Politopos de planificación cromática asociados al problema de asignación de frecuencias en sistemas de radio punto a multipunto

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Doctoral thesis

Chromatic Scheduling Polytopes Coming from the Bandwidth Allocation Problem in Point-To-Multipoint Radio Access Systems

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Contents

Abstract

Point-to-Multipoint systems are a kind of radio systems supplying wireless access to voice/data communication networks. Such systems have to be run using a certain frequency spectrum, which typically causes capacity problems. Hence it is, on the one hand, necessary to reuse frequencies but, on the other hand, no interference must be caused thereby. This leads to the *bandwidth allocation problem*, a special case of so-called chromatic scheduling problems. Both problems are $\mathcal{NP}$-hard, and there exist no polynomial time algorithms with a fixed approximation ratio for these problems. As algorithms based on cutting planes have shown to be successful for many other combinatorial optimization problems, the goal is to apply such methods to the bandwidth allocation problem. For that, knowledge on the associated polytopes is required. The present thesis contributes to this issue.

We present an integer programming formulation for the bandwidth allocation problem and define the associated *chromatic scheduling polytopes*. We first study the combinatorial structure of these polytopes, discussing the different stages –emptyness, non-emptyness but low-dimensionality, full-dimensionality but combinatorial instability, and combinatorial stability– as the frequency span increases. Moreover, we explore the relations of chromatic scheduling polytopes to the linear ordering polytope.

From a geometrical point of view, chromatic scheduling polytopes are of particular interest due to their symmetry. Outgoing from this symmetry, we develop an important tool for identifying facet-defining inequalities without any knowledge on the dimension of the polytopes. This enables us to identify the facet-inducing constraints from the integer programming model. The other model constraints need to be strengthened with the help of clique-based structures in order to yield facets. In particular, the so-called *covering-clique inequalities* generate a broad number of facets, and we also present several classes of facets coming from generalizations and variations of these inequalities. We introduce further classes of facet-inducing inequalities based on different concepts, and study the complexity of the associated separation problems.

KEYWORDS: bandwidth allocation, polyhedral combinatorics

Resumen

Los sistemas de radio punto a multipunto son conjuntos de antenas de radio que proveen acceso inalámbrico a redes de comunicación de voz y datos. Este tipo de sistemas debe ser operado utilizando un cierto espectro de frecuencias de radio, lo cual normalmente produce problemas de capacidad. Por lo tanto es necesario reutilizar frecuencias, pero este reuso no debe generar interferencia entre las señales. El problema de determinar las frecuencias para los enlaces se conoce como el *problema de asignación de frecuencias*, y en este tipo de sistemas es un caso especial de los problemas de planificación cromática. Estos problemas son $\mathcal{NP}$-hard, y no existen algoritmos aproximados polinomiales con una garantía de calidad fija. Como los métodos de planos de corte han demostrado ser efectivos para muchos otros problemas de optimización combinatoria, el objetivo es aplicar estos métodos al problema de asignación de frecuencias en sistemas punto a multipunto. Para esto, es necesario estudiar previamente los politopos asociados con el problema. El presente trabajo contribuye a este estudio.

Introducimos una formulación del problema de asignación de frecuencias en sistemas punto a multipunto como un problema de programación lineal entera, y definimos los *politopos de planificación cromática* asociados a esta formulación. Estudiamos en primer lugar la estructura combinatoria de estos politopos, analizando los distintos estados –vacuidad, no vacuidad pero dimensión incompleta, dimensión completa pero inestabilidad combinatoria, y estabilidad combinatoria– a medida que el ancho de banda disponible aumenta. Por otra parte, exploramos las relaciones de los politopos de planificación cromática con el politopo de ordenamiento lineal.

Desde el punto de vista geométrico, los politopos de planificación cromática son de un interés particular debido a su simetría. Como consecuencia de esta propiedad, desarrollamos una importante herramienta para identificar desigualdades que definen facetas sin requerir información sobre la dimensión del politopo. Esto nos permite identificar las restricciones del modelo de programación lineal entera que definen facetas del politopo asociado. Las restantes restricciones del modelo deben ser reforzadas mediante estructuras basadas en cliques del grafo de interferencia para obtener desigualdades que definen facetas. En particular, las *desigualdades de clique en cubrimiento* generan una gran familia de facetas, y además presentamos varias clases de facetas que provienen de generalizaciones y variaciones de estas desigualdades. Introducimos clases adicionales de facetas basadas en distintos conceptos, y estudiamos la complejidad de los problemas de separación asociados.

Palabras clave: asignación de frecuencias, combinatoria poliedral

Introduction

For practical purposes the difference between algebraic and exponential order is often more crucial than the difference between finite and non-finite.

– Jack Edmonds (1965)

Since the advent of wireless communications, the electromagnetic spectrum has been widely explored for many applications, the most popular today being cellular phone networks. The development of new wireless services led to scarcity of usable frequencies in the radio spectrum, and this introduced the need to reuse frequencies. A crucial problem in this kind of communication is the interference incurred whenever two nearby transmitters operate at close frequencies. Depending on many factors (including the power and orientation of the signal, geographical constraints and even wheather conditions), the received signal may be of unacceptable poor quality. Therefore, interference must be avoided by a careful assignment of frequencies to each transmitter operating in the same area. It turned out that such assignments are computationally difficult to find, and this fact has motivated a steady interest on this topic [1, 2, 9, 17, 34, 35].

Point-to-Multipoint radio access systems (PMP-Systems) are one kind of wireless networks providing voice/data access to a set of customers. Base stations form the access points to the backbone network, and customer terminals are linked to the base stations by means of radio signals. In contrast to cellular phone networks, each customer has a fixed location on a certain sector and is therefore served by a prespecified base antenna. Moreover, each customer must be assigned a frequency interval instead of single channels, subject to the constraint that no interference is originated by the use of overlapping frequencies. In this setting there are two sources of possible interference, given by (i) customers allocated to the same sector and (ii) certain pairs of potentially interfering customers in different sectors. To guarantee an interference-free communication, a particular *bandwidth allocation problem* must be solved when operating a PMP-System.

This kind of problems is known as *chromatic scheduling problem* [15] or, in some particular cases, as *consecutive coloring problem* [16] and *interval coloring problem* [22, 36]. Such problems are $\mathcal{NP}$-complete and cannot be polynomially approximated with a guaranteed quality [36]. Small and medium-sized instances could be successfully handled by greedy-like heuristics [7], but in order to tackle real world instances, algorithms have to be designed that rely on a deeper insight of the problem structure. *Cutting plane methods* have shown to be very effective at solving hard combinatorial optimization problems [6, 30, 42, 45]. For that, knowledge of the polyhedra arising in connection to an integer programming formulation of the problem is needed. This thesis is devoted to the study of the polytopes defined by the integer programming formulation of the bandwidth allocation problem in PMP-Systems. Such a polyhedral study is the starting point for the practical computational solution of real-sized instances based on cutting planes.

The thesis is organized as follows. Chapter 1 gives an overview of wireless communication and frequency assignment problems, and introduces PMP-Systems and the associated bandwidth allocation problem in detail. Chapter 2 presents an integer programming formulation for this problem, and provides the definition of the associated polytopes, called *chromatic scheduling polytopes*. Chapter 3 discusses the different combinatorial stages of these polyhedra, as well as some relations to the linear ordering polytope. Finally, Chapter 4, Chapter 5 and Chapter 6 concentrate on the search for valid inequalities and facets, and address the corresponding separation problems, the cornerstone of a successful implementation of a cutting plane approach.

Outline

Chapter 1 starts with a brief survey of the history and main applications of wireless communications. Section 1.1 introduces the frequency assignment problem (FAP) and presents a number of relevant models for different kinds of applications. In all these models we are given a set of customers and a set of channels (frequencies) for each customer, and the objective is to assign a certain number of channels to each customer, either avoiding or minimizing interference. In the *feasibility FAP* the objective is to find an assignment providing each customer with the exact number of channels that he demands. This problem may be infeasible, and in this case the *maximum service FAP* model is of interest. This model asks for an assignment providing to every customer at most the demanded number of channels, maximizing the total number of assigned channels. On the other hand, if feasible solutions to the feasibility FAP exist, one is usually interested in the assignments minimizing the total number of used channels (*minimum order FAP*) or the span of the assignment (*minimum span FAP*). We finally introduce the *minimum interference FAP*, which considers a more realistic scenario by seeking an assignment that minimizes the total amount of interference. This model is useful in situations where interference-free frequency plans do not exist, and hence the objective is to minimize the quality loss due to interference.

Section 1.2 introduces PMP-Systems in detail. We give a precise definition of the bandwidth allocation model and state this problem in graph-theoretical terms by introducing the weighted interference graph (G, d). The node set of this graph represents the customer terminals, and edges join pairs of interfering customers. In this particular model we have two types of edges, representing the two sources of possible interference (i.e., interference among customers in the same sector, and interference between certain pairs of customers in different sectors). The customers do not have a uniform communication demand but individual ones, hence we consider a node weighting d reflecting these demands. We further have the available radio frequency spectrum $[0, s]$, with $s \in \mathbf{Z}$, where all the frequency intervals have to be placed in. Finally, a guard distance $g \in \mathbf{Z}_+$ must be kept between the intervals of interfering customers in different sectors, due to technical reasons. Thus, every instance of the bandwidth allocation problem is given by a quadruple (G, d, s, g). This problem may be interpreted as a special scheduling problem, where the sectors correspond to machines and the frequency intervals to the jobs to be scheduled. In this setting, the assignment of jobs to machines is fixed in advance, and we have *antiparallelity requirements* with changeover times instead of the usual precedence constraints. We prove that this problem is $\mathcal{NP}$-complete by providing a straightforward reduction from Graph coloring, and alternatively by a reduction from Open shop scheduling. The chapter closes with a discussion motivating the study of chromatic scheduling polytopes in the forthcoming chapters.

Chapter 2 introduces a natural integer programming formulation for the bandwidth allocation problem in PMP-Systems. This formulation contains two integer variables for each customer –the interval bounds– representing the interval assigned to the customer, and a binary variable for each pair of interfering customers –the ordering variables– representing the ordering among the intervals. The latter are needed to describe the feasible solutions, since otherwise the convex hull of all integer feasible solutions would contain infeasible but integral points. Section 2.1 closes with the definition of the associated polytopes. For any instance (G, d, s, g), we define the *chromatic scheduling polytope* $P(G, d, s, g)$ to be the convex hull of all the integer vectors corresponding to feasible solutions. A special case of this problem is of particular interest, namely, the case where each customer receives an interval which has precisely the length of its demand. We also define the *fixed-length chromatic scheduling polytope* $R(G, d, s, g)$ to be the convex hull of the feasible solutions satisfying this additional condition.

Section 2.2 presents some preliminary computational studies regarding the complete linear description of the easier case $R(G, \mathbf{1}, s, 0)$ for several small graphs. On the one hand, these experiments show that simple instances of the bandwidth allocation problem generate polytopes with a rather complex structure, admitting huge numbers of extreme points and facets. On the other hand, the reported results also suggest that chromatic scheduling polytopes pass through several stages as the frequency span s increases: from a nonempty but low-dimensional stage to full-dimensionality and, finally, to a combinatorially steady state.

The purpose of Chapter 3 is to discuss these different combinatorial stages. A first important issue is to find conditions for the existence/nonexistence of feasible solutions resp. for the nonemptyness/emptyness of the polytopes, as knowing one feasible solution enables us to run a PMP-System properly. We define $s_{\min}(G, d, g)$ to be the minimum frequency span making the polytopes nonempty, and Section 3.1 provides some straightforward bounds on this threshold. Note that the $\mathcal{NP}$-completeness of the bandwidth allocation problem implies that the exact calculation of $s_{\min}(G, d, g)$ is an $\mathcal{NP}$-hard problem. We combine the weighted clique number of the weighted graph (G, d) with sectorization arguments to devise a certificate of infeasibility, whereas a lower bound on s for feasibility arises from the chromatic number of G.

We explore in Section 3.2 the dimension of chromatic scheduling polytopes, a crucial property for deciding which valid inequalities are facets (and, therefore, the best possible cutting planes). It turns out that the dimension of these polytopes is hard to characterize, because it strongly depends on the graph structure, the node weighting and the available frequency spectrum $[0, s]$. It is not difficult to verify that the dimension is a nondecreasing function of the frequency span and that $P(G, d, s, g)$ and $R(G, d, s, g)$ are full-dimensional if $s \gg \omega(G, d)$. We thus introduce the threshold $s_{\text{full}}(G, d, g)$ defined as the minimum frequency span s making $P(G, d, s, g)$ full-dimensional. Section 3.2.1 presents further results related to full-dimensionality. In particular, we give a lower bound $\gamma(G, d, g)$ on s guaranteeing full-dimensionality of both polytopes based on coloring arguments. The section closes with a special analysis of the dimension of uniform instances, providing better bounds in terms of the chromatic number of the interference graph.

In Section 3.2.2 we discuss the computational complexity of the problem of determining the dimension of a particular instance. The main result of this section states that deciding whether a certain instance generates a full-dimensional polytope is $\mathcal{NP}$-complete. Hence, determining the dimension of chromatic scheduling polytopes is an $\mathcal{NP}$-hard task. Finally, Section 3.2.3 completely characterizes the dimension of $P(G, d, s, 0)$ and $R(G, d, s, 0)$ as a function of s for a number of graph classes. In particular, we are able to determine the dimension of both polytopes when the interference graph is a complete graph, a star, a path, and a cycle. These examples show that the dimension is a nontrivial parameter of the graph structure.

Section 3.3 explores the combinatorial steady state of chromatic scheduling polytopes. It has been experimentally observed in some instances that there exists a certain $s_{\max}(G, d, g) \in \mathbf{Z}_+$ such that the polytopes $\{R(G, d, s, g)\}_{s \geq s_{\max}(G,d,g)}$ have the same number of extreme points and facets. This led to the question whether all the polytopes $\{R(G, d, s, g)\}_{s \geq s_{\max}(G,d,g)}$ are combinatorially equivalent. In this section we give an affirmative answer by proving a more general result: the polytopes $R(G, d, s, g)$ and $R(G, d, s+1, g)$ resp. $P(G, d, s, g)$ and $P(G, d, s+1, g)$ are affinely isomorphic (and therefore combinatorially equivalent) for $s \gg \omega(G, d)$. Moreover, we give an upper bound on $s_{\max}(G, d, g)$, and this bound can be shown to be sharp when G is the disjoint union of cliques.

Section 3.4 closes the chapter establishing some relations between chromatic scheduling polytopes and the linear ordering polytope P^n_{LO}. It is not surprising that chromatic scheduling polytopes posess much of the structure of the linear ordering polytope, since the ordering variables have the same meaning in both settings. We prove that $P(K_n, d, s, 0)$ and $R(K_n, d, s, 0)$ are affinely isomorphic to P^n_{LO} when $s = \sum_{i=1}^n d_i$, and we show that $R(K_n, d, s, 0)$ is affinely isomorphic to P^{n+1}_{LO} when $s > \sum_{i=1}^n d_i$. These results imply that even simple chromatic scheduling polytopes are hard to characterize, since a complete linear description of $P(K_n, d, s, 0)$ includes all the linear ordering facets. We also study relations between the valid inequalities of these polytopes over arbitrary interference graphs, and the main result in this direction asserts that every facet-inducing inequality for the linear ordering polytope is also facet-inducing for $P(G, d, s, g)$ and $R(G, d, s, g)$ provided that $s \gg \omega(G, d)$ and the set of edges with nonzero coefficients is contained in E.

Chapter 4, Chapter 5, and Chapter 6 concentrate on the search for facet-inducing inequalities for chromatic scheduling polytopes. This issue has practical implications, since strong valid inequalities are the cornerstone of successful implementations of cutting plane methods. In order to apply such methods to a certain problem, a deep polyhedral study must be carried out, so that families of strong inequalities are found. The associated separation problems are also of interest, since good separation routines are required to efficiently detect violated inequalities in order to contribute to the process. It is worth noting that the $\mathcal{NP}$-completeness of the bandwidth allocation problem implies that finding a complete linear description of these polytopes is virtually a hopeless task, unless $\mathcal{NP}$ = co-$\mathcal{NP}$ [42].

Chapter 4 starts the search of facets of chromatic scheduling polytopes by exploring valid inequalities defining facets in all nonempty instances. To this end, Section 4.1 discusses the special symmetry of chromatic scheduling polytopes, which is a particular property of these polyhedra. Recall that we do not have precedence constraints given in advance, but only antiparallelity constraints. Hence, for every feasible solution, there is a symmetric feasible solution obtained by swapping all the intervals. The polytopes $P(G, d, s, g)$ and $R(G, d, s, g)$ clearly reflect this symmetry. The fixed-length polytope $R(G, d, s, g)$ is even symmetric with respect to a certain point, and due to this symmetry there exists, for every face, a parallel face of the same dimension. There is a simple formula to compute this parallel face, using the knowledge of the symmetry point. A similar construction can be even given for $P(G, d, s, g)$, although there is no symmetry point in this case.

This special symmetry also provides a theoretical tool for identifying facet-inducing inequalities. Consider a face F of $R(G, d, s, g)$ such that any integer solution lies in F if and only if its symmetrical solution does not belong to F. The main result of Section 4.1.3 shows that such a face is a facet of $R(G, d, s, g)$ as long as this polytope is nonempty –regardless of its dimension and particular structure. This is a powerful tool for identifying facet-defining inequalities, since no knowledge on the dimension is needed. We point out that this theorem only relies on symmetry considerations. A similar result holds for $P(G, d, s, g)$ under some further technical assumptions.

Based on these results, Section 4.2 explores facets coming from the integer programming constraints. We show that the binary bounds on the ordering variables are facet-inducing for every nonempty instance, and we present a further class of valid inequalities –the triangle inequalities– that possess the same property. This section also characterizes the polytopes which admit facets coming from the demand constraints. The remaining integer programming constraints, i.e., the bounds on the interval variables and the antiparallelity constraints, do not define facets in general and the purpose of Chapter 5 is to explore facet-inducing strenghtenings of these constraints.

If s is close to the weighted clique number $\omega(G, d)$ of the interference graph (G, d), it is usually difficult to place all the intervals interference-free within the available frequency spectrum; thus such settings are the hardest ones in practice. Section 4.3 presents three classes of valid inequalities for instances with small frequency spans, and we prove by symmetry arguments that they are facet-inducing regardless of the dimension of the polytope.

Chapter 5 presents a number of classes of facets arising from strenghtenings of the interval bound and the antiparallelity constraints. A natural way to generalize the interval bounds is to consider a clique in the neighborhood of the corresponding node of the interference graph, but we show that the resulting valid inequalities, called the *clique inequalities*, only are facet-inducing for particular cases. In order to devise stronger inequalities, a so-called *covering clique* must be considered instead of an arbitrary clique. Section 5.1 presents this construction and some algorithmic results concerning the identification of covering cliques.

Afterwards we prove that the so-called *covering-clique inequalities* are facet inducing for both polytopes if $s \geq s_{\min}(G, d, g) + 3(g + d_{\max})$. Interestingly, these inequalities are not facet-inducing for every instance, and we present a (rather involved) example. Finally, we also discuss the associated separation problem, showing $\mathcal{NP}$-completeness.

Based on similar ideas, Section 5.2 explores a strenghtening of antiparallelity constraints that gives rise to a class of facet-inducing inequalities, the *double covering-clique inequalities*. It is interesting that the same construction of covering cliques used for strenghtening the interval bounds can successfully be applied to the antiparallelity constraints. We prove that the resulting inequalities are valid for every instance and induce facets if $g = 0$ and $s \geq s_{\min}(G, d, 0) + 4d_{\max}$. However, many examples can be found where these inequalities are not facet-defining for both polytopes. We also explore the complexity of the associated separation problem, showing $\mathcal{NP}$-completeness. Finally, Section 5.2.3 presents the construction of double covering-clique inequalities for the case $g > 0$, that establishes that the resulting inequalities define facets of both polytopes.

Section 5.3 presents a number of further classes of facets arising as variations and generalizations of covering-clique inequalities and double covering-clique inequalities. Section 5.3.1 and Section 5.3.2 provide two generalizations of these families, originating two broader classes of facets. Section 5.3.3 presents three further classes of facet-inducing inequalities reinforcing the double covering-clique inequalities. These new families show an interesting balance in the coefficients of double covering-clique inequalities: when we try to strengthen the left-hand side, we have to adjust the right-hand side in order to maintain both validity and facetness. This interplay is well exemplified by the reinforced inequalities introduced in this section.

Chapter 6 presents further families of facet-inducing inequalities based on other structures of the interference graph. Section 6.1 presents the so-called *4-cycle inequalities*, arising from a combination of a 4-cycle and a clique, and constraining the relation between the left interval bounds of two nonadjacent nodes and the left border of the frequency spectrum $[0, s]$. A constructive proof of facetness is given for the uniform case $d = \mathbf{1}$ and $g = 0$.

Section 6.2 considers the *cycle-order inequalities*, defined over the ordering variables corresponding to cycles on the interference graph. The main result of this section asserts that, in the case $s \geq s_{\min}(G, d, g) + O(1)d_{\max}$, a cycle-order inequality is facet-inducing if and only if the associated cycle does not contain a chord. We prove that the cycle-order inequalities can be separated in polynomial time.

Cycles in the interference graph also originate valid inequalities over the interval bounds, and Section 6.3 presents a construction over odd holes (i.e., odd cycles with no chords). The *odd hole inequalities* are valid for arbitrary instances, and we prove that they define facets of $P(C_{2k+1}, \mathbf{1}, s, 0)$. We also provide conditions guaranteeing facetness for $P(G, \mathbf{1}, s, 0)$, and we prove that a superclass of the odd hole inequalities can be separated in polynomial time.

The analysis of the polytope $P(K_n, d, s, g)$, defined over a complete graph, is of theoretical interest and can also lead to facets for the general case. Sections 6.4 and 6.5 close the chapter with two classes of facets for this polytope, along with the corresponding generalizations for arbitrary interference graphs. We also prove that the associated separation problems are $\mathcal{NP}$-complete.

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Chapter 1

Frequency assignment

> The structural problems involving combinatorial considerations have only recently been studied in an intensive manner. They involve mathematical difficulties of the highest order even in what seem to be the simplest cases.
>
> – Richard Bellman (1956)

Wireless communication via radio waves dates back to the pioneering work of the french physicist Edouard Branly and the italian physicist Guglielmo Marconi. As early as 1889, Branly was able to transmit signals over small distances, reaching on open air receivers located 100 meters away from the transmitter. Based on this and his own experience, Marconi successfully transmitted in 1897 a Morse-coded message to a ship at sea over a distance of 29 kilometers. A couple of years later a regular communication was established across the English Channel, and already in 1902 it was possible to transmit signals across the Atlantic Ocean. The continuous development led to the first installations of telegraphic equipment on ships crossing the Atlantic Ocean, and a few years later every ship was using wireless telegraphy to communicate with other ships and shore stations. The following comment from the 1921 addendum to the W. M. Jackson Encyclopaedia [29] remarkably records the extent of the new invention:

> Whatever the future of this kind of long-distance direct communication between the two Continents is, it is by now well-known that passengers on board can establish communication with New York and London, and all the ships that make the aforementioned route are equipped with wireless telegraph machines (...). This way it is possible to daily print on board a newspaper with the Stock Exchange records and the most important news from all over the world. [Even more,] the captains of different ships have fun by playing chess over the telegraph.

In the 1920s the first experimental transmissions of television signals were made, resulting in the first official television broadcast in 1927. Radio broadcast became popular after World War I, and television was successfully introduced to the mass since the end of the 1940s. Today, the radio spectrum is not only used for cellular telephony and mass broadcasting, but also for navigational systems, space communication, radio astronomy and military applications.

Wireless communication between two points is established with the use of a transmitter and a receiver. The transmitter generates electrical oscillations at a certain radio frequency, which can be modulated either via the amplitude or the frequency itself. The receiver detects these oscillations and decodes them back to recover the original signal. Every application uses a certain part of the frequency spectrum, and the availability of frequencies is regulated worldwide by the International Telecommunication Union (ITU) and locally by the national governments.

A crucial problem in wireless communication is the interference between transmitters. If two nearby transmitters use the same frequency, then the signals may interfere. The level of interference depends on the distance between them, the geographical position of the transmitters, the power and direction of the signal, and even weather conditions. When the level of interference is high, the received signal may have an unacceptable poor quality. Hence there is a need for avoiding interference.

Operators of wireless services are licensed to use one or more frequency bands in specific parts of a country. The development of new wireless services and the addition of more and more customers led to scarcity of usable frequencies in the radio spectrum. This introduced the need for operators to develop frequency plans that not only avoided high interference levels but also minimized the licensing costs. As a consequence, an operator should carefully choose the frequencies on which each station transmits. This selection of frequencies is called the *frequency assignment problem* or *bandwidth allocation problem*. The conditions that should be satisfied by the frequency plan may vary depending on the application. Therefore, many different approaches have been suggested in the literature to solve this problem. Section 1.1 briefly surveys the most recent models, and in Section 1.2 we introduce Point-To-Multipoint radio access systems and the associated bandwidth allocation problem that motivated the work of this thesis.

1.1 Frequency assignment models

This section briefly surveys alternative models for frequency assignment. For a more thorough treatment, we refer to [2, 17, 34, 35]. In a typical frequency assignment problem, a set of wireless links is given and frequencies must be assigned such that the data transmission between the two endpoints of each link is possible. Such frequencies must lie within a certain frequency spectrum $[f_{\min}, f_{\max}]$ available to the provider. This spectrum is usually partitioned into a set of intervals, all with the same bandwidth, determining an integer number of so-called channels that each link can use. A transmission may be subject to interference if a geographically nearby link uses frequencies close on the electromagnetic spectrum, and the proposed models handle this situation in different ways.

1.1.1 Feasibility and maximum service FAP

In the *feasibility frequency assignment problem*, or shortly F-FAP, we are given a set of customers along with an interference relationship, and the objective is to assign a number of single frequencies to each customer while satisfying certain interference and availability constraints.

Problem input. Let F denote the (discrete) set of available channels from the frequency spectrum, and consider a set V of customers (equivalently, a set of antennae). Each customer $i \in V$ can only be assigned a channel from a subset $F(i)$ of F due to geographical reasons. Moreover, each customer $i \in V$ must receive $m(i)$ different channels from $F(i)$. Interference is modeled by an *interference graph* $G = (V, E)$ representing the pairs of customers that may interfere each other. Each pair of potentially interfering customers is joined by an edge in G. Finally, with each edge $ij \in E$ we associate a set T_{ij} of forbidden distances between the channels assigned to customers i and j.

Problem output. The desired output of F-FAP is an assignment $t : V \to 2^F$ such that

(i) $|t(i)| = m(i)$ for every $i \in V$,

(ii) $t(i) \subseteq F(i)$ for every $i \in V$, and

(iii) if $f \in t(i)$ and $g \in t(j)$ then $|f - g| \notin T_{ij}$ for every $ij \in E$.

For each pair of interfering customers $ij \in E$, this model specifies a set of forbidden distances between the channels assigned to each one. A common setting is to take $T_{ij} = \{0, \ldots, D\}$ for every $ij \in E$, thus specifying a minimum distance that must be obeyed between channels used by interfering antennae. Note that F-FAP reduces to the standard graph coloring problem by setting $F(i) = F$ and $m(i) = 1$ for every $i \in V$, and $T_{ij} = \{0\}$ for every $ij \in E$. Therefore, F-FAP is $\mathcal{NP}$-complete.

Alternative formulations consider different interference measures. One possibility is to define $p_{ij}(f, g)$ as the interference level between the customers i and j if they use the frequencies f and g, respectively. The interference condition $|f - g| \notin T_{ij}$ is then replaced by the condition $p_{ij}(f, g) > p_{\min}$, where $p_{\min}$ is a threshold for the acceptable level of interference.

In practice, it might happen that feasible solutions to this problem are difficult to find. In this case, we can decide to look for a partial solution assigning as many frequencies to the nodes as possible. Under the same problem input as before, the desired output is now an assignment $t : V \to 2^F$ satisfying $|t(i)| \leq m(i)$ for every $i \in V$ along with conditions (ii) and (iii), and such that the total number of assigned channels $\sum_{i \in V} |t(i)|$ is maximized. This problem is known as the *maximum service frequency assignment problem* or, shortly, Max-FAP.

1.1.2 Minimum order FAP

The objective of F-FAP is to find a feasible frequency assignment. However, when many feasible solutions exist, we could try to find the best one regarding the usage of frequencies. This model is called the *minimum order frequency assignment problem*, or MO-FAP, and asks for minimizing the total number of assigned channels. The problem input is the same as for F-FAP.

Problem output. The desired output of MO-FAP is an assignment $t : V \to 2^F$ such that

(i) $|t(i)| = m(i)$ for every $i \in V$,

(ii) $t(i) \subseteq F(i)$ for every $i \in V$,

(iii) if $f \in t(i)$ and $g \in t(j)$ then $|f - g| \notin T_{ij}$ for every $ij \in E$, and

(iv) the assignment minimizes $| \cup_{i \in V} t(i)|$.

The MO-FAP is the first frequency assignment problem that was discussed in the literature [41]. Again, this problem is a direct generalization of the standard graph coloring problem and is, therefore, $\mathcal{NP}$-complete. The well-known T-coloring and list coloring problems [17] are also restricted versions of MO-FAP. It is worth noting that the latter is $\mathcal{NP}$-complete even for interval graphs [5], a class that can be colored in polynomial time.

1.1.3 Minimum span FAP

In the *minimum span frequency assignment problem* (MS-FAP) the objective is to minimize the length of the frequency band needed to accomodate all the channels. The difference between the highest and the lowest used frequencies is called the solution's *span*; the objective is to minimize the span in order to keep the licensing costs for the used frequency span low. The problem output is, therefore, the following.

Problem output. The desired output of MS-FAP is an assignment $t : V \to 2^F$ such that

(i) $|t(i)| = m(i)$ for every $i \in V$,

(ii) $t(i) \subseteq F(i)$ for every $i \in V$,

(iii) if $f \in t(i)$ and $g \in t(j)$ then $|f - g| \notin T_{ij}$ for every $ij \in E$, and

(iv) the assignment minimizes $\max \cup_{i \in V} t(i) - \min \cup_{i \in V} t(i)$.

Note that MO-FAP asks for minimizing the number of used frequencies (which are not necessarily consecutive), whereas the objective of MS-FAP is to minimize the span of the assignment. It is worth noting that there exist general instances such that an optimal assignment for MO-FAP does not have minimum span and, in turn, an optimal solution to MS-FAP does not use the minimum possible number of channels.

1.1.4 Minimum interference FAP

All the previous models ask for an assignment with no interference at all. However, this may be impossible in some situation for which, moreover, the approach proposed by Max-FAP may be infeasible as well. In this setting a more realistic model –the *minimum interference frequency assignment problem*, or MI-FAP– can be stated, looking for an assignment with the minimum possible interference.

Problem input. As in the F-FAP, we are given a set F of available channels and a set V of customers. Each customer $i \in V$ can only be assigned a channel from a subset $F(i)$ of F and must receive $m(i)$ channels. Finally, for every pair of interfering customers $ij \in E$ and for each $f \in F(i)$ and $g \in F(j)$ we have a penalty value $p_{ij}(f,g)$ that is incurred when the customers i and j receive the interfering channels f and g, respectively. These penalties model the interference caused by the assignment.

Problem output. The desired output of MI-FAP is an assignment $t : V \to 2^F$ such that

(i) $|t(i)| = m(i)$ for every $i \in V$,

(ii) $t(i) \subseteq F(i)$ for every $i \in V$, and

(iii) the assignment minimizes $\sum_{ij \in E} \sum_{f \in t(i)} \sum_{g \in t(j)} p_{ij}(f,g)$.

As for all penalties $p_{ij}(f,g) > 0$ holds if and only if $|f - g| \in T_{ij}$, the optimum assignment has objective value equal to 0 if and only if F-FAP is feasible. Hence this model generalizes F-FAP and is, therefore, an $\mathcal{NP}$-hard optimization problem as well. A usual extension of this model arising from some instances from the CALMA benchmark [4] adds penalties for the choices of certain frequencies for each customer. This leads to an extra term in the objective function. It is worth noting that MI-FAP has been widely used in recent years to model real-world applications such as GSM Frequency Planning [18].

1.2 Bandwidth allocation in Point-to-Multipoint systems

We now turn our attention to Point-to-Multipoint radio access systems and the associated bandwidth allocation problem. This section describes in detail the assignment model that must be solved when operating such a system, also addressing complexity issues concerning this problem.

The purpose of a *Point-to-Multipoint radio access system* (PMP-System) is to supply wireless access to voice/data communication networks [7]. Base stations form the access points to the network. Each base station is located on a fixed position and serves a certain geographical area. This area served by the base station is divided into *sectors*. Figure 1.1 shows an example with three base stations, each serving two, three and two sectors, respectively.

Customer terminals are linked to base stations by means of radio signals, where some specific part of the radio frequency spectrum has to be used to maintain the links. In contrast to the usual setting for the previously mentioned FAPs, each customer is provided a fixed antenna and is therefore assigned to a certain sector of a base station (for example, in Figure 1.1 the customers t_1 and t_2 are assigned to sector A within the first base station). A characteristic feature of PMP-Systems is that each customer has an individual communication demand, implying that each customer needs a particular bandwidth within the available frequency spectrum. Hence the task is to assign *frequency intervals* instead of single channels.

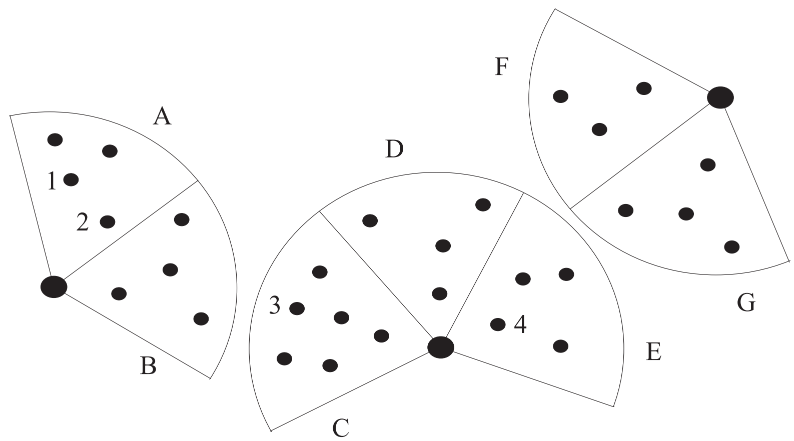


Figure 1.1: Sectorization by base stations in PMP-Systems.

A central problem is that a link connecting a customer terminal and a base station may be subject to interference from another link, provided that the same frequencies are used. We consider two sources of interference in this model. Firstly, links to customers in the same sector must not use the same frequency. Secondly, some of the links to customers in different sectors may also cause interferences. This second source of interference identifies certain pairs of customers that even being in different sectors might interfere each other due to the power of the transmitted signals and geographical reasons (for example, in Figure 1.1 the customers t_3 and t_4 are served by different antennae but still may interfere each other due to the alignment with the base station).

Moreover, in base stations oscillators provide the different frequencies with a possible difference Δ to the required frequency. Hence, between the frequency intervals of possibly interfering links in different sectors, a *guard distance* of length $g = 2\Delta$ has to be obeyed. This makes it necessary to distinguish between "in-sector" and "inter-sector" interference. To guarantee an interference-free communication, a particular *bandwidth allocation problem* has to be solved when operating a PMP-System.

Problem input. The input of this problem is given as follows. Let $\mathcal{T} = \{t_1, \ldots, t_n\}$ be the set of all customer terminals, and $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a partition of $\mathcal{T}$ into sectors, providing the information to which sector S_j the terminal $t_i \in \mathcal{T}$ belongs. Let $d = (d_1, \ldots, d_n)$ be the vector of communication demands associated with the customer terminals, indicating that customer $t_i \in \mathcal{T}$ has demand $d_i \in \mathbf{Z}_+$. Additionaly, we have a set $\mathcal{E}_X$ of unordered pairs (t_i, t_j) of terminals in *different* sectors that must not use the same frequency due to possible interference.

This setting can be viewed as a weighted graph $(G, d) = (V, E, d)$, where

- $V = \{i : t_i \in \mathcal{T}\}$ is the node set,
- $E = E_X \cup E_I$ is the edge set with

$$\begin{aligned} E_I &= \{ij : t_i, t_j \text{ in the same sector } S_l \in \mathcal{S}\}, \\ E_X &= \{ij : (t_i, t_j) \in \mathcal{E}_X\}, \end{aligned}$$

- $d = (d_1, \ldots, d_n)$ is the node weighting.

Thus, the node set represents customer terminals, the node weights reflect the communication demands, and the edge set indicates potential interference between the customer terminals. The edge set is given by the set of external interferers $\mathcal{E}_X$ and the partition of the node set V corresponding to the sectorization of $\mathcal{T}$. In graph theoretical terms, the partition of $\mathcal{T}$ into sectors $\mathcal{S} = \{S_1, \ldots, S_k\}$ corresponds to a clique covering of G, i.e., to a partition of V into k subsets $V_1, \ldots, V_k$ such that the nodes in every V_i are pairwise adjacent. We define this weighted graph (G, d) to be the *interference graph* associated with the particular instance of the bandwidth allocation problem.

Notation. Throughout this work we shall always denote by $(G, d) = (V, E, d)$ the interference graph. We also denote by $n = |V|$ resp. $m = |E|$ the number of nodes resp. edges of G.

Moreover, a guard distance $g \in \mathbf{Z}_+$ is given that must be kept between intervals of terminals $(t_i, t_j) \in \mathcal{E}_X$. Finally, we have the available radio frequency spectrum $[0, s]$, with $s \in \mathbf{Z}_+$, where all the frequency intervals have to be placed in. Thus, every instance of the bandwidth allocation problem is given by a quadruple (G, d, s, g).

Problem output. The task is to provide, for each customer $t_i \in \mathcal{T}$, a certain part[1] of the available frequency spectrum meeting the following two conditions. Firstly, the individual communication demand d_i is satisfied. Secondly, the assignment does not cause interference, i.e., no terminal within the same sector uses the same frequencies, and the guard distance is obeyed for each external interferer t_j, $(t_i, t_j) \in \mathcal{E}_X$. The desired output is, therefore, an assignment of an interval $I(i) = [l_i, r_i]$ with $l_i, r_i \in \mathbf{Z}_+$ to each customer $t_i \in \mathcal{T}$ such that:

(i) $r_i - l_i \geq d_i$ for every $i \in V$,

(ii) $[l_i, r_i] \subseteq [0, s]$ for every $i \in V$,

(iii) $\max\{l_i, l_j\} - \min\{r_i, r_j\} \geq \begin{cases} 0 & \text{if } t_i \text{ and } t_j \text{ belong to the same sector} \\ g & \text{if } (t_i, t_j) \in \mathcal{E}_X. \end{cases}$

Figure 1.2 shows a fragment of a feasible assignment. Note that customers t_1 and t_2 are assigned intervals of different lengths (the demand of customer t_1 being larger than the

[1] The frequency interval assigned to a customer is typically composed by several consecutive channels. The length of an interval corresponds to the number of those channels; the demand of a customer as well as the bounds of the assigned intervals are, therefore, represented as integers.

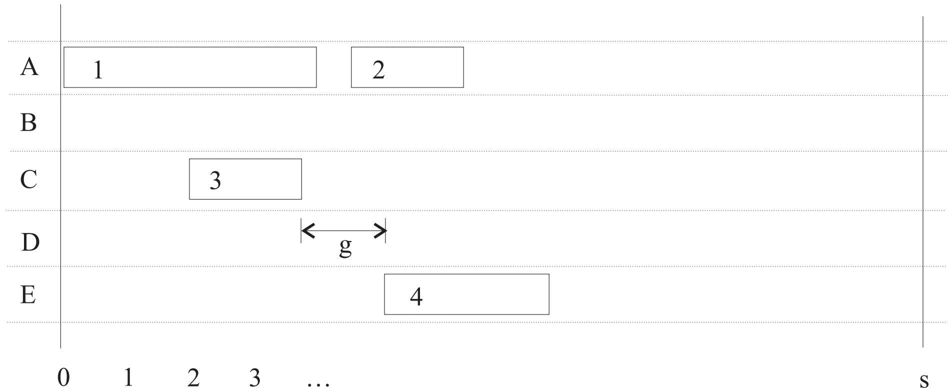


Figure 1.2: Fragment of a feasible assignment.

demand of customer t_2). These intervals do not overlap since both belong to the same sector of the same base station. On the other hand, customers t_3 and t_4 are located in different sectors but are identified in $\mathcal{E}_X$ as interfering customers; the corresponding intervals are, therefore, separated by a distance of at least g.

Remark. This setting may be interpreted as a *k-machine scheduling problem*, where the k sectors correspond to the k machines, and the customer terminals to the jobs. In our case, the assignment of jobs to machines is fixed in advance. The processing time of a job corresponds to the communication demand of the customer terminal. That no machine can process two jobs at the same time is given by E_I (recall that $\mathcal{S}$ corresponds to a clique covering of G by k cliques), where E_X gives antiparallelity requirements between jobs processed on different machines. Moreover, g can be interpreted as changeover time, and s as upper bound on the allowed makespan $\text{span}(y) = \max\{r_i : i \in V\} - \min\{l_j : j \in V\}$ with respect to a feasible schedule y (for more information on general scheduling problems see, e.g., [10]).

This particular kind of a scheduling problem does not contain the usual precedence constraints, but antiparallelity constraints are present instead. These constraints prevent certain pairs of tasks from overlapping, with a changeover time between them. The actual order among the tasks is not important, as long as the antiparallelity constraints are satisfied. This model can be applied as well to the construction of integrated circuits, the assembling of handcrafts and certain timetabling problems. $\lhd$

Since every graph is an interference graph, this model is a generalization of the *chromatic scheduling problem* [15] and, if $g = 0$, of the *consecutive coloring problem* [16] and the *interval coloring problem* [22, 36]. All of these models, in turn, generalize the standard graph coloring problem, defined as follows:

	GRAPH COLORING
Instance:	A graph $G = (V, E)$ and an integer $k \in \mathbf{Z}_+$.
Question:	Does there exist a k-coloring of G, i.e., a function $f : V \to \{1, \ldots, k\}$ such that $f(i) \neq f(j)$ for every $ij \in E$?

Theorem 1.1 *Let $g = 0$ and $d_i = 1$ for every $i \in V$. The bandwith allocation problem in PMP-Systems is feasible if and only if the associated interference graph G admits an s-coloring.*

Proof. Let $f : V \to \{1, \ldots, s-1\}$ be a coloring of G, and construct a feasible schedule by assigning the interval $I(i) = [f(i)-1, f(i)]$ to the customer $t_i \in \mathcal{T}$. Since f is a coloring, then no interfering intervals overlap (and the guard distance $g = 0$ is trivially satisfied), hence this construction is feasible. Conversely, any feasible schedule assigns an interval $I(i) = [l_i, r_i]$ to the customer $t_i \in \mathcal{T}$, such that all pairs of interfering customers receive disjoint intervals. This induces an s-coloring $f(i) = r_i$ for every $i \in V$. $\Box$

Corollary 1.2 *The bandwidth allocation problem in PMP-Systems is $\mathcal{NP}$-complete.*

This equivalence between GRAPH COLORING and the bandwidth allocation problem in PMP-Systems for the case $g = 0$ and $d = \mathbf{1}$ also shows that the latter problem cannot be approximated by a polynomial-time algorithm with a fixed approximation ratio [20]. Furthermore, consider the OPEN SHOP problem, defined as follows.

	OPEN SHOP
Instance:	A number $p \in \mathbf{Z}_+$ of processors, a set J of jobs, each job $j \in J$ consisting of p tasks $t_{1j}, \ldots, t_{pj}$ (with t_{ij} to be executed by processor i), a length $l(t_{ij}) \in \mathbf{Z}_+$ for each such task, and an overall deadline $k \in \mathbf{Z}_+$.
Question:	Is there a schedule for J that meets the deadline k?

OPEN SHOP is $\mathcal{NP}$-complete even for $p = 3$ [20]. A straightforward reduction from OPEN SHOP to the bandwidth allocation problem in PMP-Systems can be given, and this reduction provides a second proof of Corollary 1.2. Given an instance of OPEN SHOP, defined as above, construct an interference graph $(G, d) = (V, E, d)$ with one node for each task and such that two nodes are joined by an edge in E if and only if the corresponding tasks either belong to the same job or must be executed by the same processor. The demand of each node is defined to be the length of the corresponding task. Further, set $g = 0$ and $s = k$. There is a schedule meeting the deadline k if and only if this instance of the bandwidth allocation problem is feasible.

Solving the bandwidth allocation problem is a crucial task when operating a PMP-System, but we have seen that this is a demanding computational issue, since this problem generalizes difficult coloring resp. scheduling problems. Suitable heuristics based on greedy arguments have been developed, and these heuristics were able to produce span-minimal resp. feasible solutions for small resp. medium-sized problems [7]. In order to tackle problem sizes of real-world instances, algorithms have to be designed that rely on a deeper insight of the problem structure.

Cutting plane methods have turned out to be successful for many other applications [6, 30, 42, 45]. In this framework, the convex hull of the incidence vectors of all feasible solutions is studied in order to derive facets or, more modestly, valid inequalities for this polyhedron representing the solution space of the problem. A strong knowledge of these polyhedra provides the cornerstone of successful implementations of this approach. Therefore, we propose to investigate the polytopes arising from this bandwidth allocation problem, as a starting point for the practical solution to optimality of real-world instances. This thesis contributes to this polyhedral issue.

Chapter 2

Chromatic scheduling polytopes

> We hope that the feasibility of attacking problems involving a moderate number of points has been successfully demonstrated, and that perhaps some of the ideas can be used in problems of similar nature.
>
> – G. Dantzig, R. Fulkerson and S. Johnson (1954)

The study of chromatic scheduling polytopes is the topic of this thesis; the main purpose of this chapter is to introduce these polytopes and to discuss some basic properties. Section 2.1 gives an integer programming formulation for the bandwidth allocation problem in PMP-systems (BAP). We define the chromatic scheduling polytope $P(G,d,s,g)$ to be the convex hull of all feasible solutions of this integer program and the fixed-length chromatic scheduling polytope $R(G,d,s,g)$ as the special case where no demand is oversatisfied.

Section 2.2 reports some experiments regarding the complete linear description of the easier case $R(G,\mathbf{1},s,g)$ for several small graphs G and increasing values of the frequency span s. These experiments show that, on the one hand, the polytopes pass through several stages as s increases and, on the other hand, that even simple instances of the problem give rise to polytopes with a complex structure, as the number of facets and extreme points is already huge for small graphs. This adds support to the belief that chromatic scheduling polytopes are hard to characterize by means of facet-defining inequalities.

2.1 Integer programming formulation for BAP

We now present an integer programming formulation for the bandwidth allocation problem in PMP-Systems. To represent a solution, we use two groups of variables. Firstly, for each node $i \in V$ we introduce the *interval bounds* l_i and r_i, such that $I(i) = [l_i, r_i]$ represents the frequency interval assigned to the corresponding customer. Both variables are constrained to be integer and nonnegative. In addition, for each edge $ij \in E$ with $i < j$ we define the binary *ordering variable*

$$x_{ij} = \begin{cases} 1 & \text{if } r_i \leq l_j \\ 0 & \text{otherwise,} \end{cases}$$

asserting whether the interval $I(i)$ is located before the interval $I(j)$ or not. In every feasible solution, the antiparallelity requirements for intervals corresponding to potential interferers are realized by a precedence relation (i.e., a partial order) on the set of intervals. This precedence relation is represented by the ordering variables. Note that we need one ordering variable for every $ij \in E$, namely x_{ij} if $i < j$. For notational convenience, we shall use x_{ji} as a shorthand for $1 - x_{ij}$. According to the variable definitions, the incidence vector of a solution S is given by:

$$\chi^S = (\underbrace{l_1, \ldots, l_n}_{n}, \underbrace{r_1, \ldots, r_n}_{n}, \underbrace{x_{1i}, \ldots, x_{jn}}_{m}).$$

A feasible solution is, therefore, an assignment of values to l_i, r_i $\forall i \in V$ and x_{ij} $\forall ij \in E$ such that the following constraints are satisfied:

$$\begin{array}{rcll}
d_i & \leq & r_i - l_i \qquad \forall i \in V & (2.1) \\
0 & \leq & l_i \ \leq \ r_i \ \leq \ s \qquad \forall i \in V & (2.2) \\
r_i & \leq & l_j + s(1 - x_{ij}) \qquad \forall ij \in E_I,\ i < j & (2.3) \\
r_i + g & \leq & l_j + s(1 - x_{ij}) \qquad \forall ij \in E_X,\ i < j & (2.4) \\
r_j & \leq & l_i + s x_{ij} \qquad \forall ij \in E_I,\ i < j & (2.5) \\
r_j + g & \leq & l_i + s x_{ij} \qquad \forall ij \in E_X,\ i < j & (2.6) \\
x_{ij} & \in & \{0, 1\} \qquad \forall ij \in E,\ i < j & (2.7) \\
l_i,\ r_i & \in & \mathbf{Z} \qquad \forall i \in V & (2.8)
\end{array}$$

The *demand constraints* (2.1) and the *bound constraints* (2.2) assert that the interval $I(i) = [l_i, r_i]$ must satisfy the demand d_i and fit within the available frequency spectrum $[0, s]$. Inequalities (2.3) to (2.6) realize the *antiparallelity constraints*, which prevent interfering pairs of intervals from overlapping. Note that the intervals corresponding to the pairs of customers in E_I (located in the same sector) must not overlap, and there must be a distance of at least g between the intervals corresponding to pairs of interfering customers in different sectors (i.e., pairs of customers from E_X). Finally, the *integrality constraints* (2.7) resp. (2.8) force the x-variables to be binary resp. the interval bounds to be integral.

Remark. It is necessary to include the ordering variables x_{ij}, for $ij \in E$, $i < j$ in order to encode a solution. A feasible schedule can certainly be described by the interval bounds only, but then the convex hull of the incidence vectors of all feasible schedules may contain infeasible integral points. Consider, e.g., the problem given by the graph $(G, d) = (V, E, d)$ with $V = \{1, 2\}$, $E = \{12\}$, and $d = (1, 2)$ and the frequency spectrum $[0, 4]$. Then the set of all feasible solutions consists of the following ten points.

	l_1	l_2	r_1	r_2	x_{12}
p_0	0	1	1	3	1
p_1	0	1	1	4	1
p_2	0	2	1	4	1
p_3	0	2	2	4	1
p_4	1	2	2	4	1
p_5	2	0	3	2	0
p_6	2	0	4	2	0
p_7	3	0	4	2	0
p_8	3	0	4	3	0
p_9	3	1	4	3	0

Dropping the information given by x_{12}, the convex hull of even the two points $p'_0 = (0, 1, 1, 3)$ and $p'_9 = (3, 1, 4, 3)$ would contain two infeasible but integral points, namely $x = (1, 1, 2, 3)$ and $y = (2, 1, 3, 3)$, as Figure 2.1 shows. The ordering variables guarantee that the convex hull of the incidence vectors of all feasible schedules does not contain any such point. Hence these binary variables are essential to describe the solution space of the problem. $\triangleleft$

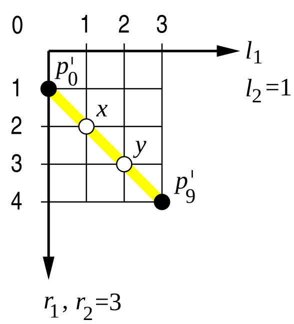


Figure 2.1: Convex hull of two feasible solutions

In order to run a Point-to-Multipoint system, one is mainly interested in finding *feasible* solutions satisfying all the constraints above. It is not difficult to verify that the weighted clique number $\omega(G, d)$ is a canonical lower bound for the makespan $\text{span}(y)$ of any feasible solution y. An instance of the bandwidth allocation problem is, therefore, hard to solve if the gap between $\omega(G, d)$ and the available frequency span s is small. This causes the interest in finding span-minimal solutions, i.e., we have to solve the combinatorial optimization problem $\min \text{span}(y)$, where $y = (l, r, x)$ is taken over all feasible solutions satisfying the constraints (2.1)-(2.8).

Small and mid-size instances of the bandwidth allocation problem can be solved by greedy-like heuristics as in [7]; large real-world instances require algorithms using deeper methods. Algorithms based on cutting planes have shown to be successful for many other combinatorial optimization problems [6, 30, 42, 45]. In order to apply such methods to the bandwidth allocation problem, we are interested in investigating the convex hull of all feasible solutions satisfying these constraints. Recall that $n = |V|$ resp. $m = |E|$ denotes the number of nodes resp. edges of the interference graph G.

Definition 2.1 (chromatic scheduling polytope) *Let* $(G, d) = (V, E, d)$ *be a graph with node weights* d*, let* $[0, s]$ *be the available frequency spectrum, and let* $g \in \mathbf{Z}_+$ *be the guard distance. The* chromatic scheduling polytope $P(G, d, s, g) \subseteq \mathbf{R}^{2n+m}$ *is defined as the convex hull of all integer solutions* $(l, r, x) \in \mathbf{R}^{2n+m}$ *satisfying constraints (2.1)-(2.8).*

A special case of the bandwidth allocation problem is of particular interest, namely the case where each customer receives an interval $I(i) = [l_i, r_i]$ which has *precisely* the length of its demand, i.e., $r_i - l_i = d_i$ for every $i \in V$. This case is in practice easier to solve and the solution space has lower dimension since the right interval bounds are no longer necessary. Hence only the l- and x-variables are required, and every solution vector has only $n + m$ entries instead of the $2n + m$ entries in the general case. Therefore, the incidence vector of a feasible schedule S_R is, in this case:

$$\chi^{S_R} = (\underbrace{l_1, \ldots, l_n}_{n}, \underbrace{x_{1i}, \ldots, x_{jn}}_{m}).$$

Definition 2.2 (fixed-length chromatic scheduling polytope) *Let* $(G, d) = (V, E, d)$ *be a graph with node weights* d*, let* $[0, s]$ *be the available frequency spectrum, and let* $g \in \mathbf{Z}_+$ *be the guard distance. The* fixed-length chromatic scheduling polytope $R(G, d, s, g) \subseteq \mathbf{R}^{n+m}$ *is defined as the convex hull of all integer solutions* $(l, x) \in \mathbf{R}^{n+m}$ *such that there exists some* $r \in \mathbf{R}^n$ *satisfying* $r_i = l_i + d_i$ *and constraints (2.2)-(2.8).*

The bandwidth allocation problem in PMP-Systems was first introduced in [7], where greedy-like heuristics were developed for solving small and mid-sized instances. A first study of the fixed-length polytope $R(G, d, s, g)$ for the special case with two sectors was carried out in [21]. Moreover, [26] presents initial results for the general polytope $P(G, d, s, g)$.

Notation. If $z = (l_1, \ldots, l_n, r_1, \ldots, r_n, x_{1i}, \ldots, x_{jn}) \in \mathbf{R}^{2n+m}$ is a feasible solution, we denote by z_{l_i} resp. z_{r_i} its i-th resp. $(n+i)$-th coordinate. For $ij \in E$, $i < j$, we denote by $z_{x_{ij}}$ the entry of z corresponding to the ordering variable associated to the edge ij and, as noted previously, we define $z_{x_{ji}} = 1 - z_{x_{ij}}$ as a notational shorthand. We also define the projections of z onto the spaces of each group of variables as

$$\begin{aligned} z_l &= (l_1, \ldots, l_n) \in \mathbf{R}^n \\ z_r &= (r_1, \ldots, r_n) \in \mathbf{R}^n \\ z_x &= (x_{1i}, \ldots, x_{jn}) \in \mathbf{R}^m \end{aligned}$$

Note that $z = (z_l, z_r, z_x) \in \mathbf{R}^{2n+m}$. The same definitions apply to the fixed-length case. Here, if $y \in \mathbf{R}^{n+m}$ is a feasible solution, then y_{l_i} resp. $y_{x_{ij}}$ denotes the left interval bound of the interval $I(i)$ resp. the ordering variable associated with the edge $ij \in E$, $i < j$. The projections y_l and y_x are defined accordingly.

2.2 Computational experiments

This section presents some preliminary computational experiments generating the complete linear description of the polytopes $R(G, \mathbf{1}, s, 0)$ associated with small graphs G and increasing frequency spans s in order to have an idea of the number of extreme points and facets involved. These experiments were carried out with PORTA [11, 12] in combination with an ad hoc program for efficiently generating the feasible solutions. All the experiments were performed on a Silicon Graphics Origin 200 machine, with a 1024 MB RAM and four R12000 processors running at 400 MHz. The experiments were run with a CPU time limit of 5 days.

Tables 2.1 and 2.2 show the number of facets and extreme points of the fixed-length chromatic scheduling polytope $R(K_n, \mathbf{1}, s, 0)$ defined over complete interference graphs, for different values of the number n of nodes and the frequency spectrum length s (the empty spaces show the infeasible cases). The number of facets is remarkably huge even for small instances, although the number of extreme points seems to grow more modestly. Moreover, the total number of feasible solutions is huge already for the smallest instances, e.g., there exist 4410 solutions for $n = 3$ and $s = 6$, and 38976 solutions for $n = 4$ and $s = 6$.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$s=1$	–						
$s=2$	2	–					
$s=3$	8	8	–				
$s=4$	8	20	20	–			
$s=5$	8	20	40	40	–		
$s=6$	8	20	40	910	910	–	
$s=7$	8	20	40	910	87472	87472	–
$s=8$	8	20	40	910	87472	$> 480 \times 10^6$	$> 480 \times 10^6$
$s=9$	8	20	40	910	87472	$> 480 \times 10^6$	?

Table 2.1: Number of facets of $R(K_n, \mathbf{1}, s, 0)$.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$s=1$	–						
$s=2$	2	–	–				
$s=3$	6	6	–				
$s=4$	6	24	24	–			
$s=5$	6	24	120	120	–		
$s=6$	6	24	120	720	720	–	
$s=7$	6	24	120	720	5040	5040	–
$s=8$	6	24	120	720	5040	40320	40320
$s=9$	6	24	120	720	5040	40320	362880

Table 2.2: Number of extreme points of $R(K_n, \mathbf{1}, s, 0)$.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$s=1$	–	–	–	–	–	–	–
$s=2$	2	2	2	2	2	2	2
$s=3$	8	24	48	72	96	120	144
$s=4$	8	24	54	110	222	454	?
$s=5$	8	24	54	116	?	?	?
$s=6$	8	24	54	?	?	?	?

Table 2.3: Number of facets of $R(P_n, \mathbf{1}, s, 0)$.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$s=1$	–	–	–	–	–	–	–
$s=2$	2	2	2	2	2	2	2
$s=3$	6	12	24	48	96	192	384
$s=4$	6	18	50	138	378	1034	2826
$s=5$	6	18	58	172	528	1586	4802
$s=6$	6	18	58	182	570	1782	5566

Table 2.4: Number of extreme points of $R(P_n, \mathbf{1}, s, 0)$.

These tables also suggest that the polytopes from the family $\{R(K_n, \mathbf{1}, s, g)\}_{s \geq n+1}$ have the same number of extreme points and facets. The same holds for the polytopes $R(K_n, \mathbf{1}, n+1, 0)$ and $R(K_{n+1}, \mathbf{1}, n+1, 0)$, for $n \geq 2$. These computational results in fact reflect a deep relationship between chromatic scheduling polytopes and the linear ordering polytope, and will be explained by the results of Section 3.4. It must be noted that the results for $n \geq 6$ and $s \geq 7$ were not generated in the computational environment described previously, but were derived from the results in Section 3.4 and the computational experiments reported in [13] for the linear ordering polytope.

Tables 2.3 and 2.4 show the number of facets and extreme points for chromatic scheduling polytopes defined over paths. Again, the number of feasible solutions is huge even for small instances (98620 feasible solutions for $n = 4$ and $s = 6$, and 179150 solutions for $n = 6$ and $s = 4$). Finally, we present in Tables 2.5 and 2.6 the experiments on chromatic scheduling polytopes defined over cycles, showing a similar behavior. The number of facets is more modest in these cases, although it is worth to mention that the computation time exceeded the time limit of 5 days even for $n = 7$ and $s = 4$. All cases which could not be computed within this time limit are indicated by a question tag within the tables.

The latter experiments imply again that the polytopes defined over the same interference graph admit the same number of facets and extreme points for $s \geq n$ (but clearly different numbers of feasible solutions). Similar observations were obtained in [21] for co-bipartite interference graphs. This motivated our investigations on the combinatorial equivalence of polytopes over the same interference graph, explored in Section 3.3.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$s=1$	–	–	–	–	–	–	–
$s=2$	2	2	2	2	2	2	2
$s=3$	8	8	72	274	816	8768	26634
$s=4$	8	20	160	644	9848	?	?
$s=5$	8	20	242	1556	?	?	?
$s=6$	8	20	242	?	?	?	?

Table 2.5: Number of facets of $R(C_n, \mathbf{1}, s, 0)$.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$s=1$	–	–	–	–	–	–	–
$s=2$	2	2	2	2	2	2	2
$s=3$	6	6	18	30	64	126	258
$s=4$	6	24	46	160	414	1120	3134
$s=5$	6	24	78	250	726	2296	6790
$s=6$	6	24	78	300	858	2940	8750

Table 2.6: Number of extreme points of $R(C_n, \mathbf{1}, s, 0)$.

Chapter 3

General properties of chromatic scheduling polytopes

> It is interesting to point out that these applications rely on the deep theorems characterizing facets of the corresponding polytope. This is in quite a contrast to previously known algorithms, which typically do not use these characterizations but quite often give them as a by-product.
>
> – M. Grötschel, L. Lovasz and A. Schrijver (1981)

Chromatic scheduling polytopes admit interesting properties from a combinatorial point of view. As observed from the experiments in Section 2.2, the chromatic scheduling polytopes are empty if the frequency span s is too small and pass through several stages as s increases: from a nonempty but low-dimensional stage to full-dimensionality and, finally, to a combinatorially steady state. We discuss these different stages and the corresponding "thresholds" $s_{\text{min}}(G,d,g)$, $s_{\text{full}}(G,d,g)$, and $s_{\text{max}}(G,d,g)$ ensuring nonemptyness, full-dimensionality, and combinatorial stability, respectively.

Section 3.1 treats the problem of proving nonemptyness for the polytopes. This is an important task as knowing one feasible solution enables us to run a PMP-System properly. We present lower (resp. upper) bounds on $s_{\text{min}}(G,d,g)$ ensuring emptyness (resp. nonemptyness). Interestingly, the weighted clique number of the weighted graph (G,d) gives a certificate of infeasibility, whereas a lower bound on $s_{\text{min}}(G,d,g)$ arising from coloring arguments guarantees feasibility.

Section 3.2 deals with the nonempty case and addresses the problem of calculating the dimension of chromatic scheduling polytopes. As the best cutting planes are facets, i.e., inequalities defining a face with dimension one less than the dimension of the polytope itself, the search for facets must usually be preceded by the study of the dimension. Unfortunately, determining the dimension of chromatic scheduling polytopes is $\mathcal{NP}$-complete in general, as shown in this section. However, partial results and bounds for $s_{\text{full}}(G,d,g)$ could be achieved.

Section 3.3 is devoted to the combinatorial steady state, i.e., to the fundamental issue that full-dimensional chromatic scheduling polytopes maintain, from a certain value $s \geq s_{\max}(G,d,g)$ of the frequency span on, the same number of facets and extreme points. We present such a lower bound $s_{\max}(G,d,g)$ for s, give a characterization of the extreme points of $R(G,d,s,g)$ resp. $P(G,d,s,g)$ and, for $s \geq s_{\max}(G,d,g)$, a natural bijection between the extreme points of $R(G,d,s,g)$ and $R(G,d,s+1,g)$ resp. $P(G,d,s,g)$ and $P(G,d,s+1,g)$ implying combinatorial equivalence.

The chapter closes with a discussion relating chromatic scheduling polytopes with linear ordering polytopes. In Section 3.4 we prove that chromatic scheduling polytopes defined over complete interference graphs are affinely isomorphic to linear ordering polytopes, implying that even these simple instances are hard to characterize. We also present some relations between the valid inequalities and facets of these polytopes, that can be exploited in a practical framework for solving the bandwidth allocation problem in PMP-Systems.

3.1 On emptyness/nonemptyness

The characterization of conditions that guarantee feasibility of the bandwidth allocation problem is a central issue. Clearly, if the frequency spectrum $[0,s]$ is too small, there exists no feasible schedule for the frequency intervals at all, and so the polytopes $P(G,d,s,g)$ and $R(G,d,s,g)$ are empty. The results presented in this section provide straightforward bounds on the frequency span s that guarantee emptyness and nonemptyness. It is worth noting that upper bounds for infeasibility arise from maximum weighted clique arguments, whereas lower bounds for feasibility come from coloring assertions. We first establish the following definitions, which provide us a notation to make conversions back and forth between feasible solutions of $P(G,d,s,g)$ and $R(G,d,s,g)$.

Definition 3.1 *Let $y \in R(G,d,s,g)$. We define the extension of y to be $\operatorname{ext}(y) \in P(G,d,s,g)$ such that*

$$\begin{array}{rcll} \operatorname{ext}(y)_{l_i} & = & y_{l_i} & \forall i \in V \\ \operatorname{ext}(y)_{r_i} & = & y_{l_i} + d_i & \forall i \in V \\ \operatorname{ext}(y)_{x_{ij}} & = & y_{x_{ij}} & \forall ij \in E \end{array}$$

Conversely, the reduction of a point $z \in P(G,d,s,g)$ is $\operatorname{red}(z) \in R(G,d,s,g)$ defined by

$$\begin{array}{rcll} \operatorname{red}(z)_{l_i} & = & z_{l_i} & \forall i \in V \\ \operatorname{red}(z)_{x_{ij}} & = & z_{x_{ij}} & \forall ij \in E \end{array}$$

The schedule represented by $\operatorname{red}(z)$ (for $z \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$) is obtained by shrinking each interval $I(i)$ to an interval of length d_i (and projecting down the vector to $\mathbf{R}^{n+m}$). Conversely, if $y \in R(G,d,s,g) \cap \mathbf{Z}^{n+m}$ is a feasible solution, then $\operatorname{ext}(y)$ represents the same schedule than y, but in a space of higher dimension that also contains the r-variables. Note that $\operatorname{red}(\operatorname{ext}(y)) = y$ for every $y \in R(G,d,s,g)$, but $\operatorname{ext}(\operatorname{red}(z))$ differs from z if $z_{r_i} - z_{l_i} > d_i$ for some $i \in V$.

As a first simple observation, we may point out that $P(G,d,s,g) \neq \emptyset$ if and only if $R(G,d,s,g) \neq \emptyset$, implying that the feasibility problems for $P(G,d,s,g)$ and $R(G,d,s,g)$ are equivalent. We call $\mathrm{proj}_{l,x}(P(G,d,s,g)) = \{\mathrm{red}(z) : z \in P(G,d,s,g)\} \subseteq \mathbf{R}^{n+m}$ to the projection of $P(G,d,s,g)$ onto the space of the l- and x-variables.

Proposition 3.1 $R(G,d,s,g) = \mathrm{proj}_{l,x}(P(G,d,s,g))$.

Proof. If $y \in R(G,d,s,g) \cap \mathbf{Z}^{n+m}$ is an integer feasible solution of $R(G,d,s,g)$, then $\mathrm{ext}(y)$ belongs to $P(G,d,s,g)$, and thus $R(G,d,s,g) \subseteq \mathrm{proj}_{l,x}(P(G,d,s,g))$. Conversely, if $z \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ is a feasible integer solution of $P(G,d,s,g)$, then $\mathrm{red}(z)$ belongs to $R(G,d,s,g)$, implying the converse inclusion. □

Corollary 3.2 *$P(G,d,s,g)$ is nonempty if and only if $R(G,d,s,g)$ is nonempty.*

It is worth noting that Corollary 1.2 implies that determining whether $R(G,d,s,g)$ is empty or not is a computationally difficult task. Observe that if $R(G,d,s_0,g)$ is nonempty, then $R(G,d,s,g)$ is nonempty for every $s \geq s_0$. Similarly, if $R(G,d,s_0,g)$ is empty, then also is $R(G,d,s,g)$ for every $s \leq s_0$.

Definition 3.2 (nonemptyness threshold) *We denote by $s_{\min}(G,d,g)$ the minimum frequency span s such that $P(G,d,s,g)$ is nonempty.*

Note that $P(G,d,s,g)$ is nonempty if and only if $s \geq s_{\min}(G,d,g)$. Corollary 3.2 implies that $s_{\min}(G,d,g)$ is also the minimum frequency span s guaranteeing feasibility for $R(G,d,s,g)$. The exact calculation of this threshold is, by Corollary 1.2, an $\mathcal{NP}$-hard problem, hence we concentrate on deriving bounds on this value. A certificate of infeasibility can be obtained by means of the weighted clique number $\omega(G,d)$ of (G,d) (i.e., the weight of a largest weighted clique of G), as Proposition 3.3 shows.

Proposition 3.3 *If $s < \omega(G,d)$, then $R(G,d,s,g)$ and $P(G,d,s,g)$ are empty.*

Proof. Let $K \subseteq V$ be a largest weighted clique of G (i.e., a clique K such that $d(K) = \omega(G,d)$). The intervals $\{I(i) : i \in K\}$ cannot overlap in any feasible solution, since all vertices in K are pairwise adjacent. Hence we need at least a span of $d(K) = \omega(G,d)$ for scheduling these intervals, and since the length of the available spectrum $[0,s]$ is strictly less than this lower bound, the problem is infeasible. □

However, $s \geq \omega(G,d)$ does not provide a certificate for feasibility, as there exist graphs (G,d) such that $\omega(G,d)$ is strictly smaller than the span of any feasible solution. Such instances clearly exist for the special case $(G,\mathbf{1},s,0)$ of usual graph coloring problems, e.g., $R(C_{2k+1},\mathbf{1},2,0)$ is empty for every odd hole C_{2k+1} with $k \geq 2$, since $\omega(C_{2k+1},\mathbf{1}) = 2 < 3 = \chi(C_{2k+1})$ holds. Moreover, [7] reports real-world instances $(G,d,s,0)$ with $d \neq \mathbf{1}$, containing critical configurations $G' \subseteq G$ with $\omega(G',d) < s_{\min}(G',d,0)$.

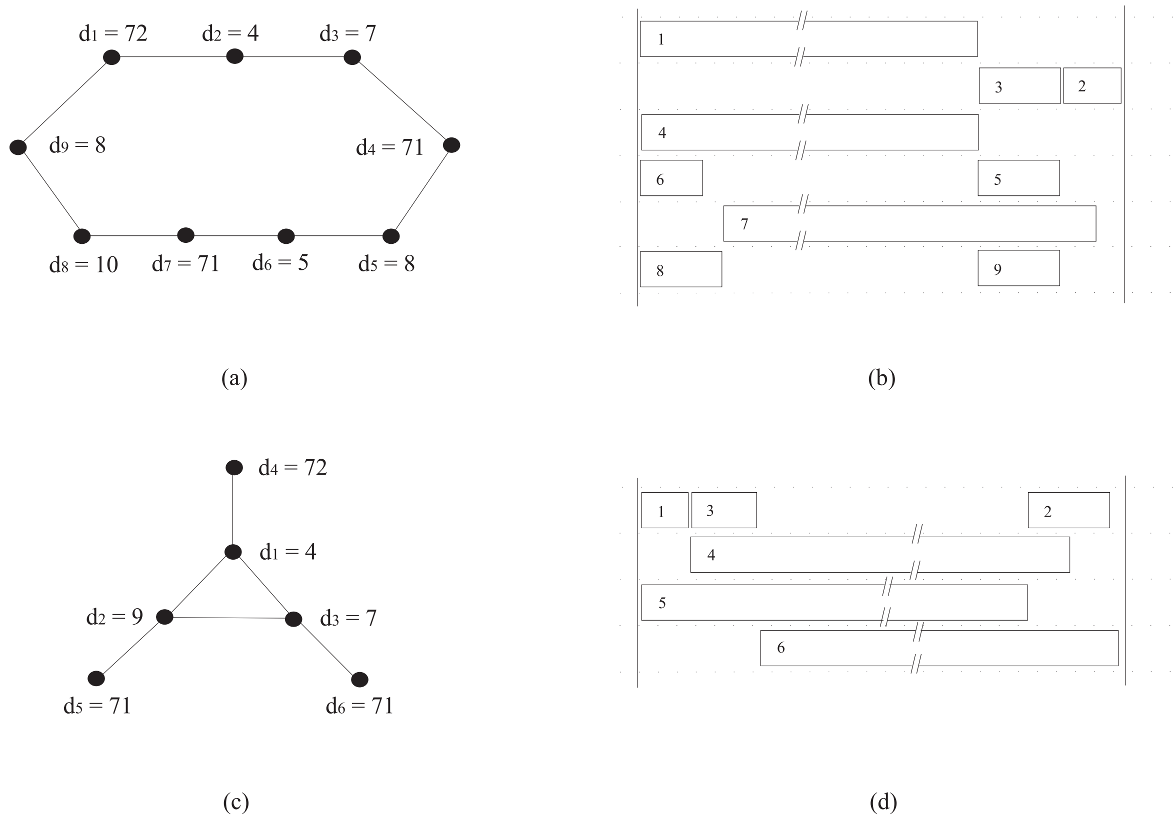


Figure 3.1: Critical configurations from two real-world instances.

Example 3.1 Consider the instance depicted in Figure 3.1(a), with $G = C_9$ and the customer demands presented in the figure. This interference graph has $\omega(G, d) = 81$ but $s_{\min}(G, d, 0) = 82$ (see Figure 3.1(b)). Further, the weighted asteroidal tripel (G, d) presented in Figure 3.1(c) has $\omega(G, d) = 80$, but $s_{\min}(G, d, 0) = 82$, as Figure 3.1(d) shows. $\triangleleft$

Remark. Graphs G with $\omega(G, d) = s_{\min}(G, d, 0)$ for all possible demand vectors d are introduced by Golumbic [22] as *superperfect graphs*. The previous example shows that interference graphs arising from PMP-Systems are not superperfect in general. $\triangleleft$

Additionally, in the case $g > 0$ we must also obey the guard distance between pairs of adjacent intervals in different sectors. This setting is more restrictive, and Proposition 3.4 gives a straightforward generalization of Proposition 3.3.

Definition 3.3 (clique bound) *If $K \subseteq V$ is a clique, define $p_K = |\{i : S_i \cap K \neq \emptyset\}|$ to be the number of sectors with nonempty intersection with K. Let $K(G)$ denote the set of all cliques of G, and define the* clique bound *$\omega(G, d, g)$ to be*

$$\omega(G, d, g) \;=\; \max_{K \in K(G)} \Big(d(K) + g\ (p_K - 1)\Big).$$

Proposition 3.4 *If $s < \omega(G, d, g)$, then $P(G, d, s, g)$ and $R(G, d, s, g)$ are empty.*

Proof. Let $K \subseteq V$ be a clique such that $d(K) + g\ (p_K - 1) = \omega(G, d, g)$. Since K is a clique, then the intervals $\{I(i) : i \in K\}$ must be disjoint. Moreover, in every feasible solution there are at least $p_K - 1$ adjacent intervals belonging to different sectors, and since K is a clique they must obey the guard distance, hence at least $p_K - 1$ guard distances must occur between the intervals assigned to the nodes of K. Therefore, we need a frequency span of at least $d(K) + g\ (p_K - 1)$ to assign all these intervals. □

Again, $s \geq \omega(G, d, g)$ does not imply that the polytopes are nonempty. In the opposite direction, we can derive an upper bound for $s_{\min}(G, d, g)$ that guarantees feasibility.

Definition 3.4 (chromatic bound) *Let $d_{\max} = \max\{d_i : i \in V\}$ denote the maximum node weight of (G, d). We define the* chromatic bound *$\chi(G, d, g)$ to be*

$$\chi(G, d, g) \ = \ (d_{\max} + g)\ \chi(G) - g.$$

Proposition 3.5 *If $s \geq \chi(G, d, g)$, then $R(G, d, s, g)$ and $P(G, d, s, g)$ are nonempty.*

Proof. Let $k = \chi(G)$ and let $c : V \to \{1, \ldots, k\}$ be a coloring of G (i.e., a partition of V into disjoint independent subsets). Construct a feasible solution $y \in R(G, d, s, g) \cap \mathbf{Z}^{n+m}$ by setting $y_{l_i} = (c(i) - 1)(d_{\max} + g)$, where $c(i)$ is the color assigned to i by c. Note that this assignment is feasible and fits in the frequency spectrum $[0, s]$. Thus $R(G, d, s, g)$ is nonempty and, by Corollary 3.2, $P(G, d, s, g)$ is also nonempty. □

Note that the weighted chromatic number $\chi(G, d)$ (i.e., the minimum number of stable sets covering every node i at least d_i times) cannot be used to obtain a better bound than $\chi(G, d, g)$ since the colors assigned to each node cannot be expected to be consecutive. Now, Proposition 3.4 and Proposition 3.5 imply that $s_{\min}(G, d, g)$ can be bounded by the clique bound and the chromatic bound:

$$\omega(G, d, g) \ \leq \ s_{\min}(G, d, g) \ \leq \ \chi(G, d, g).$$

In the uniform case $d = \mathbf{1}$ with null guard distance (i.e., $g = 0$), we obtain $s_{\min}(G, \mathbf{1}, 0) = \chi(G, \mathbf{1}, 0) = \chi(G)$ and $\omega(G, \mathbf{1}, 0) = \omega(G)$.

3.2 On the dimension of the polytopes

A common way of proving that a valid inequality is facet-defining for a certain polytope is to construct as many affinely independent points in the particular hyperplane as the dimension of the polytope is. However, determining the dimension of chromatic scheduling polytopes turns out to be a difficult task. This section presents partial results on this issue. We point out as a first observation that nonempty polytopes may not be full-dimensional when the available frequency spectrum $[0, s]$ is not large.

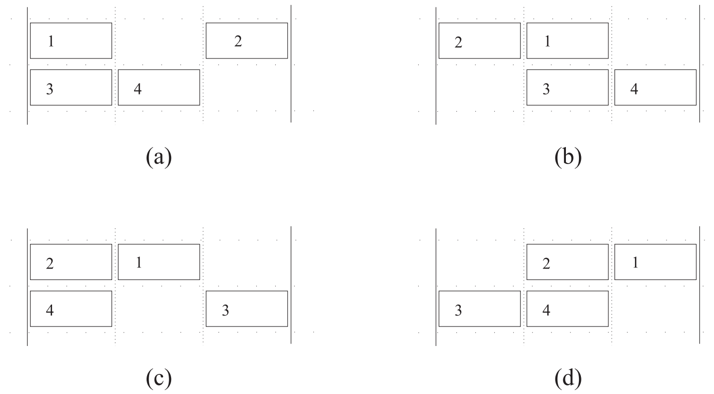


Figure 3.2: The polytope $R(C_4, \mathbf{1}, 3, 0)$ is not full-dimensional.

Example 3.2 Consider the polytope $R(C_4, \mathbf{1}, 3, 0) \subseteq \mathbf{R}^8$. Every integer feasible solution in this polytope assigns the unit intervals $I(1), \ldots, I(4)$ within the frequency span [0,3], and thus we have that $I(1) = I(3)$ or $I(2) = I(4)$ (or both). Note that $I(i) = I(j)$ implies that $x_{ik} = x_{jk}$ for every $k \in V \backslash \{i, j\}$. We claim that every feasible schedule satisfies $x_{14} - x_{12} = x_{34} - x_{32}$.

- If $I(1) = I(3)$, then the previous observation implies that $x_{14} = x_{34}$ and $x_{12} = x_{32}$ (see Figure 3.2(a) and Figure 3.2(b), along with the symmetrical constructions). Subtracting these equations we obtain $x_{14} - x_{12} = x_{34} - x_{32}$.
- If $I(2) = I(4)$, then $x_{12} = x_{14}$ and $x_{32} = x_{34}$ (see Figure 3.2(c) and Figure 3.2(d), and the symmetrical constructions). These two equations imply $x_{14} - x_{12} = 0 = x_{34} - x_{32}$.

Thus, every feasible point satisfies $x_{14} - x_{12} = x_{34} - x_{32}$, hence $\dim(R(C_4, \mathbf{1}, 3, 0)) \leq 7$ (in fact, the dimension is exactly 7). As we shall verify in Section 3.2.3, the polytopes $R(C_4, \mathbf{1}, s, 0)$ for $s \geq 4$ are full-dimensional. $\triangleleft$

The polytopes $P(G, d, s, g)$ and $R(G, d, s, g)$ are nonempty if and only if $s \geq s_{\min}(G, d, g)$. The previous example shows that they may not be full-dimensional, even if $s > s_{\min}(G, d, g)$. However, as the frequency span s increases, the dimension of both polytopes also increases (although not strictly), since every feasible solution of $R(G, d, s, g)$ is also feasible for $R(G, d, s+1, g)$. This observation implies the following.

Proposition 3.6 *If $s \geq s_{\min}(G, d, g)$, then $R(G, d, s, g) \subseteq R(G, d, s+1, g)$ and $P(G, d, s, g) \subseteq P(G, d, s+1, g)$.*

Corollary 3.7 *If $s \geq s_{\min}(G, d, g)$, then $\dim(R(G, d, s, g)) \leq \dim(R(G, d, s+1, g))$ and $\dim(P(G, d, s, g)) \leq \dim(P(G, d, s+1, g))$.*

Hence the dimension is a nondecreasing function of the frequency span s. When $s \gg \omega(G, d)$, both polytopes are full-dimensional. We prove this fact in the next subsection, where we provide a lower bound on s that guarantees full-dimensionality. Section 3.2.2 completes the analysis by showing that the exact calculation of the dimension is an $\mathcal{NP}$-hard problem. Finally, Section 3.2.3 closes with characterizations of the dimension for special families of interference graphs.

3.2.1 The full-dimensional case

It has been previously observed [26] that $P(G, d, s, g)$ and $R(G, d, s, g)$ are full-dimensional when $[0, s]$ is large enough. This subsection presents some results related to full-dimensionality. In particular, we provide a lower bound $\gamma(G, d, g)$ on s such that $P(G, d, s, g)$ and $R(G, d, s, g)$ are full-dimensional if $s \geq \gamma(G, d, g)$. We present some examples where this bound is indeed tight.

Next, we analyze the dimension in the uniform case $d = \mathbf{1}$ with $g = 0$, where the bound simplifies to $\gamma(G, \mathbf{1}, 0) = \chi(G) + 2$. We provide a characterization of full-dimensionality for bipartite graphs and $s = \chi(G) + 1$, proving that for a bipartite interference graph G, the polytope $P(G, \mathbf{1}, \chi(G) + 1, 0)$ is full-dimensional if and only if G does not contain any 4-hole. Based on this result, we also provide a partial characterization of full-dimensionality for arbitrary graphs.

Lemma 3.8 *Let $\lambda \in \mathbf{R}^{n+m}$ and $\lambda_0 \in \mathbf{R}$ such that $\lambda^T y = \lambda_0$ for every $y \in R(G, d, s, g)$. If $s > s_{\min}(G, d, g)$, then $\lambda_{l_j} = 0$ for every $j \in V$.*

Proof. Let $y \in R(G, d, s, g) \cap \mathbf{Z}^{n+m}$ be an integer feasible solution such that all the intervals are contained in $[0, s_{\min}(G, d, g)]$. Construct a digraph $D = (V, E_D)$ such that $ij \in E_D$ if and only if $ij \in E$ and $I(j)$ is located before $I(i)$. Note that D is acyclic. Now, let $i_1, \ldots, i_n$ be a topological ordering of the nodes of D and construct n feasible solutions $y^1, \ldots, y^n$ as follows. Point y^k is obtained from y by shifting the intervals $I(i_j)$ for $j = 1, \ldots, k$ one unit to the right.

These new points are feasible solutions. Indeed, if the interval $I(i_j)$ has been shifted to the right in y^k, then all the possible interfering intervals to the right of $I(I_j)$ have also been shifted, since the corresponding nodes are before i_j in any topological ordering of D. Moreover, the pair of solutions y^k and y^{k+1} for $k = 0, \ldots, n-1$ (where we consider $y^0 = y$) only differ in their l_{i_k}-coordinate, hence the l_{i_k}-coordinate of λ must be zero. Therefore, $\lambda_{l_j} = 0$ for every $j \in V$. □

Definition 3.5 *Let $F_s(G, d)$ denote the set of nodes i such that $P(G, d, s, g)$ contains some feasible schedule such that the interval $I(i)$ has length strictly greater than d_i. That is,*

$$F_s(G, d) \;=\; \{i \in V : z_{r_i} - z_{l_i} > d_i \text{ for some } z \in P(G, d, s, g)\}.$$

Note that Lemma 3.8 implies $F_s(G, d) = V$ for $s > s_{\min}(G, d, g)$. However, when $s = s_{\min}(G, d, g)$ we may have $F_s(G, d) \subset V$. In both cases, $F_s(G, d)$ states a relation between the dimension of $P(G, d, s, g)$ and the dimension of $R(G, d, s, g)$.

Lemma 3.9 *If $s \geq s_{\min}(G,d,g)$ then* $\dim(P(G,d,s,g)) = \dim(R(G,d,s,g)) + |F_s(G,d)|$.

Proof. For each $i \in F_s(G)$ let $y^i \in P(G,d,s,g)$ be a solution such that $y^i_{r_i} - y^i_{l_i} > d_i$ and $y^i_{r_j} - y^i_{l_j} = d_j$ for $j \neq i$ (such a solution exists by the definition of $F_s(G,d)$). Now, if $w^0, \ldots, w^k \in R(G,d,s,g)$ is a set of affinely independent points, then $\text{ext}(w^0), \ldots, \text{ext}(w^k)$ are also affinely independent, and moreover each of these new points satisfies $r_i + l_i = d_i$ for every $i \in V$. This implies that the point y^i is affinely independent w.r.t. $\text{ext}(w^0), \ldots, \text{ext}(w^k)$, for every $i \in F_s(G,d)$. Hence the set $\{\text{ext}(w^i)\}_{i=0}^k \cup \{y^i\}_{i \in F_s(G,d)}$ is composed by $k + |F_s(G,d)|$ affinely independent points of $P(G,d,s,g)$, and thus $\dim(R(G,d,s,g)) + |F_s(G,d)| \leq \dim(P(G,d,s,g))$.

For the reverse inequality, let $A \in \mathbf{R}^{k \times n}$, $B \in \mathbf{R}^{k \times m}$ and $b_0 \in \mathbf{R}^k$ such that $Al + Bx = b_0$ is a maximal system of equations for $R(G,d,s,g)$, implying $\dim(R(G,d,s,g)) = n + m - k$. By Proposition 3.1, we have that $Al + Bx = b_0$ is also a (possibly nonmaximal) system of k equations for $P(G,d,s,g)$ and, in addition, every feasible solution $z \in P(G,d,s,g)$ satisfies $z_{r_i} - z_{l_i} = d_i$ for each $i \notin F_s(G,d)$. Hence we construct $k + (n - |F_s(G,d)|)$ linearly independent equations satisfied by every feasible solution of $P(G,d,s,g)$. Since $P(G,d,s,g) \subseteq \mathbf{R}^{2n+m}$, we conclude that

$$\begin{aligned} \dim(P(G,d,s,g)) &\leq (2n+m) - (k + n - |F_s(G,d)|) \\ &= (n + m - k) + |F_s(G,d)| \\ &= \dim(R(G,d,s,g)) + |F_s(G,d)|. \end{aligned}$$

□

Lemma 3.10 *Let $\lambda^T z = \lambda_0$ for every $z \in P(G,d,s,g)$. If $s > s_{\min}(G,d,g)$, then $\lambda_{l_i} = 0$ and $\lambda_{r_i} = 0$ for every $i \in V$.*

Proof. Lemma 3.8 implies $F_s(G,d) = V$, hence $\dim(P(G,d,s,g)) = \dim(R(G,d,s,g)) + n$. Moreover, we have that $\text{proj}_x(P(G,d,s,g)) = \text{proj}_x(R(G,d,s,g))$, and thus $\lambda_{l_i} = \lambda_{r_i} = 0$ for every $i \in V$. □

We are now able to provide a lower bound on s that ensures full-dimensionality in the general case.

Definition 3.6 (coloring bound) *We define the* coloring bound *to be*

$$\gamma(G,d,g) \;=\; s_{\min}(G,d,g) + \max_{jk \in E}(d_j + d_k) + 2g.$$

Theorem 3.11 *If $s \geq \gamma(G,d,g)$ then $R(G,d,s,g)$ and $P(G,d,s,g)$ are full-dimensional.*

Proof. Let $\lambda^T z = \lambda_0$ for every $z \in P(G,d,s,g)$. By Lemma 3.10, we have $\lambda_{l_i} = \lambda_{r_i} = 0$ for every $i \in V$. Now, let $z \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ be a feasible solution such that

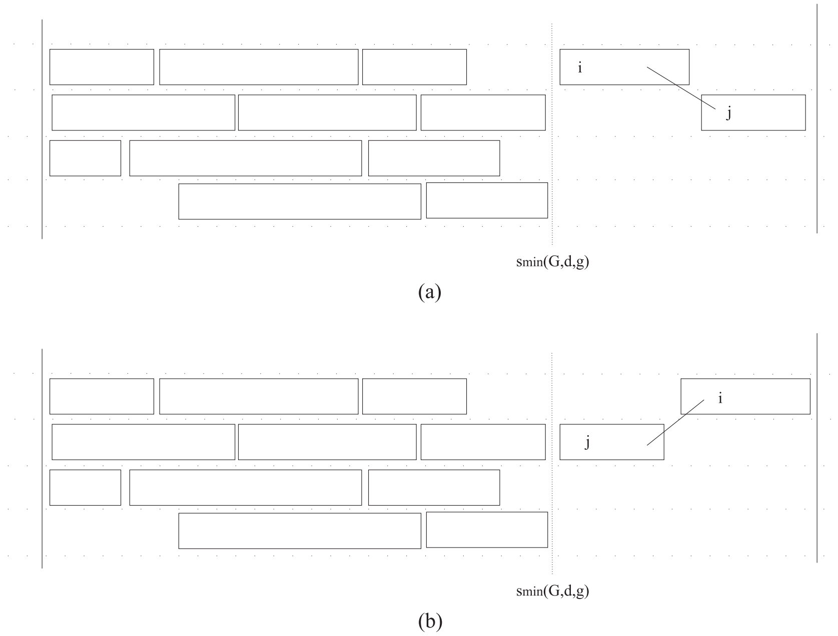


Figure 3.3: Constructions for the proof of Theorem 3.11.

$\max_{i \in V} z_{r_i} = s_{\min}(G, d, g)$ (such a solution exists by the definition of the nonemptyness threshold $s_{\min}(G, d, g)$). Consider an arbitrary edge $ij \in E$ and construct the feasible solution z^1 as follows:

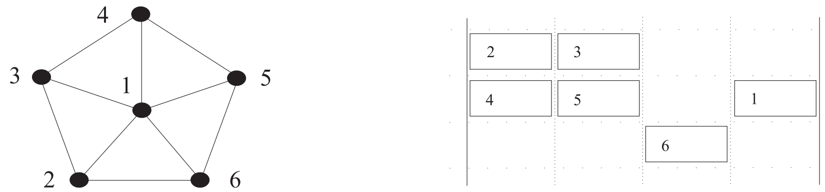
$$z^1_{l_k} = \begin{cases} s_{\min}(G, d, g) + g & \text{if } k = i \\ s_{\min}(G, d, g) + d_i + 2g & \text{if } k = j \\ z_{l_k} & \text{otherwise} \end{cases}$$

Define further $z^1_{r_k} = z^1_{l_k} + d_k$ for every $k \in V$. Now construct a new feasible solution z^2 from z^1 by swapping the intervals $I(i)$ and $I(j)$ (see Figure 3.3). These solutions only differ in their l_i-, r_i-, l_j-, r_j- and x_{ij}-coordinates and, therefore, $\lambda_{x_{ij}} = 0$. Since ij is an arbitrarily chosen edge, we have $\lambda = \mathbf{0}$, and so we conclude that $P(G, d, s, g)$ is full-dimensional. Since $F_s(G, d) = V$, Lemma 3.9 implies that $R(G, d, s, g)$ is also full-dimensional. $\square$

Theorem 3.11 implies that for every instance (G, d, s, g) there exists a frequency span s' such that the polytopes $\{P(G, d, s, g)\}_{s \geq s'}$ are full-dimensional. Hence we can introduce the following threshold for full-dimensionality.

Definition 3.7 (full-dimensionality threshold) *We denote by $s_{\text{full}}(G, d, g)$ the minimum frequency span s such that the polytope $P(G, d, s, g)$ is full-dimensional.*

Under this definition, Theorem 3.11 can be restated as $s_{\text{full}}(G, d, g) \leq \gamma(G, d, g)$. This bound is sharp, in the sense that there exist infinitely many graphs G such that $P(G, d, s-1, g)$, for $s = \gamma(G, d, g)$, has not full dimension. For example, if the interference graph is a 4-cycle, we have $s_{\text{full}}(C_4, \mathbf{1}, 0) = \gamma(C_4, \mathbf{1}, 0) = 4$ but Example 3.2 shows that the polytope $R(C_4, \mathbf{1}, 3, 0) \subseteq \mathbf{R}^8$ has dimension 7, thus not being full-dimensional. In Section 3.2.3 we shall present further instances illustrating the same situation.



(a) (b)

Figure 3.4: $R(W_6, \mathbf{1}, 4, 0)$ is full-dimensional whereas $P(W_6, \mathbf{1}, 4, 0)$ is not.

Note that $s_{\text{full}}(G, d, g)$ is the minimum frequency span guaranteeing full-dimensionality for $P(G, d, s, g)$ but not for the fixed-length polytope $R(G, d, s, g)$. If $P(G, d, s, g)$ has full dimension, then clearly $R(G, d, s, g)$ is full-dimensional, but the converse is not true as the following example shows.

Example 3.3 Consider the wheel W_6 depicted in Figure 3.4(a), composed by a 5-cycle plus a universal node. Figure 3.4(b) shows $s_{\min}(W_6, \mathbf{1}, 0) = 4$. It is not difficult to verify by inspection that $R(W_6, \mathbf{1}, 4, 0)$ is full-dimensional. However, $P(W_6, \mathbf{1}, 4, 0)$ does not have full dimension, since $r_1 - l_1 = 1$ for every feasible solution. Moreover, for this particular instance we have $s_{\text{full}}(W_6, \mathbf{1}, 0) = 5$. $\lhd$

Hence the threshold $s_{\text{full}}(G, d, g)$ for full-dimensionality in the general case cannot be directly applied to the fixed-length case. We obtain instead the following about the dimension of the two polytopes.

Corollary 3.12 *Consider an instance (G, d, s, g).*

(i) If $s < s_{\min}(G, d, g)$ then both polytopes $P(G, d, s, g)$ and $R(G, d, s, g)$ are empty.

(ii) If $s = s_{\min}(G, d, g)$ then $P(G, d, s, g)$ is full-dimensional only if $R(G, d, s, g)$ is full-dimensional.

(iii) If $s > s_{\min}(G, d, g)$ then $P(G, d, s, g)$ is full-dimensional if and only if $R(G, d, s, g)$ is full-dimensional, by $\dim(P(G, d, s, g)) = n + \dim(R(G, d, s, g))$.

Thus, we can express the minimum frequency span such that $R(G, d, s, g)$ has full dimension in terms of $s_{\text{full}}(G, d, g)$ as follows.

Corollary 3.13 *Let s_R be the minimum frequency span s such that the polytope $R(G, d, s, g)$ has full-dimension. Then, $s_R = s_{\text{full}}(G, d, g)$ if $F_{s_R}(G, d) = V$ and $s_R = s_{\text{full}}(G, d, g) - 1$ otherwise.*

In the remaining part of this section, we discuss better bounds fo $s_{\text{full}}(G, d, g)$ in the case of usual graph coloring, i.e., if we assume $d = \mathbf{1}$ and $g = 0$.

Corollary 3.14 *The polytopes $R(G, \mathbf{1}, s, 0)$ and $P(G, \mathbf{1}, s, 0)$ are full-dimensional if and only if $s \geq \chi(G) + 2$.*

Corollary 3.14 provides a small range for incomplete dimensionality in the uniform case. Indeed, $P(G, \mathbf{1}, s, 0)$ is empty if $s < \chi(G)$ and full-dimensional if $s \geq \chi(G) + 2$. So we are left to analyze the cases $s = \chi(G)$ and $s = \chi(G) + 1$. In what follows, our objective is to give a partial characterization of full-dimensionality in the case $s = \chi(G) + 1$. As we shall see, incomplete dimension is related to the existence of induced 4-cycles in the interference graph. We first analyze the case of bipartite graphs.

Theorem 3.15 *If G is a bipartite graph, then $P(G, \mathbf{1}, 3, 0)$ is full-dimensional if and only if G does not contain C_4 as an induced subgraph.*

Proof. Assume first that G does not contain any 4-hole as induced subgraph, and suppose $\lambda^T y = \lambda_0$ for every $y \in P(G, \mathbf{1}, 3, 0)$. Lemma 3.10 implies that $\lambda_{l_i} = \lambda_{r_i} = 0$ for every $i \in V$. We will now verify that the same holds for the ordering variables, thus proving the full-dimensionality of the polytope.

Fix an edge $ij \in E$ and let $c : E \to \{1, 2\}$ be a 2-coloring of G. Assume w.l.o.g. that $c(i) = 1$ and $c(j) = 2$. Define the node subsets $A = N(i)$ and $B = N(j)$ (see Figure 3.5). Note that $c(k) = 2$ for every $k \in A$ and $c(t) = 1$ for every $t \in B$, hence $A \cap B = \emptyset$. Moreover, $E(A, B) = \emptyset$, otherwise a 4-hole would be created. Partition now the remaining nodes as $C \cup D$, where

$$
\begin{aligned}
C &= \{k \notin A \cup B \cup \{i, j\} : c(k) = 1\} \\
D &= \{k \notin A \cup B \cup \{i, j\} : c(k) = 2\}
\end{aligned}
$$

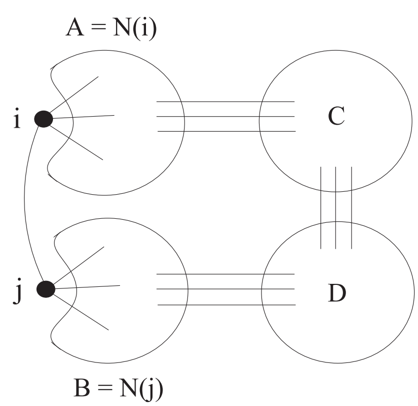


Figure 3.5: Partition of V into subsets.

These sets define the partition of V depicted in Figure 3.5. Notice that the sets A, B, C and D are stable sets. Moreover, $E(A,D)=\emptyset$ since the nodes of A and D admit the same color. The same argument shows $E(B,C)=\emptyset$.

We now define the following subsets of edges:

$$\begin{aligned} E_1 &= E(\{i\},A) \\ E_2 &= E(A,C) \\ E_3 &= E(C,D) \\ E_4 &= E(B,D) \\ E_5 &= E(\{j\},B) \end{aligned}$$

By the previous observations, we have $E=\{ij\}\cup E_1\cup\ldots\cup E_5$. We now construct the sequence of feasible solutions $y^0,\ldots,y^6$ depicted in Figure 3.6. For $k=1,\ldots,6$, consider the pair of solutions y^0 and y^k. Both solutions are feasible, and thus $\lambda^T y^0=\lambda^T y^k$, implying the following equations.

$$\begin{array}{lcrcl} k=1 & \Rightarrow & 0 & = & \lambda(E_1)+\lambda(E_2) \\ k=2 & \Rightarrow & 0 & = & \lambda(E_2)+\lambda(E_3) \\ k=3 & \Rightarrow & 0 & = & \lambda(E_3)+\lambda(E_4) \\ k=4 & \Rightarrow & 0 & = & \lambda(E_4)+\lambda(E_5) \\ k=5 & \Rightarrow & 0 & = & \lambda(E_5)+\lambda_{x_{ji}} \\ k=6 & \Rightarrow & 0 & = & \lambda(E_3)+\lambda(E_4)+\lambda(E_5) \end{array}$$

Solving these equations leads to $\lambda_{x_{ji}}=0$ and $\lambda(E_k)=0$ for $k=1,\ldots,5$ (note that this does not imply $\lambda=\mathbf{0}$). Thus, we have shown $\lambda_{x_{ji}}=0$. Since ij is an arbitrary edge of G, this procedure shows $\lambda=\mathbf{0}$. Therefore, the polytope is full-dimensional.

Now let us turn to the converse. Let $C\subseteq V$ be an induced 4-hole in G. The projection of $P(G,\mathbf{1},3,0)$ over the variables l_i, r_i for $i\in C$ and x_{ij} for $ij\in E(C)$ equals $P(C,\mathbf{1},3,0)$, and we already know that this polytope is not full-dimensional. Hence, $P(G,\mathbf{1},3,0)$ does not have full dimension as well. □

Corollary 3.16 *If G is a tree, then $P(G,\mathbf{1},\chi(G)+1,0)$ is full-dimensional.*

Based on the previous results, we now provide a partial characterization of full-dimensionality for arbitrary graphs in the case $s=\chi(G)+1$. Theorem 3.17 gives a sufficient condition for $P(G,\mathbf{1},\chi(G)+1,0)$ to be full-dimensional, whereas Theorem 3.18 provides a sufficient condition ensuring incomplete dimension. Although these conditions are similar, they are not the converse of each other and so the characterization given here is only partial.

Theorem 3.17 *If there exists a k-coloring of G with $k\le\chi(G)+1$ and color classes $I_1,\ldots,I_k$ such that $G_{I_i\cup I_j}$ does not contain a 4-hole for every $i\ne j$, then $P(G,\mathbf{1},\chi(G)+1,0)$ is full-dimensional.*

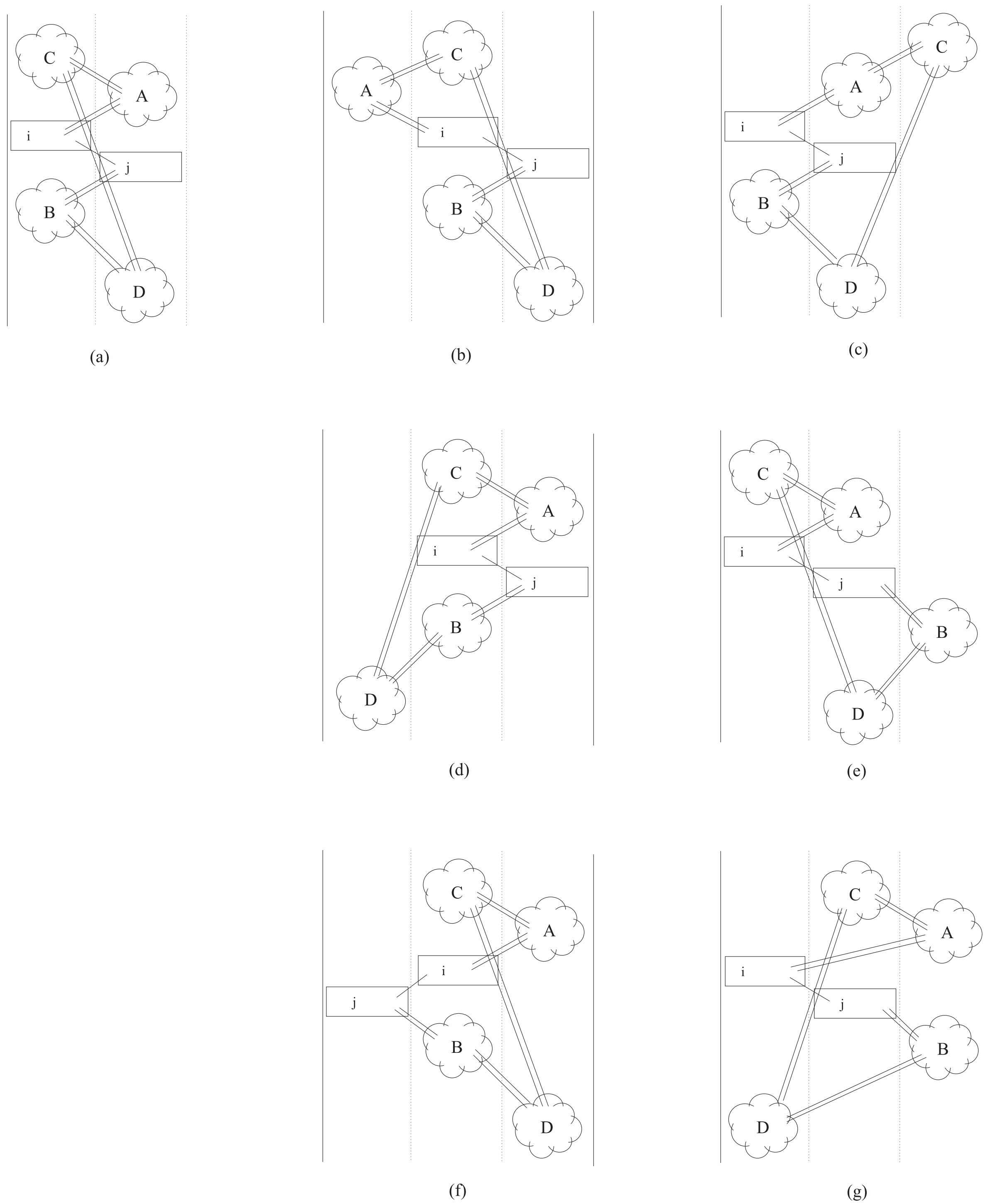


Figure 3.6: Feasible solutions $y^0, \ldots, y^6$.

Proof. Suppose that $\lambda^T y = \lambda_0$ for every $y \in P(G, \mathbf{1}, \chi(G)+1, 0)$. Lemma 3.10 implies that $\lambda_{l_i} = \lambda_{r_i} = 0$ for every $i \in V$. Now, for every pair I_i, I_j of color classes, with $i \neq j$, consider the induced subgraph $G_{ij} = G_{I_i \cup I_j}$. By Theorem 3.15, the polytope $P(G_{ij}, \mathbf{1}, 3, 0)$ is full-dimensional. Moreover, $P(G_{ij}, \mathbf{1}, 3, 0) \subseteq \mathrm{proj}_{I_i \cup I_j}\, P(G, \mathbf{1}, \chi(G)+1, 0)$ implies $\lambda_{x_e} = 0$ for every $e \in G_{ij}$. Thus, $\lambda_x = \mathbf{0}$ and so $P(G, \mathbf{1}, \chi(G)+1, 0)$ has full dimension. □

Theorem 3.18 *If there exists a 4-hole $C = \{1, 2, 3, 4\} \subseteq V$ such that every k-coloring c, with $k \leq \chi(G)+1$, has $c(1) = c(3)$ or $c(2) = c(4)$, then $P(G, \mathbf{1}, \chi(G)+1, 0)$ is not full-dimensional.*

Proof. Since every feasible schedule (l, r, x) has either $I(1) = I(3)$ or $I(2) = I(4)$, then $x_{14} - x_{12} - x_{34} - x_{32}$, hance $P(G, \mathbf{1}, \chi(G)+1, 0)$ is not full-dimensional. □

3.2.2 Determining the dimension is $\mathcal{NP}$-complete

The results of Section 3.2.1 suggest that the dimension of chromatic scheduling polytopes is hard to characterize. The purpose of this section is to show that its calculation is also a computationally hard problem, by proving that the associated decision problems are $\mathcal{NP}$-complete. As a starting point of our analysis, consider the problem of deciding whether $P(G, d, s, 0)$ has full dimension:

	FULL-DIMENSIONALITY
Instance:	A weighted graph (G, d) and an integer $s \in \mathbf{Z}_+$.
Question:	Has $P(G, d, s, 0)$ full dimension?

Theorem 3.19 FULL-DIMENSIONALITY *is $\mathcal{NP}$-complete.*

Proof. It is not hard to verify that this problem belongs to $\mathcal{NP}$, since we can nondeterministically generate a set of integer feasible solutions and verify whether this set is a set of affinely independent points with the required number of elements or not. Note we can check in polynomial time whether a set of vectors is affinely independent or not [42]. To complete the proof, we shall reduce GRAPH COLORING to FULL-DIMENSIONALITY. Let $G = (V, E)$ be an arbitrary graph and construct a graph $H = (V_H, E_H)$ from G by taking:

$$\begin{array}{rcrcl} V_H & = & V & \cup & \{v_1, v_2, v_3, v_4\} \\ E_H & = & E & \cup & \{v_i\, w : w \in V,\ i = 1, \ldots, 4\} \\ & & & \cup & \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\} \end{array}$$

We claim that $\chi(G) \leq s$ if and only if $P(H, \mathbf{1}, s+4, 0)$ has full dimension. For the forward direction, if $\chi(G) \leq s$ then $\gamma(H, d, 0) = \chi(H) + 2 \leq \chi(G) + 4 \leq s + 4$, and $P(H, \mathbf{1}, s+4, 0)$ is full-dimensional by Theorem 3.11. For the converse direction, suppose that $\chi(G) \geq s+1$. We shall prove that in this case every integer feasible solution satisfies

$$x_{v_1v_2} - x_{v_1v_4} \ = \ x_{v_3v_2} - x_{v_3v_4}, \tag{3.1}$$

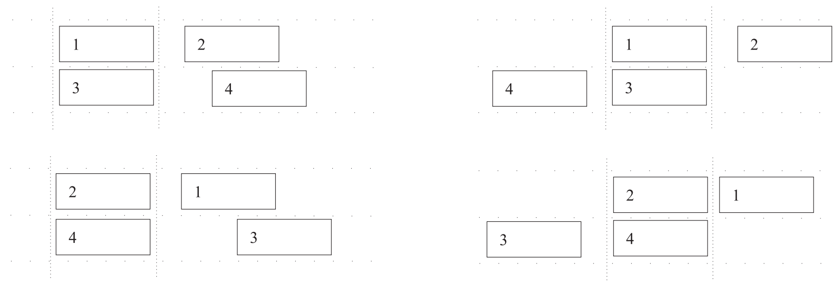


Figure 3.7: Illustration for the proof of Theorem 3.19.

thus verifying that $P(H, \mathbf{1}, s+4, 0)$ is not full-dimensional. Consider any feasible solution $y \in P(H, \mathbf{1}, s+4, 0) \cap \mathbf{Z}^{2|V_H|+|E_H|}$. This solution must have at least $s+1$ colors occupied by intervals corresponding to nodes in V, and this leaves at most three colors left for the nodes $\{v_1, \ldots, v_4\}$. Thus, either v_1 and v_3 or v_2 and v_4 have the same color, and only the four configurations depicted in Figure 3.7 (along with their symmetrical solutions) are possible. All of them satisfy (3.1), hence $P(H, \mathbf{1}, s+4, 0)$ is not full-dimensional. □

Corollary 3.20 FULL-DIMENSIONALITY *for* $R(G, d, s, 0)$ *is* $\mathcal{NP}$*-complete.*

Proof. Given a graph G, repeat the construction from the proof of Theorem 3.19 to obtain a new graph H. The same argumentation can be applied in this case to show that $\chi(G) \leq s$ if and only if $R(G, \mathbf{1}, s+4, 0)$ has full dimension. □

The complexity of the general problem of calculating the dimension of chromatic scheduling polytopes can now be addressed as a corollary to the previous results. To this end, consider the associated decision problem:

	CHROMATIC SCHEDULING POLYTOPE'S DIMENSION
Instance:	A weighted graph (G, d), and integers $k, s, g \in \mathbf{Z}_+$.
Question:	Has $P(G, d, s, g)$ dimension greater or equal than k?

Corollary 3.21 CHROMATIC SCHEDULING POLYTOPE'S DIMENSION *is* $\mathcal{NP}$*-complete.*

3.2.3 Dimension for special interference graphs

This subsection provides results about the dimension of chromatic scheduling polytopes for special classes of interference graphs. We present characterizations of the dimension of instances defined over complete graphs K_n, stars $K_{1,t}$, paths P_n, and holes C_n, the last one being the most involved case. These theorems give the hint that formulating the dimension in terms of standard graph parameters may be a nontrivial task. We start by analyzing the dimension of polytopes defined over complete interference graphs.

Theorem 3.22 *Call $D = \sum_{i=1}^{n} d_i$. Then,*

$$\dim\Big(R(K_n, d, s, 0)\Big) = \begin{cases} m & \text{if } s = D \\ n+m & \text{if } s > D \end{cases}$$
$$\dim\Big(P(K_n, d, s, 0)\Big) = \begin{cases} m & \text{if } s = D \\ n+2m & \text{if } s > D \end{cases}$$

Proof. Clearly, $R(K_n, d, s, 0)$ is nonempty if and only if $s \geq D$. When $s = D$, there are no empty spaces among the intervals, hence every feasible solution satisfies the following n equations:

$$l_i = \sum_{j \neq i} d_j \, x_{ji} \qquad i = 1, \ldots, n \tag{3.2}$$

This implies $\dim(R(K_n, d, D, 0)) \leq m$. Conversely, $s = D$ allows every linear ordering among the intervals, so $\mathrm{proj}_x(R(K_n, d, D, 0))$ contains exactly m affinely independent points. Hence we conclude $\dim(R(K_n, d, D, 0)) = m$. Moreover, $F_D(K_n, d) = \emptyset$, and thus Proposition 3.7 implies that $R(K_n, d, D, 0)$ and $P(K_n, d, D, 0)$ have the same dimension.

To complete the proof, we verify that both polytopes are full-dimensional when $s > D$. Suppose $\lambda^T y = \lambda_0$ for every point $y \in R(G, d, s, 0)$. By Lemma 3.8, $\lambda_{l_i} = 0$ follows for every $i \in V$. Moreover, note that every point in $R(K_n, d, D, 0)$ also belongs to $R(K_n, d, s, 0)$, and $\dim(\mathrm{proj}_x(R(K_n, d, D, 0)) = m$, hence $\lambda_x = \mathbf{0}$. Therefore, $\lambda = \mathbf{0}$ and $R(K_n, d, s, 0)$ is full-dimensional. Since $F_s(K_n, d) = \{1, \ldots, n\}$, then $P(K_n, d, s, 0)$ also has full dimension. □

The following theorem provides a characterization of the dimension of chromatic scheduling polytopes defined over complete and bipartite interference graphs with no induced 4-cycles. This result enables us to fully understand the dimension of chromatic scheduling polytopes defined over stars, paths, and even holes.

Theorem 3.23 *Let G be a connected and bipartite graph with at least two nodes, and such that G does not contain any 4-hole. Then, the polytopes $R(G, \mathbf{1}, s, 0)$ and $P(G, \mathbf{1}, s, 0)$ have dimension 1 if $s = 2$ and are full-dimensional if $s \geq 3$.*

Proof. Let $c : V \to \{1, 2\}$ be a 2-coloring of G. Since G is connected and bipartite, then this coloring is unique up to color renamings. Construct a feasible solution $y \in R(G, \mathbf{1}, 2, 0) \cap \mathbf{Z}^{n+m}$ by setting $y_{l_i} = c(i) - 1$ for every $i \in V$. By the uniqueness of c, there only exist two feasible solutions, namely y and $\mathrm{sym}(y)$, hence $\dim(R(G, \mathbf{1}, 2, 0)) = 1$. Since every node in G has at least one neighbor, then no feasible solution $z \in P(G, \mathbf{1}, 2, 0)$ can have $z_{r_i} - z_{l_i} > 1$, hence $F_2(G, \mathbf{1}) = \emptyset$ and Lemma 3.9 implies $\dim(P(G, \mathbf{1}, 2, 0)) = 1$.

Consider now the case $s \geq 3$. Since G is a bipartite graph with no induced 4-cycle, Theorem 3.15 implies that $R(G, \mathbf{1}, s, 0)$ is full-dimensional. Since $s > s_{\min}(G, \mathbf{1}, 0) = 2$, then $F_s(G, \mathbf{1}) = V$, implying that $P(G, \mathbf{1}, s, 0)$ also has full dimension. □

Corollary 3.24

$$\dim\left(R(K_{1,t},\mathbf{1},s,0)\right) = \begin{cases} 1 & \textit{if } s=2 \\ 2t+1 & \textit{if } s\geq 3 \end{cases}$$

$$\dim\left(P(K_{1,t},\mathbf{1},s,0)\right) = \begin{cases} 1 & \textit{if } s=2 \\ 3t+2 & \textit{if } s\geq 3 \end{cases}$$

Corollary 3.25

$$\dim\left(R(P_n,\mathbf{1},s,0)\right) = \begin{cases} 1 & \textit{if } s=2 \\ 2n-1 & \textit{if } s\geq 3 \end{cases}$$

$$\dim\left(P(P_n,\mathbf{1},s,0)\right) = \begin{cases} 1 & \textit{if } s=2 \\ 3n-1 & \textit{if } s\geq 3 \end{cases}$$

Corollary 3.26 *Let $n \geq 6$ be an even integer. Then,*

$$\dim\left(R(C_n,\mathbf{1},s,0)\right) = \begin{cases} 1 & \textit{if } s=2 \\ 2n & \textit{if } s\geq 3 \end{cases}$$

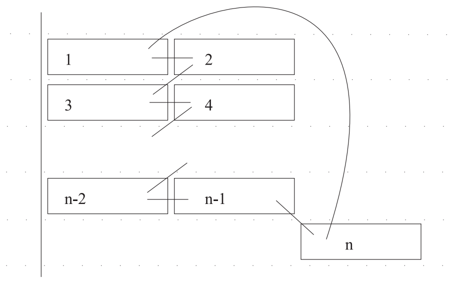
$$\dim\left(P(C_n,\mathbf{1},s,0)\right) = \begin{cases} 1 & \textit{if } s=2 \\ 3n & \textit{if } s\geq 3 \end{cases}$$

To close this section, we prove a similar result for odd cycles. The previous examples may suggest that $P(G,\mathbf{1},s,0)$ is not full-dimensional for $s = s_{\min}(G,\mathbf{1},0)$, but Theorem 3.27 shows full-dimensionality for infinitely many instances. Indeed, chromatic scheduling polytopes defined over odd cycles are empty if $s \leq 2$ and full-dimensional otherwise. In order to prove this result, we introduce the following definition.

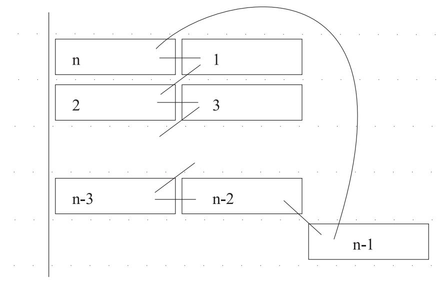
Definition 3.8 *Given a linear ordering $S = (i_1,\ldots,i_n)$ of V, the* greedy solution associated with S *is the feasible solution constructed by the following procedure:*

> ***For*** $j = 1,\ldots,n$ *do:*
>
> > ***Set*** $I(i_j) = [t_j, t_j + d_{i_j}]$, *where t_j is the minimum feasible starting time for the interval $I(i_j)$, according to the previous assignments.*
>
> ***End*** *(for)*

For example, Figure 3.8 shows two such solutions for odd cycles, associated with the sequences $(1,\ldots,n)$ and $(n,1,\ldots,n-1)$, respectively.



(a)



(b)

Figure 3.8: Examples of greedy solutions

Theorem 3.27 *Let $n \geq 5$ be an odd integer. The polytopes $R(C_n, \mathbf{1}, s, 0)$ and $P(C_n, \mathbf{1}, s, 0)$ are empty if $s \leq 2$ and have full dimension otherwise.*

Proof. Since odd cycles are nonbipartite, we have that $R(C_n, \mathbf{1}, 2, 0)$ and $P(C_n, \mathbf{1}, 2, 0)$ are empty. To complete the proof, we show that $P(C_n, \mathbf{1}, 3, 0)$ has full dimension (this implies that $R(C_n, \mathbf{1}, s, 0)$ and $P(C_n, \mathbf{1}, s, 0)$ are full-dimensional for $s \geq 3$). Suppose $\lambda^T z = \lambda_0$ for every $z \in P(C_n, \mathbf{1}, 3, 0) \cap \mathbf{Z}^{3n}$. We shall verify $\lambda = \mathbf{0}$, implying that this polytope is full-dimensional.

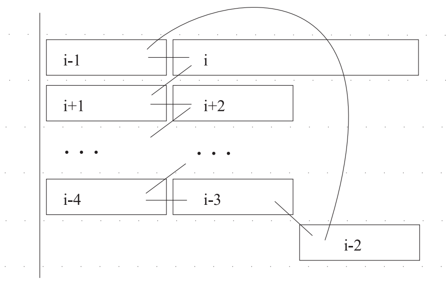
For $i = 1, \ldots, n$, construct the two feasible solutions z^i and $\bar{z}^i$ presented in Figure 3.10(a) and Figure 3.10(b). Since $\lambda^T z^i = \lambda_0 = \lambda^T \bar{z}^i$, we have that $\lambda_{l_i} = 0$. A similar construction shows $\lambda_{r_i} = 0$.

It remains to verify that $\lambda_x = \mathbf{0}$. For $i = 1, \ldots, n$, define the sequence $S_i = (i, i+1, \ldots, n, 1, \ldots, i-1)$, and let y^i be the associated greedy solution. Also define the opposite sequence $\bar{S}_i = (i, i-1, \ldots, 1, n, n-1, \ldots, i+1)$ and let $\bar{y}^i$ denote the associated greedy solution. For $i = 1, \ldots, n$, we have that $\lambda^T y^i = \lambda^T \bar{y}^i$. These n equations define an $(n \times n)$-system $D_n \lambda_x = \mathbf{0}$ of linear equations. The matrix D_n has two consecutive diagonals with ones, and the remaining diagonals are alternatively composed by -1 and 1 (see Figure 3.9 for an example).

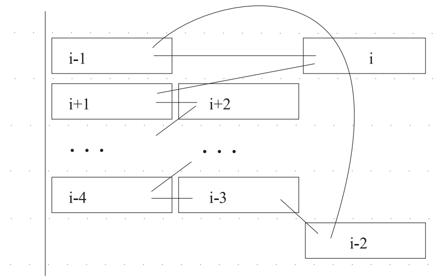
$$D_7 = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Figure 3.9: A matrix arising from greedy solutions and its triangulation.

It is not difficult to verify that D_n is a nonsingular matrix (recall that n is an odd integer). To this end, for $i = n, \ldots, 2$ in decreasing order, add row $i-1$ to row i. The resulting matrix



(a)



(b)

Figure 3.10: Feasible solutions of $P(C_n, \mathbf{1}, 3, 0)$ showing $\lambda_{l_i} = 0$.

is upper triangular (see Figure 3.9 for an example with $n = 7$), thus proving that the only solution to $D_n\lambda_x = \mathbf{0}$ is $\lambda_x = \mathbf{0}$. Hence $\lambda = \mathbf{0}$ and $P(C_n, \mathbf{1}, 3, 0)$ is full-dimensional. □

Remark. Consider the vectors $\{\mathrm{proj}_x(y^i)\}_{i=1}^n$ of the ordering variables corresponding to the greedy solutions associated with the n ascending sequences $S_1, \ldots, S_n$ introduced in the proof of Theorem 3.27. Let A be the quadratic 0/1-matrix with these vectors as rows. Then A has a special structure, with the first two diagonals filled with ones, and the remaining diagonals alternating between zeros and ones, respectively. It is worth noting that A is nonsingular and has determinant $(n-1)/2$ (since n is odd). $\triangleleft$

3.3 The combinatorial steady state

This section explores a fundamental issue concerning the combinatorial structure of chromatic scheduling polytopes. It has been experimentally observed in [21] for some instances $(G, d, s, 0)$ that, from a certain value $s_{\max}(G, d, 0)$ on, the polytopes $\{R(G, d, s, 0)\}_{s \geq s_{\max}(G,d,0)}$ reach a combinatorial steady state with the same number of extreme points and facets. This led to the question whether the polytopes $\{R(G, d, s, g)\}_{s \geq s_{\max}(G,s,g)}$ are pairwise combinatorially equivalent. In this section we give an affirmative answer by proving a more general result: the polytopes $R(G, d, s, g)$ and $R(G, d, s+1, g)$ resp. $P(G, d, s, g)$ and $P(G, d, s+1, g)$ are affinely isomorphic (and therefore combinatorially equivalent) for $s \gg \omega(G, d)$. Moreover, we give a lower bound on s ensuring this isomorphism, and this bound can be shown to be sharp when G is the union of disjoint cliques.

3.3.1 A characterization of the extreme points

We start by providing a simple characterization of the extreme points of chromatic scheduling polytopes. For any valid ordering $\bar{x} \in \mathrm{proj}_x(R(G, d, s, g) \cap \mathbf{Z}^{n+m})$, define the lower and upper bounds for the interval $I(i)$ assigned to customer $i \in V$ as follows:

$$
\begin{aligned}
L_i(\bar{x}, s) &= \min\{y_{l_i} : y \in R(G,d,s,g) \cap \mathbf{Z}^{n+m} \text{ and } y_x = \bar{x}\} \\
U_i(\bar{x}, s) &= \max\{y_{l_i} : y \in R(G,d,s,g) \cap \mathbf{Z}^{n+m} \text{ and } y_x = \bar{x}\}
\end{aligned}
$$

For every $ij \in E$, let δ_{ij} be the minimum gap required between the intervals $I(i)$ and $I(j)$, i.e.,

$$
\delta_{ij} = \begin{cases} g & \text{if } ij \in E_X \\ 0 & \text{otherwise} \end{cases}
$$

Definition 3.9 (fixed-length adjacency graph) *Let $y \in R(G,d,s,g) \cap \mathbf{Z}^{n+m}$ be a feasible schedule. The* adjacency graph *associated with this schedule is $G(y) = (V', E')$, with $V' = V$ and $E' = \{\, ij \in E : y_{l_i} + d_i + \delta_{ij} = y_{l_j},\ \text{or}\ y_{l_j} + d_j + \delta_{ij} = y_{l_i}\}$.*

Nodes i and j are adjacent in $G(y)$ if they are adjacent in G and there is a space of exactly δ_{ij} between the intervals $I(i)$ and $I(j)$. For example, if H is the interference graph depicted in Figure 3.11(a), then Figure 3.11(b) shows a feasible schedule and Figure 3.11(c) presents its associated adjacency graph.

Definition 3.10 *A connected component C of $G(y)$ is called a* border component *if there exists some $i \in C$ with $y_{l_i} = 0$ or $y_{l_i} = s - d_i$.*

Theorem 3.28 *The vector $y \in R(G,d,s,g) \cap \mathbf{Z}^{n+m}$ is an extreme point of $R(G,d,s,g)$ if and only if every connected component of $G(y)$ is a border component.*

Proof. Only if. Consider a feasible solution y and its fixed-length adjacency graph $G(y)$. Suppose that $G(y)$ has a component C such that every node $i \in C$ has $y_{l_i} > 0$ and $y_{l_i} < s - d_i$. Then, we can construct two feasible points $y^1, y^2 \in R(G,d,s,g)$ by shifting all the intervals assigned to nodes in C one unit to the left and one unit to the right, respectively:

$$
\begin{aligned}
y^1_{l_i} &= \begin{cases} y_{l_i} & \text{if } i \notin C \\ y_{l_i} - 1 & \text{if } i \in C \end{cases} && \forall\, i \in V \\
y^2_{l_i} &= \begin{cases} y_{l_i} & \text{if } i \notin C \\ y_{l_i} + 1 & \text{if } i \in C \end{cases} && \forall\, i \in V \\
y^1_{x_{ij}} &= y_{x_{ij}} && \forall\, ij \in E \\
y^2_{x_{ij}} &= y_{x_{ij}} && \forall\, ij \in E
\end{aligned}
$$

Note that $0 \leq y^i_{l_j} \leq s - d_i$ $(i = 1, 2)$, since $0 < y_{l_j} < s - d_j$ for all $j \in C$. Moreover, this shifting does not cause interval overlappings. Any such overlapping in y^1 would be $y^1_{l_j} + d_j + \delta_{ij} > y^1_{l_i}$ for $i \in C$ and $j \notin C$, but then $y_{l_j} + d_j + \delta_{ij} = y_{l_i}$, and thus $j \in C$. A similar analysis shows that y^2 is feasible.

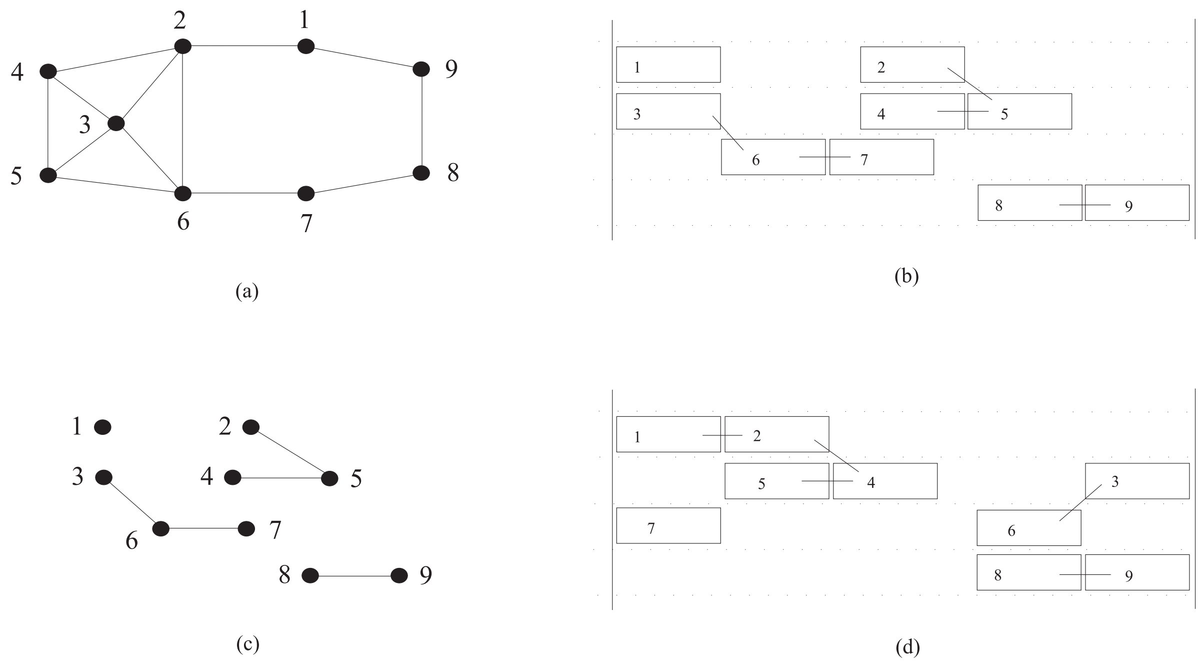


Figure 3.11: Examples for Section 3.3.1.

But now we have that $y = \frac{1}{2}y^1 + \frac{1}{2}y^2$, and thus y is not an extreme point of $R(G,d,s,g)$, contradicting the hypothesis.

If. Let y be a feasible solution such that every connected component of $G(y)$ is a border component. Further, suppose that $z^1, \ldots, z^k \in R(G,d,s,g) \cap \mathbf{Z}^{n+m}$ are k extreme points of $R(G,d,s,g)$, such that $y = \sum_{i=1}^{k} \alpha_i z^i$, with $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i > 0$ for $i = 1, \ldots, k$. Since $0 \leq y_{x_e}, z^i_{x_e} \leq 1$ for every edge $e \in E$, then $y_{x_e} = z^i_{x_e}$. This implies that y and z^i $(i = 1, \ldots, k)$ have the same ordering among the intervals.

Consider now any connected component C of $G(y)$, and assume without loss of generality that $y_{l_t} = 0$ for some $t \in C$. Define $C_L = \{i \in C : y_{l_i} = L_i(y_x, s)\}$, which is nonempty since $t \in C_L$. For each node $i \in C$, let γ_i denote the distance from i to C_L (i.e., the length of the shortest path from i to some node in C_L). Note that $\gamma_i = 0 \Leftrightarrow i \in C_L$.

Claim: $\boldsymbol{z^i_{l_j} = y_{l_j}}$ for every $\boldsymbol{j \in C}$ and $\boldsymbol{i = 1, \ldots, k}$. We shall prove this claim by induction on the distance γ_j from j to C_L.

- $\gamma_j = 0$: Then $j \in C_L$, and so $y_{l_j} = L_j(y_x, s)$. But z^i has the same ordering among the intervals than y, and thus $z^i_{l_j} \geq L_j(y_x, s)$, for $i = 1, \ldots, k$. Thus, $z^i_{l_j} = L_i(y_x, s)$, since otherwise $\sum_i \alpha_i z^i_{l_j} > L_j(y_x, s) = y_{l_j}$.
- $\gamma_j > 0$: Then $y_{l_j} + d_j + \delta_{jp} = y_{l_p}$ or $y_{l_p} + d_p + \delta_{jp} = y_{l_j}$ for some $p \in C$ in the path from j to C_L (assume without loss of generality that the former holds). By the induction hypothesis, $z^i_{l_p} = y_{l_p}$ for $i = 1, \ldots, k$, so

$$z^i_{l_j} + d_j + \delta_{jp} \leq z^i_{l_p} = y_{l_p}.$$

But $y_{l_j} + d_j + \delta_{jp} = y_{l_p}$, and thus $z^i_{l_j} = y_{l_j}$. $\diamond$

Hence $z^i = y$ for $i = 1, \dots, k$, implying that y is an extreme point of $R(G, d, s, g)$. $\square$

Theorem 3.28 states that a feasible solution $y \in R(G, d, s, g) \cap \mathbf{Z}^{n+m}$ is an extreme point if and only if every connected component of $G(y)$ has at least one interval located either to the left or to the right bound of the spectrum $[0, s]$. In the example above, the feasible schedule depicted in Figure 3.11(b) is not an extreme point of $R(H, \mathbf{1}, s, g)$, whereas Figure 3.11(d) presents a solution whose incidence vector is an extreme point of $R(H, \mathbf{1}, s, g)$. Note that, in a border component C, not every node $i \in C$ has to satisfy $l_i = L_i(x, s)$ or $l_i = U_i(x, s)$ (i.e., attain its leftmost or rightmost position). For example, consider the border component $C = \{1, 2, 4, 5\}$ from the schedule depicted in Figure 3.11(d). The intervals $I(1)$, $I(2)$ and $I(4)$ are located in their leftmost position, but the interval $I(5)$ is not, despite the fact that it belongs to C since $l_5 + d_5 = l_4$.

A similar construction can be given for the general case $r_i - l_i \geq d_i$, $i \in V$. In this case, the adjacency graph contains two nodes for each interval $I(i) = [l_i, r_i]$, representing the left and the right bound, respectively. For $i \in V$, the nodes l_i and r_i are adjacent if the interval $I(i)$ has lenght exactly d_i. For $ij \in E$, the nodes l_i and r_j are adjacent if there exists a space of exactly δ_{ij} between $I(i)$ and $I(j)$.

Definition 3.11 (adjacency graph) *Let $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be a feasible schedule. The* adjacency graph *associated with this schedule is $H(z) = (V', E')$, with*

$$\begin{aligned} V' &= \{\, l_i : i \in V\} \;\cup\; \{\, r_i : i \in V\} \\ E' &= \{\, l_i\, r_i : i \in V \text{ and } z_{r_i} - z_{l_i} = d_i\} \;\cup \\ &\quad\; \{\, r_i\, l_j : ij \in E \text{ and } z_{r_i} + \delta_{ij} = z_{l_j}\}. \end{aligned}$$

Definition 3.12 *A connected component C of $H(z)$ is called a* border component *if there exists some $l_i \in C$ with $z_{l_i} = 0$ or some $r_i \in C$ with $z_{r_i} = s$.*

Theorem 3.29 *The point $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ is an extreme point of $P(G, d, s, g)$ if and only if every connected component of $H(z)$ is a border component.*

Proof. *Only if.* Consider a feasible solution z and its adjacency graph $H(z)$. Suppose that $H(z)$ has a nonborder component C, and construct two feasible schedules z^1, $z^2 \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ from z by shifting the bounds in C one unit to the left resp. to the right, i.e.,

$$z^1_{l_j} = \begin{cases} z_{l_j} - 1 & \text{if } l_j \in C \\ z_{l_j} & \text{if } l_j \notin C \end{cases} \qquad z^2_{l_j} = \begin{cases} z_{l_j} + 1 & \text{if } l_j \in C \\ z_{l_j} & \text{if } l_j \notin C \end{cases}$$

$$z^1_{r_j} = \begin{cases} z_{r_j} - 1 & \text{if } r_j \in C \\ z_{r_j} & \text{if } r_j \notin C \end{cases} \qquad z^2_{r_j} = \begin{cases} z_{r_j} + 1 & \text{if } r_j \in C \\ z_{r_j} & \text{if } r_j \notin C \end{cases}$$

Claim: $z^1, z^2 \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$. We first verify that $z^1_{r_j} - z^1_{l_j} \geq d_j$ for every $j \in V$. Suppose that $r_j \in C$ but $l_j \notin C$. The construction of $H(z)$ implies $z_{r_j} - z_{l_j} > d_j$, since otherwise l_j would belong to C. Hence z^1 satisfies the demand constraints. It is not difficult to verify that $0 \leq z^1_{l_j}$ for every $j \in V$, since the left interval bound l_j is shifted to the left only when l_j belongs to a nonborder component, implying $z_{l_j} > 0$. The opposite constraints $z^1_{l_j} \leq s - d_j$ are clearly satisfied.

To complete the proof of the claim we show that z^1 satisfies the antiparallelity constraints, by verifying that no overlappings are produced by the shifting. In this setting, an overlapping can occur only when $z_{x_{jk}} = 1$ (for $jk \in E$) and z_{l_k} is shifted but z_{r_j} remains unchanged. By construction, this implies $l_k \in C$ and $r_j \notin C$, hence $z_{r_j} + \delta_{jk} < z_{l_k}$ and so $z^1_{r_j} + \delta_{ij} \leq z^1_{l_k}$. The schedule z^2 is defined similarly, and the same arguments show that it is feasible. $\diamond$

But now we have $z = \frac{1}{2}(z^1 + z^2)$ and, therefore, z is not an extreme point.

If. Let z be a feasible solution such that every connected component of $H(z)$ is a border component. Further, suppose that $z^1, \ldots, z^p \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ are p extreme points of $P(G,d,s,g)$ such that $z = \sum_{i=1}^p \alpha_i z^i$, with $\sum_{i=1}^p \alpha_i = 1$ and $\alpha_i > 0$ for $i = 1, \ldots, p$. Since $z_{x_e}, z^i_{x_e} \in \{0,1\}$ for every edge $e \in E$, then $z_{x_e} = z^i_{x_e}$.

Let C be a connected component of $H(z)$. Since C is a border component, then either (a) $l_t \in C$ and $z_{l_t} = 0$ or (b) $r_t \in C$ and $z_{r_t} = s$, for some $t \in V$. Assume w.l.o.g. that the former holds. For $k \in C$, define γ_k to be the distance from node k to l_t in $H(z)$ (note that $\gamma_{l_t} = 0$). We now verify by induction on γ that $z_{l_j} = z^i_{l_j}$ for every $l_j \in C$ and $z_{r_j} = z^i_{r_j}$ for every $r_j \in C$. Let $k \in C$. If $\gamma_k = 0$ then $k = l_t$, so $z_{l_t} = 0$. But $z^i_{l_t} \geq 0$ for $i = 1, \ldots, p$, implying $z^i_{l_t} = 0$. On the other hand, if $\gamma_k > 0$, then either $k = l_j$ or $k = r_j$ for some $j \in V$. Suppose w.l.o.g. the former and consider the following cases:

- If there exists some $r_l \in C$ such that $z_{l_j} + \delta_{jl} = z_{r_l}$ and $\gamma_{r_l} = \gamma_{l_j} - 1$, by the induction hypothesis we have $z_{r_l} = z^i_{r_l}$ for $i = 1, \ldots, p$. Since z and z^i have the same ordering among the intervals, then $z^i_{l_j} \geq z^i_{r_l} - \delta_{jl} = z_{r_l} - \delta_{jl} = z_{l_j}$, implying $z^i_{l_j} = z_{l_j}$ for $i = 1, \ldots, p$.

- On the other hand, if $z_{r_j} - z_{l_j} = d_j$ and $\gamma_{r_j} = \gamma_{l_j} - 1$, the induction hypothesis implies $z^i_{r_j} = z_{r_j}$ for $i = 1, \ldots, p$. Since $z^i_{l_j} \leq z^i_{r_j} - d_j = z_{r_j} - d_j = z_{l_j}$, then $z^i_{l_j} = z_{l_j}$ for $i = 1, \ldots, p$.

The same arguments apply to the case $k = r_j$. This way we show that $z = z^i$ for $i = 1, \ldots, p$ and, therefore, z is an extreme point of $P(G,d,s,g)$. $\square$

3.3.2 Combinatorial equivalence for large frequency spans

The main result of this subsection asserts that for every interference graph (G,d) and every guard distance g there exists a value $s_{\max}(G,d,g) \in \mathbf{Z}_+$ such that the polytopes from the families $\{R(G,d,s,g)\}_{s \geq s_{\max}(G,d,g)}$ resp. $\{P(G,d,s,g)\}_{s \geq s_{\max}(G,d,g)}$ are pairwise affinely isomorphic, hence being combinatorially equivalent. We also provide an upper bound on $s_{\max}(G, d, g)$.

Definition 3.13 *The polytopes $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^m$ are* affinely isomorphic*, denoted by $P \cong Q$, if there is a bijective affine map $f : \mathbf{R}^n \to \mathbf{R}^m$ between the two polytopes.*

Note that the definition asks for an affine bijection between all the points of the polytopes, and this is equivalent to finding an affine bijection between the extreme points of P and Q, since affine bijections preserve convex combinations of points. Moreover, if f is a bijection in the ambient spaces, then P and Q are basically "the same polytope" with respect to an affine change of coordinates. From the combinatorial point of view, if P and Q are affinely isomorphic, then they share the same facial structure. In particular, the affine map gives an isomorphism between their extreme points, and between their facets [46].

Definition 3.14 *Let $\tau(G, d, g)$ denote the minimum frequency spectrum length s such that $R(G, d, s, g)$ admits a solution for every possible ordering among the intervals.*

In order to prove the equivalence of $R(G, d, s, g)$ and $R(G, d, s + 1, g)$, we define now a different representation for feasible schedules, in terms of binary variables. For every node $i \in V$ and every $k \in \{0, \ldots, s - 1\}$, define the binary *position variable* q_{ik} as:

$$q_{ik} = \begin{cases} 1 & \text{if } l_i \geq k \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

We also consider the ordering variables x_{ij}, for $ij \in E$, with the usual meaning. If P is a polytope, we denote by $\mathrm{vert}(P)$ the set of extreme points of P. Therefore, to every extreme point $y = (l, x) \in \mathrm{vert}(R(G, d, s, g))$ we can associate a point $z^y = (q, x) \in \mathbf{Z}^{ns+m}$ with $z^y_x = y_x$ and z^y_q defined by (3.3).

Definition 3.15 $\mathcal{R}(G, d, s, g) = \mathrm{conv}\{z^y : y \in \mathrm{vert}(R(G, d, s, g))\}$.

Since the extreme points $y_1, \ldots, y_t$ of $R(G, d, s, g)$ are pairwise distinct, then $z^{y_1}, \ldots, z^{y_t}$ are pairwise distinct as well. Moreover, $z^{y_1}, \ldots, z^{y_t}$ are binary vectors and, therefore, none of them can be written as a convex combination of the remaining ones. Hence $\mathcal{R}(G, d, s, g)$ has exactly $t = |\mathrm{vert}(R(G, d, s, g))|$ extreme points.

Lemma 3.30 *$R(G, d, s, g) \cong \mathcal{R}(G, d, s, g)$.*

Proof. Let $\mathbf{0}_d \in \mathbf{R}^{1\times d}$ resp. $\mathbf{1}_d \in \mathbf{R}^{1\times d}$ denote the d-dimensional row vector with only 0-entries resp. 1-entries. Consider the affine map $f : \mathrm{vert}(\mathcal{R}(G, d, s, g)) \to \mathrm{vert}(R(G, d, s, g))$

defined by $f(z) = Bz$, where:

$$B = \left(\begin{array}{cccc|c} \mathbf{1}_s & \mathbf{0}_s & \dots & \mathbf{0}_s & \mathbf{0}_n \\ \mathbf{0}_s & \mathbf{1}_s & \dots & \mathbf{0}_s & \mathbf{0}_n \\ \vdots & \vdots & \ddots & \vdots & \mathbf{0}_n \\ \mathbf{0}_s & \mathbf{0}_s & \dots & \mathbf{1}_s & \mathbf{0}_n \\ \hline \mathbf{0}_s & \mathbf{0}_s & \dots & \mathbf{0}_s & I_n \end{array}\right)$$

This function maps the point (q, x) to the point $B(q, x) = (l, x)$, with $l_i = \sum_{k=1}^{s-1} q_{ik}$. Therefore, f maps extreme points of $\mathcal{R}(G, d, s, g)$ onto extreme points of $R(G, d, s, g)$. This mapping is clearly injective and, since the sets of the extreme points of both polytopes have the same cardinality, it follows that f is a bijection between these sets. Since f is an affine bijection between $\mathrm{vert}(\mathcal{R}(G, d, s, g))$ and $\mathrm{vert}(R(G, d, s, g))$, then f is a bijection between $\mathcal{R}(G, d, s, g)$ and $R(G, d, s, g)$ and, therefore, these polytopes are affinely isomorphic. $\square$

Lemma 3.31 *If $s > 2\tau(G, d, g)$, then $\mathcal{R}(G, d, s, g) \cong \mathcal{R}(G, d, s + 1, g)$.*

Proof. Let y be an extreme point of $R(G, d, s, g)$, and let C be a connected component of $G(y)$. Since C is a border component, there there exists some $i \in C$ such that either $y_{l_i} = 0$ or $y_{l_i} = s - d_i$ holds. If $y_{l_i} = 0$, $s > 2\tau(G, d, g)$ implies $\max_{j \in C} y_{l_j} < s/2$. Similarly, if $y_{l_i} = s - d_i$, $s > 2\tau(G, d, g)$ implies $\min_{j \in C} y_{l_j} > s/2$. Hence the interval set can be partitioned into two subsets, namely the intervals located in $[0, s/2]$ and the intervals located in $[s/2, s]$.

Now, if z^y is a feasible solution of $\mathcal{R}(G, d, s, g)$, we denote by $\mathrm{shift}(z^y)$ the corresponding extreme point of $\mathcal{R}(G, d, s+1, g)$, which has the same configuration, but the intervals located in $[s/2, s]$ are now shifted one unit to the right (i.e., these intervals are located in the right part of the new frequency spectrum $[0, s + 1]$). The point $\mathrm{shift}(z^y)$ can be written as:

$$\begin{aligned} \mathrm{shift}(z^y)_{q_{ik}} &= \begin{cases} y_{q_{ik}} & \text{if } k < \lfloor s/2 \rfloor \\ y_{q_{i,k-1}} & \text{if } k \geq \lfloor s/2 \rfloor \end{cases} \\ \mathrm{shift}(z^y)_{x_{ij}} &= y_{x_{ij}} \end{aligned}$$

This mapping shifts the intervals of y that are located in $[s/2, s]$ (and therefore have $q_{i,s/2} = 1$) one unit to the right, and lets the remaining intervals unchanged. Moreover, it is an affine bijection between the extreme points of $\mathcal{R}(G, d, s, g)$ and $\mathcal{R}(G, d, s + 1, g)$ implying that they are affinely isomorphic. $\square$

Theorem 3.32 *If $s > 2\tau(G, d, g)$, then $R(G, d, s, g) \cong R(G, d, s + 1, g)$.*

Proof. From Lemma 3.30 and Lemma 3.31 follows $R(G, d, s, g) \cong \mathcal{R}(G, d, s, g) \cong \mathcal{R}(G, d, s + 1, g) \cong R(G, d, s + 1, g)$. $\square$

Remark. The definition of $\mathcal{R}(G, d, s, g)$ presented in this section was inspired by the construction given in [37] for characterizing the integer hull of a general polytope. It is also worth noting that an alternative proof of a weaker version of Theorem 3.32 was found by proving that the Fourier-Motzkin elimination method [43, 44, 46] performs the same operations on $R(G,d,s,g)$ and $R(G,d,s+1,g)$ when $s \gg \omega(G,d)$. $\triangleleft$

The same construction can be applied to prove a similar result for the polytope $P(G,d,s,g)$. To this end, we consider a new set of binary variables u_{ik} for $i \in V$ and $k \in \{1,\ldots,s\}$, defined by

$$u_{ik} = \begin{cases} 1 & \text{if } r_i \geq k \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

To every extreme point $z = (l,r,x) \in \text{vert}(P(G,d,s,g))$ we can associate a point $w^z = (q,u,x) \in \mathbf{Z}^{2ns+m}$ with $w^z_x = z_x$ and w^z_q resp. w^z_u defined by (3.3) resp. (3.4). We define $\mathcal{P}(G,d,s,g) \subseteq \mathbf{R}^{2ns+m}$ to be the convex hull of all the points constructed this way. The same techniques from the previous lemmas can be applied to show the following result.

Theorem 3.33 *If $s > 2\tau(G,d,g)$, then $P(G,d,s,g) \cong P(G,d,s+1,g)$.*

Hence, there exists a certain value of the frequency span which ensures combinatorial stability for the general polytope $P(G,d,s,g)$. We thus introduce the corresponding threshold for combinatorialy stability of chromatic scheduling polytopes, which is well-defined by Theorem 3.33.

Definition 3.16 (combinatorial stability threshold) *We denote by $s_{\max}(G,d,g)$ the minimum frequency span s such that the polytopes $P(G,d,s,g)$ and $P(G,d,s+1,g)$ are combinatorially equivalent.*

Theorem 3.33 implies $2\tau(G,d,g) \leq s_{\max}(G,d,g)$, but the computational experiments from Section 2.2 suggest $s_{\max}(G,d,g) = \tau(G,d,g)+1$. Moreover, this computational evidence suggest that $s_{\max}(G,d,g)$ is also the minimum frequency span ensuring combinatorial stability for the fixed-length polytope $R(G,d,s,g)$.

3.3.3 A better bound for the case $E_X = \emptyset$

If $E_X = \emptyset$ (i.e., we have no inter-sector edges), then G is the disjoint union of cliques $T_1,\ldots,T_t$, each one corresponding to one sector. In this case, we can prove the combinatorial equivalence of $R(G,d,s,g)$ and $R(G,d,s+1,g)$ for $s > \tau(G,d,g)$, thus giving a better bound for $s_{\max}(G,d,g)$ in this particular setting.

In order to state this result, we define another representation for feasible solutions. For each node $i \in V$, consider the *gap variable* p_i measuring the total gap to the left of the interval $I(i)$ (not just the gap between $I(i)$ and its immediate predecessor, but the sum of all gaps located to the left of $I(i)$). We also consider the ordering variables x_{ij}, for $ij \in E$, with the usual meaning. In this setting, a feasible solution is any assignment of integer values to these variables such that the following constraints are satisfied:

$$\begin{aligned}
p_j &\leq p_i + s\,x_{ij} && \forall ij \in E,\ i < j && (3.5)\\
p_i &\leq p_j + s\,(1 - x_{ij}) && \forall ij \in E,\ i < j && (3.6)\\
0 &\leq p_i \leq s - \sum_{j \in T_k} d_j x_{ij} && \forall k = 1, \ldots, t,\ \forall i \in T_k && (3.7)\\
2 &\geq x_{ij} + x_{jk} + (1 - x_{ik}) && \forall ij, jk \in E,\ i < j,\ j < k && (3.8)\\
x_{ij} &\in \{0, 1\} && \forall ij \in E,\ i < j && (3.9)
\end{aligned}$$

Definition 3.17 *Let $\bar{R}(G,d,s,g) \subseteq \mathbf{R}^{n+m}$ denote the convex hull of all feasible solutions $(p,x) \in \mathbf{Z}^{n+m}$ satisfying constraints (3.5)-(3.9).*

Lemma 3.34 $R(G,d,s,g) \cong \bar{R}(G,d,s,g)$.

Proof. We show that both polytopes are affinely isomorphic by verifying that the gap variables p can be obtained from the interval bounds l and the ordering variables x by an affine map. If $i \in T_k$, then

$$p_i = l_i - \sum_{j \in T_k \setminus \{i\}} d_j x_{ji} \tag{3.10}$$

Given any integer solution $(l,x) \in R(G,d,s,g) \cap \mathbf{Z}^{n+m}$, we can find its associated solution $(p,x) \in \bar{R}(G,d,s,g)$ using 3.10. We can write this mapping in matrix form as $(p,x)^T = A(l,x)^T$, with $A \in \mathbf{R}^{(n+m)\times(n+m)}$:

$$\begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} I_n & M \\ 0 & I_m \end{pmatrix} \begin{pmatrix} l \\ x \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix and M is a $(m \times m)$-matrix with integer entries. Given this structure, it can be seen that A is nonsingular, and thus this mapping is an isomorphism on the ambient spaces. Therefore, $R(G,d,s,g) \cong \bar{R}(G,d,s,g)$. $\square$

Lemma 3.35 *The point $z \in \bar{R}(G,d,s,g)$ is an extreme point of $\bar{R}(G,d,s,g)$ if and only if each clique T_k of G can be partitioned as $T_k = T_k' \cup T_k''$ in such a way that $z_{p_i} = 0$ for $i \in T_k'$ and $z_{p_i} = s - \omega(T_k)$ for $i \in T_k''$.*

Proof. Only if. If $0 < z_{p_i} < s - \omega(T_k)$ for some $i \in T_k$, then the set of intervals associated with nodes in T_k having no gap between them and including $I(i)$ can be shifted one unit to the left and one unit to the right, thus constructing two feasible solutions z_1 and z_2 such that $z = \frac{1}{2}(z_1 + z_2)$.

If. Suppose that $z = \sum_{i=1}^{p} \alpha_i z^i$, with $\sum_{i=1}^{p} \alpha_i = 1$ and $\alpha_i > 0$. Since $x \in \{0,1\}^m$, then $z_x^i = z_x$ for $i = 1, \ldots, p$. Moreover, if $j \in T_k'$ then $z_{p_j}^i \geq 0 = z_{p_j}$, and if $j \in T_k''$ then $z_{p_j}^i \leq s - \omega(T_k) = z_{p_j}$, for every $i = 1, \ldots, p$. Thus, $z_{p_j}^i = z_{p_j}$ for all $j \in V$, and then z is an extreme point of $\bar{R}(G,d,s,g)$. $\square$

Lemma 3.36 *If $s > \tau(G,d,g)$, then $\bar{R}(G,d,s,g) \cong \bar{R}(G,d,s+1,g)$.*

Proof. Note first that $s > \tau(G,d,g)$ if and only if $s \geq \omega(T_k)+1$ for every $k = 1,\ldots,t$. For each clique T_k of G, define $n_k = |T_k|$ and let $M_k \in \mathbf{R}^{n_k \times n_k}$ be the matrix

$$M_k \;=\; \frac{s+1-\omega(T_k)}{s-\omega(T_k)} I_{n_k}.$$

We now define an affine map $f : \mathbf{R}^{n+m} \to \mathbf{R}^{n+m}$ as $f(y) = By$, with

$$B \;=\; \begin{pmatrix} M_1 & \mathbf{0}_{n_2\times n_2} & \ldots & \mathbf{0}_{n_t\times n_t} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n_1\times n_1} & M_2 & \ldots & \mathbf{0}_{n_t\times n_t} & \mathbf{0}_{m\times m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{n_1\times n_1} & \mathbf{0}_{n_2\times n_2} & \ldots & M_t & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n_1\times n_1} & \mathbf{0}_{n_2\times n_2} & \ldots & \mathbf{0}_{n_t\times n_t} & I_m \end{pmatrix}.$$

Let z be an extreme point of $\bar{R}(G,d,s,g)$. By Lemma 3.35, each clique $T_k \subseteq G$ has a partition $T_k = T'_k \cup T''_k$ such that $z_{p_i} = 0$ for $i \in T'_k$ and $z_{p_i} = s - \omega(T_k)$ for $i \in T''_k$. Thus, $f(z)_{p_i} = 0$ for $i \in T'_k$ and $g(z)_{p_i} = s+1-\omega(T_k)$. Moreover, $f(z)_x = z_x$, and so $f(z)$ is the same point than z, but with the intervals corresponding to $\cup_k T''_k$ shifted one unit to the right (i.e., at the right of the new frequency spectrum $[0,\ s+1]$).

Since $s \geq \omega(T_k)+1$ for $k = 1,\ldots,t$, we have that f maps every extreme point of $\bar{R}(G,d,s,g)$ onto its corresponding extreme point of $\bar{R}(G,d,s+1,g)$. Note that the lower bound on s ensures that all orderings among the intervals are feasible in $\bar{R}(G,d,s,g)$ and thus no new interval ordering is introduced in $\bar{R}(G,d,s+1,g)$. Since B is nonsingular, then $\bar{R}(G,d,s,g) \cong \bar{R}(G,d,s+1,g)$. $\square$

Theorem 3.37 *If $s > \tau(G,d,g)$, then $R(G,d,s,g) \cong R(G,d,s+1,g)$.*

Proof. By Lemmas 3.34 and 3.36, we have that $R(G,d,s,g) \cong \bar{R}(G,d,s,g) \cong \bar{R}(G,d,s+1,g) \cong R(G,d,s+1,g)$. Hence $R(G,d,s,g) \cong R(G,d,s+1,g)$. $\square$

Corollary 3.38 *If $s > \tau(G,d,g)$, then the polytopes $R(G,d,s,g)$ and $R(G,d,s+1,g)$ are combinatorially equivalent.*

3.4 Relations to the linear ordering polytope

A *linear ordering* of a finite set $V = \{1,\ldots,n\}$ is a bijective mapping $\sigma : V \to \{1,\ldots,n\}$. For $i \in V$ and $j \in V$, we say that i is *before* j in σ if $\sigma(i) < \sigma(j)$. Given a linear ordering σ of V, we can define an acyclic tournament $T = (V,A)$ with arc set $A = \{ij : \sigma(i) < \sigma(j)\}$ and, conversely, every acyclic tournament $T = (V,A)$ induces a linear ordering of V. For every two elements $i,j \in V$ two values $c_{ij} \in \mathbf{R}$ and $c_{ji} \in \mathbf{R}$ are given, measuring the profit we obtain from having i before j resp. j before i in a linear ordering. The weight of a linear ordering σ is defined to be $c(\sigma) = \sum_{\sigma(i)<\sigma(j)} c_{ij}$, and the problem of finding a linear ordering of maximum weight is called the *linear ordering problem.* This problem is $\mathcal{NP}$-hard [20] and it is closely related to the so-called feedback arc set problem and the acyclic subgraph problem [24]. It has applications in economics (triangulation of input-output matrices), scheduling (minimizing average weighted completion time), sports (ranking of teams), mathematical psychology, archeology and anthropology.

We can associate with each linear ordering σ a characteristic vector $x^\sigma \in \mathbf{R}^{n(n-1)}$, defined as follows.

$$x_{ij}^\sigma = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

The *linear ordering polytope* P_{LO}^n on n nodes is the convex hull of the characteristic vectors of all linear orderings of $\{1, \ldots, n\}$. This polytope has attracted much attention. Several classes of facet-defining inequalities are known [8, 19, 23, 38], and the complexity of the associated separation problems has been studied in detail [39]. Complete descriptions of P_{LO}^n are known for $n \leq 7$, with 87.472 facets for $n = 7$. A conjectured complete description for $n = 8$ contains over 480 million facets [13].

Chromatic scheduling polytopes share many structural properties with the linear ordering polytope, since the ordering variables have the same meaning in both settings. Not surprisingly, some of the simplest cases of chromatic scheduling polytopes, namely the instances defined over complete graphs, are equivalent to P_{LO}^n. We show that $R(K_n, d, s, 0)$ and $P(K_n, d, s, 0)$ are affinely isomorphic to P_{LO}^n when $s = \sum_{i=1}^n d_i$, and afterwards we present a generalization of this result for the fixed-length case when $s > \sum_{i=1}^n d_i$.

Recall that two polytopes $P \in \mathbf{R}^n$ and $Q \in \mathbf{R}^m$ are affinely isomorphic, denoted $P \cong Q$, if there is an affine bijection $f : \mathbf{R}^n \to \mathbf{R}^m$ between the points of the two polytopes.

Theorem 3.39 *If $s = \sum_{i=1}^n d_i$, then $P(K_n, d, s, 0) \cong P_{LO}^n$ and $R(K_n, d, s, 0) \cong P_{LO}^n$.*

Proof. Since $s = \omega(K_n, d)$ then $P(K_n, d, s, 0)$ is nonempty. Moreover, all intervals $I(i)$ have exactly length d_i and there is no gap between two intervals left; thus the feasible solutions distinguish only in the order of the intervals. Therefore, the following linear equations are satisfied by every feasible solution of $P(K_n, d, s, 0)$:

$$\begin{array}{rcll} l_i & = & \sum_{j \neq i} d_j \; x_{ji} & \qquad i = 1, \ldots, n \\ r_i & = & \sum_{j \neq i} d_j \; x_{ji} + d_i & \qquad i = 1, \ldots, n \end{array}$$

Hence the interval bound variables can be written as affine combinations of the ordering variables, which are precisely the linear ordering variables. Moreover, this affine mapping is a bijection, since every linear ordering generates a feasible schedule in $P(K_n, d, s, 0)$ and conversely. Thus, $P(K_n, d, s, 0) \cong P_{LO}^n$. Since every feasible schedule $z \in P(K_n, d, s, 0) \cap \mathbf{Z}^{2n+m}$ has $z_{r_i} - z_{l_i} = d_i$, then $P(K_n, d, s, 0) \cong R(K_n, d, s, 0)$, implying $R(K_n, d, s, 0) \cong P_{LO}^n$. $\square$

When $s = \omega(K_n, d)$, every feasible solution of $P(K_n, d, s, 0)$ *is* a linear ordering. The affine mapping is possible since there cannot be empty spaces between the intervals. If

$s > \omega(K_n, d)$, there will be some empty space between the intervals or there exist intervals $I(i)$ with $r_i > l_i + d_i$. We can still give a characterization of $R(K_n, d, s, 0)$ in terms of the linear ordering polytope, but not for $P(K_n, d, s, 0)$ anymore.

Theorem 3.40 *If $s > \sum_{i=1}^n d_i$, then $R(K_n, d, s, 0) \cong P_{LO}^{n+1}$.*

Proof. By Theorem 3.28, every extreme point y of $R(K_n, d, s, 0)$ has the following structure. The node set is partitioned into $V = L_y \cup R_y$ such that

$$\begin{aligned} y_{l_i} &= \sum_{j \in L_y} y_{x_{ji}} d_j \qquad \forall i \in L_y \\ y_{l_i} &= s - \sum_{j \in R_y} y_{x_{ij}} d_j \qquad \forall i \in R_y \end{aligned}$$

That is, the intervals corresponding to nodes in L_y resp. R_y are located in the left resp. right part of the frequency spectrum, and there is only one empty interval in between, namely $[d(L_y), s - d(R_y)]$. We can regard this unique empty interval as a new interval with length $s - \sum_{i=1}^n d_i$, and so every extreme point of $R(K_n, d, s, 0)$ represents a linear ordering on $n+1$ nodes. Hence, given an extreme point $x \in \mathrm{vert}(P_{LO}^{n+1})$ we can construct an extreme point of $R(K_n, d, s, 0)$ by

$$l_i = \sum_{j=1}^n d_j x_{ji} + \Big(s - \sum_{j=1}^n d_j\Big) x_{n+1,i} \qquad i = 1, \ldots, n$$

Since $\mathrm{vert}(R(K_n, d, s, 0))$ includes every linear ordering among the $n+1$ considered intervals, then this mapping is an isomorphism and, therefore, $R(K_n, d, s, 0) \cong P_{LO}^{n+1}$. $\square$

These results imply that even simple chromatic scheduling polytopes, namely those defined over complete graphs, are hard to characterize. A complete description of $R(K_n, d, s, 0)$ in terms of its facets should include all the linear ordering facets, which amount to several millions of valid inequalities even for small instances [13]. One may expect that similar relationships may hold for chromatic scheduling polytopes over arbitrary graphs, and this is indeed the case. The remaining of this section is devoted to presenting these results.

Definition 3.18 *If $\pi^T x \leq \pi_0$ is a valid inequality of P_{LO}^n, let S_π denote the set of directed arcs having nonzero coefficients in the inequality (i.e., $S_\pi = \{e \in E : \pi_e \neq 0\}$).*

Proposition 3.41 *Let $\pi^T x \leq \pi_0$ be a valid inequality of P_{LO}^n with $S_\pi \subseteq E$. Then the inequality $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leq \pi_0$ is valid for $P(G, d, s, g)$ and $R(G, d, s, g)$.*

Proof. Let $(l, r, x) \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be an integer feasible solution. The vector x specifies a partial ordering among the intervals, and can be extended into a linear ordering $x' \in P_{LO}^n$ satisfying $\pi^T x' \leq \pi_0$. Since $S_\pi \subseteq E$, then $\pi^T x' = \sum_{ij \in S_\pi} \pi_{ij} x'_{ij} = \sum_{ij \in S_\pi} \pi_{ij} x_{ij}$, implying that $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leq \pi_0$ is valid for $P(G, d, s, g)$. Since this inequality only involves the ordering variables, it is also valid for $R(G, d, s, g)$. $\square$

Theorem 3.42 *Let $\pi^T x \leq \pi_0$ be a facet-defining inequality of P_{LO}^n with $S_\pi \subseteq E$. If $s \gg \omega(G,d)$, then $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leq \pi_0$ defines a facet of $P(G,d,s,g)$ and $R(G,d,s,g)$.*

Proof. Since the equations $x_{ij} + x_{ji} = 1 \ \forall i \neq j$ are a maximal equation system for P_{LO}^n, there exist $k = n(n-1)/2$ affinely independent integer points $x^1, \ldots, x^k \in P_{LO}^n$ such that $\pi^T x^i = \pi_0$ for $i = 1, \ldots, k$. These points have $n(n-1)/2$ coordinates, one for each edge of K_n. Delete the coordinates corresponding to the edges that are not present in G. That way we obtain the new points $\mathrm{proj}_x(x^1), \ldots, \mathrm{proj}_x(x^k) \in \mathbf{R}^m$, and we can find m affinely independent points among them. Since $s \gg \omega(G,d)$, we can extend $\bar{x}^i = \mathrm{proj}_x(x^i)$ to a feasible schedule $z^i \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$, by assigning the intervals in such a way that the precedence relation indicated by $\bar{x}^i$ is satisfied, i.e., $z^i_{l_j} = L_j(\bar{x}^i, s)$ and $z^i_{r_j} = L_j(\bar{x}^i, s) + d_j$ for $j \in V$. By construction, this schedule is feasible.

We now construct $2n$ more affinely independent points from z^1 as follows. Let $D = (V, E_D)$ be a digraph such that $ij \in E_D$ if and only if $ij \in E$ and $I(j)$ is located before $I(i)$ in z^1. Let $i_1, \ldots, i_n$ be a topological ordering of D, and construct n feasible solutions $u^1, \ldots, u^n \in P(G,d,s,g)$ by setting

$$
\begin{aligned}
u^i_{l_j} &= \begin{cases} z^1_{l_j} + 1 & \text{if } j = i_t, \text{ for } t \leq i \\ z^1_{l_j} & \text{if } j = i_t, \text{ for } t > i \end{cases} \\
u^i_{r_j} &= u^i_{l_j} + d_j
\end{aligned}
$$

Now, for $j = 1, \ldots, n$, construct a point $w^j \in P(G,d,s,g)$ from u^j by enlarging the interval $I(i_j)$ one unit to the left. These new schedules are affinely independent with respect to $z^1, \ldots, z^n$. This way we complete a set of $2n + m$ affinely points and, therefore, $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leq \pi_0$ defines a facet of the (full-dimensional) polytope $P(G,d,s,g)$. The construction of the schedules $z^1, \ldots, z^k$ and $u^1, \ldots, u^n$ shows that this inequality also defines a facet of $R(G,d,s,g)$. $\square$

Chapter 4

Facets for all nonempty instances coming from symmetry arguments

> An algorithm which is good in the sense used here is not necessarily very good from a practical viewpoint. However, the good versus not-good dichotomy is useful. (...) The classes of problems which are respectively known and not known to have good algorithms are very interesting theoretically.
>
> – Jack Edmonds (1967)

Chromatic scheduling polytopes also admit interesting properties from a geometrical point of view. The main reason is that there are only antiparallelity requirements on the jobs but no prescribed partial orders, implying strong symmetry properties as addressed in Section 4.1. The main consequence is a powerful tool for identifying facet-defining inequalities for nonempty polytopes without any knowledge on the dimension. This is of particular interest as determining the dimension of chromatic scheduling polytopes is $\mathcal{NP}$-complete.

Based on this tool, we analyze in Section 4.2 the demand constraints, the binary bounds on the ordering variables, and a further class of valid inequalities showing that they induce facets whenever the polytopes are nonempty. We also observe that the remaining integer programming constraints, i.e., the bounds on the interval variables and the antiparallelity constraints, do not define facets in general.

Section 4.3 presents three classes of facet-defining inequalities for the polytopes $P(G, d, s, g)$ where the frequency span s is small compared to the weighted clique number $\omega(G, d)$. This setting is the hardest case in practice, since we cannot expect to find feasible solutions in a straightforward manner. We explore three classes of inequalities being valid only in low-dimensional polytopes, but being facet-inducing due to symmetry arguments.

4.1 Symmetry of chromatic scheduling polytopes

Chromatic scheduling polytopes admit a particular property: they are symmetric. Recall that we only have antiparallelity constraints for potential interferers $ij \in E$ but no precedence relation given in advance. Hence, in a feasible solution either the interval $I(i)$ has to be scheduled before the interval $I(j)$ or $I(j)$ comes before $I(i)$. Thus, for every feasible schedule S, there is a feasible schedule symmetric to S w.r.t. the available spectrum $[0, s]$, obtained by swapping all intervals of S. This is obviously not true for scheduling problems in general. Clearly, the polytopes $P(G, d, s, g)$ and $R(G, d, s, g)$ reflect the symmetry of the schedules.

This was first observed in [21] and further explored in [26]. In this section we discuss this property in more detail and study how it affects the search for valid inequalities. We first state the main results concerning the symmetry of $R(G, d, s, g)$ resp. $P(G, d, s, g)$ in Section 4.1.1 resp. Section 4.1.2. This special symmetry provides tools for identifying facet-defining inequalities without any knowledge of the dimension of the polytopes, see Section 4.1.3. We shall apply these theorems in Section 4.2 and Section 4.3 to some classes of valid inequalities showing that they define facets whenever the polytopes are nonempty.

4.1.1 Symmetry results for $R(G, d, s, g)$

In the fixed-length case, the polytope admits a symmetry point as observed in [21, 26].

Theorem 4.1 ([26]) *The polytope $R(G, d, s, g)$ is symmetric with respect to the point*

$$p = \Big(\underbrace{\frac{s-d_1}{2}, \ldots, \frac{s-d_n}{2}}_{n}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{m}\Big).$$

Proof. Let S be a feasible schedule, representing an assignment of an interval $I(i) = [l_i, l_i + d_i]$ to each customer $i \in V$. We obtain a symmetric assignment of intervals $I'(i) = [l'_i, l'_i + d_i] = [s - l_i - d_i, s - l_i]$ in the reverse order if we mirror the interval $I(i)$ with respect to the available spectrum $[0, s]$ for every $i \in V$. Thus the schedule S' given by the left interval bounds $l'_i = s - l_i - d_i$ $\forall i \in V$ and the precedence variables $x'_{ij} = 1 - x_{ij}$ $\forall ij \in E$, $i < j$ describes a feasible schedule symmetric to S. Hence

$$\frac{l_i + l'_i}{2} = \frac{l_i + s - l_i - d_i}{2} = \frac{s - d_i}{2} \quad \text{and} \quad \frac{x_{ij} + x'_{ij}}{2} = \frac{x_{ij} + 1 - x_{ij}}{2} = \frac{1}{2}$$

implies that

$$p = \Big(\underbrace{\frac{s-d_1}{2}, \ldots, \frac{s-d_n}{2}}_{n}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{m}\Big)$$

is the symmetry point of $R(G, d, s, g)$. □

Definition 4.1 *If $y = (l, x) \in R(G, d, s, g) \cap \mathbf{Z}^{n+m}$ is a feasible integer solution, then* $\mathrm{sym}(y) = 2p - y$ *denotes its symmetrical solution, i.e.,*

$$\mathrm{sym}\begin{pmatrix} l \\ x \end{pmatrix} = \begin{pmatrix} s\,\mathbf{1} - d \\ \mathbf{1} \end{pmatrix} - \begin{pmatrix} l \\ x \end{pmatrix}.$$

Due to the symmetry of the polytope $R(G, d, s, g)$, to every face exists a parallel face of the same dimension and there is a simple formula to compute this parallel face.

Theorem 4.2 ([26]) *Let $b \leq a^T x$ be a valid (facet-inducing) inequality of $R(G, d, s, g)$. Then $a^T x \leq 2a^T p - b$ is also valid (facet-inducing) for $R(G, d, s, g)$.*

Proof. We first prove that $a^T x \leq 2a^T p - b$ is valid for $R(G, d, s, g)$. Let y be a feasible solution and let $y' = \mathrm{sym}(y) = 2p - y$. Then $a^T y = a^T(2p - y') = 2a^T p - a^T y' \leq 2a^T p - b$ (since y' is feasible and $a^T y' \geq b$). Now, if there are k affinely independent points in $H = \{y \in R(G, d, s, g) : a^T y = 2a^T p - b\}$, there are obviously k affinely independent points in $H' = \{y \in R(G, d, s, g) : a^T y = b\}$. Thus, if $b \leq a^T x$ is facet-inducing for $R(G, d, s, g)$, then $a^T x \leq 2a^T p - b$ is facet-defining too. □

4.1.2 Symmetry results for $P(G, d, s, g)$

In the general case, every feasible schedule is represented by the interval bounds $l, r \in \mathbf{R}^n$ and the ordering variables $x \in \mathbf{R}^m$. Swapping all the intervals of a feasible solution $z = (l, r, x)$ with respect to the spectrum $[0, s]$ constructs a new point z' which is also feasible and symmetric to the original one. Thereby, the swapping maps the left interval bounds l_i of z to the right interval bounds r'_i of z', and reverses the order of the intervals:

$$\begin{array}{lcll} l_i & \rightarrow & r'_i = s - l_i & \forall i \in V \\ r_i & \rightarrow & l'_i = s - r_i & \forall i \in V \\ x_{ij} & \rightarrow & x'_{ij} = 1 - x_{ij} & \forall ij \in E \end{array}$$

Hence, swapping the intervals yields

$$(l_1, \ldots, l_n, r_1, \ldots, r_n, x_{1i}, \ldots, x_{jn}) \rightarrow (r'_1, \ldots, r'_n, l'_1, \ldots, l'_n, x'_{1i}, \ldots, x'_{jn}).$$

The point $\bar{p}$ with entries

$$\begin{array}{lclclcl} \bar{p}_{l_i} & = & \frac{l_i + r'_i}{2} & = & \frac{l_i + s - l_i}{2} & = & \frac{s}{2}\ \forall i \in V \\ \bar{p}_{r_i} & = & \frac{r_i + l'_i}{2} & = & \frac{r_i + s - r_i}{2} & = & \frac{s}{2}\ \forall i \in V \\ \bar{p}_{x_{ij}} & = & \frac{x_{ij} + x'_{ij}}{2} & = & \frac{x_{ij} + 1 - x_{ij}}{2} & = & \frac{1}{2}\ \forall ij \in E \end{array}$$

is, therefore, the symmetry point for every pair of symmetric feasible solutions z and z'. Since $\bar{p}$ is independent of the special choice of z, it can be seen as the symmetry point of $P(G, d, s, g)$ with respect to swapping schedules.

Definition 4.2 *Let* $\mathrm{sym}(z)$ *denote the symmetrical point of an integer solution* $z = (l, r, x) \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$*, where*

$$\mathrm{sym}\begin{pmatrix} l \\ r \\ x \end{pmatrix} = \begin{pmatrix} s\,\mathbf{1} - r \\ s\,\mathbf{1} - l \\ \mathbf{1} - x \end{pmatrix} = \begin{pmatrix} s\,\mathbf{1} \\ s\,\mathbf{1} \\ \mathbf{1} \end{pmatrix} - \begin{pmatrix} r \\ l \\ x \end{pmatrix}.$$

We again benefit from the symmetry of the polytope in order to find, for every inequality valid for $P(G, d, s, g)$, a symmetric valid inequality. For that, let S be a feasible schedule and let $z^S \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be its associated vector. Let $b \leq a^T x$ be a valid inequality of $P(G, d, s, g)$. The straight line through z^S and the symmetry point $\bar{p}$ meets the hyperplane $H = \{x \in \mathbf{R}^{2n+m} : a^T x = b\}$ in a point, say z_H^S. Let $z^{S'}$ and $z_H^{S'}$ be the images of z^S and z_H^S obtained by the swapping. Then $z_H^{S'}$ lies on the hyperplane $H' = \{x' \in \mathbf{R}^{2n+m} : a^T x' = b'\}$ with

$$x' = (x_{r_1}, \ldots, x_{r_n}, x_{l_1}, \ldots, x_{l_n}, x_{x_{1i}}, \ldots, x_{x_{jn}}).$$

Observe that $a^T x' = a'^T x$ holds by

$$(a_l, a_r, a_x)\begin{pmatrix} x_r \\ x_l \\ x_x \end{pmatrix} = (a_r, a_l, a_x)\begin{pmatrix} x_l \\ x_r \\ x_x \end{pmatrix}.$$

Thus we may represent the hyperplane $H' = \{x \in \mathbf{R}^{2n+m} : a'^T x = b'\}$ with

$$a' = (a_{r_1}, \ldots, a_{r_n}, a_{l_1}, \ldots, a_{l_n}, a_{x_{1i}}, \ldots, a_{x_{jn}}).$$

By $P(G, d, s, g) \subseteq \{x \in \mathbf{R}^{2n+m} : b \leq a^T x\}$ and the symmetry of the polytope, $P(G, d, s, g) \subseteq \{x \in \mathbf{R}^{2n+m} : -b' \leq -a'^T x\}$ follows, i.e., $a'^T x \leq b'$ is valid for $P(G, d, s, g)$. We have to determine b'. The previous observations imply $z_H^{S'} = 2\bar{p} - z_H^S$. Thus, from $a^T z_H^S = b$ and $a^T\ z_H^{S'} = b'$ follows

$$b' = a^T z_H^{S'} = a^T(2\bar{p} - z_H^S) = 2a^T\bar{p} - a^T z_H^S = 2a^T\bar{p} - b$$

and $a'^T x \leq 2a^T\bar{p} - b$ is, therefore, the valid upper bound inequality of $P(G, d, s, g)$ symmetric to $b \leq a^T x$. (Note $a^T\bar{p} = a'^T\bar{p}$.) Further, if there are k affinely independent points in $H \cap P(G, d, s, g)$, there are obviously k affinely independent points in $H' \cap P(G, d, s, g)$. Thus, if $b \leq a^T x$ is facet-inducing for $P(G, d, s, g)$, so is $a'^T x \leq 2a^T\bar{p} - b$ and we have obtained the following theorem:

Theorem 4.3 ([26]) *Let* $b \leq a^T x$ *be a valid (facet-inducing) inequality of* $P(G, d, s, g)$ *and let* $\bar{p}$ *be the symmetry point of* $P(G, d, s, g)$ *with respect to swapping schedules. Then* $a'^T x \leq 2a^T\bar{p} - b$ *is also valid (facet-inducing) for* $P(G, d, s, g)$ *where*

$$a' = (\underbrace{a_{r_1}, \ldots, a_{r_n}}_{n}, \underbrace{a_{l_1}, \ldots, a_{l_n}}_{n}, \underbrace{a_{x_{1i}}, \ldots, a_{x_{jn}}}_{m}).$$

4.1.3 Facets arising from symmetry arguments

The symmetry of chromatic scheduling polytopes provides us an important tool for identifying facet-defining inequalities, where no knowledge on the dimension is required. The results of this subsection show that if F is a face such that $y \in F \Leftrightarrow \mathrm{sym}(y) \notin F$, then F is a facet of $R(G,d,s,g)$. With some other minor assumptions, the same result applies to $P(G,d,s,g)$.

Theorem 4.4 *Let F be a face of $R(G,d,s,g)$ such that $y \in F \Leftrightarrow \mathrm{sym}(y) \notin F$ for every $y \in R(G,d,s,g) \cap \mathbf{Z}^{n+m}$. Then F is a facet of $R(G,d,s,g)$.*

Proof. Assume that $\dim(F) = k$, and let $y_0, \ldots, y_k$ be a maximal set of affinely independent points in F. Let $y_{k+1} \notin F$ be any feasible solution outside F. Then, $y_0, \ldots, y_k, y_{k+1}$ are affinely independent, because $y_0, \ldots, y_k$ satisfy the equation which defines F and y_{k+1} does not.

Now let $y_{k+2} \notin F$ be some other feasible solution not in F. Note that $\mathrm{sym}(y_{k+1})$ and $\mathrm{sym}(y_{k+2})$ are in F, and thus they can be written as affine combinations of $y_0, \ldots, y_k$. Then,

$$
\begin{aligned}
y_{k+2} - y_{k+1} &= \begin{pmatrix} s\,\mathbf{1} - d \\ \mathbf{1} \end{pmatrix} - y_{k+1} - \begin{pmatrix} s\,\mathbf{1} - d \\ \mathbf{1} \end{pmatrix} + y_{k+2} \\
&= \mathrm{sym}(y_{k+1}) - \mathrm{sym}(y_{k+2}) \\
&= \sum_{i=0}^{k} \alpha_i y_i - \sum_{i=0}^{k} \beta_i y_i \\
&= \sum_{i=0}^{k} (\alpha_i - \beta_i) y_i,
\end{aligned}
$$

where $\sum_i \alpha_i = \sum_i \beta_i = 1$. But then

$$
y_{k+2} = y_{k+1} + \sum_{i=0}^{k} (\alpha_i - \beta_i) y_i
$$

implies that y_{k+2} is an affine combination of the points $y_0, \ldots, y_k, y_{k+1}$. This proves that $\dim(R(G,d,s,g)) = \dim(F) + 1$ holds, and thus F is a facet of $R(G,d,s,g)$. □

The symmetry for the general case provides some tools for identifying facet-defining inequalities as well. In order to state these results, recall Lemma 3.9, which relates the dimension of $R(G,d,s,g)$ and $P(G,d,s,g)$ by means of the node subset $F_s(G,d)$.

Theorem 4.5 *Let $F = \{y \in R(G,d,s,g) : a^T y = b\}$ be a face of $R(G,d,s,g)$ such that $\mathrm{red}(z) \in F \Leftrightarrow \mathrm{red}(\mathrm{sym}(z)) \notin F$ for every $z \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$. Then $F' = \{z \in P(G,d,s,g) : a^T \mathrm{red}(z) = b\}$ is a facet of $P(G,d,s,g)$.*

Proof. If $y \in R(G,d,s,g)$, then $\mathrm{ext}(y) \in P(G,d,s,g)$. By the hypothesis, we have that either $\mathrm{red}(\mathrm{ext}(y)) \in F$ or $\mathrm{red}(\mathrm{sym}(\mathrm{ext}(y))) \in F$ (but not both). But $\mathrm{red}(\mathrm{ext}(y)) = y$ and

$\mathrm{red}(\mathrm{sym}(\mathrm{ext}(y))) = \mathrm{sym}(y)$ imply $y \in F \Leftrightarrow \mathrm{sym}(y) \notin F$. Therefore, F is a facet of $R(G,d,s,g)$ by Theorem 4.4. Let $r = \dim(R(G,d,s,g))$, then there exist r affinely independent vectors $y^1,\ldots,y^r$ in the facet F (i.e., $a^T\ y^k = b$ for $k = 1,\ldots,r$). Then, $\mathrm{ext}(y^1),\ldots,\mathrm{ext}(y^r)$ are affinely independent points satisfying $a^T\ \mathrm{red}(\mathrm{ext}(y^k)) = b$ by definition.

Now, for each $k \in F_s(G)$ let $z^k \in P(G,d,s,g)$ be a solution such that $z^k_{r_k} - z^k_{l_k} > d_k$ and $z^k_{r_l} - z^k_{l_l} = d_l$ for $l \neq k$. We can assume that $\mathrm{red}(z^k) \in F'$ (otherwise, consider the reduction of its symmetrical point $\mathrm{sym}(z^k)$). Define the following set of feasible solutions:

$$A \ = \ \{\mathrm{ext}(y^1),\ldots,\mathrm{ext}(y^r)\} \ \cup \ \{z^k : k \in F_s(G)\}.$$

For every $k \in F_s(G)$, z^k is affinely independent w.r.t. the points in $A\backslash\{z^k\}$, since all the points in $A\backslash\{z^k\}$ satisfy $r_k - l_k = d_k$, but z_k does not. This way we have by Lemma 3.9 $|A| = \dim(R(G,d,s,g)) + |F_s(G)| = \dim(P(G,d,s,g))$ affinely independent points in $P(G,d,s,g)$ satisfying $a^T\ \mathrm{red}(z) \leq b$ at equality and this inequality defines, therefore, a facet of $P(G,d,s,g)$. □

Corollary 4.6 *Let $F = \{y \in R(G,d,s,g) : a^T y = b\}$ be a face of $R(G,d,s,g)$ such that $y \in F \Leftrightarrow \mathrm{sym}(y) \notin F$ for every $y \in R(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ and $\mathrm{proj}_l(a) = 0$ (i.e. only x-variables have nonnegative coefficients in $a^T y \leq b$). Then $F' = \{z \in P(G,d,s,g) : a^T\ \mathrm{red}(z) = b\}$ is a facet of $P(G,d,s,g)$.*

Proof. We verify that the assumptions of Theorem 4.5 are satisfied. Consider any feasible solution $z \in P(G,d,s,g)$. By the hypothesis, we know that $\mathrm{red}(z) \in F \Leftrightarrow \mathrm{sym}(\mathrm{red}(z)) \notin F$. Moreover,

$$\begin{aligned} a^T\ \mathrm{red}(\mathrm{sym}(z)) &= \mathrm{proj}_x(a)\ \mathrm{proj}_x(\mathrm{red}(\mathrm{sym}(z))) \\ &= \mathrm{proj}_x(a)\ \mathrm{proj}_x(\mathrm{sym}(\mathrm{red}(z))) \\ &= a^T\ \mathrm{sym}(\mathrm{red}(z)). \end{aligned}$$

Then, we have that

$$\begin{aligned} \mathrm{red}(z) \in F &\Leftrightarrow a^T\ \mathrm{red}(z) = b \\ &\Leftrightarrow a^T\ \mathrm{sym}(\mathrm{red}(z)) < b \\ &\Leftrightarrow a^T\ \mathrm{red}(\mathrm{sym}(z)) < b \\ &\Leftrightarrow \mathrm{red}(\mathrm{sym}(z)) \notin F. \end{aligned}$$

So, the hypotheses of Theorem 4.5 are satisfied, and thus F' is a facet of $P(G,d,s,g)$. □

4.2 Facets coming from the model constraints

With the help of the results from the previous section, we are now able to determine which model constraints define facets of chromatic scheduling polytopes. In this section we show that the lower and upper bounds on the ordering variables $0 \leq x_{ij} \leq 1\ \forall ij \in E$ implied by the binary constraints $x_{ij} \in \{0,1\}$ are always facet-defining whenever the polytopes are nonempty, and we present a further class of valid inequalities which admits the same property. We also give a characterization of the cases where the demand constraints define facets of $P(G,d,s,g)$. We start with the bounds on the ordering variables.

Theorem 4.7 *If $ij \in E$, then $x_{ij} \geq 0$ and $x_{ij} \leq 1$ define facets of $R(G,d,s,g)$ and $P(G,d,s,g)$, whenever the polytopes are nonempty.*

Proof. Let $F = \{y \in R(G,d,s,g) : y_{x_{ij}} = 1\}$ be the face defined by $x_{ij} \leq 1$, i.e., the convex hull of the set of points having $I(i)$ before $I(j)$. A point has $I(i)$ before $I(j)$ if and only if its symmetrical point has $I(j)$ before $I(i)$, and thus $y \in F \Leftrightarrow \mathrm{sym}(y) \notin F$. Theorem 4.4 shows that F is a facet of $R(G,d,s,g)$, and Corollary 4.6 implies that $F' = \{z \in P(G,d,s,g) : z_{x_{ij}} = 1\}$ is a facet of $P(G,d,s,g)$. The same argumentation applies to $x_{ij} \geq 0$. □

Definition 4.3 (triangle inequalities) *Consider a triangle $T = \{i,j,k\}$ of G, i.e., a set of three pairwise adjacent nodes of G. We define*

$$x_{ij} + x_{jk} + x_{ki} \leq 2 \tag{4.1}$$

to be the triangle inequality *associated with T.*

It is easy to verify that triangle inequalities are valid for both polytopes, since $x_{ij} = x_{jk} = x_{ki} = 1$ is obviously not possible in any feasible solution. We now apply the results of Section 4.1.3 to prove facetness.

Theorem 4.8 *The triangle inequalities define facets of $R(G,d,s,g)$ and $P(G,d,s,g)$ whenever the polytopes are nonempty.*

Proof. Let $y \in R(G,d,s,g)$ be an integer solution. Since $\{i,j,k\}$ is a complete subgraph, the intervals $I(i)$, $I(j)$ and $I(k)$ cannot overlap in y. Thus y contains one of the six configurations depicted in Figure 4.1. Note that the cases (a), (b), and (c) satisfy (4.1) at equality, whereas the cases (d), (e), and (f) do not. Moreover, the cases (a), (b), resp. (c) are the symmetric cases of (d), (e), resp. (f). Thus, if F is the face defined by (4.1), then $y \in F \Leftrightarrow \mathrm{sym}(y) \notin F$ holds. Theorem 4.4 resp. Corollary 4.6 implies that F is a facet of $R(G,d,s,g)$ resp. $P(G,d,s,g)$. □

Corollary 4.9 *If $T = \{i,j,k\}$ is a triangle of G, then the inequality $1 \leq x_{ij} + x_{jk} + x_{ki}$ symmetric to (4.1) is facet-inducing for $P(G,d,s,g)$ and $R(G,d,s,g)$ whenever the polytopes are nonempty.*

Let us now analyze the demand constraints $l_i + d_i \leq r_i$ for $P(G,d,s,g)$ (recall that these constraints are replaced by equalities in $R(G,d,s,g)$). Let $i \in V$. If $i \notin F_s(G,d)$, i.e., if

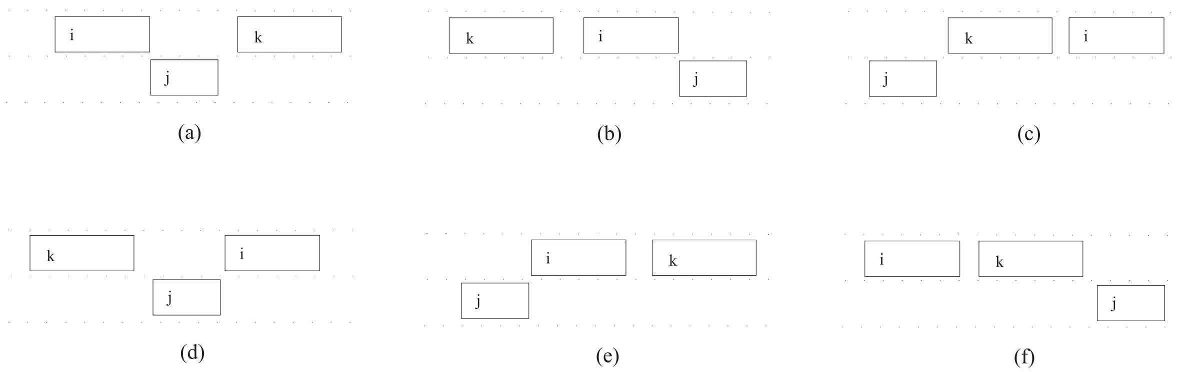


Figure 4.1: Possible cases for y.

every point in $P(G,d,s,g)$ satisfies $l_i + d_i = r_i$, then $P(G,d,s,g) \subseteq \{y : y_{l_i} + d_i = y_{r_i}\}$. On the other hand, if $i \in F_s(G,d)$, i.e., if there exists a feasible solution $z \in P(G,d,s,g)$ with $z_{l_i} + d_i < z_{r_i}$, then the demand constraint for the node i defines a proper face of $P(G,d,s,g)$ and, moreover, this face is a facet.

Theorem 4.10 *If $i \in F_s(G,g)$, then the demand constraint $l_i + d_i \leq r_i$ defines a facet of $P(G,d,s,g)$.*

Proof. Call $\dim(P(G,d,s,g)) = k$, and let $y^0, \ldots, y^k \in P(G,d,s,g)$ be $k+1$ affinely independent points in P ($y^j \in \mathbf{R}^{2n+m}$). For $i = 0, \ldots, k$, consider the vector $\bar{y}^j$ obtained from y^j by replacing its r_i-entry by $y^j_{l_i} + d_i$. Note that this shrinks the interval $I(i)$ to its minimum length d_i in every y^j, leaving the remaining intervals unchanged, and thus keeping feasibility. These new points lie in the face F of $P(G,d,s,g)$ defined by $l_i + d_i \leq r_i$. Moreover, from $\dim\{y^0, \ldots, y^k\} = k$ follows $\dim\{\bar{y}^0, \ldots, \bar{y}^k\} \geq k-1$. But there is a point $z \in P(G,d,s,g)$ which does not satisfy the demand constraint $l_i + d_i \leq r_i$ at equality, and thus $\dim\{\bar{y}^0, \ldots, \bar{y}^k\} = k-1$, implying that this inequality defines a facet of $P(G,d,s,g)$. □

It is natural to ask whether the remaining model constraints, i.e., the bounds on the interval variables and the antiparallelity constraints, induce facets. In Chapter 5 we shall see that these constraints do not induce facets in general, and we shall devise strengthenings of the corresponding inequalities providing facet-inducing families of inequalities.

4.3 Facet-defining inequalities for small frequency spans

If s is close to the weighted clique number $\omega(G,d)$ of the interference graph, then the frequency spectrum $[0,s]$ does not allow every possible ordering among the intervals. This setting is the hardest case in practice since we cannot expect to find feasible solutions easily. This section presents valid inequalities that arise in this situation. The main idea is to identify structures on the interference graph that preclude every possible ordering, and to state a valid inequality asserting this constraint. The inequalities devised in this section are amenable of

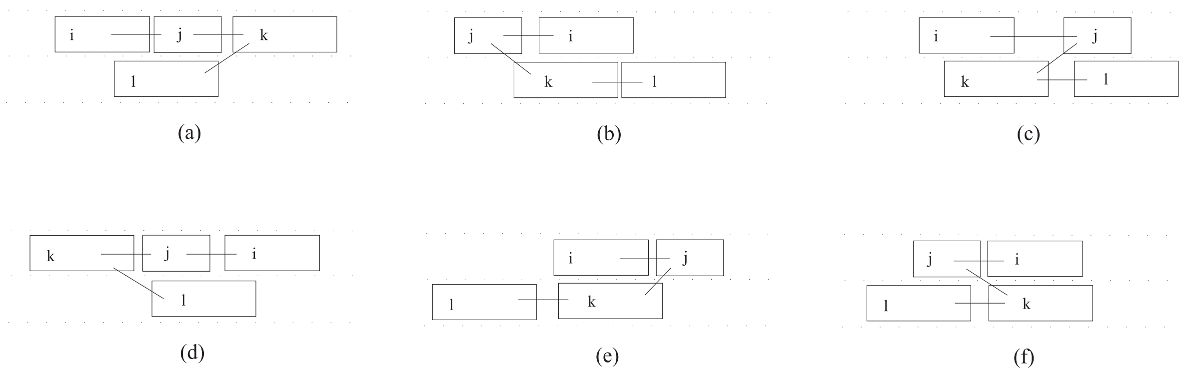


Figure 4.2: Possible configurations of a feasible solution in the proof of Theorem 4.12.

being analyzed with symmetry arguments, and we will use the results presented in Section 4.1.3 to show that these inequalities are facet-defining as long as the polytopes are nonempty.

Definition 4.4 (4-path inequalities) *Let $i, j, k, l \in V$ be four nodes of G such that ij, jk, $kl \in E$ and no feasible solution of $P(G, d, s, g)$ has the ordering $i \to j \to k \to l$. We define*

$$x_{ij} + x_{jk} + x_{kl} \;\leq\; 2 \tag{4.2}$$

to be the 4-path inequality *associated with the path $\{i, j, k, l\}$.*

Proposition 4.11 *If no feasible solution has the ordering $i \to j \to k \to l$, then the 4-path inequality (4.2) is valid for $R(G, d, s, g)$ and $P(G, d, s, g)$.*

Proof. The 4-path inequality can only be violated by a solution $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ such that $z_{x_{ij}} = z_{x_{jk}} = z_{x_{kl}} = 1$, but this implies that z has the ordering $i \to j \to k \to l$, which is excluded by the hypothesis. Hence (4.2) is valid for $P(G, d, s, g)$ and, since it does not involve the interval bounds, it is also valid for $R(G, d, s, g)$. □

Theorem 4.12 *If no feasible solution has the ordering $i \to j \to k \to l$, then the 4-path inequality (4.2) is facet-inducing for $R(G, d, s, g)$ and $P(G, d, s, g)$.*

Proof. Let $y \in R(G, d, s, g) \cap \mathbf{Z}^{n+m}$ be an integer feasible solution. Since the ordering $i \to j \to k \to l$ is not allowed, then y has one of the six forms depicted in Figure 4.2. Note that cases 4.2(a), 4.2(b) and 4.2(c) satisfy (4.2) at equality, whereas cases 4.2(d), 4.2(e) and 4.2(f) do not. Moreover, cases 4.2(a) and 4.2(d) are symmetrical, cases 4.2(b) and 4.2(e) are symmetrical, as well as 4.2(c) and 4.2(f). Thus, if F is the face defined by (4.2), then $y \in F \Leftrightarrow \mathrm{sym}(y) \notin F$, and by Theorem 4.4 and Corollary 4.6, the inequality (4.2) defines a facet of $R(G, d, s, g)$ and $P(G, d, s, g)$. □

Remark. The 4-path inequality appears only for small values of s preventing a linear ordering of the nodes $\{i, j, k, l\}$. This ordering is not feasible if

$$d_i + d_j + d_k + d_l + g\ (\delta_{ij} + \delta_{jk} + \delta_{kl}) > s, \tag{4.3}$$

where δ_{ij} denotes the minimum possible distance between $I(i)$ and $I(j)$. Note that the converse is not true in general, i.e., it may happen that (4.3) is not satisfied but still the structure of G does not allow the ordering $i \to j \to k \to l$. This is the situation in the example depicted in Figure 4.3, which has $g = 0$ and $d_i + d_j + d_k + d_l \le s$, but does not allow the ordering in question. $\triangleleft$

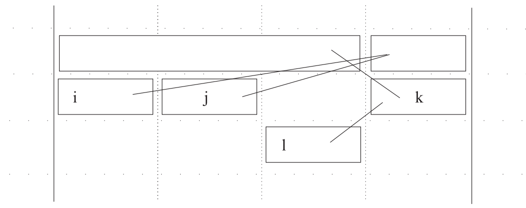


Figure 4.3: The ordering $i \to j \to k \to l$ is not feasible but (4.3) does not hold.

The 4-path inequalities cannot be trivially generalized to facet-inducing inequalities associated with paths on more than 4 nodes. For example, let $j_1, \dots, j_k$ be a path in G on $k > 4$ nodes, such that no feasible solution has $x_{j_i, j_{i+1}} = 1$ for $i = 1, \dots, k-1$. Then, the inequality

$$\sum_{i=1}^{k-1} x_{j_i, j_{i+1}} \;\le\; k - 1 \tag{4.4}$$

is valid but may not define a facet if s is too small.

Definition 4.5 (paw inequalities) *Let $i, j, k, l \in V$ be four distinct nodes of G such that $\{i, j, k\}$ induces a triangle and $jl \in E$. Furthermore, suppose that no feasible solution of $P(G, d, s, g)$ has the ordering $i \to j \to k$ and $j \to l$. We define*

$$x_{jk} + x_{jl} \;\le\; 1 + x_{ji} \tag{4.5}$$

to be the paw inequality *associated with the nodes $\{i, j, k, l\}$.*

Remark. Note that the definition of the paw inequalities allows $il \in E$ and $kl \in E$, i.e., the node set $\{i, j, k, l\}$ is not supposed to define an induced paw. $\triangleleft$

Proposition 4.13 *If no feasible solution has the ordering $i \to j \to k$ and $j \to l$, then the paw inequality (4.5) is valid for $R(G, d, s, g)$ and $P(G, d, s, g)$.*

Proof. The only combination of values for variables x_{jk}, x_{jl} and x_{ji} violating inequality (4.5) is $x_{jk} = x_{jl} = 1$ and $x_{ji} = 0$, which amounts to the forbidden ordering $i \to j \to k$ and $j \to l$. Thus, (4.5) is a valid inequality for $R(G, d, s, g)$ and $P(G, d, s, g)$. $\square$

Theorem 4.14 *If no feasible solution has the ordering $i \to j \to k$ and $j \to l$, then the paw inequality (4.5) is facet-defining for $R(G, d, s, g)$ and $P(G, d, s, g)$.*

Proof. To show that this inequality defines a facet of these polytopes, it is enough to verify that y is in the face defined by (4.5) if and only if $\mathrm{sym}(y)$ is not, and then applying Theorem 4.4 and Corollary 4.6. □

To close this section, we now present a facet-defining inequality for a 5-node structure.

Definition 4.6 (extended paw inequalities) *Let $1, \ldots, 5 \in V$ be five distinct nodes such that $12,\ 23 \in E$ and $\{3, 4, 5\}$ form a triangle in G. Moreover, assume that no feasible solution has the orderings $1 \to 2 \to 3 \to 4$, $1 \to 2 \to 3 \to 5$ and $2 \to 3 \to 4 \to 5$. We define*

$$x_{34} + x_{35} - x_{21} \ \leq \ 2x_{32} \tag{4.6}$$

to be the extended paw inequality *associated with the nodes $\{1, \ldots, 5\}$.*

Remark. Again, note that the definition of the extended paw inequalities allows $14, 15 \in E$ and $24, 25 \in E$. ◁

Proposition 4.15 *If no feasible solution has the orderings $1 \to 2 \to 3 \to 4$, $1 \to 2 \to 3 \to 5$ and $2 \to 3 \to 4 \to 5$, the extended paw inequality (4.6) is valid for $R(G, d, s, g)$ and $P(G, d, s, g)$.*

Proof. Since the LHS of (4.6) is bounded by 2, this inequality is satisfied by any feasible solution y with $y_{x_{32}} = 1$. So, let y be an integer solution with $y_{x_{32}} = 0$. In this case, (4.6) can only be violated in one of the following cases:

- **LHS = 1:** This can only happen in one of the following three situations:
  - $y_{x_{34}} = 1$, $y_{x_{35}} = 0$ and $y_{x_{21}} = 0$, but this amounts to the ordering $1 \to 2 \to 3 \to 4$, which is forbidden by the hypotheses.
  - $y_{x_{34}} = 0$, $y_{x_{35}} = 1$ and $y_{x_{21}} = 0$, but this yields the ordering $1 \to 2 \to 3 \to 5$, which again is forbidden by the hypotheses.
  - $y_{x_{34}} = 1$, $y_{x_{35}} = 1$ and $y_{x_{21}} = 1$, but this corresponds to the ordering $2 \to 3 \to 4 \to 5$, which cannot appear in a feasible solution.
- **LHS = 2:** This can only happen with $y_{x_{34}} = y_{x_{35}} = 1$ and $y_{x_{21}} = 0$, but this implies that y has the orderings $1 \to 2 \to 3 \to 4$ and $1 \to 2 \to 3 \to 5$, which are both forbidden by the hypotheses.

So, we can only have RHS = 0 when LHS = 0, thus verifying that (4.6) is a valid inequality for $P(G, d, s, g)$ and $R(G, d, s, g)$. □

Theorem 4.16 *If no feasible solution has the orderings $1 \to 2 \to 3 \to 4$, $1 \to 2 \to 3 \to 5$ and $2 \to 3 \to 4 \to 5$, the extended paw inequality (4.6) is facet-inducing for $R(G, d, s, g)$ and $P(G, d, s, g)$.*

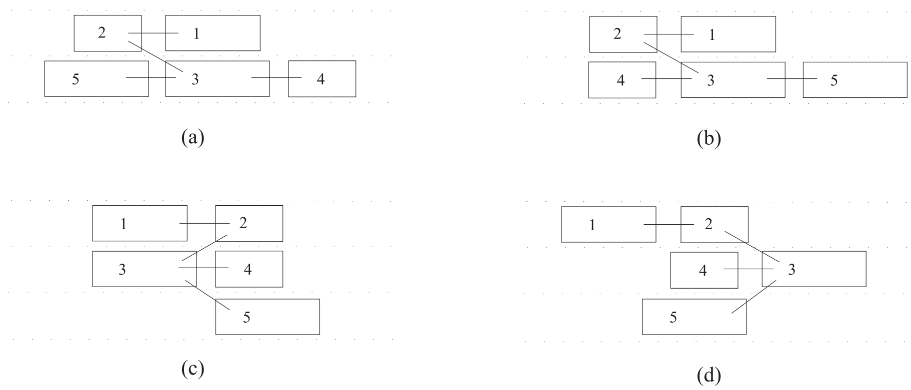


Figure 4.4: Feasible configurations for the proof of Theorem 4.16.

Proof. Consider all the possible configurations for the nodes 1 to 5 (i.e., excluding the forbidden orderings given by the hypotheses). There are 8 possible configurations, 4 of which satisfy (4.6) at equality and are depicted in Figure 4.4. The remaining 4 configurations (which do not satisfy (4.6) at equality) are exactly the symmetrical configurations, so Theorem 4.4 and Corollary 4.6 imply that this inequality defines a facet of $P(G, d, s, g)$ and $R(G, d, s, g)$. □

Chapter 5

Clique inequalities and facet-defining variants

> For a class of discrete problems, formulated in a natural way, one may hope then that equivalent linear constraints are pleasant enough though they are not explicit in the discrete formulation
>
> – Jack Edmonds (1965)

This chapter provides constructions of valid and facet-defining classes of inequalities derived from the interval bound constraints and the antiparallelity constraints, respectively. Section 5.1 presents the construction of the *clique inequalities* as a strengthening of the bound constraints for the interval variables. We prove that these new inequalities are facet-defining for $R(G, \mathbf{1}, s, 0)$ and $P(G, \mathbf{1}, s, 0)$ if $s \geq s_{\min}(G, d, 0) + 3$, and analyze a particular subclass, the *covering-clique inequalities*, that induces facets of nonuniform instances. We also address the associated separation problem.

Section 5.2 analyzes the antiparallelity constraints, showing that these inequalities do not define facets in general. We strengthen these inequalities with a clique structure, obtaining the so-called *double covering-clique inequalities*, being valid for $P(G, d, s, g)$ and $R(G, d, s, g)$. These inequalities are facet-inducing for $s \geq s_{\min}(G, d, 0) + 4d_{\max}$ but not for instances with small frequency span in general. We present further examples suggesting that instances with small frequency spans can have facet-defining inequalities with unusual structures.

Section 5.3 presents generalizations and extensions of the standard covering-clique inequalities. Section 5.3.1 and Section 5.3.2 provide two classes of facet-inducing inequalities generalizing the covering-clique inequalities, i.e., containing the covering-clique inequalities as special cases. Finally, we discuss in Section 5.3.3 three classes of facet-defining inequalities arising as variations of the double covering-clique inequalities.

5.1 Clique inequalities and covering-clique inequalities

The integer programming model for the bandwidth allocation problem in PMP-Systems includes the bound constraints, asserting $0 \leq l_i$ and $r_i \leq s$ for $i \in V$. The inequality $0 \leq l_i$ does not define a facet in general, since any feasible schedule $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ satisfying $z_{l_i} = 0$ must have $z_{x_{ij}} = 1$ for every $j \in N(i)$, implying that the corresponding face cannot have the required dimension for being a facet if the polytope is full-dimensional. The same argumentation applies to the opposite constraint.

However, we can strengthen the interval bound $0 \leq l_i$ by considering a neighbor of the node i. Let $j \in N(i)$ be such a neighbor and consider the following simple inequality:

$$d_j \; x_{ji} \;\leq\; l_i. \tag{5.1}$$

This inequality is clearly valid for $R(G, d, s, g)$ and $P(G, d, s, g)$, since $x_{ji} = 1$ implies that the interval $I(j)$ is located before the interval $I(i)$, and thus $l_i \geq d_j$. We can generalize this inequality by considering a clique K in $N(i) = \{j \in V : \; ij \in E\}$. As we shall see below, the resulting inequality is facet-inducing for $P(G, \mathbf{1}, s, 0)$ and $R(G, \mathbf{1}, s, 0)$ if K is maximal and s is large enough. However, this inequality does not define a facet of chromatic scheduling polytopes in the general case $d \geq \mathbf{1}$.

Definition 5.1 (clique inequalities) *If $i \in V$ and $K \subseteq N(i)$ is a clique of G, then we define*

$$\sum_{k \in K} d_k \; x_{ki} \;\leq\; l_i \tag{5.2}$$

to be the clique inequality *associated with i and K.*

Proposition 5.1 *The clique inequalities are valid for $R(G, d, s, g)$ and $P(G, d, s, g)$.*

Proof. Let $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be an integer feasible solution of $P(G, d, s, g)$. Let $L \subseteq K$ be the set of nodes $k \in K$ such that the interval $I(k)$ is located before $I(i)$. Since K is a clique, the intervals $\{I(k)\}_{k \in K}$ are pairwise disjoint, implying $z_{l_i} \geq \sum_{k \in L} d_k = \sum_{k \in K} z_{x_{ki}} d_k$. Hence the clique inequality (5.2) is valid for $P(G, d, s, g)$. Moreover, since this inequality does not involve the r-variables, it is also valid for $R(G, d, s, g)$. $\square$

Theorem 5.2 *Let $K \subseteq N(i)$ be a maximal clique in $N(i)$. If $s \geq s_{\min}(G, \mathbf{1}, 0) + 3$, then the clique inequality (5.2) defines a facet of $R(G, \mathbf{1}, s, 0)$ and $P(G, \mathbf{1}, s, 0)$.*

Proof. We already know that (5.2) is valid for $P(G, \mathbf{1}, s, 0)$ and $R(G, \mathbf{1}, s, 0)$, so it remains to show that the corresponding face F is maximal. To this end, suppose $\lambda^T z = \lambda_0$ for every $z \in P(G, d, s, 0)$ satisfying (5.2) at equality. We will show that (λ, λ_0) is in fact a multiple of (5.2), thus proving that this inequality induces a facet of $P(G, d, s, 0)$.

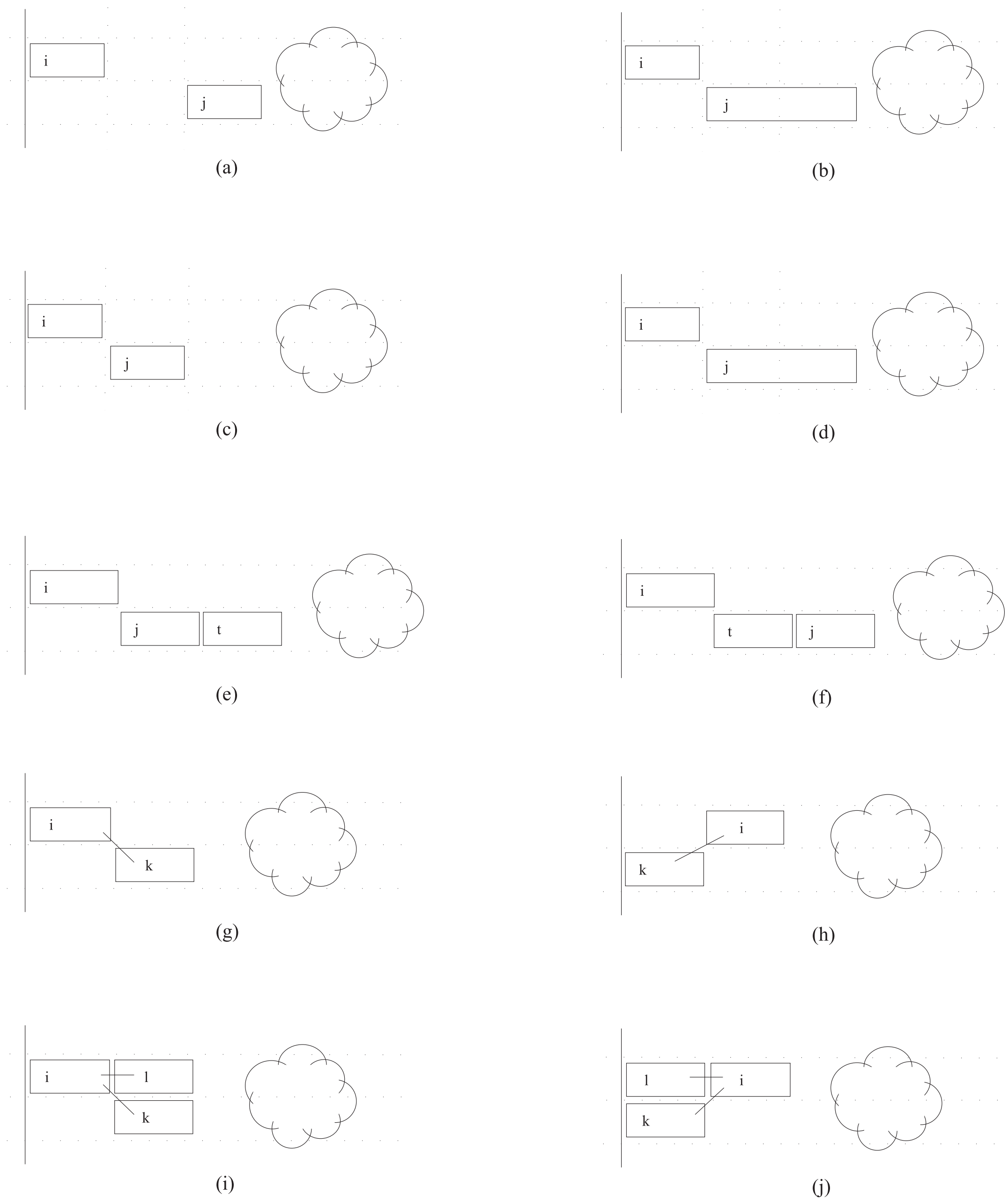
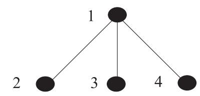
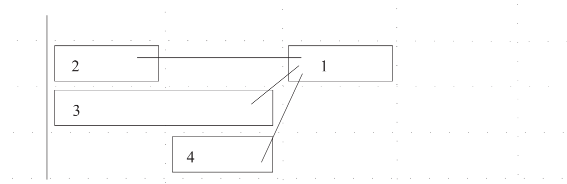


Figure 5.1: Constructions for the proof of Theorem 5.2.



(a)



(b)

Figure 5.2: Clique inequalities do not define facets in general.

Claim 1: $\lambda_{l_j} = 0$ for $j \neq i$. Consider the feasible schedules z and z' presented in Figure 5.1(a) and Figure 5.1(b), respectively. It is not difficult to verify that $z, z' \in F$ and, therefore, $\lambda^T z = \lambda_0 = \lambda^T z'$. Since these points only differ in their l_j-coordinate, $\lambda_{l_j} = 0$ follows. $\diamond$

Claim 2: $\lambda_{r_j} = 0$ for every $j \in V$. The feasible schedules presented in Figure 5.1(c) and Figure 5.1(d) satisfy (5.2) at equality, implying $\lambda_{r_j} = 0$. $\diamond$

Claim 3: $\lambda_{x_{jt}} = 0$ for every $jt \in E \backslash \delta(i)$. Consider now the feasible solutions presented in Figure 5.1(e) and Figure 5.1(f). Note that this construction is possible since $s \geq s_{\min}(G, \mathbf{1}, 0) + 3$. We know from the previous claims that $\lambda_{l_j} = \lambda_{r_j} = 0$ and $\lambda_{l_t} = \lambda_{r_t} = 0$, thus $\lambda_{x_{jt}} = 0$. $\diamond$

Claim 4: $\lambda_{x_{ik}} = -d_k \lambda_{l_i}$ for every $k \in K$. The feasible integer solutions depicted in Figure 5.1(g) and Figure 5.1(h) satisfy (5.2) at equality. Hence, $\lambda_{x_{jk}} = 0$. $\diamond$

Claim 5: $\lambda_{x_{il}} = 0$ for every $l \in N(i) \backslash K$. Since K is a maximal clique in $N(i)$, there exists some node in K, say node k, such that $lk \notin E$. Consider the feasible schedules in Figure 5.1(i) and Figure 5.1(j). Both lie in the face F defined by (5.2) and, therefore, $\lambda_{x_{il}} = 0$. $\diamond$

This sequence of claims shows that λ is a multiple of the coefficient vector of (5.2), hence this clique inequality induces a facet of $P(G, \mathbf{1}, s, 0)$. The same argumentation (omitting Claim 2) applies to $R(G, \mathbf{1}, s, 0)$. $\square$

If $A \subseteq V$, we denote by G_A the subgraph of G induced by A. Notice that $K \cup \{i\}$ is a maximal clique of G if and only if K is a maximal clique of $G_{N(i)}$. The inequality (5.2) is stronger than the inequality (5.1), but does not define a facet of the polytopes in the general case $d \geq \mathbf{1}$, even if K is a maximal clique.

Example 5.1 Consider the graph $K_{1,3}$ in Figure 5.2(a) (called "claw"), with node weights $d_1 = d_2 = d_4 = 1$ and $d_3 = 2$. The inequality $l_1 \geq x_{21}$ is a clique inequality (take $i = 1$ and $K = \{2\}$). No feasible solution satisfying this inequality at equality can have $x_{13} = 0$, since in this case we would have $l_1 \geq d_3 = 2 > x_{21}$ (see Figure. 5.2(b)). Therefore, $x_{13} = 1$ in every integer solution in the face defined by this inequality, and this shows that $l_1 \geq x_{21}$ is not facet-defining for $s \geq 4$. $\triangleleft$

In order to construct a class of facet-defining inequalities for the general case $d \geq \mathbf{1}$, we shall introduce the following definition.

Definition 5.2 (covering clique) *Let $A \subseteq V$, and let $K \subseteq A$ be a clique. We say that K* covers *A if every node $k \in A \backslash K$ satisfies $d_k \leq \sum_{i \in K \backslash N(k)} d_i$.*

Proposition 5.3 *Every node subset admits a covering clique, and such a clique can be found in polynomial time.*

Proof. Let $A \subseteq V$, and let $i_1, i_2, \ldots, i_n$ be an ordering of the nodes in A such that $d_{i_k} \geq d_{i_{k+1}}$. Consider every node in this sequence and construct K iteratively as follows. At step k, we must decide whether i_k has to be inserted into K or not. If there is some $i_t \in K$ with $i_k i_t \notin E$, then do not insert i_k into K. Otherwise, insert i_k into K. Note that in both cases K is a covering clique of $\{i_1, \ldots, i_k\}$ due to the ordering of the nodes, so upon termination of the algorithm K is a clique covering A. This procedure gives an $O(m + n \log n)$ algorithm. □

Definition 5.3 (covering-clique inequalities) *Let $i \in V$ be a node of G, and let K be a clique covering $N(i)$. We define*

$$\sum_{k \in K} d_k \ x_{ki} \ \leq \ l_i \tag{5.3}$$

to be the covering-clique inequality *associated with i and K.*

Covering-clique inequalities are, as special clique inequalities, valid for $P(G, d, s, g)$ and $R(G, d, s, g)$ by Lemma 5.1 and define facets if s is large enough.

Theorem 5.4 *If $s \geq s_{\min}(G, d, 0) + 3d_{\max}$, then the covering-clique inequalities (5.3) define facets of $P(G, d, s, 0)$ and $R(G, d, s, 0)$.*

Proof. To prove that covering-clique inequalities are facet-inducing, suppose that $\lambda^T z = \lambda_0$ for every $z \in P(G, d, s, 0) \cap \mathbf{Z}^{2n+m}$ satisfying (5.3). Claims 1, 2 and 3 from the proof of Theorem 5.2 imply $\lambda_{l_j} = 0$ for every $j \neq i$, $\lambda_{r_j} = 0$ for every $j \in V$, and $\lambda_{x_{jt}} = 0$ for $jt \notin \delta(i)$. Moreover, Claim 4 from Theorem 5.2 implies $\lambda_{x_{ik}} = -d_k \lambda_{l_i}$ for every $k \in K$.

So it is left to verify $\lambda_{x_{ij}} = 0$ for every $j \in S = N(i) \backslash K$. To this end, consider a node set $U_j \subseteq K \backslash N(j)$ such that $d_j \leq d(U_j)$ (note that such a set U_j exists by the construction of the covering clique K). The feasible schedule z resp. z' depicted in Figure 5.3(a) resp. Figure 5.3(b) satisfies (5.3) at equality. Hence

$$\begin{aligned} 0 &= \lambda_{x_{ji}} + \sum_{k \in U_j} \lambda_{x_{ki}} + z'_{l_i} \lambda_{l_i} \\ &= \lambda_{x_{ji}} + \sum_{k \in U_j} (-d_k \lambda_{l_i}) + \sum_{k \in U_j} d_k \lambda_{l_i} \\ &= \lambda_{x_{ji}} \end{aligned}$$

shows that (λ, λ_0) is a multiple of the coefficient vector of inequality (5.3) and, therefore, this inequality defines a facet of $P(G, d, s, 0)$. The same argumentation applies to $R(G, d, s, 0)$. □

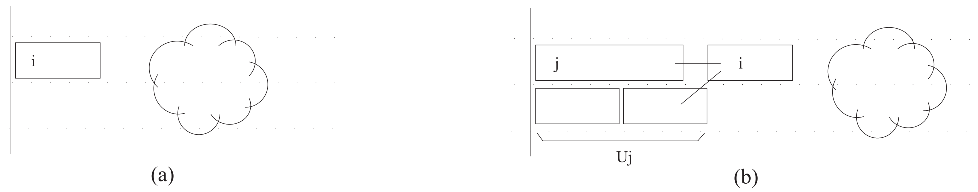


Figure 5.3: Constructions for the proof of Theorem 5.4.

Remark. An alternative proof can be given for Theorem 5.4 by considering the interval bound constraint $0 \leq l_i$ and lifting the variables x_{ki} for (a) $k \in K$ and (b) $k \in N(i)\backslash K$. The interval bound is facet-inducing for $P(G,d,s,0) \cap \{z \in \mathbf{R}^{2n+m} : z_{ki} = 0 \ \forall k \in N(i)\}$. Moreover, the maximum lifting coefficient for the variable x_{ki} is d_k if $k \in K$ and 0 otherwise, implying that the resulting covering-clique inequality is facet-inducing for $P(G,d,s,0)$. These maximum lifting coefficients are independent of the order in which the variables are lifted, provided the variables x_{ki} with $k \in K$ are lifted before the variables x_{ki} for $k \notin K$. This procedure provides a natural view of covering-clique inequalities as a strengthening of the interval bound constraints. $\lhd$

Recall from Section 4.1 that the symmetric inequality of a facet-inducing inequality is again a facet-inducing inequality. The following corollary presents the symmetric construction of covering-clique inequalities.

Corollary 5.5 *Let $i \in V$ be a node of G, and let K be a clique covering $N(i)$. The following inequality is valid for $P(G,d,s,0)$:*

$$r_i \ \leq \ s - \sum_{k \in K} d_k \ x_{ik} \tag{5.4}$$

Moreover, if $s \geq s_{\min}(G,d,0) + 3d_{\max}$, then this inequality defines a facet of $P(G,d,s,0)$. The same holds for $R(G,d,s,0)$ if we replace r_i by $l_i + d_i$ in (5.4).

Remark. The covering-clique inequalities (5.3) describe the relation between the left bound of the interval $I(i)$ and the left bound on the frequency span $[0,s]$. The corresponding symmetric inequalities (5.4) describe the opposite relation between the right bound of $I(i)$ and the right bound of the frequency span. $\lhd$

Covering-clique inequalities are facet-defining in many cases, but unfortunately there are instances where they do not induce facets, as the following example shows. The construction presented in this counterexample is rather involved, suggesting that instances without covering-clique facets may be unusual.

Example 5.2 Let G be the graph depicted in Figure 5.4, with node set $V = \{1,\ldots,13\}$ and the following node weights:

$$\begin{aligned} d_1,\ldots,d_5 &= 1 \\ d_6, d_9 &= 4 \end{aligned}$$

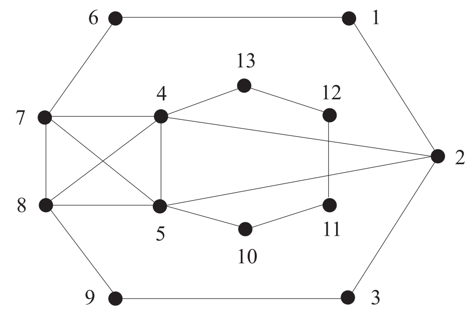


Figure 5.4: Counterexample for general facetness of covering-clique inequalities.

$$
\begin{aligned}
d_7, d_8 &= 1 \\
d_{10}, d_{13} &= 3 \\
d_{11}, d_{12} &= 2
\end{aligned}
$$

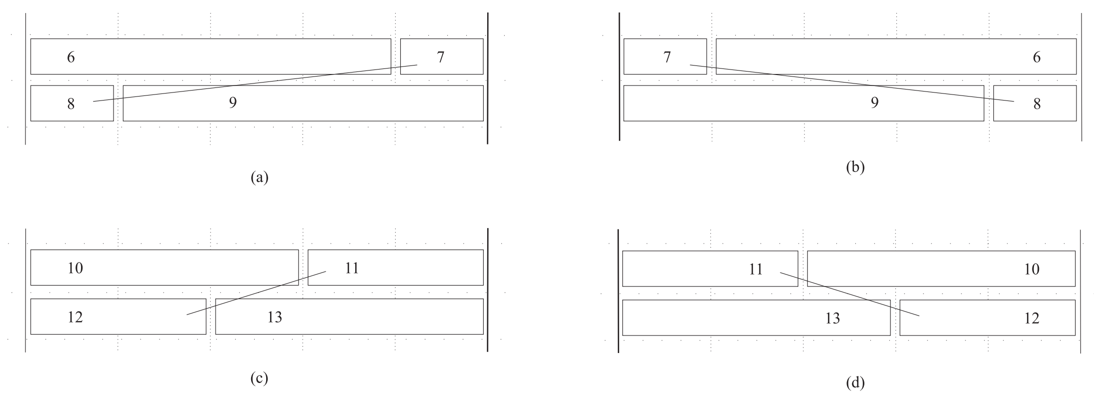


Figure 5.5: Possible configurations for intervals $I(6)$ to $I(13)$.

Consider the nonempty polytope $P(G, d, 5, 0)$. Let $y \in P(G, d, 5, 0) \cap \mathbf{Z}^{31}$ be a feasible solution. Due to $d_6 + d_7 = 5$ and $67 \in E$, there are only two possible assignments for the interval $I(6)$, namely $l_6 = 0$ or $l_6 = 5$. Moreover, each of these assignments completely determines the positions of the intervals $I(7)$, $I(8)$ and $I(9)$. Thus, the intervals $I(6), \ldots, I(9)$ only admit the two possible configurations depicted in Figure 5.5(a) and Figure 5.5(b). A similar analysis applies to the intervals $I(10), \ldots, I(13)$, which only admit the two possible configurations presented in Figure 5.5(c) and Figure 5.5(d).

Hence we can assign the intervals $I(6), \ldots, I(9)$ according to two possible configurations, and the intervals $I(10), \ldots, I(13)$ according to two other configurations. Moreover, these configurations uniquely determine the positions of intervals $I(1), \ldots, I(5)$, so that $P(G, d, 5, 0)$ has only the 4 feasible solutions presented in Figure 5.6.

Consider now the following covering-clique inequality, being valid for $P(G, d, 5, 0)$:

$$
l_2 \geq x_{42} + x_{52} \tag{5.5}
$$

Having listed all the feasible solutions of $P(G, d, 5, 0)$, it is not difficult to verify that the polytope $P(G, d, 5, 0)$ has dimension 2, but only the feasible schedule presented in Figure

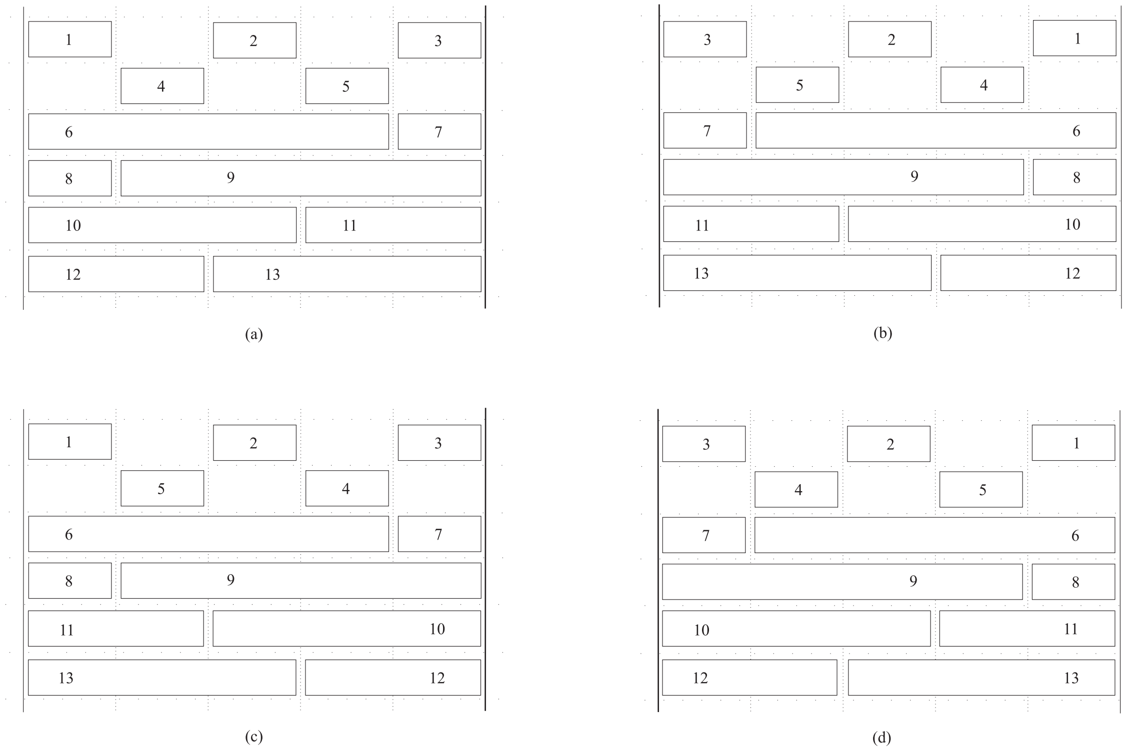


Figure 5.6: Feasible solutions of $P(G, d, 5, 0)$.

5.6(a) satisfies (5.5) at equality, and thus the face defined by (5.5) has dimension 0. Therefore, this covering-clique inequality does not define a facet of $P(G, d, 5, 0)$. $\triangleleft$

5.1.1 Complexity of the separation problem

Given a point in the linear relaxation of an integer programming model, the separation problem for a family of valid inequalities consists in deciding whether this point violates some inequality belonging to the family or not. This problem is of practical interest, since efficient separation procedures are required for the implementation of cutting plane methods. This section explores the separation problem for covering-clique inequalities, and the main theorem states the negative result that this problem is $\mathcal{NP}$-complete. If $P_{LP}(G, d, s, g)$ denotes the linear relaxation of $P(G, d, s, g)$, i.e., the solution space of constraints (2.1)-(2.6), then the separation problem for covering-clique inequalities can be defined as follows.

	COVERING-CLIQUE INEQUALITIES SEPARATION
Instance:	A point $y \in P_{LP}(G, d, s, 0)$
Question:	Does y violate some covering-clique inequality?

Note that the separation problem takes as input a point in the linear relaxation of the integer programming model, since this is the common situation within a branch&cut framework.

Moreover, note that the separation of the constraints (2.1)-(2.6) can be performed in $O(n+m)$ time by exhaustive inspection. The proof of $\mathcal{NP}$-completeness for this separation problem involves MAX-CLIQUE and a special case of this problem, called MAX MAJORITY-CLIQUE.

	MAX-CLIQUE
Instance:	A graph G on n nodes, and an integer $k \geq 0$
Question:	Does G contain a clique of size k or greater?

	MAX MAJORITY-CLIQUE
Instance:	A graph G on n nodes, and an integer $k \geq n/2+1$ (we may assume w.l.o.g. that $n \geq 2$ and $k \leq n$)
Question:	Does G contain a clique of size k or greater?

We denote by $\omega(G)$ the clique number of G, i.e., the size of a clique of G of maximum cardinality. Note that MAX-CLIQUE and MAX MAJORITY-CLIQUE consist in deciding whether $\omega(G) \geq k$ or not, but under different conditions. MAX-CLIQUE is a well-known $\mathcal{NP}$-complete problem [20], and we now prove that MAX MAJORITY-CLIQUE is also $\mathcal{NP}$-complete.

Lemma 5.6 MAX MAJORITY-CLIQUE *is $\mathcal{NP}$-complete.*

Proof. Note that the set of instances of MAX MAJORITY-CLIQUE is contained in the set of instances of MAX-CLIQUE, and since the latter belongs to $\mathcal{NP}$, then MAX MAJORITY-CLIQUE also belongs to $\mathcal{NP}$. To prove $\mathcal{NP}$-completeness, we construct a polynomial reduction from MAX-CLIQUE. Let (H,t) be an instance of MAX-CLIQUE, and define an instance (G,k) of MAX MAJORITY-CLIQUE as follows. The graph G is constructed from H by adding $m+2$ universal nodes $u_1,\ldots,u_{m+2}$, and k is defined as $k=t+m+2$. Note that G has $n=2m+2$ nodes and $k > n/2+1$. We finally verify that $\omega(H) \geq t$ if and only if $\omega(G) \geq k$.

- $\Rightarrow$) If $\omega(H) \geq t$, then H has a t-clique K, and it can be extended to the $(t+m+2)$-clique $K \cup \{u_1,\ldots,u_{m+2}\}$ of G. Hence G has a k-clique and so $\omega(G) \geq k$.
- $\Leftarrow$) Conversely, suppose that $\omega(G) \geq k$ and let K be a k-clique of G. Therefore, the node set $K \backslash \{u_1,\ldots,u_{m+2}\}$ is a clique of H with at least $k-(m+2)=t$ nodes, so $\omega(H) \geq t$.

Thus, MAX MAJORITY-CLIQUE is $\mathcal{NP}$-complete. □

Theorem 5.7 COVERING-CLIQUE INEQUALITIES SEPARATION *is $\mathcal{NP}$-complete.*

Proof. It is not difficult to verify that the problem belongs to the class $\mathcal{NP}$, since we can nondeterministically generate a clique K and verify in polynomial time whether K is a covering clique and the clique inequality defined by K is violated by y. To complete the proof we construct a polynomial reduction from MAX MAJORITY-CLIQUE. Let (H,k) be an instance of MAX MAJORITY-CLIQUE, given by a graph H on n nodes and an integer $k > n/2$. Define a new weighted graph $(G, d) = (V, E, \mathbf{1})$ from H by the addition of a universal node, i.e.,

$$\begin{aligned} V &= V_H \cup \{i\} \\ E &= E_H \cup \{ij : j \in V_H\} \end{aligned}$$

Set further $g = 0$ and $s = n/2 + 1$. Finally, construct the point $y \in P_{LP}(G, d, s, 0)$ as follows:

$$\begin{aligned} y_{l_j} &= \begin{cases} \frac{n}{2} & \text{if } j \neq i \\ \frac{k-1}{2} & \text{if } j = i \end{cases} \qquad \forall\, j \in V \\ y_{r_j} &= y_{l_j} + d_j \qquad \forall\, j \in V \\ y_{x_e} &= 1/2 \qquad \forall\, e \in E \end{aligned}$$

This construction is polynomial in the size of H. To show that (G, k) is a well-defined instance of COVERING-CLIQUE INEQUALITIES SEPARATION we must verify that $y \in P_{LP}(G, d, s, 0)$ by checking that y satisfies all the constraints of this relaxed polytope.

a) We first verify that the antiparallelity constraints $l_j + d_j \leq l_k + s x_{kj}$ are satisfied by y, considering the following three cases:

Case 1: $\boldsymbol{j, k \neq i}$. (recall that $n \geq 2$)

$$y_{l_j} + d_j \;=\; \frac{n}{2} + 1 \;\leq\; \frac{n}{2} + \frac{n/2+1}{2} \;=\; y_{l_k} + s y_{x_{kj}}$$

Case 2: $\boldsymbol{j \neq i}$ and $\boldsymbol{k = i}$. (recall that the hypothesis of MAX MAJORITY-CLIQUE asserts $k > n/2 + 1$)

$$y_{l_j} + d_j \;=\; \frac{n}{2} + 1 \;\leq\; \frac{k-1}{2} + \frac{n/2+1}{2} \;=\; y_{l_i} + s y_{x_{ij}}$$

Case 3: $\boldsymbol{j = i}$ and $\boldsymbol{k \neq i}$.

$$y_{l_i} + d_i \;=\; \frac{k-1}{2} + 1 \;\leq\; \frac{n}{2} + \frac{n/2+1}{2} \;=\; y_{l_k} + s y_{x_{ki}}$$

b) The bounds $0 \leq l_k \leq s - d_k$ on variables l_k are trivially satisfied, since

$$\max\{y_{l_k} : k \in V\} \;=\; \frac{n}{2} \;\leq\; \frac{n}{2} + 1 \;=\; s - d_k.$$

c) The relaxed constraints $0 \leq x_e \leq 1$ are also satisfied, since $y_{x_e} = 1/2$ for all $e \in E$.

To complete the proof, we show that $\omega(H) \geq k$ if and only if there exists some covering-clique inequality violated by y.

$\Rightarrow$) If $\omega(H) \geq k$, let $K \subseteq V_H$ be a maximum k-clique of H. Since i is a universal node of G, then $K \subseteq N_G(i)$, and moreover $d = \mathbf{1}$ implies that K covers $N_G(i)$. Hence the covering-clique inequality defined by K is violated by y:

$$\sum_{k \in K} d_k y_{x_{ki}} = \frac{|K|}{2} > \frac{k-1}{2} = y_{l_i}$$

$\Leftarrow$) Conversely, suppose that the covering-clique inequality defined by the node j and the covering clique $K \subset N_G(j)$ is violated by y, i.e.,

$$\sum_{k \in K} d_k y_{x_{kj}} \; > \; y_{l_j} \tag{5.6}$$

holds. Note that the LHS of this inequality is $\sum_{k \in K} d_k y_{x_{kj}} = \frac{1}{2}|K|$. This implies $j = i$, for otherwise $l_j = \frac{n}{2}$, and thus (5.6) would not be violated (because $|K| \leq n$). Hence $j = i$ and thus $K \subseteq N_G(i)$, implying that K is a clique of H. But $y_{l_i} = \frac{k}{2}$ and, therefore, (5.6) reads:

$$\frac{|K|}{2} = \sum_{k \in K} d_k y_{x_{ki}} \; > \; y_{l_i} = \frac{k-1}{2}$$

Thus, $|K| \geq k$, and so $\omega(H) \geq k$.

This finally shows that the polynomial transformation maps affirmative instances of MAX MAJORITY-CLIQUE onto affirmative instances of COVERING-CLIQUE INEQUALITIES SEPARATION and conversely. Therefore, the latter is $\mathcal{NP}$-complete. □

5.1.2 Covering-clique inequalities in the case $g > 0$

The covering-clique inequalities (5.3) are valid for every instance, but Theorem 5.4 shows facetness only if $g = 0$. In the case $g > 0$ these inequalities remain valid but may no longer be facet-defining if the associated covering clique covers nodes in more than one sector. In this setting a more general version of covering-clique inequalities can be given, and this section is devoted to presenting these general inequalities.

Definition 5.4 *For $i \in V$, let $a(i)$ denote the sector to which node i belongs (i.e., $i \in S_{a(i)}$).*

Definition 5.5 (general covering-clique inequalities) *Fix an arbitrary node $i \in V$ and let K be a clique covering $N(i)$. Assume w.l.o.g. that $K = \{1, \ldots, t\}$ and, for $k = 1, \ldots, t$, let $A_k = \{i\} \cup \{1, \ldots, k-1\}$. Partition the clique K into $K = N \cup C$, with*

$$\begin{aligned} N &= \{k \in K : a(k) \neq a(t) \text{ for every } t \in A_k\} \\ C &= \{k \in K : a(k) = a(t) \text{ for some } t \in A_k\} \end{aligned}$$

We define

$$\sum_{k \in N} (d_k + g) x_{ki} + \sum_{k \in C} d_k x_{ki} \; \leq \; l_i \tag{5.7}$$

to be the general covering-clique inequality *associated with the node i, the clique K and the ordering $K = \{1, \ldots, t\}$.*

The proof of facetness for the general covering-clique inequalities goes along the argumentation of the proof of facetness for the standard covering-clique inequalities presented in Theorem 5.4.

Theorem 5.8 *The general covering-clique inequalities (5.7) are valid for* $P(G, d, s, g)$ *and* $R(G, d, s, g)$*, and define facets for both polytopes if* $s \geq s_{\min}(G, d, g) + 3(d_{\max} + g)$*.*

Under the same setting as before, the following symmetric inequality

$$r_i \;\leq\; s - \sum_{k \in N} (d_k + g) x_{ik} - \sum_{k \in C} d_k x_{ik}$$

is valid for $P(G, d, s, g)$ and facet-inducing if $s \geq s_{\min}(G, d, g) + 3(d_{\max} + g)$. The same result holds for $R(G, d, s, g)$ if we replace r_i by $l_i + d_i$.

Remark. These general inequalities arise as a natural strengthening of the interval bound constraints $0 \leq l_i$ for every $i \in V$, by lifting the variables x_{ki}, for $k \in N(i)$. In the case $g = 0$, we first lift the variables x_{ki} for $k \in K$, and afterwards we lift the variables x_{ki} for $k \notin K$. The lifting of variables x_{ki} for $k \in K$ resp. $k \notin K$ is sequence-independent and originates the standard covering-clique inequalities (5.3). In the case $g > 0$, however, the lifting is not independent of the sequence, requiring different definitions for the coefficients for $k \in N$ and $k \in C$. $\lhd$

5.2 Double covering-clique inequalities

We now turn to the antiparallelity constraints. Recall that these constraints are given by the following inequalities:

$$\begin{array}{rcll} r_i & \leq & l_j + s(1 - x_{ij}) & \forall ij \in E_I,\ i < j \qquad (2.4) \\ r_i + g & \leq & l_j + s(1 - x_{ij}) & \forall ij \in E_X,\ i < j \qquad (2.5) \\ r_j & \leq & l_i + s x_{ij} & \forall ij \in E_I,\ i < j \qquad (2.6) \\ r_j + g & \leq & l_i + s x_{ij} & \forall ij \in E_X,\ i < j \qquad (2.7) \end{array}$$

Proposition 5.9 *Every point* $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ *satisfying the antiparallelity constraint (2.4) at equality must have* $z_{x_{ik}} - z_{x_{jk}} = -z_{x_{ji}}$ *for every* $k \in N(i) \cap N(j)$*.*

Proof. Let $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be a point satisfying (2.4) at equality, and let $k \in N(i) \cap N(j)$.

Case 1: $z_{x_{ij}} = 1$. Since z satisfies (2.4) at equality, we have $z_{r_i} = z_{l_j}$, implying $z_{x_{ki}} = z_{x_{kj}}$ and hence $z_{x_{ik}} - z_{x_{jk}} = 0 = -z_{x_{ji}}$.

Case 2: $z_{x_{ij}} = 0$. In this case, we have $z_{r_i} = s$ and $z_{l_j} = 0$, implying $z_{x_{ik}} = 1$ and $z_{x_{jk}} = 0$. Therefore, $z_{x_{ik}} - z_{x_{jk}} = 1 = -z_{x_{ji}}$. $\square$

If $P(G, d, s, g)$ is full-dimensional, then this proposition shows that the face defined by (2.4) cannot have the required dimension for being a facet. The same is true for the other antiparallelity constraints, showing that these inequalities do not define facets of $P(G,d,s,g)$ for arbitrary instances if $N(i) \cap N(j) \neq \emptyset$.

Fortunately, we can strengthen these inequalities by considering a covering clique in the common neighborhood of the nodes whose intervals are separated by the constraint. This process can be viewed as a lifting from the antiparallelity constraints into a new class of facet-defining inequalities, resembling the covering-clique inequalities presented in the previous section. The resulting inequalities describe the interaction between these two nodes, involving many similarities with the construction of covering-clique inequalities.

Definition 5.6 (double covering-clique inequalities) *Let $ij \in E$ be an edge of G, and let K be a clique covering $N(i) \cap N(j)$. We define*

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_j + (s - d(K))x_{ji} \tag{5.8}$$

to be the double covering-clique inequality *associated with ij and K, where $d(K) = \sum_{k \in K} d_k$.*

Proposition 5.10 *The double covering-clique inequalities (5.8) are valid for $P(G,d,s,g)$.*

Proof. Let $y \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ be a feasible integer solution, and consider two cases:

Case 1: $\boldsymbol{y_{x_{ji}} = 0}$. In this case, the interval $I(i)$ is located to the left of $I(j)$. Let $M \subseteq K$ be the set of nodes k such that the interval $I(k)$ is between the intervals $I(i)$ and $I(j)$, i.e., $M = \{k \in K : y_{x_{ik}} = 1 \text{ and } y_{x_{jk}} = 0\}$. Since $K \cup \{i,j\}$ is a clique, then the corresponding intervals cannot overlap, and thus $y_{l_j} - y_{r_i} \geq d(M)$, implying that y satisfies (5.8). $\diamond$

Case 2: $\boldsymbol{y_{x_{ji}} = 1}$. Here, the interval $I(j)$ is before $I(i)$. Partition $K = L \cup M \cup R$ as follows:

$$\begin{aligned} L &= \{k \in K : y_{x_{jk}} = 0\} \\ M &= \{k \in K : y_{x_{jk}} = 1 \text{ and } y_{x_{ik}} = 0\} \\ R &= \{k \in K : y_{x_{ik}} = 1\} \end{aligned}$$

Note that $d(L) \leq y_{l_j}$ and $y_{r_i} \leq s - d(R)$. Moreover, $\sum_{k \in K} d_k(y_{x_{ik}} - y_{x_{jk}}) = -d(M)$. These observations imply

$$\begin{aligned} y_{r_i} - y_{l_j} + \sum_{k \in K} d_k(y_{x_{ik}} - y_{x_{jk}}) &\leq s - d(R) - d(L) - d(M) \\ &= s - d(K). \diamond \end{aligned}$$

Since y was arbitrarily chosen, (5.8) is valid for $P(G,d,s,g)$. $\square$

Theorem 5.11 *If $s \geq s_{\min}(G,d,0) + 4d_{\max}$, then the double covering-clique inequalities (5.8) define facets of $P(G,d,s,0)$.*

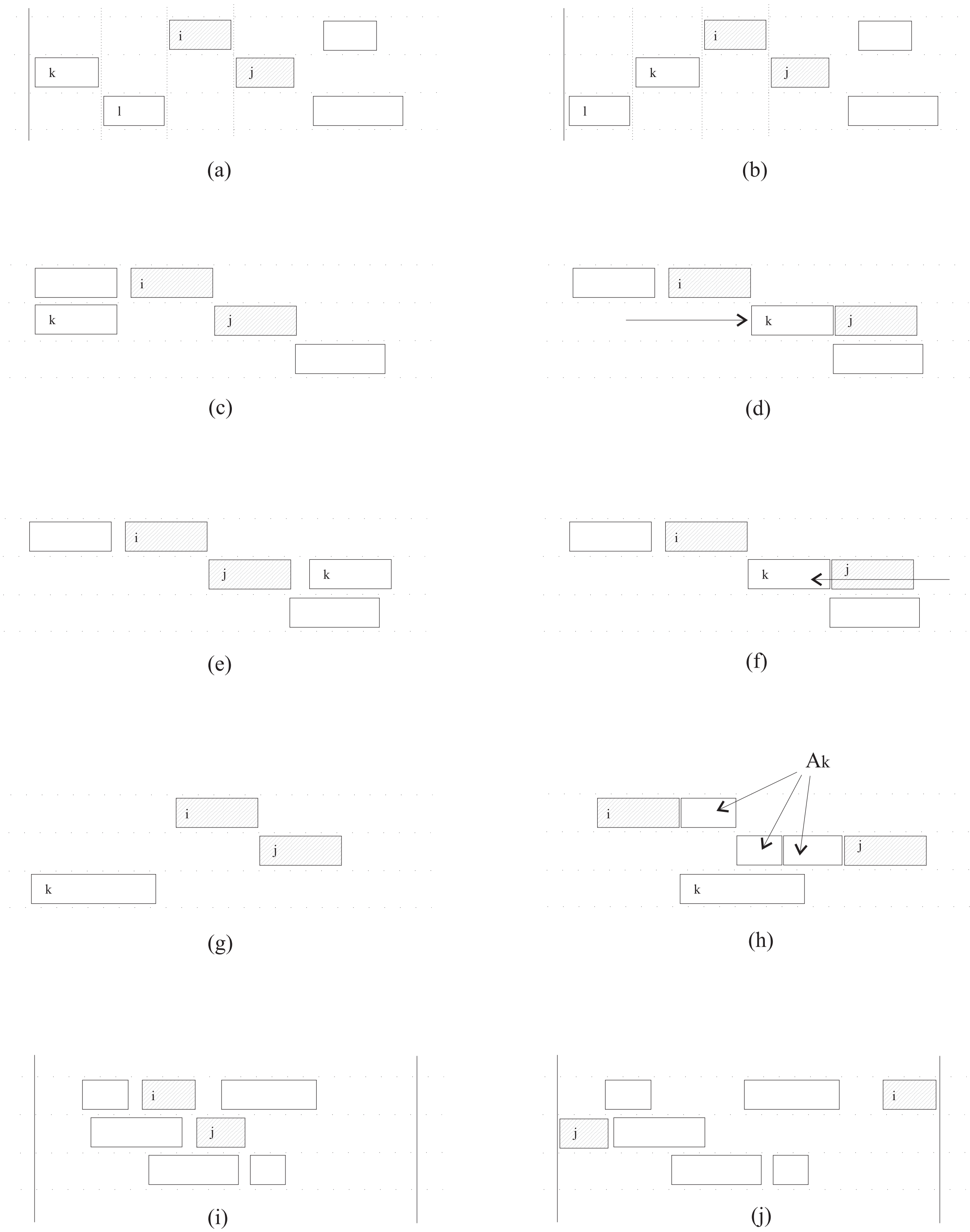


Figure 5.7: Constructions for the proof of Theorem 5.11.

Proof. By Proposition 5.10, the double covering-clique inequalities are valid for $P(G, d, s, 0)$. We now prove that, under these hypotheses, they define facets of this polytope. Note first that any feasible solution satisfying $l_j = r_i$ is tight for inequality (5.8). Such points exist whenever $s \geq s_{\min}(G, d, 0) + 2d_{\max}$, hence this inequality defines a nonempty face in this case. Let F be the face of $P(G, d, s, 0)$ defined by (5.8), and suppose that $\lambda^T y \leq \lambda_0$ defines a facet containing F. We will show that (λ, λ_0) is in fact a multiple of (5.8), thus proving that this inequality is facet-inducing, i.e., that F is not contained in any other facet. To this end, we prove the following sequence of claims:

Claim 1: $\boldsymbol{\lambda_{l_k} = 0}$ for $\boldsymbol{k \neq j}$. Let $k \neq j$ and let $y \in F$ be an integer solution with $y_{r_k} - y_{l_k} > d_k$ (which exists because $s > s_{\min}(G, d, 0) + 2d_{\max}$). Define y' to be the solution obtained from y by just setting $y'_{l_k} = y_{l_k} + 1$. Note that this new solution is feasible. Both points lie in F, implying $\lambda^T y = \lambda^T y' = \lambda_0$. Moreover, they only differ in their l_k-coordinates, hence

$$\lambda_{l_k} y_{l_k} = \lambda_{l_k} y'_{l_k} = \lambda_{l_k}(y_{l_k} + 1).$$

Thus $\lambda_{l_k} = 0$, proving the claim. $\diamond$

Claim 2: $\boldsymbol{\lambda_{r_k} = 0}$ for $\boldsymbol{k \neq i}$. A similar construction, with points $y, y' \in F$ such that $y_{r_k} - y_{l_k} > d_k$ and $y'_{r_k} = y_{r_k} - 1$ shows that $\lambda_{r_k} = 0$ for $k \neq i$. $\diamond$

Claim 3: $\boldsymbol{\lambda_{x_{kt}} = 0}$ if both $\boldsymbol{k, t}$ differ from $\boldsymbol{i, j}$. Let $y \in F$ be a feasible solution with $y_{l_k} = 0$, $y_{l_t} = d_k$, and all the remaining intervals to the right of $I(k)$ (such a y exists by $s \geq s_{\min}(G, d, 0) + 4d_{\max}$). Let y' be a new feasible solution obtained from y by switching the intervals $I(k)$ and $I(l)$ (see Figure 5.7(a), (b)). Both solutions are in F, and thus $\lambda^T y = \lambda^T y'$. These two feasible solutions only differ in their l_k-, l_t-, r_k-, r_t- and x_{kt}-coordinates. Moreover, we know from the previous claims that $\lambda_{l_k} = \lambda_{r_k} = \lambda_{l_t} = \lambda_{r_t} = 0$, implying $\lambda_{x_{kt}} = 0$. $\diamond$

Claim 4: $\boldsymbol{\lambda_{r_i} = -\lambda_{l_j}}$. Let $y \in F$ be a feasible solution with $y_{r_i} = y_{l_j}$, such that both intervals $I(i)$ and $I(j)$ can be moved one unit to the right (this is possible since $s > s_{\min}(G, d, 0) + 2d_{\max}$). Let y' be the solution obtained by this shifting. Since both solutions are in F and $\lambda_{l_i} = \lambda_{r_j} = 0$, we obtain

$$\lambda_{r_i} y_{r_i} + \lambda_{l_j} y_{l_j} \;=\; \lambda_{r_i}(y_{r_i} + 1) + \lambda_{l_j}(y_{l_j} + 1).$$

This implies that $\lambda_{r_i} + \lambda_{l_j} = 0$, thus justifying the claim. $\diamond$

Claim 5: $\boldsymbol{\lambda_{x_{ik}} = d_k \lambda_{r_i}}$ for $\boldsymbol{k \in K}$. Let y be an integer point in F with $y_{r_i} = y_{l_j}$, and let y' be a feasible solution with only intervals k and j changed in such a way that $y'_{l_k} = y_{r_i}$ and $y'_{l_j} = y'_{r_k} = y'_{l_k} + d_k$ (see Figure 5.7(c) and Figure 5.7(d)). This construction is possible since $s > s_{\min}(G, d, 0) + d_i + d_j + d_k$. Both solutions lie in F, so $\lambda^T y = \lambda^T y' = \lambda_0$, and thus

$$\lambda_{l_k} y_{l_k} + \lambda_{l_j} y_{l_j} \;=\; \lambda_{l_k} y'_{l_k} + \lambda_{l_j} y'_{l_j} + \lambda_{x_{ik}}.$$

But $\lambda_{l_k} = 0$ and $y'_{l_j} = y_{r_i} + d_k$ imply $\lambda_{x_{ik}} = d_k \lambda_{r_i}$, proving the claim. $\diamond$

Claim 6: $\boldsymbol{\lambda_{x_{jk}} = -d_k \lambda_{r_i}}$ for $\boldsymbol{k \in K}$. A similar construction verifies this claim, by considering the solutions presented in Figure 5.7(e), (f). $\diamond$

Claim 7: $\boldsymbol{\lambda_{x_{ik}} = \lambda_{x_{jk}} = 0}$ for $\boldsymbol{k \in [N(i) \cap N(j)] \backslash K}$. Let $A_k \subset K$ be a set of nodes not adjacent to k such that $d(A_k) \geq d_k$. Such a set exists by the definition of the covering clique

K of $N(i) \cap N(j)$. The two feasible solutions depicted in Figure 5.7(g) and Figure 5.7(h) show that $\lambda_{x_{ik}} = 0$, and the opposite construction implies $\lambda_{x_{jk}} = 0$. $\diamond$

Claim 8: $\boldsymbol{\lambda_{x_{ik}} = 0}$ for $\boldsymbol{k \in N(i) \backslash N(j)}$. Let $y \in F$ be a solution with $y_{l_i} = 0$, $y_{l_j} = d_i$, $y_{l_k} = d_i + d_j$ and $y_{l_t} \geq d_i + d_j + d_k$ for $t \notin \{i, j, k\}$. Construct a new solution $y' \in F$ from y by setting $y'_{l_k} = 0$, $y'_{l_i} = d_k$ and $y'_{l_j} = d_k + d_i$. Since both solutions are tight for F, we conclude that $\lambda_{x_{ik}} = 0$. $\diamond$

Claim 9: $\boldsymbol{\lambda_{x_{jk}} = 0}$ for $\boldsymbol{k \in N(j) \backslash N(i)}$. If k is adjacent to j and not adjacent to i, the construction applied in Claim 8 also shows $\lambda_{x_{jk}} = 0$. $\diamond$

Claim 10: $\boldsymbol{\lambda_0 = 0}$ and $\boldsymbol{\lambda_{x_{ji}} = -(s - d(K))\lambda_{r_i}}$. Let $y \in F$ be any integer solution with $y_{r_i} = y_{l_j}$, and let y' be a solution with $y'_{l_i} = s - d_i$ and $y'_{l_j} = 0$ (and thus $y'_{x_{ji}} = 1$), as in Figure 5.7(i) and Figure 5.7(j). Note that $y_{x_{ik}} - y_{x_{jk}} = 0$, $y'_{x_{ik}} = 0$, and $y'_{x_{jk}} = 1$ for $k \in N(i) \cap N(j)$. This implies that y' satisfies (5.8) at equality, and, therefore, $y' \in F$. Moreover, we have that

$$\begin{aligned}
\lambda_0 \;=\; \lambda^T y \quad &= \quad \lambda_{r_i} y_{r_i} + \lambda_{l_j} y_{l_j} + \sum_{k \in K} (\lambda_{x_{ik}} y_{x_{ik}} + \lambda_{x_{jk}} y_{x_{jk}}) \;= \\
&= \quad \lambda_{r_i} \underbrace{(y_{r_i} - y_{r_i})}_{=\,0} + \sum_{k \in K} d_k \lambda_{r_i} \underbrace{(y_{x_{ik}} - y_{x_{jk}})}_{=\,0} \;=\; 0 \\
\lambda^T y' \quad &= \quad \lambda_{r_i} y'_{r_i} + \lambda_{l_j} y'_{l_j} + \sum_{k \in K} \lambda_{x_{jk}} y'_{x_{jk}} + \lambda_{x_{ji}} y'_{x_{ji}} \;= \\
&= \quad \lambda_{r_i} s + \sum_{k \in K} \lambda_{x_{jk}} + \lambda_{x_{ji}} \;= \\
&= \quad \lambda_{r_i} \Big(s + \sum_{k \in K} (-d_k) \Big) + \lambda_{x_{ji}}
\end{aligned} \tag{5.9}$$

We conclude $\lambda_{x_{ji}} = -(s - d(K))\lambda_{r_i}$, proving the claim. $\diamond$

This way, we have that

$$\lambda^T y \;=\; \Big[y_{r_i} - y_{l_j} + \sum_{k \in K} d_k (y_{x_{ik}} - y_{x_{jk}}) - (s - d(K)) y_{x_{ji}} \Big] \lambda_{r_i}.$$

Then λ is a multiple of the LHS of inequality (5.8), implying that $\lambda_0 = 0$. Thus, the face F defined by (5.8) cannot be contained in any other facet of $P(G, d, s, 0)$ and defines, therefore, itself a facet of the (full-dimensional) polytope $P(G, d, s, 0)$. $\square$

Remark. An alternative proof can be given for Theorem 5.11 by considering the antiparallelity constraint $r_i \leq l_j + s x_{ji}$ and lifting the variables x_{ik} and x_{jk}, for $k \in N(i)$. We first lift the variables x_{ik} and x_{jk} for $k \in K$, and afterwards lift the remaining variables. The antiparallelity constraint is facet-inducing for $P(G, d, s, 0) \cap \{z \in \mathbf{R}^{2n+m} : z_{ik} = z_{jk} = 0\}$. Moreover, the maximum lifting coefficient for variable x_{ik} resp. x_{jk} is d_k resp. $-d_k$ and, therefore, the resulting double covering-clique inequality is facet-inducing for $P(G, d, s, 0)$. Thus, we naturally arise double covering-clique inequalities as a strengthening of the antiparallelity constraints. $\lhd$

Corollary 5.12 *Let $ij \in E$ be an edge of G such that $N(i) \cap N(j) = \emptyset$. If $s \geq s_{\min}(G, d, 0) + 4d_{\max}$, then the antiparallelity constraints (2.4)-(2.7) define facets of $P(G, d, s, 0)$.*

Corollary 5.13 *Let $ij \in E$. The double covering-clique inequality*

$$l_i + d_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \ \leq \ l_j + (s - d(K))x_{ji} \tag{5.10}$$

is valid for $R(G, d, s, g)$ and defines a facet of $R(G, d, s, 0)$ if $s \geq s_{\min}(G, d, 0) + 4d_{\max}$.

Proposition 5.14 *The symmetric inequality of a double covering-clique inequality is again a double covering-clique inequality.*

Proof. Consider first the polytope $R(G, d, s, g)$. Let $a^T y \leq b$ be the double covering-clique inequality (5.8) associated with (K, S). Recall that the symmetric inequality of $a^T y \leq b$ is $2a^T p - b \leq a^T y$, where $p = \frac{1}{2}(s\mathbf{1} - d, \mathbf{1})$ is the symmetry point of $R(G, d, s, g)$. We have that

$$\begin{aligned} 2a^T p - b \quad &= \quad 2\left(\frac{(s - d_i)}{2} - \frac{(s - d_j)}{2} + \sum_{k \in K}(\frac{d_k}{2} - \frac{d_k}{2}) + \frac{d(K) - s}{2}\right) + d_i \\ &= \quad d_j + d(K) - s \end{aligned}$$

holds. This implies that $2a^T p - b \leq a^T y$ is the inequality:

$$d_j + d(K) - s \ \leq \ l_i - l_j + \sum_{k \in K} d_k(x_{ik} - x_{jk}) - (s - d(K))x_{ji},$$

which can be rewritten as

$$l_j + d_j + \sum_{k \in K} d_k(x_{jk} - x_{ik}) \ \leq \ l_i + (s - d(K))(1 - x_{ji}). \tag{5.11}$$

Recalling the notation $x_{ij} = 1 - x_{ji}$, we obtain that (5.11) is again a double covering-clique inequality. A similar argumentation applies to $P(G, d, s, g)$. □

5.2.1 Double covering-clique inequalities are not always facet-defining

As we have seen previously, the covering-clique inequalities presented in Section 5.1 are not always facet-defining, although they do induce facets in many instances. Example 5.1 suggests that it is difficult to construct instances in which these inequalities do not induce facets. We shall see in this section that double covering-clique inequalities do not always induce facets, but the counterexamples are more straightforward.

Example 5.3 Let $(G, d) = (V, E, d)$ be the weighted graph depicted in Figure 5.8, and consider the polytope $R(G, d, 4, 0)$. By inspection, this polytope has dimension 4. We shall verify that the double covering-clique inequality $l_4 + d_4 \leq l_2 + 4x_{24}$ does not induce a facet. All

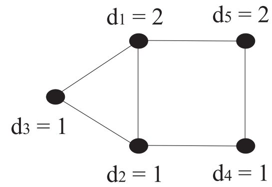


Figure 5.8: Interference graph for Example 5.3.

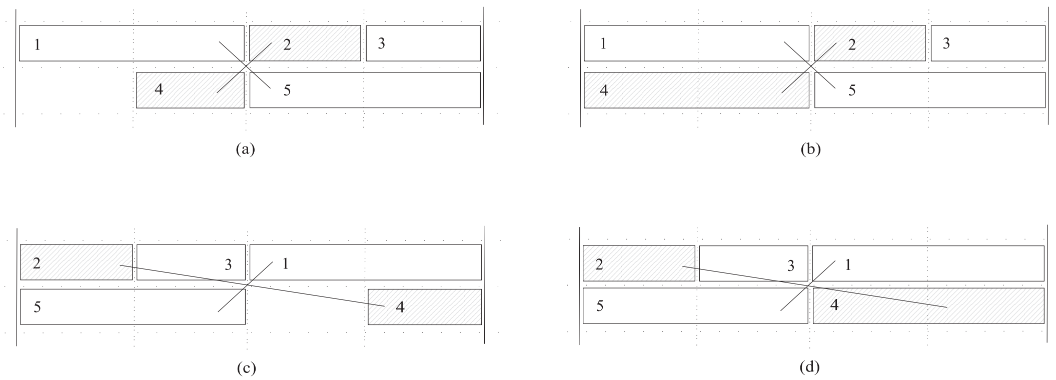


Figure 5.9: The only four feasible solutions in the double covering-clique face.

the feasible solutions satisfying this inequality at equality are the 4 points $y_1, \ldots, y_4$ depicted in Figure 5.9, and it is not difficult to verify $y_4 = y_1 - y_2 + y_3$. Hence y_4 is an affine combination of the other three solutions, and so the dimension of the face defined by the inequality is at most 2, implying that this face is not a facet of the polytope. $\lhd$

Double covering-clique inequalities may not define facets even if the polytope is full-dimensional. The following counterexample shows an instance inducing a full-dimensional polytope with a double covering-clique inequality that does not define a facet.

Example 5.4 Consider the weighted graph $(G, d) = (V, E, d)$ presented in Figure 5.10, and consider the polytope $P(G, d, 9, 0)$. It is straigthforward to verify that this polytope has full dimension.

Consider now the edge $26 \in E$. The face F defined by the double covering-clique inequality $r_2 \leq l_6 + 9x_{62}$ is the convex hull of all feasible solutions satisfying it at equality, which either have (i) $x_{26} = 1$ and $r_2 = l_6$ or (ii) $x_{26} = 0$, $l_6 = 0$ and $r_2 = 9$. Every point of group (i) has intervals $I(2)$ and $I(6)$ in parallel, and therefore:

- It cannot hold $x_{12} = x_{32} = 1$ because there would be no space left for the interval $I(6)$ after the interval $I(2)$, as required by $x_{26} = 1$.
- If $x_{12} = x_{32} = 0$, then $x_{46} \neq x_{56}$ (see Figure 5.11b).
- If $x_{12} \neq x_{32}$, then $l_2 \geq 2$ and thus $l_6 = r_2 \geq 5$. This implies that $l_6 \geq 7$, and therefore $x_{46} = x_{56} = 1$.

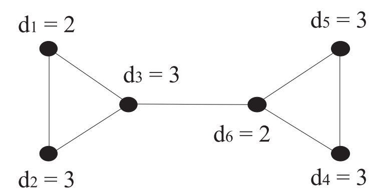


Figure 5.10: Interference graph for Example 5.4.

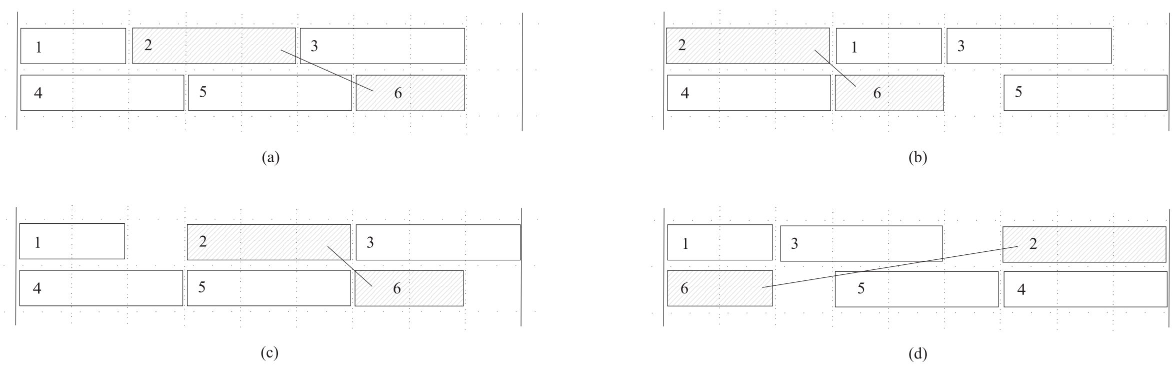


Figure 5.11: Instance for Example 5.4.

Hence every point of group (i) either has $x_{12} = x_{32} = 0$ and $x_{46} \neq x_{56}$, or $x_{12} \neq x_{32}$ and $x_{46} = x_{56} = 1$. Consider now any point of group (ii). Such a point has $x_{26} = 0$, implying that intervals $I(1)$ and $I(3)$ are located before the intervals $I(2)$ and $I(4)$, and the intervals $I(4)$ and $I(5)$ are located after $I(6)$ (see Figure 5.11(d)). Thus, $x_{12} = x_{32} = 1$ and $x_{46} = x_{56} = 0$. Having enumerated all the possible cases, we can now verify that every feasible solution in F satisfies

$$x_{45} + x_{56} + 3(1 - x_{26}) = 1 + (x_{12} + x_{32}).$$

This shows $\dim(F) < 18$, and since $P(G, d, 9, 0) \subseteq \mathbf{R}^{19}$ has full dimension, F is not a facet of this polytope. $\lhd$

The final example shows an instance where a certain double covering-clique inequality defines a facet of $P(G, d, s, 0)$ but not of $P(G, d, s+1, 0)$. At first sight, one would expect that a facet-inducing inequality for $P(G, d, s, 0)$ should also be facet-inducing for $P(G, d, s+1, 0)$, but the following example shows that this is, surprisingly, not the case.

Example 5.5 Let $(G, d) = (V, E, d)$ be the weighted graph depicted in Figure 5.12. The polytope $P(G, d, 5, 0)$ has only 4 integer solutions, and has dimension 2. It is not difficult to verify by inspection that $r_2 \leq l_5 + 5x_{52}$ defines a face of $P(G, d, 5, 0)$ of dimension 1, which is a facet.

Consider now the polytope $P(G, d, 6, 0)$ and the feasible solution depicted in Figure 5.13(a)). Starting from this solution, alternatively shift the interval bounds to the right (repeating the proof of Lemma 3.8 and Lemma 3.9) to construct 10 affinely independent points. Moreover, Figure 5.13(b), Figure 5.13(c) and Figure 5.13(d) present three affinely independent points w.r.t. the preceding constructions, showing that $\dim(P(G, d, 6, 0)) \geq 13$. Conversely, it is not hard to prove that every feasible solution satisfies the equations:

$$x_{13} = x_{23} \tag{5.12}$$
$$x_{25} = x_{45} \tag{5.13}$$
$$x_{23} = x_{43} \tag{5.14}$$

Since $P(G, d, 6, 0) \in \mathbf{R}^{16}$, then $\dim(P(G, d, 6, 0)) \leq 16 - 3 = 13$, and thus $\dim(P(G, d, 6, 0)) = 13$.

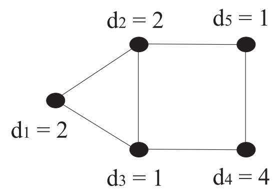


Figure 5.12: Interference graph for Example 5.5.

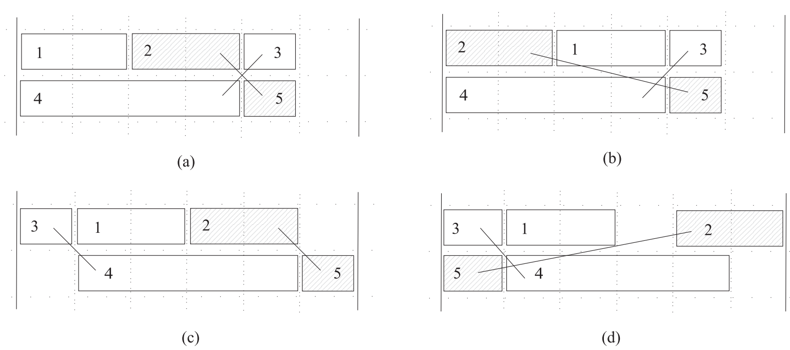


Figure 5.13: Feasible solutions for Example 5.5.

Let F denote the face of $P(G, d, 6, 0)$ defined by $r_2 \leq l_5 + 6x_{52}$. Every feasible solution in F satisfies this inequality at equality, by definition. Since $F \subseteq P(G, d, 6, 0)$, the feasible solutions lying on F also satisfy (5.12), (5.13) and (5.14). We now claim that every integer point in F also has interval $I(1)$ before interval $I(2)$:

(i) If $x_{25} = 1$, then $x_{45} = 1$ and so $r_2 = l_5 \geq 4$. This leaves no space to assign $I(1)$ after $I(2)$.

(ii) If $x_{25} = 0$ then $r_2 = 6$, hence $I(1)$ must be before $I(2)$.

Therefore, every feasible solution in F satisfies $x_{12} = 1$, and we have 6 equations for every point in F. This proves that $\dim(F) \leq 11$ (in fact, $\dim(F) = 11$), and thus F is not a facet of $P(G, d, 6, 0)$. $\triangleleft$

5.2.2 Complexity of the separation problem

This section addresses the computational complexity of the separation problem for double covering-clique inequalities. Recall that $P_{LP}(G,d,s,g)$ denotes the linear relaxation of $P(G,d,s,g)$. With this definition, the separation problem for this class of inequalities can be defined as follows:

	DOUBLE COVERING-CLIQUE INEQUALITIES SEPARATION
Instance:	A point $y=(l,r,x)\in P_{LP}(G,d,s,g)$
Question:	Does y violate some double covering-clique inequality?

Theorem 5.15 DOUBLE COVERING-CLIQUE INEQUALITIES SEPARATION *is $\mathcal{NP}$-complete.*

Proof. We can easily check that this problem belongs to the class $\mathcal{NP}$, since we can nondeterministically generate an edge $ij\in E$ and a clique $K\subseteq N(i)\cap N(j)$ and verify in deterministic polynomial time whether K covers $N(i)\cap N(j)$ and the double covering-clique inequality associated with ij and K is violated by the point y. To complete the proof, we construct a polynomial reduction from MAX-CLIQUE. An instance of the latter is given by a pair (H,p), where $H=(V_H,E_H)$ is a graph and $p\in\mathbf{Z}_+$ is an integer such that $1\le p\le |V_H|$, and consists in deciding whether H has a clique of size at least p. Assume w.l.o.g. $|V_H|\ge 2$ and that H is noncomplete. We construct a graph $G=(V,E)$ from H by adding two universal nodes i and j, thus

$$\begin{aligned} V &= V_H\cup\{i,j\} \\ E &= E_H\cup\{ti,tj : t\in V_H\}\cup\{ij\} \end{aligned}$$

Also set $d=\mathbf{1}$, $g=0$ and $s=2n$, where $n=|V|$. Finally, define a point y as follows:

$$\begin{aligned} y_{l_t} &= \begin{cases} 0 & \text{if } t\ne j \\ \frac{p+1}{2} & \text{if } t=j \end{cases} \qquad \forall t\in V \\ y_{r_t} &= y_{l_t}+1 \qquad \forall t\in V \\ y_{x_e} &= \begin{cases} 1 & \text{if } e=tj \text{ for some } t\in V \\ \frac{1}{2} & \text{otherwise} \end{cases} \qquad \forall e\in E \end{aligned}$$

This construction is polynomial in the size of H. We first verify that $y\in P_{LP}(G,\mathbf{1},2n,0)$ by checking that the point y satisfies all the constraints of this relaxed polytope. The demand constraints, the interval bounds and the relaxed constraints $0\le x_e\le 1$ for every $e\in E$ are trivially satisfied by construction. So we are left to verify that the antiparallelity constraints $l_k+d_k\le l_t+sx_{tk}$ are also satisfied. Consider the following cases:

1. If $k,t\ne j$, then $y_{x_{tk}}=1/2$ and, therefore,

$$y_{l_k}+d_k=1\le n=y_{l_t}+sy_{x_{tk}}.$$

2. If $k = j$, then $y_{x_{tk}} = 1$ and we have that

$$y_{l_j} + d_j = \frac{p+1}{2} + 1 \leq 2n = y_{l_t} + s y_{x_{tj}}.$$

3. If $t = j$, then $y_{x_{tk}} = 0$ and

$$y_{l_k} + d_k = 1 \leq \frac{p+1}{2} = y_{l_j} + s y_{x_{jk}}.$$

Therefore, $y \in P_{LP}(G, \mathbf{1}, 2n, 0)$. To complete the proof, we must show that the prescribed transformation maps affirmative instances of MAX-CLIQUE onto affirmative instances of DOUBLE COVERING-CLIQUE INEQUALITIES SEPARATION and conversely, i.e., $\omega(H) \geq p$ if and only if y violates some double covering-clique inequality.

$\Rightarrow$) Let $K \subseteq V_H$ be a maximal clique of H of size at least p. Since i and j are universal nodes, then $K \subseteq N_G(i) \cap N_G(j)$. Moreover, $d = \mathbf{1}$ implies that K covers $N_G(i) \cap N_G(j) = V_H$. The construction of y implies that the double covering-clique inequality associated with $(K, V_H \backslash K)$ is violated by this point:

$$y_{l_i} + d_i + \sum_{k \in K} d_k (y_{x_{ik}} - y_{x_{jk}}) = 1 + \frac{d(K)}{2} > \frac{p+1}{2} = y_{l_j} + (s - d(K)) y_{x_{ji}}.$$

$\Leftarrow$) Conversely, suppose that the double covering-clique inequality defined by the nodes k and t and the clique $K \subseteq N_G(k) \cap N_G(t)$ is violated, i.e.,

$$y_{l_k} + d_k + \sum_{l \in K} d_l (y_{x_{kl}} - y_{x_{tl}}) \ > \ y_{l_t} + (s - d(K)) y_{x_{tk}}. \tag{5.15}$$

Claim: $t = j$. Suppose $t \neq j$ and consider two cases.

- If $k \neq j$, then $y_{x_{kl}} - y_{x_{tl}} = 0$ for every $l \in V \backslash \{k, t\}$, and therefore (5.15) has LHS $= 1$ and RHS $= \frac{1}{2}(s - d(K)) \geq \frac{1}{2}(2n - \omega(H)) \geq 1$. Hence (5.15) does not hold, a contradiction.
- On the other hand, if $k = j$ then LHS $= 1 + \frac{1}{2}(p + 1 - |K|)$ and RHS $= 2n - d(K)$. Again, we have LHS $\leq$ RHS, contradicting the fact that(5.15) holds. $\diamond$

This claim proves that, in this setting, violated double covering-clique inequalities must have $I(j)$ as the right hand side interval. Since $t = j$, then $y_{l_t} = \frac{p+1}{2}$ and $y_{x_{kl}} - y_{x_{tl}} = 1/2$ follows for every $l \in K$. Hence (5.15) reads $1 + \frac{|K|}{2} \ > \ \frac{p+1}{2}$, implying $|K| \geq p$. Therefore K is a clique of G with at least p nodes. Now, if $i \notin K$ then $K \subseteq V_H$ and $\omega(H) \geq p$. On the other hand, if $i \in K$ then $(K \backslash \{i\}) \cup \{k\}$ is a clique of H on p nodes, also implying $\omega(H) \geq p$.

Hence the transformation maps affirmative instances of MAX-CLIQUE onto affirmative instances of DOUBLE COVERING-CLIQUE INEQUALITIES SEPARATION and conversely. Therefore, the latter is $\mathcal{NP}$-complete. $\square$

5.2.3 Double covering-clique inequalities in the case $g > 0$

Theorem 5.11 shows that the double covering-clique inequalities (5.8) are facet-defining when $g = 0$. Clearly, these inequalities are still valid if $g > 0$, but may not define facets in this case since the set of feasible solutions can be much smaller. This section presents a generalization of double covering-clique inequalities for this case, such that the resulting inequalities are valid for every instance, and facet-inducing if $s \geq \omega(G,d) + 4(g + d_{\max})$. Recall that we denote by $a(i)$ the sector to which the node i belongs, for $i \in V$.

Definition 5.7 (general double covering-clique inequalities) *Let $ij \in E$, and let K be a clique covering $N(i) \cap N(j)$. Fix $K = \{1, \ldots, t\}$ as order of the nodes in K and, for $k \in K$, let $A_k = \{i, j\} \cup \{1, \ldots, k-1\}$. We define*

$$r_i + \sum_{k \in K} \varphi_k (x_{ik} - x_{jk}) + \delta_{ij} \;\leq\; l_j + \Big(s + \delta_{ij} + \sum_{k \in K} \varphi_k\Big) x_{ji} \tag{5.16}$$

to be the general double covering-clique inequality *associated with the edge ij, the clique K and the ordering $K = \{1, \ldots, t\}$, where the coefficients φ_k are defined as follows. Let*

$$\begin{aligned} N &= \{\, k \in K : a(k) \neq a(t) \text{ for all } t \in A_k \,\} \\ C &= \{\, k \in K : a(k) = a(t) \text{ for some } t \in A_k \,\} \end{aligned}$$

and consider two cases. If $N = \emptyset$, then $\varphi_k = d_k$ for every $k \in K$. On the other hand, if $N \neq \emptyset$, let k_0 be some fixed node of N and, for every $k \in K$,

$$\varphi_k = \begin{cases} d_k + 2g & \text{if } k = k_0 \\ d_k + g & \text{if } k \in N \setminus \{k_0\} \\ d_k & \text{if } k \in C \end{cases}$$

The proof of facetness for the general double covering-clique inequalities goes along the argumentation of the proof of facetness for the standard double covering-clique inequalities presented in Theorem 5.11.

Theorem 5.16 *The general double covering-clique inequalities (5.16) are valid for the polytope $P(G,d,s,g)$, and define facets if $s \geq s_{\min}(G,d,g) + 4(d_{\max} + g)$.*

Remark. A similar result holds for $R(G,d,s,g)$ if we replace r_i by $l_i + d_i$ in (5.16). Notice that these inequalities arise as a natural strengthening of the antiparallelity constraints by lifting the variables x_{ik} and x_{jk}, for $k \in K$. In the case $g = 0$, this lifting is sequence-independent and originates the standard double covering-clique inequalities (5.8). In the case $g > 0$, however, the lifting depends on the sequence, requiring the different definitions of the coefficients φ_k for $k = k_0$, $k \in N \setminus \{k_0\}$ and $k \in C$. $\triangleleft$

5.3 Generalizations and extensions of clique inequalities

This section presents families of facet-defining inequalities arising from the covering-clique inequalities as generalizations (containing the covering-clique inequalities as particular cases) or extensions (defined over slightly different structures). The first family, introduced in Section 5.3.1, fixes a clique in $N(i)$ and considers a clique covering the remaining nodes of this neighborhood. We also provide a generalization of double covering-clique inequalities based on these ideas. The second family, presented in Section 5.3.2, considers a subset of nodes from $N[N(i)]$, introducing coefficients for the edges linking $N(i)$ to these nodes. We show that both classes of valid inequalities are facet-inducing for $s > s_{\min}(G, d, g) + O(1)d_{\max}$, and that they contain the covering-clique inequalities as special cases. Finally, we discuss in Section 5.3.3 three classes of facet-defining inequalities arising as variations of the double covering-clique inequalities.

5.3.1 Reinforced covering-clique inequalities

Definition 5.8 *If $K \subseteq V$ and $j \in V \backslash K$, we define $c_K(j) = \max\{0,\ d_j - \sum_{k \in K \backslash N(j)} d_k\}$ (see Figure 5.14).*

Definition 5.9 (reinforced covering-clique inequalities) *Let $i \in V$ be a node of G and fix a clique $K' \subseteq N(i)$. Furthermore, let K be a clique covering $N(i) \backslash K'$. We define*

$$\sum_{k \in K} d_k x_{ki} + \sum_{k \in K'} c_K(k) x_{ki} \ \leq \ l_i \tag{5.17}$$

to be the reinforced covering-clique inequality *associated with K and K'.*

Note that the existence of a clique K covering $N(i) \backslash K'$ is guaranteed by Proposition 5.3. The standard covering-clique inequalities discussed in Section 5.1 can be obtained as a special case of these reinforced covering-clique inequalities by setting $K' = \emptyset$.

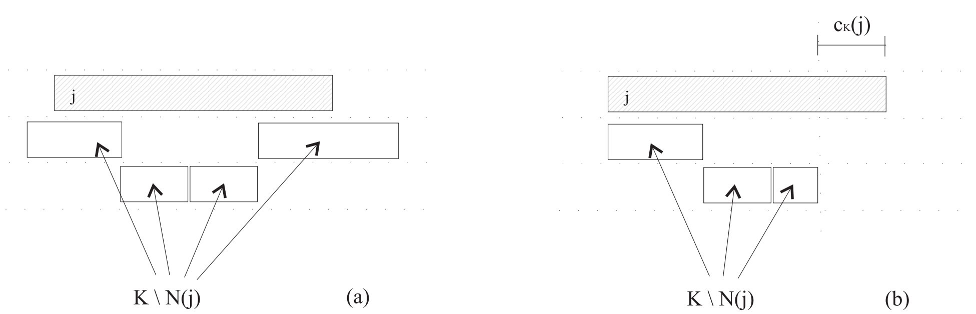


Figure 5.14: (a) Example of $c_K(j) = 0$, and (b) example of $c_K(j) > 0$.

Proposition 5.17 *The reinforced covering-clique inequalities are valid for $R(G,d,s,g)$ and $P(G,d,s,g)$.*

Proof. Let $y \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ be an arbitrary schedule, and define the node sets $A = \{k \in K' : y_{x_{ki}} = 1 \text{ and } c_K(k) > 0\}$ and $B = \{t \in K : y_{x_{ti}} = 1\}$. Since K resp. K' is a clique, the intervals corresponding to nodes in K resp. K' do not overlap. Moreover, define $Q = \{t \in K : \ tk \in E \ \forall k \in A\}$. Note that $A \cup Q$ is a clique, hence $A \cup (B \cap Q)$ is also a clique. The following chain of inequalities establishes the validity of (5.17):

$$
\begin{aligned}
y_{l_i} &\geq \sum_{k \in A} d_k + \sum_{t \in B \cap Q} d_t \\
&= \sum_{k \in A} d_k + \sum_{t \in B \cap Q} d_t - \sum_{t \in B \setminus Q} d_t + \sum_{t \in B \setminus Q} d_t \\
&= \sum_{k \in A} d_k - \sum_{t \in B \setminus Q} d_t + \Big(\sum_{t \in B \cap Q} d_t + \sum_{t \in B \setminus Q} d_t \Big) \\
&\geq \sum_{k \in A} (d_k - \sum_{t \in K \setminus N(k)} d_t) + \sum_{t \in B} d_t \\
&= \sum_{k \in A} c_K(k) + \sum_{t \in B} d_t \\
&= \sum_{k \in K'} c_K(k) \ y_{x_{ki}} + \sum_{k \in K} d_k \ y_{x_{ki}}
\end{aligned}
$$

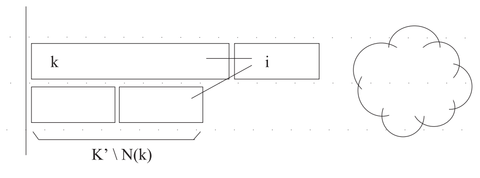
□

Theorem 5.18 *The reinforced covering-clique inequalities induce facets of $R(G,d,s,g)$ and $P(G,d,s,g)$ if $s \geq s_{\min}(G,d,g) + 3d_{\max}$.*

Proof. Suppose $\lambda^T z = \lambda_0$ for every feasible schedule $z \in P(G,d,s,0) \cap \mathbf{Z}^{2n+m}$ satisfying (5.17) at equality. Claims 1, 2 and 3 from the proof of Theorem 5.2 show $\lambda_{l_j} = 0$ for every $j \neq i$, $\lambda_{r_j} = 0$ for every $j \in V$ and $\lambda_{x_{jt}} = 0$ for $jt \notin \delta(i)$. Moreover, Claim 4 from the proof of Theorem 5.2 implies $\lambda_{x_{ik}} = -d_k \lambda_{l_i}$ for every $k \in K$ and Theorem 5.4 implies $\lambda_{x_{ik}} = 0$ for every $k \in N(i) \setminus (K \cup K')$. So it is left to prove that $\lambda_{x_{ki}} = -c_K(k)\lambda_{l_i}$ for every $k \in K'$. To this end, consider two cases.

Case 1: $\boldsymbol{c_K(k) > 0}$. Let $z \in P(G,d,s,0) \cap \mathbf{Z}^{2n+m}$ be a feasible solution with $z_{l_i} = 0$. Now construct a feasible solution $z' \in P(G,d,s,0) \cap \mathbf{Z}^{2n+m}$ by setting $z_{l_k} = 0$ and $z_{l_i} = d_k$, and assigning every interval $I(t)$, for $t \in K \setminus N(k)$, to the left of the interval $I(i)$ (see Figure 5.15(a)). These two feasible solutions satisfy (5.17) at equality and, therefore, $\lambda_{x_{ki}} = -c_K(k)\lambda_{l_i}$. ◇

Case 2: $\boldsymbol{c_K(k) = 0}$. As in the previous case, let $z \in P(G,d,s,0) \cap \mathbf{Z}^{2n+m}$ be a feasible solution with $z_{l_i} = 0$. Now construct a feasible solution $z' \in P(G,d,s,0) \cap \mathbf{Z}^{2n+m}$ by setting $z_{l_k} = 0$, $z_{l_i} = \sum_{l \in K \setminus N(k)} d_l$, and assigning every interval $I(t)$, for $t \in K \setminus N(k)$, to the left of the interval $I(i)$ (see Figure 5.15(b)). Again, these two points satisfy (5.17) at equality, implying $\lambda_{x_{ki}} = 0$. ◇



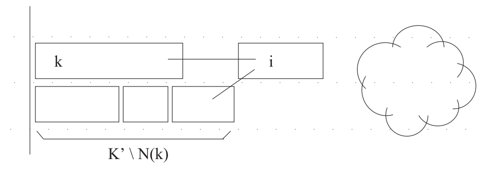


Figure 5.15: Constructions for the proof of Theorem 5.18.

Hence we verify that λ is a multiple of the coefficient vector of (5.17) and thus this inequality induces a facet of $P(G,d,s,0)$. Since both $P(G,d,s,0)$ and $R(G,d,s,0)$ are full-dimensional, and the inequality does not involve the r-variables, it is also facet-inducing for $R(G,d,s,0)$. □

The symmetric inequalities of the reinforced covering-clique inequalities describe the interaction between the right bound of the interval $I(i)$ and the right bound of the frequency spectrum $[0,s]$. Under the same setting as in Theorem 5.18, the symmetric inequality

$$r_i \;\leq\; s - \sum_{k\in K} d_k x_{ik} - \sum_{k\in K'} c_K(k) x_{ik}$$

is valid and facet-inducing for $P(G,d,s,0)$, and the same holds for $R(G,d,s,0)$ if we replace r_i by $l_i + d_i$. Note that this result generalizes Corollary 5.5 for covering-clique inequalities.

Definition 5.10 (reinforced double covering-clique inequalities) *Let $i,j \in V$ be two adjacent nodes of G and fix a clique $K' \subseteq N(i)\cap N(j)$. Furthermore, let K be a clique covering $[N(i)\cap N(j)]\backslash K'$. Finally, for $k \in K'$, let $U_k = \{l \in K : lk \notin E\}$ (i.e., the set of nodes in K not adjacent to k). We define*

$$r_i + \sum_{k\in K} d_k(x_{ik} - x_{jk}) + \sum_{k\in K'} c_K(k)(x_{ik} - x_{jk}) \;\leq\; l_j + \Big(s - \sum_{k\in K} d_k - \sum_{k\in K'} c_K(k)\Big) x_{ij} \quad (5.18)$$

to be the reinforced double covering-clique inequality *associated with K and K'.*

The proof of facetness for the reinforced double covering-clique inequalities is similar to the proof of Theorem 5.11.

Theorem 5.19 *The reinforced double covering-clique inequalities are valid for $P(G,d,s,0)$, and define facets if $s \geq s_{\min}(G,d,0) + 4d_{\max}$.*

5.3.2 Replicated covering-clique inequalities

Definition 5.11 (replicated covering-clique inequalities) *Fix a node $i \in V$ and let K be a clique covering $N(i)$. Consider a clique $Q = \{q_1,\ldots,q_t\} \in V\backslash N(i)$ and a subset $K' = \{k_1,\ldots,k_t\} \subseteq K$ such that $k_j\, q_j \in E$ for $j = 1,\ldots,t$ (see Figure 5.16). We define*

$$\sum_{k\in K} d_k x_{ki} + \sum_{k\in K'} c_K(p_k)(x_{p_k k} - x_{ik}) \;\leq\; l_i \quad (5.19)$$

to be the replicated covering-clique inequality *associated with the cliques K and Q.*

Note that the definition of the replicated covering-clique inequalities allows edges between K and Q other than $k_j\, q_j$, $j = 1, \ldots, t$. In the case $Q = \emptyset$, the replicated covering-clique inequality (5.19) is equivalent to the standard covering-clique inequality (5.3). Moreover, when both K and Q are singletons, these inequalities are equivalent to the *path inequalities* introduced in [21].

Proposition 5.20 *The replicated covering-clique inequalities (5.19) are valid for $R(G,d,s,g)$ and $P(G,d,s,g)$.*

Proof. Let $y \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ denote an arbitrary integer solution, and define $A = \{k \in K : y_{x_{ki}} = 1\}$ and $B = \{k \in K' : y_{x_{p_k k}} = 1,\ y_{x_{ki}} = 1,\ c_K(p_k) > 0\}$. Also define $T = \{k \in K : kt \in E\ \forall t \in Q\}$, and note that $Q \cup T$ is a clique. The following chain of inequalities establishes the validity of (5.19):

$$
\begin{aligned}
y_{l_i} &\geq \sum_{k \in B} d_{p_k} + \sum_{k \in T \cap A} d_k \\
&= \sum_{k \in B} d_{p_k} + \sum_{k \in T \cap A} d_k + \sum_{k \in A \setminus T} d_k - \sum_{k \in A \setminus T} d_k \\
&= \sum_{k \in B} d_{p_k} - \sum_{k \in A \setminus T} d_k + \Big(\sum_{k \in T \cap A} d_k + \sum_{k \in A \setminus T} d_k \Big) \\
&\geq \sum_{k \in B} (d_{p_k} - \sum_{t \in K \setminus N(p_k)} d_t) + \sum_{k \in A} d_k \\
&= \sum_{k \in B} c_K(p_k) + \sum_{k \in A} d_k \\
&\geq \sum_{k \in B} c_K(p_k)(y_{x_{p_k k}} - y_{x_{ik}}) + \sum_{k \in K} d_k y_{x_{ki}}
\end{aligned}
$$

□

Theorem 5.21 *If $s \geq s_{\min}(G,d,0) + 3d_{\max}$, then the replicated covering-clique inequality (5.19) defines a facet of $P(G,d,s,0)$ and $R(G,d,s,0)$.*

Proof. Let F be the face of $P(G,d,s,0)$ defined by (5.19), and suppose that every point $y \in F$ satisfies $\lambda^T y \leq \lambda_0$. We will show that λ is a multiple of the coefficient vector of (5.19), implying that this inequality induces a facet.

We show first $\lambda_{l_j} = \lambda_{r_j} = 0$ with the help of the constructions illustrated in Figure 5.17(a) and Figure 5.17(b). Points y_1 and y_2 (Figure 5.17(a) and Figure 5.17(b), respectively) are constructed with $l_i = 0$, and thus $x_{ki} = 0$ for all $k \in K$. We also take care of assigning every $k \in K'$ after its associated node p_k, so that $x_{p_k k} - x_{ik} = 0$. This implies that y_1 and y_2 are in F, and thus $\lambda^T y_1 = \lambda_0 = \lambda^T y_2$. These points only differ in their l_j-coordinates, hence $\lambda_{l_j} = 0$ for $j \neq i$. A similar argument shows $\lambda_{r_j} = 0$ for every j (including node i).

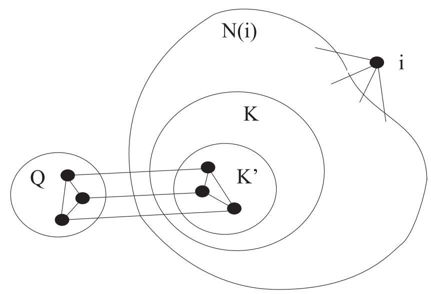


Figure 5.16: Structure for replicated covering-clique inequalities.

Consider now any edge $jl \in E$ such that $jl \neq ik$ for $k \in K$ and $jl \neq p_k k$ for $k \in K'$. We construct the points depicted in Figure 5.17(c) and Figure 5.17(d), which belong to F. Since $\lambda_{l_j} = \lambda_{r_j} = \lambda_{l_l} = \lambda_{r_l} = 0$, we have $\lambda_{x_{jl}} = 0$.

It remains to prove that the nonzero coefficients of λ can be obtained as a multiple of (5.19). To this end, we rewrite (5.19) as

$$\sum_{k \in K'} \Big(d_k + c_K(p_k)\Big) x_{ki} + \sum_{k \in K'} c_K(p_k) x_{p_k k} + \sum_{k \notin K'} d_k x_{ki} \leq l_i + \sum_{k \in K'} c_K(p_k).$$

Let $k \in K'$, and suppose $K \cap \bar{N}(p_k) = \{k_1, \ldots, k_t\}$, so that $c_K(p_k) = d_{p_k} - \sum_{1 \leq v \leq t} d_{k_v}$. Consider the pair of points depicted in Figure 5.17(e) and Figure 5.17(f). Since both points belong to F they satisfy $\lambda^T x = \lambda_0$ at equality, and we have

$$(d_k + d_{k_1} + \ldots + d_{k_t}) \lambda_{l_i} = \lambda_{x_{p_k k}} + (d_{p_k} + d_k) \lambda_{l_i},$$

implying

$$\begin{aligned} \lambda_{x_{p_k k}} &= (d_{k_1} + \ldots + d_{k_t} - d_{p_k}) \lambda_{l_i} \\ &= -c_K(p_k) \lambda_{l_i}. \end{aligned} \tag{5.20}$$

Now, for any $k \in K$, consider the two following cases:

Case 1: $\boldsymbol{k \notin K}$. The points depicted in Figure 5.17(g) and Figure 5.17(h) satisfy (5.19) at equality, hence $\lambda_{x_{ki}} + d_k \lambda_{l_i} = 0$. $\diamond$

Case 2: $\boldsymbol{k \in K}$. The two points depicted in Figure 5.17(i) and Figure 5.17(j) satisfy (5.19). Since $\lambda_{l_j} = \lambda_{r_j} = 0$, we have $\lambda_{x_{p_k k}} = \lambda_{x_{k_i}} + d_k \lambda_{l_i}$. From (5.2) we have $\lambda_{x_{p_k k}} = -c_K(p_k)\lambda_{l_i}$, implying $\lambda_{x_{ki}} = -\Big(d_k + c_K(p_k)\Big)\lambda_{l_i}$. $\diamond$

Therefore, we have $\lambda = -\lambda_{l_i} \pi$, where π denotes the coefficient vector of (5.19). Hence the replicated covering-clique inequality (5.19) defines a facet of $P(G, d, s, 0)$. The same argumentation applies to $R(G, d, s, 0)$ $\square$

The symmetric inequalities of the replicated covering-clique inequalities describe the interaction between the interval $I(i)$ and the cliques K and K' with the right bound of the

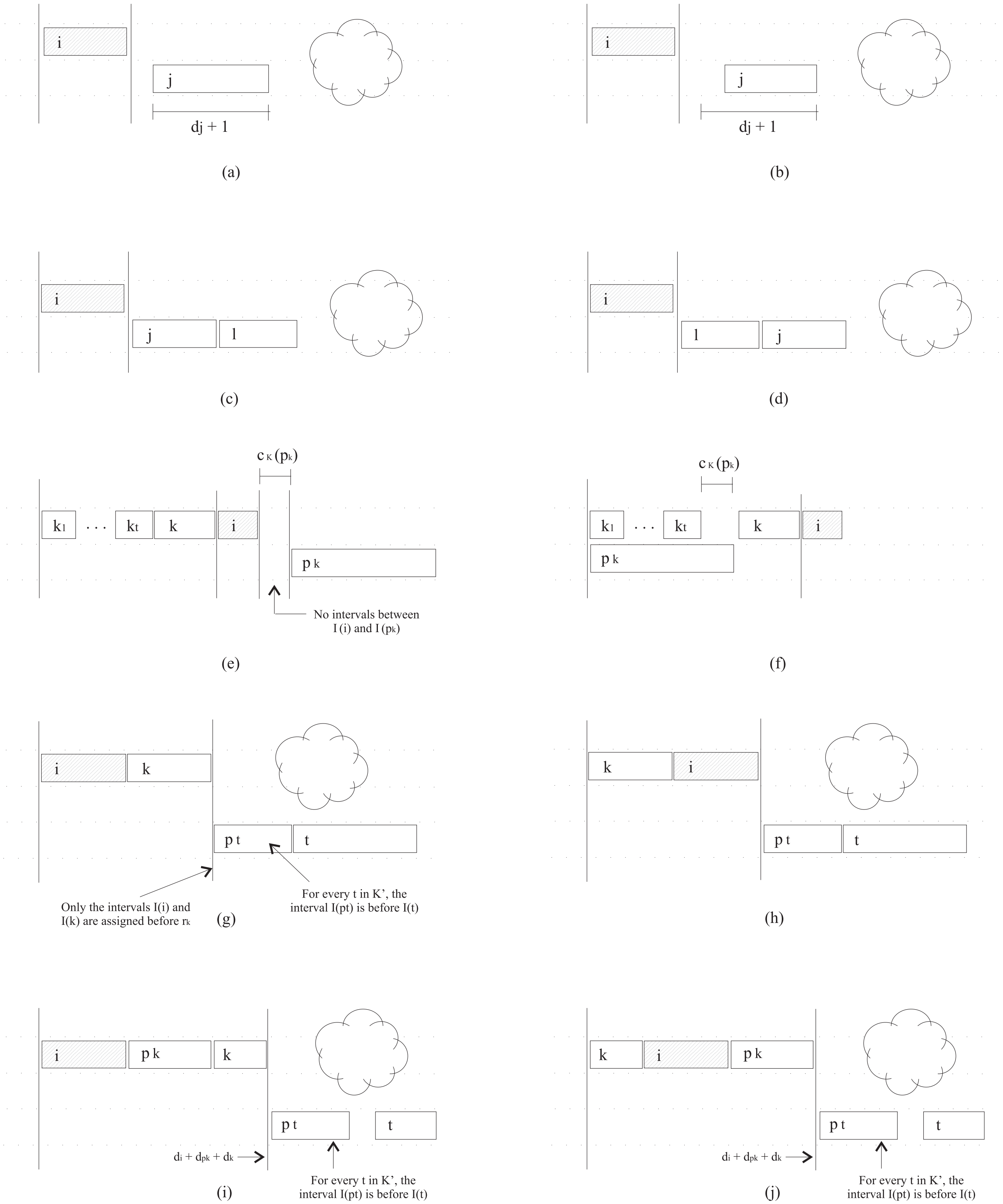


Figure 5.17: Constructions for the proof of Theorem 5.21.

frequency spectrum [0,s]. Under the same setting as in Theorem 5.21, the following symmetric inequality is valid and facet-inducing for $P(G,d,s,0)$:

$$r_i \;\leq\; s - \sum_{k\in K} d_k x_{ik} + \sum_{k\in K'} c_K(p_k)(x_{kp_k} - x_{ki}).$$

If we replace r_i by $l_i + d_i$ in this inequality, the resulting inequality is valid and facet-defining for $R(G,d,s,0)$.

5.3.3 Extensions of double covering-clique inequalities

The ideas involved in the development of double covering-clique inequalities do not restrict to that particular family of inequalities, but can be further exploited to find new classes of facet-inducing inequalities based on similar concepts. In this section we explore facet-defining valid inequalities over slightly different structures, analyzing the effect of these structure changes in the resulting inequalities. The constructions presented in this section resemble the development of the *reinforced fence inequalities* from the *fence inequalities* for the linear ordering polytope, adding a node to the subgraph that supports the inequality and adjusting the coefficients to maintain validity while enforcing facetness [38].

Definition 5.12 *If $K \subseteq V$ and $t \in V$, we define $A(K,t)$ to be the set*

$$A(K,t) \;=\; \text{argmax}\{d(B) : B \subseteq K\backslash N(t) \text{ and } d(B) \leq d_t\}.$$

That is, $A(K,t) \subseteq V$ is the maximum demand of a node subset of K that can be assigned inside the interval $[0,d_t]$ in a schedule with $l_t = 0$. Note that the exact calculation of $A(K,t)$ is $\mathcal{NP}$-hard, since this problem generalizes the feasibility problem for chromatic scheduling polytopes.

Definition 5.13 (extended double covering-clique inequalities) *Let $i,j \in V$ be two adjacent nodes, and let K be a clique covering $N(i)\cap N(j)$. Furthermore, fix some node $t \in N(j)\backslash N(i)$ (see Figure 5.18(a)). We define*

$$r_i + \sum_{k\in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_j + \varphi x_{ji} + \varphi_t x_{jt} \tag{5.21}$$

to be the extended double covering-clique inequality *associated with K and t, where $\varphi = s - d(K\backslash A(K,t))$ and $\varphi_t = d_t - d(A(K,t))$.*

Proposition 5.22 *The extended double covering-clique inequalities (5.21) are valid for the polytope $P(G,d,s,g)$.*

Proof. Let $y \in P(G,d,s,g)$ be an integer solution. If $y_{x_{ji}} = 0$, then the inequality (5.21) is dominated by the standard double covering-clique inequality (5.8), and thus is satisfied by y. On the other hand, if $y_{x_{ji}} = 1$ consider two cases:

Case 1: $y_{x_{jt}} = 1$. In this case, the inequality (5.21) admits the form

$$\begin{aligned} r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) &\leq y_{l_j} + \varphi' y_{x_{ji}} + \varphi_t y_{x_{jt}} \\ &= y_{l_j} + \varphi + \varphi_t \\ &= y_{l_j} + (s - \sum_{k \in K} d_k) \\ &= y_{l_j} + (s - \sum_{k \in K} d_k) y_{x_{ji}} \end{aligned}$$

Thus, the inequality reads as a standard double covering-clique inequality, and is therefore satisfied by y. $\diamond$

Case 2: $y_{x_{jt}} = 0$. In this case, the interval $I(j)$ is located before $I(i)$, which in turn is located before $I(t)$. Note that $y_{r_i} \leq s - d_t$ and $y_{l_j} \geq d(\{k \in K : \ y_{x_{kj}} = 1\})$. Moreover, for every $k \in K$ we have $y_{x_{ik}} - y_{x_{jk}} = -1$ only if $I(j)$ is located before $I(k)$ and $I(k)$ is located before $I(i)$, and $y_{x_{ik}} - y_{x_{jk}} = 0$ otherwise. Combining these observations, we get

$$\begin{aligned} & y_{r_i} + \sum_{k \in K} d_k (y_{x_{ik}} - y_{x_{jk}}) - y_{l_j} \\ \leq\ & (s - d_t) - d(\{k \in K : \ y_{x_{jk}} = y_{x_{ki}} = 1\}) - d(\{k \in K : \ y_{x_{kj}} = 1\}) \\ =\ & (s - d_t) - d(K \backslash A(K, t)) \\ =\ & \varphi \\ =\ & \varphi y_{x_{ji}} + \varphi_t y_{x_{jt}} \ \diamond \end{aligned}$$

Since y is an arbitrary integer solution, we conclude that the extended double covering-clique inequality (5.21) is valid for $P(G, d, s, g)$. $\square$

The proofs of all the facetness results in this section go along the argumentation of the proof of facetness for the standard double covering-clique inequalities presented in Theorem 5.11.

Theorem 5.23 *If $s \geq s_{\min}(G, d, 0) + 4d_{\max}$, then the extended double covering-clique inequalities (5.21) induce facets of $P(G, d, s, 0)$, and the same holds for $R(G, d, s, 0)$ if we replace r_i by $l_i + d_i$ in (5.21).*

It is interesting to compare the standard double covering-clique inequalities (5.8) with the extended inequalities (5.21). The coefficient of x_{ji} is smaller in the extended inequality, which in turn has a new positive coefficient in the RHS, corresponding to x_{jt}. This means that we cannot reinforce the original inequalities with a nonnegative coefficient in x_{jt} for free: when we force this variable to have a nonzero coefficient, variable x_{ji} decreases its coefficient to maintain validity.

Moreover, it is worthwhile to compute the dual inequality of this new class. The dual of a double covering-clique inequality is again a double covering-clique inequality, but the dual

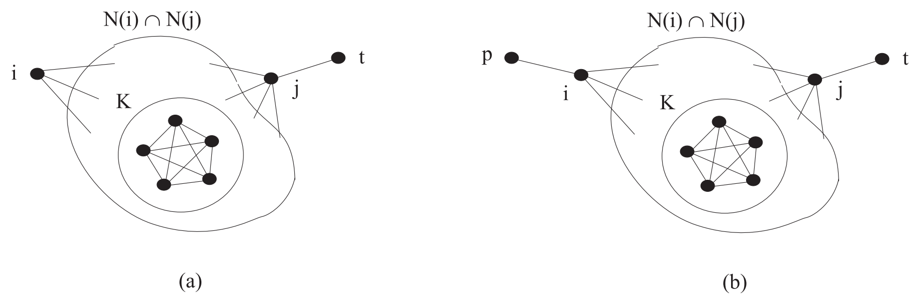


Figure 5.18: Supports for extended double covering-clique inequalities

of this extension is a new valid inequality:

$$r_j + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_i + \varphi x_{ij} + \varphi_t x_{tj}. \tag{5.22}$$

In this case, the inequality is reinforced by adding a coefficient associated with the edge $tj \in E$, but now the interval $I(j)$ is the left interval in the inequality. These inequalities can be generalized to the case $g > 0$. In this setting, a more general definition for the coefficients accompanying variables x_{ji} and x_{jt} must be given.

Definition 5.14 (2-extended double covering-clique inequalities) *Let $i, j \in V$ be two adjacent nodes of G, and let K be a clique covering $N(i) \cap N(j)$. Moreover, let $p \in N(i) \backslash N(j)$ and $t \in N(j) \backslash N(i)$ (see Figure 5.18(b)). We define*

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_j + \varphi' x_{ji} + \varphi_p x_{pi} + \varphi_t x_{jt} \tag{5.23}$$

to be the 2-extended double covering-clique inequality *associated with K and nodes t and p, where*

$$\begin{aligned} \varphi' &= s - d(K \backslash (A(K,t) \cup A(K,p))) - d_t - d_p \\ \varphi_t &= d_t - d(A(K,t)) \\ \varphi_p &= d_p - d(A(K,p)) \end{aligned}$$

Note that the 2-reinforced double covering-clique inequalities are obtained by "combining" inequalities (5.21) and (5.22) into a new valid one. Now we have two new nodes, namely p and t, adjacent to nodes i and j, respectively. The standard double covering-clique inequality is reinforced with nonzero coefficients associated with the variables x_{ip} and x_{jt}.

Theorem 5.24 *If $s \geq s_{\min}(G, d, 0) + 4d_{\max}$, then the 2-extended double covering-clique inequalities are facet-inducing for $P(G, d, s, 0)$, and the same holds for $R(G, d, s, 0)$ if we replace r_i by $l_i + d_i$.*

Definition 5.15 (closed double covering-clique inequalities) *Let $i, j \in V$ be two adjacent nodes of G, and let K be a clique covering $N(i) \cap N(j)$. Moreover, let $p \in N(i) \backslash N(j)$ and $t \in N(j) \backslash N(i)$ such that $pt \in E$ and $pk, tk \in E$ for all $k \in K$. We define*

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_j + \varphi'' x_{ji} + \varphi_p x_{pi} + \varphi_t x_{jt} - \varphi_{pt} x_{pt} \tag{5.24}$$

to be the closed double covering-clique inequality *associated with K and nodes t and p, where*

$$\begin{aligned} \varphi'' &= s - d(K) - (d_p + d_t) \\ \varphi_t &= d_t + \min\{d_p, d_t\} \\ \varphi_p &= d_p \\ \varphi_{pt} &= \min\{d_p, d_t\} \end{aligned}$$

Theorem 5.25 *If $s \geq s_{\mathrm{min}}(G, d, 0) + 4d_{\mathrm{max}}$, then the closed double covering-clique inequalities (5.24) induce facets of $P(G, d, s, 0)$, and the same holds for $R(G, d, s, 0)$ if we replace r_i by $l_i + d_i$.*

Example 5.6 It is worth comparing the inequalities presented in this section arising from the same graph structure. Suppose $N(i) \cap N(j) = \emptyset$ (so that $K = \emptyset$) and take $d = \mathbf{1}$. Moreover, set $s = 4$ and suppose that $P(G, d, 4, 0)$ is nonempty. In this setting, the standard and the extended double covering-clique inequalities have the following form:

$$\begin{array}{ccccccccccccc} \text{standard} & \rightarrow & r_i & \leq & l_j & + & 4x_{ji} & & & & & & \\ \text{extended} & \rightarrow & r_i & \leq & l_j & + & 3x_{ji} & + & x_{jt} & & & & \\ \text{extended (symm.)} & \rightarrow & r_i & \leq & l_j & + & 3x_{ji} & & & + & x_{pi} & & \\ \text{2-extended} & \rightarrow & r_i & \leq & l_j & + & 2x_{ji} & + & x_{jt} & + & x_{pi} & & \\ \text{closed} & \rightarrow & r_i & \leq & l_j & + & 2x_{ji} & + & 2x_{jt} & + & x_{pi} & - & x_{pt} \\ \text{closed} & \rightarrow & r_i & \leq & l_j & + & 2x_{ji} & + & x_{jt} & + & 2x_{pi} & - & x_{pt} \end{array}$$

These inequalities show an interesting interplay among the coefficients of the ordering variables involving the new nodes t and p. The RHS of the extended inequalities gets more and more strengthened and, at the same time, the coefficient of x_{ji} decreases to maintain facetness (but not too much in order to keep validity).

It is remarkable that all these inequalities are facet-inducing for $P(G, d, s, 0)$, showing that the ideas leading to the covering-clique inequalities appear in many different facet-defining inequalities of this polytope. These results give another hint of the hardness of chromatic scheduling polytopes, since so many variations of a same idea are present as facets. It would be interesting to search for further variations of covering-clique inequalities involving more than two nodes outside the standard clique structure. $\triangleleft$

Chapter 6

Further classes of valid inequalities

> The results of this paper suggest that, in applying linear programming to a combinatorial problem, the number of relevant inequalities is not important but their combinatorial structure is.
>
> – Jack Edmonds (1965)

Chapter 5 presented facet-inducing inequalities coming from strengthenings and variations of the interval bound constraints and the antiparallelity contraints, mainly based on covering cliques of the interference graph. We now turn our attention to the development of facet-inducing inequalities based on different graph structures.

Section 6.1 opens the chapter with the so-called 4-cycle inequalities, an interesting class with an unusual structure. These inequalities combine a 4-cycle with a clique in the interference graph, involving two interval bounds and a number of ordering variables. A constructive proof of facetness is given for the uniform case $d = \mathbf{1}$. Section 6.2 analyses valid inequalities over cycles of the interference graph involving the ordering variables only. The main result of this section asserts that a cycle inequality is facet-inducing if and only if the associated cycle does not contain a chord, and it is worth noting that this result does not depend on the parity of the cycle.

Cycles in the interference graph also allow to construct inequalities only involving the interval variables. Section 6.3 presents a class of valid inequalities defined over odd holes of G. These inequalities are valid for every interference graph, and we prove that they define facets of $P(C_{2k+1}, \mathbf{1}, s, 0)$ whenever the polytope is nonempty. We also devise sufficient conditions for this inequality to be facet-inducing for arbitrary graphs.

The analysis of the polytope $P(K_n, d, s, 0)$, defined over a complete graph, is of theoretical interest and can also lead to facets for the general case. Sections 6.4 and 6.5 present two classes of facets for this polytope, along with the corresponding generalizations for arbitrary interference graphs. We also prove that the associated separation problems are $\mathcal{NP}$-complete.

6.1 4-Cycle inequalities

Chromatic scheduling polytopes over cycles are interesting and complex objects. For example, the polytope $R(C_4, \mathbf{1}, 4, 0)$ has 2.738 feasible solutions and 160 facets, whereas the polytope $R(C_5, \mathbf{1}, 4, 0)$ admits 17.500 feasible solutions and 644 facets. The following example presents a remarkable inequality that originated the results of this section.

Example 6.1 Consider the interference graph $(C_4, \mathbf{1})$ and suppose $s \geq 4$. The following inequality is valid for the polytope $P(C_4, \mathbf{1}, s, 0)$:

$$2x_{34} - 2x_{14} + 1 \;\leq\; l_1 + l_2 \tag{6.1}$$

This inequality can be viewed as a strenghtening of $1 \leq l_1 + l_2$, which is trivially valid if $12 \in E$, but does not define a facet if this edge 12 belongs to a larger clique. It is interesting to analyze the validity of inequality (6.1). The only nontrivial case is $x_{34} = 1$ and $x_{14} = 0$, where we have the two possible situations illustrated by Figure 6.1, depending on whether $x_{23} = 0$ or $x_{23} = 1$. In both cases, inequality (6.1) is satisfied. Furthermore, this inequality defines a facet of the full-dimensional polytope $P(C_4, \mathbf{1}, 4, 0)$, implying that it is facet-defining for all polytopes $P(C_4, \mathbf{1}, s, 0)$ with $s \geq 4$. It is remarkable that a valid inequality having only these nontrivial cases for validity still defines a facet of full-dimensional polytopes. $\lhd$

Figure 6.1: Possible cases for $x_{34} = 1$ and $x_{14} = 0$.

In the remaining of this section we construct a class of valid inequalities containing (6.1), and we prove that they are facet-defining when $g = 0$ and $s \geq s_{\min}(G, d, 0) + O(1)d_{\max}$. The construction of these inequalities takes a 4-cycle and replaces one of its nodes by a clique (see Figure 6.2). Recall that $d_{\max}$ stands for the maximum demand in the weighted interference graph.

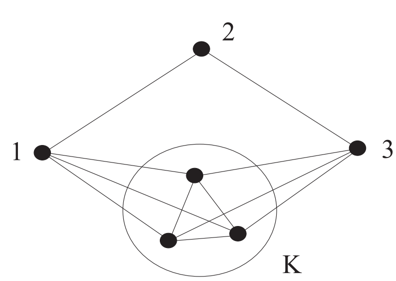


Figure 6.2: Structure for 4-cycle inequalities.

Definition 6.1 (4-cycle inequalities) *Let $1, 2, 3 \in V$ be three nodes such that $12, 23 \in E$ and $13 \notin E$. Let K be a clique covering $N(1) \cap N(3)$, and assume w.l.o.g. that $K = \{4, \ldots, t\}$. We define*

$$l_1 + l_2 \;\geq\; \sum_{k \in K} \alpha_k (x_{3k} - x_{1k}) + \beta. \tag{6.2}$$

to be the 4-cycle inequality *associated with these nodes, where* $\alpha_k = \begin{cases} d_k + d_3 & \text{if } k = 4 \\ d_k & \text{if } k > 4 \end{cases}$ *and* $\beta = \min\{d_1, d_2, d_3\}$.

We now prove that the 4-cycle inequalities are always valid but facet-inducing only if there are no edges between node 2 and the clique K.

Proposition 6.1 *The 4-cycle inequalities are valid for $P(G, d, s, g)$ and $R(G, d, s, g)$.*

Proof. Let $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be an integer feasible solution, and consider the following cases:

Case 1: $z_{r_3} \leq z_{l_1}$. Let $A = \{k \in K : z_{x_{3k}} = 1 \text{ and } z_{x_{1k}} = 0\}$. By definition, $A \cup \{3\}$ is a clique in G, and so the corresponding intervals do not overlap, hence $z_{l_1} \geq z_{l_3} + d_3 + \sum_{k \in A} d_k$. Moreover, $12 \in E$ implies $z_{l_1} + z_{l_2} \geq \min\{d_1, d_2\} \geq \beta$. Adding these two inequalities we get

$$\begin{aligned} z_{l_1} + z_{l_2} \;&\geq\; d_3 + \sum_{k \in A} d_k + \beta \\ &\geq\; \sum_{k \in K} \alpha_k (z_{x_{3k}} - z_{x_{1k}}) + \beta \;\diamond \end{aligned}$$

Case 2: $z_{r_3} > z_{l_1}$. In this case, $z_{x_{3k}} - z_{x_{1k}} \leq 0$, and thus the inequality (6.2) is dominated by $\beta \leq z_{l_1} + z_{l_2}$, which holds because the intervals $I(1)$ and $I(2)$ do not overlap in a feasible schedule. $\diamond$

In both cases the 4-cycle inequality (6.2) is satisfied, so it is valid for $P(G, d, s, g)$ and $R(G, d, s, g)$. $\square$

Theorem 6.2 *Assume that $N(1) \cap N(2) \cap N(3) = \emptyset$. If $s \geq s_{\min}(G, \mathbf{1}, 0) + 4$, then the 4-cycle inequality (6.2) defines a facet of $P(G, \mathbf{1}, s, 0)$.*

Proof. Let F be the face of $P(G, \mathbf{1}, s, 0)$ defined by (6.2). To prove that F is a facet, we shall construct the required number of affinely independent points in F.

1. Let H be the graph obtained from G by deleting the nodes 1, 2 and 3. Consider a feasible schedule $z \in P(H, \mathbf{1}, s-2, 0)$, and construct a point $y \in P(G, \mathbf{1}, s, 0) \cap \mathbf{Z}^{2n+m}$ as follows.

$$y_{l_i} \;=\; \begin{cases} z_{l_i} + 2 & \text{if } i \neq 1, 2, 3 \\ 1 & \text{if } i = 1, 3 \\ 0 & \text{if } i = 2 \end{cases} \qquad\qquad y_{r_i} \;=\; \begin{cases} z_{r_i} + 2 & \text{if } i \neq 1, 2, 3 \\ 2 & \text{if } i = 1, 3 \\ 1 & \text{if } i = 2 \end{cases}$$

Figure 6.3(a) shows this construction. This new solution is feasible and satisfies (6.2) at equality. We can construct many such solutions. In fact, there is a bijection between this set of solutions and the feasible integer solutions of $P(H, \mathbf{1}, s-2, 0)$. Since $s \geq s_{\min}(G, \mathbf{1}, 0) + 4$, the polytope $P(H, \mathbf{1}, s-2, 0)$ is full-dimensional, hence there are $2(n-3) + (m - |E(\{1,2,3\})|)$ such affinely independent points.

Notice that these points satisfy the following conditions:

$$
\begin{aligned}
y_{x_{21}} &= 1 && && (6.3)\\
y_{x_{23}} &= 1 && && (6.4)\\
y_{x_{1k}} &= 1 && \text{for } k \in N(1) && (6.5)\\
y_{x_{3k}} &= 1 && \text{for } k \in N(3) && (6.6)\\
y_{x_{2k}} &= 1 && \text{for } k \in N(2) && (6.7)\\
y_{r_i} - y_{l_i} &= 1 && \text{for } i = 1,2,3 && (6.8)
\end{aligned}
$$

For each of these equations in sequence, we now construct a feasible schedule in F not satisfying it at equality but satisfying the remaining ones, thus showing that F is a facet of $P(G, \mathbf{1}, s, 0)$.

2. The feasible solution depicted in Figure 6.3(b) satisfies (6.2) at equality and has $x_{21} = 0$, thus violating (6.3). Note that this solution satisfies conditions (6.4) to (6.8).

3. Similarly, the feasible solution in Figure 6.3(c) satisfies (6.2) at equality and has $x_{23} = 0$, thus violating (6.4) and being affinely independent w.r.t. the previous points. This solution satisfies conditions (6.5) to (6.8).

4. We now construct feasible solutions violating condition (6.5). To this end, for every $k \in N(1)$ construct a feasible solution according to the following cases:

   - If $k = 4$, consider the solution of Figure 6.3(d). Note that this construction is feasible since there are no edges between node 2 and K.
   - If $k \in K$ but $k \neq 4$, construct the feasible solution depicted in Figure 6.3(e).
   - If $k \in N(3) \backslash K$, consider the feasible solution presented in Figure 6.3(f). Note that $2k \notin E$ since $N(1) \cap N(2) \cap N(3) = \emptyset$ and $4k \notin E$ by the definition of the covering clique K.
   - Finally, if $k \notin N(3)$, consider the feasible solution presented in Figure 6.3(g).

   Each of these feasible points satisfies (6.2) at equality but does not satisfy condition (6.5), thus being affinely independent w.r.t. the previous points. Note that conditions (6.6) to (6.8) hold for these solutions.

5. For every $k \in N(3)$, we now construct a feasible solution in F not satisfying (6.6). If $k \notin N(2)$ consider the solution depicted in Figure 6.3(h), and if $k \notin N(1)$ consider Figure 6.3(i). Note that k must satisfy one of these conditions, for otherwise $k \in N(1) \cap N(2) \cap N(3)$, contradicting the hypothesis. Moreover, these solutions are in F and violate condition (6.6), thus being affinely independent w.r.t. the preceding points. Note that these points satisfy conditions (6.7) and (6.8).

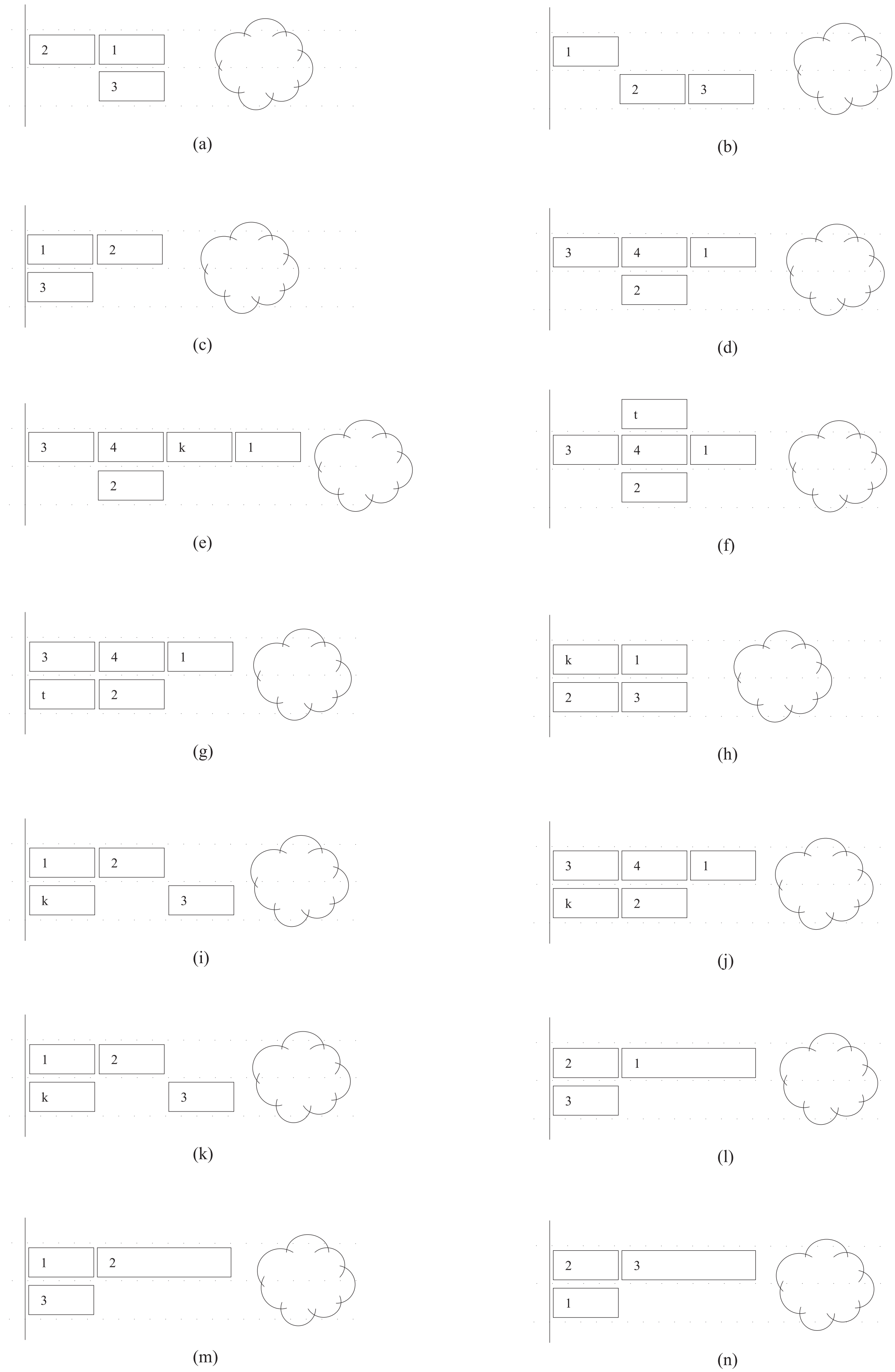


Figure 6.3: Feasible points for the proof of Theorem 6.2.

6. Now, for each $k \in N(2)$ we shall construct a feasible solution with $x_{2k} = 0$, hence violating (6.7). If $k \notin N(3)$ construct the solution presented in Figure 6.3(j), otherwise consider Figure 6.3(k) (in this case we have $k \notin N(1)$ by our hypothesis $N(1) \cap N(2) \cap N(3) = \emptyset$). These points do not satisfy condition (6.7), and therefore are affinely independent with the previous points. Moreover, note that these points satisfy (6.8).

7. To construct a feasible solution $y \in F$ with $y_{r_i} - y_{l_i} > d_i$ for $i = 1, 2, 3$ (thus finally violating condition (6.8)), we can consider any of the previous constructions having the interval $I(i)$ to the right of intervals $\{1, 2, 3\} \backslash \{i\}$, and extend the interval $I(i)$ one unit to the right. Figure 6.3(l), Figure 6.3(m) and Figure 6.3(n) show three feasible solutions that can be constructed that way. These three solutions are obviously affinely independent w.r.t. the previous points.

This way we construct the required number of affinely independent points in the face F of $P(G, d, s, 0)$ defined by (6.2). Thus, this inequality induces a facet of both $P(G, d, s, 0)$ and $R(G, d, s, 0)$. □

6.2 Cycle-order inequalities

Definition 6.2 (cycle-order inequalities) *Let $C = \{1, \ldots, k\}$ be a k-cycle in G. The following inequality is the* cycle-order inequality *associated with C:*

$$x_{12} + x_{23} + \ldots + x_{k-1,k} + x_{k1} \;\leq\; k - 1 \tag{6.9}$$

Note that the triangle inequalities 4.1 are a special kind of cycle-order inequalities. It is not difficult to verify that cycle-order inequalities are valid for both $P(G, d, s, g)$ and $R(G, d, s, g)$, since they are valid for the linear ordering polytope and every partial ordering given by the ordering variables can be extended to a linear ordering (which satisfies the cycle-order inequalities by definition).

However, these inequalities are facet-defining for the linear ordering polytope only if $k = 3$, due to the equality constraints $x_{ij} + x_{ji} = 1$ [23]. Due to this fact, we cannot expect cycle-order inequalities to be facet-defining for chromatic scheduling polytopes in general. This section shows that the cycle-order inequalities are facet-defining if and only if C is a chordless cycle, provided the frequency spectrum $[0, s]$ is large enough. These results do not depend on the parity of the number of nodes of C. It is worth noting that cycle-order inequalities do define facets of the associated acyclic subdigraph polytope, where the weaker constraints $x_{ij} + x_{ji} \leq 1$ are imposed [24].

Definition 6.3 *If $C = \{1, \ldots, k\} \subseteq V$ is a cycle, we define $\nu(C) = \#\{ij \in E(C) : i$ and j belong to different sectors$\}$.*

Theorem 6.3 *If C is a chordless cycle and $s > s_{\min}(G \backslash C, d, g) + d(C) + g\,\nu(C) + d_{\max}$, then the cycle-order inequality (6.9) defines a facet of $P(G, d, s, g)$ and $R(G, d, s, g)$.*

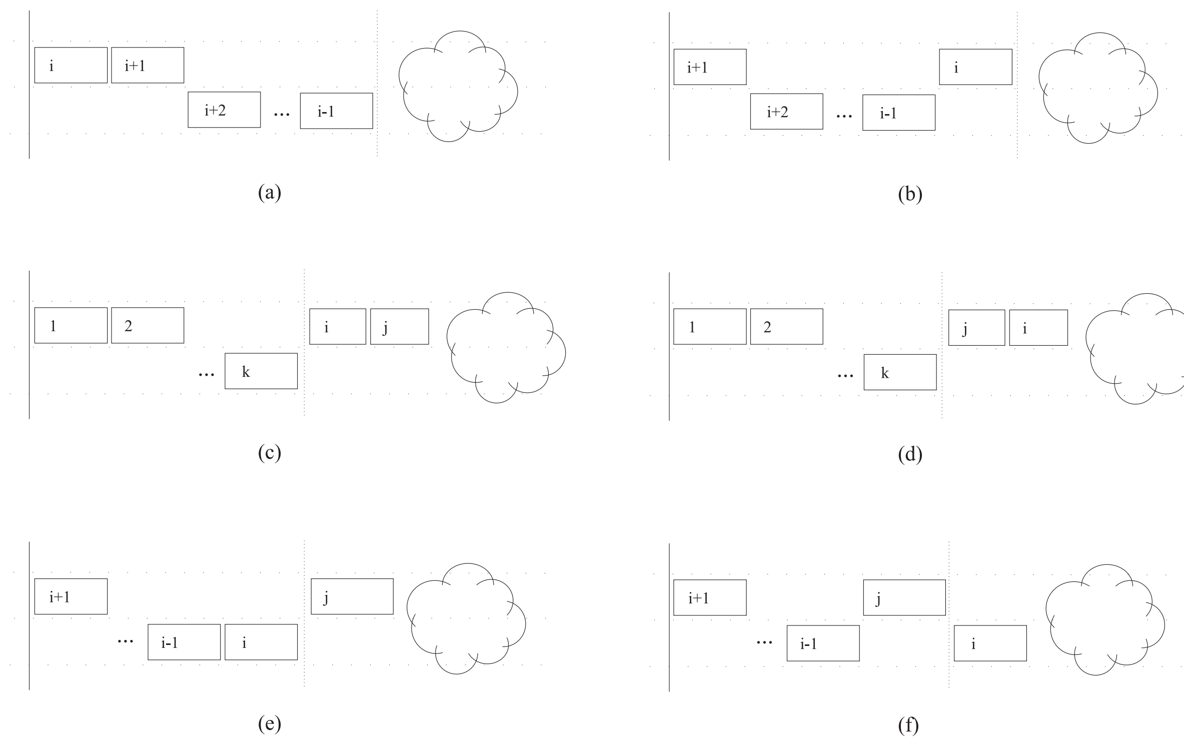


Figure 6.4: Constructions for the proof of Theorem 6.3.

Proof. Let F be the face of $R(G,d,s,g)$ defined by (6.9), suppose $\lambda^T z = \lambda_0$ for every $z \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$. Since $s > s_{\min}(G,d,g)$, we have $\lambda_{l_i} = \lambda_{r_i} = 0$ for every $i \in V$ by Lemma 3.8 and Lemma 3.10. To complete the proof, we show that $\lambda_{x_{ij}} = 0$ for every $ij \in E$.

Claim 1: $\boldsymbol{\lambda_{x_{i,i+1}} = 0}$ for $\boldsymbol{i, i+1 \in E(C)}$. Consider the feasible schedules z^1 and z^2 depicted in Figure 6.4(a) and Figure 6.4(b) respectively, where the intervals $\{I(k)\}_{k \in C}$ are assigned within the interval $[0, kd(C) + \nu(C)]$. Both points belong to F, hence $\lambda^T z^1 = \lambda_0 = \lambda^T z^2$ and thus $\lambda_{x_{i,i+1}} = 0$. $\diamond$

Claim 2: $\boldsymbol{\lambda_{x_{ij}} = 0}$ for $\boldsymbol{ij \notin E(C)}$. The feasible solutions presented in Figure 6.4(c) and Figure 6.4(d) show that $\lambda_{x_{ij}} = 0$. Note that these constructions are feasible since $s > s_{\min}(G \backslash C, d, g) + s_{\min}(C, d, g) + d_{\max}$. $\diamond$

Claim 3: $\boldsymbol{\lambda_{x_{ij}} = 0}$ if $\boldsymbol{i \in C}$ and $\boldsymbol{j \notin C}$. To prove this claim, consider the feasible solutions depicted in Figure 6.4(e) and Figure 6.4(f). Both points belong to F, hence $\lambda_{x_{ij}} = 0$. $\diamond$

This sequence of claims shows $\lambda = \mathbf{0}$, hence F is a facet of $P(G,d,s,g)$ and $R(G,d,s,g)$. $\square$

Proposition 6.4 *If C has a chord and $P(G,d,s,g)$ resp. $R(G,d,s,g)$ is full-dimensional, then the cycle-order inequality (6.9) does not define a facet of $P(G,d,s,g)$ resp. $R(G,d,s,g)$.*

Proof. Let $ij \in E$ be a chord of C (i.e., $1 \leq i < j \leq k$ and $j \neq i+1 \pmod{k}$), and consider an arbitrary point $z \in P(G,d,s,g) \cap \mathbf{Z}^{2n+m}$ satisfying (6.9) at equality. This implies $z_{x_{12}} + \ldots + z_{x_{k1}} = k-1$, hence all variables $z_{x_{12}}, \ldots, z_{x_{k1}}$ but one are set to 1. Let $t \in \{1, \ldots, k\}$ such that $z_{x_{t,t+1}} = 0$. Therefore, the intervals corresponding to the nodes in C are assigned in the order $t+1 \to t+2 \ldots k \to 1 \to 2, \ldots, t$. Let $P = \{i, i+1, \ldots, j-1, j\}$ denote the path from i to j in C. We shall show that z satisfies

$$z_{x_{ij}} = \sum_{e \in E(P)} z_{x_e} - (|E(P)| - 1). \tag{6.10}$$

Case 1: $z_{x_{ij}} = 1$. In this case, $I(i)$ is located before $I(j)$. But this means that $I(i)$ is located before $I(i+1)$, $I(i+1)$ is located before $I(i+2)$, ..., and $I(j-1)$ is located before $I(j)$, implying $z_{x_e} = 1$ for every edge $e \in E(P)$. Hence $\sum_{e \in E(P)} z_{x_e} = |E(P)|$, so we conclude that z satisfies (6.10). $\diamond$

Case 2: $z_{x_{ij}} = 0$. Here, $I(j)$ is located before $I(i)$, and thus we have $z_{x_{i,i+1}} = \ldots = z_{x_{t-1,t}} = 1$, $z_{x_{t,t+1}} = 0$ and $z_{x_{t+1,t+2}} = \ldots = z_{x_{j-1,j}} = 1$. But now we have $\sum_{e \in E(P)} z_{x_e} = |E(P)| - 1$ and so (6.10) is again satisfied. $\diamond$

Therefore, the point z satisfies (6.10) and (6.9) at equality, and it is not difficult to check that these equations are linearly independent. Hence the dimension of the face of $P(G,d,s,g)$ defined by (6.9) is at most $2n+m-2$. Since $P(G,d,s,g)$ is full-dimensional, (6.9) does not define a facet. The same argumentation applies to the fixed-length case. $\square$

Corollary 6.5 *If $s > s_{\min}(G \backslash C, d, g) + d(C) + g\,\nu(C) + d_{\max}$, then the cycle-order inequality associated with a cycle C is facet-defining if and only if C is chordless.*

It is interesting to generate the symmetric inequalities of cycle-order inequalities. By Theorem 4.3 we can verify that the symmetric inequality of (6.9) is given by

$$1 \leq x_{12} + x_{23} + \ldots + x_{k-1,k} + x_{k1}.$$

It is worth noting that this symmetric inequality gives the opposite lower bound on the ordering variables along the cycle. By Theorem 4.3, this new inequality is facet-defining for $s > s_{\min}(G \backslash C, d, g) + k d_{\max}$ if and only if C is a chordless cycle.

6.2.1 Complexity of the separation problem

We now address the complexity of the separation problem for the cycle-order inequalities. Given a point $z \in P_{LP}(G,d,s,g)$, this problem consists in deciding whether there exists some cycle-order inequality violated by z or not.

	CYCLE-ORDER INEQUALITIES SEPARATION
Instance:	A point $z = (l, r, x) \in P_{LP}(G,d,s,g)$
Question:	Does z violate some cycle-order inequality?

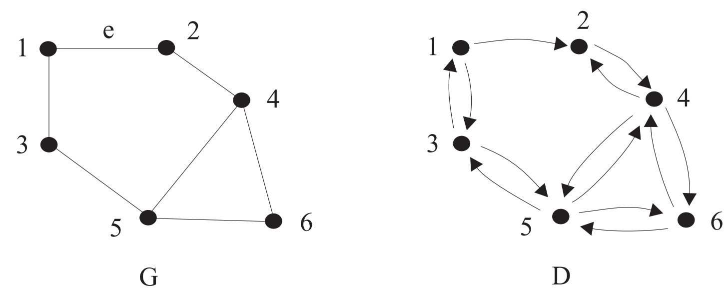


Figure 6.5: Construction of D from G.

The main result of this section asserts that this problem is polynomially solvable, by providing a number of reductions to the minimum mean cycle problem [3, 33]. The latter takes as input a directed graph D with edge costs $c : E_D \to \mathbf{R}$ and consists in finding a directed cycle C such that $\frac{1}{|C|}\sum_{ij \in E(C)} c_{ij}$ is minimum among all directed cycles in D. Such a cycle is called a *minimum mean cycle* of D. The minimum mean cycle problem arises as a special case of the minimum cost-to-time ratio problem [3] and can be solved in $O(nm)$ time [31, 32].

Theorem 6.6 *The cycle-order inequalities can be separated in $O(nm^2)$ time.*

Proof. Let $e \in E$ be a directed edge of the interference graph, and construct a digraph $D = (V, E_D)$ by replacing every (nondirected) edge of G by two directed edges with the same endpoints and opposite directions. The only exception is the edge e, which is transformed into only one directed edge in D:

$$E_D \;=\; \{ij, ji : ij \in E \text{ and } e \neq ij\} \cup \{e\}.$$

Figure 6.5 shows this construction. Now define edge costs $c : E_D \to \mathbf{R}$ as the values of the ordering variables in z, according to the orientation of the corresponding directed edge (again, the edge e is an exception):

$$c_{ij} \;=\; \begin{cases} -(1 + z_{x_{ij}}) & \text{if } ij = e \\ -z_{x_{ij}} & \text{otherwise} \end{cases}$$

Claim: The point $z \in P_{LP}(G, d, s, g)$ violates a cycle-order inequality such that the associated cycle contains the edge e if and only if the digraph D has a directed cycle C such that $\frac{1}{|C|}\sum_{ij \in E(C)} c_{ij} < -1$.

$\Rightarrow$) Let C be a directed cycle with $\frac{1}{|C|}\sum_{ij \in E(C)} c_{ij} < -1$ and call $k = |C|$. Such a cycle contains e, since otherwise $c_{ij} \geq -1$ for every edge $ij \in E(C)$, implying $\sum_{ij \in E(C)} \frac{c_{ij}}{k} \geq -1$. Consider now the cycle-order inequality associated with the directed cycle C. We have $\sum_{ij \in E(C)} c_{ij} < -k$, and moreover $-\sum_{ij \in E(C)} c_{ij} = 1 + \sum_{ij \in E(C)} z_{x_{ij}}$, hence the cycle-order inequality associated with C is violated by the point z.

$\Leftarrow$) Let $C \subseteq V$ be a directed k-cycle such that $e \in E(C)$ and $\sum_{ij \in E(C)} z_{x_{ij}} > k - 1$. By the construction of D, it is not difficult to verify that C is a cycle with mean strictly less than -1:

$$\begin{aligned}
\sum_{ij \in E(C)} \frac{c_{ij}}{k} &= \frac{1}{k}\Big(c_e + \sum_{ij \in E(C)\setminus\{e\}} c_{ij}\Big) \\
&= -\frac{1}{k}\Big(1 + z_{x_e} + \sum_{ij \in E(C)\setminus\{e\}} z_{x_{ij}}\Big) \\
&= -\frac{1}{k}\Big(1 + \sum_{ij \in E(C)} z_{x_{ij}}\Big) \\
&< -\frac{1}{k}\left(1 + (k-1)\right) = -1 \diamond
\end{aligned}$$

Now, for each $ij \in E$, apply the preceding procedure twice to decide whether some cycle-order inequality containing ij resp. ji violates the point $z \in P_{LP}(G, d, s, g)$. The overall running time of this algorithm is clearly $O(nm^2)$. $\square$

6.3 Odd hole inequalities

This section presents a class of valid inequalities defined over odd holes of the interference graph. The integer solutions in the face of $R(G, d, s, 0)$ defined by these inequalities have a very particular combinatorial structure that can be exploited to show that these inequalities induce facets of $R(C_{2k+1}, \mathbf{1}, s, 0)$ for $k \geq 2$. Throughout this section we assume $g = 0$.

Definition 6.4 (odd hole inequalities) *Let $C = \{1, \ldots, 2k+1\}$ be an induced odd cycle, called an* odd hole*, of the interference graph. We define*

$$\sum_{i=1}^{2k+1} l_i \geq k + 2 \qquad (6.11)$$

to be the odd hole inequality *associated with C.*

Proposition 6.7 *The odd hole inequalities are valid for $P(G, \mathbf{1}, s, 0)$ and $R(G, \mathbf{1}, s, 0)$.*

Proof. Let $z \in P(G, \mathbf{1}, s, 0) \cap \mathbf{Z}^{2n+m}$ be a feasible schedule. Since C is a nonbipartite graph, we have $z_{l_i} \geq 2$ for at least one node $i \in C$ (otherwise we would be able to assign all the intervals $I(j)$, with $j \in C$, within the frequency spectrum $[0, 2]$, a contradiction). Assume w.l.o.g. that $C = \{1, \ldots, 2k+1\}$ and $z_{l_{2k+1}} \geq 2$. For $t = 1, \ldots, k$, the inequality $z_{l_{2t}} + z_{l_{2t+1}} \geq 1$ holds, since $2t$ and $2t+1$ are adjacent nodes. Summing up these inequalities, we obtain $\sum_{i=1}^{2k} z_{l_i} \geq k$. Combining this last inequality with $z_{l_{2k+1}} \geq 2$ we get $\sum_{i=1}^{n} z_{l_i} \geq k + 2$, hence z satisfies the odd hole inequality associated with C. Since (6.11) does not involve the r-variables, it is also valid for $R(G, \mathbf{1}, s, 0)$. $\square$

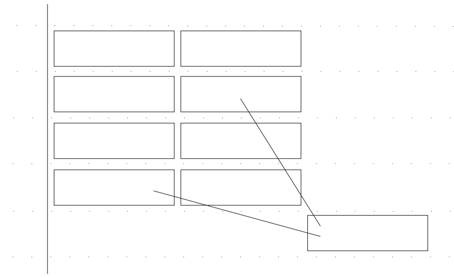


Figure 6.6: Feasible solution satisfying the odd hole inequality at equality.

We now analyze the faces induced by the odd hole inequalities. The feasible schedules in these faces must satisfy $\sum_{i \in C} l_i = k + 2$. This implies that k nodes of C are assigned the interval $[0, 1]$, and k distinct nodes receive the interval $[1, 2]$ in the schedule. In order to maintain feasibility, the remaining node must be assigned the interval $[2, 3]$ (see Figure 6.6 for an example). This combinatorial structure was used in Section 3.2.3 to provide a proof of full-dimensionality of $R(C_{2k+1}, \mathbf{1}, 3, 0)$ for $k \geq 2$. The same arguments can be applied to prove that the odd hole inequalities induce facets of chromatic scheduling polytopes.

Theorem 6.8 *Let $C_{2k+1} = \{1, \ldots, 2k+1\}$ be a hole on $2k+1$ nodes. The odd hole inequality associated with C_{2k+1} induces facets of $R(C_{2k+1}, \mathbf{1}, s, 0)$ and $P(C_{2k+1}, \mathbf{1}, s, 0)$ if $k \geq 2$ and $s \geq 3$.*

Proof. For $i = 1, \ldots, 2k+1$, define an order of the nodes by $S_i = (i, i+1, \ldots, 2k+1, 1, \ldots, i-1)$ and let y^i be the greedy solution associated with this sequence (see Section 3.2.3 for the definition). Further define the opposite order $\bar{S}_i = (i, i-1, \ldots, 1, 2k+1, \ldots, i+1)$ and let $\bar{y}^i$ be the associated greedy solution. It is not difficult to verify that these solutions lie in the face of $R(C_{2k+1}, \mathbf{1}, s, 0)$ defined by the odd hole inequality associated with C_{2k+1}. Moreover, following the proof of Theorem 3.27 in Section 3.2.3 we obtain that the solutions $\{y^i, \bar{y}^i : i \in C_{2k+1}\}$ are affinely independent. Since $R(C_{2k+1}, \mathbf{1}, s, 0) \subseteq \mathbf{R}^{4k+2}$, the existence of these $4k + 2$ affinely independent points shows that the odd hole inequality associated with C_{2k+1} induces a facet of this polytope. Now, for $i = 1, \ldots, 2k + 1$, construct the two feasible solutions presented in Figure 3.10(a), (b). These feasible schedules, together with the previous constructions, show that the odd hole inequality associated with C_{2k+1} induces a facet of $P(C_{2k+1}, \mathbf{1}, s, 0)$. $\square$

Now we turn to arbitrary interference graphs. Let $C \subseteq V$ be an odd hole of G, and suppose w.l.o.g. that $C = \{1, \ldots, 2k + 1\}$. We say that $i \notin C$ is *parity nonadjacent* to the cycle C if i is nonadjacent to a stable set of size k in C_{2k+1}. If this does not hold, we say that i is *parity adjacent* to the cycle C.

Corollary 6.9 *Let $C \subseteq V$ be an odd hole and suppose $s \geq s_{\min}(G, \mathbf{1}, 0) + 4$. The odd hole inequality associated with C defines a facet of $R(G, \mathbf{1}, s, 0)$ if and only if every node $i \notin C$ is parity nonadjacent to C.*

Proof. Since $s \geq s_{\min}(G, \mathbf{1}, 0) + 4$, $R(G, \mathbf{1}, s, 0)$ and $P(G, \mathbf{1}, s, 0)$ are full-dimensional by Theorem 3.11. If $i \notin C$ is parity adjacent to C, then every feasible solution satisfying the odd hole inequality at equality has $x_{ji} = 1$ for every $j \in C \cap N(i)$, hence the face defined by this inequality cannot have the required dimension for being a facet.

Conversely, suppose that every node $i \notin C$ is parity nonadjacent to C, and let $\lambda \in \mathbf{R}^{n+m}$ and $\lambda_0 \in \mathbf{R}$ such that $\lambda^T y = \lambda_0$ for every $y \in R(G, \mathbf{1}, s, 0)$. For every feasible schedule $y \in R(C, \mathbf{1}, 3, 0)$ and every feasible schedule $y' \in R(G \backslash C, \mathbf{1}, s, 0)$, for $s = s_{\min}(G, \mathbf{1}, 0)$, construct a new schedule $z \in R(G, \mathbf{1}, s, 0)$ by setting

$$z_{l_i} = \begin{cases} y_{l_i} & \text{if } i \in C \\ y'_{l_i} + 3 & \text{if } i \notin C \end{cases}$$

This set of feasible solutions shows $\lambda_{l_i} = 1$ for $i \in C$, $\lambda_{l_i} = 0$ for $i \in V \backslash C$, and $\lambda_{x_{ij}} = 0$ for $ij \in E(C) \cup E(V \backslash C)$. To complete the proof, it remains to show $\lambda_{x_{ij}} = 0$ for every $ij \in E$ with $i \notin C$ and $j \in C$. For every such edge, construct a feasible solution satisfying the odd hole inequality associated with C, such that $I(j) = [2, 3]$ and $I(i) = [1, 2]$. Such a solution exists since i is parity nonadjacent to C. This new feasible solution shows $\lambda_{x_{ij}} = 0$, hence λ is a multiple of the coefficient vector of the odd hole inequality associated with C which, therefore, defines a facet of $R(G, \mathbf{1}, s, 0)$. A similar argumentation applies to $P(G, \mathbf{1}, s, 0)$. □

We can devise a similar inequality for the nonuniform case $d \geq \mathbf{1}$. If $C = \{1, \ldots, 2k+1\}$ is an odd hole of G, then

$$\sum_{i=1}^{2k+1} l_i \geq d_{\min}(C)(k+2) \tag{6.12}$$

is valid for $P(G, d, s, 0)$ and $R(G, d, s, 0)$, where $d_{\min}(C) = \min_{i \in C} d_i$ is the minimum demand among the nodes in C. Note that this inequality generalizes (6.11), since $d_{\min}(C) = 1$ if $d = \mathbf{1}$. However, this inequality does not induce facets for arbitrary instances, since $d_i < d_{i+1}$ implies $x_{i,i+1} = 1$ for every feasible schedule satisfying (6.12) at equality.

6.3.1 Complexity of the separation problem

It is not difficult to verify that a superclass of the odd hole inequalities can be separated in polynomial time, provided $l_i + l_j \geq 1$ for every $ij \in E$. Consider a fractional solution $z \in P_{LP}(G, \mathbf{1}, s, 0)$ and assume $z_{l_i} + z_{l_j} \geq 1$ for every $ij \in E$ (if this assumption is not satisfied, we have detected the violated inequality $l_i + l_j \geq 1$). Consider the interference graph $G = (V, E)$ with edge weights $c : E \to \mathbf{R}_+$ defined as $c_{ij} = z_{l_i} + z_{l_j} - 1$ (note that $c_{ij} \geq 0$ by the initial assumption). Under these assumptions, the odd hole inequality (6.11) is equivalent to

$$\sum_{i=1}^{2k+1} c_{i,i+1} \geq 3,$$

where indices are taken modulo $2k+1$. Therefore, there is a violated odd cycle inequality (associated with a not necessarily chordless cycle) if and only if there exists an odd hole with weight strictly less than 3. The problem of finding a minimum odd cycle in an undirected graph with nonnegative edge weights can be polynomially solved by successive applications of the shortest path algorithm [25]. Hence the odd hole inequalities can be separated in $O(m\ SP(n,m))$ time, where $SP(n,m)$ is the running time of a shortest path algorithm in a graph with n nodes and m edges.

6.4 Interval-sum inequalities

This section presents a canonical valid inequality that constrains the total interval length in the nonfixed case $P(G,d,s,0)$. This inequality is facet-inducing for $P(K_n,d,s,0)$ if and only if $s > \sum_{i=1}^n d_i$, and is also facet-inducing for $P(G,d,s,0)$ when $s \gg \omega(G,d)$.

Assumption. Throughout this section we shall assume $g = 0$.

6.4.1 Interval-sum inequalities for complete interference graphs

Definition 6.5 (interval-sum inequalities) *Let K_n be the complete graph on n nodes, and consider the polytope $P(K_n,d,s,0)$. We define*

$$\sum_{k=1}^{n}(r_k - l_k) \le s \tag{6.13}$$

to be the interval-sum inequality *associated with this instance.*

Note that this inequality does not apply to the fixed-length polytope $R(K_n,d,s,0)$ since the natural replacement $r_i = l_i + d_i$ for the fixed-length case would yield the trivial inequality $\sum_{i\in V} d_i \le s$. It is not difficult to verify that (6.13) is valid for $P(K_n,d,s,0)$, since the intervals $\{I(i)\}_{i=1}^n$ cannot overlap. If $s = \sum_{i=1}^n d_i$, then every feasible schedule of $P(K_n,d,s,0)$ satisfies (6.13) at equality, and so the corresponding face is not proper. On the other hand, if $s > \sum_{i=1}^n d_i$ then this inequality induces a facet of $P(K_n,d,s,0)$ as Theorem 6.10 shows.

Theorem 6.10 *If $s > \sum_{i=1}^n d_i$ then (6.13) defines a facet of $P(K_n,d,s,0)$.*

Proof. Since $s > \sum_{i=1}^n d_i$, Theorem 3.11 implies that $P(K_n,d,s,0)$ is full-dimensional. Let F be the face of this polytope defined by (6.13), and suppose $\lambda^T y = \lambda_0$ for every point $y \in F$. We shall prove that $\lambda = \alpha\pi$, where π is the coefficient vector of the inequality (6.13), thus showing that this inequality induces a facet.

Let i and j be two different nodes and consider the points y^1 and y^2 depicted in Figure 6.7(a) and Figure 6.7(b). These points are in F and thus $\lambda^T y^1 = \lambda_0 = \lambda^T y^2$. Since y^1 and y^2 only differ in their r_i- and l_j-coordinates, we have

$$d_i\lambda_{r_i} + d_i\lambda_{l_j} = (d_i+1)\lambda_{r_i} + (d_i+1)\lambda_{l_j}$$

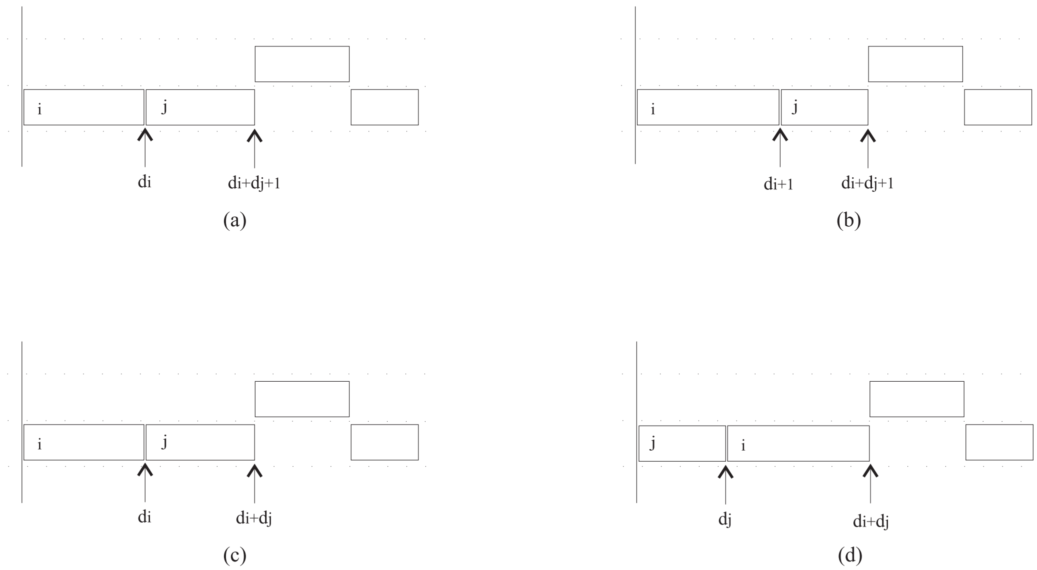


Figure 6.7: Constructions for the proof of Theorem 6.10.

and, therefore, $\lambda_{r_i} = -\lambda_{l_j}$. Since i and j are arbitrary, there exists some $\alpha \in \mathbf{R}$ such that

$$\lambda_{r_k} = \alpha \qquad k = 1, \ldots, n \tag{6.14}$$

$$\lambda_{l_k} = -\alpha \qquad k = 1, \ldots, n \tag{6.15}$$

Consider now the two points depicted in Figure 6.7(c) and Figure 6.7(d). Again, these points are in F, and thus we have

$$d_i \lambda_{r_i} + d_i \lambda_{l_j} + (d_i + d_j)\lambda_{r_j} = d_j \lambda_{r_j} + d_j \lambda_{l_i} + (d_i + d_j)\lambda_{r_i} + \lambda_{x_{ji}}.$$

But we know that $\lambda_{r_i} = -\lambda_{l_j}$, and so $d_i \lambda_{r_i} + d_i \lambda_{l_j} = 0$. We obtain $d_j \lambda_{r_j} + d_j \lambda_{l_i} = 0$ in a similar way, and thus

$$\lambda_{x_{ji}} = (d_i + d_j)(\lambda_{r_j} - \lambda_{r_i}) = 0.$$

Since i and j are arbitrarily chosen, we have $\lambda_{x_e} = 0$ for every edge e of K_n. Hence $\lambda = \alpha\pi$, and this implies $\lambda_0 = \alpha s$. Therefore, the inequality (6.13) defines a facet of $P(K_n, d, s, 0)$. $\square$

6.4.2 Interval-sum inequalities for arbitrary interference graphs

We now analyze the interval-sum inequalities in the general case $P(G, d, s, 0)$ for an arbitrary interference graph G. If $K \subseteq V$ is a clique (recall that a *clique* is not necessarily a maximal complete subgraph), then

$$\sum_{k \in K} (r_k - l_k) \leq s \tag{6.16}$$

is valid for $P(G, d, s, 0)$. We are interested in characterizing the cases for which this inequality is facet-inducing. To this end, note that if K is not a maximal clique then no feasible schedule can satisfy (6.16) at equality, hence the associated face is empty. So K must be maximal if (6.16) is supposed to define a facet of $P(G, d, s, 0)$.

However, the maximality of K is necessary but not sufficient for facetness. If there exists some node $i \notin K$ having a unique nonneighbor $k \in K$, then $y_{x_{il}} = y_{x_{it}}$ $\forall\ l, t \in K \backslash \{k\}$ for every integer point y in the face defined by (6.16), so this face is not maximal if $P(G, d, s, 0)$ is full-dimensional. Therefore, if K is not maximal or if there exists some $i \notin K$ with $|N(i) \cap K| = |K| - 1$, then (6.16) does not define a facet of $P(G, d, s, 0)$. Theorem 6.11 shows that the converse is also true.

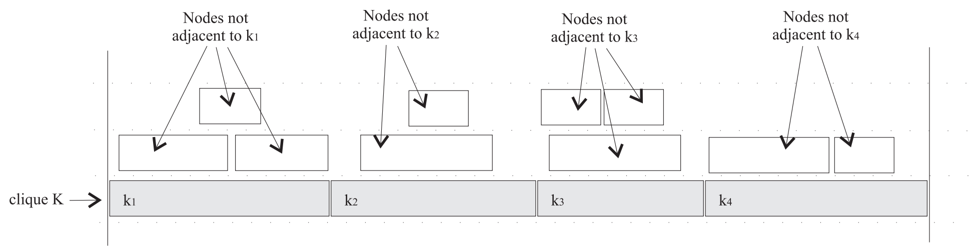


Figure 6.8: Construction of feasible solutions in F.

Theorem 6.11 *If $s \geq \sum_{i \in V} d_i$, $K \subseteq V$ is a clique, and every node $i \notin K$ has at least two nonneighbors $p(i), p'(i) \in K$, then (6.16) defines a facet of $P(G, d, s, 0)$.*

Proof. Let F denote the face of $P(G, d, s, 0)$ defined by (6.16), and suppose $\lambda^T y = \lambda_0$ for every point $y \in F$. We shall prove $\lambda = \alpha\pi$, for some $\alpha \in \mathbf{R}$, where π is the coefficient vector of the inequality. Note first that we can construct a feasible solution $y \in F$ by covering $[0, s]$ with nonoverlapping intervals corresponding to the nodes in K, and assigning every node $i \notin K$ inside the interval $[y_{l_{p(i)}}, y_{r_{p(i)}}]$ (see Figure 6.8). The intervals assigned to the nodes in K must be large enough to allow this construction (note that this construction is feasible since we are considering the general polytope $P(G, d, s, 0)$ and $s \geq \sum_{i \in V} d_i$).

Similar configurations as in Figure 6.7(a) and Figure 6.7(b) can be used to show $\lambda_{r_i} = \lambda_{l_j}$ for $i, j \in K$. We construct two points in F, assigning $I(k)$, for $k \notin K$, "inside" the interval $I(p(k))$ or $I(p'(k))$, as in Figure 6.9(a). If $\{p(k), p'(k)\} = \{i, j\}$, then we assign $I(k)$ in $[0, y_{r_j}]$, as in Figure 6.9(b). This way we show $\lambda_{r_i} = -\lambda_{l_i} = \alpha \in \mathbf{R}$ $\forall i \in K$. Similarly, the construction of Figure 6.7(c) and Figure 6.7(d) can be adapted to this case to prove $\lambda_{x_{ij}} = 0$ for $i, j \in K$.

It only remains to show $\lambda_{l_k} = \lambda_{r_k} = 0$ for $k \notin K$, and $\lambda_{x_{ij}} = 0$ for $i \notin K$ or $j \notin K$ (or both). Figure 6.9(c) and Figure 6.9(d) show how to construct two points in F that only differ in their r_k-coordinate, thus proving $\lambda_{r_k} = 0$. We can show $\lambda_{l_k} = 0$ for every $k \notin K$ similarly.

Finally, we verify that $\lambda_{x_e} = 0$ holds for every edge $e \notin E$, by considering two cases. If $e = ik$ with $i \in K$ and $k \notin K$, define y^1 and y^2 as in Figure 6.9(e) and Figure 6.9(f) respectively, and if $e = kr$ with $k, r \notin K$, define y^1 and y^2 as depicted in Figure 6.9(g) and Figure 6.9(h), respectively. The points y^1 and y^2 are in F, so $\lambda^T y^1 = \lambda^T y^2$ and thus

$$y^1_{l_k}\lambda_{l_k} + y^1_{r_k}\lambda_{r_k} + \lambda_{x_e} = y^2_{l_k}\lambda_{l_k} + y^2_{r_k}\lambda_{r_k}.$$

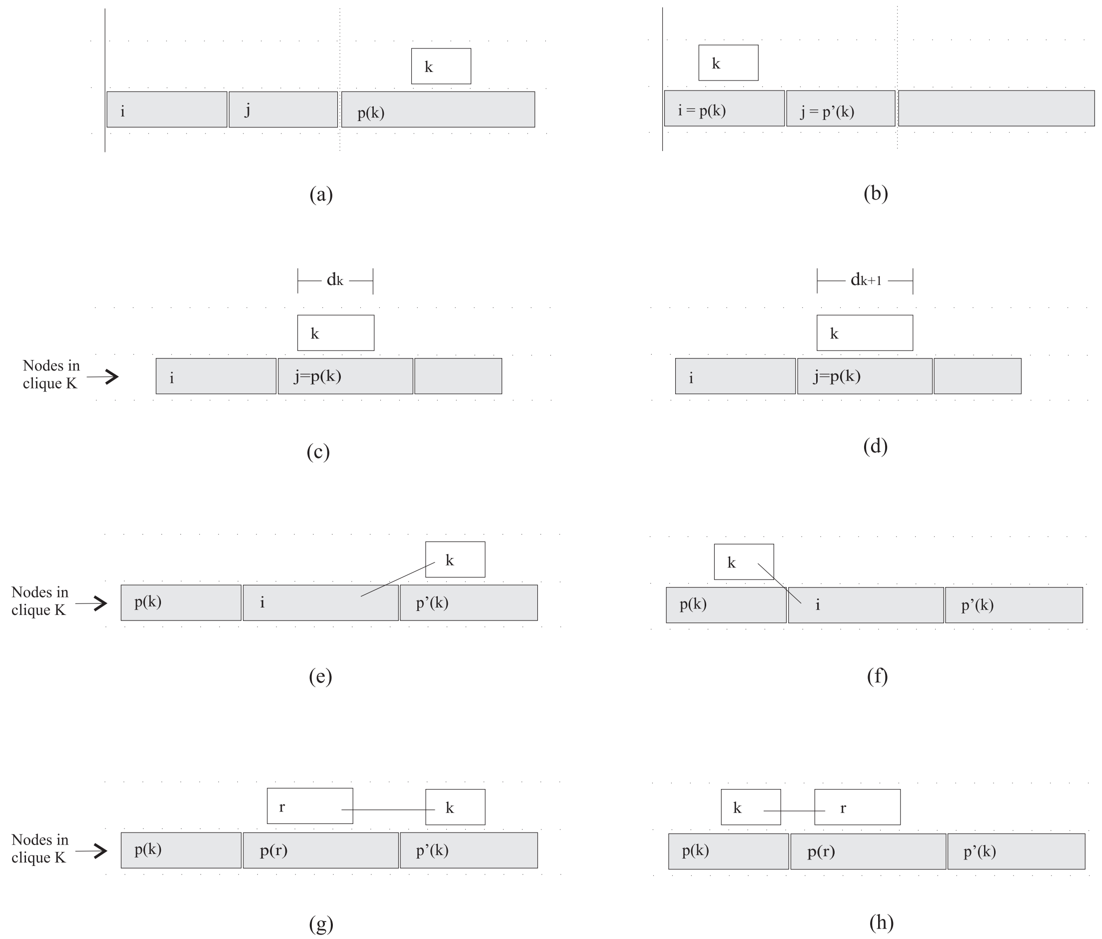


Figure 6.9: Constructions for the proof of Theorem 6.11.

But $\lambda_{l_k} = \lambda_{r_k} = 0$ for $k \notin K$, hence $\lambda_{x_e} = 0$. Therefore, the inequality (6.16) defines a facet of $P(G, d, s, 0)$. $\square$

As we have already noted, if there exists some $i \in K$ with at most one nonneighbor in K (which implies that K is a maximal clique), then (6.16) is not facet-inducing for $P(G, d, s, 0)$. Combining this observation with Theorem 6.11 yields the following result.

Corollary 6.12 *Let $s \geq \sum_{i \in V} d_i$. Then, the interval-sum inequality (6.16) defines a facet of $P(G, d, s, 0)$ if and only if $|K \backslash N(i)| \geq 2$ for every $i \notin K$.*

Remark. Suppose that $|K \backslash N(i)| \geq 2$ for every $i \notin K$, and partition $V \backslash K$ into $V \backslash K = \cup_{k \in K} V_k$ such that $V_k \cap N(k) = \emptyset$. Moreover, let G_k be the subgraph of G induced by V_k. Under these definitions, we can strengthen the bound $s \geq \sum_{i \in V} d_i$ from Theorem 6.11. Under these definitions, the interval-sum inequality (6.16) defines a facet of $P(G, d, s, 0)$ if $s > \max\{d(K), \sum_{k \in K} s_{\min}(G_k, d, 0)\}$. $\triangleleft$

6.4.3 Complexity of the separation problem

The separation problem for the interval-sum inequalities takes as input a point in the linear relaxation $P_{LP}(G, d, s, 0)$, and consists in deciding whether this point is violated by some interval-sum inequality or not. We may state this problem as follows:

	INTERVAL-SUM INEQUALITIES SEPARATION
Instance:	A point $y \in P_{LP}(G, d, s, 0)$.
Question:	Is there any maximal clique K such that $\sum_{i \in K} y_{r_i} - y_{l_i} > s$?

Theorem 6.13 INTERVAL-SUM INEQUALITIES SEPARATION *is $\mathcal{NP}$-complete in the strong sense.*

Proof. Consider the WEIGHTED MAX-CLIQUE problem, defined as follows:

Instance:	A graph $H = (V_H, E_H)$, a weight $w_i \in \mathbf{Z}_+$ for each $i \in V_H$, and an integer k (me way assume $k \geq 3$ and $1 \leq w_i \leq k-1$).
Question:	Is there a clique K of H with weight at least k?

WEIGHTED MAX-CLIQUE is $\mathcal{NP}$-complete in the strong sense [20], and we will construct a pseudopolynomial reduction from this problem to INTERVAL-SUM INEQUALITIES SEPARATION. Given an instance (H, w, k) of WEIGHTED MAX-CLIQUE, we construct an instance of the separation problem as follows. Let $D = \{i \in V_H : w_i > \frac{k-1}{2}\}$. We define a new graph $G = (V, E)$ by taking H and splitting the nodes in D.

$$\begin{aligned} V &= \{i : i \in V_H\} \ \cup \ \{i' : i \in D\} \\ E &= E_H \ \cup \ \{i'j : ij \in E_H, i \in D\} \\ &\quad \cup \{i'j' : ij \in E_H \text{ and } i, j \in D\} \ \cup \ \{ii' : i \in D\} \end{aligned}$$

We take $s = k - 1$ and set $d = \mathbf{1}$. Now, define the point $y \in P_{LP}(G, d, s, 0)$ by setting $y_{l_i} = 0$ for every $i \in V$ and

$$y_{r_i} = \begin{cases} w_i & \text{for } i \notin D \\ w_i/2 & \text{for } i \in D \end{cases}$$

Furthermore, let $y_{r_{i'}} = w_i/2$ for $i \in D$, and $y_{x_{ij}} = 1/2$ for every $ij \in E$. Note that $0 \leq y_{l_i} \leq y_{r_i} \leq s$ and $y_{r_i} - y_{l_i} \geq 1 = d_i$, so the bound constraints and the demand constraints are satisfied. Moreover, $y_{r_j} = w_j \leq \frac{k-1}{2}$ for $j \notin D$, and $y_{r_j} = y_{r_{j'}} = \frac{w_j}{2} \leq \frac{k-1}{2}$ if $j \in D$, and thus

$$\begin{aligned} y_{r_j} &\leq \frac{k-1}{2} = 0 + s/2 = y_{l_i} + s x_{ij}, \\ y_{r_i} &\leq \frac{k-1}{2} = 0 + s/2 = y_{l_j} + s(1 - x_{ij}). \end{aligned}$$

Hence the antiparallelity constraints are also satisfied and, therefore, $y \in P_{LP}(G, d, s, 0)$. We finally show that H has a clique of weight k or greater if and only if there is some clique $K \subseteq V$ such that the inequality (6.16) defined by K is violated by y.

If. Let $K \subseteq V_H$ be a clique with weight at least k and define $K' = K \cup \{i' : i \in K \cap D\}$. The construction of G implies that K' is a clique of G, and moreover

$$\begin{aligned}
& \sum_{i \in K'} (y_{r_i} - y_{l_i}) \\
= \quad & \sum_{i \in K \setminus D} (y_{r_i} - y_{l_i}) + \sum_{i \in K \cap D} (y_{r_i} - y_{l_i}) + (y_{r_{i'}} - y_{l_{i'}}) \\
= \quad & \sum_{i \in K \setminus D} w_i + \sum_{i \in K \cap D} (w_i/2 + w_i/2) \\
= \quad & \sum_{i \in K} w_i \geq k = s + 1 > s.
\end{aligned}$$

Hence the inequality $\sum_{i \in K'} r_i - l_i \leq s$ is violated by y.

Only if. Suppose that $\sum_{i \in K} y_{r_i} - y_{l_i} > s$ for some clique $K \subseteq V$. Define $K' = \{i : i \in K$ or $i' \in K\} \subseteq V_H$. Again, we have $\sum_{i \in K'} w_i \geq \sum_{i \in K} y_{r_i} - y_{l_i} > s = k - 1$, and since $w_i \in \mathbf{Z}$, we conclude that $w(K') \geq k$.

This reduction from Weighted Max-Clique to the separation problem for (6.16) is polynomial, and thus it is also pseudopolynomial. Therefore, Interval-sum inequalities separation is strongly $\mathcal{NP}$-complete. □

6.5 Clique-interval inequalities

This section introduces an interesting class of valid inequalities, namely the *clique-interval inequalities* as a combination of the clique inequalities and the interval-sum inequalities. The full potential of the ideas giving rise to this family appears in chromatic scheduling polytopes defined over complete interference graphs, and Section 6.5.1 is devoted to these results. It is worth noting that although complete interference graphs are not interesting in practice, chromatic scheduling polytopes defined over complete interference graphs admit a complex combinatorial structure. Unfortunately, a generalization of the clique-interval inequalities to arbitrary instances is not straightforward, involving coefficients whose exact calculation is $\mathcal{NP}$-hard. Sections 6.5.2 and 6.5.3 present this generalization, together with some preliminary results for heuristically generating bounds on these coefficients.

6.5.1 Clique-interval inequalities for complete interference graphs

Definition 6.6 *For $j = 1, \ldots, n$, define $\bar{d}_j = s - \sum_{k \neq j} d_k$. Note that every integer feasible solution $y \in P(K_n, d, s, 0) \cap \mathbf{Z}^{2n+m}$ has $y_{r_j} - y_{l_j} \leq \bar{d}_j$.*

Definition 6.7 (clique-interval inequalities) *Consider a complete interference graph (K_n, d). Fix a node $i \in V = \{1, \ldots, n\}$, and partition $V \setminus \{i\} = K \cup K'$ arbitrarily, where K*

or K' may be empty. We define

$$\sum_{j \in K} (r_j - l_j) + \sum_{j \in K'} d_j x_{ji} \;\leq\; l_i + \sum_{j \in K} \bar{d}_j x_{ij}. \tag{6.17}$$

to be the clique-interval inequality *associated with K and K'.*

Example 6.2 Consider the polytope $P(K_4, \mathbf{1}, 5, 0)$, associated with a uniform complete interference graph on 4 nodes. Take $i = 1$ and define $K = \{2\}$ and $K' = \{3, 4\}$. Then,

$$(r_2 - l_2) + (x_{31} + x_{41}) \;\leq\; l_1 + 2x_{12}$$

is the clique-interval inequality associated with this partition. It is not difficult to verify that this inequality is valid for this particular instance. $\triangleleft$

Proposition 6.14 *The clique-interval inequalities are valid for $P(K_n, d, s, 0)$.*

Proof. Let $y \in P(K_n, d, s, 0) \cap \mathbf{Z}^{2n+m}$ be a feasible solution, and define the following sets:

$$\begin{aligned} A &= \{j \in K : y_{x_{ij}} = 0\}, \\ B &= \{j \in K' : y_{x_{ij}} = 0\}, \\ C &= \{j \in K : y_{x_{ij}} = 1\}, \end{aligned} \tag{6.18}$$

Since the intervals do not overlap, $\sum_{j \in A}(y_{r_j} - y_{l_j}) + \sum_{j \in B} d_j \leq y_{l_i}$ holds. Moreover, each $j \in C$ has $y_{r_j} - y_{l_j} \leq \bar{d}_j$ (by definition of $\bar{d}_j$), and so $\sum_{j \in C}(y_{r_j} - y_{l_j}) \leq \sum_{j \in C} \bar{d}_j$. Combining these two inequalities, we obtain

$$\begin{aligned} & \sum_{j \in K}(y_{r_j} - y_{l_j}) + \sum_{j \in K'} d_j y_{x_{ji}} \\ = \;& \sum_{j \in A}(y_{r_j} - y_{l_j}) + \sum_{j \in C}(y_{r_j} - y_{l_j}) + \sum_{j \in B} d_j \\ = \;& \Big[\sum_{j \in A}(y_{r_j} - y_{l_j}) + \sum_{j \in B} d_j\Big] + \Big[\sum_{j \in C}(y_{r_j} - y_{l_j})\Big] \\ \leq \;& y_{l_i} + \sum_{j \in C} \bar{d}_j \\ = \;& y_{l_i} + \sum_{j \in K} \bar{d}_j y_{x_{ij}}. \end{aligned}$$

Therefore, the clique-interval inequality (6.17) is valid for $P(K_n, d, s, 0)$. $\square$

If $s = \sum_{i=1}^{n} d_i$, then every feasible solution satisfies (6.17) at equality, and so this inequality does not define a proper face of $P(K_n, d, s, 0)$. On the other hand, if $s > \sum_{i=1}^{n} d_i$ we can show that the clique-interval inequalities define facets of $P(K_n, d, s, 0)$. Theorem 6.15 can be proved in a similar way as the facetness results presented in the previous sections. Note that these results do not apply to the fixed-length polytope $R(K_n, d, s, 0)$.

Theorem 6.15 *The clique-interval inequality (6.17) defines a facet of $P(K_n, d, s, 0)$ if and only if $s > \sum_{i=1}^{n} d_i$.*

Remark. It is worth noting that the separation of the clique-interval inequalities over a complete interference graph is a polynomially solvable problem. Given a point $z \in P_{LP}(K_n, d, s, g)$ and a fixed node $i \in \{1, \ldots, n\}$, we partition $V \backslash \{i\} = K \cup K'$ as follows. For each $j \in V \backslash \{i\}$, insert j into K if $z_{r_j} - z_{l_j} - \bar{d}_j z_{x_{ij}} \geq d_j z_{x_{ji}}$, otherwise insert j into K'. Repeating the procedure for $i = 1, \ldots, n$, we construct n clique-interval inequalities. If the point z violates some clique-interval inequality then it must violate some of the constructed inequalities, and conversely. $\lhd$

6.5.2 Clique-interval inequalities for arbitrary interference graphs

The purpose of this section is to provide a generalization of the clique-interval inequalities (6.17) for arbitrary interference graphs. Proposition 6.16 presents a straightforward generalization giving valid inequalities for this case, but unfortunately these inequalities are not facet-inducing for $P(G, d, s, 0)$. The same arguments from the proof of Proposition 6.14 can be applied to establish this result.

Proposition 6.16 *Let $i \in V$ and consider disjoint cliques $K, K' \subseteq N(i)$ (K or K' may be empty). The inequality*

$$\sum_{j \in K} (r_j - l_j) + \sum_{j \in K'} d_j x_{ji} \ \leq \ l_i + \sum_{j \in K} \bar{d}_j x_{ij} \tag{6.19}$$

is valid for $P(G, d, s, g)$.

Unfortunately, inequality (6.19) does not necessarily define a facet of $P(G, d, s, g)$ since we may not be able to find feasible solutions satisfying it at equality with some interval $I(j)$, with $j \in K$, located to the right of $I(i)$. The rest of this section provides a stronger inequality for this case, by applying lifting procedures for the coefficients on the variables x_{ij}, for $j \in K$. As we shall see, the calculation of these coefficients is a difficult task, and we devise in Section 6.5.3 a procedure for heuristically bounding their values.

Theorem 6.17 *Let $i \in V$ and consider disjoint cliques $K, K' \subseteq N(i)$ such that for every node $j \notin K \cup K' \cup \{i\}$ there exists some node $k \in K$ with $jk \notin E$. Then, the inequality*

$$\sum_{j \in K} (r_j - l_j) + \sum_{j \in K'} d_j x_{ji} \ \leq \ l_i \tag{6.20}$$

defines a facet of $P(G, d, s, 0) \cap \{y \in \mathbf{R}^{2n+m} : y_{x_{ji}} = 1 \ \forall j \in K\}$ if $s \gg \omega(G, d)$.

Proof. Let $P' = \{y \in P(G, d, s, 0) : y_{x_{ji}} = 1 \ \forall j \in K\}$, and let F be the face of P' defined by (6.20). Suppose $\lambda^T y = \lambda_0$ for every point $y \in F$. We will prove that (λ, λ_0) is a multiple of (6.20), thus showing that this inequality induces a facet of P'.

The technique applied in the proof of Theorem 6.10 can be used to prove that there exists some $\alpha \in \mathbf{R}$ such that $\lambda_{r_j} = -\lambda_{l_j} = \alpha$ for $j \in K$, and $\lambda_{x_{jk}} = 0$ for $j, k \in K$. Moreover, it is not hard to see that $\lambda_{l_i} = -\alpha$.

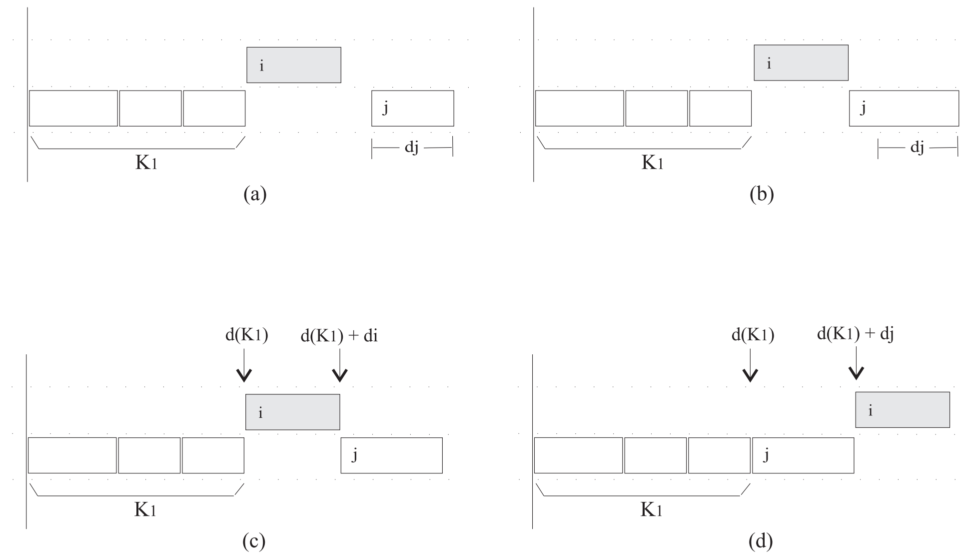


Figure 6.10: Constructions for the proof of Theorem 6.17.

We now prove $\lambda_{l_j} = \lambda_{r_j} = 0$ for $j \notin K \cup \{i\}$. To this end, consider the points y^1 and y^2 defined in Figure 6.10(a) and Figure 6.10(b), respectively. These points are in F, hence $\lambda_{l_j} = 0$. A similar argumentation yields $\lambda_{r_j} = 0$ for $j \notin K$ (note that $\lambda_{r_i} = 0$).

For any node $j \in K'$, consider now the two points depicted in Figure 6.10(c) and Figure 6.10(d). Both points satisfy (6.20) at equality, and we know $\lambda_{r_i} = \lambda_{l_j} = \lambda_{r_j} = 0$, implying

$$d(K)\lambda_{l_i} = [d(K) + d_j]\lambda_{l_i} + \lambda_{x_{ji}}.$$

Since $\lambda_{l_i} = -\alpha$, we conclude $\lambda_{x_{ji}} = \alpha d_j$.

To complete the proof, we must show $\lambda_{x_{jk}} = 0$ for the remaining edges jk:

Case 1: $\boldsymbol{j, k \notin K \cup \{i\}}$. As in the previous cases, we can construct a point in F with K to the left of $I(i)$, K' to the right of $I(i)$, and no space between the intervals $I(j)$ and $I(k)$, $I(j)$ being before $I(k)$. If we now swap these two intervals, we get another point in F, showing $\lambda_{x_{jk}} = 0$. $\diamond$

Case 2: $\boldsymbol{j = i}$ and $\boldsymbol{k \notin K \cup K}$. By the hypothesis, there exists some $k_0 \in K$ such that $k_0 k \notin E$. We can construct a feasible solution $y \in P'$ with $y_{r_{k_0}} - y_{l_{k_0}} \geq d_k$ (Figure 6.11(a)), so that we can put $I(k)$ "inside" K_0 (Figure 6.11(b)). These two points satisfy (6.20) at equality, hence $\lambda_{x_{jk}} = 0$. $\diamond$

Case 3: $\boldsymbol{j \notin K}$ and $\boldsymbol{k \in K}$. Applying the same procedure used in the previous case, we can construct two points with $I(j)$ located to the left and to the right of $I(k)$, respectively. Case 2 implies $\lambda_{x_{ij}} = 0$, hence $\lambda_{x_{jk}} = 0$. $\diamond$

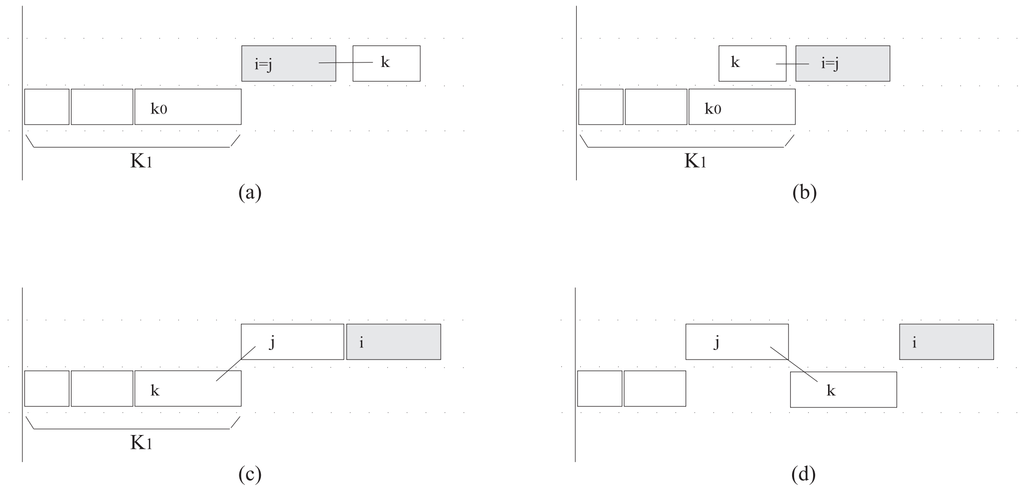


Figure 6.11: Constructions for the proof of Theorem 6.17.

Case 4: $j \in K$, $k \in K$. Consider the two points depicted in Figure 6.11(c) and Figure 6.11(d). These points are in F, and we know $\lambda_{l_j} = \lambda_{r_j} = 0$, so

$$\lambda_{l_k} \sum_{l \in K \setminus \{k\}} d_l + \lambda_{r_k} \sum_{l \in K} d_l = \lambda_{l_k}\Big(d_j + \sum_{l \in K \setminus \{k\}} d_l\Big) + \lambda_{r_k}\Big(d_j + \sum_{l \in K} d_l\Big) + \lambda_{x_{jk}}.$$

But $\lambda_{l_k} = -\alpha$ and $\lambda_{r_k} = \alpha$, hence $\lambda_{x_{jk}} = 0$. $\diamond$

Therefore, we show $\lambda = \alpha\pi$, proving that (6.20) defines a facet of P'. $\Box$

Note that we do not need a covering clique in order to establish Theorem 6.17. To obtain a valid and facet-defining inequality for $P(G, d, s, 0)$ from (6.20), we can consider a lifting procedure over the variables x_{ij} ($j \in K$), that are set to 0 in $P(G, d, s, 0) \cap \{y \in \mathbf{R}^{2n+m} : y_{x_{ji}} = 1 \ \forall j \in K\}$. Consider any fixed lifting sequence, and let α_j denote the maximum lifting coefficient for x_{ij} with $j \in K$. We then get the following inequality, defining a facet of $P(G, d, s, 0)$:

$$\sum_{j \in K} (r_j - l_j) + \sum_{j \in K'} d_j x_{ji} \ \leq \ l_i + \sum_{j \in K} \alpha_j x_{ij}. \tag{6.21}$$

Unfortunately, the calculation of these lifting coefficients is $\mathcal{NP}$-hard. Consider the first lifted variable x_{ij}, and define the decision problem associated with α_j as follows:

	CLIQUE-INTERVAL INEQUALITY LIFTING
Instance:	A graph $G = (V, E)$ and integers k and s. A node $i \in V$, node sets $K, K' \subseteq V$ as above, and some node $j \in K$.
Question:	Is α_j (defined as above) greater or equal than k?

Theorem 6.18 CLIQUE-INTERVAL INEQUALITY LIFTING *is* $\mathcal{NP}$*-hard.*

Proof. Consider the feasibility problem for chromatic scheduling polytopes:

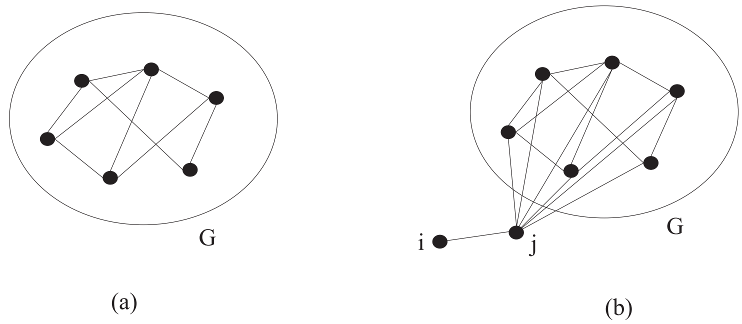


Figure 6.12: Construction of H from G.

CHROMATIC SCHEDULING FEASIBILITY
Instance: A weighted graph (G, d) and an integer s'.
Question: Is $P(G, d, s', 0)$ nonempty?

Recall that Corollary 1.2 implies that CHROMATIC SCHEDULING FEASIBILITY is $\mathcal{NP}$-complete. We shall construct a reduction of this problem to CLIQUE-INTERVAL INEQUALITY LIFTING. Given (G, d) and s', construct a new graph $H = (V_H, E_H)$ with $V_H = V \cup \{i, j\}$ and $E_H = E \cup \{jk : k \in V\} \cup \{ij\}$ (see Figure 6.12). Define $K = \{j\}$ and $K' = \emptyset$, and take $s = s' + d_j$ and $k = d_j$. We claim that $P(G, d, s', 0) \neq \emptyset$ if and only if $\alpha_j \geq k$.

If. Suppose that $\alpha_j \geq k$. If we define $P_j = \{y \in P(G, d, s, 0) : y_{x_{ij}} = 1\}$, the maximum lifting coefficient α_j for x_{ij} is:

$$\alpha_j \;=\; \max_{y \in P_j} \Big[\sum_{t \in K} (y_{r_t} - y_{l_t}) + \sum_{t \in K'} d_t y_{x_{ti}} - y_{l_i} \Big] \;=\; \max_{y \in P_j} [y_{r_j} - y_{l_j} - y_{l_i}].$$

Suppose that y^* realizes this maximum, and that $y^*_{r_j} - y^*_{l_j} - y^*_{l_i} \geq k = d_j$. This solution must have $y^*_{l_i} = 0$, otherwise we could shift $I(i)$ to the left, obtaining a better value for α_j (note that this shifting is feasible since the only neighbor of the node i is j, and $I(j)$ is located to the right of $I(i)$). Since $y^*_{r_j} - y^*_{l_j} \geq d_j$ and $jk \in E_H$ for all $k \in V$, we can construct a feasible solution y' of $P(G, d, s', 0)$ in the following way (see Figure 6.13):

$$\begin{aligned}
y'_{l_k} &= \begin{cases} y^*_{l_k} & \text{if } y^*_{x_{jk}} = 0 \\ y^*_{l_k} - (y^*_{r_j} - y^*_{l_j}) & \text{otherwise} \end{cases} \\
y'_{r_k} &= \begin{cases} y^*_{r_k} & \text{if } y^*_{x_{jk}} = 0 \\ y^*_{r_k} - (y^*_{r_j} - y^*_{l_j}) & \text{otherwise} \end{cases} \\
y'_{x_{kl}} &= y^*_{x_{kl}}
\end{aligned}$$

This construction shifts the intervals located to the right of $I(j)$ at least d_j units to the left. Now $\max_{k \in V}(y^*_{r_k}) \leq s$ implies $\max_{k \in V}(y'_{r_k}) \leq s - d_j = s'$, hence $y' \in P(G, d, s', 0)$ and so $P(G, d, s', 0)$ is nonempty.

Only if. If $P(G, d, s', 0)$ is nonempty, then we can transform any feasible solution into a point $y \in P_j$ by adding the interval $I(i)$ with $l_i = 0$ and $r_i = d_i$, and interval j with $l_j = s'$ and $r_j = s$. This new solution y' has $y'_{r_j} - y'_{l_j} - y'_{l_i} = s - s' = d_j = k$, showing that $\alpha_j \geq k$.

Therefore, CLIQUE-INTERVAL INEQUALITY LIFTING is $\mathcal{NP}$-complete. □

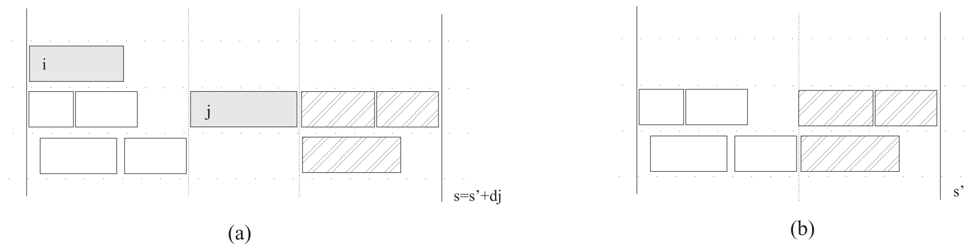


Figure 6.13: Construction of y' (fig. (b)) from y^* (fig. (a)).

6.5.3 Upper bounds for the lifting coefficients

Since the lifting coefficients α_j introduced above are difficult to calculate, we can consider to replace each coefficient by an upper bound, thus maintaining validity (although not necessarily facetness). This section shows a simple procedure for calculating such upper bounds. Note that this is a priori a nontrivial issue, since the generation of upper bounds for these coefficients is in a sense the dual of the lifting maximization problem. This section develops, by combinatorial arguments, a dual for this problem whose feasible solutions are easy to calculate, so they can be used for heuristically generating upper bounds for the lifting coefficients.

Lemma 6.19 *$\alpha_j \geq 0$ for every $j \in K$.*

Proof. Suppose that the variables x_{il} for $l \in L$ have already been lifted, and define $P_L = \{y \in P(G,d,s,0) : y_{x_{il}} = 0 \text{ for } l \in K \backslash L\}$. Then, $\alpha_j = \max_{y \in P_{L \cup \{j\}}} g(y)$, with

$$g(y) = \sum_{k \in K} (y_{r_k} - y_{l_k}) + \sum_{k \in K'} d_k y_{x_{ik}} - y_{l_i} - \sum_{k \in L} \alpha_k y_{x_{ik}}.$$

We now construct a point $\bar{y}$ with $g(\bar{y}) \geq 0$, thus proving $\alpha_j \geq 0$. The point $\bar{y}$ has all intervals corresponding to $K' \cup K \backslash \{j\}$ located to the left of $I(i)$, each with length equal to its demand (i.e., $\bar{y}_{r_k} - \bar{y}_{l_k} = d_k$). Furthermore, we leave no empty space between them, and no empty space between the last interval and $I(i)$ (see Figure 6.14), so that

$$\sum_{k \in K \backslash \{j\}} (\bar{y}_{r_k} - \bar{y}_{l_k}) + \sum_{k \in K'} d_k \bar{y}_{x_{ij}} \leq \bar{y}_{l_i}.$$

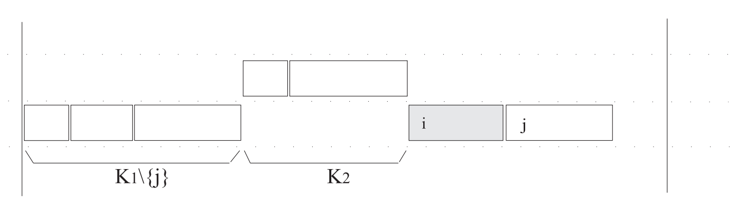


Figure 6.14: Construction of $\bar{y}$.

Moreover, we have $\bar{y}_{x_{it}} = 0$ for every $t \in L$, and so $\sum_{t \in L} \alpha_t \bar{y}_{x_{it}} = 0$. Thus,

$$\begin{aligned} g(\bar{y}) &= \sum_{k \in K} (\bar{y}_{r_k} - \bar{y}_{l_k}) + \sum_{k \in K'} d_k \bar{y}_{x_{ik}} - \bar{y}_{l_i} - \sum_{t \in L} \alpha_t \bar{y}_{x_{it}} \\ &= (\bar{y}_{r_j} - \bar{y}_{l_j}) + \sum_{k \in K \setminus \{j\}} (\bar{y}_{r_k} - \bar{y}_{l_k}) + \sum_{k \in K'} d_k \bar{y}_{x_{ik}} - \bar{y}_{l_i} \\ &= \bar{y}_{r_j} - \bar{y}_{l_j} \;\geq\; 0 \end{aligned}$$

Therefore $g(\bar{y}) \geq 0$, implying $\alpha_j \geq 0$. $\Box$

Using Lemma 6.19 we can now obtain a lower bound for each α_j. As in the previous proof, assume that the variables x_{il} for $l \in L$ have been lifted and let $y \in P_{L \cup \{j\}} \cap \mathbf{Z}^{2n+m}$ be a point with $y_{x_{ij}} = 1$. Partition $K = A_y \cup B_y$ such that $A_y = \{t \in K : y_{x_{ti}} = 1\}$ and $B_y = \{t \in K : y_{x_{ti}} = 0\}$ (note that $j \in B_y$). Then,

$$\begin{aligned} & \sum_{t \in K} (y_{r_t} - y_{l_t}) + \sum_{t \in K'} d_t y_{x_{ti}} - y_{l_i} - \sum_{t \in L} \alpha_t x_{it} \qquad (6.22) \\ \leq\; & \sum_{t \in K} (y_{r_t} - y_{l_t}) + \sum_{t \in K'} d_t y_{x_{ti}} - y_{l_i} \\ =\; & \Big[\sum_{t \in A_y} (y_{r_t} - y_{l_t}) + \sum_{t \in K'} d_t y_{x_{ti}} - y_{l_i} \Big] + \sum_{t \in B_y} (y_{r_t} - y_{l_t}) \\ \leq\; & \sum_{t \in B_y} (y_{r_t} - y_{l_t}) \end{aligned}$$

The first inequality holds because $\alpha_j \geq 0$ (by Lemma 6.19), and the last inequality holds since $A_y \cup K'$ is a clique and all its corresponding intervals are allocated to the left of $I(i)$, hence

$$\sum_{t \in A_y} (y_{r_t} - y_{l_t}) + \sum_{t \in K'} d_t y_{x_{ti}} \leq y_{l_i}.$$

Let $C(y) = \{T \subseteq V : B_y \subseteq T$ and T is a clique$\}$ and consider any $T \in C(y)$. We obtain

$$\begin{aligned} \sum_{t \in B_y} (y_{r_t} - y_{l_t}) &\leq s - \sum_{t \in V \setminus B_y} (y_{r_t} - y_{l_t}) \\ &\leq s - \sum_{t \in T \setminus B_y} (y_{r_t} - y_{l_t}) \qquad (6.23) \\ &\leq s - \sum_{t \in T \setminus B_y} d_t \end{aligned}$$

This last inequality is valid for any $T \in B_y$, so

$$g(y) \leq \min_{y \in P_{L \cup \{j\}}} \Big(s - d(T \setminus B_y) \Big)$$

Define $S = \{T \subseteq V : T$ is a clique and $T \cap K \neq \emptyset\}$. For every $T \in S$, we have that $T \in C(z)$ for some point z such that $z_{x_{it}} = 1$ for $t \in T \cap K$. Moreover, $s - d(T \setminus B_y) \leq s - d(T \setminus K)$, since $B_y \subseteq K$. Then,

$$\min_{y \in P_{L \cup \{j\}}} \Big(s - d(T \setminus B_y) \Big) \leq \min_{T \in S} \Big(s - d(T \setminus K) \Big). \qquad (6.24)$$

Thus, by combining (6.22), (6.23) and (6.24), we get:

$$\alpha_j \;\leq\; \min_{T \in S} \Big[\; s - d(T \backslash K) \;\Big]. \tag{6.25}$$

We can compute an upper bound on α_j by heuristically generating cliques in S and taking the minimum of $s - d(T \backslash K)$ over all the generated cliques.

6.5.4 Complexity of the separation problem

To conclude our analysis of the clique-interval inequalities, we state in this section a negative result concerning the complexity of the associated separation problem. Since the proof of this fact is similar to the complexity analyses presented previously for other families of inequalities, we only give the reduction that establishes this result.

	CLIQUE-INTERVAL INEQUALITIES SEPARATION
Instance:	A point $z \in P_{LP}(G, d, s, g)$.
Question:	Does z violate a clique-interval inequality?

Theorem 6.20 CLIQUE-INTERVAL INEQUALITIES SEPARATION *is $\mathcal{NP}$-complete.*

Sketch of proof. Let (H, k) be an instance of MAX-CLIQUE (that consists in deciding whether $\omega(H) \geq k$ or not). Construct a graph $G = (V, E)$ from $H = (V_H, E_H)$ by the addition of a universal node i, i.e., $V = V_H \cup \{i\}$ and $E = E_H \cup \{ij : j \in V_H\}$. Furthermore, set $s = 2n + 1$ and define the point $z \in P_{LP}(G, \mathbf{1}, s, g)$ by $z_{l_j} = n$ for $j \in V_H$ and $z_{l_i} = k/2$. Moreover, set $z_{r_j} = z_{l_j} + 1$ for every $j \in V$ and $z_{x_{jk}} = 1/2$ for every $jk \in E$. The point z violates some clique-interval inequality if and only if $\omega(H) \geq k$. $\square$

Chapter 7

Concluding remarks and open problems

> Very recent mathematical work on the traveling salesman problem (...) indicates that the problem is fundamentally complex. It seems very likely that quite a different approach from any yet used may be required for successful treatment of the problem. In fact, there may well be no general method for treating the problem and impossibility results would also be valuable.
>
> – M. Flood (1956)

This thesis contributes an initial study of chromatic scheduling polytopes by partially uncovering their combinatorial structure, presenting first classes of valid and facet-defining inequalities, and addressing the associated separation problems. We briefly review now the results presented in the preceding chapters and point out some important open problems in this topic.

Emptyness/nonemptyness

Solving the bandwidth allocation problem in PMP-Systems amounts to determining whether the polytopes are empty or not, hence emptyness/nonemptyness is a crucial issue with strong practical implications. The clique bound resp. chromatic bound gives a certificate of emptyness resp. nonemptyness, but it would be interesting to strengthen or refine these bounds in order to have more precise conditions ensuring feasibility/infeasibility of the associated bandwidth allocation problem.

Dimension

A central issue in polyhedral combinatorics is to calculate the dimension of the polytopes in question. As we have seen, obtaining the dimension of chromatic scheduling polytopes is a difficult task, both computationally and theoretically. We know that the dimension is a nondecreasing function of the frequency span and that $P(G, d, s, g)$ and $R(G, d, s, g)$ are full-dimensional if $s \geq \gamma(G, d, g)$, but there are many open questions concerning the case $s < \gamma(G, d, g)$. Section 3.2 provides partial results for the uniform case and for particular classes of interference graphs. One important case is given by the instances with uniform demand $d = \mathbf{1}$, but even in this setting we do not have a complete characterization of the dimension yet (note that this case corresponds to the usual graph coloring problem, which is already a hard problem). Recall that $\gamma(G, \mathbf{1}, 0) = \chi(G) + 2$ holds in this setting.

Problem 1 *Can we characterize the dimension of the polytopes $R(G, \mathbf{1}, s, 0)$ and $P(G, \mathbf{1}, s, 0)$ for $s = \chi(G)$ and $s = \chi(G) + 1$?*

We know that both polytopes have full dimension if $s \geq \chi(G)+2$ and, furthermore, Section 3.2 provides a partial characterization of the dimension of $R(G, \mathbf{1}, s, 0)$ when $s = \chi(G) + 1$. However, a complete characterization of the dimension in the uniform case is still not known. A more modest problem is to provide conditions ensuring full-dimensionality in the uniform case. Here, the following question remains unanswered.

Problem 2 *For which interference graph G are $R(G, \mathbf{1}, \chi(G), 0)$ and $P(G, \mathbf{1}, \chi(G), 0)$ full-dimensional?*

These open questions are particular cases of a more general unsolved problem concerning chromatic scheduling polytopes, namely the existence of a formula for the dimension of the polytopes for arbitrary interference graphs and general node weights. The most general question is the following.

Problem 3 *Do there exist formulas for the dimension of $P(G, d, s, g)$ and $R(G, d, s, g)$ in terms of standard graph parameters? How does the node weighting affect such a formula?*

It is not clear whether this question can be answered affirmatively, since calculating the dimension proves to be a difficult issue even for uniform instances. Having a complete characterization of the dimension would help to establish facetness properties of valid inequalities for these polytopes. Based on the bounds given in Section 3.2, we have been able to provide facetness results for a number of valid inequalities in the case $s \geq s_{\min}(G, d, g) + O(1)d_{\max}$. However, full knowledge of the dimension would help to give complete characterizations of the facet-defining cases of each valid inequality.

Combinatorial stability

Section 3.3 shows that the polytopes $R(G, d, s, g)$ and $R(G, d, s + 1, g)$ resp. $P(G, d, s, g)$ and $P(G, d, s + 1, g)$ are affinely isomorphic if $s > 2\tau(G, d, g)$, but empirical evidence suggests that only $s > \tau(G, d, g)$ is needed to establish this isomorphism. As shown in that section, if every connected component of G is a clique, then $R(G,d,s,g) \cong R(G,d,s+1,g)$ if and only if $s > \tau(G,d,g)$. Therefore, it is natural to ask whether this is the case for arbitrary graphs.

Problem 4 *Is $R(G,d,s,g) \cong R(G,d,s+1,g)$ for $s > \tau(G,d,g)$?.*

The proof technique presented in Section 3.3 constrains the condition to be $s > 2\tau(G,d,g)$, so a different idea should be employed to prove this more general assertion.

Symmetry

The symmetry of chromatic scheduling polytopes is a very particular theoretical property. The most remarkable aspect of this property is that it provides results for proving facetness independently of the dimension of the associated polytopes. This turns out to be a valuable tool for identifying facet-inducing inequalities in a context where the dimension of the polytopes is still unknown. It would be interesting to develop further implications of symmetry related to the search for facets.

Problem 5 *Can we further exploit the special symmetry of $P(G,d,s,g)$ and $R(G,d,s,g)$ to provide theoretical tools for identifying facet-defining inequalities?*

Valid inequalities and facets

Since the bandwidth allocation problem in PMP-Systems is $\mathcal{NP}$-complete, we cannot expect a complete characterization of chromatic scheduling polytopes unless $\mathcal{NP} = \text{co-}\mathcal{NP}$[42]. However, many families of facet-inducing inequalties are obtained here, which encourages the use of cutting plate methods for solving this problem. Covering cliques prove to be a useful construction for the development of facets, and Chapter 5 introduces several classes of facet-inducing inequalities arising from such structures in the interference graph. Hence, developing these ideas further seems to be a promising line for future studies of chromatic scheduling polytopes.

Problem 6 *Can we devise further generalizations (as in Section 5.3) of covering-clique inequalities?*

Problem 7 *Can we devise further extensions (as in Section 5.3.3) of the standard double covering-clique inequalities?*

On the other hand, Chapter 6 presents a number of classes of facet-inducing inequalities based on different structures of the interference graph. Some families arise as variations of inequalities from the linear ordering polytope, whereas the remaining ones seem to be particular to chromatic scheduling polytopes. The families presented in Section 4.3, which only are valid for small frequency spectrums, are of practical importance as they could serve as cutting planes for the hardest instances in practice.

Problem 8 *Find new classes of facet-inducing inequalities, either arising as variations of known facets for related polyhedra or being particular to chromatic scheduling polytopes.*

Problem 9 *Find classes of valid inequalities for small frequency spectrums, and characterize the cases where these inequalities induce facets.*

The last issue seems to be a difficult one, since facetness is hard to analyze when the frequency spectrum is small. When $[0, s]$ is large, we can easily construct feasible solutions and prove facetness this way. However, when $s = \omega(G, d) + O(1)$, the construction of feasible solutions becomes more involved and, therefore, it is more difficult to prove facetness in this case. The only known way to accomplish this task relies on symmetry arguments. This shows how important the special symmetry of chromatic scheduling polytopes is for our purposes.

Separation problems

The practical implementation of a cutting plane approach involves routines for efficiently identifying violated valid inequalities. Therefore, the separation problem for the known classes of inequalities is not only of theoretical interest but also of practical importance in a cutting plane environment. Throughout this work we proved that many of the nontrivial families of valid inequalities have $\mathcal{NP}$-complete separation problems. This implies that a more detailed study must be carried out concerning these separation problems.

Problem 10 *For each class of valid inequalities, identify particular cases where the separation problem is polynomially solvable.*

Problem 11 *For each class of valid inequalities with $\mathcal{NP}$-complete separation problems, develop effective and fast heuristics for the corresponding separation problem.*

Problem 12 *Find polynomially separable superclasses of valid inequalities with $\mathcal{NP}$-complete separation problems.*

* * *

The recent progress at exactly solving combinatorial optimization problems by integer programming techniques and the consequent interest that these activities have generated are a motivation to multiply the efforts within this field. This work constitutes a contribution in this direction, by continuing the polyhedral study of a problem with important applications, namely the bandwidth allocation problem in PMP-Systems. Such polyhedral investigations are the first steps for the successful implementation of cutting plane approaches, and we hope that this work may contribute to the practical solution to optimality of real-world instances of this problem in a near future.

Appendix A

Summary of valid inequalities

> This problem is of course a linear programming problem, and hence may be solved by Dantzig's simplex algorithm. However, for the flow problem, we shall describe what appears to be a considerably more efficient algorithm; it is, moreover, readily learned by a person with no special training, and may easily be mechanized for handling large networks.
>
> – L. Ford and D. Fulkerson (1955)

This appendix summarizes the facet-inducing inequalities presented in Chapter 4, Chapter 5, and Chapter 6. We also provide a short comment on facetness results and the complexity of the associated separation problems, for the families where this information is known.

Triangle inequalities. Let $T = \{i, j, k\}$ be a triangle of G. The following are the *triangle inequality* associated with T and its symmetric inequality, respectively.

$$\begin{aligned} x_{ij} + x_{jk} + x_{ki} &\leq 2 \\ x_{ij} + x_{jk} + x_{ki} &\geq 1 \end{aligned}$$

If $P(G, d, s, g) \neq \emptyset$, then both inequalities define facets of $R(G, d, s, g)$ and $P(G, d, s, g)$, independently of the dimension of the polytopes (see Section 4.2). The separation problem for triangle inequalities by complete enumeration is clearly polynomial.

4-path inequalities. Let $i, j, k, l \in V$ be four nodes of G such that ij, jk, $kl \in E$ and no feasible solution of $P(G, d, s, g)$ has the ordering $i \to j \to k \to l$. The inequality

$$x_{ij} + x_{jk} + x_{kl} \leq 2$$

is the *4-path inequality* associated with the path $\{i, j, k, l\}$, and is valid and facet-inducing for $R(G, d, s, g)$ and $P(G, d, s, g)$ (see Section 4.3). The separation problem for 4-path inequalities can be solved in polynomial time by complete enumeration.

Paw inequalities. Let $i, j, k, l \in V$ be four distinct nodes of G such that $\{i, j, k\}$ induces a triangle and $jl \in E$. Furthermore, suppose that no feasible solution of $P(G, d, s, g)$ has the ordering $i \to j \to k$ and $j \to l$. The inequality

$$x_{jk} + x_{jl} \;\leq\; 1 + x_{ji}$$

is the *paw inequality* associated with the nodes $\{i, j, k, l\}$, and is valid and facet-inducing for $R(G, d, s, g)$ and $P(G, d, s, g)$ (see Section 4.3). Again, the separation problem for paw inequalities is polynomially solvable by complete enumeration.

Extended paw inequalities. Let $1, \ldots, 5 \in V$ be five distinct nodes such that $12,\ 23 \in E$ and $\{3, 4, 5\}$ form a triangle in G. Moreover, assume that no feasible solution has the orderings $1 \to 2 \to 3 \to 4$, $1 \to 2 \to 3 \to 5$ and $2 \to 3 \to 4 \to 5$. The inequality

$$x_{34} + x_{35} - x_{21} \;\leq\; 2x_{32} \tag{A.1}$$

is the *extended paw inequality* associated with the nodes $\{1, \ldots, 5\}$. The extended paw inequalities are valid and facet-inducing for $R(G, d, s, g)$ and $P(G, d, s, g)$, and the corresponding separation problem can be solved in polynomial time by complete enumeration (see Section 4.3).

Covering-clique inequalities. Let $i \in V$ be a node of G, and let K be clique covering $N(i)$. The *covering-clique inequality* associated with i and K, and its symmetrical inequality are

$$\begin{aligned} \sum_{k \in K} d_k \; x_{ki} \;&\leq\; l_i \\ s - \sum_{k \in K} d_k \; x_{ik} \;&\geq\; r_i \end{aligned}$$

If $s \geq s_{\min}(G, d, 0) + 3d_{\max}$, the covering-clique inequalities define facets of $P(G, d, s, 0)$ (see Section 5.1). The same result holds for $R(G, d, s, 0)$ if we replace r_i by $l_i + d_i$ in the symmetric inequality. The separation problem for covering-clique inequalities is $\mathcal{NP}$-complete (see Section 5.1.1). These inequalities are also valid if $g > 0$ but may not define facets in this case. A generalization of covering-clique inequalities for the case $g > 0$ such that the resulting inequalities are facet-inducing is presented in Section 5.1.2.

Double covering-clique inequalities. Let $ij \in E$ be an edge of G, and let K be a clique covering $N(i) \cap N(j)$. The *double covering-clique inequality* associated with ij and K is

$$r_i + \sum_{k \in K} d_k (x_{ik} - x_{jk}) \;\leq\; l_j + (s - d(K)) x_{ji}.$$

If $s \geq s_{\min}(G, d, 0) + 4d_{\max}$, the double covering-clique inequalities define facets of $P(G, d, s, 0)$, and the same holds for $R(G, d, s, 0)$ if we replace r_i by $l_i + d_i$ (see Section 5.2). The symmetric inequality of a double covering-clique inequality is again a double covering-clique inequality. Again, this construction can be generalized for the case $g > 0$, and the resulting facet-inducing inequalities are presented in Section 5.2.3. The separation problem for double covering-clique inequalities is $\mathcal{NP}$-complete (see Section 5.2.2).

Reinforced covering-clique inequalities. Let $i \in V$ be a node of G and fix a clique $K \subseteq N(i)$. Furthermore, let K' be a clique covering $N(i) \backslash K$. The inequality

$$\sum_{k \in K} d_k x_{ki} + \sum_{k \in K'} c_K(k) x_{ki} \;\leq\; l_i$$

is the *reinforced covering-clique inequality* associated with K and K'. These inequalities induce facets of $P(G,d,s,0)$ and $R(G,d,s,0)$ if $s \geq s_{\min}(G,d,0)+3d_{\max}$ (see Section 5.3.1). The *reinforced double covering-clique inequalities* are defined similarly.

Replicated covering-clique inequalities. Fix a node $i \in V$ and let K be a clique covering $N(i)$. Consider a clique $Q \in V \backslash N(i)$ and a subset $K' \subseteq K$ with $|K'| = |Q|$ such that every node $k \in K'$ is adjacent to some node $p_k \in Q$, and such that these adjacencies form a bijection between K' and Q. The inequality

$$\sum_{k \in K} d_k x_{ki} + \sum_{k \in K'} c_K(p_k)(x_{p_k k} - x_{ik}) \leq l_i$$

is the *replicated covering-clique inequality* associated with the cliques K and Q. If $s \geq s_{\min}(G,d,0)+3d_{\max}$, the replicated covering-clique inequalities define facets of $P(G,d,s,0)$ and $R(G,d,s,0)$ (see Section 5.3.2).

Extended double covering-clique inequalities. Let $i,j \in V$ be two adjacent nodes, and let K be a clique covering $N(i) \cap N(j)$. Furthermore, fix some node $t \in N(j) \backslash N(i)$. The inequality

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_j + \varphi x_{ji} + \varphi_t x_{jt}$$

is the *extended double covering-clique inequality* associated with K and t where $\varphi = s - d(K \backslash A(K,t))$ and $\varphi_t = d_t - d(A(K,t))$. If $s \geq s_{\min}(G,d,0)+4d_{\max}$, then this inequality induces a facet of $P(G,d,s,0)$, and the same holds for $R(G,d,s,0)$ if we replace r_i by $l_i + d_i$ (see Section 5.3.3). The symmetric family is a new family of facets.

2-extended double covering-clique inequalities. Let $i,j \in V$ be two adjacent nodes of G, and let K be a clique covering $N(i) \cap N(j)$. Moreover, let $p \in N(i) \backslash N(j)$ and $t \in N(j) \backslash K$. The following is the *2-extended double covering-clique inequality* associated with K and nodes t and p

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_j + \varphi' x_{ji} + \varphi_p x_{pi} + \varphi_t x_{jt},$$

where the coefficients φ', φ_t and φ_p are defined in Section 5.3.3. If $s \geq s_{\min}(G,d,0)+4d_{\max}$, then the 2-extended double covering-clique inequalities are facet-inducing for $P(G,d,s,0)$, and the same holds for $R(G,d,s,0)$ if we replace r_i by $l_i + d_i$.

Closed double covering-clique inequalities. Let $i,j \in V$ be two adjacent nodes of G, and let K be a clique covering $N(i) \cap N(j)$. Moreover, let $p \in N(i) \backslash N(i)$ and $t \in N(j) \backslash K$ such that $pt \in E$ and $pk, tk \in E$ for all $k \in K$. The following is the *closed double covering-clique inequality* associated with K and the nodes t and p

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \;\leq\; l_j + \varphi'' x_{ji} + \varphi_p x_{pi} + \varphi_t x_{jt} - \varphi_{pt} x_{pt},$$

where the coefficients for the ordering variables in the RHS are defined in Section 5.3.3. If $s \geq s_{\min}(G,d,0)+4d_{\max}$, then these inequalities (5.24) induce facets of $P(G,d,s,0)$, and the same is true for $R(G,d,s,0)$ if we replace r_i by $l_i + d_i$.

4-cycle inequalities. Let $1, 2, 3 \in V$ be three nodes such that $12, 23 \in E$, and let K be a clique covering $N(1) \cap N(3)$. Assume w.l.o.g. that $K = \{4, \ldots, t\}$. The inequality

$$l_1 + l_2 \;\geq\; \sum_{k \in K} \alpha_k (x_{3k} - x_{1k}) + \beta$$

is the *4-cycle inequality* associated with these nodes, where $\alpha_k = d_k + d_3$ if $k = 4$ and $\alpha_k = d_k$ otherwise, and $\beta = \min\{d_1, d_2, d_3\}$. If $N(1) \cap N(2) \cap N(3) = \emptyset$ and $s \geq s_{\min}(G, d, 0) + O(1) d_{\max}$, then these inequalities define facets of $P(G, \mathbf{1}, s, 0)$ and $R(G, \mathbf{1}, s, 0)$ (see Section 6.1).

Cycle-order inequalities. Let $C = \{1, \ldots, k\}$ be a k-cycle in G. The following inequalities are the *cycle-order inequality* associated with C and its symmetrical inequality, respectively.

$$\begin{aligned} x_{12} + x_{23} + \ldots + x_{k-1,k} + x_{k1} &\;\leq\; k - 1 \\ x_{12} + x_{23} + \ldots + x_{k-1,k} + x_{k1} &\;\geq\; 1 \end{aligned}$$

These inequalities are facet-defining for $s > s_{\min}(G, d, g) + O(1) d_{\max}$ if and only if C is a chordless cycle (see Section 6.2). The separation problem for cycle-order inequalities can be solved in $O(m^2 n)$ time.

Odd hole inequalities. Let $C = \{1, \ldots, n\}$ be an odd hole of the interference graph. The following inequalities are the *odd hole inequality* associated with C and its symmetrical inequality, respectively.

$$\begin{aligned} \sum_{i=1}^{n} l_i &\;\geq\; \frac{n+3}{2} \\ \sum_{i=1}^{n} r_i &\;\leq\; s - \frac{n+3}{2} \end{aligned}$$

Both inequalities induce facets of $P(G, \mathbf{1}, s, 0)$ for $s \geq s_{\min}(G, d, 0) + 3$. In the particular case $G = C_n$ (with $n \geq 5$ an odd integer), the odd hole associated with C_n induces facets of $P(C_n, \mathbf{1}, s, 0)$ for $s \geq 3$ (see Section 6.3). The same results apply to the fixed-length polytope $R(G, \mathbf{1}, s, 0)$ if we replace r_i by $l_i + d_i$ in the second inequality. A superclass of the odd hole inequalities can be separated in polynomial time.

Interval-sum inequalities. If $K \subseteq V$ is a not necessarily maximal clique, then the inequality

$$\sum_{k \in K} r_k - l_k \;\leq\; s$$

is the *interval-sum inequality* associated with K. If the interference graph is complete and we take $K = V$, then this inequality induces a facet of $P(K_n, d, s, 0)$ if and only if $s > \sum_{i=1}^{n} d_i$. For arbitrary interference graphs and $s \gg \omega(G, d)$, the interval-sum inequality defines a facet of $P(G, d, s, 0)$ if and only if K is a maximal clique and $|K \backslash N(i)| \geq 2$ for every $i \notin K$ (see Section 6.4). The separation problem for the interval-sum inequalities is $\mathcal{NP}$-complete.

Clique-interval inequalities. Assume that G is a complete graph. Fix any node $i \in V$ and partition $V = K \cup K' \cup \{i\}$, where K or K' may be empty. The inequality

$$\sum_{j \in K} (r_j - l_j) + \sum_{j \in K'} d_j x_{ji} \leq l_i + \sum_{j \in K} \bar{d}_j x_{ij}$$

is the *clique-interval inequality* associated with K and K'. This inequality is valid for $P(G, d, s, 0)$ and it is facet-inducing if and only if $s > \sum_{i=1}^{n} d_i$. If G is an arbitrary graph we can generalize this inequality, but this construction involves coefficients whose calculation is $\mathcal{NP}$-hard (see Section 6.5).

Appendix B

Basics

> The largest example tried was a 20×20 optimal assignment problem. For this example, the simplex method required well over an hour, the present method about thirty minutes of hand computation.
>
> – L. Ford and D. Fulkerson (1956)

B.1 Graph theory

A *graph* $G = (V, E)$ consists of a finite nonempty set V of *nodes* and a finite set E of unordered pairs of distinct points of V, called *edges*. If $e = \{i, j\} \in E$ is an edge, we say that e *joins* the nodes i and j, and we briefly write $e = ij$. Two nodes that are joined by an edge are called *adjacent* or *neighbors*. The *neighborhood* of a node $i \in V$ is $N_G(i) = \{j \in V : ij \in E\}$. If there is no danger of confusion, we just denote this neighborhood by $N(i)$. A node $i \in V$ is *universal* if $N(i) = V \setminus \{i\}$, i.e., if it is adjacent to all the remaining nodes.

If $A \subseteq V$, we define the *neighborhood* of A as $N(A) = \{j \in V : ij \in E \text{ for some } i \in A\}$. We also define the edge sets $E(A) = \{ij \in E : i \in A \text{ and } j \in A\}$ and $\delta(A) = \{ij \in E : i \in A \text{ and } j \notin A\}$. We also use the notation $\delta(i) = \delta(\{i\})$. If $A, B \subseteq V$ are disjoint node sets, we define $E(A, B) = \{ij \in E : i \in A \text{ and } j \in B\}$. A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The subgraph of G *induced* by a node set $A \subseteq V$ is $G_A = (A, E')$, with $E' = E(A)$. Such a graph is called an *induced subgraph* of G.

A sequence of distinct nodes $v_1, \ldots, v_k$ is a *path* in G if $v_i v_{i+1} \in E$ for $i = 1, \ldots, k-1$. The number k is the *length* of this path. For $n \geq 1$, we denote by $P_n = (V, E)$ the graph on n nodes such that $V = \{1, \ldots, n\}$ and $E = \{i, i+1 : i = 1, \ldots, n-1\}$. A sequence of distinct nodes $v_1, \ldots, v_k$ is a *cycle* in G if $v_i v_{i+1} \in E$ for $i = 1, \ldots, k-1$ and $v_1 v_k \in E$. The number k is the *length* of this cycle. A cycle with length 3 is called a *triangle*. A cycle is *odd* resp. *even* if its length is odd resp. even. Every edge $v_i v_j$ in the subgraph of G induced by the nodes $v_1, \ldots, v_k$ with $j \neq i+1$ is a *chord* of the cycle. A cycle with no chords is called a *chordless* or *induced cycle* or a *hole*, if it has length at least 4. An odd chordless cycle is called an *odd hole*. For $n \geq 1$, we denote by $C_n = (V, E)$ the graph on n nodes such that $V = \{1, \ldots, n\}$

and $E = \{i, i+1 : i = 1, \ldots, n-1\} \cup \{1n\}$. A graph is called a *wheel* if it is composed by a cycle with the addition of a universal node. We denote by W_n the wheel on n nodes.

A graph is called *complete* if every two nodes are joined by an edge. A *clique* in a graph G is a set of nodes inducing a complete subgraph of G (note that we do not require this set to be maximal). We denote by $\omega(G)$ the size of a largest clique of G, also called the *clique number* of G. We denote by K_n the complete graph on n nodes. A *stable set* is a set of nodes any two of which are nonadjacent. A *coloring* of G is a partition of V into disjoint stable sets. We call a coloring using k stable sets a *k-coloring*, and denote by $\chi(G)$ the minimum number of stable sets needed for such a partition of V. This number is also called the *chromatic number* of G.

A *weighted graph* is a pair (G, d) such that $G = (V, E)$ is a graph and $d \in \mathbf{R}^{|V|}$ is a *node weighting*, associating a number d_i to every node $i \in V$. This number is called the *weight* of the node i. The *weight* of a node subset $A \subseteq V$ is $d(A) = \sum_{i \in A} d_i$. The *weighted clique number* $\omega(G, d)$ is the largest weight of a clique in G.

A *directed graph* or *digraph* $D = (V, A)$ consists of a finite nonempty set V of *nodes* and a finite set A of ordered pairs of distinct points of V, called *arcs*. If $e = (i, j) \in A$ is an arc of D, we simply write $e = ij$, and we refer to node i resp. j as the *tail* resp. *head* of the arc. The arc ij is an *outgoing* arc of node i and an *incoming* arc of node j.

A *directed cycle* is a sequence of nodes $v_1, \ldots, v_k$ such that $v_i v_{i+1} \in A$ for $i = 1, \ldots, k-1$ and $v_k v_1 \in A$. A digraph which admits no cycles is called *acyclic*. A *tournament* is a complete digraph, i.e., a digraph such that all of its nodes are pairwise adjacent. A tournament with no cycles is called an *acyclic tournament*. A *topological ordering* of a digraph $D = (V, A)$ is an ordering $v_1, \ldots, v_n$ of D such that $i < j$ whenever $v_i v_j \in A$. Such an ordering can be found in linear time [3].

A *node-weighted digraph* is a pair (D, w) such that $D = (V, A)$ is a digraph and $w \in \mathbf{R}^{|V|}$ is a *node weighting*, associating a number w_i to every node $i \in V$. An *arc-weighted digraph* is a pair (D, u) such that $D = (V, A)$ is a digraph and $u \in \mathbf{R}^{|A|}$ is an *arc weighting*, associating a number u_{ij} to every arc $ij \in A$. For further definitions and results on graph theory, we refer to [28].

B.2 Polyhedral theory

A vector set K is *convex* if for any two points $x, y \in K$ it also contains the straight line segment $[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$ between them. For any vector set K, the *convex hull* of K, denoted by $\mathrm{conv}(K)$, is the smallest (w.r.t. set inclusion) convex set containing K, i.e., $\mathrm{conv}(K) = \cap\{K' \subseteq \mathbf{R}^n : K \subseteq K' \text{ and } K' \text{ is convex}\}$. If $K = \{x_1, \ldots, x_k\}$ is finite, we can equivalently write $\mathrm{conv}(K)$ as the convex combinations of its vectors:

$$\mathrm{conv}(K) \;=\; \Big\{ \sum_{i=1}^{k} \lambda_i x_i : \lambda \geq 0 \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \Big\}.$$

A *cone* $C \subseteq \mathbf{R}^n$ is a nonempty set of vectors such that for any finite set of vectors of C it also contains all their linear combinations with nonnegative coefficients. For an arbitrary subset $K \subseteq \mathbf{R}^n$, we define its *conical hull* $\mathrm{cone}(K)$ to be the intersection of all cones in $\mathbf{R}^n$ containing K. If $K = \{x_1, \ldots, x_k\}$ is finite, we can write:

$$\mathrm{cone}(K) \;=\; \Big\{ \sum_{i=1}^{k} \lambda_i x_i : \lambda \geq 0 \Big\}.$$

The *Minkowsi sum* or *vector sum* of two sets $P, Q \subseteq \mathbf{R}^n$ is defined to be $P + Q = \{x + y : x \in P,\ y \in Q\}$.

A *polyhedron* $P \subseteq \mathbf{R}^n$ is the intersection of a finite number of closed halfspaces, i.e., $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ for a matrix $A \in \mathbf{R}^{m \times n}$ and a vector $b \in \mathbf{R}^m$. Equivalently, polyhedra can be described by the Minkowski sum of a finitely generated convex hull and a finitely generated conical hull, i.e., $P = \mathrm{conv}(K) + \mathrm{cone}(W)$ for finite vector sets $K, W \subseteq \mathbf{R}^n$. A *polytope* is a bounded polyhedron. A polytope P can just be described by the convex hull of a finite set of vectors, i.e., $P = \mathrm{conv}(K)$ for a finite set $K \in \mathbf{R}^n$.

The vectors $x_1, \ldots, x_k \in \mathbf{R}^n$ are *affinely independent* if $\sum_{i=1}^{k} \alpha_i x_i = 0$ and $\sum_{i=1}^{k} \alpha_i = 0$ implies $\alpha_i = 0$ for $i = 1, \ldots, k$. If $P \subseteq \mathbf{R}^n$ is a polyhedron and $\{x_0, \ldots, x_k\} \subseteq P$ is a maximal subset of affinely independent vectors of P, then we denote by $\dim(P) = k$ the *dimension* of P. If $\dim(P) = n$, we say that P has *full dimension* or that P is a *full-dimensional* polytope. The polytope P has dimension k if and only if a maximal system of linear equations for P has exactly $n - k$ linearly independent equations.

A linear inequality $cx \leq c_0$ is *valid* for a polyhedron P if it is satisfied by all vectors $x \in P$. A *face* of P is any set of the form $F = P \cap \{x \in \mathbf{R}^n : cx = c_0\}$, where $cx \leq c_0$ is a valid inequality for P. A face F is called *proper* if $F \neq \emptyset$ and $F \neq P$. The faces of dimensions 0, 1, $\dim(P) - 2$ and $\dim(P) - 1$ are called *extreme points*, *edges*, *ridges* and *facets*, respectively. In particular, the vertices are the minimal nonempty faces and the facets are the maximal proper faces. The set of all extreme points of P is denoted by $\mathrm{vert}(P)$. Every polytope is the convex hull of its vertices, and if $P = \mathrm{conv}(K)$ then $\mathrm{vert}(P) \subseteq K$.

Two polytopes $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^m$ are *affinely isomorphic*, denoted by $P \cong Q$, if there exists an affine map $f : \mathbf{R}^n \to \mathbf{R}^m$ that is a bijection between the points of the two polytopes. The polytopes P and Q are *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. This is equivalent to a bijection between $\mathrm{vert}(P)$ and $\mathrm{vert}(Q)$ such that the extreme points of faces of P correspond (under this bijection) to the extreme points of faces of Q. If two polytopes are affinely isomorphic then they are combinatorially equivalent. For a more thorough treatment of this topic we refer to [46].

B.3 Computational complexity

A *decision problem* Π consists of a set D_Π of *instances* and a subset $Y_\Pi \subseteq D_\Pi$ of *affirmative instances.* The set of instances is usually described by a general definition of all its parameters, and the affirmative instances are defined by a yes-no question asked in terms of the problem parameters. In this setting, an instance of the problem is obtained by specifying particular values for all the problem parameters. We assume that each problem has an associated *encoding scheme*, which maps problem instances into finite strings from a given alphabet. The *input length* of an instance $I \in D_\Pi$ is defined to be the number of symbols in the description obtained from the encoding scheme for the problem, and is denoted by Length(I). The length function Length : $D_\Pi \to \mathbf{Z}_+$ is used as the formal measure of the instance size.

The *time complexity function* $T_A : \mathbf{Z}_+ \to \mathbf{Z}_+$ of an algorithm A expresses its time requirements by giving, for each possible input length, the largest amount of time needed by the algorithm to solve a problem of that size. An algorithm A is called a *polynomial-time algorithm* if there exists a polynomial $p : \mathbf{R} \to \mathbf{R}$ such that $T_A(n) \leq p(n)$ for all $n \in \mathbf{Z}_+$. The class $\mathcal{P}$ is composed by the problems solvable by a polynomial-time algorithm.

A *nondeterministic algorithm* is an algorithm composed of a *guessing stage* and a *checking stage*. Given an instance of the problem, the guessing stage nondeterministically generates some structure. We then provide this structure to the checking stage, which computes in a normal deterministic manner and halts either with the answer "yes" or with the answer "no". A nondeterministic algorithm *solves* a decision problem if there exists some guessed structure such that the checking stage answers "yes" if and only if the instance is affirmative. A nondeterministic algorithm is said to *operate in polynomial time* if for every affirmative instance there is some guessed structure that leads the checking stage to an affirmative answer within time bounded by a polynomial in the input size. The class $\mathcal{NP}$ is defined to be the class of all decision problems solvable by nondeterministic algorithms operating in polynomial time. Clearly $\mathcal{P} \subseteq \mathcal{NP}$, but it is not known whether this inclusion is strict or not.

A *polynomial transformation* from a decision problem Π to a decision problem Π' is a function $f : D_\Pi \to D_{\Pi'}$ such that f is computable by a polynomial time deterministic algorithm and, for every $I \in D_\Pi$, $I \in Y_\Pi$ if and only if $f(I) \in Y_{\Pi'}$. If there is a polynomial transformation from Π to Π', we write $\Pi \propto \Pi'$. It is not difficult to verify that the relation induced by $\propto$ is transitive and reflexive. A decision problem Π is defined to be *$\mathcal{NP}$-complete* if $\Pi \in \mathcal{NP}$ and $\Pi' \propto \Pi$ for all $\Pi' \in \mathcal{NP}$. To prove that a certain decision problem Π is $\mathcal{NP}$-complete, it suffices to show that $\Pi \in \mathcal{NP}$ and that $\Pi' \propto \Pi$ for some $\mathcal{NP}$-complete problem Π'. If Π is $\mathcal{NP}$-complete, then there exists a polynomial-time algorithm solving Π if and only if $\mathcal{P}=\mathcal{NP}$.

If Π is a decision problem, we define the function Max : $D_\Pi \to \mathbf{Z}_+$ such that Max(I) denotes the magnitude of the largest number in I. An algorithm that solves a problem is called a *pseudo-polynomial time algorithm* if its time complexity is bounded by a polynomial on Length(I) and Max(I). A problem Π is a *number problem* if there exists no polynomial $p : \mathbf{R} \to \mathbf{R}$ such that Max$(I) \leq p(\text{Length}(I))$ for all $I \in D_\Pi$. For any decision problem Π and any polynomial $p : \mathbf{Z} \to \mathbf{Z}$, let Π_p denote the subproblem of Π obtained by restricting Π to only those instances I satisfying Max$(I) \leq p(\text{Length}(I))$. The decision problem Π is *$\mathcal{NP}$-complete in the strong sense* if Π belongs to $\mathcal{NP}$ and there exists a polynomial $p : \mathbf{Z} \to \mathbf{Z}$ such that Π_p is $\mathcal{NP}$-complete. If Π is $\mathcal{NP}$-complete in the strong sense, then there does not exist any pseudo-polynomial time algorithm solving Π unless $\mathcal{P}=\mathcal{NP}$.

Let Π and Π' denote arbitrary decision problems, with instance functions Length and Max, resp. Length$'$ and Max$'$, A *pseudo-polynomial transformation* from Π to Π' is a function $f : D_\Pi \to D_{\Pi'}$ such that

(a) for all $I \in D_\Pi$, $I \in Y_\Pi$ if and only if $f(I) \in Y_{\Pi'}$,

(b) f can be computed in time polynomial in the two variables Max(I) and Length(I),

(c) there exists a polynomial q_1 such that $q_1(\text{Length}'(f(I)) \leq \text{Length}(I)$ for all $I \in D_\Pi$, and

(d) there exists a two-variable polynomial q_2 such that $\text{Max}'(f(I)) \leq q_2(\text{Max}(I), \text{Length}(I))$ for all $I \in D_\Pi$.

Every polynomial transformation is a pseudo-polynomial transformation. If Π is $\mathcal{NP}$-complete in the strong sense, $\Pi' \in \mathcal{NP}$, and there exists a pseudo-polynomial transformation from Π to Π', then Π' is $\mathcal{NP}$-complete in the strong sense.

A *search problem* Π consists of a set D_Π of *instances* and, for each instance $I \in D_\Pi$, a set $S_\Pi(I)$ of *solutions*. An algorithm is said to *solve* a search problem Π if, given as input any instance $I \in D_\Pi$, it returns some solution belonging to $S_\Pi(I)$ whenever this set is nonempty. A *polynomial-time reduction* from a search problem Π to a search problem Π' is an algorithm A that solves Π by using a hypothetical subroutine S for solving Π' such that, if S is a polynomial-time algorithm for Π' then A is a polynomial-time algorithm for Π. If there exists a polynomial-time reduction from Π to Π', we write $\Pi \propto_R \Pi'$. A search problem Π is $\mathcal{NP}$-hard if there exists some $\mathcal{NP}$-complete problem Π' such that $\Pi' \propto_R \Pi$. An $\mathcal{NP}$-hard search problem cannot be solved in polynomial time unless $\mathcal{P}$=$\mathcal{NP}$.

Notation index

$\mathbf{R}$	the set of real numbers
$\mathbf{Z}$	the set of integer numbers
$\mathbf{Z}_+$	the set of non-negative integer numbers
2^F	power set of F
$\mathbf{1}$	vector $(1, \ldots, 1)$
$\mathcal{T}$	set of customers
$\mathcal{S}$	partition of $\mathcal{T}$ into sectors
$\mathcal{E}_X$	interfering pairs of customers in different sectors
$G = (V, E)$	interference graph
E_I	set of pairs of nodes in the same sector
E_X	interfering pairs of nodes in different sectors
n	number of nodes of G
m	number of edges of G
d	demand vector
g	guard distance
s	length of the frequency spectrum
$a(i)$	sector node i belongs to
$N(i)$	neighbor set of node i
$N(A)$	neighbor set of the node set A
l_i, r_i	interval bound variables
x_{ij}	ordering variables
$I(i) = [l_i, r_i]$	interval assigned to customer i
χ^S	incidence vector of a schedule S
$P(G, d, s, g)$	chromatic scheduling polytope
$R(G, d, s, g)$	fixed-length chromatic scheduling polytope
$P_{LP}(G, d, s, g)$	linear relaxation of $P(G, d, s, g)$
$R_{LP}(G, d, s, g)$	linear relaxation of $R(G, d, s, g)$
z_{l_i}	variable l_i from the incidence vector z
z_{r_i}	variable r_i from the incidence vector z
$z_{x_{ij}}$	variable x_{ij} from the incidence vector z
z_l	vector $(z_{l_1}, \ldots, z_{l_n})$
z_r	vector $(z_{r_1}, \ldots, z_{r_n})$
z_x	vector $(z_{x_{1i}}, \ldots, z_{x_{jn}})$

$\mathrm{ext}(y)$	extension of a solution $y \in R(G,d,s,g)$
$\mathrm{red}(z)$	reduction of a solution $z \in P(G,d,s,g)$
$\mathrm{sym}(z)$	symmetric solution
$s_{\mathrm{min}}(G,d,g)$	minimum frequency span such that $P(G,d,s,g) \neq \emptyset$
$s_{\mathrm{full}}(G,d,g)$	lower bound ensuring full-dimensionality
$s_{\mathrm{max}}(G,d,g)$	lower bound ensuring combinatorial stability
d_{max}	maximum demand $\max_{i \in V} d_i$
$d_{\mathrm{min}}(C)$	minimum demand $\max_{i \in C} d_i$
$d(K)$	summation $\sum_{i \in K} d_i$
p_K	number of sectors with nonempty intersection with K
$\nu(C)$	number of sector changes in the cycle C
δ_{ij}	minimum distance between $I(i)$ and $I(j)$
$\chi(G)$	chromatic number of G
$\omega(G)$	clique number of G
$\omega(G,d)$	weighted clique number of (G,d)
$\tau(G,d,g)$	minimum span generating a solution for each ordering
C_n	cycle on n nodes
P_n	path on n nodes
K_n	complete graph on n nodes
$K_{n,m}$	complete (n,m)-bipartite graph
G_A	subgraph induced by the node subset A
$E(A)$	set of edges with both endpoints in A
$E(A,B)$	set of edges with endpoints in A and B respectively
$F_s(G,d)$	set of nodes i with intervals greater than d_i
$\dim(P)$	dimension of the polyhedron P
$L_i(x,s)$	lower bound for l_i in $[0,s]$ under the ordering x
$U_i(x,s)$	upper bound for l_i in $[0,s]$ under the ordering x
$G(y)$	fixed-length adjacency graph
$H(z)$	general adjacency graph
$\cong$	affine isomorphism
$\mathrm{vert}(P)$	extreme points of P
P_{LO}^n	linear ordering polytope on n nodes
S_π	support of the inequality $\pi x \leq \pi_0$

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