \overline{[\mathfrak{m}, \mathfrak{m}]}$. *In particular,* G_M *has a polar decomposition*

$$G_M \simeq M \times K = P(G_M) \times U_{G_M}$$

(c) $M = P(G_M) \simeq G_M / K$.

(d) M *has nonpositive sectional curvature.*

(e) For $g \in G_M$, *consider* $I_g(r) = grg^*$. *Then* $I : G_M \to I_0(M)$.

(f) Take $p, q \in M$, *and set* $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}} q p^{-\frac{1}{2}})^{\frac{1}{2}} p^{-\frac{1}{2}} \in G_M$. *Then* I_g *is an isometry in* $I_0(M)$ *which sends* p *to* q, *namely* G_M *acts transitively and isometrically on* M.

In section VI we state and prove the main result about uniqueness and existence of the minimizing geodesic (VI.9):

Theorem 5 *Let* M *be a geodesically convex, closed submanifold of* Σ_∞. *Then for every point* $p \in \Sigma_\infty$, *there is a unique normal geodesic* γ_p *joining* p *to* M *such that* $L(\gamma_p) = \mathrm{dist}\,(p, M)$.
This geodesic is orthogonal to M, *and if* $\Pi_M : \Sigma_\infty \to M$ *is the map that assigns to* p *the end-point of* γ_p, *then* Π_M *is a contraction for the geodesic distance.*

As a corollary (VI.13), we obtain a polar descomposition relative to any fixed convex submanifold. This decomposition resembles the Iwasawa decomposition of (finite dimensional) Lie groups, see [Hel62]:

Theorem 6 *Assume* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *is a closed, convex submanifold. Then for any* $g \in GL(\mathcal{H}_\mathbb{C})$ *there exists a unique factorization of the form*

$$g = pe^v u, \text{ where } p \in M, v \in \mathfrak{m}^\perp, u \in \mathcal{U}(\mathcal{H}_\mathbb{C}) \text{ is a unitary operator.}$$

The map $g \mapsto (p, e^v, u)$ *is an analytic bijection which gives an isomorphism*

$$GL(\mathcal{H}_\mathbb{C}) \simeq M \times \exp(\mathfrak{m}^\perp) \times \mathcal{U}(\mathcal{H}_\mathbb{C})$$

Section VII deals with a main application of the factorization theorem. When applied to the manifold of diagonal operators (VII.2), provides a decomposition of positive operators as a product of a diagonal positive operator and the exponential of a codiagonal, selfadjoint operator:

Theorem 7 *Take any selfadjoint Hilbert-Schmidt operator* A *such that* $1 + A > 0$. *Then there exist a diagonal, strictly positive Hilbert-Schmidt perturbation of the identity* D *and a selfadjoint Hilbert-Schmidt operator* V *with null diagonal such that the following factorization holds:*

$$1 + A = D\, e^V D$$

Moreover, D *and* V *are unique and the map* $1 + A \mapsto (D, V)$ *is real analytic.*

A straightforward application (VII.4) of the last theorem is an alternative proof to an already known decomposition for matrices (Theorem 3 of the paper [Mos55] by G.D. Mostow, see also Theorem 1 of the paper [CPR91] by Corach, Porta and Recht)

Result *Fix a positive invertible matrix* $A \in M_n^+$. *Then there exist* unique *matrices* $D, V \in M_n$, *such that* D *is diagonal and strictly positive,* V *is symmetric and with null diagonal, and the following formula holds*

$$A = D\, e^V D$$

Moreover, the maps $A \mapsto D$ *and* $A \mapsto V$ *are real analytic.*

A nice corollary (VII.3) of Theorems 6 and 7 is the following; this decomposition is close to the Iwasawa decomposition [Hel62] of (finite dimensional) Lie groups:

Theorem 8 *For any* $g \in \mathrm{GL}(\mathcal{H}_{\mathbb{C}})$*, there is a unique factorization*

$$g = de^{w}u,$$

where d *is a positive invertible diagonal operator of* $\mathcal{H}_{\mathbb{C}}$*,* w *is a selfadjoint operator with null diagonal in* $\mathcal{H}_{\mathbb{C}}$ *and* u *is a unitary operator of* $\mathcal{H}_{\mathbb{C}}$*.*

In section VIII, we discuss a foliation of codimension one of the total space by totally geodesic, closed leaves. The tangent space of each leaf is the Banach space of selfadjoint Hilbert-Schmidt operators (shortly, HS^h). The leaves are also parallel in the sense that geodesics that have minimal length among those which join them are orhogonal to both of them (Proposition VIII.4).

We prove that sectional curvature is trivial along vertical 2-planes (Proposition VIII.5), and also (VIII.6) that Σ_∞ is isometric to the direct product of the complete and totally geodesic submanifolds $\Sigma_1 = \exp(\mathrm{HS}^h)$ and Λ (the positive scalars), *i.e.*

$$\Sigma_\infty \simeq \Sigma_1 \times \Lambda$$

The leaf Σ_1 contains the identity and its tangent space is the set of selfadjoint Hilbert-Schmidt operators, so whenever it is possible, we work inside Σ_1 to avoid the manipulation of scalars.

The intrinsic version of the decomposition theorem takes a simpler form; based upon the results of section V, it reads:

Theorem 9 *Assume* $\mathfrak{m} \subset \mathrm{HS}^h$ *is a closed subspace such that*

$$[x,[x,y]] \in \mathfrak{m} \qquad \textit{for any } \; x, y \in \mathfrak{m}$$

Then for any $a \in \mathrm{HS}^h$ *there is a unique decomposition of the form*

$$e^a = e^x e^v e^x$$

where $x \in \mathfrak{m}$*, and* $v \in \mathrm{HS}^h$ *is such that* $\mathrm{tr}(vz) = 0$ *for any* $z \in \mathfrak{m}$*. The operator* x *is the unique minimizer in* $\mathfrak{m}$ *of the map*

$$y \mapsto \mathrm{tr}\left(\ln^2(e^{a/2}e^{-y}e^{a/2})\right)$$

This is an infinite dimensional analogue of a theorem of G.D. Mostow for matrices [Mos55].

In section VIII.2, we embed the space M_n^+ of positive invertible $n \times n$ matrices in Σ_1 (this embedding can also be found in [AV03]). This embedding is closed and geodesically convex; in (VIII.10) we only consider elements $p \in \Sigma_1$ and show another application of the main theorem:

Theorem 10 *If we identify M_n^+ with the first block of the matrix representation of the Hilbert-Schmidt operators (in any fixed orthornomal basis), for any positive invertible operator e^b (b is Hilbert-Schmidt and selfadjoint) there is a unique factorization of the form*

$$e^b = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} e^{-A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{O}_{n\times n} & Y^* \\ Y & X \end{pmatrix} \right\}$$

In section IX, we sketch the proof of the inclusion of symmetric manifolds of the noncompact type in M_n^+ (this result is due to P. Eberlein, see [Eb85]). This result together with the embedding of M_n^+ in Σ_1 (see section VIII.2) gives us

Theorem 11 *For any symmetric manifold M of the noncompact type there is an embedding into Σ_1 which is a diffeomorphism betwen M and a closed, geodesically convex submanifold of Σ_1. This map preserves the metric tensor in the following sense: if we pull back the inner product of Σ_1 to M, then this inner product is a (positive) constant multiple of the inner product of M (on each irreducible de Rham factor of M). Assuming we identify M with its image, M factorizes Σ_∞* via *the contractive map Π_M.*

In section X, for fixed $e^a \in \Sigma_1$, we consider the action of the full unitary group of $L(H)$ by means of the conjugation $g \mapsto g e^a g^*$, and also the action of the unitaries that are Hilbert-Schmidt perturbations of a scalar multiple of the identity. The orbit acting with either group is not necessarily the same set (Example X.5). Throughout, $\mathcal{U}(\mathcal{B})$ stands for the unitary group of the involutive Banach algebra $\mathcal{B}$.

We discuss whether the orbit Ω can be given an analytic structure of submanifold; this question is partially answered by Theorem 12 (X.6) and Theorem 13 (X.3), which state:

Theorem 12 *If the* C^**-algebra generated by* a *and* 1 *is finite dimensional, then the orbit of* e^a *with the action of the Hilbert-Schmidt unitaries can be given an analytic submanifold structure.*

Theorem 13 *The orbit of* e^a *under the action of the full unitary group of* $L(H)$ *can be given an analytic submanifold structure if and only if the* C^**-algebra generated by* a *and* 1 *is finite dimensional.*

The results of section X.2 are related to the study of the geodesics of the orbit Ω, with different Riemannian metrics. In section X.2.1 we immerse the orbit in the Euclidean space of Hilbert-Schmidt operators and we give it the induced metric: we show that for any selfadjoint h, the curve

$$\gamma(t) = e^{ith} e^a e^{-ith}$$

is a geodesic of the orbit whenever $e^a - 1$ is an orthogonal projector and h is codiagonal in the representation associated to $e^a - 1$. This is Proposition X.11.

We also show that, for any e^a, these curves are the usual geodesics of Σ_1 only if they are constant curves (this is Proposition X.9); in particular, when the orbit is regarded as a submanifold of the Euclidean space of Hilbert-Schmidt operators, this submanifold is not geodesic in any of its points whenever $e^a - 1$ is an orthogonal projector.

In section X.2.2 we take a peak at the geodesics of the orbit of e^a as a Riemannian submanifold $\Omega \subset \Sigma_1$; the main result is Theorem XII below (X.14). Throughout $[\ ,\ ]$ stands for the usual commutator of operators, and these results are valid for the action of any of the groups $\mathcal{U}(L(H))$ or $\mathcal{U}(\mathcal{H}_{\mathbb{C}})$ because they induce the same manifold in Σ_1 (this is proved in Lemma X.7):

Theorem 14 *Assume* $e^a = 1 + A$ *with* A *an orthogonal projector, and* $\Omega \subset \Sigma_1$ *is the unitary orbit of* e^a*. Then*

1. Ω *is a Riemannian submanifold of* Σ_1*.*

2. $T_p\Omega = \{i[x,p] : x \in HS^h\}$ *and* $T_p\Omega^{\perp} = \{x \in HS^h : [x,p] = 0\}$.

3. *The action of the unitary group is isometric, namely*

$$\mathrm{dist}^{\Omega}(upu^*, uqu^*) = \mathrm{dist}^{\Omega}(p,q)$$

for any unitary operator $u \in L(H)$.

4. *For any* $v = i[x,p] \in T_p\Omega$, *the exponential map is given by*

$$\mathrm{Exp}_p^{\Omega}(v) = e^{ighg^*} p\, e^{-ighg^*}$$

where $p = ge^a g^*$ *and* h *is the codiagonal part of* g^*xg *(in the matrix representation of Proposition X.11). In particular, the exponential map is defined in the whole tangent space.*

5. *If* $p = ge^a g^*$, $q = we^a w^*$, *and* h *is a selfadjoint, codiagonal operator such that* w^*ge^{ih} *commutes with* e^a, *then the curve*

$$\gamma(t) = e^{itghg^*} p e^{-itghg^*}$$

is a geodesic of $\Omega \subset \Sigma_1$, *which joins* p *to* q.

6. *If we assume that* $h \in HS^h$, *then* $L(\gamma) = \frac{\sqrt{2}}{2}\|h\|_2$

7. *The exponential map* $\mathrm{Exp}_p^{\Omega} : T_p\Omega \to \Omega$ *is surjective.*

In the last section we end the exposition with some open questions and remarks.

I.3 Precedents

- In his 1955 PhD. Thesis "Infinite Dimensional Manifolds and Morse Theory" [McA65], J. McAlpin set the foundations of the Riemannian geometry in infinite dimensions. Among other relevant results, he proved that a non-positively curved Hilbert manifold M has a Riemannian exponential which is an isomorphism for each $p \in M$, and that this exponential has an expansive differential at any point. This result is deeply connected with the convexity of the Jacobi fields and the geodesic distance, two facts that lay deeply in the core of this manuscript.

- Convexity of Jacobi fields and the geodesic distance (in manifolds modeled on $L(H)$) was studied by G. Corach, H. Porta and L. Recht [CPR92], [CPR94]. In this context of positive invertible operators of $L(H)$, convexity can be thought of as a reinterpretation of Segal's classical inequality for operators: $\|e^{x+y}\| \leq \|e^{x/2}e^{y}e^{x/2}\|$.

- Based upon the classical construction of a Riemannian structure on the set M_n^+ of positive invertible $n \times n$ matrices, (the first publication on the subject seems to be the paper [Mos55] by G.D. Mostow), E. Andruchow and A. Varela show in a recent paper [AV03] how the Hilbert-Schmidt operators HS with inner product given by the trace provide a convenient framework for the construction of a Hilbert manifold Σ_∞ modeled on the real Hilbert space HS^h, that turns out to be a Hadamard manifold in the classical (Riemannian) sense of the term. This manuscript is based upon the mentioned construction. See also [Har72] by P. de la Harpe.

- The decomposition theorems have obvious precedents in the polar decomposition of operators, but we should also mention the splitting of the positive set of a matrix algebra (see [Mos55] by Mostow, [CPR91] by Corach *et al.*) and the paper by Porta and Recht [PR94] which deals with C^*-algebras and conditional expectations.

- In [Eb85], Patrick Eberlein shows that any symmetric manifold M of the noncompact type can be embedded in $P(\mathfrak{g})$ (the positive invertible operators acting in the Lie algebra of the group of isometries of M) as a closed, geodesically convex submanifold. This embedding is isometric in the following sense: if g^* is the pull back of the metric of $P(\mathfrak{g})$ on M, then g^* is a constant multiple of the metric of M on each irreducible de Rham factor of M.

- The relationship between the spectrum of an operator, and the existence of a homogeneous reductive structure for the orbit of that operator has been systematically studied through the years by diverse authors, including Andruchow, Deckard, Fialkow, Raeburn and Stojanoff in [DF79], [AFHS90], [Rae77], [AS89], [AS91], [Fial79] and [AS94]. In particular, [DF79] seems to be the first systematic approach to the subject.

- The geometry of the homogeneous reductive spaces which appear naturally in Banach and C^*-algebras has been extensively studied, and we should mention a few articles: Corach, Porta and Recht study the space of idempotents in ([PR87a], [PR87b], [CPR93b], [CPR90b]), the set of positive invertible operators is treated in the papers [CPR92], [CPR93a] and [AV03], and the space of relatively regular elements in a Banach Algebra in [CPR90a]. Generalized flags (grassmanians, spectral measures, etc) are also studied by Andruchow, Durán, Mata-Lorenzo, Recht, Stojanoff, and Wilkins in [ARS92], [DMR00], [DMR04a], [DMR04b] and [Wilk90]. Partial isometries are studied in [AC04], the sphere of a Hilbert module is treated in [ACS99], and weights on von Neumann algebras are studied by Andruchow and Varela in [AV99].

II The Main Objects Involved

The framework of this manuscript is the von Neumann algebra $\mathsf{L}(H)$ of bounded operators acting on a complex, separable Hilbert space H.

II.1 Hilbert-Schmidt operators

Throughout, HS stands for the bilateral ideal of Hilbert-Schmidt operators of $\mathsf{L}(H)$: this is the ideal of compact operators with singular values lying in ℓ_2. Recall that HS is a Banach algebra without unit when given the norm

$$\|a\|_2 = 2\,\mathrm{tr}(a^*a)^{\frac{1}{2}} = 2\left(\sum_{i\geq 1}\langle ae_i, ae_i\rangle\right)^{\frac{1}{2}}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is any given orthonormal basis of H. The reader can find many of the statements we will use about trace operators and trace ideals in [Simon89].

In $L(H)$ we consider some Fredholm operators:

$$\mathcal{H}_{\mathbb{C}} = \{a+\lambda : \ a \in \mathrm{HS},\, \lambda \in \mathbb{C}\},$$

the complex linear subalgebra consisting of Hilbert-Schmidt perturbations of scalar multiples of the identity. This algebra is not norm closed, in fact, its closure in the uniform norm of $L(H)$ is the set of compact perturbations of scalar multiples of the identity.

There is a natural Hilbert space structure for this subspace, where scalar operators are orthogonal to Hilbert-Schmidt operators, which is given by the inner product

$$\langle a+\lambda, b+\beta\rangle_2 = 4\mathrm{tr}(ab^*) + \lambda\overline{\beta}$$

The algebra $\mathcal{H}_{\mathbb{C}}$ is complete with this norm, for the Hilbert-Schmidt operators are complete with the trace inner product.

Remark II.1. Another natural (but not quadratic) norm is given by the formula

$$\|a+\lambda\|_1 = 2\,\mathrm{tr}(a^*a)^{\frac{1}{2}} + |\,\lambda\,|$$

With this norm $\mathcal{H}_{\mathbb{C}}$ becomes a Banach algebra, that is

$$\|(a+\lambda)(b+\beta)\|_1 \le \|a+\lambda\|_1\|b+\beta\|_1$$

However, we will use the norm defined by the inner product, that is

$$\|a+\lambda\|_2 = \sqrt{\|a\|_2^2 + |\,\lambda\,|^2} = \left(4\mathrm{tr}(a^*a) + |\,\lambda\,|^2\right)^{\frac{1}{2}}$$

Both norms are equivalent, but $\|\cdot\|_2$ provides an Euclidean structure for $\mathcal{H}_{\mathbb{C}}$.

We also use the term Banach algebra for a normed algebra $\mathcal{B}$ where the sum and product are continuous operations; this is slightly different from the usual definition (see Rickart [Rick60] or Guichardet [Guich67]).

The model space that we are interested in is the real part of $\mathcal{H}_{\mathbb{C}}$:

$$\mathcal{H}_{\mathbb{R}} = \{a+\lambda : \ a^* = a, \ a \in \mathrm{HS}, \ \lambda \in \mathbb{R}\},$$

which inherits the structure of real Banach space, and with the same inner product, becomes a real Hilbert space.

Remark II.2. For this inner product, we have (by cyclicity of the trace)

$$\langle XY, Y^*X^* \rangle_2 = \langle YX, X^*Y^* \rangle_2 \qquad \text{for any } X, Y \in \mathcal{H}_{\mathbb{C}}, \text{ and also}$$

$$\langle ZX, YZ \rangle_2 = \langle XZ, ZY \rangle_2 \qquad \text{for } X, Y \in \mathcal{H}_{\mathbb{C}} \text{ and } Z \in \mathcal{H}_{\mathbb{R}}$$

We will use HS^h to denote the closed subspace of selfadjoint Hilbert-Schmidt operators. In $\mathcal{H}_{\mathbb{R}}$, consider the subset

$$\Sigma_\infty := \{A > 0, A \in \mathcal{H}_{\mathbb{R}}\}$$

This is the set of invertible operators $a+\lambda$ such that $\sigma(a+\lambda) \subset (0,+\infty)$, with a selfadjoint and Hilbert-Schmidt, $\lambda \in \mathbb{R}$.
Note that, since a is compact, then $0 \in \sigma(a)$, which forces $\lambda > 0$ because

$$\sigma(a+\lambda) \subset (0,+\infty) \iff \sigma(a) \subset (-\lambda,+\infty)$$

Our main reference for standard facts about functional analysis, operator algebras and functional calculus is the four volume treatise of Functional Analysis by Michael Reed and Barry Simon, [RS79].

II.2 Some basic geometrical facts

The following result is elementary, but we will give a proof anyway to get a taste of the nature of the objects involved, see also Corollary V.12:

Proposition II.3. *Σ_∞ is an open set of $\mathcal{H}_{\mathbb{R}}$.*

Proof. Consider the analytic exponential map $\exp : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ that assigns

$$A \mapsto e^A = \sum \frac{A^n}{n!}$$

The restriction of exp to $\mathcal{H}_{\mathbb{R}}$ is well defined because for every Hilbert-Schmidt, selfadjoint operator and every real λ we can write $e^{a+\lambda} = b + \beta$, where

$$b = e^{\lambda} \sum \frac{a^k}{k!} \qquad \text{and} \qquad \beta = e^{\lambda}$$

Obviously, β is real and b is selfadjoint; moreover b lies in HS because the latter is a bilateral ideal in $L(H)$, and $b = a \cdot c$ for a bounded operator c.

We claim that $\Sigma_\infty = \exp(\mathcal{H}_{\mathbb{R}})$. One inclusion has already been proved. To prove the other, apply the functional calculus to the function $g(x) = \ln(x)$ and the operator $b + \beta \in \Sigma_\infty$. Since this operator is positive, the logarithm has the form of a series; an argument similar to the one we used for the exponential shows that $\ln(\beta + b) = \lambda + a$, with λ real and a Hilbert-Schmidt (and selfadjoint). This proves that the logarithm gives a local analytic inverse of exp, so exp maps onto.

The proof of our initial assertion follows from general results about Banach algebras and analytic maps: any analytic map (from a Banach algebra into itself) with an analytic local inverse is locally open, and as a consequence, $\Sigma_\infty = \exp(\mathcal{H}_{\mathbb{R}}) \subset \mathcal{H}_{\mathbb{R}}$ is open. $\square$

Remark II.4. For $p \in \Sigma_\infty$, we identify $T_p\Sigma_\infty$ with $\mathcal{H}_{\mathbb{R}}$, and endow this manifold with a (real) Riemannian metric by means of the formula

$$\langle X, Y\rangle_p = \left\langle p^{-1}X, Yp^{-1}\right\rangle_2 = \left\langle Xp^{-1}, p^{-1}Y\right\rangle_2$$

where $\langle \alpha + a, \beta + b\rangle_2 = \alpha\overline{\beta} + 4tr(b^*a)$. Throughout, $\|X\|_p^2 := \langle X, X\rangle_p$, which can be rewritten as

$$\|X\|_p^2 = \|p^{-\frac{1}{2}}Xp^{-\frac{1}{2}}\|_2 = \left\langle Xp^{-1}, p^{-1}X\right\rangle_2 = \left\langle p^{-1}X, Xp^{-1}\right\rangle_2,$$

and is the norm of tangent vectors $X \in T_p\Sigma_\infty$.

Lemma II.5. *Covariant derivative in Σ_∞ (for the metric introduced in Remark II.4) is given by the expression*

$$(\nabla_X Y)_p = \{X(Y)\}_p - \frac{1}{2}\left(X_p p^{-1}\, Y_p + Y_p p^{-1}\, X_p\right) \tag{1}$$

where $X(Y)$ *denotes derivation of the vector field* Y *in the direction of* X *(performed in the linear space* $\mathcal{H}_{\mathbb{R}}$*).*

Proof. Note that ∇ is clearly symmetric and verifies all the formal identities of a connection; the proof that this is the Levi-Civita connection relays on the compatibility condition between the connection and the metric,

$$\frac{d}{dt}\langle X, Y\rangle_\gamma = \left\langle \nabla_{\dot\gamma} X, Y\right\rangle_\gamma + \left\langle X, \nabla_{\dot\gamma} Y\right\rangle_\gamma$$

where γ is a smooth curve in Σ_∞ and X, Y are tangent vector fields along γ. This identity is straightforward from the definitions for both terms and the cyclicity of the trace. □

Euler's equation $\nabla_{\dot\gamma}\dot\gamma = 0$ reads

$$\ddot\gamma - \dot\gamma\gamma^{-1}\dot\gamma = 0, \tag{2}$$

and

$$\gamma_{pq}(t) = p^{\frac{1}{2}}\left(p^{-\frac{1}{2}} q p^{-\frac{1}{2}}\right)^t p^{\frac{1}{2}} \tag{3}$$

is the unique solution of Euler's equation with $\gamma(0) = p$ and $\gamma(1) = q$.

These curves look formally equal to the geodesics between positive definite matrices (regarded as a symmetric space), and we will prove (Theorem IV.9) that the unique minimizing geodesic (i.e. the shortest path) joining p to q is given by the curve above.

Lemma II.6. *The metric in* Σ_∞ *is invariant under the action of the group of invertible elements: if* g *is an invertible operator in* $\mathcal{H}_{\mathbb{C}}$*, then* $I_g(p) = gpg^*$ *is an isometry of* Σ_∞*.*

Proof. Note that $d_r I_g(X) = gXg^*$ for any $X \in T_r\Sigma_\infty$, so

$$\|gXg^*\|^2_{grg^*} = \left\langle gXg^*(g^*)^{-1}r^{-1}g^{-1}, (g^*)^{-1}r^{-1}g^{-1}gXg^*\right\rangle_2 =$$

$$= \left\langle gXr^{-1}g^{-1}, (g^*)^{-1}r^{-1}Xg^*\right\rangle_2 = \left\langle Xr^{-1}, r^{-1}X\right\rangle_2 = \|X\|^2_r$$

where the third equality in the above equation follows from Remark II.2 (naming $X = gXr^{-1}$, $Y = g^{-1}$). $\square$

III Local Structure

III.1 The curvature tensor

Proposition III.1. *The manifold Σ_∞ has a curvature tensor given by the following commutant of operators:*

$$\mathcal{R}_p(X,Y)Z = -\frac{1}{4}\, p\left[\left[p^{-1}X, p^{-1}Y\right], p^{-1}Z\right] \tag{4}$$

Proof. This follows from the definition $\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ (here ∇ stands for the covariant derivative) together with the formula for ∇ given in Lemma 1. □

Definition III.2. *A Riemannian submanifold $M \subset \Sigma_\infty$ is* flat *at $p \in M$ if sectional curvature vanishes for any 2-subspace of T_pM. M is a flat manifold* *if it is flat at any* $p \in M$.

A Riemannian submanifold $M \subset \Sigma_\infty$ *(with the induced metric) is* geodesic *at* $p \in M$ *if geodesics of the ambient space starting at* p *which have initial velocity in* T_pM*, are also geodesics of* M. M *is a* totally geodesic *manifold if it is geodesic at any* $p \in M$. *Equivalently, any geodesic of* M *is also a geodesic of the ambient space* Σ_∞.

Obviously, in this context curvature and commutativity are related; the following proposition makes this relation explicit:

Proposition III.3. *Assume* $M \subset \Sigma_\infty$ *is a submanifold. Assume further that* M *is flat and geodesic at* $p \in M$. *Then, if* $X, Y \in T_pM$, $p^{-\frac{1}{2}}Xp^{-\frac{1}{2}}$ *commutes with* $p^{-\frac{1}{2}}Yp^{-\frac{1}{2}}$

Proof. Since M is geodesic at p, the curvature tensor is the restriction of the curvature tensor of Σ_∞. Set $X = p^{-\frac{1}{2}}Xp^{-\frac{1}{2}}$, $Y = p^{-\frac{1}{2}}Yp^{-\frac{1}{2}}$. Then a straightforward computation shows that

$$\langle \mathcal{R}_p(X,Y)Y,X\rangle_p = -\frac{1}{4}\left\{\left\langle XY^2,X\right\rangle_2 - 2\langle YXY,X\rangle_2 + \left\langle Y^2X,X\right\rangle_2\right\}$$

Now $X, Y \in \mathcal{H}_\mathbb{R}$, so $X = \lambda + a$, $Y = \beta + b$, and the equation reduces to

$$\langle \mathcal{R}_p(X,Y)Y,X\rangle_p = -\frac{1}{2}\left\{\mathrm{tr}(a^2b^2) - \mathrm{tr}((ab)^2)\right\} \tag{5}$$

The Cauchy-Schwarz inequality for the trace tells us that curvature at $p \in \Sigma_\infty$ is always nonpositive, and it is zero if and only if a and b commute. Hence whenever M is flat, X and Y commute for any pair of tangent vectors $X, Y \in T_pM$ as stated. $\square$

Corollary III.4. *Sectional curvature of* Σ_∞ *is everywhere nonpositive.*

III.2 Convexity of the Jacobi fields

Let $J(t)$ be a Jacobi field along a geodesic γ of Σ_∞. That is, J is a solution of the differential equation

$$\frac{D^2 J}{dt^2} + \mathcal{R}_\gamma(J,\dot\gamma)\dot\gamma = 0 \tag{6}$$

Next we show that the norm of a Jacobi field is convex. If $X,Y \in \mathcal{H}_\mathbb{R}$ are regarded as tangent vectors of Σ_∞ at the point p, then the following condition (which is a non positive sectional curvature condition) holds (see Proposition III.3 above): $\langle \mathcal{R}_p(X,Y)Y,X\rangle_p \le 0$. Then

$$\frac{1}{2}\frac{d^2}{dt^2}\langle J,J\rangle_\gamma = \left\langle \frac{D^2 J}{dt^2}, J\right\rangle_\gamma + \left\langle \frac{DJ}{dt},\frac{DJ}{dt}\right\rangle_\gamma =$$

$$= -\langle R_\gamma(J,\dot\gamma)\dot\gamma, J\rangle_\gamma + \left\langle \frac{DJ}{dt},\frac{DJ}{dt}\right\rangle_\gamma \ge 0$$

In other words, the smooth function $t \mapsto \langle J,J\rangle_\gamma$ is convex.

But more can be said: the *norm* of the Jacobi field (and not of the *square* of the norm just noted) is a convex function.

Proposition III.5. *Let γ be a geodesic of Σ_∞ and J a Jacobi field along γ. The map $t \mapsto \|J\|_\gamma = \langle J,J\rangle_\gamma^{\frac{1}{2}}$ is convex.*

Proof. Clearly, is suffices to prove this assertion for a field J which does not vanish. By the invariance of the connection and the metric under the action of the group of invertible operators, it suffices to consider the case of a geodesic $\gamma(t) = e^{tX}$ starting at $1 \in \Sigma_\infty$ ($X \in \mathcal{H}_\mathbb{R}$). For the field $K(t) = e^{-tX/2}J(t)e^{-tX/2}$ the Jacobi equation translates into

$$4\ddot K(t) = K(t)X^2 + X^2K(t) - 2XK(t)X. \tag{7}$$

We may assume (since scalars are orthogonal to Hilbert-Schmidt operators) that $J \subset HS^h$. In this case, $\langle J,J\rangle_\gamma^{1/2} = tr(\gamma^{-1}J\gamma^{-1}J)^{1/2} = tr(K^2)^{1/2} = \|K\|_2$
Let us prove therefore that the map $t \mapsto f(t) = \|K(t)\|_2$ is convex, for any (non vanishing) solution K of (7). Note that $f(t)$ is smooth, and $\dot f = tr(K^2)^{-1/2}tr(K\dot K)$.

Then

$$\ddot{f} = -\mathrm{tr}(K^2)^{-3/2}\mathrm{tr}(K\dot{K})^2 + \mathrm{tr}(K^2)^{-1/2}\{\mathrm{tr}(\dot{K}^2) + \mathrm{tr}(K\ddot{K})\}.$$

We multiply this expresion by $\mathrm{tr}(K^2)^{3/2}$ to obtain

$$\mathrm{tr}(K^2)^{3/2}\ddot{f}(t) = -\mathrm{tr}(K\dot{K})^2 + \mathrm{tr}(K^2)\mathrm{tr}(\dot{K}^2) + \mathrm{tr}(K^2)\mathrm{tr}(K\ddot{K}). \tag{8}$$

The first two terms add up to a non negative number. Indeed,

$$\mathrm{tr}(K\dot{K})^2 \leq \mathrm{tr}(K^2)\mathrm{tr}(\dot{K}^2)$$

by the Cauchy-Schwarz inequality for the trace. Therefore, it suffices to show that $\mathrm{tr}(K\ddot{K})$ is non negative. Using (7),

$$\mathrm{tr}(K\ddot{K}) = \frac{1}{4}\{\mathrm{tr}(K^2X^2) + \mathrm{tr}(KX^2K) - 2\mathrm{tr}(KXKX)\} = \frac{1}{2}\{\mathrm{tr}(K^2X^2) - \mathrm{tr}(KXKX)\}.$$

This number is positive, again by the Cauchy-Schwarz inequality:

$$\mathrm{tr}(KXKX) = \mathrm{tr}((XK)^*KX) \leq \mathrm{tr}((XK)^*XK)^{1/2}\mathrm{tr}((KX)^*KX)^{1/2} = \mathrm{tr}(K^2X^2) \quad \square$$

Corollary III.6. *If γ and δ are geodesics, the map $f : t \mapsto \mathrm{dist}(\gamma(t), \delta(t))$ is a convex function of t.*

Proof. Distance between $\gamma(t)$ and $\delta(t)$ is given by the geodesic $\alpha_t(s)$, obtained as the s variable ranges in a geodesic square $h(s,t)$ with vertices in the starting and ending points of γ and δ, namely $\{\gamma(t_0), \delta(t_0), \gamma(t_1), \delta(t_1)\}$
Taking the partial derivative along the direction of s gives a Jacobi field $J(s,t)$ along the geodesic $\beta_s(t) = h(s,t)$ and it also gives the speed of α_t. Hence

$$f(t) = \int_0^1 \|\frac{\partial \alpha_t}{\partial s}(s)\|_{\alpha_t(s)} ds = \int_0^1 \|J(s,t)\|_{h(s,t)} ds$$

This equation states that $f(t)$ can be written as the limit of a convex combination of convex functions $u_i(t) = \|J(s_i,t)\|_{h(s_i,t)}$, so f must be convex itself. $\square$

Lemma III.7. *The following inequality holds for any* $X, Y \in \mathcal{H}_{\mathbb{R}}$*:*

$$\operatorname{dist}(e^X, e^Y) = \|\ln\left(e^{X/2}e^{-Y}e^{X/2}\right)\|_2 \geq \|X-Y\|_2 \tag{9}$$

Proof. Take $\gamma(t) = e^{tx}$, $\delta(t) = e^{ty}$ and f as in the previous Corollary (by the orthogonality of scalars we may assume that $x, y \in HS^h$). Note that $f(0) = 0$, hence $f(t)/t \leq f(1)$ for any $0 < t \leq 1$; hence $\lim_{t\to 0^+} f(t)/t \leq f(1)$. We assert that

$$f(t)/t = \frac{1}{t}\|\ln\left(e^{tx/2}e^{-ty}e^{tx/2}\right)\|_2 = \operatorname{tr}\left(\left[\frac{1}{t}\ln\left(e^{tx/2}e^{-ty}e^{tx/2}\right)\right]^2\right)^{1/2},$$

since

$$\frac{1}{t}\ln\left(e^{tx/2}e^{-ty}e^{tx/2}\right) = \frac{d}{dt}\,|_{t=0} \ln\left(e^{tx/2}e^{-ty}e^{tx/2}\right)$$

and the logarithm of $\beta(t) = e^{tx/2}e^{-ty}e^{tx/2}$ can be approximated uniformly by polinomials $p_n(\beta) = \sum_k \alpha_{n,k}\beta^k$ for t close enough to zero ($\beta(0) = 1$). Then $\frac{d}{dt}\beta\,|_{t=0} = x - y$, and we obtain the desired inequality. $\square$

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IV Global Structure

IV.1 The exponential map

Remark IV.1. We will use the notation $\mathrm{Exp}_p : T_p\Sigma_\infty \to \Sigma_\infty$ to denote the exponential map of Σ_∞. Note that $\mathrm{Exp}_p(V) = p^{\frac{1}{2}}\, e^{\,p^{-\frac{1}{2}}Vp^{-\frac{1}{2}}} p^{\frac{1}{2}}$. Rearranging the exponential series we get a simpler expression

$$\mathrm{Exp}_p(V) = p\, e^{p^{-1}V} = e^{Vp^{-1}}\, p$$

A straightforward computation also shows that for $p, q \in \Sigma_\infty$ we have

$$\mathrm{Exp}_p^{-1}(q) = p^{\frac{1}{2}} \ln(p^{-\frac{1}{2}}\, q\, p^{-\frac{1}{2}}) p^{\frac{1}{2}}$$

We will prove that the differential of the Riemannian exponential is an analytic isomorphism; this is a standard result and we follow [Lang95], [McA65]:

Lemma IV.2. *Let γ be a geodesic such that $\gamma(0) = p$, $\dot{\gamma}(0) = V$. Let J be a Jacobi field along a geodesic γ, with $J(0) = 0$ and $\dot{J}(0) = \nabla_{\dot{\gamma}} J(0) = W \in T_p \Sigma_\infty = T_V (T_p \Sigma_\infty)$. Then, for any $t \in \mathbb{R}_{>0}$*

$$\frac{1}{t} J(t) = d \left(\mathrm{Exp}_p\right)_{tV} (W)$$

Proof. Take $F(t,s) = \mathrm{Exp}_p (t(V + sW))$. Then F_s is a geodesic for each s and $F_0 = \gamma$. Let K be the Jacobi field along γ given by $\frac{d}{ds}|_{s=0} F$. Then

$$K(t) = d \left(\mathrm{Exp}_p\right)_{tV} (tW) = t \, d \left(\mathrm{Exp}_p\right)_{tV} (W)$$

Clearly $K(0) = 0$; on the other hand if we divide by t and take limit for $t \to 0^+$, we get $\dot{K}(0) = d \left(\mathrm{Exp}_p\right)_0 (W) = \mathrm{Id}(W) = W$. By the uniqueness of the Jacobi fields along geodesics, it must be that $K = J$. ☐

Remark IV.3. If J, K are Jacobi fields along a geodesic γ, then

$$\left\langle \left(\nabla_{\dot{\gamma}} J\right)(t), K(t) \right\rangle_{\gamma(t)} = \left\langle J(t), \left(\nabla_{\dot{\gamma}} K\right)(t) \right\rangle_{\gamma(t)} + C$$

for some real constant C. This follows easily differentiating the above expression and using the derivation property of the covariant derivative.

Lemma IV.4. *The exponential map $\mathrm{Exp}_p : T_p \Sigma_\infty \to \Sigma_\infty$ has an expansive differential, namely*

$$\| d \left(\mathrm{Exp}_p\right)_V (W) \|_{\mathrm{Exp}_p(V)} \geq \|W\|_p$$

for any $p \in \Sigma_\infty$, $V \in T_p \Sigma_\infty$ and $W \in T_V (T_p \Sigma_\infty) = T_p \Sigma_\infty$.

Proof. From the definition of the exponential map and the metric, together with Lemma IV.2 and the convexity of the Jacobi fields. ☐

Lemma IV.5. *Take* $p \in \Sigma_\infty$*,* $V \in T_p\Sigma_\infty$*. Set* $q = \mathrm{Exp}_p(V)$*,* $Y = -P_p^q(V)$ *(namely* $Y = -\dot\gamma(1)$ *if* $\gamma(t) = \mathrm{Exp}_p(tV)$*). Then*

$$\left\langle d\left(\mathrm{Exp}_p\right)_V(W), Z\right\rangle_q = \left\langle W, d\left(\mathrm{Exp}_q\right)_Y(Z)\right\rangle_p$$

Proof. Let $J(t)$ be the Jacobi field along γ such that $J(1) = 0$ and $\nabla_{\dot\gamma}J(1) = Z$; let $K(t)$ be the Jacobi field along γ such that $K(0) = 0$, $\dot K(0) = \nabla_{\dot\gamma}K(0) = W$. By Remark IV.3,

$$\left\langle \left(\nabla_{\dot\gamma}J\right)(1), K(1)\right\rangle_{\gamma(1)} = \left\langle J(1), \left(\nabla_{\dot\gamma}K\right)(1)\right\rangle_{\gamma(1)} + C = 0 + C = C$$

where C equals

$$\left\langle \left(\nabla_{\dot\gamma}J\right)(0), K(0)\right\rangle_{\gamma(0)} - \left\langle J(0), \left(\nabla_{\dot\gamma}K\right)(0)\right\rangle_{\gamma(0)} = -\left\langle J(0), \left(\nabla_{\dot\gamma}K\right)(0)\right\rangle_{\gamma(0)}$$

Take L the unique Jacobi field along $\beta(t) = \gamma(1-t)$ such that $L(0) = 0$ and $\nabla_{\dot\beta}L(0) = Z$. Then $L(t) = J(1-t)$, so $J(0) = L(1)$ and we get

$$\langle Z, K(1)\rangle_{\gamma(1)} = -\langle L(1), W\rangle_{\gamma(0)}$$

Since $\gamma(0) = p$ and $\gamma(1) = q = \mathrm{Exp}_p(V)$, Lemma IV.2 gives the result. □

Corollary IV.6. *The Riemannian exponential* $\mathrm{Exp}_p : T_p\Sigma_\infty \to \Sigma_\infty$ *has a* C^ω *diffeomorphism for any* $p \in \Sigma_\infty$*; in particular,* $\exp : \mathcal{H}_\mathbb{R} \to \Sigma_\infty$ *is a* C^ω *diffeomorphism.*

Proof. By Lemma IV.4, for each $p \in \Sigma_\infty$ and each $V \in T_p\Sigma_\infty$, the differential map $A = d\left(\mathrm{Exp}_p\right)_V$ is injective, has closed range and a bounded inverse on its range. By Lemma IV.5, the adjoint map $A^* = d\left(\mathrm{Exp}_p\right)_V^*$ equals $d\left(\mathrm{Exp}_q\right)_{-Y}$, which has the same property. Now $Ran(A) = \overline{Ran(A)} = Ker(A^*)^\perp = \{0\}^\perp$, hence A is surjective. Using the inverse map theorem for Banach manifolds [Lang95], we obtain the result. □

Corollary IV.7. *If* $d\exp_X$ *denotes the differential at* x *of the usual exponential map* $X \mapsto e^X$*, then the following inequality holds for any* $X, Y \in \mathcal{H}_{\mathbb{R}}$*:*

$$\|d\exp_X(Y)\|_{e^X} = \|e^{-\frac{X}{2}} d\exp_X(Y) e^{-\frac{X}{2}}\| \geq \|Y\|_2$$

Proof. Rewriting the inequality of Lemma IV.4 above for $p = 1$, $V = X$ and $W = Y$ we obtain the result. □

Corollary IV.8. *For any* $X \in HS^h$*, the map* $T_X : Y \mapsto e^{-X/2} d\exp_X(Y) e^{-X/2}$ *is bounded, selfadjoint for the 2-inner product (when restricted to* HS^h*) and invertible. The inverse is contractive i.e*

$$\|T_X^{-1}(Z)\|_2 \leq \|Z\|_2$$

Proof. The map is clearly bounded and invertible, the bound for the inverse follows from the proof of the previous Lemma. To prove that it is selfadjoint, note that

$$< T_X(Y), Z >_2 = tr(Z T_X(Y)) = tr\left(e^{-X/2} \sum_{n\geq 0} \frac{1}{n!} \sum_{p+q=n-1} X^p Y X^q e^{-X/2} Z \right) =$$

$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{p+q=n-1} tr(e^{-X/2} X^p Y X^q e^{-X/2} Z) =$$

$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{p+q=n-1} tr(X^p e^{-X/2} Y e^{-X/2} X^q Z) =$$

$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{p+q=n-1} tr(e^{-X/2} X^q Z X^p e^{-X/2} Y) = tr(T_X(Z) Y) = < Y, T_X(Z) >_2$$ □

IV.2 The shortest path and the geodesic distance

We measure curves in Σ_∞ using the norms in each tangent space, namely

$$L(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_{\alpha(t)}\, dt \tag{10}$$

We define the distance between two points $p, q \in \Sigma_\infty$ as the infimum of the lenghts of smooth curves in Σ_∞ joinint p to q, namely

$$\operatorname{dist}(p,q) = \inf\{L(\alpha) : \alpha \subset \Sigma_\infty,\ \alpha \text{ is smooth},\ \alpha(0) = p,\ \alpha(1) = q\}$$

For any pair of elements $p, q \in \Sigma_\infty$, we have the smooth curve $\gamma_{pq} \subset \Sigma_\infty$

$$\gamma_{pq}(t) = p^{1/2}\left(p^{-1/2} q p^{-1/2}\right)^t p^{1/2}$$

joining p to q (which is the unique solution of the Euler equation in Σ_∞). A straightforward computation shows that

$$\|\dot{\gamma_{pq}}(t)\|_{\gamma_{pq}(t)} \equiv \|\ln(p^{1/2} q^{-1} p^{1/2})\|_2 = L(\gamma_{pq})$$

Theorem IV.9. *Let $a, b \in \Sigma_\infty$. Then the geodesic γ_{ab} is the shortest curve joining a and b in Σ_∞, if the length of curves is measured with the metric defined above (10).*

Proof. Let α be a smooth curve in Σ_∞ with $\alpha(0) = a$ and $\alpha(1) = b$. We must compare the length of α with the length of γ_{ab}. Since the invertible group acts isometrically for the metric, it preserves the lengths of curves. Thus we may act with $a^{-1/2}$, and suppose that both curves start at 1, or equivalently, $a = 1$. Therefore $\gamma_{1b}(t) := \gamma(t) = e^{tX}$, with $X = \ln b$. The length of γ is then $\|X\|_2$. The proof follows easily from the inequality proved in Corollary IV.7. Indeed, since α is a smooth curve in Σ_∞, it can be written as $\alpha(t) = e^{\beta(t)}$, with $\beta = \ln \alpha$. Then β is a smooth curve of selfadjoint operators with $\beta(0) = 0$ and $\beta(1) = X$. Moreover,

$$L(\gamma) = \|X\|_2 = \|X - 0\|_2 = \|\int_0^1 \dot\beta(t)\, dt\|_2 \leq \int_0^1 \|\dot\beta(t)\|_2\, dt$$

On the other hand, by the mentioned inequality,

$$\|\dot\beta(t)\|_2 \leq \|e^{-\frac{\beta(t)}{2}}\, d\exp_{\beta(t)}(\dot\beta(t)) e^{-\frac{\beta(t)}{2}}\|_2 = \|d\exp_{\beta(t)}(\dot\beta(t))\|_{e^{\beta(t)}} = \|\dot\alpha(t)\|_{\alpha(t)}$$

Thus,

$$L(\gamma) \leq \int_0^1 \|\dot{\alpha}(t)\|_{\alpha(t)}\, dt = L(\alpha) \quad \square$$

Remark IV.10. The geodesic distance induced by the metric is given by

$$\operatorname{dist}(a,b) = \|\ln\big(a^{-1/2}ba^{-1/2}\big)\|_2$$

Corollary IV.11. *The sum of the inner angles of any geodesic triangle in Σ_∞ is less or equal than π*

Proof. The previous remark, inequality (9) in Lemma III.7, together with the invariance of the metric for the action of the group of invertible operators, leads to

$$l_i^2 \geq l_{i+1}^2 + l_{i-1}^2 - 2l_{i+1}l_{i-1}\cos(\alpha_i) \tag{11}$$

squaring both sides of the inequality. Here l_i (i=1,2,3) are the sides of any geodesic triangle and α_i is the angle opposite to l_i. These inequalities show that we can construct an Euclidian triangle in the affine plane with sides l_i. For this Euclidian triangle with angles β_i (opposite to the side l_i) we have

$$l_i^2 = l_{i+1}^2 + l_{i-1}^2 - 2l_{i+1}l_{i-1}\cos(\beta_i)$$

This equation together with inequality (11) imply that the angle β_i is bigger than α_i for $i = 1,2,3$. Adding the three angles we have

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \beta_1 + \beta_2 + \beta_3 = \pi \quad \square$$

As a corollary of these inequalities we obtain the completeness of the metric space $(\Sigma_\infty, \operatorname{dist})$, where dist is the geodesic distance:

Proposition IV.12. *Σ_∞ is a complete metric space with the distance induced by the minimizing geodesics.*

Proof. Consider a Cauchy sequence $\{p_n\} \subset \Sigma_\infty$. Again by virtue of inequality (9) of Lemma III.7, $X_n = \ln(p_n)$ is a Cauchy sequence in $\mathcal{H}_\mathbb{R}$. Since Hilbert-Schmidt operators are complete with the trace norm, there is a vector $X \in \mathcal{H}_\mathbb{R}$ such that $X_n \to X$ in the trace norm. As the inverse map, the exponential map, the product and the logarithm are all analytic maps with respect to the trace norm, $\mathrm{dist}(p_n, e^X) = \| \ln(e^{X/2} e^{-X_n} e^{X/2}) \|_2 \to 0$ when $n \to \infty$. $\square$

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V Convex Submanifolds

Convex sets are particulary useful in geometry, and play a major role in the theory of hyperbolic (*i.e.* nonpositevely curved) spaces.

V.1 Definitions

Definition V.1. *A set* $M \subset \Sigma_\infty$ *is* geodesically convex *(also* totally convex*, or* convex*) if given any two points* $p, q \in M$*, the unique geodesic of* Σ_∞ *joining* p *to* q *lays entirely in* M*.*

Note that convex sets are connected. We refer the reader to Chapter IV, Section 5 of [SakT96] for a discussion of the different kinds of convex (strong, local, total) Riemannian objects. However, in our context, all definitions agree, because Σ_∞

is complete and for any two points there exists a unique normal (*i.e.* unit speed) geodesic joining them (which is clearly minimizing).

Definition V.2. *A Riemannian submanifold* $M \subset \Sigma_\infty$ *is* complete *at* $p \in M$ *if* Exp_p^M *is defined in the whole tangent space and maps onto* M. *We say that* M *is a* complete manifold *if it is complete at any pont.*

Remark V.3. Note that M is geodesic at p if and only if $\mathrm{Exp}_p^M = \mathrm{Exp}_p$. In particular Exp_p^M is defined in the whole T_pM. So if M is geodesic at p, then M is complete at p if and only if for any point $q \in M$, there is a geodesic γ of M joining p to q (in other words, if $\mathrm{Exp}_p^M = \mathrm{Exp}_p$ maps *onto* M).

Remark V.4. Σ_∞ is complete; moreover, Exp_p is a diffeomorphism onto Σ_∞ for each $p \in \Sigma_\infty$. The reader should be careful with other notions of completeness, because, as C.J. Atkin shows in [Atkin75] and [Atkin97], Hopf-Rinow's theorem does not necessarily hold in (infinite dimensional) Banach manifolds.

These previous notions are strongly related, as the following proposition shows:

Proposition V.5. *Let* $M \subset \Sigma_\infty$ *be a Riemannian submanifold of* Σ_∞ *(with the induced metric). Then*

$$M \textit{ geodesically convex } \iff M \textit{ complete and totally geodesic}$$

Proof. The proof of ($\Leftarrow$) is trivial; let's prove ($\Rightarrow$). To see that M is complete, take $p, q \in M$. Then there exists a geodesic α of Σ_∞ joining p to q, $\alpha \subset M$. Among curves in M joining p to q, α is the shortest. So α is a critical point of the variational problem in M, hence a geodesic of M. To see that M is totally geodesic, take γ a geodesic of M joining p to q. By virtue of the convexity, there is a geodesic α of Σ_∞ joining p to q; by the preceding argument α is also a geodesic of M. We can assume that q is close enough to p for the exponential map of M to be an isomorphism, and in this situation, geodesics are unique, so $\alpha = \gamma$ is a geodesic of Σ_∞. □

Remark V.6. The reader should be aware of the fact that the concept of convexity is strong, and completely general (M does not need to have the induced submanifold metric, in fact, for the definition of geodesically convex to make sense, it is not necessary for M to have any manifold structure at all).

V.2 An intrinsic characterization of convexity

As always, $[\,,\,]$ denotes the usual commutator of operators in $L(H)$. To deal with convex sets the following definition will be useful:

Definition V.7. *We say that a subspace* $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ *is a* Lie triple system *if* $[[A,B],C] \in \mathfrak{m}$ *for any* $A,B,C \in \mathfrak{m}$.

Remark V.8. Note that whenever a,b,c are selfadjoint operators, $d=[a,[b,c]]$ is also a selfadjoint operator. So, for any algebra of operators $\mathfrak{a} \subset \mathcal{H}_{\mathbb{C}}$, $\mathfrak{m} = \mathfrak{Re}(\mathfrak{a})$ is a Lie triple system in $\mathcal{H}_{\mathbb{R}}$. This is also true for a Lie algebra of operators $\mathfrak{a}$.

Remark V.9. Assume $M \subset \Sigma_{\infty}$ is a submanifold such that $1 \in M$, and M is geodesic at $p=1$. Then $T_1 M$ is a Lie triple system, because the curvature tensor at $p=1$ is the restriction to $T_1 M$ of the curvature tensor of Σ_{∞}, and $\mathcal{R}_1(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$.

This particular condition on the tangent space turns out to be strong enough to ensure convexity; this result is standard:

Theorem V.10. *Assume* $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ *is a closed subspace, set* $M = \exp(\mathfrak{m}) \subset \Sigma_{\infty}$ *with the induced topology and Riemannian metric.*

$$\textit{If } \mathfrak{m} \textit{ is a Lie triple system, then } p,q \in M \Rightarrow pqp \in M$$

Proof. As Pierre de la Harpe pointed out, the proof of G.D. Mostow for matrices in [Mos55] can be translated to Hilbert-Schmidt operators without any modification: we give a sketch of the proof here. We assume $p = e^X$, $q = e^Y$ with $X,Y \in HS^h$.

Set $D_X : HS \to HS$, $D_X = L_X - R_X$, the difference between left and right multiplication by X in HS (which is clearly a bounded linear operator from HS to HS). First we establish the identity

$$T_X(Y) = \sinh c(D_X/2)(Y) \tag{12}$$

where T_X is the map from Corollary IV.8, and $\sinh c(Z) = \frac{\sinh(Z)}{Z} = \sum_{n\geq 0} \frac{Z^{2n}}{(2n+1)!}$ is an entire function. Note that $\sinh c(Z) = \frac{e^Z - e^{-Z}}{2Z}$. To prove (12), we take derivative with respect to t in the identity $X(t)e^{X(t)} = e^{X(t)}X(t)$, where $X(t) = X + tY$; after rearranging the terms we come up with

$$\left(e^{D_X/2} - e^{-D_X/2}\right) Y = D_X \circ T_X(Y)$$

Note that if D_X were invertible, we would be set; this is clearly not the case. However, $D_X^2 = D_X \circ D_X$ is selfadjoint when restricted to HS^h, and since T_X is also sefaldjoint (cf Corollary IV.8), the operator $T = T_X - \sinh c(D_X/2)$ is selfadjoint on HS^h (note that $\sinh c(Z)$ involves only even powers of Z). The equation above says that we have proved that $D_X \circ T(Y) = 0$ for any $Y \in HS$; in other words T maps HS^h into $\{X\}' = \{b \in HS^h : bX = Xb\}$. A straightworward computation shows that $Tb = 0$ for any $b \in \{X\}'$, which proves equation (12) since T is selfadjoint.

Now for $X, Y \in \mathfrak{m}$ consider the curve $e^{\alpha(t)} = e^{tX}e^Y e^{tX}$. Clearly $\alpha(0) = Y \in \mathfrak{m}$; we will prove that α obeys a differential equation in HS^h which has a flow that maps $\mathfrak{m}$ into $\mathfrak{m}$, and with that we will have $e^{\alpha(1)} = e^X e^Y e^X \in e^{\mathfrak{m}} = M$.

Differentiating at $t = t_0$ the equation yields to

$$Xe^{\alpha(t_0)} + e^{\alpha(t_0)}X = d\exp_{\alpha(t_0)}(\dot{\alpha}(t_0)) = \frac{d}{ds}|_{s=0} \exp(\alpha(t_0) + s\dot{\alpha}(t_0)) =$$

$$= e^{\alpha(t_0)/2} T_{\alpha(t_0)}(\dot{\alpha}(t_0)) e^{\alpha(t_0)/2} = e^{\alpha(t_0)/2} \cdot \sinh c(D_{\alpha(t_0)}/2)(\dot{\alpha}(t_0)) \cdot e^{\alpha(t_0)/2}$$

Note that $\sinh c(Z)$ is invertible whenever Z is a bounded linear operator, and also that the power series for $Z\coth(Z/2)$ involves only even powers of Z; hence

$$\dot{\alpha} = \mathrm{senhc}^{-1}(D_\alpha/2) \circ (e^{-\alpha/2}Xe^{\alpha/2} + e^{\alpha/2}Xe^{-\alpha/2}) =$$

$$= \mathrm{senhc}^{-1}(D_\alpha/2) \circ (R_{e^{\alpha/2}}L_{e^{-\alpha/2}} + R_{e^{-\alpha/2}}L_{e^{\alpha/2}})X =$$

$$
\begin{aligned}
&= \operatorname{senhc}^{-1}(D_{\alpha/2})\circ(D_{e^{\alpha/2}}+D_{e^{-\alpha/2}})(X) = \\
&= \operatorname{senhc}^{-1}(D_{\alpha/2})\circ(e^{D_{\alpha/2}}+e^{-D_{\alpha/2}})(X) = \\
&= D_\alpha \coth(D_{\alpha/2})(X) = \sum_n c_n D_\alpha^{2n} X = \\
&= \sum_n c_n D_\alpha^2 \circ \cdots \circ D_\alpha^2(X) = F(\alpha)
\end{aligned}
$$

Since $D_Z^2(X) = [Z,[Z,X]]$, $F(Z) = \sum_n c_n D_Z^{2n}(X)$ can be regarded as a map from $\mathfrak{m}$ to $\mathfrak{m}$, and since it is clearly an analytic map of HS into HS, it fulfills a Lipschitz condition. Now the unique solution must be $\alpha(t) = \ln(e^{tX}e^Y e^{tX}) \subset \mathfrak{m}$. Hence $e^{\alpha(1)} = pqp \in M$ and the claim follows. $\square$

Corollary V.11. *Assume* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *as above, and* $\mathfrak{m}$ *is a Lie triple system. Then* M *is geodesically convex.*

Proof. Take $p, q \in M$. Then $p = e^X$, $q = e^Y$ with $X, Y \in \mathfrak{m}$. If we set $r = e^{-X/2}e^Y e^{-X/2}$, then $r \in M$ because $e^{-X/2}$ and e^Y are in M. Moreover, $Z = \ln(r) \in \mathfrak{m}$. But the only geodesic of Σ_∞ joining p to q is

$$\gamma(t) = e^{X/2}e^{tZ}e^{X/2}, \quad \text{so} \quad \gamma \subset M \quad \square$$

Corollary V.12. *Assume* $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ *is a closed abelian subalgebra of operators. Then the manifold* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *is a closed, convex and flat Riemannian submanifold. Moreover,* M *is an open subset of* $\mathfrak{m}$ *and an abelian Banach-Lie group.*

Proof. The first assertion follows from the fact that $\mathfrak{m}$ is a Lie triple system. Curvature is given by commutators, hence M is flat. Since $\mathfrak{m}$ is a closed subalgebra, $e^X = \sum \frac{X^n}{n!} \in \mathfrak{m}$ for any $X \in \mathfrak{m}$, so $M \subset \mathfrak{m}$. That M is open in $\mathfrak{m}$ follows from Corollary IV.6. $\square$

Corollary V.13. *Assume* $M = \exp(\mathfrak{m})$ *is closed and flat. If* M *is geodesic at* $p = 1$*, then* M *is a convex submanifold. Moreover,* M *is an abelian Banach-Lie group and* M *is an open subset of* $\mathfrak{m}$*.*

Proof. If M is geodesic and flat at $p = 1$, $T_1 M = \mathfrak{m}$ is abelian (by Proposition III.3). Now we can apply the previous corollary. $\square$

The definition of symmetric space we adopt is the usual definition for Riemannian manifolds, see the book [Hel62] by Sigurdur Helgason:

Definition V.14. *A Hilbert manifold* M *is called a* globally symmetric space *if each point* $p \in M$ *is an isolated fixed point of an involutive isometry* $s_p : M \to M$. *The map* s_p *is called the* geodesic symmetry at p.

Theorem V.15. *Assume* $M = \exp(\mathfrak{m})$ *is closed and geodesically convex. Then* M *is a symmetric space; the geodesic symmetry at* $p \in M$ *is given by* $s_p(q) = pq^{-1}p$ *for any* $q \in M$. *In particular,* Σ_∞ *is a symmetric space.*

Proof. Observe that, for $p = e^X$, $q = e^Y$, $s_p(q) = e^X e^{-Y} e^X$; this shows that s_p maps M into M. To prove that s_p is an isometry, consider the geodesic α_V of M such that $\alpha(0) = q$ and $\dot{\alpha}(0) = V$. Then $\alpha(t) = qe^{t\,q^{-1}V}$ and

$$d_q(s_p)(V) = \frac{d}{dt}|_{t=0}(s_p \circ \alpha_V) = -pq^{-1}Vq^{-1}p$$

Since M has the induced metric, $\|pq^{-1}Vq^{-1}p\|^2_{pq^{-1}p} = \|V\|^2_q$ by Lemma II.6 (with $g = pq^{-1}$). In particular, $d_p s_p = -id$, so p is an isolated fixed point of s_p for any $p \in M$. $\square$

Theorem V.10 and its corollaries imply that Σ_∞ (as any symmetric space) contains plenty of convex sets; in particular

Remark V.16. We can embed isometrically any k-dimensional plane in Σ_∞ as a geodesically convex, closed submanifold: take an orthonormal set of k commuting operators (for instance, fix an orthonormal basis $\{e_i\}_{i \in \mathbb{M}}$ of H, and take $p_i = e_i \otimes e_i$, $i = 1, \cdots, k$), now take the exponential of the linear span of this set . In the languaje of symmetric spaces, we are saying that $rank(\Sigma_\infty) = +\infty$.

Following the usual notation for symmetric spaces, we set $I_0(M)$=the connected component of the identity of the group of isometries of M.

Remark V.17. Assume $1 \in M \subset \Sigma_\infty$ is closed and convex. Then, since any isometry φ is uniquely determined by its value at $1 \in M$ and its differential $d_1\varphi$, $I_0(M)$ carries a natural structure of Banach-Lie group (this result was proved by J. Eells in the mid 60's, [Eells66]). Moreover, the Lie algebra of $I_0(M)$ identifies naturally whith the Killing vectors of M. We can be more precise in this context: take $\varphi \in I_0(M)$, and consider

$$\overline{\varphi}\,(q) = \varphi(1)^{\frac{1}{2}} \cdot \varphi(q) \cdot \varphi(1)^{\frac{1}{2}}.$$

Note that $d_1\overline{\varphi}$ is a unitary operator of $T_1M = \mathfrak{m}$ (with the natural Hilbert-space structure), so there is an inclusion $J : I_0(M) \hookrightarrow M \times \mathcal{U}(L(\mathfrak{m}))$ given by $\varphi \mapsto (\varphi(1), d_1\overline{\varphi})$. We will see later that the unitary operators of the form $x \mapsto gxg^*$ (inner automorphisms) are enough to act transitively on M (g must be in G_M, see Theorem V.29).

Theorem V.18. *Assume* $M = \exp(\mathfrak{m})$ *is closed and geodesically convex. Then* $I_0(M)$ *acts transitively on* M.

Proof. Take $p = e^X$, $q = e^Y$ two points in M and $\gamma(t) = pe^{tp^{-1}V}$ the geodesic joining p to q. Note that $p = \gamma(1) = pe^{p^{-1}V} = e^{\nu p^{-1}}p$. If we consider the curve of isometries $\varphi_t = s_{\gamma(t/2)} \circ s_p$, since $\varphi_0 = id$, then $\varphi_t \subset I_0(M)$. Now

$$\varphi_1(p) = e^{\frac{1}{2}Ve^{-X}} e^X e^{-X} e^{\frac{1}{2}Ve^{-X}} e^X = e^{Ve^{-X}} e^X = q$$

which proves that $I_0(M)$ acts transitively on M. □

Remark V.19. If $M = \exp(\mathfrak{m})$ is closed and convex, in particular it is geodesic at p for any $p \in M$, so $T_pM = \mathrm{Exp}_p^{-1}(M) = \{p^{\frac{1}{2}} \ln(p^{-\frac{1}{2}}\, q\, p^{-\frac{1}{2}}) p^{\frac{1}{2}} : q \in M\}$ (see Remark IV.1). This observation together with Theorem V.10 proves the identification

$$T_pM = p^{\frac{1}{2}}\,(T_1M)\, p^{\frac{1}{2}} = p^{\frac{1}{2}}\,\mathfrak{m}\, p^{\frac{1}{2}}$$

From previous identifications of the tangent space it follows easily (see Remark II.2) that an operator $V \in \mathcal{H}_{\mathbb{R}}$ is orthogonal to M at p (that is, $V \in T_pM^{\perp}$) if and only if

$$\left\langle p^{-\frac{1}{2}} Z p^{-\frac{1}{2}}, V \right\rangle_2 = \left\langle p^{-\frac{1}{2}} V p^{-\frac{1}{2}}, Z \right\rangle_2 = 0 \quad \text{for any} \quad Z \in \mathfrak{m}$$

In particular, $T_1 M^{\perp} = \mathfrak{m}^{\perp} = \{V \in \mathcal{H}_{\mathbb{R}} : \langle V, Z \rangle_2 = 0 \text{ for any } Z \in \mathfrak{m}\}$.

Remark V.20. Note that when $\mathfrak{m}$ is a closed commutative associative subalgebra of $\mathcal{H}_{\mathbb{R}}$, $p^{\frac{1}{2}} = e^{X/2} \in \mathfrak{m}$, which also iplies that the map $Y \mapsto p^{\frac{1}{2}} \cdot Y \cdot p^{\frac{1}{2}}$ is a linear automorphism of $\mathfrak{m}$; so $T_p M = \mathfrak{m} = T_1 M$ in this case (for any $p \in M$). This also follows easily from Corollary V.12. Clearly,

$$T_p M^{\perp} = T_1 M^{\perp} = \mathfrak{m}^{\perp} \quad \text{for any } p \in M$$

Remark V.21. Assume $M \subset \Sigma_\infty$ is geodesically convex. Then, if γ is the geodesic joining p to q, the isometry $\varphi_t = s_{\gamma(t/2)} \circ s_p$ translates along the curve γ, namely

$$\varphi_t(\gamma(u)) = p\, e^{\frac{t}{2} p^{-1} V} \cdot p^{-1} \cdot p\, e^{u p^{-1} V} \cdot p^{-1} \cdot p\, e^{\frac{t}{2} p^{-1} V} =$$

$$= p\, e^{\frac{t}{2} p^{-1} V} \cdot e^{u p^{-1} V} \cdot e^{\frac{t}{2} p^{-1} V} = p\, e^{(u+t) p^{-1} V} = \gamma(u+t)$$

Now take any tangent vector $W \in T_{\gamma(u)} M$, and set

$$W(t) := (d\varphi_t)_{\gamma(u)}(W) = e^{\frac{t}{2} V p^{-1}} \cdot W \cdot e^{\frac{t}{2} p^{-1} V}$$

Then $W(t)$ is the parallel translation of W from $\gamma(u)$ to $\gamma(u+t)$; namely $\nabla_{\dot\gamma} W \equiv 0$ (this follows from a straightforward computation using equation (II.5))

We conclude that the map $(d\varphi_t)_{\gamma(u)} : T_{\gamma(u)} M \to T_{\gamma(u+t)} M$ gives parallel translation along γ, namely $(d\varphi_t)_{\gamma(u)} = P_u^{t+u}(\gamma)$. In particular, since $q = \gamma(1) = p^{\frac{1}{2}}\, e^{p^{-\frac{1}{2}} V p^{\frac{1}{2}}} p^{\frac{1}{2}}$,

$$W \mapsto p^{\frac{1}{2}} (p^{-\frac{1}{2}} q p^{-\frac{1}{2}})^{\frac{1}{2}} p^{-\frac{1}{2}} \cdot W \cdot p^{-\frac{1}{2}} (p^{-\frac{1}{2}} q p^{-\frac{1}{2}})^{\frac{1}{2}} p^{\frac{1}{2}}$$

gives parallel translation from $T_p M$ to $T_q M$.

Remark V.22. It should also be noted that the exponential map of M (whenever

M is a convex submanifold) is the restriction to T_pM of the exponential map $\mathrm{Exp}_p : T_p\Sigma_\infty \to \Sigma_\infty$, hence it is a C^ω diffeomorphism from T_pM to M for any $p \in M$; in particular, when $M = \exp(\mathfrak{m})$, $\exp : \mathfrak{m} \to M$ is a C^ω diffeomorphism.

V.2.1 A few examples of convex sets

We list several Lie triple systems of $\mathcal{H}_\mathbb{R}$; for some of them we show in this manuscript an explicit factorization theorem. The general factorization theorems (Theorem VI.11, Theorem VI.12 and Theorem VI.13) apply for any of these (to be precise, to their closures in the trace norm):

1. For any subspace $\mathfrak{s} \subset \mathcal{H}_\mathbb{R}$, the subspace $\mathfrak{m}_\mathfrak{s} = \{X \in \mathcal{H}_\mathbb{R} : [X,Y] = 0 \ \forall\, Y \in \mathfrak{s}\}$ is a Lie triple system.
2. In particular, for any $Y \in \mathcal{H}_\mathbb{R}$, $\mathfrak{m}_Y = \{X \in \mathcal{H}_\mathbb{R} : [X,Y] = 0\}$ is a Lie triple system.
3. The family of operators in $\mathcal{H}_\mathbb{R}$ which act as endomorphisms of a closed subspace $S \subset H$ form a Lie triple system in $\mathcal{H}_\mathbb{R}$.
4. Any norm closed abelian subalgebra of $\mathcal{H}_\mathbb{R}$ is a Lie triple system, in particular
   (a) The diagonal operators (see section VII). This is a maximal abelian closed subspace of $\mathcal{H}_\mathbb{R}$, hence the manifold Δ (which is the exponential of this set) is a maximal flat submanifold of Σ_∞.
   (b) The scalar manifold $\Lambda = \{\lambda \cdot 1 : \lambda \in \mathbb{R}_{>0}\}$ is the exponential of the Lie triple system $\mathbb{R} \cdot 1 \subset \mathcal{H}_\mathbb{R}$.
   (c) For fixed $a \in HS^h$, the real part of the closed algebra generated by a, which is the closure in the 2-norm of the set of polynomials in a.
5. The real part of any Lie subalgebra of $\mathcal{H}_\mathbb{C}$ is a Lie triple system (in particular: the real part of any associative Banach subalgebra).
6. Any real Banach-Lie algebra $\mathfrak{g}$ with a compatible Riemannian product invariant under inner automorphisms has a complexification which leads to

the structure of an L^*-algebra, and any L^*-algebra can be embedded as a closed Lie subalgebra of HS (see [CGM90] and [Neh93]).

7. If G is a simply connected semisimple locally compact Lie group, then any irreducible representation of $C^*(G)$ into $L(H)$ maps $C_K(G)$ (the continous functions with compact support) into HS (see [Bag69]). This inclusion is also true for any irreducible subrepresentation of the left regular representation of a unimodular group G.

V.3 Convex manifolds as homogeneous spaces

Definition V.23. *A Banach-Lie group is a group G together with a compatible Banach manifold structure. If G is a Banach-Lie group, we say $K \subset G$ is a* Lie subgroup *if K is a subgroup of G which is also a split-embedded submanifold (hence a closed subgroup) of G.*

We recall (for a proof, see for instance [Lang95] or [Lar80]) a result for quotients of Banach-Lie groups:

Theorem V.24. *Let G be an analytic Banach-Lie group, and K a Banach-Lie subgroup. Then on the left cosets space G/K there exists a unique analytic manifold strutcture such that the projection is a submersion. The canonical action $G \times G/K \to G/K$ is analytic.*

For any Banach algebra $\mathcal{B}$, we will denote $GL(\mathcal{B})$ the group of invertible elements. Note that this group has a natural structure of manifold as an open set of the algebra, so $GL(\mathcal{B})$ is always a Banach-Lie group with Lie algebra $\mathcal{B}$.

Remark V.25. The group $GL(\mathcal{H}_{\mathbb{C}})$, having the homotopy type of the inductive limit of the groups $GL(n, \mathbb{C})$ (see [Har72], section II.6) is connected; moreover, there is a homotopy equivalence

$$GL(\mathcal{H}_{\mathbb{C}}) \simeq S^1 \times S^1 \times SU(\infty)$$

Here $SU(\infty)$ stands for the inductive limit of the groups $SU(n, \mathbb{C})$

The following result is standard in finite dimension (see for instance, [Hel62]); we say that G is a selfadjoint subgroup of $GL(\mathcal{H}_{\mathbb{C}})$ ($G^* = G$) if $g^* \in G$ whenever $g \in G$. Note that G is selfadjoint iff $\mathfrak{g}^* = \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of the Lie group G.

We will use $|\, x \,| = \sqrt{xx^*}$ to denote the modulus of an element $x \in \mathcal{B}$ (as usual, $\mathcal{B}$ is an involutive Banach algebra).

Theorem V.26. *Fix a connected Lie subgroup $G \subset GL(\mathcal{H}_{\mathbb{C}})$ such that $G^* = G$. Let P be the analytic map*

$$P : GL(\mathcal{H}_{\mathbb{C}}) \to GL(\mathcal{H}_{\mathbb{C}}) \qquad \textit{where} \qquad g \overset{P}{\longmapsto} gg^* = |\, g \,|^2$$

If K denotes the isotropy group of P (namely $K = P^{-1}(1) \cap G$ with the induced analytic structure), then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ are the selfadjoint elements of $\mathfrak{g}$. In particular, K is a Lie subgroup of G.

Proof. Note that $\sigma(g) = g^*$ is involutive so its differential at $g = 1$ gives an involution Θ of $\mathfrak{g}$ that induces the desired splitting of the Lie algebra of G. Now K is a Lie subgroup because the Lie algebra splits. $\square$

Remark V.27. For $M = \exp(\mathfrak{m})$ a geodesically convex closed manifold in Σ_∞, consider

$$[\mathfrak{m}, \mathfrak{m}] = span\{[A, B] : A, B \in \mathfrak{m}\} = \left\{ \sum_{i \in F} [A_i, B_i] \,:\, A_i, B_i \in \mathfrak{m};\, F \text{ a finite set} \right\}$$

Note that all the operators in $[\mathfrak{m}, \mathfrak{m}]$ are skewadjoint. Set $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m}, \mathfrak{m}]}$. Then $\mathfrak{g}_M$ is a closed Lie subalgebra of $\mathcal{H}_{\mathbb{C}}$ because $\mathfrak{m}$ is a Lie triple system (see [Hel62]). Since $\mathcal{H}_{\mathbb{C}}$ is a Hilbert space and $\mathfrak{g}_M$ is closed, the Lie algebra splits: it follows that $\mathfrak{g}_M$ is integrable (see [Lang95]). Let G_M be the connected Lie subgroup of $GL(\mathcal{H}_{\mathbb{C}})$ corresponding to the Lie algebra $\mathfrak{g}_M$.

Since $(A + [B, C])^* = A + [C, B]$ for any $A, B, C \in \mathfrak{m}$, then $M \subset G_M$ and $G_M^* = G_M$. It is also clear that $\mathfrak{k} = \overline{[\mathfrak{m}, \mathfrak{m}]}$ (in the notation of Theorem V.26). G_M is the smallest Lie group containing M.

The elements of M are indeed the positive elements of G_M, and the elements of K the unitary operators of G_M; we prove it below. Note that when $\mathfrak{m}$ is an abelian Lie subalgebra, $\mathfrak{g}_M = \mathfrak{m}$ and also $G_M = M \subset \mathfrak{m}$ is an open set.

Lemma V.28. *With the notation of the previous remark and the hypothesis of Theorem V.26, we have* $P(G_M) \subset M$.

Proof. Since $\mathfrak{g}_M$ splits, there are neighbourhoods of zero $U_\mathfrak{m} \subset \mathfrak{m}$ and $U_\mathfrak{k} \subset \mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$ such that the map $X_\mathfrak{m} + Y_\mathfrak{k} \mapsto e^{X_\mathfrak{m}} e^{Y_\mathfrak{k}}$ is an isomorphism from $U_\mathfrak{m} \oplus U_\mathfrak{k}$ onto an open neighbourhood V_M of $1 \in G_M$. Clearly, the group generated by V_M is open (and closed) in G_M, and so is the whole of G_M. So, for any $g \in G_M$,

$$g = (e^{X_1} e^{Y_1})^{\alpha_1} \cdots (e^{X_n} e^{Y_n})^{\alpha_n}$$

for some selfadjoint operators $X_i \in U_\mathfrak{m}$, some skewadjoint operators $Y_i \in U_\mathfrak{k}$, and $\alpha_i = \pm 1$.

Now $e^X e^Y e^X \in M$ whenever $X, Y \in \mathfrak{m}$ (see Theorem V.10), so mere inspection of the expression for $P(g) = gg^*$ shows that $P(g)$ will be in M if we can prove that $e^Y e^X e^{-Y} \in M$ whenever $X \in \mathfrak{m}$ and $Y \in \mathfrak{k}$ (namely, if we can prove that $kMk^* \subset M$ for any $k \in K$). It will be enough to show this holds for $X \in \mathfrak{m}$ and $Y = \sum_i [A_i, B_i] \in [\mathfrak{m},\mathfrak{m}]$ because M is closed. We assert that this is true, but to avoid cumbersome notations we write the proof for $Y = [A,B]$. The proof of the general case is identical.

Consider the map $F : \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$ given by $F(z) = [[A,B],z]$. Since F maps $\mathfrak{m}$ into $\mathfrak{m}$, the flow of F in $\mathfrak{m}$ stays in $\mathfrak{m}$, so the ordinary differential equation $\dot{X}(t) = F(X(t))$ has unique solution in $\mathfrak{m}$ if $X(0) \in \mathfrak{m}$ is given (see [Lang95]). Take $\alpha(t) = e^{\,t[A,B]} X\, e^{-t[A,B]}$. Then $\alpha(0) = X \in \mathfrak{m}$; moreover

$$\dot{\alpha}(t) = e^{\,t[A,B]}[[A,B],X]\, e^{-t[A,B]} = [[A,B], e^{\,t[A,B]} X\, e^{-t[A,B]}] = F(\alpha(t))$$

which proves that $\alpha(t) \in \mathfrak{m}$ for any $t \geq 0$. In particular,

$$\alpha(1) = e^{\,[A,B]} X\, e^{-[A,B]} \in \mathfrak{m}$$

As $e^{[A,B]}$ is a unitary operator, exponentiating both sides leads to

$$e^{[A,B]}e^{X}\,e^{-[A,B]} \in M \quad \square$$

The previous lemma will be used to prove the first of the following results:

Theorem V.29. *If* $M = \exp(\mathfrak{m})$ *is convex and closed, and* $G_M \subset GL(\mathcal{H}_{\mathbb{C}})$ *is the connected Lie subgroup with Lie algebra* $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m},\mathfrak{m}]}$*, then*

(a) $P(G_M) = M$*, so* M *is a homogeneous space for* G_M*.*

(b) For any $g = \mid g \mid u_g$ *(Cauchy polar decomposition) in* G_M*, we have*

$$\mid g \mid = \sqrt{gg^*} \in M \subset G_M,$$

and also $u_g \in K \subset G_M$ *where* K *is the isotropy Lie subgroup*

$K = \{g \in G_M \,:\, gg^* = 1\}$ *with Lie algebra* $\mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$

In particular, G_M *has a polar decomposition*

$$G_M \simeq M \times K = P(G_M) \times U(G_M)$$

(c) $M = P(G_M) \simeq G_M/K$

(d) M *has nonpositive sectional curvature.*

(e) For $g \in G_M$*, consider* $I_g(r) = grg^*$*. Then* $I : G_M \to I_0(M)$*.*

(f) Take $p,q \in M$*, and set* $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{-\frac{1}{2}} \in G_M$*. Then* I_g *is an isometry in* $I_0(M)$ *which sends* p *to* q*, namely* G_M *acts transitively and isometrically on* M*.*

Proof. Since any $p \in M$ is the exponential of some $X \in \mathfrak{m}$, we get $p = P(e^{X/2})$, which proves that $M \subset P(G_M)$; the other inclusion is given by Lemma V.28.

To prove (*b*), note that $P(G_M) = M = \exp(\mathfrak{m})$; namely for any $g \in G_M$, $gg^* = P(g) \in M$; hence $gg^* = e^{X_0}$ for some $X_0 \in \mathfrak{m}$ which implies that $\mid g \mid = e^{X_0/2} \in M \subset G_M$. By definition, $u_g = \mid g \mid^{-1} \cdot$ is an element of G_M (and clearly $u_g \in K$).

Statement (c) follows from Theorem V.26, Remark V.27 and statement (b).

The assertion in (d) follows from (a) and the fact that M is totally geodesic, together with equation (5) in the proof of Proposition III.3.

To prove (e) , note that I_g is an isometry of M because M has the induced metric so Lemma II.6 applies; from Lemma V.28 we deduce that $kMk^* \subset M$ for any $k \in K$; from Theorem V.10 and statement (b) follows easily that I_g maps M into M; since G_M is connected, we have the assertion.

Statement (f) follows from statement (e) and the proof of Theorem V.18. $\square$

From folk results (or from the classification of L^*-algebras, see [Neh93] and [CGM90]) follows that

$$\overline{[HS, HS]} = HS \quad \text{and} \quad \overline{[HS^h, HS^h]} = iHS^h,$$

so taking $\mathfrak{m} = \mathcal{H}_{\mathbb{R}} = \mathbb{R} \oplus HS^h$ we get $\mathfrak{k} = iHS^h$, hence $\mathfrak{g}_M = \mathbb{R} \oplus HS = \mathcal{H}_{\mathbb{C}}/i\mathbb{R}$, hence

$$G_{\Sigma_\infty} = GL(\mathcal{H}_{\mathbb{C}})/S^1 = \{\alpha + a\,;\, \alpha \in \mathbb{R}_{>0},\ a \in HS \text{ and } -\alpha \notin \sigma(a)\}$$

In the preceding line $\sigma(a)$ denotes the spectrum of a as an element of $L(H)$. Clearly $P(G_{\Sigma_\infty}) = P(GL(\mathcal{H}_{\mathbb{C}})) = \Sigma_\infty$ since any positive invertible operator has an invertible square root. On the other hand it is also obvious that the isotropy group K equals $\mathcal{U}(\mathcal{H}_{\mathbb{C}})$ (the unitary group of $\mathcal{H}_{\mathbb{C}}$, see section X), so

Corollary V.30. *There is an analytic isomorphism given by polar decomposition*

$$\Sigma_\infty \simeq GL(\mathcal{H}_{\mathbb{C}})/\mathcal{U}(\mathcal{H}_{\mathbb{C}})$$

The manifold of positive invertible operators Σ_∞ is a homogeneous space for the group of invertible operators $GL(\mathcal{H}_{\mathbb{C}})$, which acts isometrically and transitively on Σ_∞.

This last statement is well known, and Theorem V.29 can be seen as a natural generalization in this context.

VI Factorization Theorems

Combining the usual theory of Hadamard manifolds with some ad-hoc techniques for the infinite-dimensional context, we shall prove that given a geodesically convex closed submanifold M of Σ_∞, there is a unique geodesic γ joining p and M, such that the length of γ is exactly the distance between p and M.

VI.1 Geodesic projection

We will use the first and second variation formulas for curves in Riemannian manifolds; we refer the reader to [Lang95].

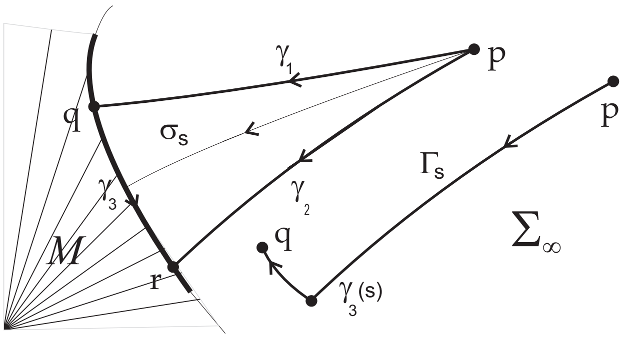
Proposition VI.1. *Let M be a geodesically convex subset of Σ_∞, and let $p \in \Sigma_\infty$. Then there is at most one normal geodesic γ of Σ_∞ joining p and*

M *such that* $L(\gamma) = \text{dist}(p, M)$. *In other words, there is at most one point* $q \in M$ *such that* $\text{dist}(p, q) = \text{dist}(p, M)$.

Proof. Suppose there are two such points, q and $r \in M$, joined by a geodesic $\gamma_3 \in M$, such that $L(\gamma_1) = \text{dist}(p, q) = L(\gamma_2) = \text{dist}(p, r) = d(p, M)$. We construct a proper variation of $\gamma \equiv \gamma_1$, which we call Γ_s.
The construction follows the figure below, where

$$\sigma_s(t) = \sigma(s,t) = p^{\frac{1}{2}} \left[p^{-\frac{1}{2}} q^{\frac{1}{2}} \left(q^{-\frac{1}{2}} r q^{-\frac{1}{2}} \right)^s q^{\frac{1}{2}} p^{-\frac{1}{2}} \right]^t p^{\frac{1}{2}}$$

is the minimal geodesic joining p with $\gamma_3(s)$.



$$\Gamma(s,t) = \begin{cases} \sigma(s,t) & \text{if} \quad 0 \le t \le 1 \\ \gamma_3(s(2-t)) & \text{if} \quad 1 \le t \le 2 \end{cases}$$

Note that

$$\gamma(t) = \Gamma(0,t) = \begin{cases} \sigma(t,0) & \text{if} \quad 0 \le t \le 1 \\ q & \text{if} \quad 1 \le t \le 2 \end{cases} = \begin{cases} \gamma_1(t) & \text{if} \quad 0 \le t \le 1 \\ q & \text{if} \quad 1 \le t \le 2 \end{cases}$$

so

$$\dot{\gamma}(t) = \begin{cases} \dot{\gamma}_1(t) & \text{if} \quad 0 \le t \le 1 \\ 0 & \text{if} \quad 1 \le t \le 2 \end{cases}$$

Also note that the variation vector field (which is a piecewise Jacobi field for the curve γ) is given by equations

$$V(t) = \frac{\partial \Gamma}{\partial s}(t,0) = \begin{cases} \frac{\partial \sigma}{\partial s}(t,0) & \text{if} \quad 0 \le t \le 1 \\ \\ (2-t)\dot{\gamma}_3(0) & \text{if} \quad 1 \le t \le 2 \end{cases}$$

If $\Delta_i \dot{\gamma}$ denotes the jump of the tangent vector field to γ at t_i, namely $\dot{\gamma}(t_i^+) - \dot{\gamma}(t_i^-)$, and Γ is a proper variation of γ, then the first variation formula for

$\gamma : [a,b] \to \Sigma_\infty$ reads

$$\|\dot\gamma\| \, \frac{d}{ds}|_{s=0} L(\Gamma_s) = -\int_a^b \langle V(t), D_t\dot\gamma(t)\rangle \, dt - \sum_{i=1}^{k-1} \langle V(t_i), \Delta_i \dot\gamma\rangle$$

where D_t stands for the covariant derivative.
In this case, $D_t\dot\gamma$ is zero in the whole $[0,2]$, because γ consists (piecewise) of geodesics. The jump points are $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$, so the formula reduces to

$$\|\dot\gamma\| \, \frac{d}{ds}|_{s=0} L(\Gamma_s) = -\langle V(1), \Delta_1 \dot\gamma\rangle$$

Thus, we get

$$\langle \dot\gamma_3(0), \dot\gamma_1(1)\rangle = \frac{d}{ds}|_{s=0} L(\Gamma_s) \, \|\dot\gamma\|$$

Recall that $\gamma_3 \subset M$, and that γ_1 is minimizing. Then the right hand term is zero, which proves that γ_1 and γ_3 are orthogonal at q. Similarly, γ_2 and γ_3 are orthogonal at r.
Hence, the sum of the three inner angles of this geodesic triangle is *at least* π. It follows from Lemma IV.11 that the angle at p must be null, which proves that γ_1 and γ_2 are the same geodesic, and uniqueness follows. □

Now, we consider the problem of the existence of the minimizing geodesic. We can rephrase the problem in the following way:

Theorem VI.2. *Let M be a geodesically convex submanifold of Σ_∞, and p a point of Σ_∞ not in M. Then existence of a geodesic joining p with M such that $L(\gamma) = \mathrm{dist}(p, M)$ is equivalent to the existence of a geodesic joining p with M with the property that γ is orthogonal to M.*

Proof. In fact, the existence of such a geodesic is equivalent to the existence of a point $q_p \in M$ such that $\mathrm{dist}(p, M) = \mathrm{dist}(p, q_p)$, and we will show that if $q \in M$ is a point such that γ_{qp} is orthogonal to M at q, then $\mathrm{dist}(q, p) = \mathrm{dist}(M, p)$. The other implication follows from the uniqueness theorem.
For this, consider the geodesic triangle generated by p, q and d, where d is any point in M different form q. As γ_{qp} is orthogonal to T_qM, it is, in particular,

orthogonal to γ_{qd}. Then, by virtue of the hyperbolic Cosine Law (equation (11)), we have

$$L(\gamma_{dp})^2 \geq L(\gamma_{qp})^2 + L(\gamma_{qd})^2 > L(\gamma_{qp})^2$$

which implies $\text{dist}(q,p) < \text{dist}(d,p)$. $\square$

We conclude that the existence problem is equivalent to the question:

- Is NM, the normal bundle of M, diffeomorphic to the whole Σ_∞, *via* the exponential map?

The answer to the local version of this question is yes, by virtue of the inverse function theorem. The reader can find the Banach space version of such theorem in [Lang95].

Lemma VI.3. *Set $E : NM \to \Sigma_\infty$, the map which assigns $(q,V) \mapsto \text{Exp}_q(V)$. Then there is an open neighbourhood $M \subset \Omega_\epsilon \subset \Sigma_\infty$ such that $\Omega_\epsilon \subset E(NM)$ and Ω_ϵ is C^ω diffeomorphic to the open tube $\{(p,V) : \|V\|_p < \epsilon\} \subset NM$.*

Proof. With the proper identifications, the differential of E at $(1,0) \in NM$ is the identity map because $T_1 M \oplus T_1 M^\perp = T_1 \Sigma_\infty$. The inverse map theorem gives a local neighbourhood, and the invariance of the metric for the maps $I_p : X \mapsto p^{\frac{1}{2}} X p^{\frac{1}{2}}$ $(p \in M)$ gives the desired tube. $\square$

Remark VI.4. Clearly $E(NM) \subset \Sigma_\infty$ is the set of points in Σ_∞ with the following property: there is a point $q \in M$ such that $\text{dist}(q,p) = \text{dist}(M,p)$.

Note that the map $\Pi_M : E(NM) \to M$, which assigns to $p \in E(NM)$ the unique point $q \in M$ such that $\text{dist}(q,p) = \text{dist}(M,p)$ is injective. This map is obtained *via* a geodesic that joins p and M, and this geodesic is orthogonal to M, therefore we will call $\Pi_M(p)$ the *foot of the perpendicular* from p to M.

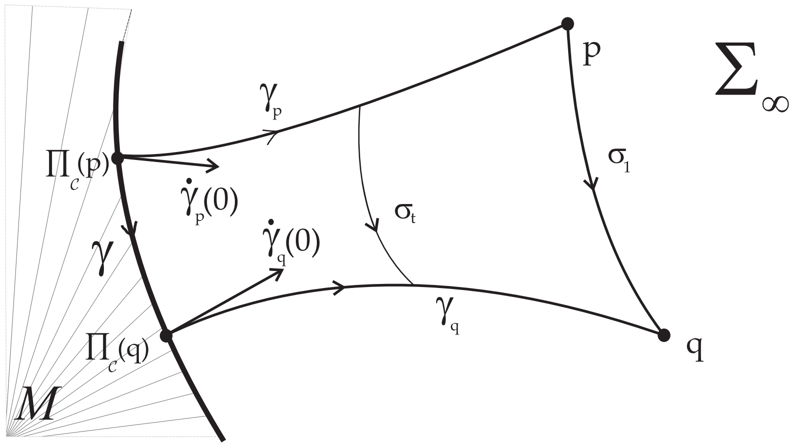
Theorem VI.5. *The map Π_M is a contraction, namely*

$$\text{dist}(\Pi_M(p), \Pi_M(q)) \leq \text{dist}(p,q)$$

Proof. We may assume that $p, q \notin M$, and that $\Pi_M(p) \neq \Pi_M(q)$. If γ_p is a geodesic that joins $\Pi_M(p)$ to p and γ_q is a geodesic that joins $\Pi_M(q)$ to q, set

$$f(t) = \operatorname{dist}(\gamma_p(t), \gamma_q(t))$$

Note that $f(0) = d(\Pi_M(p), \Pi_M(q))$ and $f(1) = \operatorname{dist}(p, q)$. We also know that f is a convex function of t (Corollary III.6). If we prove that $f'(0) \geq 0$, it will follow that f is monotone increasing, and we will have proved the assertion.
Take a variation $\sigma(t, s)$, being $\sigma_t(s)$ the unique geodesic joining $\gamma_p(t)$ to $\gamma_q(t)$. Then $\sigma(t, 0) = \gamma_p(t)$, $\sigma(t, 1) = \gamma_q(t)$, $\sigma(0, s) = \gamma(s)$ is the geodesic joining $\Pi_M(p)$ to $\Pi_M(q)$ (which is contained in M by virtue of the convexity), and finally $\sigma(1, s)$ is the geodesic joining p to q. This construction is better shown in the following figure:



Note that $f(t) = L(\sigma_t)$. We apply the first variation formula to this particular σ, to get

$$\|\dot\gamma\| \frac{d}{dt}|_{t=0} L(\sigma_t) = -\int_0^1 \langle V(s), D_s\dot\gamma(s)\rangle\, ds + \langle V(1), \dot\gamma(1)\rangle - \langle V(0), \dot\gamma(0)\rangle$$

The fact that γ is a geodesic and observation of the figure above reduces the formula to

$$\|\dot\gamma\|\, f'(0) = -\langle V(1), -\dot\gamma(1)\rangle + \langle -V(0), \dot\gamma(0)\rangle$$

Looking at the figure also shows that $V(0) = \dot\gamma_p(0)$, $V(1) = \dot\gamma_q(0)$. Recalling that the angles at M are right angles, we get $f'(0) = 0$. □

Remark VI.6. In the preceding proof, $f'(0)=0$ implies that it is exactly in this geodesic joining the projections that the distance between the projecting geodesics is minimal. This is related with the fact that Σ_∞ is a symmetric manifold. There is an alternate proof for the fact that $f'(0)=0$, which involves nothing but a little bit of the Riesz functional calculus; we include it because we think the proof shows in what extent this result can be translated to other algebras of operators such as a von Neumann algebra with a faithful trace (where the Riemannian structure is not necessarily complete in the metric sense). See Remark XI.5. For simplicity we assume that $M=\exp(\mathfrak{m})$ where $\mathfrak{m}\subset HS^h$ is a closed Lie triple system.

Proof. Let's consider the square of the distance function

$$f^2(t)=\text{dist}^2(\gamma_p(t),\gamma_q(t))=\text{dist}^2(Exp_{\Pi_M(p)}(tV),Exp_{\Pi_M(q)}(tW))$$

Naming $r=\Pi_M(p)$, $s=\Pi_M(q)$, recall that

$$\gamma_p(t)=r^{\frac{1}{2}}e^{tr^{-\frac{1}{2}}Vr^{-\frac{1}{2}}}r^{\frac{1}{2}}\qquad \gamma_q(t)=s^{\frac{1}{2}}e^{ts^{-\frac{1}{2}}Ws^{-\frac{1}{2}}}s^{\frac{1}{2}}$$

Since $V\in(T_rM)^\perp$ and $W\in(T_sM)^\perp$, we have

$$\begin{gathered}< V,r^{\frac{1}{2}}Xr^{\frac{1}{2}}>_r=\text{tr}\left(Xr^{-\frac{1}{2}}Vr^{-\frac{1}{2}}\right)=0 \text{ for any } X\in\mathfrak{m}=T_1M \text{ and}\\ < W,s^{\frac{1}{2}}Ys^{\frac{1}{2}}>_s=\text{tr}\left(Ys^{-\frac{1}{2}}Ws^{-\frac{1}{2}}\right)=0 \text{ for any } Y\in\mathfrak{m}=T_1M\end{gathered}\tag{13}$$

Now we use the formula $\text{dist}(e^A,e^B)=\|\ln(e^{A/2}e^{-B}e^{A/2})\|_2$ for $A=\ln(\gamma_p(t))$ and $B=\ln(\gamma_q(t))$, to write

$$f^2(t)=\|\ln(\gamma_p^{\frac{1}{2}}\gamma_q^{-1}\gamma_p^{\frac{1}{2}})\|_2^2=\text{tr}\left(\ln^2(\gamma_p^{\frac{1}{2}}\gamma_q^{-1}\gamma_p^{\frac{1}{2}})\right)$$

Now assume that C is a simple, positively oriented curve in $\mathbb{C}$, around the spectrum of $\alpha_0=r^{\frac{1}{2}}s^{-1}r^{\frac{1}{2}}$. Then we can use the Cauchy formula to calculate $\ln^2(\alpha)$

for any element $a \in \mathcal{H}_{\mathbb{C}}$ such that $\sigma(a) \subset \mathrm{int}(C)$, namely

$$\ln^2(a) = \frac{1}{2\pi i}\int_C \ln^2(z)(z-a)^{-1}\,dz \tag{14}$$

Naming $\alpha(t) = \gamma_p^{\frac{1}{2}}(t)\gamma_q^{-1}(t)\gamma_p^{\frac{1}{2}}(t)$, this formula holds true for $\alpha_0 = \alpha(0)$ and for $\alpha(t)$ for t sufficiently small, since α is a smooth function of t and the spectrum is a lower semicontinuous function.

Note that

$$f^2(t) = \mathrm{tr}\left(\gamma^{-\frac{1}{2}}(t)\gamma^{\frac{1}{2}}(t)\ln^2(\alpha(t))\right) = \mathrm{tr}\left(\gamma^{-\frac{1}{2}}(t)\ln^2(\alpha(t))\gamma^{\frac{1}{2}}(t)\right)$$

but for X invertible in $\mathcal{H}_{\mathbb{C}}$, $Xg(a)X^{-1} = g(XaX^{-1})$ for any element $a \in \mathcal{H}_{\mathbb{C}}$ and any analytic function g in a neighbourhood of $\sigma(a)$; this follows from formula (14) above and the identity $X(z-a)^{-1}X^{-1} = (z-XaX^{-1})^{-1}$. This leads to

$$f^2(t) = \mathrm{tr}\left(\ln^2\left[\gamma_p(t)\gamma_q^{-1}(t)\right]\right) = \frac{1}{2\pi i}\int_C \ln^2(z)\,\mathrm{tr}\left[\left(z-\gamma_p(t)\gamma_q^{-1}(t)\right)^{-1}\right]dz$$

by the linearity of the trace.

Now we compute $f'(0)$; note first that $\gamma_p(0)\gamma_q^{-1}(0) = rs^{-1}$ and also that

$$\frac{d}{dt}_{t=0}\gamma_p(t)\gamma_q^{-1}(t) = -Vs^{-1} + rs^{-1}Ws^{-1}$$

Using the ciclicity of the trace we get

$$\frac{d}{dt}_{t=0} f^2(t) = -\frac{1}{2\pi i}\int_C \ln^2(z)\,\mathrm{tr}\left[\left(z-rs^{-1}\right)^{-2}(-Vs^{-1}+rs^{-1}Ws^{-1})\right]dz =$$

$$= \mathrm{tr}\left[\left(-\frac{1}{2\pi i}\int_C \ln^2(z)\left(z-rs^{-1}\right)^{-2}dz\right)\left(-Vs^{-1}+rs^{-1}Ws^{-1}\right)\right]$$

Now we integrate by parts the first factor inside the trace, and what we obtain (since $\frac{d}{dz}\ln^2(z) = 2\ln(z)z^{-1} = 2z^{-1}\ln(z)$ and C is a closed curve) is

$$2f(0)f'(0) = \mathrm{tr}\left[\left(\frac{1}{2\pi i}\int_C 2\ln(z)z^{-1}\left(z-rs^{-1}\right)^{-1}dz\right)\left(-Vs^{-1}+rs^{-1}Ws^{-1}\right)\right] =$$

$$= -\mathrm{tr}\left[\left(\frac{1}{2\pi i}\int_C 2\ln(z)z^{-1}\left(z-rs^{-1}\right)^{-1}dz\right)Vs^{-1}\right]+$$

$$+\mathrm{tr}\left[\left(\frac{1}{2\pi i}\int_C 2\ln(z)z^{-1}\left(z-rs^{-1}\right)^{-1}dz\right)rs^{-1}Ws^{-1}\right]$$

So, by ciclicity of the trace and linearity,

$$2f(0)\,f'(0) = -\frac{1}{2\pi i}\int_C 2\ln(z)z^{-1}\,\mathrm{tr}\left[s^{-1}\left(z-rs^{-1}\right)^{-1}V\right]dz+$$

$$+\frac{1}{2\pi i}\int_C 2\ln(z)z^{-1}\,\mathrm{tr}\left[s^{-\frac{1}{2}}\left(z-rs^{-1}\right)^{-1}rs^{-1}Ws^{-\frac{1}{2}}\right]dz$$

Now we use the identities

$$r^{\frac{1}{2}}(z-r^{\frac{1}{2}}s^{-1}r^{\frac{1}{2}})^{-1}r^{-\frac{1}{2}} = (z-rs^{-1})^{-1} = s^{\frac{1}{2}}(z-s^{-\frac{1}{2}}rs^{-\frac{1}{2}})^{-1}s^{-\frac{1}{2}}$$

to arrive to the expression

$$2f(0)\,f'(0) = -\frac{1}{2\pi i}\int_C 2\ln(z)z^{-1}\,\mathrm{tr}\left[s^{-1}r^{\frac{1}{2}}\left(z-r^{\frac{1}{2}}s^{-1}r^{\frac{1}{2}}\right)^{-1}r^{-\frac{1}{2}}V\right]dz+$$

$$+\frac{1}{2\pi i}\int_C 2\ln(z)z^{-1}\,\mathrm{tr}\left[\left(z-s^{-\frac{1}{2}}rs^{-\frac{1}{2}}\right)^{-1}s^{-\frac{1}{2}}rs^{-\frac{1}{2}}s^{-\frac{1}{2}}Ws^{-\frac{1}{2}}\right]dz=$$

$$= -2\mathrm{tr}\left[s^{-1}r^{\frac{1}{2}}r^{-\frac{1}{2}}sr^{-\frac{1}{2}}\ln(r^{\frac{1}{2}}s^{-1}r^{\frac{1}{2}})r^{-\frac{1}{2}}V\right]+$$

$$+2\mathrm{tr}\left[\ln(s^{-\frac{1}{2}}rs^{-\frac{1}{2}})s^{\frac{1}{2}}r^{-1}s^{\frac{1}{2}}s^{-\frac{1}{2}}rs^{-\frac{1}{2}}s^{-\frac{1}{2}}Ws^{-\frac{1}{2}}\right) =$$

$$= -2\mathrm{tr}\left[\ln(r^{\frac{1}{2}}s^{-1}r^{\frac{1}{2}})r^{-\frac{1}{2}}Vr^{-\frac{1}{2}}\right]+2\mathrm{tr}\left[\ln(s^{-\frac{1}{2}}rs^{-\frac{1}{2}})s^{-\frac{1}{2}}Ws^{-\frac{1}{2}}\right] =$$

$$= 0+0 = 0$$

by the orthogonality relations (13), naming $X = \ln(r^{\frac{1}{2}}s^{-1}r^{\frac{1}{2}})$ (recall that M is convex), and $Y = \ln(s^{-\frac{1}{2}}rs^{-\frac{1}{2}})$. Since we assumed that $r \neq s$, we have $f(0) \neq 0$, which proves that $f'(0) = 0$. $\square$

Let's get back to our problem: we wish to prove that $E(NM) = \Sigma_\infty$. We will do that by proving that it is both open and closed in Σ_∞. First we to prove that it is open:

Lemma VI.7. *For* $\lambda \in [1, +\infty)$*, set* $\eta_\lambda : E(NM) \to E(NM)$ *by* $\eta_\lambda(\mathrm{Exp}_p(V)) = \mathrm{Exp}_p(\lambda V)$*. Then* $E(NM) = \bigcup_{\lambda \geq 1} \eta_\lambda(\Omega_\epsilon)$*, and each* $\eta_\lambda : \Omega_\epsilon \to \Sigma_\infty$ *is a* C^ω *diffeomorphism with its image.*

Corollary VI.8. *The set* $E(NM)$ *is open in* Σ_∞*.*

Let's prove the Lemma; the geometric idea of the proof is due to Porta and Recht [PR94]:

Proof. Clearly $\cup_{\lambda \geq 1} \eta_\lambda(\Omega_\epsilon) \subset E(NM)$. On the other hand, if $r = \mathrm{Exp}_p(V)$ and $\|V\|_p < \epsilon$ then $r \in \Omega_\epsilon = \eta_1(\Omega_\epsilon)$. We may assume that $\|V\|_p \geq \epsilon$; taking $\lambda = \|V\|_p / c$ for any $c < \epsilon$ does the job.
We will prove that, for any $\lambda \geq 1$ and $r \in \Omega_\epsilon$, $d(\eta_\lambda)_r : T_r\Sigma_\infty \to T_{\eta_\lambda(r)}\Sigma_\infty$ is a linear isomorphism, and this will prove the assertion. Take $\alpha \subset \Omega_\epsilon$ a geodesic such that $\alpha(0) = r$ and $\dot\alpha(0) = X$. Since α is a geodesic, we have $\mathrm{dist}(\alpha(t), r) = t\|\dot\alpha(0)\|_r$ for $t \geq 0$ (see section IV.2). Set $\beta(t) = \eta_\lambda \circ \alpha$; then $\beta(0) = \eta_\lambda(r)$ and $\dot\beta(0) = d(\eta_\lambda)_r(X)$. Clearly $\mathrm{dist}(\beta(t), \eta_\lambda(r)) \leq L_0^t(\beta) = \int_0^t \|\dot\beta(s)\|_{\beta(s)} ds$. On the other hand,

$$\mathrm{dist}(\eta_\lambda(\alpha(t)), \eta_\lambda(r)) \geq \mathrm{dist}(\alpha(t), r) = t\|X\|_r$$

where the inequality comes from the proof of Theorem VI.5, since $\lambda \geq 1$. If we put together these two inequalities and divide by t, we get

$$\frac{1}{t}\int_0^t \|\dot\beta(s)\|_{\beta(s)} ds \geq \|X\|_r$$

and taking limit for $t \to 0^+$ gives

$$\|d(\eta_\lambda)_r(X)\|_{\eta_\lambda(r)} \geq \|X\|_r$$

Now set $A_\lambda = I_{\eta_\lambda(r)}^{-1} \circ d(\eta_\lambda)_r \circ I_r$, where the maps $I_p : V \mapsto p^{\frac{1}{2}} V p^{\frac{1}{2}}$ are linear isomorphisms (see Lemma II.6). If we consider $A_\lambda : T_1\Sigma_\infty \to T_1\Sigma_\infty = \mathcal{H}_\mathbb{R}$, what the inequality above says is that $\|A_\lambda(X)\|_2 \geq \|X\|_2$ for any $X \in \mathcal{H}_\mathbb{R}$.

Clearly $\eta_1 = id_{\Omega_\varepsilon}$ and $d(\eta_1)_r = id_{T_r\Sigma_\infty}$; since the map $(\lambda, r) \mapsto \eta_\lambda(r)$ is analytic from $\mathbb{R}_{>0} \times \Omega_\varepsilon$ to Σ_∞, there is an open neighbourhood of $1 \in \mathbb{R}$ such that A_λ is an isomorphism. Assume A_λ is invertible for $\lambda \in [1, m)$: then $\|A_\lambda^{-1}\|_{L(\mathcal{H}_\mathbb{R})} \leq 1$ for any $\lambda \in [1, m)$. Since $A_m = \lim_{\lambda \to m^-} A_\lambda$ (in the operator norm of $L(\mathcal{H}_\mathbb{R})$) and $\|A_m A_\lambda^{-1} - 1\| \leq \|A_m - A_\lambda\| < 1$ if λ is close enough to m, we get that $A_m A_\lambda^{-1}$ is invertible, thus A_m is invertible. Since the maps I_p are isomorphisms, we have proved that $d(\eta_\lambda)_r$ is an isomorphism for any $\lambda \geq 1$, for any $r \in \Omega_\varepsilon$. $\square$

Now we are ready to prove the main result of this section:

Theorem VI.9. *Let M be a geodesically convex, closed submanifold of Σ_∞. Then for every point $p \in \Sigma_\infty$, there is a unique normal geodesic γ_p joining p to M such that $L(\gamma_p) = \mathrm{dist}(p, M)$.*
Moreover, this geodesic is orthogonal to M, and if $\Pi_M : \Sigma_\infty \to M$ is the map that assigns to p the end-point of γ_p, then Π_M is a contraction for the geodesic distance.

Proof. The theorem will follow once we prove that $E(NM) = \Sigma_\infty$. But since Σ_∞ is connected and $E(NM)$ is open, it is enough to prove that $E(NM)$ is also closed.
Let $p \in \overline{E(NM)}$. There exist points $q_n \in M$, $V_n \in T_{q_n}M^\perp$ such that

$$p = \lim_n p_n = \lim_n \mathrm{Exp}_{q_n}(V_n)$$

Now observe that $q_n = \Pi_M(p_n)$, so $\mathrm{dist}(q_n, q_m) \leq \mathrm{dist}(p_n, p_m)$. As $\{p_n\}$ converges to p, it is a Cauchy sequence. It follows that $\{q_n\}$ is also a Cauchy sequence; since M is closed (and then complete), there must exist a point $q \in M$ such that $q = \lim_n q_n$. We assert that $\mathrm{dist}(p, q) = \mathrm{dist}(p, M)$. First observe that

$$\mathrm{dist}(p, q_n) \leq \mathrm{dist}(p, p_n) + \mathrm{dist}(p_n, q_n)$$

and $\mathrm{dist}(p_n, q_n) = \mathrm{dist}(p_n, M)$, so

$$\mathrm{dist}(p, q_n) \leq \mathrm{dist}(p, p_n) + \mathrm{dist}(p_n, M)$$

Taking limits gets us to the inequality $\mathrm{dist}(p, q) \leq \mathrm{dist}(p, M)$, which shows that the distance between p and M is given by $\mathrm{dist}(p, q)$. This concludes the proof. $\square$

Note that Σ_∞ decomposes as a direct product: with the contraction Π_M, we can decompose Σ_∞ by picking, for fixed p,

1. the unique point $q = \Pi_M(p)$ such that $\mathrm{dist}(p, q) = \mathrm{dist}(p, M)$
2. a vector V_p normal to $T_q M$ such that the geodesic in Σ_∞ with this initial velocity starting at q passes through p.

Note that $V_p = \mathrm{Exp}^{-1}_{\Pi_M(p)}(p)$, and also $\|V_p\|_p = \mathrm{dist}(p, M)$.

Since the exponential map is an analytic function on both of its variables (recall that, for any $q \in \Sigma_\infty$, and any $V \in \mathcal{H}_\mathbb{R}$, $\mathrm{Exp}_q(V) = q e^{q^{-1}V}$), we get

Theorem VI.10. *The map* $p \mapsto (\Pi_M(p), V_p)$ *is the inverse of the exponential map* $(q, V_q) \mapsto \mathrm{Exp}_q(V_q)$*, and it is, in fact, a real-analytic isomorphism between the manifolds* NM *and* Σ_∞*.*

This is a remarkable global analogue of the (linear) orthogonal decomposition of tangent spaces; we can read the theorem in a different fashoin if we recall that all points and tangent vectors are operators. This theorem is inspired mainly by the results on C^*-algebra decompositions [CPR91]

Theorem VI.11. *Fix a closed, geodesically convex submanifold* M *of* Σ_∞*. Take any operator* $A \in \Sigma_\infty$*. Then there exist operators* $C \in \Sigma_\infty, V \in \mathcal{H}_\mathbb{R}$ *such that* $C \in M$*,* $V \in T_C M^\perp$*, and:*

$$A = C e^{C^{-1}V} \tag{15}$$

Moreover, C *and* V *are unique, and the map* $A \mapsto (C, V)$ *(which maps* $\Sigma_\infty \to NM$*) is a real analytic isomorphism between manifolds.*

Naming $B = C^{\frac{1}{2}}$, $W = C^{-\frac{1}{2}} V C^{-\frac{1}{2}}$, equation (15) reads

$$A = B e^{W} B$$

for unique B, W.

Using the tools of section V, we can restate the theorem in terms of intrinsic operator equations (see [Mos55] for the finite dimensional analogue):

Theorem VI.12. *Assume $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ is a closed Lie triple system. Then for any operator $A \in \mathcal{H}_{\mathbb{R}}$, there exist unique operators $X \in \mathfrak{m}$, $V \in \mathfrak{m}^{\perp}$ such that the following decomposition holds:*

$$e^{A} = e^{X} e^{V} e^{X}$$

The operator X is the unique minimizer in $\mathfrak{m}$ of the map

$$Y \mapsto \| \ln(e^{A/2} e^{-Y} e^{A/2}) \|_2$$

As a corollary, we obtain a polar descomposition relative to any fixed convex submanifold. This decomposition resembles the Iwasawa decomposition of (finite dimensional) Lie groups, see [Hel62]:

Theorem VI.13. *Assume $M = \exp(\mathfrak{m}) \subset \Sigma_{\infty}$ is a closed, convex submanifold. Then for any $g \in GL(\mathcal{H}_{\mathbb{C}})$ there exists a unique factorization of the form $g = e^{X} e^{V} u$ where $X \in \mathfrak{m}$, $V \in \mathfrak{m}^{\perp}$ and $u \in \mathcal{U}(\mathcal{H}_{\mathbb{C}})$ is a unitary operator.*
The map $g \mapsto (e^{X}, e^{V}, u)$ is an analytic bijection which gives an isomorphism

$$GL(\mathcal{H}_{\mathbb{C}}) \simeq M \times \exp(\mathfrak{m}^{\perp}) \times \mathcal{U}(\mathcal{H}_{\mathbb{C}})$$

Proof. Since $gg^* \in \Sigma_{\infty}$, we can write $gg^* = e^{X} e^{2V} e^{X}$ with $X \in \mathfrak{m}$ and $V \in \mathfrak{m}^{\perp}$. Setting $u = (e^{X} e^{V})^{-1} g = e^{-V} e^{-X} g$ we have

$$uu^* = e^{-V} e^{-X} g g^* e^{-X} e^{-V} = 1 \quad \text{and also} \quad u^* u = g^* e^{-X} e^{-V} e^{-V} e^{-X} g = 1$$

Hence u is a unitary operator and $g = e^{X} e^{V} u$. This factorization is unique because if $g = e^{X_1} e^{V_1} u_1 = e^{X_2} e^{V_2} u_2$, then $gg^* = e^{X_1} e^{2V_1} e^{X_1} = e^{X_2} e^{2V_2} e^{X_2}$, so $X_1 = X_2$, $V_1 = V_2$ and then $u_1 = u_2$. $\square$

VII Some Applications

VII.1 Preliminaries

We will use the factorization theorem in several ways; for convenience we first state the following lemma, which we will be useful on several ocasions:

Lemma VII.1. *(the generalized exponential formula): for the exponential map in* Σ_∞*, we have*

$$\mathrm{Exp}_{\alpha+a}(\beta+b) = (\alpha+a)e^{(\alpha+a)^{-1}(\beta+b)} = (\alpha+a)[1+(\alpha+a)^{-1}(\beta+b)+\cdots]$$

$$= (\alpha+a)\left[1+\frac{\beta}{\alpha}+\frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2+\cdots+k\right] = \alpha\, e^{\beta/\alpha}+k$$

where $\alpha\, e^{\beta/\alpha} \in \mathbb{R}$ *and* k *is a Hilbert-Schmidt operator.*

We need some remarks before we proceed with the main applications. Fix an orthonormal basis of H.

1. The diagonal manifold $\Delta \subset \Sigma_\infty$ we define below is closed and geodesically convex.

   $$\Delta = \{d + \alpha > 0 : d \text{ is a diagonal Hilbert-Schmidt operator and } \alpha \in \mathbb{R}\}$$

   This is due to the fact that the diagonal operators form a closed abelian subalgebra (see Propositions III.3 and Corollary V.12).

2. If $p_0 \in \Delta$, then $T_{p_0}\Delta = \{\alpha + d;\ \alpha \in \mathbb{R},\ d \text{ a diagonal operator}\} = T_1\Delta$ by Remarks V.19 and V.20.

3. Consider the map $A \mapsto A^D$ = the diagonal part of A. Then

   (a) For Hilbert-Schmidt operators we have $A^D = \sum\limits_i p_i A p_i$ where convergence is in the 2-norm (and hence in the operator norm); here $p_i = e_i \otimes e_i = \langle e_i, \cdot \rangle\, e_i$ is the orthogonal projection in the real line generated by e_i

   (b) $(A^D)^D = A^D$

   (c) $\operatorname{tr}\left(A^D A\right) = \operatorname{tr}((A^D)^2)$

   (d) $\operatorname{tr}(A^D B) = \operatorname{tr}(AB)$ if B is diagonal

4. The scalar manifold $\Lambda = \{\lambda \cdot 1 \ : \lambda \in \mathbb{R}_{>0}\}$ is geodesically convex and closed in Σ_∞, with tangent space $\mathbb{R} \cdot 1 \subset \mathcal{H}_{\mathbb{R}}$

5. A vector $V = \mu + u \in T_{p_0}\Delta^\perp$ if and only if $\mu = 0$ and $u^D = 0$. This follows from: Remarks V.19 and V.20, the fact that $\mu + u^D \in T_{p_0}\Delta$, and Remark (3) of this list.

VII.2 The factorization itself

Theorem VII.2. *(infinite dimensional diagonal factorization): Take any selfadjoint Hilbert-Schmidt operator* a*. Then there exist a real scalar* $\lambda >$

0, a positive invertible Hilbert-Schmidt diagonal operator d *and a Hilbert-Schmidt selfadjoint operator with null diagonal* V *such that the following formula holds:*

$$a+\lambda=(d+\lambda)e^{(d+\lambda)^{-1}V}=(d+\lambda)^{\frac{1}{2}}e^{(d+\lambda)^{-\frac{1}{2}}V(d+\lambda)^{-\frac{1}{2}}}(d+\lambda)^{\frac{1}{2}}$$

Moreover (for fixed λ*)* d *and* V *are unique and* $a+\lambda\mapsto(d,V)$ *(wich maps* $\Sigma_\infty\to N\Delta$*) is a real analytic isomorphism between manifolds.*

Proof. Let $\lambda=\|a\|_\infty+\epsilon$, for any $\epsilon>0$. Then $p=a+\lambda\in\Sigma_\infty$, and $\Pi_\Delta(p)=d+\alpha$. The operator d satisfies our requirements. Now pick the unique $V\in T_{d+\alpha}\Delta^\perp$ such that $\mathrm{Exp}_{d+\alpha}(V)=p$, this operator V has zero diagonal because of remark (5) above. As a consequence of the 'exponential formula' (Lemma VII.1), $\alpha=\lambda$, for in this case, $\beta=0$. □

This theorem can be rephrased saying that, given a selfadjoint Hilbert-Schmidt operator a, for any $\lambda\in\mathbb{R}_{>0}$ such that $a+\lambda>0$, one has a unique factorization

$$a+\lambda=D\,e^{W}D$$

where $D=(\lambda+d)^{\frac{1}{2}}>0$ is a diagonal operator and $W=D^{-1}VD^{-1}$ is a selfadjoint operator and has null diagonal.

Corollary VII.3. *For any* $g\in GL(\mathcal{H}_\mathbb{C})$*, there is a unique factorization*

$$g=de^{W}u,$$

where d *is a positive invertible diagonal operator of* $\mathcal{H}_\mathbb{C}$*,* W *is a selfadjoint operator with null diagonal in* $\mathcal{H}_\mathbb{C}$ *and* u *is a unitary operator of* $\mathcal{H}_\mathbb{C}$*.*

Proof. It is a consequence of the previous remark together with Theorem VI.13. □

We now observe that, for finite (strictly positive) matrices, we could choose $\lambda=0$ (in a sense we will make precise) because any matrix has finite spectrum. With this observation in mind, we can state and prove a finite dimensional analogue of the factorization theorem, which has a simpler form. We should remark that (in this particular case) this result is **exactly** Theorem 3 of [Mos55] by G.D. Mostow (see also [CPR91] by Corach, Porta and Recht). The only thing to remark is that the geometric intrepretation of the splitting is crystal clear in this context, because the diagonal matrix D is the closest diagonal matrix to A, and V is the initial direction of the geodesic starting at D which joins D to A. We will use the standard notation

$$M_n^+ = \{M \in \mathbb{C}^{n\times n} : M^t = M^\dagger,\ \sigma(M) \subset (0,+\infty)\}$$

The dagger (†) stands for complex conjugation of the coefficients of M.
We will use M_n to denote the tangent space of M_n^+ at Id; recall that M_n^+ is open in M_n, and also that M_n can be identified with the hermitian matrices of $\mathbb{R}^{n\times n}$.

Theorem VII.4. *(finite dimensional diagonal factorization): Fix a positive invertible matrix* $A \in M_n^+$. *Then there exist* unique *matrices* $D, V \in M_n$, *such that* D *is diagonal and strictly positive,* V *is symmetric and with null diagonal, which make the following formula hold:*

$$A = D\, e^V D$$

Moreover, the maps $A \mapsto D$ *and* $A \mapsto V$ *are real analytic.*

Proof. We will prove the result using block products. For this, choose an orthonormal basis of H, and write Hilbert-Schmidt operators as infinite matrices. In this way we can embed M_n^+ in Σ_∞, by means of the map that sends A to the first $n \times n$ block:

$$A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = a + 1, \text{ where } a = \begin{pmatrix} A-1 & 0 \\ 0 & 0 \end{pmatrix} \in HS^h$$

Note that $a + 1 = A + P_{(KerA)^\perp}$, and that $a + 1 > 0$ because $A > 0$.
Using the infinite dimensional theorem, we can factorize $a + 1 = de^V d$, where $V = d^{-1} W d^{-1}$ and W is orthogonal to the diagonal submanifold $\Delta \subset \Sigma_\infty$. Obviously,

$$d = \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix}$$

but note that $V = \ln\left(d^{-1}(a+1)d^{-1}\right)$ so

$$V = \ln\begin{pmatrix} D^{-1}AD^{-1} & 0 \\ 0 & D_\infty^{-2} \end{pmatrix} = \begin{pmatrix} \ln(D^{-1}AD^{-1}) & 0 \\ 0 & \ln(D_\infty^{-2}) \end{pmatrix}$$

which shows that the desired V (the first block of V) has the desired properties, as V has them. Now $a+1 = de^V d$ reads

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} \exp\left\{\begin{pmatrix} V & 0 \\ 0 & \ln(D_\infty^{-2}) \end{pmatrix}\right\} \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} =$$

$$= \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} \begin{pmatrix} e^V & 0 \\ 0 & D_\infty^{-2} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} = \begin{pmatrix} D\,e^V D & 0 \\ 0 & 1 \end{pmatrix}$$

and comparing the first blocks, we have the claim. □

Remark VII.5. In [AV03] Andruchow and Varela prove that there is a natural, flat embedding of $M = M_n^+$ into Σ_∞ (Proposition 4.1 and Remark 4.2). This embedding makes M_n^+ a closed, geodesically convex submanifold of Σ_∞. We will postpone a projection theorem for this submanifold for the sake of simplicity. See Theorem VIII.10

.

VIII A Foliation of Codimension One

In this section we describe a foliation of the total manifold, and show how to translate the results from previous sections to a particular leaf (the submanifold Σ_1). We begin with a description of the leaves.

VIII.1 The leaves Σ_λ

Recall that we write HS^h (hermitian Hilbert-Schmidt operators) to denote the closed vector space of operators in $\mathcal{H}_\mathbb{R}$ with no scalar part. We define the following family of submanifolds (for fixed $\lambda \in \mathbb{R}_{>0}$):

$$\Sigma_\lambda = \{a + \lambda \in \Sigma_\infty,\ a \in \mathrm{HS}^h\}$$

Observe that $\Sigma_\lambda \cap \Sigma_\beta = \emptyset$ when $\lambda \neq \beta$, since $a + \lambda = b + \beta$ implies $a - b = \beta - \lambda$.

In this way, we can decompose the total space by means of these leaves,

$$\Sigma_\infty = \coprod_{\lambda>0} \Sigma_\lambda$$

Theorem VIII.1. *The leaves Σ_λ are closed and geodesically convex submanifolds of Σ_∞.*

Proof. The fact that the projection Π_Λ is a contractive map implies that Σ_λ is closed, one must only observe that $\Sigma_\lambda = \Pi_\Lambda^{-1}(\lambda)$.
To show that Σ_λ is convex we recall that, by virtue of the 'exponential formula' (Lemma VII.1), for any real $\lambda > 0$ and any $p \in \Sigma_\lambda$, there is an identification *via* the inverse exponential map at p, $T_p\Sigma_\lambda = HS^h$. □

Remark VIII.2. Take $\delta + c \in T_{a+\lambda}\Sigma_\lambda{}^\perp$. Since $T_{a+\lambda}\Sigma_\lambda$ can be identified with HS^h, condition

$$\langle \delta + c, d\rangle_{a+\lambda} = 0 \qquad \forall \quad d \in HS^h$$

immediately translates into

$$\mathrm{tr}\left[(a+\lambda)^{-1}\left[(\delta+c)(a+\lambda)^{-1} - \frac{\delta}{\lambda}\right] d\right] = 0 \qquad \forall \quad d \in HS^h$$

This says that $T_{a+\lambda}\Sigma_\lambda^\perp = span(a+\lambda)$; shortly $T_p\Sigma_\lambda^\perp = span(p)$ for any $p \in \Sigma_\lambda$.

Proposition VIII.3. *Fix real $\alpha, \lambda > 0$. Set $\Pi_{\alpha,\lambda} = \Pi_{\Sigma_\lambda}|_{\Sigma_\alpha} : \Sigma_\alpha \to \Sigma_\lambda$. Then*

1. $\Pi_{\alpha,\lambda}(p) = \frac{\lambda}{\alpha}p$*, so* $\Pi_{\alpha,\lambda}(p)$ *commutes with* p
2. $\Pi_{\alpha,\lambda}$ *is an isometric bijection between* Σ_α *and* Σ_λ*, with inverse* $\Pi_{\lambda,\alpha}$.
3. $\Pi_{\alpha,\lambda}$ *gives parallel translation (see Remark V.21) along 'vertical' geodesics joining both leaves.*

Proof. Notice that for a point $b + \alpha \in \Sigma_\alpha$ to be the endpoint of the geodesic γ starting at $a + \lambda \in \Sigma_\lambda$ such that $L(\gamma) = \mathrm{dist}\,(b + \alpha, \Sigma_\lambda)$, we must have

$$b + \alpha = \mathrm{Exp}_{a+\lambda}(x + c) = \mathrm{Exp}_{a+\lambda}(k.(a+\lambda)) = e^k(a+\lambda)$$

because $x+c \in T_{a+\lambda}\Sigma_\lambda{}^\perp$. From Lemma VII.1, we deduce that $k = \ln\left(\frac{\alpha}{\lambda}\right)$, and $a = \frac{\lambda}{\alpha}b$. So, $b+\alpha = \frac{\alpha}{\lambda}(a+\lambda)$ and also

$$\gamma(t) = (a+\lambda)\left(\tfrac{\alpha}{\lambda}\right)^t$$

Now it is obvious that $\Pi_\lambda(b+\alpha) = \frac{\lambda}{\alpha}(b+\alpha)$ and commutes with $b+\alpha$.
To prove that Π is isometric, observe that

$$\operatorname{dist}\left(\Pi_{\alpha,\lambda}(p), \Pi_{\alpha,\lambda}(q)\right) = \operatorname{dist}\left(\frac{\lambda}{\alpha}p, \frac{\lambda}{\alpha}q\right) = \operatorname{dist}(p,q)$$

by inspection of the geodesic equation (2) of section II and Remark II.4.
That Π gives parallel translation along γ follows from $q = \frac{\lambda}{\alpha}p$ and Remark V.21.

□

Proposition VIII.4. *The leaves Σ_α, Σ_λ are also parallel in the following sense: any minimizing geodesic joining a point in one of them with its projection in the other is orthogonal to both of them. For any $b+\alpha \in \Sigma_\alpha$,*

$$\operatorname{dist}(b+\alpha, \Sigma_\lambda) = \operatorname{dist}(\Sigma_\alpha, \Sigma_\lambda) = \left| \ln\left(\tfrac{\alpha}{\lambda}\right) \right|$$

In particular, the distance between α, λ in the scalar manifold Λ is given by the Haar measure of the open interval (α, β) on $\mathbb{R}_{>0}$. (This was remarked by E. Vesentini in his paper [Ves76]).

Proof. It is a straightforward computation that follows from the previous results.

□

Since Σ_∞ is a symmetric space, curvature is preserved when we parallel-translate bidimensional planes; note also that vertical planes are commuting sets of operators, so

Proposition VIII.5. *For any point $p \in \Sigma_\lambda$, sectional curvature of vertical 2-planes is trivial.*

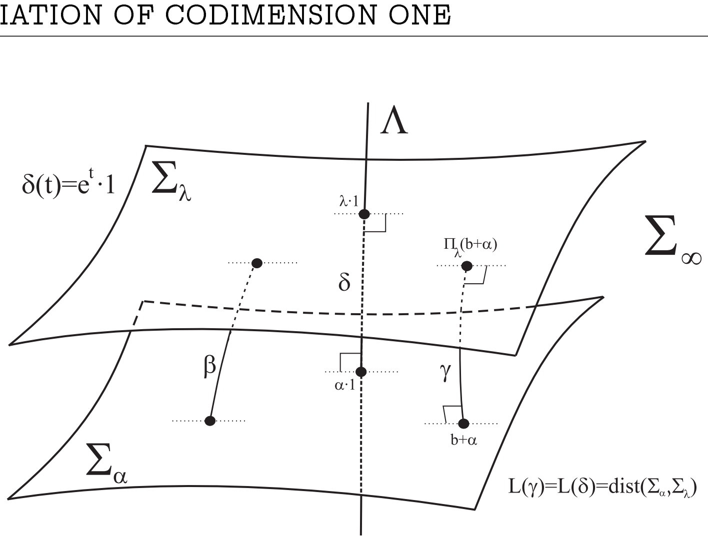


Figure 1: The geodesics γ and δ are minimizing, the geodesic β is not

Proof. We know that p generates $T_p\Sigma_\lambda^\perp$; take any other vector $V \in T_p\Sigma_\lambda = HS^h$. Equation (4) of section II says

$$\langle \mathcal{R}_p(p,V)V,p\rangle_p = -\frac{1}{4}\left\langle \left[\left[p^{-1}p,p^{-1}V\right]p^{-1}V\right],pp^{-1}\right\rangle_2 = 0 \quad \square$$

Theorem VIII.6. *The map* $T : \Sigma_\infty \to \Sigma_1 \times \Lambda$*, which assigns*

$$a+\alpha \mapsto \left(\frac{1}{\alpha}(a+\alpha),\alpha\right)$$

is bijective and isometric (Σ_1 *and* Λ *have the induced submanifold metric). In other words, there is a Riemannian isomorphism*

$$\Sigma_\infty \simeq \Sigma_1 \times \Lambda$$

Proof. Another straightforward computation. $\square$

The previous theorems show that the geometry of Σ_∞ is essentially the geometry of Σ_1; in particular, the factorization theorem inside Σ_1 has a simpler form; we state it below

Theorem VIII.7. *Fix a closed, geodesically convex submanifold* M *of* Σ_1*. For any* $a+1 \in \Sigma_1$*, there is a selfadjoint Hilbert-Schmidt operator* d *such* *that* $d+1 \in M$, *and a selfadjoint Hilbert-Schmidt operator* V, *such that* $V \in T_{d+1}M^{\perp}$, *which make the following formula hold:*

$$1+a = [1+d]\; e^{(1+d)^{-1} V}$$

Moreover d *and* V *are unique, and the map* $1+a \mapsto (1+d, V)$ *(which maps* Σ_1 *to* NM*) is a real analytic isomorphism between manifolds. Equivalently,*

$$1+a = [1+d]^{\frac{1}{2}}\; e^{(1+d)^{-\frac{1}{2}} V (1+d)^{-\frac{1}{2}}}\, [1+d]^{\frac{1}{2}}$$

The intrinsic version of the theorem reads (see Theorem V.11):

Theorem VIII.8. *Assume* $\mathfrak{m} \subset HS^h$ *is a closed subspace such that*

$$[x,[x,y]] \in \mathfrak{m} \quad \textit{for any} \quad x, y \in \mathfrak{m}$$

Then for any $a \in HS^h$ *there is a unique decomposition of the form*

$$e^a = e^x e^v e^x$$

where $x \in \mathfrak{m}$ *and* $v \in HS^h$ *is such that* $\operatorname{tr}(vz) = 0$ *for any* $z \in \mathfrak{m}$*. The operator* x *is the unique minimizer in* $\mathfrak{m}$ *of the map*

$$y \mapsto \operatorname{tr}\left(\ln^2(e^{a/2} e^{-y} e^{a/2})\right)$$

VIII.2 The embedding of M_n^+ in Σ_1

We are ready to state and prove a projection theorem for $M = M_n^+$ (the positive invertible $n \times n$ matrices with complex coefficients).

First note that we can embed $M_n^+ \hookrightarrow \Sigma_1$ for any $n \in \mathbb{N}$ (see the proof of Theorem VII.4). Fix an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of H, set $p_{ij} = e_i \otimes e_j$, and identify

M_n with the set

$$\mathcal{T} = \left\{ \sum_{i,j=1}^{n} a_{ij}\, p_{ij} \;:\; a_{ij} = a_{ji} \in \mathbb{R} \right\} \subset HS^h$$

In this way, we can identify isometrically the manifolds M_n^+ with the set

$$\mathcal{P} = \left\{ e^T : T \in \mathcal{T} \right\} \subset \Sigma_1$$

and the tangent space at each $e^T \in \mathcal{P}$ is $\mathcal{T}$. $\mathcal{P}$ is closed and geodesically convex in Σ_1 by Corollary V.11

Let's call $S = \operatorname{span}(e_1, \cdots, e_n)$, $S^\perp = \operatorname{span}(e_{n+1}, e_{n+2} \cdots)$. The operator P_s is the orthogonal projection to S and $Q_s = 1 - P_s$ is the orthogonal projection to $S^\perp$.

Using matrix blocks, for any operator $A \in L(S)$, we identify

$$\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and } \mathcal{P} = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix}$$

Remark VIII.9. There is a direct sum decomposition of $HS^h = \mathcal{T} \oplus \mathcal{J}$ where operators in $J \in \mathcal{J}$ are such that $P_s\, J\, P_s = 0$. A straightforward computation using the matrix-block representation shows that $\operatorname{tr}(ab) = 0$ for any $a \in \mathcal{T}, b \in \mathcal{J}$, which says $\mathcal{T}^\perp = \mathcal{J}$.

So the manifolds $\exp(\mathcal{J})$ and $\mathcal{P} = \exp(\mathcal{T})$ are orthogonal at 1, the unique intersection point.

Theorem VIII.10. *(projection to positive invertible $n \times n$ matrices) : Set $\mathcal{P} \simeq M_n^+ \subset \Sigma_1$ with the above identification. Then for any positive invertible operator $e^b \in \Sigma_1$, ($b \in HS^h$) there is a unique factorization of the form*

$$e^b = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} e^{-A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{O}_{n \times n} & Y^* \\ Y & X \end{pmatrix} \right\}$$

where $e^a = e^A P_s + Q_s \in \mathcal{P} \simeq M_n^+$, ($a \in \mathcal{T}$), $X^* = X$ acts on the subspace $S^\perp$ and $Y \in L(S, S^\perp)$.

An equivalent expression for the factorization is

$$e^b = \begin{pmatrix} e^{A/2} & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{\begin{pmatrix} \mathbb{O}_{n\times n} & e^{-A/2}Y^* \\ Ye^{-A/2} & X \end{pmatrix}\right\}\begin{pmatrix} e^{A/2} & 0 \\ 0 & 1 \end{pmatrix}$$

Yet another form is the following: for any $p \in \Sigma_1$ exist unique $V \in HS^h$ such that $P_S V P_S = 0$, and unique $q \in \Sigma_1$ such that $P_S\, q\, Q_S = Q_S\, q\, P_S = 0$ and $Q_S\, q\, Q_S = Q_S$ which make the following equation valid

$$p = q\, e^V q$$

Proof. From previous theorems and the observations we made, we know that

$$e^b = \begin{pmatrix} e^{A/2} & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{\begin{pmatrix} e^{-A/2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V_{11} & V_{21}^* \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} e^{-A/2} & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} e^{A/2} & 0 \\ 0 & 1 \end{pmatrix}$$

for some $A \in L(S)$ and some $V \in HS^h$. That $V_{11} = 0$ follows from the fact (see Remark VIII.9) that $\mathcal{T}^\perp = \mathcal{J}$, and $V \in T_{e^a}\mathcal{P}^\perp$ iff $tr(e^{-A}Be^{-A}V_{11}) = 0$ for any $B \in \mathcal{T}$. This says that V has the desired form. $\square$

Remark VIII.11. Since V is orthogonal to $\mathcal{P}$ at any point, in particular it is orthogonal to $\mathcal{P}$ at 1; so 1 is the foot of the perpendicular from e^V to $\mathcal{P}$, or, in other words, 1 is the point in $\mathcal{P}$ closest to e^V; the distance between 1 and e^V is exactly $\|V\|_2$.

In the notation of Theorem VIII.10, $e^a = 1$ if and only if $A = 0$, if and only if $V = b$, and we conclude that for any $b \in HS^h$ such that $P_S\, b\, P_S = 0$, the point in $\mathcal{P}$ closest to e^b is 1. This is nothing but Remark VIII.9 in disguise.

Remark VIII.12. For any $b \in HS^h$, it holds true that the operator

$$e^a = e^A P_{S_n} + P_{S_n^\perp} = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

is the 'first block' $n \times n$ matrix which is closest to e^b in Σ_∞, and with a slight abuse of notation for the traces of $L(S_n)$ and $L(S_n^\perp)$, we have

$$dist(\mathcal{P}, e^b) = dist(e^a, e^b) = \left\|\begin{pmatrix} \mathbb{O}_{n\times n} & Y^* \\ Y & X \end{pmatrix}\right\|_{e^a} = \sqrt{\|Y e^{-A/2}\|_2^2 + \|X\|_2^2}$$

.

IX Embedding Symmetric Spaces of the Noncompact Type

IX.1 A classical result

In a series of notes devoted to the geometry of manifolds of nonpositive sectional curvature (in particular, [Eb85]), Patrick Eberlein puts together a result which 'does not seem to be stated in the literature in precisely this form' (*sic*).

Eberlein shows that every symmetric (real, finite dimensional) manifold M of noncompact type can be realized isometrically as a complete, totally geodesic sumanifold of $M_n^+(\mathbb{R})$, where $n = \dim(M)$, with the precaution that one multiplies the metric on each irreducible de Rham factor of M by a suitable constant. If $I_0(M)$ denotes the connected component of the isometry group of M that contains the identity, then $G = I_0(M)$ is a Lie group when given the compact-open topology; if $\mathfrak{g}$ is the Lie algebra of G, the idea of this result is based in the representation of M into $\mathrm{End}(\mathfrak{g})$. In the following paragraphs we outline the main tools an ennunciate the result.

We state the de Rham decomposition theorem; for a proof see Theorem 6.11 of Chapter III in [SakT96]

Theorem IX.1. *Let M be a complete simply connected Riemannian manifold. Then M is isometric to the Riemannian direct product $M_0 \times M_1 \times \cdots \times M_k$, where M_0 is Euclidean space and the other M_i are complete simply connected irreducible Riemannian manifolds. Moreover, this decomposition is unique up to order.*

Let $(M, \langle\ ,\ \rangle_M)$ be a symmetric space of noncompact type (*i.e.* simply connected, with no Euclidean de Rham factor and nonpositive sectional curvature). For these manifolds, $G = I_0(M)$ is a semisimple Lie group (see [Eb85]).

Fix a point $p \in M$. Since M is symmetric, the geodesic symmetry s_p generates an involutive automorphism σ_p of $I_0(M)$, where $\sigma_p(g) = s_p \circ g \circ s_p$. The differential of this map gives an involutive Lie algebra automorphism (see section V of this manuscript, [Eb85], or [Hel62]) $\Theta_p = d_p \sigma_p : \mathfrak{g} \to \mathfrak{g}$; this map is characterized by the equation

$$\sigma_p\left(e^{tX}\right) = e^{t\,\Theta_p(X)} \qquad \text{for all } X \in \mathfrak{g} \text{ and all } t \in \mathbb{R}$$

and gives a canonical decomposition of $\mathfrak{g}$ where $\mathfrak{m}$ identifies with the tangent space T_pM and $\mathfrak{k} = \mathrm{Fix}(\Theta_p)$.

Let's denote with $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the *Killing form* of G, which maps

$$(X, Y) \mapsto \mathrm{trace}(\mathrm{ad}X \circ \mathrm{ad}Y)$$

We define an inner product on $\mathfrak{g}$ using the Killing form:

$$\langle X, Y\rangle_{\mathfrak{g}} = -B\left[\Theta_p(X), Y\right] = -\mathrm{trace}\left(\mathrm{ad}\Theta_p(X) \circ \mathrm{ad}Y\right)$$

Now we ennunciate a few facts that we prove only partially, because they can be deduced from the general theory of representations (see, for instance, section 6.2, Chapter IV of [SakT96]), or can be found in Eberlein's paper [Eb85]. See also section V.2 of this manuscript.

- The fact that G is semisimple ensures that B is nondegenerate.
- By definition, Ad_g is the differential at $Id \in G$ of the g-inner automorphism α_g, that is, the map which sends $\phi \mapsto g\phi g^{-1}$; since this map fixes Id, its differential is an endomorphism of $\mathfrak{g}$.
- G acts transitively on M, by means of the symmetries s_{p_q}, where p_q is the middle point of the minimal geodesic joining p to q.
- This inner product makes $\mathfrak{m} \perp \mathfrak{k}$; adX is symmetric relative to this inner product for any $X \in \mathfrak{m}$, and adX is skew-symmetric for any $X \in \mathfrak{k}$.
- Recall that $adX(Z) = [X, Z]$. Then $adX = \frac{d}{dt}\Big|_{t=0} \alpha\left(e^{tX}\right)$ and also $Ad_{e^X} = e^{\,adX}$.
- $tr(adX) = 0$ for any $X \in \mathfrak{g}$. This is due to the following:
  1. We can span $\mathfrak{g}$ with a basis $\{E_i\}$, such that $\Theta_p(E_i) = \epsilon_i E_i$, $\epsilon_i = \pm 1$ and $B[E_i, E_j] = \epsilon_i \delta_{ij}$, so $\langle E_i, E_j \rangle_{\mathfrak{g}} = \delta_{ij}$
  2. $\langle adX(E_i), E_i \rangle_{\mathfrak{g}} = -B\,[adX(E_i), \Theta_p(E_i)] = -\epsilon_i B\,[adX(E_i), E_i]$
  3. $tr(adX) = \sum_{i=1}^{n} \langle adX(E_i), E_i \rangle_{\mathfrak{g}} = -\sum_{i=1}^{n} \epsilon_i B\,[adX(E_i), E_i]$
  4. $B\,[adZ(X), Y] = -B\,[X, adZ(Y)]$ (this can be deduced using the Jacobi identity twice)
- $Ad_G \subset GL(\mathfrak{g})$, in fact $Ad_G \subset SL(\mathfrak{g})$. This a consecuence of:
  1. The image of the exponential map $e : \mathfrak{g} \to G$ generates G; in other words
  $$G = \bigcup_n e(\mathfrak{g})^n$$
  2. $det(e^A) = e^{trA}$ for any linear operator A
  3. The two previous observations

- If we denote with a dagger the adjoint whith respect to the inner product introduced above, then $\mathrm{Ad}_G^\dagger \subset SL(\mathfrak{g})$ also.

- The inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is invariant under Ad_K, that is

$$\langle \mathrm{Ad}_k X, \mathrm{Ad}_k Y\rangle_{\mathfrak{g}} = \langle X, Y\rangle_{\mathfrak{g}} \quad \text{for any} \quad k \in K$$

  (K is the isotropy group of p).

- If $q \in M$ is such that $q = g_1(p) = g_2(p)$ ($g_i \in G$) then calling $u = g_1^{-1}g_2$, u is in the isotropy group K of p, and using that the inner product is Ad_K-invariant, we get

$$\mathrm{Ad}_{g_1}^\dagger \mathrm{Ad}_{g_1} = \mathrm{Ad}_{g_2}^\dagger \mathrm{Ad}_{g_2}$$

- Moreover, $\mathrm{Ad} : G \to SL(\mathfrak{g})$ is injective. This is a consecuence of the fact that M has no Euclidean de Rham factor (see [Wolf64]).

Theorem IX.2. *Fix a point p in any symmetric (real, finite dimensional) manifold M of noncompact type. Then the map $F_p : M \to GL^+(\mathfrak{g})$ given by*

$$q = g(p) \mapsto \mathrm{Ad}_g^\dagger \mathrm{Ad}_g$$

is a diffeomorphism with a closed, totally geodesic submanifold of $GL^+(\mathfrak{g})$ Moreover, if we pull back the inner product on $GL^+(\mathfrak{g})$ to M, this inner product differs only by a constant positive factor from the inner product of M, on each irreducible de Rham factor of M.

Proof. That $\mathrm{Ad}_g^\dagger \mathrm{Ad}_g$ is positive and invertible in $\mathrm{End}(\mathfrak{g})$, and the map is well defined is a consequence of the previous observations.

The proof of the theorem can be found in Eberlein's survey, Proposition 19 of [Eb85]. □

IX.2 A new result

Now fix an orthonormal basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ and identify X_i with e_i in $\mathbb{R}^n$. Then we obtain an embedding $\mathcal{F}_p : M \to \Sigma_\infty$ which is the composition of the previous map, the identification of $GL(\mathfrak{g})$ with $GL(n, \mathbb{R})$, and the isometric, closed and geodesically convex embedding of $M_n^+(\mathbb{R})$ in Σ_∞ (see section VIII.2 of this manuscript and also section 4 of [AV03] by Andruchow *et al.*).

In this way, we can identify M with a subset of the first $n \times n$ block in the matrix representation of Σ_∞; in the notation of section VIII.2, M can be identified with a closed and geodesically convex submanifold of $\mathcal{P}$; remember that operators in $\mathcal{P}$ have a matrix representation of the form

$$\begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem IX.3. *For any (finite dimensional, real) symmetric manifold M of the noncompact type (that is, with no Euclidean de Rham factor, simply connected and with nonpositive sectional curvature), there is an embedding $\mathcal{F}_M : M \longrightarrow \Sigma_\infty$ which is a diffeomorphism betwen M and a closed, geodesically convex submanifold of Σ_∞. This map preserves the metric tensor in the following sense: if we pull back the inner product on Σ_∞ to M, then this inner product is a (positive) constant multiple of the inner product of M (on each irreducible de Rham factor of M). Moreover, $\mathcal{F}_M(M) \subset \Sigma_1$.*

This theorem together with the general factorization theorem says that, for any finite dimensional symmetric manifold M of the noncompact type, we can project operators in Σ_∞ using the contraction Π_M (assuming we identify M with its image $\mathcal{F}_M(M)$).

.

X Unitary Orbits

There is a distinguished leaf in the foliation we defined in Section VIII, namely Σ_1, which contains the identity. Moreover, $\Sigma_1 = \exp(HS^h)$. We will focus on this submanifold since the nontrivial part of the geometry of Σ_∞ is, by Theorem VIII.6 contained in the leaves. We won't have to deal with the scalar part of tangent vectors, and some computations will be less involved.

X.1 The action of the unitary groups $\mathcal{U}(\mathcal{H}_\mathbb{C})$ and $\mathcal{U}(\mathsf{L}(\mathsf{H}))$

We are interested in the orbit of an element $1 + a \in \Sigma_1$ under the action of some group of unitary operators.

We first consider the group of unitaries of the complex Banach algebra of 'unitized' Hilbert-Schmidt operators. To be precise, let's call

$$\mathcal{U}(\mathcal{H}_{\mathbb{C}}) = \{g = \lambda + a : a \in \mathrm{HS}, \lambda \in \mathbb{C}, g^* = g^{-1}\}$$

It is apparent from the definition that $|\lambda| = 1$ (so we can write $\lambda = e^{i\theta}$), and also that a must be a normal operator; this definition can be restated (naming $g = a + \lambda = u + iv + e^{i\theta}$) in the form of the following operator equation:

$$(u + \cos(\theta))^2 + (v + \sin(\theta))^2 = 1$$

It will be apparent from the definition of the action that we will be always able to choose $\theta = 0$, so $g = 1 + x$ with x a normal operator and $\sigma(x) \subset S^1 - 1$ (here -1 denotes translation in the complex plane).

The Lie algebra of this Lie group consists of the operators of the form $i(x + r1)$ where x is a Hilbert-Schmidt, selfadjoint operator, and r is a real number, that is

$$T_1\left(\mathcal{U}(\mathcal{H}_{\mathbb{C}})\right) = i\mathcal{H}_{\mathbb{R}} = \{a + \lambda : \ a^* = -a \text{ and } \lambda \in \ i\,\mathbb{R}\}$$

Since these are the antihermitian operators of the unitized Hilbert-Schmidt algebra, we have $T_1\left(\mathcal{U}(\mathcal{H}_{\mathbb{C}})\right) = \mathcal{H}_{\mathbb{C}}^{ah}$. But we mentioned early that it will be enough to consider unitaries $\lambda + x$ with $\lambda = 1$; in this case, with a slight abuse of notation, we have an identification

$$T_1\left(\mathcal{U}(\mathcal{H}_{\mathbb{C}})\right) = i\mathrm{HS}^h$$

Remark X.1. The problem of determining whether a set in Σ_1 can be given the structure of submanifold (or not) can be translated into the tangent space by taking logarithms; to be precise, note that

$$\exp(uau^*) = ue^a u^*$$

for any $a \in \mathrm{HS}^h$ and any unitary operator u, and that this map is an analytic isomorphism between Σ_1 and its tangent space. We will state the problem in this context.

We fix an element a in the tangent space (that is, $a \in \mathrm{HS}^h$) and make the unitary group act *via* the map

$$\pi_a : \mathcal{U}(\mathcal{H}_{\mathbb{C}}) \to \mathrm{HS}^h \qquad g \mapsto gag^*$$

Definition X.2. *Let* $S_a := \{gag^* : g \in \mathcal{U}(\mathcal{H}_{\mathbb{C}})\}$ *be the orbit of a selfadjoint Hilbert-Schmidt operator.*

- When is the orbit S_a of a selfadjoint Hilbert-Schmidt operator a submanifold of HS^h?

The answer to this question can be partially answered in terms of the spectrum:

Theorem X.3. *If the algebra* $C^*(a)$ *generated by* a *and* 1 *is finite dimensional, then the orbit* $S_a \subset \mathrm{HS}^h$ *can be given an analytic submanifold structure.*

Proof. We give the tools for constructing the proof, and refer the reader to [AS89] and [AS91]. A local section for the map π_a is a pair (U_a, φ_a) where U_a is an open neighbourhood of a in HS^h and φ_a is an analytic map from U_a to $\mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$ such that:

- $\varphi_a(a) = 1$
- φ_a restricted to $U_a \cap S_a$ is a section for π_a, that is

$$\pi_a \circ \varphi_a|_{U_a \cap S_a} = id_{U_a \cap S_a}$$

A section for π_a provides us with sufficient conditions to give the orbit the structure of immerse submanifold of HS^h (see Propostion 2.1 of [AS89]). The section φ_a can be constructed by means of the finite rank projections in the matrix algebra where $C^*(a)$ is represented. The finite dimension of the algebra is key to the continuity (and furthermore analyticity) of all the maps involved. To fix some notation, as in Theorem 1.3 of [AS91], suppose $n = \sum_{i=1}^{p} n_i = \dim C^*(a)$ and τ is the *-isomorphism

$$\tau : C^*(a) \to M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_p}(\mathbb{C})$$

Consider the set of systems of projections (here $p_i^2 = p_i = p_i^*, p_i p_j = 0$ for any $i \neq j$):

$$P_n = \{(p_1, \cdots, p_n) \in \mathcal{H}_{\mathbb{C}}^n : \sum_{i=1}^{m} p_i = 1\}$$

Denote $e_{jk}^i \in M_{n_i}(\mathbb{C})$ the elementary matrix with 1 in the (j,k)-entry and zero elsewhere, but embedded in the direct sum; take $p_{jk}^i(X)$ the polynomial which makes $e_{jk}^i = p_{jk}^i(\tau(a))$, and consider the following element in HS^h: $e_{jk}^i = p_{jk}^i(a)$ There is a neighbourhood U_a of a in HS^h such that $1 - \left[e_{11}^i - p_{11}^i(x)\right]^2$ has strictly positive spectrum, because $r(x) = \|x\| \leq \|x\|_2$ and $\mathcal{H}_{\mathbb{C}}$ is a Banach algebra (here $r(x)$ denotes spectral radius). The map

$$\varphi_a(x) = \sum_{i=1}^{p} \sum_{j=i}^{n_i} p_{j1}^i(x) E_{11}^i \left[1 - \left(E_{11}^i - p_{11}^i(x)\right)^2\right]^{-\frac{1}{2}} E_{1j}^i$$

is a cross section for π_a, and it is analytic from $U_a \subset HS^h \to \mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$ since the p_{jk}^i are multilinear and all the operations are taken inside the Banach algebra $\mathcal{H}_{\mathbb{C}}$ (the computation that proves that φ_a is in fact a cross section for π_a is straightforward and can be found in the article by Andruchow *et al.*, [AFHS90]). $\square$

Remark X.4. At first sight, it is not obvious if this strong restriction (on the spectrum of a) is necessary for S_a to be a submanifold of HS^h. The main difference with the work done so far by Deckard and Fialkow in [DF79], Raeburn in [Rae77], and Andruchow *et al.* in [AS89], [AS91] is that the Hilbert-Schmidt operators (with any norm equivalent to the $\|\cdot\|_2$-norm) are not a C^*-algebra. A remarkable byproduct of Voiculescu's theorem [Voic76] says that, for the unitary orbit of an operator a with the action of the full group of unitaries of $L(H)$, it is indeed necessary that a has finite spectrum. For the time being, we don't know if this is true for the algebra $\mathcal{B} = \mathcal{H}_{\mathbb{C}}$.

Let's examine what happens when we act with the full group $\mathcal{U}(L(H))$ by means of the same action. For convenience let's fix the notation

$$\mathfrak{S}_a = \{uau^* \,:\, u \in \mathcal{U}(\mathrm{L}(\mathrm{H}))\}$$

We will develop an example that shows that the two orbits (S_a and $\mathfrak{S}_a$) are, in general, not equal when the spectrum of a is infinite. Since a is compact and selfadjoint, we can assume that a is a diagonal operator; that is, there's an orthonormal basis $\{e_k\}$ of H such that

$$a = \sum_k \alpha_k \, e_k \otimes e_k, \qquad \text{where} \qquad \sum_k |\, \alpha_k \,|^2 = \mathrm{tr}(a^* a) < +\infty$$

Example X.5. Take $\mathrm{H} = l_2(\mathbb{Z})$, $S \in \mathrm{L}(\mathrm{H})$ the right shift ($Se_k = e_{k+1}$). Then S is a unitary operator with $S^* e_k = e_{k-1}$. Pick any a of the form

$$a = \sum_{k \in \mathbb{Z}} r_k \, e_k \otimes e_k \qquad \text{and} \qquad \sum_k |\, r_k \,|^2 < +\infty$$

where all the r_k are different. (For instance, $r_k = \frac{1}{|k|+1}$ would do). Obviously, $a \in \mathrm{HS}^h$. We affirm that there is no Hilbert-Schmidt unitary such that $SaS^* = waw^*$

Proof. To prove this, suppose that there is an $w \in \mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$ such that $SaS^* = waw^*$. From this equation we deduce that $S^* w$ commutes with a, and given the particular a and the fact that $S^* w$ is unitary, we have

$$S^* w = \sum_{k \in \mathbb{Z}} \omega_k \, e_k \otimes e_k \qquad \text{with } \;|\, \omega_k \,| = 1$$

because $C^*(a)$ is maximal abelian. Multiplying by S we get to

$$w = \sum_{k \in \mathbb{Z}} \omega_k \, (Se_k) \otimes e_k = \sum_{k \in \mathbb{Z}} \omega_k \, e_{k+1} \otimes e_k$$

or, in other terms, $we_k = \omega_k e_{k+1}$. Since w is a compact perturbation of a scalar operator, w must have a nonzero eigenvector x, with eigenvalue $\alpha = e^{i\theta}$ (since

w is also unitary); comparing coefficients the equation $\alpha x = wx$ reads

$$\alpha x_k = \omega_{k-1} x_{k-1}, \qquad \text{where } x = \sum_k x_k e_k$$

This is impossible because $x \in l_2(\mathbb{Z})$, but the previous equation leads to $|x_k| = |x_j|$ for any $k, j \in \mathbb{Z}$. $\square$

As we see from the previous example, the two orbits do not coincide in general. For the action of the full group of unitaries we have the following:

Theorem X.6. *The set $\mathfrak{S}_a \subset \mathrm{HS}^h$ (the orbit of the Hilbert-Schmidt operator a under the action of $\mathcal{U}(L(H))$, the full unitary group) can be given an analytic submanifold structure if and only if the C^*-algebra generated by a and 1 is finite dimensional.*

Proof. The 'only if' part goes in the same lines of the proof of the previous theorem but being careful about the topologies involved, since now we must take an open set $U_a \subset \mathrm{HS}^h$ such that the map $\phi : U_a \to \mathcal{U}(L(H))$ is analytic. But this can be done since the polynomials p^i_{jk} are now taken from U_a to $L(H)^n$, and the maps $+$ and $\cdot$ are analytic since $\|x.y\|_{L(H)} \leq \|x\|_2 \|y\|_2$.
The relevant part of this theorem is the 'if' part. Suppose we can prove that the orbit $\mathfrak{S}_a$ is closed in $L(H)$. Then Voiculescu's theorem (see [Voic76], Proposition 2.4) would tell us that $C^*(a)$ is finite dimensional. This is a deep result about $*-$representations, and the argument works in the context of $L(H)$, but not in $\mathcal{H}_{\mathbb{C}}$ because the latter is not a C^*-algebra.
To prove that $\mathfrak{S}_a$ is closed in $L(H)$, we first prove that it is closed in $\mathcal{H}_{\mathbb{C}}$. To do this, observe that if $\mathfrak{S}_a$ is an analytic submanifold of HS^h, then $\mathfrak{S}_a$ must be locally closed in the $\|\cdot\|_2$ norm. Since the action of the full unitary group is isometric, the neighbourhood can be chosen uniformly, that is, there is an $\epsilon > 0$ such that for all $c \in \mathfrak{S}_a$, the set $N_c = \{d \in \mathfrak{S}_a : \|c - d\|_2 \leq \epsilon\}$ is closed in HS^h (with the 2-norm, of course). This is another way of saying that $\mathfrak{S}_a$ is closed in HS^h.
Now suppose $a_n = u_n a u_n^* \to y$ in $L(H)$. We claim that $\|a_n - y\|_2 \to 0$, which follows from a dominated convergence theorem for trace class operators (see

[Simon89], Theorem 2.17). The theorem states that whenever $\|a_n - y\|_\infty \to 0$ and $\mu_k(a_n) \leq \mu_k(a)$ for some $a \in \mathrm{HS}$, and all k (here $\mu_k(x)$ denotes the non zero eigenvalues of $|x|$), then $\|a_n - y\|_2 \to 0$.
Observe that $|a_n| = u_n |a| u_n^*$ so we have in fact equality of eigenvalues. This proves that $\mathfrak{S}_a$ is closed in $L(H)$ since it is closed in HS^h. □

We proved that, when the spectrum of a is finite, S_a and $\mathfrak{S}_a$ are submanifolds of Σ_1. But more can be said: S_a and $\mathfrak{S}_a$ are the *same subset* of HS^h (compare with Example X.5):

Lemma X.7. *If $a \in \mathrm{HS}^h$ has finite spectrum, then the orbit under both unitary groups are the same submanifold.*

Proof. The main idea behind the proof is the fact that, when $\sigma(a)$ is finite, a and gag^* act on a finite dimensional subspace of H (for any $g \in \mathcal{U}(L(H))$). To be more precise, let's call $S = Ran(a)$, $V = Ran(b)$, where $b = gag^*$. Note that $V = g(S)$ so S and V are isomorphic, finite dimensional subspaces of H. Naming $T = S + V$ this is another finite dimensional subspace of H, and clearly a and b act on T, since they are both selfadjoint operators. For the same reason, there exist unitary operators $P, Q \in L(T)$ and diagonal operators $D_a, D_b \in L(T)$ such that

$$a = PD_aP^*, \quad b = QD_bQ^*$$

But $\sigma(b) = \sigma(gag^*) = \sigma(a)$, so $D_a = D_b := D$. This proves that $b = QP^*aPQ^*$ (the equality should be interpreted in T). Now take P_T the orthogonal projector in $L(H)$ with rank T, and set $u = 1 + (QP^* - 1_T)P_T$ (note the slight abuse of notation). Then clearly $u \in \mathcal{U}(\mathcal{H}_\mathbb{C})$ and $uau^* = b$. □

X.2 Riemannian structures for the orbit Ω

Suppose that there is, in fact, a submanifold structure for $\mathfrak{S}_a$ (resp. S_a). Then the tangent map ($= d_1\pi_a$) has image

$$\{va - av : v \in \mathcal{B}^{ah}\},$$

where $\mathcal{B}$ stands for $L(H)$ (resp. $\mathcal{H}_{\mathbb{C}}$). So, in this case

$$T_a S_a(\text{ or } T_a\mathfrak{S}_a) = \{va - av : v \in \mathcal{B}^{ah}\}$$

We can go back to the manifold Σ_1 *via* the usual exponential of operators; we will use the notation

$$\Omega = e^{S_a} \quad \text{or} \quad \Omega = e^{\mathfrak{S}_a}$$

without further distinction, since the meaning will be clear from the context. Note that $\Omega = \{ue^a u^* : u \in \mathcal{U}(\mathcal{B})\} \subset \Sigma_1$ and we can identify

$$T_{e^a}\Omega = \{ve^a - e^a v : v \in \mathcal{B}^{ah}\} = \{ i(he^a - e^a h) : h \in \mathcal{B}^h\}$$

Remark X.8. For any $p \in \Omega$, we have

$$T_p\Omega = \{vp - pv : v \in \mathcal{B}^{ah}\} \quad \text{and} \quad T_p\Omega^\perp = \{x \in HS^h : [x,p] = 0\}$$

These two identifications follow from the definition of the action, and the equality

$$\langle x, vp - pv\rangle_p = 4\,\mathrm{tr}\left[(p^{-1}x - xp^{-1})V\right]$$

The submanifold Ω is connected: the curves indexed by $w \in \mathcal{B}^{ah}$,

$$\gamma_w(t) = e^{tw}e^a e^{-tw}$$

join e^a to $ue^a u^*$, assuming that $u = e^w$.

We can ask whether the curves γ_w will be the familiar geodesics of the ambient space (equation (3) of section II). Of course they are trivial geodesics if a and w commute. We will prove that this is the only case, for any a:

Proposition X.9. *For any $a \in HS^h$, the curve γ_w is a geodesic of Σ_1 if and only if w commutes with a. In this case the curve reduces to the point e^a.*

Proof. The (ambient) covariant derivative for γ_w (equations (II.5) and (2) of section II) simplifies up to $we^a we^{-a} = e^a we^{-a}w$ or, writing $w = ih$ (h is self-adjoint)

$$he^a he^{-a} = e^a he^{-a}h \tag{16}$$

Consider the Hilbert space $(H, \langle\ ,\ \rangle_a)$ with inner product

$$\langle x, y\rangle_a = \left\langle e^{-a/2}x, e^{-a/2}y \right\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product of H. The norm of an operator x is given by

$$\|x\|_a = \sup_{\|z\|_a = 1} \|xz\|_a = \sup_{\|e^{-a/2}z\|=1} \|e^{-a/2}xz\|_\infty = \|e^{-a/2}xe^{a/2}\|_\infty$$

because $e^{-a/2}$ is an isomorphism of H. This equation also shows that the Banach algebras $(L(H), \|\cdot\|_\infty)$ and $\mathcal{B} = (L(H), \|\cdot\|_a)$ are topologically isomorphic and, as a byproduct, $\sigma_{\mathcal{B}}(h) \subset \mathbb{R}$. From the very definition it also follows easily that $\mathcal{B}$ is indeed a C^*-algebra.
A similar computation shows that $X^{*\mathcal{B}} = e^a X^* e^{-a}$. Note that e^a is $\mathcal{B}$-selfadjoint, moreover, it is $\mathcal{B}$-positive. We can restate equation (16) as

$$hh^{*\mathcal{B}} = h^{*\mathcal{B}}h,$$

This equations says that h is $\mathcal{B}$-normal, so a theorem of Weyl and von Neumann (see [Dav96]) says it can be aproximated by diagonalizable operators with the same spectrum; since h has real spectrum, h turns out to be $\mathcal{B}$-selfadjoint . That h is $\mathcal{B}$-selfadjoint reads, by definition, $e^a h e^{-a} = h^{*\mathcal{B}} = h$; this proves that a and h (and also a and w) commute. □

X.2.1 The orbit Ω as a Riemannian submanifold of HS^h

We've shown earlier that the orbit of an element $a \in HS^h$ has a structure of analytic submanifold of HS^h (which is a flat Riemannian manifold) if and only if $\Omega = e^a$ has a structure of analytic submanifold of Σ_1.
Since the inclusion $\Omega \subset HS^h$ is an analytic embedding, we can ask whether the curves

$$\gamma_w(t) = e^{tw}e^a e^{-tw}$$

will be geodesics of Ω as a Riemannian submanifold of HS^h (with the induced metric).

For this, we notice that the geodesic equation reads $\ddot{\gamma}_w(t) \perp T_{\gamma_w(t)}\Omega$, and we use the elementary identities $\dot{\gamma} = w\gamma - \gamma w$, $\ddot{\gamma} = w^2\gamma - 2w\gamma w + \gamma w^2$; we get to the following necessary and sufficient condition using the caracterization of the normal space at $\gamma(t)$ of the previous section:

$$w^2\gamma^2 - 2w\gamma w\gamma + 2\gamma w\gamma w - \gamma^2 w^2 = 0$$

But observing that $e^{-wt}\gamma^{\pm 1}\, e^{wt} = e^{\pm a}$, this equation transforms in the operator condition

$$w^2 e^{2a} - 2we^a we^a + 2e^a we^a w - e^{2a} w^2 = 0 \tag{17}$$

Let's fix some notation: set $e^a = 1 + A$ with $A \in HS^h$; then the tangent space at e^a can be thought of as the subspace

$$T_{e^a}\Omega = \{\, i(Ah - hA) : h \in \mathcal{B}^h\} \subset HS^h$$

and its orthogonal complement in HS^h is (see Remark X.8)

$$T_{e^a}\Omega^\perp = \{\, x \in \mathcal{B}^h : [x, A] = 0\}$$

It should be noted that both subspaces are closed by hypothesis. Then equation (17) can be restated as

$$h^2A^2 - 2hAhA + 2AhAh - A^2h^2 = 0 \tag{18}$$

where h is the hermitian generating the curve

$$\gamma(t) = 1 + e^{ith}Ae^{-ith} = e^{ith}e^a e^{-ith}$$

Let's consider the case when $A^2 = A$:

Remark X.10. If $A^2 = A$, A must be a finite rank orthogonal projector (since $A = e^a - 1$ and a is a Hilbert-Schmidt operator). Hence, $\sigma(a)$ must be a finite set, and in this case (Lemma X.7) the orbit with the full unitary group and the orbit with the Hilbert-Schmidt unitary group are the same set.

To solve the problem of the geodesics completely, we review the work of Corach,

Porta and Recht ([PR96] or, more specifically [CPR93a]); we follow the idea of section 4 of that article and put the result in context.
Observe that when A is a projector, we have a matrix decomposition of the tangent space of Σ_1, namely $HS^h = A_0 \oplus A_1$, where

$$A_0 = \left\{ \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} \right\} \quad \text{and} \quad A_1 = \left\{ \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix} \right\}$$

In this decomposition, $x_{11} = AhA$, $x_{22} = (1-A)h(1-A)$ are selfadjoint operators (since h is) and also $x_{12}^* = x_{21} = (1-A)hA$ for the same reason.

Theorem X.11. *Whenever $A = e^a - 1$ is a projector, any curve of the form $\gamma(t) = e^{ith}e^a e^{-ith}$ with h selfadjoint and codiagonal is a geodesic of $\Omega \subset HS^h$*

Proof. Note that $A_0 = T_{e^a}\Omega^\perp$, and $A_1 = T_{e^a}\Omega$; note also that equation (18) translates in this context to $x_{11}x_{12} = x_{12}x_{22}$, a condition which is obviously fullfilled by $h \in A_1$. $\square$

Remark X.12. Equation (18) translates exactly in 'h_0 commutes with h_1' whenever $h = h_0 + h_1 \in HS^h$, and we have

$$[A_0, A_1] \subset A_1 \quad [A_0, A_0] \subset A_0 \quad [A_1, A_1] \subset A_0$$

Since the orbit under both unitary groups coincide (Remark X.10), assume that we are acting with $G = \mathcal{U}(\mathcal{B})$; since the tangent space at the identity of this group can be identified with $\mathcal{B}^{ah}$, the above commutator relationships say that $iA_0 \oplus iA_1$ is a Cartan decomposition of the Lie algebra $\mathfrak{g} = \mathcal{B}^{ah}$. It is apparent that iA_0 is the vertical space, and iA_1 is the horizontal space (see section IX). Moreover,

$$A_0 \cdot A_0 \subset A_0 \qquad A_1 \cdot A_1 \subset A_0 \qquad A_0 \cdot A_1 \subset A_1 \qquad A_1 \cdot A_0 \subset A_1$$

Corollary X.13. *If $e^a - 1$ is an orthogonal projector, there is no point $p \in \Omega$ such that Ω is geodesic at p.*

X.2.2 The orbit Ω as a Riemannian submanifold of Σ_1

In this section we give Ω the induced Riemannian metric as a submanifold of Σ_1, and discuss shortly the form of the geodesics and the sectional curvature.

Recall that covariant derivative in the ambient space is given by

$$\nabla_{\dot\gamma}\dot\gamma = \ddot\gamma - \dot\gamma\gamma^{-1}\dot\gamma$$

and the orthogonal space to $p \in \Omega$ are the operators commuting with p, so $\nabla_{\dot\gamma}\dot\gamma \perp T_\gamma\Omega$ if and only if

$$\ddot\gamma\gamma - \gamma\ddot\gamma + \gamma\dot\gamma\gamma^{-1}\dot\gamma - \dot\gamma\gamma^{-1}\dot\gamma\gamma = 0 \tag{19}$$

This is an odd equation; we know that any curve in Ω starting at $p = e^a$ must be of the form $\gamma(t) = g(t)e^a g(t)^*$ for some curve of unitary operators g.

For the particular curves $\gamma(t) = e^{ith}e^a e^{-ith}$, $h(t) = ith$, so $\dot h(t) = ih$, and $\ddot h(t) \equiv 0$; equation (19) reduces to the operator equation

$$he^a he^{-a} + he^{-a}he^a = e^{-a}he^a h + e^a he^{-a}h \tag{20}$$

or $X^* = X$, where $X = he^a he^{-a} + he^{-a}he^a$.

Recall that, when the spectrum of e^a is finite, the unitary groups $\mathcal{U}(\mathsf{L}(\mathsf{H}))$ and $\mathcal{U}(\mathcal{H}_{\mathbb{C}})$ induce the same manifold $\Omega \subset \Sigma_1$.

Theorem X.14. *Assume* $e^a = 1 + A$ *with* A *an orthogonal projector, and* $\Omega \subset \Sigma_1$ *is the unitary orbit of* e^a. *Then (throughout* $[\ ,\ ]$ *stands for the usual commutator of operators)*

(1) Ω *is a Riemannian submanifold of* Σ_1.

(2) $T_p\Omega = \{i[x,p] : x \in HS^h\}$ *and* $T_p\Omega^\perp = \{x \in HS^h : [x,p] = 0\}$.

(3) The action of the unitary group is isometric, namely

$$\mathrm{dist}^{\Omega}(upu^*, uqu^*) = \mathrm{dist}^{\Omega}(p,q)$$

for any unitary operator $u \in \mathsf{L}(\mathsf{H})$.

(4) For any $v = i[x,p] \in T_p\Omega$*, the exponential map is given by*

$$\mathrm{Exp}_p^{\Omega}(v) = e^{ighg^*} p\, e^{-ighg^*}$$

where $p = ge^{a}g^*$ *and* h *is the codiagonal part of* g^*xg *(in the matrix representation of Proposition X.11). In particular, the exponential map is defined in the whole tangent space.*

(5) If $p = ge^{a}g^*$*,* $q = we^{a}w^*$*, and* h *is a selfadjoint, codiagonal operator such that* w^*ge^{ih} *commutes with* e^{a}*, then the curve*

$$\gamma(t) = e^{itghg^*} p e^{-itghg^*}$$

is a geodesic of $\Omega \subset \Sigma_1$*, which joins* p *to* q*.*

(6) If we assume that $h \in HS^h$*, then* $L(\gamma) = \frac{\sqrt{2}}{2}\,\|h\|_2$

(7) The exponential map $\mathrm{Exp}_p^{\Omega} : T_p\Omega \to \Omega$ *is surjective.*

Proof. Statements (1) and (2) are a consequence of Remark X.10 and Theorems X.3 and X.6. Statement (3) is obvious because the action of the unitary group is isometric for the 2-norm (see Lemma II.6). To prove statement (4), take $x \in HS^h$, and set

$$v = i[x,p] = i(xgAg^* - gAg^*x) = ig[g^*xg, e^{a}]g^*$$

Observe that

$$e^{-a} = (1+A)^{-1} = 1 - \frac{1}{2}A$$

Rewriting equation (20), we obtain

$$h^2A - Ah^2 + 2AhAh - 2hAhA = 0$$

Now if $y = g^*xg$, take $h =$ the codiagonal part of y; clearly $hA - Ah = yA - Ay$, so

$$\gamma_1(t) = e^{ith}e^{a}e^{-ith}$$

is a geodesic of Ω starting at $r = e^{a}$ with initial speed $w = i[y, e^{a}] = g^*vg$ (see Proposition X.11). Now consider $\gamma = g\gamma_1 g^*$. Clearly γ is a geodesic of Ω starting at $p = ge^a g^*$ with initial speed v. To prove (5), note that

$$\gamma(t) = ge^{iht}e^a e^{-iht}g^* = e^{itghg^*}ge^a g^* e^{itghg^*} = e^{itghg^*}pe^{itghg^*}$$

which shows that $\gamma(0) = p$ and $\gamma(1) = q$ because $w^* ge^{ih}e^a = e^a w^* ge^{ih}$. To prove (6), we can assume that $p = e^a$, and then

$$L(\gamma)^2 = \|[h,p]\|_p^2 = \|[h,e^a]\|_{e^a}^2 = 4 \cdot \mathrm{tr}(2he^a he^{-a} - 2h^2)$$

Now write h as a matrix operator $[0, Y^*, Y, 0] \in A_1$ (see Proposition X.11), to obtain

$$\mathrm{tr}(2he^a he^{-a} - 2h^2) = \mathrm{tr}(Y^*Y) = \tfrac{1}{2}\,\mathrm{tr}(h^2),$$

hence $L(\gamma)^2 = 2\,\mathrm{tr}(h^2) = \frac{1}{2}\|h\|_2^2$ as stated. The assertion in (7) can be deduced from folk results (see [Br93]) because $q = we^a w^*$ and $p = ge^a g^*$ are finite rank projectors acting on a finite dimensional space (see the proof of Lemma X.7). $\square$

XI Concluding Remarks

Remark XI.1. Theorem X.3 doesn't answer whether is it necessary that the spectrum of a should be finite for the orbit to be a submanifold, when we act with $\mathcal{U}(\mathcal{H}_{\mathbb{C}})$ (see Remark X.4). The problem can be stated in a very simple form:

- Choose any involutive Banach algebra with identity $\mathcal{B}$, take $a = a^* \in \mathcal{B}$.
- Name S_a the image of the map $\pi_a : \mathcal{U}(\mathcal{B}) \to \mathcal{B}$ which assigns $u \mapsto uau^*$
- Is the condition "a has finite spectrum" necessary for the set $S_a \subset \mathcal{B}$ to be closed?

Remark XI.2. The standard representation of $L(H)$ (acting on the Hilbert-Schmidt operators by left or right product) induces a morphism of the latter operators into the state space of $L(H)$. Hyperbolic geometry of states seems to be possible in this context.

Remark XI.3. It should be interesting to find an applicaction of the factorization theorem in the theory of integral equations. Such a nonlinear factorization should take the following form: if k is the symmetric kernel of the equation

$$(Kf)(t) = \int_I k(s,t)\, f(s)ds$$

(namely $f \mapsto Kf$ is a selfadjoint operator of $L^2(I)$), then find $\lambda > 0$ such that K is a positive invertible operator and find a convenient LTS to project to (for instance: diagonal operators) then write

$$K = D\exp(Y)D - \lambda$$

with $D^{\frac{1}{2}}$ the diagonal invertible operator closest to $K+\lambda$ in the geodesic distance, and Y a codiagonal operator. The equation should take the form

$$(Kf)(t) = \int_I\int_I\int_I d(v,t)j(u,v)d(s,u)\, f(s)dsdudv - \lambda f(t)$$

If Y is small, the original equation could be replaced by

$$(Df)(t) = \int_I d(s,t)f(s)ds - \lambda f(t)$$

with an error term that can be bounded using the inequalities of section III.

Remark XI.4. In several recent papers (the latest at the moment we write these lines is [CGM]), R. Cirelli, M. Gatti and A. Manià propose a delinearization program for quantum mechanics based in identifying the pure state space with a convenient homogeneous manifold (the infinite projective space). The manifold Σ_∞ seems to be another convenient setting for a delinearization program.

Remark XI.5. Assume $\mathcal{A}$ is a von Neumann algebra with a faithful trace τ (for instance, the reduced group algebra of a unimodular locally compact group), and $\mathcal{A}_h$ for stands for the selfadjoint elements of $\mathcal{A}$. If we use $GL(\mathcal{A})$ to denote the group of invertible elements of $\mathcal{A}$, and $\mathcal{A}^+$ to denote the set of positive invertible elements, then a construction similar to the one we made for Hilbert-Schmidt operators can be made in order to construct a nonpositively curved manifold $\Sigma := \mathcal{A}^+$ with an invariant metric (under the action of the group $GL(\mathcal{A})$ with action $g \mapsto gpg^*$), setting

$$\langle x, y\rangle_2 := \tau(y^* x) \qquad \text{and} \qquad \langle x, y\rangle_p := \tau(y^* p^{-1} x p^{-1})$$

The tangent space at any point $p \in \Sigma$ can be naturally identified with $\mathcal{A}_h$. All the results concerning curvature, convexity of the geodesic distance, minimality of the geodesics, geodesic triangles, and algebraic characterization of convex submanifolds of sections III, and V of this manuscript hold true with proofs that can be translated almost verbatim.

One technical obstacle that should be remarked is the following: with this inner product given by a faithfull trace, the induced pre-Hilbert space that we construct in $\mathcal{A}_h$ is not complete. This is an obstacle for the construction of the projections, but it can be saved with a refinement [PR94] of the argument we used in section VI, when we proved that the set of points in Σ that can be projected to a convex submanifold is open and closed in the norm topology.

The natural subsets where one would be able to project are the hermitian part of subalgebras of $\mathcal{A}$. By a result of Takesaki [Tak72], for any subalgebra $\mathcal{M}$ of $\mathcal{A}$ there is a conditional expectation $E : \mathcal{A} \to \mathcal{M}$ compatible with the trace, namely $\tau(E(x)y) = \tau(xy)$ for any $y \in \mathcal{M}$ and any $x \in \mathcal{A}$. In this way the kernel of the conditional expectation acts as an 'orthogonal complement' of $\mathcal{M}_h$ (with respect to the trace inner product): $\mathcal{M}_h^\perp$ is a closed involutive subspace of $\mathcal{A}_h$.

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