Revising the AGM postulates

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Abstract

The logic of theory became a major subject in philosophical logic and artificial intelligence in the middle of the 1980's. The initial step was provided by Levi [Lev67, Lev80] and Alchourrón, Gärdenfors and Makinson in [AGM85] (the commonly called AGM model). In the AGM there are three types of change: *expansion, contraction* and *revision*. One way of defining the AGM functions is by means of postulates. Among these postulates, *recovery*, in contraction, and *success*, in revision, have provoked the greatest number of criticisms.

The present dissertation analyzes both these postulates in detail (see **Chapters 3** and **5**) and proposes alternative models of *contraction* (**Chapter 4**) and *revision* (**Chapter 6**). In **Chapters 7** and (**8** we introduce the notion of Credibility-Limited operators and define contraction and revision functions in terms of it. In the **Appendix** we introduce a battery of alternative postulates that allows us to construct several different change functions.

The background needed to read the dissertation is presented in **Chapters 1** and **2**.

List of Papers

The major part of the new results reported in the present thesis can also be found in the following list of papers. In the case of joint papers, they are used with the co-authors' kind permission. The results taken from each paper are specified in each chapter.

[•] Carlos Areces, Verónica Becher, Eduardo Fermé, and Ricardo Rodríguez. Observaciones a la teoría AGM. In *Proceedings Primer Encuentro en Temas de Lógica no Standard. Vaquerías - Córdoba*, 1996.

[•] Eduardo Fermé. Five faces of recovery. In H.Rott and M-A Williams, editors, *Frontiers in Belief Revision*. Kluwer Academic Publisher, 1999. to appear.

[•] Eduardo Fermé. A little note about maxichoice and epistemic entrenchment. 1998. (manuscript).

[•] Eduardo Fermé. On the logic of theory change: Contraction without recovery. *Journal of Logic, Language and Information*, 7:127–137, 1998.

[•] Eduardo Fermé. Technical note: Irrevocable belief revision and epistemic entrenchment. 1998. (manuscript).

[•] Eduardo Fermé and Sven Ove Hansson. Selective revision. *Studia Logica*, 1998. In press.

[•] Eduardo Fermé and Sven Ove Hansson. Shielded contraction. In H.Rott and M-A Williams, editors, *Frontiers in Belief Revision*. Kluwer Academic Publisher, 1999. to appear.

[•] Eduardo Fermé and Ricardo Rodríguez. Semi-contraction: Axioms and construction. 1997. (manuscript).

[•] Eduardo Fermé and Ricardo Rodriguez. A brief note about the Rott contraction. *Logic Journal of the IGPL*, 6(6):835–842, 1998.

[•] Sven Ove Hansson, Eduardo Fermé, John Cantwell, and Marcelo Falappa. Credibility-limited revision. 1998. (manuscript).

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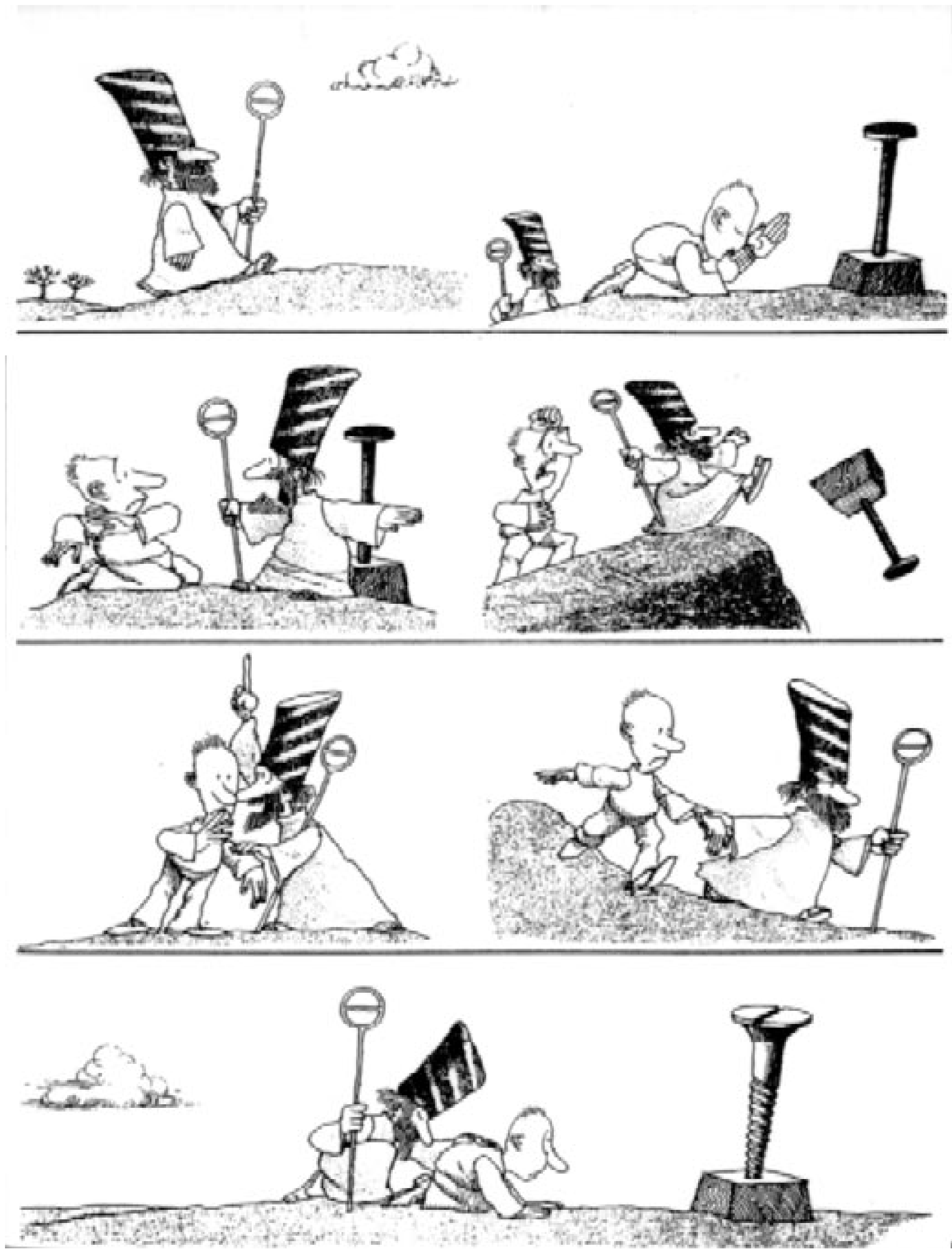
To Angelita, Eduardo, Graciela and Juanita,
with love and gratitude.

To Alchy and Manolo,
in memoriam.

"Todo se construye y se destruye
tan rápidamente,
que no puedo dejar de sonreir..."

[Charly García]



Quino. Déjenme Inventar. Ediciones de La Flor. Buenos Aires. Argentina. 1983

Preface

In **Chapter 6** we present an operation for belief revision where the new information is not always accepted. This operation is based on the AGM revision, but contradicts the *success* postulate of the AGM approach. If you agree with our proposal, you accept the idea that not always the new information is better than the old one; otherwise, rejecting our proposal is a clear example of the primacy of the old information.

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Part I

Introduction

Chapter 1

The Logic of Theory Change

In this chapter we introduce the notion of change and some possible ways of modeling changes. We also describe the elements of an epistemological theory.
Parts of the first section were extracted from:

[•] Carlos Areces, Verónica Becher, Eduardo Fermé and Ricardo Rodríguez. Observaciones a la teoría AGM. In *Primer Encuentro en Temas de Lógica no Standard. Vaquerías - Córdoba* (1996).

1.1 Preliminaries

1.1.1 An example of the change problem

We consider the following set of sentences in natural language[1]: *"Juan was born in Puerto Carreño"* (α), *"José was born in Puerto Ayacucho"* (β), *"Two people are compatriots if they were born in the same country "* (γ). We assume that this set represents all the currently available information

[1] This example is simply a modified version of the usual examples.

about Juan and José. Suppose that we receive the following piece of new information: *"Juan and José are compatriots"* (δ). If we add the new information to our corpus of beliefs we obtain a new set of beliefs that contains the sentences α, β, γ and δ. We can define an addition operation as an operation that takes a sentence and a set and returns the minimal set that includes both the previous beliefs and the new sentence. This addition operation exemplifies the simplest way of changing a set of sentences. There are other types of change that are not so simple.

For example, suppose that upon consulting an atlas we discover with surprise that Puerto Carreño is in Colombia (ϵ) and Puerto Ayacucho is in Venezuela (ϕ). If we add ϵ and ϕ to the set $\{\alpha, \beta, \gamma, \delta\}$, the result will be a set with contradictory information: Juan and José are compatriots but Puerto Carreño and Puerto Ayacucho do not belong to the same country. The addition does not satisfactorily reflect the notion of a *consistent update*. If we wish to retain consistency, some subset of the original set must be discarded or perhaps a part of the new information has to be rejected. In our example, there are several possible alternatives. The information about the Juan or José' birthplace could be wrong, and so could the atlas. Finally the fact that Juan and José are compatriots could be wrong. Any of these three options, either individually or combined, will allow us to solve the problem of the incompatibility among the original and the new information or beliefs. Consequently, we can specify an update operation that takes a set and a sentence and returns a new consistent set. The new set includes part (or all) of the beliefs of the original set and the new sentence (if we are willing to accept it). The outcome of an update can be expressed as a consistent subset of the outcome of the addition. We have shown that the update operation is based on two notions: *consistency* and a *selection* among the possible ways to perform the change.

There are other possible ways to change. Suppose that we discover that γ is incorrect, and therefore wish to discard it from our set, the result of which will be a new set where γ is absent. We want the fact that Juan and José are compatriots to remain indeterminate. Note that this is different from accepting as a fact that Juan and José are not compatriots. We can ask if the process of discarding information should behave as the inverse of the process of adding information: If after discarding information we proceed to add it again, will we obtain the original set or not? As the update operation, the operation of discarding requires the selection of some of the several possible results.

1.1.2 Some questions about the change problem

Before formalizing belief change, we should consider several points: Any formalization of change requires the selection of a representation language. In our previous example the information about Juan and José is represented by a set of sentences in natural language. The selection of a language implies the acceptance of important idealizations. Whatever language is chosen, the question that emerges is how to use the language to represent the information corpus or epistemic state: should it be represented by a single sentence or by a set (perhaps infinite) of sentences? In the last case, should the set be closed under some notion of logical consequence or only a simple enumeration of facts? This second option implies the need to obtain in some way the consequences of these facts and to differentiate between implicit and explicit information.

Can the corpus be updated spontaneously or does updating require an external stimulus? In other words, is the corpus internally stable? When an information corpus suffers changes only in response to external stimuli: Should the corpus and the information that provokes the change be represented by the same or different types of formal structures? Should both be sentences or both be sets of sentences? How should the sentences of the corpus be interpreted? If an epistemic interpretation of the sentences is chosen: What are the possible statuses of the sentences?: Acceptance, rejection, indetermination, or perhaps degrees of acceptability? Which types of information can be represented in the corpus?

On the other hand, it seems to be fundamental to define operations that answer to the minimal change notion, or maximum preservation of the information corpus. That is to say, it is required in some way to "calculate the value" of the information to be discarded. Does there exist a preference order that represents the credibility or informational value of expressions of the language? Is this order included in the information corpus or is it intrinsic to the change operation? Must the minimal change be quantitative or qualitative?

How many and which are the different ways in which an information corpus can be modified? Are they independent or interdefinible? What is the relationship between the original and the updated corpus? Does a function to update the original corpus exist? Which are the parameters of this function: The original corpus, the new information, or any other possibly parameters? Should the change operations take into account the history of the produced changes, or is each new operation performed independently of those performed earlier?

These kinds of questions encourage several authors [AGM85, DP92, Lev67, Lev91, KM91, MS88, Seg97, Spo87] to propose different beliefs change models and to assume some of the above options and discard others.

1.1.2.1 Some applications of the Logic of Theory Change

1.1.2.1.1 Artificial Intelligence A knowledge base (KB) is a structured collection of pieces of information. This information is represented by a finite set of sentences in a given language $\mathcal{L}$, and an inference engine to capture some or all of the logical consequences of the KB.

A database (DB) is a special case of a KB. It contains simple expressions in a more restricted language, that only allows atomic formulae (facts).

The problem of updating a KB is essential to make intelligent systems. The updating process is not a trivial operation, as we will see in the following example:

> **Example 1.1.1** [FUV83] "Consider for example a relational database with the ternary relations SUPPLIES, where a tuple $< a, b, c, >$ means that suppliers a supplies part b to project c. Suppose now that the relation contains the tuple <*Hughes, tiles, Space Shuttle*>, and that the user asks to delete this tuple. A simpleminded approach would to just go ahead and delete the tuple from the relation. However, while it is true that Hughes does not supply tiles to the Space Shuttle project anymore, it is not clear what to do about three other facts that were implied by the above tuple, i.e. that Hughes supplies tiles, that Hughes

> supplies parts to the Space Shuttle project, and that the Space Shuttle project uses tiles. In some circumstances it might not be a bad idea to replace the deleted tuple by three tuples with *null* values:
>
> *<Hughes, tiles, NULL>*
>
> *<Hughes, NULL, Space Shuttle>*
>
> *<NULL, tiles, Space Shuttle>*
>
> The database is not viewed merely as a collection of atomic facts, but rather as a collection of facts from which other facts can be derived. It is the interaction between the updated facts and the derived facts that is the source of the problems."

A knowledge base and its consequences can be used as a model of an epistemic state; and, the mentioned changes can correspond to the expansion, contraction and revision of an epistemic state. Consequently, the problem of updating a database and its consequences can be solved by means of change functions analogous to the functions for theories. The main problem consists in that not all the beliefs of the KB are registered explicitly; there are beliefs that are derived from a set, necessarily finite, which composes the KB.

1.1.2.1.2 Legal Codes A legal code can be represented as a set of propositions. When applying the code we make use not only of these propositions but also of their logical consequences. Some laws are added and others are discarded. New laws may contradict the previous ones. An amendment can be represented as a process that discards part of the old norms and adds new norms. Interesting works about legal codes and belief revision are [AB71, AM81, Gär89, HM97] and [Han98b, Chapter 4].

The relationship between belief revision, legal codes and defeseable conditional was studied by the late Carlos Alchourrón in the last years of his life [Alc93, Alc95, Alc96]. In these papers he proposed a philosophical elucidation of the notion of defeasibility and applied it to clarify deontic concepts such as that of a *prima facie* duty. We summarized his last works in [BFL+99].

1.2 The elements of an Epistemological Theory [Gär88]

An *epistemological theory* provides a conceptual apparatus for investigating changes in knowledge and belief, a representation of the *epistemic elements* and a criterion of rationality that governs the dynamics. The elements that compose an epistemological theory are:

Epistemic states: The epistemic states are used to represent an current or possible cognitive state of a rational agent in a certain moment. An epistemic state is "in equilibrium" if it is consistent and satisfies the rationality criteria.

Epistemic attitudes: These are the status of the *pieces of belief* included in an epistemic state. For example in a model based on propositions the epistemic attitudes may be: *accepted*, *rejected*, *indetermined*. In a probabilistic model possible epistemic attitudes are: *probable*, *likely*; in a possibilistic model, *possible* etc.

Epistemic inputs: If we assume that the corpus or epistemic state is internally stable, updates require external stimuli: the epistemic inputs. These inputs provoke "belief changes" and the transformation of the original epistemic state into a new epistemic state.

Criteria of rationality: They are situated on the metalevel of the epistemological theory and are used to determine the behaviour of the "belief change". For example: minimal change of the previous beliefs, consistency, primacy of the new information, etc.

Chapter 2

The AGM Account

The purpose of this chapter is to introduce the AGM account of belief change, originally developed by Alchourrón, Gärdenfors and Makinson [AGM85]. In **Sections 2.1** to **2.3** we introduce the formal apparatus of belief sets. In **Sections 2.4** to **2.6** we introduce five approaches to the AGM model. The relations among the five approaches are summarized in **Section 2.7**. Some results and analyses of this chapter appeared in:

[•] Carlos Areces, Verónica Becher, Eduardo Fermé and Ricardo Rodríguez. Observaciones a la teoría AGM. In *Primer Encuentro en Temas de Lógica no Standard. Vaquerías - Córdoba* (1996).

[•] Eduardo Fermé. A little note about Maxichoice and epistemic entrenchment. (submitted), 1998.

2.1 Formal Preliminaries

We shall primarily consider a propositional language $\mathcal{L}$. We assume that $\mathcal{L}$ may be either finite or infinite, unless we explicitly specify that it is finite.

We also assume that the language contains the usual truth functional connectives: negation($\neg$), conjunction($\wedge$), disjunction ($\vee$), implication ($\rightarrow$). $\bot$ denotes an arbitrary contradiction and $\top$ an arbitrary tautology. $\mathcal{L}$ is closed under truth-functional operations (for example, if $\alpha \in \mathcal{L}$ and $\beta \in \mathcal{L}$, then $\alpha \vee \beta \in \mathcal{L}$, etc.). We identify $\mathcal{L}$ with the set of all well-formed formulae. Lower case Greek letters α, β, δ, ... denote sentences. Upper case Latin letters A, B, C, ... denote sets of sentences. Boldface upper case Latin letters **K**, **H** are reserved for belief sets. $\mathcal{K}$ is the set of all belief sets. Upper case Greek letters , , Δ, Π, ... denote sets of sets of sentences (for example $\Delta = \{A, B, C\}$).

We say that two sentences α and β are logically independent if and only if all combinations of truth values are logically possible for them.

2.2 The consequence operator

Definition 2.2.1 [Tar56] A *consequence operation* on a language $\mathcal{L}$ is a function Cn that takes each subset of $\mathcal{L}$ to another subset of $\mathcal{L}$, such that:

Inclusion $A \subseteq Cn(A)$.

Iteration $Cn(A) = Cn(Cn(A))$.

Monotony If $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$.

To simplify the notation, we write $Cn(\alpha)$ for $Cn(\{\alpha\})$ when $\alpha \in \mathcal{L}$.
We are going to assume that Cn satisfies the following three properties:

Supraclassicality If α can be derived from A by classical truth-functional logic, then $\alpha \in Cn(A)$.

Deduction $\beta \in Cn(A \cup \{\alpha\})$ if and only if $(\alpha \to \beta) \in Cn(A)$.

Compactness If $\alpha \in Cn(A)$, then $\alpha \in Cn(A')$ for some finite subset A' of A.

We use $\vdash \alpha$ as an alternative notation for $\alpha \in Cn(\emptyset)$, $A \vdash \alpha$ for $\alpha \in Cn(A)$ and $\alpha \vdash \beta$ for $\beta \in Cn(\alpha)$. The consequence operator satisfies the following properties [Hanss]:

2.2.2 If Cn satisfies **iteration, monotony, supraclassicality**, and **deduction** then $Cn(\alpha \vee \beta) = Cn(\alpha) \cap Cn(\beta)$.

2.2.3 If Cn satisfies **iteration, monotony, supraclassicality**, and **deduction** then: If $\beta \in Cn(A \cup \{\alpha_1\})$ and $\beta \in Cn(A \cup \{\alpha_2\})$, then $\beta \in Cn(A \cup \{\alpha_1 \vee\})$. (introduction of disjunction into premises)

2.2.4 If Cn satisfies **deduction** then: $Cn(A) \vdash \neg\alpha$ if and only if $Cn(A \cup \{\alpha\}) \vdash \bot$.[1]

2.2.5 If Cn satisfies **iteration**, and **monotony** then: If $A \subseteq B \subseteq Cn(A)$ then $Cn(A) = Cn(B)$

2.2.6 If Cn satisfies **monotony** then $Cn(A) \cap Cn(B) = Cn(Cn(A) \cap Cn(B))$.

$Cn(A) \cup Cn(B) = Cn(Cn(A) \cup Cn(B))$ is not true in general: Let α and β be logically independent sentences, $A = \{\alpha\}$ and $B = \{\alpha \to \beta\}$: $\beta \notin Cn(A) \cup Cn(B)$, but $\beta \in Cn(Cn(A) \cup Cn(B))$. With respect to $\cup$, Cn satisfies the following properties:

[1] This property has a special role in Artificial Intelligence, and is used in the refutation process (see, for example, [Nil71]).

2.2.7 If Cn satisfies **inclusion, iteration**, and **monotony** then $Cn(A \cup B) = Cn(A \cup Cn(B))$.

2.2.8 If Cn satisfies **inclusion, iteration**, and **monotony** then: $Cn(A) \cup Cn(B) = Cn(Cn(A) \cup Cn(B))$ if and only if $A \subseteq B$ or $B \subseteq A$.

The following properties relate the Cn operator to the language:

2.2.9 If Cn satisfies **inclusion, iteration, monotony**, and **supraclassicality** then: If $\alpha \in Cn(A)$ and $\neg\alpha \in Cn(A)$, then $Cn(A) = \mathcal{L}$

2.2.10 If Cn satisfies **inclusion, iteration, monotony**, and **supraclassicality** then $Cn(\{\alpha \wedge \beta\}) = Cn(\{\alpha, \beta\})$

2.2.11 If Cn satisfies **inclusion, iteration, monotony, supraclassicality**, and **compactness** then: If $\mathcal{L}$ is finite, then for all sets A there exists a sentence α, such that $Cn(A) = Cn(\{\alpha\})$

The last property mentioned will be very helpful when we assume that $\mathcal{L}$ is finite, since then every set of sentences closed under Cn can be represented by the consequences of a single sentence.

2.3 Belief Sets

Definition 2.3.1 A set of sentences $\mathbf{K}$ is a **belief set** if and only if $\mathbf{K} = Cn(\mathbf{K})$.

Belief sets are also called *theories*. $\mathbf{K}_\perp$ denotes the inconsistent belief set, and it follows from **Property 2.2.9** that $\mathbf{K}_\perp = \mathcal{L}$. Note that $\mathbf{K} \vdash \alpha$ if and only if $\alpha \in \mathbf{K}$. We will use both notations interchangeably.

2.3.1 The AGM dynamics for belief sets

In **Section 1.2** we introduced the elements of an epistemological theory. Now we will describe these elements for the AGM account:

Epistemic States and Epistemic Inputs: Every belief set represents a belief state and all belief states can be represented by a belief set. Every sentence represents a belief, and all beliefs can be represented by a sentence.

Epistemic Attitudes: For any sentence α of $\mathcal{L}$ there are three possible epistemic attitudes to α with respect to a belief set $\mathbf{K}$: α is accepted if and only if $\alpha \in \mathbf{K}$, α is rejected if and only if $\neg\alpha \in \mathbf{K}$, otherwise α is indetermined. Note that when $\mathbf{K} \neq \mathbf{K}_{\perp}$, α is accepted if and only if $\neg\alpha$ is rejected and for any sentence α there is one and only one epistemic attitude.

Rationality Criteria: In the AGM account we can identify the following rationality criteria (in order of priority): **1.** *Primacy of new information*: the new information is always accepted **2.** *Consistency*: the new epistemic state must be consistent if possible **3.** *Minimal loss of previous beliefs*: the attempt to retain as much of the old beliefs as possible[2].

The dynamics of beliefs consists basically in constructions that modify the epistemic attitude to a sentence α in the actual belief set[3]. These changes

[2]Note that **1** is given higher priority than **2** when the new information is inconsistent; and **2** is given higher priority than **3** when the new information is consistent and the belief set is inconsistent.

[3]Since the operations of change are functions, they must also include the vacuous cases (expanding or revising by a sentence previously included in the original belief set,

are defined by functions that take a belief set and a sentence as input, and return a new belief set. Note that this change can also lead to changes in the attitudes to other sentences. There are six possible ways of change:

1. From *indetermined* to *accepted*.
2. From *indetermined* to *rejected*.
3. From *accepted* to *rejected*.
4. From *rejected* to *accepted*.
5. From *accepted* to *indetermined*.
6. From *rejected* to *indetermined*.

1. and 2. are called **expansion**, and consist in the simple addition of a new sentence to the belief set.

3. and 4. are referred to as **revision** and consist again in adding a new sentence, but consistency is preserved if possible.

5. and 6. have the name of **contraction**. They consist in the elimination of a sentence from the belief set.

There are different ways of defining **expansion**, **contraction** and **revision** functions that satisfy the rationality criteria. In **Sections 2.4- 2.6** we present five different approaches.

2.4 Syntactic Approach

In this section we present the AGM functions through a set of postulates that determine the behaviour of a change function, i.e., a set of conditions or constraints that change functions must satisfy. We will only present the AGM postulates. Variations and other postulates for change functions can be found in the **Appendix**.

contracting by a sentence that is not included in the original belief set). In these cases the functions leave the original belief set unchanged.

2.4.1 Expansion

Expansion is the simplest of the three AGM operations. It consists in adding the new information to the belief set. We write $+$ to refer to an expansion function $\mathcal{K} \times \mathcal{L} \rightarrow \mathcal{K}$ and we denote by $\mathbf{K}+\alpha$ the expansion of $\mathbf{K}$ by a sentence α.

Since the result of the expansion function must be a belief set, the first postulate demands the following condition:

- **Closure:** $\mathbf{K}+\alpha$ is a belief set.

Given that we add new information, we expect this information to be accepted in the outcome of the expansion:

- **Success:** $\alpha \in \mathbf{K}+\alpha$.

According to the criterion of *informational economy*, the expansion function preserves all the previous beliefs: Formally:

- **Inclusion:** $\mathbf{K} \subseteq \mathbf{K}+\alpha$.

Another main criterion of the AGM theory is *minimal change of belief*. In the vacuous case that α is already accepted in $\mathbf{K}$, according to minimal change, to incorporate α is to do nothing, i.e.:

- **Vacuity:** If $\alpha \in \mathbf{K}$, $\mathbf{K}+\alpha = \mathbf{K}$.

The notion of expansion is additive,

- **Monotony:** If $\mathbf{K} \subseteq \mathbf{H}$, $\mathbf{K}+\alpha \subseteq \mathbf{H}+\alpha$.

The previous postulates allow to define a family of expansion operators. To obtain a full characterization we require that the original theory not to be modified beyond what is strictly necessary to include the new information:

- **Minimality:** For all belief sets $\mathbf{K}$ and all sentences α, $\mathbf{K}{+}\alpha$ is the smallest belief set that satisfies *closure, success*, and *inclusion*.

The expansion operation $+$ can be uniquely determined as follows:

THEOREM 2.4.1 Let $+$ be an operation on $\mathbf{K}$. Then $+$ satisfies *minimality* if and only if $\mathbf{K}{+}\alpha = Cn(\mathbf{K} \cup \{\alpha\})$.

This theorem confirms the intuition that, in fact, the expansion operation is quite simple: It is suffices to: (1) Add the new sentence to the theory and (2) Close the result under logical consequence. Note that we don't use the postulates of *vacuity* and *monotony*. This means that these postulates are derivable from the others.

The expansion function satisfies also the following properties:

2.4.2 If $\beta \in \mathbf{K}{+}\alpha$, then $\mathbf{K}{+}\beta \subseteq \mathbf{K}{+}\alpha$

2.4.3 $\mathbf{K}{+}\alpha = \mathbf{K}{+}\beta$ if and only if $\beta \in \mathbf{K}{+}\alpha$ and $\alpha \in \mathbf{K}{+}\beta$

2.4.4 If $\alpha \leftrightarrow \beta \in \mathbf{K}$, then $\mathbf{K}{+}\alpha = \mathbf{K}{+}\beta$

2.4.5 If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}{+}\alpha = \mathbf{K}{+}\beta$

2.4.6 $(\mathbf{K} \cap \mathbf{H}){+}\alpha = \mathbf{K}{+}\alpha \cap \mathbf{H}{+}\alpha$

2.4.7 $\mathbf{K}{+}(\alpha \vee \beta) \subseteq \mathbf{K}{+}\alpha$

2.4.8 $(\mathbf{K}{+}\alpha) + \beta = \mathbf{K}{+}(\alpha \wedge \beta)$

2.4.9 $(\mathbf{K}+\alpha)+\beta = (\mathbf{K}+\beta)+\alpha$

In this subsection we have presented a set of postulates for expansion that uniquely determine the expansion function. As it will be seen in the next two subsections, it is not possible to do the same for contraction or revision in a plausible way.

2.4.2 Contraction

A contraction of a belief set occurs when some beliefs are retracted but no new belief is added. In this subsection, we introduce the contraction postulates. We write $-$ to denote a contraction function from $\mathbf{K} \times \mathcal{L}$ to $\mathcal{K}$. Hence, $\mathbf{K}-\alpha$ denotes the contraction of $\mathbf{K}$ by a sentence α.

Again, the result of the operation should be a belief set:

- **Closure:** $\mathbf{K}-\alpha$ is a belief set.

As far as possible, the objective of the operation must be carried out; i.e., if it is possible to eliminate the sentence from the theory then it must be eliminated. The only sentences that cannot be contracted are the tautologies.

- **Success:** If $\not\vdash \alpha$, then $\mathbf{K}-\alpha \not\vdash \alpha$.

No new belief is added to the belief set:

- **Inclusion:** $\mathbf{K}-\alpha \subseteq \mathbf{K}$.

The above postulates appear to be vital for all contraction functions for belief sets, so we can take them as necessary conditions:

Definition 2.4.10 [Hanss] An operator $-$ for a belief set $\mathbf{K}$ is a contraction operator if and only if it satisfies *closure, success* and *inclusion.*

In the limiting case when the sentence to be contracted is not implied by the original belief set, to eliminate α from $\mathbf{K}$ is to do nothing:

●**Vacuity:** If $\mathbf{K} \nvdash \alpha$, then $\mathbf{K}-\alpha = \mathbf{K}$.

The contraction operation must be independent of the syntactic representation of the sentences, in other words, logically equivalent sentences must yield the same result:

●**Extensionality:**[4] If $\vdash \alpha \leftrightarrow \beta$ then $\mathbf{K}-\alpha = \mathbf{K}-\beta$.

Definition 2.4.11 [Mak87] An operator $-$ for a belief set $\mathbf{K}$ is a *withdrawal* if and only if it satisfies *closure, inclusion, success, vacuity* and *extensionality.*

The criterion of informational economy requires that $\mathbf{K}-\alpha$ be a *large* subset of $\mathbf{K}$. For example,

$$\mathbf{K}-\alpha = \begin{cases} \mathbf{K} & \text{if } \alpha \notin \mathbf{K} \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

satisfies the above postulates. However, the above function it seems extreme, since for all non-limiting cases of contraction it returns only the minimal theory. Again, it remains for us to fix a last postulate that imposes a minimal change condition.

The AGM theory proposes as a rule of minimality the postulate of

[4] This postulate is also called **Preservation.**

recovery, that states that it is enough to add (by expansion) the eliminated sentence to recover totally the original theory.

●**Recovery:** $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + \alpha$

The converse of *recovery* follows from *inclusion*:

2.4.12 Whenever $-$ satisfies *inclusion*, if $\mathbf{K} \vdash \alpha$ then $(\mathbf{K}-\alpha) + \alpha \subseteq \mathbf{K}$.

The postulates listed above are called the basic AGM (or Gärdenfors) postulates. In addition to them, the AGM trio provided postulates for contraction by a conjunction. In order to contract a conjunction $\alpha \wedge \beta$ from a theory $\mathbf{K}$, we must either cease believing α or cease believing β. Now, if α is suppressed upon contracting by $\alpha \wedge \beta$, we expect that if a sentence δ has to be removed in order to remove α then it will also be removed when $\alpha \wedge \beta$ is removed:

●**Conjunctive inclusion:** If $\mathbf{K}-(\alpha \wedge \beta) \nvdash \alpha$, then $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$.

On the other hand, if a sentence δ in $\mathbf{K}$ is not suppressed either in the contraction of $\mathbf{K}$ by α or in the contraction of $\mathbf{K}$ by β, then δ must not be suppressed in the contraction of $\mathbf{K}$ by $\alpha \wedge \beta$:

●**Conjunctive overlap:** $\mathbf{K}-\alpha \cap \mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta)$.

The last two postulates are called the supplementary AGM (or Gärdenfors) postulates. In presence of the basic postulates, the supplementary postulates are equivalent to:

●**Conjunctive factoring:** $\mathbf{K}-(\alpha \wedge \beta) = \begin{cases} \mathbf{K}-\alpha, \text{ or} \\ \mathbf{K}-\beta, \text{ or} \\ \mathbf{K}-\alpha \cap \mathbf{K}-\beta \end{cases}$

Observation 2.4.13 [AGM85] Let **K** be a belief set and $-$ an operator on **K** that satisfies *closure, inclusion, vacuity, extensionality*, and *recovery.* Then $-$ satisfies both *conjunctive overlap* and *conjunctive inclusion* if and only if $-$ satisfies *conjunctive factoring.*

The intuition behind this observation and the conjunctive factoring postulate is one of the pillars of the AGM theory. If we wish to contract the belief set by a conjunction and there exists some preference between the conjuncts, then this contraction is equivalent to contraction by the non-preferred conjuncts. In the case of indifference among the conjuncts, the outcome of contracting by the conjunction equals the intersection of the outcomes of contractions by the conjuncts.

2.4.3 Revision

The revision function is related to expansion, in the sense that it incorporates new beliefs. However, as opposed to expansion, consistency is preserved in revision (unless the new information is inconsistent itself). Consequently, the revision process must eliminate enough sentences to avoid contradiction with the new belief. Just as for contraction, it is not possible to define a revision function uniquely, but it can be constrained by a set of postulates that the revision must satisfy.
We write $*$ to refer to a revision function from $\mathbf{K} \times \mathcal{L}$ to $\mathcal{K}$. Hence $\mathbf{K}{*}\alpha$ denotes the belief set that is the outcome of the revision of **K** by a sentence α.

Again, the result of the change must be a belief set:

- **Closure:** $\mathbf{K}{*}\alpha$ is a belief set.

According to the principle of "primacy of the new information", the new sentence must be incorporated in the revision.

●**Success:** $\mathbf{K}{*}\alpha \vdash \alpha$.

The revised belief set consists in the logical consequence of the new belief and a subset of sentences of **K** that do not contradict the new belief. The following postulate guarantees this:

●**Inclusion:** $\mathbf{K}{*}\alpha \subseteq \mathbf{K}{+}\alpha$.

Note that if $\neg\alpha \in \mathbf{K}$, then $\mathbf{K}{+}\alpha$ is the inconsistent belief set. In the case that the new belief does not contradict any of the sentences in **K**, there is no reason to remove any of them:

●**Vacuity:** If $\mathbf{K} \nvdash \neg\alpha$, then $\mathbf{K}{+}\alpha \subseteq \mathbf{K}{*}\alpha$.

According to the "consistency" criteria, unless the new belief is itself inconsistent, the result of the revision must be consistent.

●**Consistency:** If $\nvdash \neg\alpha$ then $\mathbf{K}{*}\alpha \neq \mathbf{K}_{\perp}$.

Note that $\mathbf{K}{*}\alpha$ is consistent even if α is consistent. In this revision differs from *updating* [KM92].

Just as contraction, the revision operation must be independent of the syntactic representation of the sentences, in other words, logically equivalent sentences must yield the same result:

●**Extensionality:** If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}{*}\alpha = \mathbf{K}{*}\beta$.

The above are the basic AGM (Gärdenfors) postulates for contraction.[5]

Let us now analyze revision of a theory $\mathbf{K}$ by a conjunction $\alpha \wedge \beta$. The idea is that, if $\mathbf{K}$ is to be changed minimally so as to include two sentences α and β, such a change should be possible by first revising $\mathbf{K}$ with respect to α and then expanding $\mathbf{K}*\alpha$ by β, provided that β does not contradict the beliefs in $\mathbf{K}*\alpha$. This argument (extracted from [GR93]) supports the following postulates[6]:

- **Superexpansion:** $\mathbf{K}*(\alpha \wedge \beta) \subseteq (\mathbf{K}*\alpha)+\beta$.

- **Subexpansion:** If $\mathbf{K}*\alpha \nvdash \neg\beta$, then $(\mathbf{K}*\alpha)+\beta \subseteq \mathbf{K}*(\alpha \wedge \beta)$.

Note that when $\neg\beta \in \mathbf{K}*\alpha$, then $(\mathbf{K}*\alpha)+\beta = \mathbf{K}_{\perp}$. Therefore the condition $\mathbf{K}*\alpha \nvdash \neg\beta$ is not needed in *superexpansion*. *Superexpansion* and *subexpansion* are called the supplementary AGM (or Gärdenfors) postulates. They are presented in terms of revision by a conjunction, but can be "translated" to closely related postulates of revision by a disjunction.

If a sentence δ in $\mathbf{K}$ is incorporated both in the revision of $\mathbf{K}$ by α and in the revision from $\mathbf{K}$ by β, then δ must be also incorporated in the revision of $\mathbf{K}$ by $\alpha \vee \beta$:

- **Disjunctive overlap:** $(\mathbf{K}*\alpha) \cap (\mathbf{K}*\beta) \subseteq \mathbf{K}*(\alpha \vee \beta)$.

When $\mathbf{K}*(\alpha \vee \beta) \nvdash \neg\alpha$, the sentences in $\mathbf{K}$ that remain in $\mathbf{K}*(\alpha \vee \beta)$ have no reason to be suppressed in $\mathbf{K}*\alpha$, and obviously the sentences incorporated in

[5] The original AGM postulates presented in [AGM85] included the "Harper Identity", see **Subsection 2.4.4**, and a combined version of *inclusion* and *vacuity* instead of having them separately. Here we present the modified version of Gärdenfors [Gär88], since in this version revision and contraction are totally independent functions.

[6] The precise formulation is split into two postulates to relate them to the supplementary postulates for contraction (see **2.4.4**) and with *partial meet revision* (see **2.5.1.2**).

the revision by $\alpha \vee \beta$ are also incorporated in the revision by α. The following postulate expresses this idea:

●**Disjunctive inclusion:** If $\mathbf{K}{*}(\alpha \vee \beta) \nvdash \neg\alpha$, then $\mathbf{K}{*}(\alpha \vee \beta) \subseteq \mathbf{K}{*}\alpha$.

The last two postulates appear to be, at first, more intuitive than the original AGM supplementary postulates. However, there exists a direct correspondence between the original postulates and the disjunctive postulates:

Observation 2.4.14 [Gär88] Let $\mathbf{K}$ be a belief set and let $*$ be an operator for $\mathbf{K}$ that satisfies *closure, success, inclusion, vacuity, consistency* and *extensionality.* Then:

1. $*$ satisfies *disjunctive overlap* if and only if it satisfies *superexpansion.*
2. $*$ satisfies *disjunctive inclusion* if and only if it satisfies *subexpansion.*

Finally, in the presence of the basic postulates, *superexpansion* (*disjunctive overlap*) and *subexpansion* (*disjunctive inclusion*) are equivalent to the following postulate:

●**Disjunctive factoring:** $\mathbf{K}{*}(\alpha \vee \beta) = \begin{cases} \mathbf{K}{*}\alpha, \text{ or} \\ \mathbf{K}{*}\beta, \text{ or} \\ \mathbf{K}{*}\alpha \cap \mathbf{K}{*}\beta \end{cases}$

Observation 2.4.15 [Gär88] Let $\mathbf{K}$ be a belief set and let $*$ be an operator for $\mathbf{K}$ that satisfies *closure, success, inclusion, vacuity, consistency* and *extensionality.* Then $*$ satisfies both *superexpansion* and *subexpansion* if and only if $*$ satisfies *disjunctive factoring.*

The intuition behind this observation is that if we wish to contract by a disjunction and there exist some preference between the disjuncts, then this revision is equivalent to revising by the preferred member. In the case of indifference, revising by the disjunction returns the beliefs that are common to the outcomes of revising by each member of the disjunction.

2.4.4 Relations Between Contraction and Revision

We have seen that contraction and revision are characterized by two different sets of postulates. These postulates are independent in the sense that the postulates of revision do not refer to contraction and vice versa. However, it is possible to define revision functions in terms of contraction functions, and vice versa.

2.4.4.1 Contraction to Revision

We can define revision in terms of contraction by mean of the Levi identity:

Definition 2.4.16 [Mak87] Let $\mathbf{K}$ be a theory, then $\mathbb{R}(-)$ is the function such that for every operator $-$ for $\mathbf{K}$, $\mathbb{R}(-)$ is the operator for $\mathbf{K}$ such that for all α:

$$\mathbf{K}\mathbb{R}(-)\alpha = (\mathbf{K}-\neg\alpha) + \alpha$$

Here, revision consists of two sub-operations: (1) contracting the theory by the negation of the sentence (and consequently obtain, if possible, a subset of the theory consistent with the new sentence) and (2) expanding the result by the new sentence. The identity is supported by the following observation:

Observation 2.4.17 [AGM85, Mak87] Let $\mathbf{K}$ be a theory and $-$ an operator for $\mathbf{K}$ that satisfies the contraction postulates *closure, inclusion, success, vacuity* and *extensionality.* Then $\mathbb{R}(-)$ is an operator for $\mathbf{K}$ that satisfies the revision postulates *closure, success, inclusion, vacuity, consistency* and *extensionality.*

The Levi identity allows us to use a contraction function as a primitive, and treat revision as defined in terms of contraction.

In **Observation 2.4.17** the *recovery* postulate is not needed. This means that each withdrawal function generates, via the Levi identity, a revision function that satisfies the six basic AGM postulates. If $-_1$ and $-_2$ are two withdrawal functions that generate the same revision function they are called *revision equivalent.* We write $[-]$ for the class of all withdrawal functions that are revision equivalent to $-$. In [Mak87], Makinson proved the following observation:

Observation 2.4.18 [Mak87] Let $\mathbf{K}$ be a belief set and $-_1$ a withdrawal function. Then there is a unique AGM contraction function $-$ that is revision equivalent to $-_1$. Furthermore for all elements $-_i$ of $[-_1]$, $\mathbf{K} -_i \alpha \subseteq \mathbf{K}-\alpha$.

The last observation shows that the AGM contraction function is the unique withdrawal operator for $\mathbf{K}$ that eliminates as little as possible. We will return to this point in **Chapter 3**.

With respect to the supplementary postulates, the role of *recovery* is different, as we will see in the following observations:

Observation 2.4.19 [AGM85] Let $\mathbf{K}$ be a theory and $-$ an operator for $\mathbf{K}$ that satisfies the contraction postulates *closure,* *inclusion, success, vacuity, extensionality* and *conjunctive inclusion*. Then $\mathbb{R}(-)$ satisfies *subexpansion*.

Observation 2.4.20 [AGM85] Let $\mathbf{K}$ be a theory and $-$ an operator for $\mathbf{K}$ that satisfies the contraction postulates *closure, inclusion, success, vacuity, extensionality,* **recovery** and *conjunctive overlap*. Then $\mathbb{R}(-)$ satisfies *superexpansion*.

Observation 2.4.21 Let $\mathbf{K}$ be a theory and $-$ an operator for $\mathbf{K}$ that satisfies the contraction postulates *closure, inclusion, success, vacuity, extensionality* and *conjunctive overlap*, but not *recovery*. Then, $\mathbb{R}(-)$ does not in general satisfy *superexpansion*.

The last observations show that, *recovery* is not needed to prove *subexpansion*. In presence of the other contraction postulates, it is enough to guarantee *superexpansion*. However, it is possible to define a contraction operator without *recovery*, whose revision via Levi identity satisfies *superexpansion* and *subexpansion*, for example *Rott Contraction* [Rot91b] (see **Subsection 3.2.2**).

2.4.4.2 Revision to Contraction

To define revision in terms of contraction we use one of the original AGM postulates, later excluded from the axioms and called the "Harper identity" (it is possible to find it also as the "Gärdenfors identity"):

Definition 2.4.22 [Mak87] Let $\mathbf{K}$ be a theory. Then $\mathbb{C}(*)$ is the function such that for every operator $*$ for $\mathbf{K}$, $\mathbb{C}(*)$ is the operator for $\mathbf{K}$ such that for all α:

$$\mathbf{K}\mathbb{C}(*)\alpha = \mathbf{K} \cap \mathbf{K}{*}\neg\alpha$$

The AGM trio provided the following observation:

> **Observation 2.4.23** [AGM85, Mak87] Let **K** be a theory and $*$ an operator for **K** that satisfies the revision postulates *closure, success, inclusion, vacuity, consistency* and *extensionality.* Then $\mathbb{C}(*)$ is an operator for **K** that satisfies the contraction postulates *closure, success, inclusion, vacuity, consistency, recovery* and *extensionality.* If $*$ also satisfies *superexpansion* then $\mathbb{C}(*)$ satisfies *conjunctive inclusion.* If $*$ also satisfies *subexpansion* then $\mathbb{C}(*)$ satisfies *conjunctive overlap.*

2.4.4.3 Complete relation

Given the Levi and Harper identities, the following question emerges: What would happen if a revision obtained through the Levi identity is used in the Harper identity and vice versa? Makinson [Mak87] obtained the following results:

> **THEOREM 2.4.24** Let **K** be a theory and $-$ an operator for **K** that satisfies the contraction postulates *closure, success, inclusion, vacuity, recovery*, and *extensionality.* Then $\mathbb{C}(\mathbb{R}(-)) = -$.

> **THEOREM 2.4.25** Let **K** be a theory and $*$ an operator for **K** that satisfies the revision postulates *closure, success, inclusion, vacuity, consistency*, and *extensionality.* Then $\mathbb{R}(\mathbb{C}(*)) = *$.

These results show that although every withdrawal function is revision equivalent to some AGM contraction, the Harper and Levi identities are in one to one correspondence only for AGM contractions, not for withdrawals. In **Chapter 4** we introduce a new identity like Harper's that relates revision to a special kind of withdrawal, semi-contraction.

2.5 Constructive Methods

2.5.1 Partial Meet Functions

2.5.1.1 Contraction

Another approach to the AGM change functions is to construct them explicitly. According to the informational economy criterion, the contraction function must retain as large a subset of $\mathbf{K}$ as possible. The sets that satisfy this property can be identified as follows:

Definition 2.5.1 [AM81] Let $\mathbf{K}$ be a belief set and α a sentence. The set $\mathbf{K}\perp\alpha$ ($\mathbf{K}$ *remainder* α) is the set of sets such that $\mathbf{H} \in \mathbf{K}\perp\alpha$ if and only if:

$$\begin{cases} \mathbf{H} \subseteq \mathbf{K} \\ \mathbf{H} \nvdash \alpha \\ \text{There is no set } \mathbf{H}' \text{ such that } \mathbf{H} \subset \mathbf{H}' \subseteq \mathbf{K} \text{ and } \mathbf{H}' \nvdash \alpha \end{cases}$$

$\mathbf{K}\perp\alpha$ is called a *remainder set* and its elements are the *remainders of* $\mathbf{K}$ *by* α. There is a special remainder set $\mathcal{L}\perp\perp$, that consists of all the maximal consistent subsets of the language. In some contexts these sets have been called *state descriptions* or *possible worlds* as we will see in Grove's model (**Subsection 2.6**).

Remainder sets satisfy the following properties:

2.5.2 $\mathbf{K}\perp\alpha = \{\mathbf{K}\}$ if and only if $\mathbf{K} \nvdash \alpha$

2.5.3 $\mathbf{K}\perp\alpha = \emptyset$ if and only if $\vdash \alpha$

2.5.4 [AM81] If $\mathbf{H} \subseteq \mathbf{K}$ and $\mathbf{H} \nvdash \alpha$, then there exists some $\mathbf{H}' \in \mathbf{K}\perp\alpha$ such that $\mathbf{H} \subseteq \mathbf{H}'$.

2.5.5 [AM82] If $\alpha \in \mathbf{K}$ and $\nvdash \alpha$, then for all $\mathbf{H}$ in $\mathbf{K}\perp\alpha$, $\mathbf{H}+\neg\alpha$ is a maximal consistent subset of the language.

A first tentative approach to constructing a contraction function is to choose α one element of $\mathbf{K}\perp\alpha$ [AM82] for each input sentence:

Maxichoice Contraction: $\mathbf{K}-\alpha \in \mathbf{K}\perp\alpha$ when $\nvdash \alpha$, $\mathbf{K}\perp\alpha = \mathbf{K}$ otherwise.

Though it seems to be intuitive, maxichoice contraction generates belief sets that are "too large", since it satisfies the following postulate [Mak85]:

Observation 2.5.6 Let $-$ be a maxichoice contraction function on a belief set $\mathbf{K}$. Then $-$ satisfies:

- **Saturability:** If $\alpha \in \mathbf{K}$, then for any $\beta \in \mathcal{L}$, either $\alpha \vee \beta \in \mathbf{K}-\alpha$ or $\alpha \vee \neg\beta \in \mathbf{K}-\alpha$

The following example shows the implausibility of this property:

Example 2.5.7 I believe that "it is four o'clock" (α). Then I discover that my watch was stopped. After that I must contract my belief α (but not revise it by $\neg\alpha$). According to *saturability* I must retain either "it is four o'clock or life exists after death"($\alpha \vee \beta$) or "it is four o'clock or life does not exist after death"($\alpha \vee \neg\beta$), but I have no reason to do this.

As regard to this point, Makinson [Mak85, page 357] said that "in general, neither $\alpha \vee \beta$ nor $\alpha \vee \neg\beta$ should be retained in the process of eliminating α from $\mathbf{K}$, unless there is "some reason" in $\mathbf{K}$ for their continued presence"[7]. We will return to maxichoice in relation with revision.

[7] In the original, Makinson used x and w instead of α and β, and A instead of $\mathbf{K}$.

In order to relate maxichoice contraction to the AGM axioms we need to introduce the following postulate [Gär88, Hanss]:

●**Fullness:** If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}-\alpha$ then $\not\vdash \alpha$ and $\alpha \rightarrow \beta \in \mathbf{K}-\alpha$

Fullness is a stronger version of *recovery*. Using fullness we can obtain the axiomatic characterization of the maxichoice contraction:

THEOREM 2.5.8 [Gär88] Let $\mathbf{K}$ be a belief set. An operator $-$ on $\mathbf{K}$ is a maxichoice contraction if and only if $-$ satisfies *closure, success, inclusion, vacuity, extensionality*, and *fullness*.

In the other extreme, we can consider another function that returns only the propositions that are common to all of the elements of $\mathbf{K}\perp\alpha$ [AM82]:

Full Meet Contraction: [AGM85] $\mathbf{K}-\alpha = \bigcap \mathbf{K}\perp\alpha$ when $\not\vdash \alpha$, $\mathbf{K}-\alpha = \mathbf{K}$ otherwise.

Contrary to maxichoice contraction, full meet contraction generates belief sets that are "too small" as we will see:

Observation 2.5.9 Let $-$ be a full meet contraction function on a belief set $\mathbf{K}$. Then $-$ satisfies

●**Devastation:** If $\alpha \in \mathbf{K}$, then for any $\beta \in \mathcal{L}$, either $\alpha \vee \beta \notin \mathbf{K}-\alpha$ or $\vdash \alpha \vee \beta$.

Since this condition holds for any β, we retain only the β sentences such that $\vdash \alpha \vee \beta$. Note that if $\delta \in \mathbf{K}$, then $\beta = \neg\alpha \vee \delta$ is retained, since $\vdash \alpha \vee \neg\alpha \vee \delta$. Consequently, we obtain the following characterization:

THEOREM 2.5.10 [AM82] If $-$ is a full meet contraction on $\mathbf{K}$ and $\alpha \in \mathbf{K}$, then $\mathbf{K}-\alpha = Cn(\{\neg\alpha\}) \cap \mathbf{K}$.

Full meet contraction satisfies the following postulate:

●**Meet Identity:** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha \cap \mathbf{K}-\beta$

Meet identity allows us to characterize full meet contraction in terms of postulates:

THEOREM 2.5.11 [Gär88] Let $\mathbf{K}$ be a belief set. An operator $-$ on $\mathbf{K}$ is a full meet contraction if and only if $-$ satisfies *closure, success, inclusion, vacuity, extensionality*, and *meet identity.*

Although a full meet contraction is not an appropriate contraction function, it provides the *lower bound* for the *recovery* postulate [Mak85]. We can formalize this concept in the following observation:

Observation 2.5.12 Let $\mathbf{K}$ be a belief set, $-$ the operator of full meet contraction for $\mathbf{K}$ and $\sim$ an operator for $\mathbf{K}$. Then $\sim$ satisfies *recovery* if and only if $\mathbf{K}-\alpha \subseteq \mathbf{K} \sim \alpha$ for all α.

A third approach is to generate the contraction outcome by the intersection of only some of the elements of $\mathbf{K}\perp\alpha$ [AGM85]. To do this we need to define a selection function for $\mathbf{K}\perp\alpha$.

Definition 2.5.13 [AGM85] Let $\mathbf{K}$ be a belief set. A *selection function* for $\mathbf{K}$ is a function γ such that for all sentences α:

1. If $\mathbf{K}\perp\alpha$ is non-empty, then $\gamma(\mathbf{K}\perp\alpha)$ is a non-empty subset of $\mathbf{K}\perp\alpha$.
2. If $\mathbf{K}\perp\alpha$ is empty, then $\gamma(\mathbf{K}\perp\alpha) = \mathbf{K}$.

We can further specify this selection function, to ensure that the "best" elements of $\mathbf{K}\perp\alpha$ are selected. For this purpose, we need to introduce a preference relation on $\mathbf{K}\perp\alpha$:

Definition 2.5.13 (cont.) γ is *relational* if and only if there is a relation $\sqsubseteq$ such that for all sentences α, if $\mathbf{K}\perp\alpha$ is non-empty, then:

$$\gamma(\mathbf{K}\perp\alpha) = \{\mathbf{B} \in \mathbf{K}\perp\alpha \mid \mathbf{B}' \sqsubseteq \mathbf{B} \text{ for all } \mathbf{B}' \in \mathbf{K}\perp\alpha\}$$

γ is *transitively relational* if and only if $\sqsubseteq$ is a transitive relation.

Partial meet contraction is defined in terms of the selection function γ:

Definition 2.5.14 [AGM85] Let $\mathbf{K}$ be a belief set and γ a selection function for $\mathbf{K}$. The *partial meet contraction* on $\mathbf{K}$ that is generated by γ is the operation $\sim_\gamma$ such that for all sentences α:

$$\mathbf{K} \sim_\gamma \alpha = \cap\gamma(\mathbf{K}\perp\alpha)$$

An operation $-$ on $\mathbf{K}$ is a partial meet contraction if and only if there is a selection function γ for $\mathbf{K}$ such that for all sentences $\alpha : \mathbf{K}-\alpha = \mathbf{K} \sim_\gamma \alpha$. Furthermore, $-$ is (transitively) relational if and only if it can be generated from a (transitively) relational selection function.

One of the major achievements of AGM theory is the characterization of partial meet contraction, and its transitively relational variant, in terms of a set of postulates:

THEOREM 2.5.15 [AGM85] Let $\mathbf{K}$ be a set of sentences. An operator $-$ on $\mathbf{K}$ is a partial meet contraction function if and only if $-$ satisfies *closure, success, inclusion, vacuity, recovery,* and *extensionality*. Furthermore, $-$ is a *transitively relational partial meet contraction* if and only if it also satisfies *conjunctive inclusion* and *conjunctive overlap*.

2.5.1.2 Revision

As we have seen in **Subsection 2.4.4**, we can define *partial meet revision* by means of the Levi identity:

Definition 2.5.16 [AGM85] Let $\mathbf{K}$ be a belief set. Let $*$ and $-$ be operators on $\mathbf{K}$ such that for all sentences α:

$$\mathbf{K}*\alpha = (\mathbf{K}-\neg\alpha)+\alpha$$

Then:

1. $*$ is a *maxichoice revision* if and only if $-$ is a *maxichoice contraction*.
2. $*$ is a *full meet revision* if and only if $-$ is a *full meet contraction*.
3. $*$ is a *partial meet revision* if and only if $-$ is a *partial meet contraction*.
4. $*$ is a *(transitively) relational partial meet revision* if and only if $-$ is a (transitively) relational partial meet contraction.

In the last subsubsection, we have observed that *maxichoice* and *full meet* are implausible contraction functions, but useful upper and lower bounds of *partial meet contraction*. This implausibility is even more evident in revision, as can be inferred from the following observations:

Observation 2.5.17 [AM82] Let $*$ be a *maxichoice revision* for a belief set $\mathbf{K}$. If $\neg\alpha \in \mathbf{K}$, then $\mathbf{K}{*}\alpha \in \mathcal{L}\perp\perp$.

Observation 2.5.18 [AM82] Let $*$ be a *full meet revision* for a belief set $\mathbf{K}$. If $\neg\alpha \in \mathbf{K}$, then $\mathbf{K}{*}\alpha = Cn(\alpha)$.

As we did for contraction, we can characterise *partial meet revision* in terms of postulates:

THEOREM 2.5.19 [AGM85] Let $\mathbf{K}$ be a belief set. An operator $*$ on $\mathbf{K}$ is a *partial meet revision function* if and only if $*$ satisfies *closure, success, inclusion, vacuity, consistency*, and *extensionality*. It is a *transitively relational partial meet revision function* if and only if it also satisfies *superexpansion* and *subexpansion.*

2.5.2 Epistemic Entrenchment

"Even if all sentences in a knowledge set are accepted or considered as facts, this does not mean that all sentences are of equal value for planning or problem solving purposes. Certain pieces of knowledge and belief about the world are more important than others when planning future actions, conducting scientific investigations or reasoning in general. We will say that some sentences in a knowledge system have a higher degree of epistemic entrenchment than others. The degree of entrenchment will, intuitively, have a bearing on what is abandoned from a knowledge set and what is retained, when a contraction or revision is carried out." [GM88]

This is the key idea of epistemic entrenchment introduced by Gärdenfors in [Gär88] to represent formally a preference ordering among formulae in a theory. He attempted at defining the contraction of a theory by a sentence in terms of an order of the sentences, and identifying the properties that this order must satisfy for the generated contraction to satisfy the AGM postulates. (Note that this is different from the preference orderings on remainder sets that we introduced in **Definition 2.5.13**)

Gärdenfors proposed a set of five postulates for the order among sentences where we write $\alpha <_{\mathbf{K}} \beta$ to denote $\alpha \leq_{\mathbf{K}} \beta$ and $\beta \not\leq_{\mathbf{K}} \alpha$; and $\alpha =_{\mathbf{K}} \beta$ to denote $\alpha \leq_{\mathbf{K}} \beta$ and $\beta \leq_{\mathbf{K}} \alpha$ The first postulate simply states that an epistemic entrenchment ordering is transitive:

- **(EE1) Transitivity:** If $\alpha \leq_{\mathbf{K}} \beta$ and $\beta \leq_{\mathbf{K}} \gamma$, then $\alpha \leq_{\mathbf{K}} \gamma$.

The Dominance postulate is based on the fact that, whenever a formula α entails a formula β and either α and β must be given up, the smaller change would result from abandoning α. Giving up β alone it is not possible since, being a consequence of α, it would be retained in the resulting belief set. On the other hand, is possible to give up α alone. Hence, β cannot be strictly less entrenched than α:

- **(EE2) Dominance:** If $\alpha \vdash \beta$, then $\alpha \leq_{\mathbf{K}} \beta$.

Removing $\alpha \wedge \beta$ necessarily implies removing either α or β. It is therefore natural to assume that $\alpha \wedge \beta$ is at least as entrenched as α or β.

- **(EE3) Conjunctiveness:** $\alpha \leq_{\mathbf{K}} (\alpha \wedge \beta)$ or $\beta \leq_{\mathbf{K}} (\alpha \wedge \beta)$.

The minimality postulate states that non-beliefs are all minimally entrenched.

●**(EE4) Minimality:** If $\mathbf{K} \neq \mathbf{K}_\perp$, then $\alpha \notin \mathbf{K}$ if and only if $\alpha \leq_{\mathbf{K}} \beta$ for all β.

The maximality postulate, on the other hand, states that the maximally entrenched beliefs are (exactly) the logical truths.

●**(EE5) Maximality:** If $\beta \leq_{\mathbf{K}} \alpha$ for all β, then $\vdash \alpha$.

A relation satisfying (**EE1**)-(**EE5**) is a *standard entrenchment ordering.* Standard entrenchment orderings satisfy the following properties [Foo90, GM88, Hanss]:

2.5.20 $\alpha \leq_{\mathbf{K}} \beta$ or $\beta \leq_{\mathbf{K}} \alpha$. (Connectivity)

2.5.21 If $\beta \wedge \delta \leq_{\mathbf{K}} \alpha$, then $\beta \leq_{\mathbf{K}} \alpha$ or $\delta \leq_{\mathbf{K}} \alpha$.

2.5.22 $\alpha <_{\mathbf{K}} \beta$ if and only if $\alpha \wedge \beta <_{\mathbf{K}} \beta$.

2.5.23 If $\delta \leq_{\mathbf{K}} \alpha$ and $\delta \leq_{\mathbf{K}} \beta$, then $\delta \leq_{\mathbf{K}} \alpha \wedge \beta$.

2.5.24 If $\alpha \leq_{\mathbf{K}} \beta$, then $\alpha \leq_{\mathbf{K}} \alpha \wedge \beta$.

2.5.25 If $\alpha <_{\mathbf{K}} \beta$ and $\beta <_{\mathbf{K}} \delta$, then $\alpha <_{\mathbf{K}} \delta$.

2.5.26 If $\alpha \leq_{\mathbf{K}} \beta$ and $\beta <_{\mathbf{K}} \delta$, then $\alpha <_{\mathbf{K}} \delta$

2.5.27 If $\beta <_{\mathbf{K}} \delta$ and $\beta <_{\mathbf{K}} \alpha$, then $\beta <_{\mathbf{K}} \alpha \wedge \delta$.

2.5.28 If $\alpha <_{\mathbf{K}} \beta$, then $\alpha <_{\mathbf{K}} \alpha \vee \beta$.

2.5.29 If $\alpha <_{\mathbf{K}} \beta$, then $\alpha \wedge \delta <_{\mathbf{K}} \beta$ for any $\delta \in \mathcal{L}$.

2.5.30 If $\alpha \leq_{\mathbf{K}} \beta$, then $\alpha \wedge \delta \leq_{\mathbf{K}} \beta$.

2.5.31 If $\beta \wedge \delta <_{\mathbf{K}} \alpha$, then $\beta <_{\mathbf{K}} \alpha$ or $\delta <_{\mathbf{K}} \alpha$.

2.5.32 If $\alpha \leq_{\mathbf{K}} \beta$, then $\alpha \wedge \delta \leq_{\mathbf{K}} \beta \wedge \delta$.

2.5.33 If $\nvdash \alpha$ and $\vdash \beta$, then $\alpha <_{\mathbf{K}} \beta$.

2.5.34 $\alpha \notin \mathbf{K}$ if and only if $\alpha <_{\mathbf{K}} \beta$ for all $\beta \in \mathbf{K}$.

2.5.35 If $\alpha <_{\mathbf{K}} \beta$ then $\alpha =_{\mathbf{K}} \alpha \wedge \beta$.

2.5.36 $\alpha \vee \beta <_{\mathbf{K}} \alpha \vee \neg\beta$ if and only if $\neg\alpha <_{\mathbf{K}} \alpha \vee \neg\beta$.

2.5.37 If $\vdash \alpha \leftrightarrow \alpha'$ and $\vdash \beta \leftrightarrow \beta'$, then: $\alpha \leq_{\mathbf{K}} \beta$ if and only if $\alpha' \leq_{\mathbf{K}} \beta'$. (Intersubstitutivity)

2.5.38 If $\mathbf{K} \neq \mathbf{K}_{\perp}$, then: $\alpha \in \mathbf{K}$ if and only if $\perp <_{\mathbf{K}} \alpha$.

2.5.39 $\top \leq_{\mathbf{K}} \alpha$ if and only if there is no δ such that $\alpha <_{\delta}$.

2.5.2.1 Contraction

The relation $\leq_{\mathbf{K}}$ of epistemic entrenchment is independent of the change functions in the sense that it does not refer to any contraction or revision function. In addition to stating the axioms of entrenchment, Gärdenfors proposed the connections between orders of epistemic entrenchment and contraction functions. The two are connected by the following equivalences:

$(C \leq)$ $\quad \alpha \leq_{\mathbf{K}} \beta$ if and only if $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$ or $\vdash (\alpha \wedge \beta)$.

Gärdenfors' entrenchment-based contraction

$(-_{\mathrm{G}})$ $\quad \beta \in \mathbf{K}-\alpha$ if and only if $\beta \in \mathbf{K}$ and, either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$.

THEOREM 2.5.40 [Gär88, GM88] Let $\leq_{\mathbf{K}}$ be a standard entrenchment ordering on a consistent belief set $\mathbf{K}$. Furthermore let $-_{\mathrm{G}}$ be the Gärdenfors entrenchment-based contraction on $\mathbf{K}$ defined by condition $(-_{\mathrm{G}})$ from $\leq_{\mathbf{K}}$. Then $-_{\mathrm{G}}$ satisfies the eight AGM postulates, and $(C\leq)$ also holds.

THEOREM 2.5.41 [Gär88, GM88] Let $-$ be an operation on a consistent belief set $\mathbf{K}$ that satisfies the eight AGM postulates. Furthermore let $\leq_{\mathbf{K}}$ be the relation defined from $-$ by condition $(C\leq)$. Then $\leq_{\mathbf{K}}$ satisfies the standard entrenchment postulates and $(-_{\mathrm{G}})$ also holds.

In [Fer98a] we investigated the relation between maxichoice contraction and entrenchment postulates. We can relate a special kind of maxichoices contraction that satisfies the supplementary AGM postulates (for example the orderly maxichoice contraction, defined in [AM82]) to epistemic entrenchment. In order to do this, we need the following postulate:

●**(EE6) Choice:** If $\alpha \in \mathbf{K}$ and $\not\vdash \alpha$, then $\alpha <_{\mathbf{K}} \alpha \vee \beta$ or $\alpha <_{\mathbf{K}} \alpha \vee \neg\beta$

Relations between epistemic entrenchment and maxichoice are provided by the following theorems[8]:

THEOREM 2.5.42 Let $\leq_{\mathbf{K}}$ be an entrenchment ordering on a consistent belief set $\mathbf{K}$ that satisfies (**EE1**)–(**EE6**). Furthermore let $-_G$ be the Gärdenfors' entrenchment-based contraction on $\mathbf{K}$ defined by condition $(-_G)$ from $\leq_{\mathbf{K}}$. Then $-_G$ satisfies *closure,*

[8]**Note:** We were informed, in a personal communication, that the same characterization has been independently obtained by Hans Rott [Rot].

inclusion, vacuity, success, extensionality, fullness, conjunctive inclusion and *conjunctive overlap*, and ($C \leq$) also holds.

THEOREM 2.5.43 Let $-$ be an operation on a consistent belief set $\mathbf{K}$ that satisfies *closure, inclusion, vacuity, success, extensionality, fullness, conjunctive inclusion* and *conjunctive overlap*. Furthermore let $\leq_{\mathbf{K}}$ be the relation defined from $-$ by condition ($C \leq$). Then $\leq_{\mathbf{K}}$ satisfies the entrenchment postulates (**EE1**) – (**EE6**) and ($-_{\mathrm{G}}$) also holds.

2.5.2.2 Revision

In the last subsection we have seen that through **Theorems 2.5.40** and **2.5.41** we can construct an AGM contraction based on an entrenchment ordering and conversely. Then it is natural to define entrenchment-based revision via the Levi identity. However, it is also possible to define entrenchment-based revision directly from an entrenchment ordering, by means of the following equivalences [9]:

($C \leq_*$) $\alpha \leq_{\mathbf{K}} \beta$ if and only if: If $\alpha \in \mathbf{K}{*}\neg(\alpha \wedge \beta)$ then $\beta \in \mathbf{K}{*}\neg(\alpha \wedge \beta)$.

($*_{EBR}$) [LR91, Rot91a] $\beta \in \mathbf{K}{*}\alpha$ if and only if either $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$ or $\alpha \vdash \bot$.

[9]**Note:** ($*_{EBR}$) was defined in [LR91, Rot91a], but in these papers the revision function is defined via the Levi identity. Here the relation does not refer completely to the contraction function. The proof can be found in the Appendix of the Chapter.

THEOREM 2.5.44 Let $\leq_{\mathbf{K}}$ be a standard entrenchment ordering on a consistent belief set $\mathbf{K}$. Furthermore let $*$ be an entrenchment-based contraction on $\mathbf{K}$ defined by condition $(*_{EBR})$ from $\leq_{\mathbf{K}}$. Then $*$ satisfies the eight AGM revision postulates, and $(C \leq_*)$ also holds.

THEOREM 2.5.45 Let $*$ be an operation on a consistent belief set $\mathbf{K}$ that satisfies the eight AGM revision postulates. Furthermore let $\leq_{\mathbf{K}}$ be the relation defined from $*$ by condition $(C \leq_*)$. Then $\leq_{\mathbf{K}}$ satisfies the standard entrenchment postulates and $(*_{EBR})$ also holds.

2.5.3 Safe and Kernel Contraction

Safe Contraction [AM85] and its generalization Kernel Contraction [Han94a] are based on a selection among the sentences of a belief set $\mathbf{K}$ that contribute effectively to imply α; and to use this selection in contracting by α. According to this concept, we define the *kernel* set for a sentence α and a belief set $\mathbf{K}$ as follows:

Definition 2.5.46 [Han94a] Let $\mathbf{K}$ be a belief set and α a sentence. Then $\mathbf{K} \perp\!\!\!\perp \alpha$ is the set such that $A \in \mathbf{K} \perp\!\!\!\perp \alpha$ if and only if:

$$\begin{cases} A \subseteq \mathbf{K} \\ A \vdash \alpha \\ \text{If } B \subset A \text{ then } B \nvdash \alpha \end{cases}$$

$\mathbf{K} \perp\!\!\!\perp \alpha$ is called the *kernel set of* $\mathbf{K}$ *respect to* α and its elements are the α-*kernels* of $\mathbf{K}$

Basically, at least one element of each *α-kernel* of $\mathbf{K}$ must be removed in the contraction process, otherwise the sentence α would continue being implied. On the other hand, due to the minimality criterion, we only discard sentences that are included in on or more elements of the kernel set. The remaining problem is how to choose the sentences to discard. The most general case, i.e. without additional criteria about the selection, is an incision function, defined as follows:

> **Definition 2.5.47** [Han94a] An *incision function* σ for $\mathbf{K}$ is a function such that for all sentences α:
>
> $$\begin{cases} \sigma(\mathbf{K} \perp\!\!\!\perp \alpha) \subseteq \bigcup(\mathbf{K} \perp\!\!\!\perp \alpha) \\ \emptyset \neq A \in \mathbf{K} \perp\!\!\!\perp \alpha, \text{ then } A \cap \sigma(\mathbf{K} \perp\!\!\!\perp \alpha) \neq \emptyset \end{cases}$$

The next step is to eliminate the set determined by the *incision function* from $\mathbf{K}$:

> **Definition 2.5.48** [Han94a] Let $\mathbf{K}$ be a belief set and σ an incision function for $\mathbf{K}$. The *kernel contraction* $-_\sigma$ for $\mathbf{K}$ is defined as follows:
>
> $$\mathbf{K} -_\sigma \alpha = \mathbf{K} \setminus \sigma(\mathbf{K} \perp\!\!\!\perp \alpha).$$
>
> An operator $-$ for a belief set $\mathbf{K}$ is a kernel contraction if and only if there is an incision function σ for $\mathbf{K}$ such that $\mathbf{K} \sim \alpha = \mathbf{K} -_\sigma \alpha$ for all sentences α.

Kernel contraction satisfies all the basic AGM postulates except *closure*. In order to achieve the satisfaction of *closure*, Hansson defined a special case of kernel contraction as the *smooth* kernel contraction:

Definition 2.5.49 [Han94a] An incision function σ for a belief set $\mathbf{K}$ is *smooth* if and only if it holds for all subsets A of $\mathbf{K}$ that: *if* $A \vdash \beta$ and $\beta \in \sigma(\mathbf{K} \perp\!\!\!\perp \alpha)$, *then* $A \cap \sigma(\mathbf{K} \perp\!\!\!\perp \alpha) \neq \emptyset$. A kernel contraction is smooth if and only if it is based on a smooth incision function.

THEOREM 2.5.50 [Han94a] Let $\mathbf{K}$ be a belief set. Then an operation $-$ is a smooth kernel contraction for $\mathbf{K}$ if and only if it is a partial meet contraction for $\mathbf{K}$.

The above observation relates the kernel contraction to partial meet contraction and, consequently, to the basic AGM postulates. To obtain a representation theorem for the supplementary AGM postulates (and for the transitively relational partial meet contraction), we must introduce constraints on the incision function. Alchourrón and Makinson [AM85, AM86] defined safe contraction. In this contraction, the belief set $\mathbf{K}$ is ordered according to a relation $\prec$. $\beta \prec \delta$ means that δ should be retained rather that β if we have to give up one of them, and we say that "β is less safe that δ". The ordering $\prec$ helps us to choose which element to remove from each kernel. The remaining beliefs are safe and can be used to determine the safe contraction of a belief set $\mathbf{K}$ by α (modulo $\prec$). $\prec$ must be an acyclic, irreflexive and asymmetric relation. Alchourrón and Makinson referred to this relation as a "hierarchy". Formally:

Definition 2.5.51 [AM85] Any sentence β in a belief set $\mathbf{K}$ is safe with respect to α if and only if β is not minimal under $\prec$ with respect to the elements of any $\mathbf{A} \in \mathbf{K} \perp\!\!\!\perp \alpha$ The set of all *safe* sentences of $\mathbf{K}$ respect to α is denoted $\mathbf{K}/\alpha$. Safe Contraction is defined by the following identity:

$$\mathbf{K} \sim \alpha = Cn(\mathbf{K} \cap \mathbf{K}/\alpha)$$

In order to satisfy the supplementary postulates we must add conditions on the hierarchy:

Definition 2.5.52 [AM86] Let $\mathbf{K}$ be a belief set and $\prec$ a hierarchy over $\mathbf{K}$. Then, for all α, β and $\delta \in \mathbf{K}$, $\prec$ is virtually connected over $\mathbf{K}$ if and only if: *if* $\alpha \prec \beta$ *then either* $\alpha \prec \delta$ *or* $\delta \prec \beta$.

THEOREM 2.5.53 [AM86, Rot92b] Let $\mathbf{K}$ be a belief set and $-$ an operator on $\mathbf{K}$. Then $-$ is a safe contraction, based on a virtually connected hierarchy $\prec$, if and only if $-$ is a transitively relational partial meet contraction.

2.6 Semantic Approach

2.6.1 Possible worlds and Grove's Sphere-Systems

We can consider the presentation through postulates that we gave in **Section 2.4** as purely syntactic, and in certain ways equivalent to the axioms of a logical system. As for all syntactic presentations, the question remains whether we are able, through these postulates, to capture the behaviour of the operation of change. Even if it is clear that all the postulates are important and seem to be correct, what guarantees that we have not forgotten to state some indispensable and fundamental properties to obtain a total characterization? This is not a simple question to answer due to the fact that we do not know all the characteristics of the change functions.

Grove presents an alternative model for the change functions, based on a system of spheres, whose form is similar to and inspired by the fields semantics for counterfactuals proposed by Lewis [Lew73]. We consider Grove's model to be fundamental since it provides semantics to the AGM model, which allows us to obtain a sense of soundness and completeness. As Pagnucco pointed out [Pag96], Grove's model can be seen as semantic insofar as it gives a "picture" of AGM belief change; strictly speaking, however, it deals with syntactic objects.

Grove takes as his starting point the set $M_{\mathcal{L}}$ of all maximal consistent subsets of the language. This set $M_{\mathcal{L}}$ can be seen as the set of possible worlds (propositions) that can be described in the language.

Definition 2.6.1 : Let $\mathcal{L}$ be a language. Then $M_{\mathcal{L}} = \mathcal{L} \perp\perp$.

A theory $\mathbf{K}$ is represented by the set of maximal consistent sets $\|\mathbf{K}\| \subseteq M_{\mathcal{L}}$ that includes all formulas of $\mathbf{K}$. Formally:

Definition 2.6.2 Let $\mathbf{K}$ be a belief set. Then $\|\mathbf{K}\| = \{M \in M_{\mathcal{L}} : \mathbf{K} \subseteq M\}$.

Similarly, each sentence α can be represented by the set $\|\alpha\| = \|Cn(\{\alpha\})\|$. A special case of this definition is $\|\mathbf{K}\| = \emptyset$, that represents the inconsistent theory $\mathbf{K}_{\perp}$. On the other hand, all sets of possible worlds have an associated theory:

Definition 2.6.3 Let $V \subseteq M_{\mathcal{L}}$. $Th(V)$ is the associate theory of V if and only if $Th(V) = \bigcap\{M : M \in V\}$.

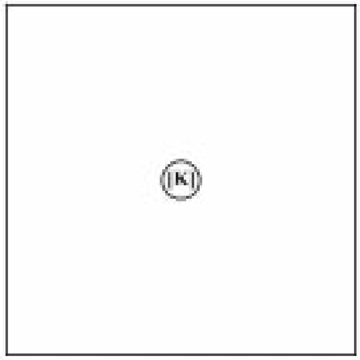


Figure 1 Possible worlds model.

It is interesting to emphasize the relationship between possible worlds and belief sets: $A \subseteq \mathbf{K}$ if and only if $\|\mathbf{K}\| \subseteq \|A\|$; i.e., having more possible worlds implies less sentences and conversely.

Let $\mathbf{K}$ and $\mathbf{H}$ be logically closed sets and α and β sentences. Then the following properties hold [Hanss]:

2.6.4 If $w \in \mathcal{L} \perp\perp$, then $w \in \|\alpha\|$ if and only if $w \notin \|\neg\alpha\|$.

2.6.5 $\|Cn(\mathbf{K} \cup \mathbf{H})\| = \|\mathbf{K}\| \cap \|\mathbf{H}\|$

2.6.6 $\|\mathbf{K}\| \cup \|\mathbf{H}\| \subseteq \|\mathbf{K} \cap \mathbf{H}\|$.

2.6.7 $\|\alpha\| \subseteq \|\beta\|$ if and only if $\vdash \alpha \rightarrow \beta$.

2.6.8 $\|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\|$

2.6.9 $\|\alpha \vee \beta\| = \|\alpha\| \cup \|\beta\|$.

2.6.10 $\|\alpha \rightarrow \beta\| = \|\neg\alpha\| \cup \|\beta\|$.

2.6.11 $\|\alpha \rightarrow \beta\| \cap \|\beta \rightarrow \alpha\| = \|\alpha \wedge \beta\| \cup \|\neg\alpha \wedge \neg\beta\|$.

These simple tools allow us to define the expansion operation. The expansion defined by postulates has a semantic counterpart: $\|\mathbf{K}+\alpha\|$ is the selection of $\mathbf{K}$ worlds that validate α. Expansion is defined as the operation that takes the common elements among $\|\mathbf{K}\|$ and $\|\alpha\|$:

$$\mathbf{K}+\alpha = Th(\|\mathbf{K}\| \cap \|\alpha\|)$$

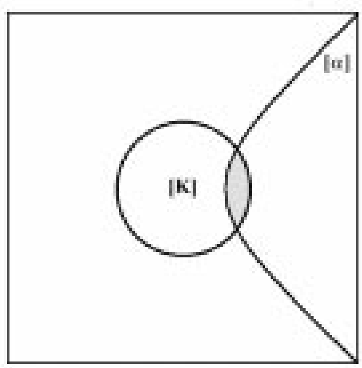


Figure 2 Expansion function.

2.6.2 Contraction

Conversely, the relation $\mathbf{K} \subseteq \mathbf{H}$ if and only if $\|\mathbf{H}\| \subseteq \|\mathbf{K}\|$ shows that to contract means to add possible worlds to our actual set of possible worlds, without discarding any previous worlds from $\|\mathbf{K}\|$. To obtain success in the process of contracting by α, we must add at least one $\neg\alpha$-world, and due to recovery, we must add *exclusively* $\neg\alpha$-worlds. Maxichoice contraction is the operation of contraction corresponding to minimal change, consequently it is easy to show that it corresponds to the addition of only one $\neg\alpha$-world to $\|\mathbf{K}\|$.

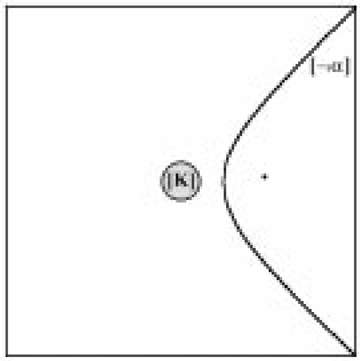


Figure 3 Maxichoice Contraction

On the other hand, Full Meet Contraction results in the minimal subset of **K** that satisfies recovery, so it corresponds to the addition of all the $\neg\alpha$-worlds to $\|\mathbf{K}\|$.

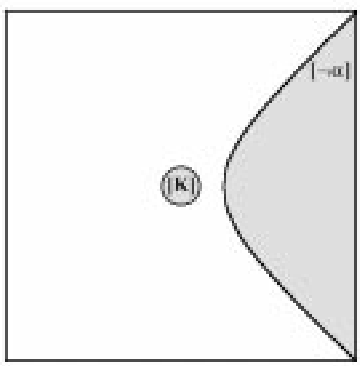


Figure 4 Full Meet Contraction

To define *partial meet contraction* in terms of possible world we introduce the following definitions:

Definition 2.6.12 [Hanss] Let $M \subseteq M_{\mathcal{L}}$. A *propositional selection function* for M is a function f such that for all sentences α:

(I) $f(\|\alpha\|) \subseteq \|\alpha\|$

(II) If $\|\alpha\| \neq \emptyset$ then $f(\|\alpha\|) \neq \emptyset$.

(III) If $M \cap \|\alpha\| \neq \emptyset$, then $f(\|\alpha\|) = M \cap \|\alpha\|$.

Definition 2.6.13 Let $M \subseteq M_{\mathcal{L}}$. An operator $\ominus$ is a *propositional contraction operator* for M if and only if there is a propositional selection function f for M such that for all α:

$$M \ominus \|\alpha\| = M \cup f(\|\neg\alpha\|)$$

THEOREM 2.6.14 [Gro88] Let $\mathbf{K}$ be a belief set and $-$ an operator for $\mathbf{K}$. Then the following conditions are equivalent:

1. $-$ satisfies *closure, inclusion, vacuity, success, extensionality*, and *recovery*.
2. There exists a propositional contraction $\ominus$ on $\|\mathbf{K}\|$ such that $\mathbf{K}-\alpha = \cap\|\mathbf{K} \ominus \alpha\|$ for all α.

To capture the supplementary postulates either in contraction or in revision, we need tools that are much more sophisticated. Grove defines a sphere-system centered around $\|\mathbf{K}\|$ as a collection $\mathbf{S}$ of subsets of $M_{\mathcal{L}}$ ordered by inclusion. Figuratively, the distance of a possible world to the center of the system reflects its plausibility related to the theory $\mathbf{K}$.

Definition 2.6.15 [Gro88] \$ is a *system of spheres* if and only if it satisfies:

\$1 $\emptyset \neq \$ \subseteq \mathcal{P}(\mathcal{L}\perp_{\perp})$,

\$2 $\cap\$ \in \$$,

\$3 If $\mathbf{G}$, $\mathbf{G}' \in \$$, then $\mathbf{G} \subseteq \mathbf{G}'$ or $\mathbf{G}' \subseteq \mathbf{G}$,

\$4 $\cup\$ \in \$$,

\$5 If $\|\alpha\| \cap (\cup\$) \neq \emptyset$, then $\mathbf{S}_\alpha \in \$$ and $\mathbf{S}_\alpha \cap \|\alpha\| \neq \emptyset$, and

\$6 $\mathcal{L} \perp\perp \in \$$,

where $\mathbf{S}_\alpha = \bigcap\{\mathbf{G} \in \$ \mid \mathbf{G} \cap \|\alpha\| \neq \emptyset\}$.

Definition 2.6.16 Let \$ be a system of spheres. Then:

$$\mathbf{K}_\$ = \bigcap \cap \$.$$

$\mathbf{S}_\alpha$ is the closest sphere that is compatible with α. $\mathbf{K}_\$$ is the belief set corresponding to \$.

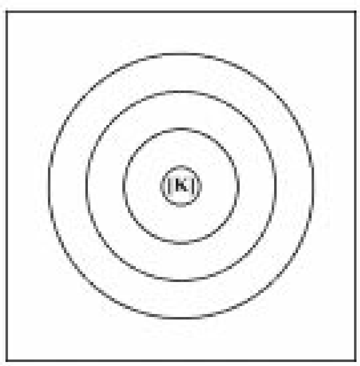


Figure 5 System of spheres centered in **K**.

Definition 2.6.17 [Gro88] A propositional selection function f for a proposition M is *sphere-based* if and only if there is a system of spheres \$ such that for all α: If $\|\alpha\| \neq \emptyset$, then $f(\|\alpha\|) = \mathbf{S}_\alpha \cap \|\alpha\|$.

Definition 2.6.18 Let $M \subseteq M_\mathcal{L}$. An operator $\ominus$ is a *sphere-based propositional contraction operator* for M if and only if there is a sphere based propositional selection function f for M such that for all α:

$$M \ominus \|\alpha\| = M \cup f(\|\neg\alpha\|)$$

THEOREM 2.6.19 [Gro88] Let $\mathbf{K}$ be a belief set and $-$ an operator for $\mathbf{K}$. Then the following conditions are equivalent:

1. $-$ satisfies *closure, inclusion, vacuity, success, extensionality, recovery, disjunctive inclusion*, and *disjunctive overlap*.
2. There exists a sphere-based propositional contraction $\ominus$ on $\|\mathbf{K}\|$ such that $\mathbf{K}-\alpha = \cap\|\mathbf{K} \ominus \alpha\|$ for all α.

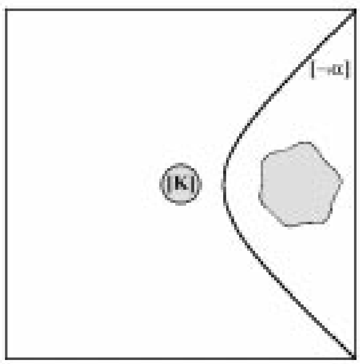


Figure 6 Partial Meet Contraction.

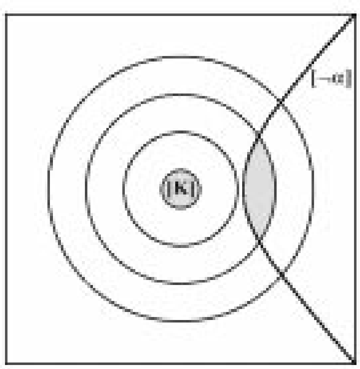


Figure 7 Transitively Relational Partial Meet Contraction.

2.6.3 Revision

Due to the Levi identity it is easy to show that the corresponding revision functions are the contraction functions without the **K**-worlds. **Figure 8** and **Figure 9** illustrate these relations.

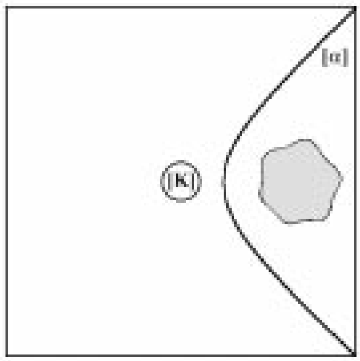


Figure 8 Partial Meet Revision.

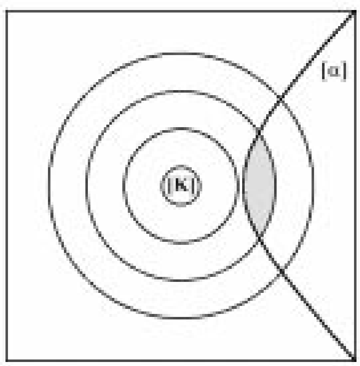


Figure 9 Transitively Relational Partial Meet Revision.

THEOREM 2.6.20 [Gro88] Let **K** be a belief set and $*$ an operator for **K**. Then the following conditions are equivalent:

1. $*$ satisfies *closure, inclusion, vacuity, success, extensionality, consistency, subexpansion*, and *superexpansion*.
2. There exists a propositional sphere-based selection function f such that for all α, $\mathbf{K}{*}\alpha = Th(f(\neg\alpha))$.

2.7 The Interconnections

In **Sections 2.4–2.6** we presented the AGM model from five different angles and some relations between them. The complete relations can be seen in **Figure 10** for the basic level, corresponding with the basic AGM postulates; and in **Figure 11** for the supplementary level, corresponding the basic plus the supplementary AGM postulates.

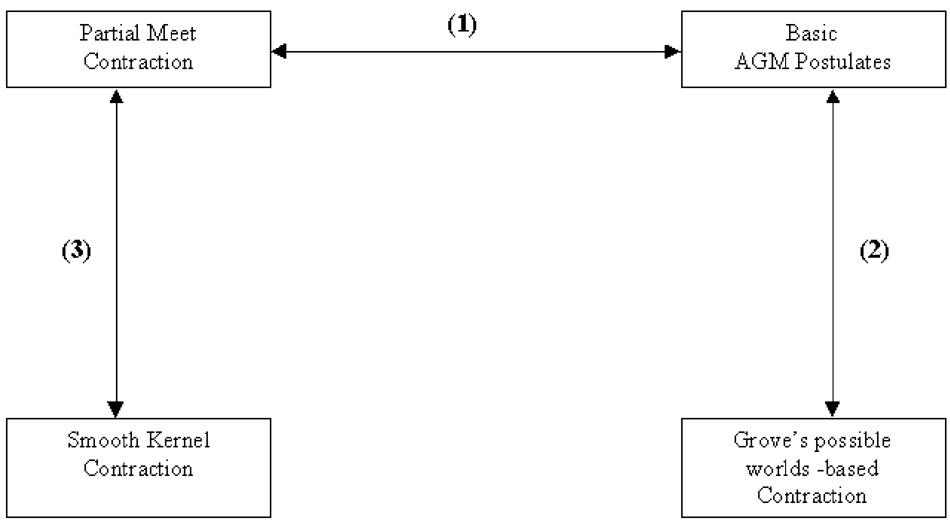


Figure 10 The interconnections on the basic level

(1) [AGM85] **(2)** [Gro88]
(3) [Han94a]

THEOREM 2.7.1 Let **K** be a belief set and $-$ an operator on **K**. Then the following conditions are equivalent:

1. $-$ satisfies *closure, inclusion, vacuity, success, extensionality* and recovery.
2. $-$ is a *partial meet contraction* function.

3. $-$ is a *smooth kernel contraction* function.

4. There exists a function f such that for all α, $f(\alpha) \subseteq \|\alpha\|$ and $\mathbf{K}-\alpha = Th(\|\mathbf{K}\| \cup f(\neg\alpha))$.

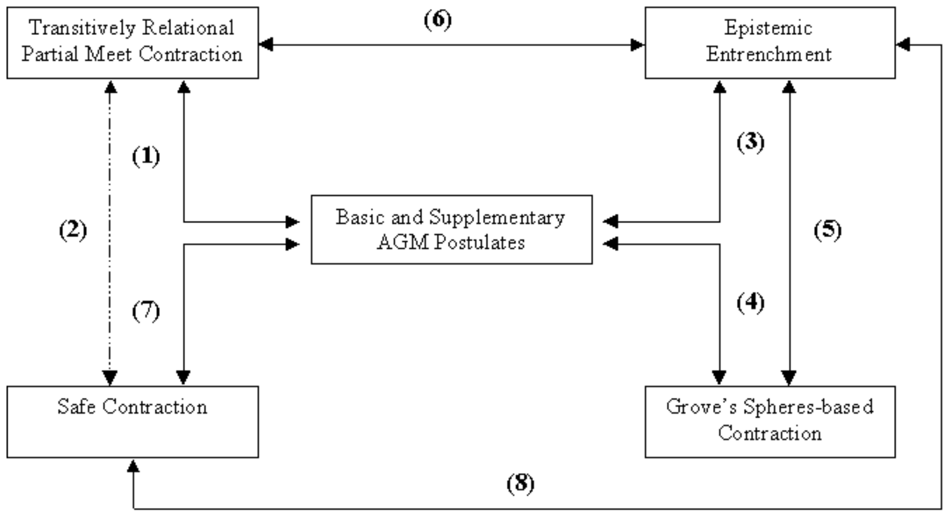


Figure 11 The interconnection in the supplementary level

(1) [AGM85]	**(2)** [AM86] (finite case)
(3) [GM88]	**(4)** [Gro88]
(5) [Gär88, Gro88]	**(6)** [Rot91b]
(7) [AM85, Rot92a]	**(8)** [Rot92a]

THEOREM 2.7.2 Let $\mathbf{K}$ be a belief set and $-$ an operator on $\mathbf{K}$. Then the following conditions are equivalent:

1. $-$ satisfies *closure, inclusion, vacuity, success, extensionality, recovery, disjunctive inclusion*, and *disjunctive overlap.*

2. $-$ is a *transitively relational partial meet contraction* function.

3. $-$ is a safe contraction function, based on a virtually connected hierarchy $\prec$.

4. $-$ is a Gärdenfors entrenchment-base contraction based on a relation $\leq_{\mathbf{K}}$ defined from $-$ by condition $(C\leq)$ and $\leq_{\mathbf{K}}$ satisfies the standard entrenchment postulates.

5. There exists a propositional sphere-based selection function f such that for all α, $\mathbf{K}{*}\alpha = Th(f(\neg\alpha))$.

2.8 Proofs of Chapter 2

Note: We only include the proofs of new results. Previous results by other authors are not included here.

Proof of Observation 2.4.21. Let $\mathcal{L} = \{\alpha, \beta, \gamma\}$ and $\mathbf{K} = Cn(\alpha \wedge \beta)$, and let $-$ be defined as follows:

$$\mathbf{K}{-}x = \begin{cases} \mathbf{K} & \text{if } x \notin \mathbf{K} \text{ or } \vdash x \\ Cn(\alpha \vee \gamma) & \text{if } \vdash x \leftrightarrow \alpha \\ Cn(\beta \vee \gamma) & \text{if } \vdash x \leftrightarrow \beta \\ Cn(\beta \vee \beta \vee \gamma) & \text{if } \vdash x \leftrightarrow \alpha \wedge \beta \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

We must prove **(a)** that $-$ satisfies *closure, inclusion, vacuity, success, extensionality* and *conjunctive overlap*, **(b)** that $-$ does not satisfy *recovery* and **(c)** that $\mathbb{R}(-)$ does not satisfy *superexpansion.*

(a) It is trivial that $-$ satisfies *closure, inclusion, success, vacuity* and *extensionality.* To prove that satisfies *conjunctive* *overlap* we show that the cases where $\mathbf{K}{-}x_1 \cap \mathbf{K}{-}x_2 \not\subseteq \mathbf{K}{-}x_1 \wedge x_2$ are not possible. Due the symmetry between x_1 and x_2 and the definition of $-$, the only possible cases are where $\mathbf{K}{-}x_1 = \mathbf{K}$, but in these cases, if $\vdash x_1$, then $\vdash x_1 \leftrightarrow x_1 \wedge x_2$, in which case $\mathbf{K}{-}x_1 \wedge x_2 = \mathbf{K}$. If $x_1 \notin \mathbf{K}$, then $x_1 \wedge x_2 \notin \mathbf{K}$, hence $\mathbf{K}{-}x_1 \wedge x_2 = \mathbf{K}$.

(b) Trivial, let $\vdash x \leftrightarrow \beta$.

(c) Due to **Observation 2.4.14**, we can prove that $\mathbb{R}(-)$ does not satisfy *disjunctive overlap*. Let $x_1 = \alpha$ and $x_2 = \beta$.
$(\mathbf{K}{-}\alpha) + \neg\alpha = Cn(\neg\alpha \wedge \gamma)$.
$(\mathbf{K}{-}\beta) + \neg\beta = Cn(\neg\beta \wedge \gamma)$.
$(\mathbf{K}{-}\neg(\neg\alpha \vee \neg\beta)) + (\neg\alpha \vee \neg\beta) = Cn((\alpha \vee \beta \vee \gamma) \wedge (\neg\alpha \vee \neg\beta))$.
It is easy to show that $\gamma \in \mathbf{K}\mathbb{R}(-)\neg\alpha \cap \mathbf{K}\mathbb{R}(-)\neg\beta$, and $\gamma \notin \mathbf{K}\mathbb{R}(-)(\neg\alpha \vee \neg\beta)$; hence $\mathbb{R}(-)$ does not satisfy *disjuntive overlap*. ■

Proof of Theorem 2.5.42. The proofs for *closure, inclusion, vacuity, success, extensionality, conjunctive inclusion, conjunctive overlap*, and $(-_{\mathrm{G}})$ can be found in [GM88]. Here, we must only prove *fullness*:

Fullness: Let $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}{-}\alpha$, then by $-_{\mathrm{G}}$, $\alpha \vee \beta \leq_{\mathbf{K}} \alpha$, then by (**EE6**), $\alpha <_{\mathbf{K}} \alpha \vee \neg\beta$. By (**EE1**) and (**EE2**), $\alpha <_{\mathbf{K}} \alpha \vee (\alpha \vee \neg\beta)$, hence, by $-_{\mathrm{G}}$, $\alpha \vee \neg\beta \in \mathbf{K}{-}\alpha$. ■

Proof of Theorem 2.5.43. Since *fullness* implies *recovery* the proofs for (**EE1**) – (**EE5**) and $(-_{\mathrm{G}})$ can be found in [GM88]. We must only prove (**EE6**).

(**EE6**): Let $\alpha \in \mathbf{K}$, $\nvdash \alpha$ and $\alpha \vee \beta \leq_{\mathbf{K}} \alpha$, then by $(C \leq)$, $\alpha \vee \beta \notin \mathbf{K} - \alpha \wedge (\alpha \vee \beta)$ and by *extensionality* $\alpha \vee \beta \notin \mathbf{K} - \alpha$. Then by *fullness* and *closure* $(\alpha \vee \beta) \rightarrow \alpha \in \mathbf{K} - \alpha$, i.e., $\neg\beta \vee \alpha \in \mathbf{K} - \alpha$, then by *extensionality* $\neg\beta \vee \alpha \in \mathbf{K} - \alpha \wedge (\alpha \vee \neg\beta)$. By *vacuity, success* and *extensionality* $\alpha \notin \mathbf{K} - \alpha \wedge (\alpha \vee \neg\beta)$; hence by $(C \leq)$, $\alpha <_{\mathbf{K}} \alpha \vee \neg\beta$. ■

Proof of Theorem 2.5.44. Due to **Observation 2.4.15** we can prove *disjunctive factoring* instead of *superexpansion* and *subexpansion.*

Closure Let $\epsilon \in \mathcal{L}$. Then, by compactness of the underlying logic, there is a finite subset $\{\beta_1, ..., \beta_n\} \subseteq \mathcal{L}$, such that $\{\beta_1, ..., \beta_n\} \vdash \epsilon$. We must prove that if $\{\beta_1, ..., \beta_n\} \subseteq \mathbf{K} * \alpha$, then $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K} * \alpha$ and $\epsilon \in \mathbf{K} * \alpha$. If $\alpha \vdash \perp$, then it follows trivially from $(*_{EBR})$ that $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K} * \alpha$ and $\epsilon \in \mathbf{K} * \alpha$. Let $\alpha \nvdash \perp$. Then:

Part 1. We are going to show that $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K} * \alpha$. For this purpose we are going to prove that if $\beta_1 \in \mathbf{K} * \alpha$ and $\beta_2 \in \mathbf{K} * \alpha$ then $\beta_1 \wedge \beta_2 \in \mathbf{K} * \alpha$. The rest follows by iteration of the same procedure. It follows from $\beta_1 \in \mathbf{K} * \alpha$ by $(*_{EBR})$ that $(\alpha \rightarrow \neg\beta_1) <_{\mathbf{K}} (\alpha \rightarrow \beta_1)$. Then by **Property 2.5.35** and **Property 2.5.36**, $\neg\alpha <_{\mathbf{K}} (\alpha \rightarrow \beta_1)$. Then it follows from $\beta_2 \in \mathbf{K} * \alpha$ that $\neg\alpha <_{\mathbf{K}} (\alpha \rightarrow \beta_2)$. By (**EE3**), either $(\alpha \rightarrow \beta_1) \leq_{\mathbf{K}} ((\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2))$ or $(\alpha \rightarrow \beta_1) \leq_{\mathbf{K}} ((\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2))$. Equivalently by **Property 2.5.37** either $(\alpha \rightarrow \beta_1) \leq_{\mathbf{K}} (\alpha \rightarrow (\beta_1 \wedge \beta_2))$ or $(\alpha \rightarrow \beta_2) \leq_{\mathbf{K}} (\alpha \rightarrow (\beta_1 \wedge \beta_2))$. In the first case, we use (**EE1**) and $\neg\alpha <_{\mathbf{K}} (\alpha \rightarrow \beta_1)$ to obtain $\neg\alpha <_{\mathbf{K}} (\alpha \rightarrow (\beta_1 \wedge \beta_2))$ and in the second one we use $\neg\alpha <_{\mathbf{K}} (\alpha \rightarrow \beta_2)$ to obtain the same result. It follows that $\beta_1 \wedge \beta_2 \in \mathbf{K}{*}\alpha$.

Part 2. By repeated use of **Part 1**, we know that $\{\beta_1 \wedge ... \wedge \beta_n\} \in \mathbf{K}{*}\alpha$. Let $\vdash \beta \leftrightarrow \beta_1 \wedge ... \wedge \beta_n$. We also have $\vdash \beta \rightarrow \epsilon$, then by $(*_{EBR})$ $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$. Since $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \epsilon)$ and $\vdash (\alpha \rightarrow \neg\epsilon) \rightarrow (\alpha \rightarrow \neg\beta)$, dominance yields $(\alpha \rightarrow \beta) \leq_{\mathbf{K}} (\alpha \rightarrow \epsilon)$ and $(\alpha \rightarrow \neg\epsilon) \leq_{\mathbf{K}} (\alpha \rightarrow \neg\beta)$. We can apply (**EE1**) to $(\alpha \rightarrow \neg\epsilon) \leq_{\mathbf{K}} (\alpha \rightarrow \neg\beta)$, $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$ and $(\alpha \rightarrow \beta) \leq_{\mathbf{K}} (\alpha \rightarrow \epsilon)$ to obtain $(\alpha \rightarrow \neg\epsilon) <_{\mathbf{K}} (\alpha \rightarrow \epsilon)$. Hence by $(*_{EBR})$, $\epsilon \in \mathbf{K}{*}\alpha$.

Success If $\alpha \vdash \bot$, then it follows trivially from $(*_{EBR})$ that $\alpha \in \mathbf{K}{*}\alpha$. Let $\alpha \nvdash \bot$. Then by **Property 2.5.33** $\neg\alpha <_{\mathbf{K}} (\neg\alpha \vee \alpha)$ or equivalently by **Property 2.5.37** $(\alpha \rightarrow \neg\alpha) <_{\mathbf{K}} (\alpha \rightarrow \alpha)$. Hence by $(*_{EBR})$, $\alpha \in \mathbf{K}{*}\alpha$.

Inclusion If $\alpha \vdash \bot$, then it follows trivially from $(*_{EBR})$ that $\mathbf{K}{*}\alpha = \mathbf{K}{+}\alpha = \mathcal{L}$. Let $\alpha \nvdash \bot$ and $\beta \in \mathbf{K}{*}\alpha$. We want to show that $\beta \in \mathbf{K}{+}\alpha$, which can be done by showing that $\alpha \rightarrow \beta \in \mathbf{K}$. By the definition of $(*_{EBR})$, since $\beta \in \mathbf{K}{*}\alpha$, $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$; hence by (**EE4**), $(\alpha \rightarrow \beta) \in \mathbf{K}$.

Vacuity Let $\neg\alpha \notin \mathbf{K}$ and $\beta \in \mathbf{K}{+}\alpha$. Then $\alpha \rightarrow \beta \in \mathbf{K}$. Due to $\neg\alpha \notin \mathbf{K}$, $\mathbf{K}{+}\alpha \neq \mathbf{K}_{\bot}$, then $\neg\beta \notin \mathbf{K}{+}\alpha$, then $\alpha \rightarrow \neg\beta \notin \mathbf{K}$; and it follows by (**EE4**) that $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$. Hence by $(*_{EBR})$, $\beta \in \mathbf{K}{*}\alpha$

Consistency Suppose that $\bot \in \mathbf{K}{*}\alpha$ and $\alpha \nvdash \bot$. Then by

$(*_{EBR})$, $(\alpha \to \neg\bot) <_{\mathbf{K}} (\alpha \to \bot)$. Then by **Property 2.5.37** $\top <_{\mathbf{K}} \bot$. Contradiction by (**EE2**).

Extensionality Let $\vdash \alpha \leftrightarrow \alpha'$. If $\alpha \vdash \bot$ then $\alpha' \vdash \bot$, hence by $(*_{EBR})$, $\mathbf{K}{*}\alpha = \mathbf{K}{*}\alpha'$. By **Property 2.5.37** it follows for all β that $(\alpha \to \neg\beta) =_{\mathbf{K}} (\alpha' \to \neg\beta)$ and $(\alpha \to \beta) =_{\mathbf{K}} (\alpha' \to \beta)$. Hence by (**EE1**) $(\alpha \to \neg\beta) <_{\mathbf{K}} (\alpha \to \beta)$ if and only if $(\alpha' \to \neg\beta) <_{\mathbf{K}} (\alpha' \to \beta)$; hence $\mathbf{K}{*}\alpha = \mathbf{K}{*}\alpha'$.

Disjunctive factoring If $\vdash \alpha$, then $\vdash (\alpha \vee \beta) \leftrightarrow \beta$ and the rest follows from the previous proof of *extensionality*. The symmetric case when $\vdash \beta$ can be handled in the same way. Let $\nvdash \alpha$ and $\nvdash \beta$. We have three subcases:

(a) $\neg\alpha <_{\mathbf{K}} \neg\beta$. Then $\nvdash \neg\alpha$. We will prove that $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\alpha$. For one direction let $\delta \in \mathbf{K}{*}\alpha$. It follows by $(*_{EBR})$ that $(\alpha \to \neg\delta) <_{\mathbf{K}} (\alpha \to \delta)$. Then by **Property 2.5.35** and **Property 2.5.36**, $\neg\alpha <_{\mathbf{K}} (\alpha \to \delta)$. It follows by $\neg\alpha <_{\mathbf{K}} \neg\beta$ and **Property 2.5.35** that $\neg\alpha =_{\mathbf{K}} (\neg\alpha \wedge \neg\beta)$. Since (**EE2**) yields $\neg\beta <_{\mathbf{K}} (\beta \to \delta)$, we use (**EE1**) to obtain both $(\neg\alpha \wedge \neg\beta) <_{\mathbf{K}} (\alpha \to \delta)$ and $(\neg\alpha \wedge \neg\beta) <_{\mathbf{K}} (\beta \to \delta)$. (**EE2**) and (**EE3**) yield $(\neg\alpha \wedge \neg\beta) <_{\mathbf{K}} ((\alpha \vee \beta) \to \delta)$. Hence $\delta \in \mathbf{K}{*}(\alpha \vee \beta)$. For the other direction, let $\delta \in \mathbf{K}{*}(\alpha \vee \beta)$. It follows by $\neg\alpha =_{\mathbf{K}} (\neg\alpha \wedge \neg\beta)$ that $\nvdash (\neg\alpha \wedge \neg\beta)$; then by $(*_{EBR})$, $(\neg\alpha \wedge \neg\beta) <_{\mathbf{K}} ((\alpha \vee \beta) \to \delta)$. By (**EE2**) $((\alpha \vee \beta) \to \delta) \leq_{\mathbf{K}} (\alpha \to \delta)$. (**EE1**) yields $\neg\alpha <_{\mathbf{K}} (\alpha \to \delta)$, hence $\delta \in \mathbf{K}{*}\alpha$.

(b) $\neg\beta <_{\mathbf{K}} \neg\alpha$: Equivalently to case **(a)**; $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\beta$.

(c) $\neg\alpha =_{\mathbf{K}} \neg\beta$. Then $\neg\alpha =_{\mathbf{K}} \neg\beta =_{\mathbf{K}} (\neg\alpha \wedge \neg\beta)$. Then $\delta \in \mathbf{K}{*}\alpha \cap \mathbf{K}{*}\beta$ iff (by $(*_{EBR})$) $\neg\alpha <_{\mathbf{K}} (\alpha \rightarrow \delta)$ and $\neg\beta <_{\mathbf{K}} (\alpha \rightarrow \delta)$ iff (by (**EE1**)) $(\neg\alpha \wedge \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \delta)$ and $(\neg\alpha \wedge \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \delta)$ iff (by (**EE2**) and (**EE3**)) $(\neg\alpha \wedge \neg\beta) <_{\mathbf{K}} ((\alpha \vee \beta) \rightarrow \delta)$ iff (by $(*_{EBR})$) $\delta \in \mathbf{K}{*}(\alpha \vee \beta)$.

$(C \leq_{*})$ For the first direction, let $\alpha \leq_{\mathbf{K}} \beta$ and let $\alpha \in \mathbf{K}{*}\neg(\alpha \wedge \beta)$. There are two subcases according to $(*_{EBR})$: If $\neg(\alpha \wedge \beta) \vdash \bot$, it follows trivially from $(*_{EBR})$ that $\beta \in \mathbf{K}{*}\neg(\alpha \wedge \beta)$. Let $\neg(\alpha \wedge \beta) \nvdash \bot$, then $(\neg(\alpha \wedge \beta) \rightarrow \neg\alpha) <_{\mathbf{K}} (\neg(\alpha \wedge \beta) \rightarrow \alpha)$, then by **Property 2.5.37**, $(\beta \vee \neg\alpha) <_{\mathbf{K}} \alpha$. By (**EE2**), $\beta \leq_{\mathbf{K}} (\beta \vee \neg\alpha)$, then it follows by (**EE1**) that $\beta <_{\mathbf{K}} \alpha$. Contradiction.

The other direction can be proved by showing that **(a)** if $\beta <_{\mathbf{K}} \alpha$, then $\alpha \in \mathbf{K}{*}\neg(\alpha \wedge \beta)$ and **(b)** if $\beta <_{\mathbf{K}} \alpha$, then $\beta \notin \mathbf{K}{*}\neg(\alpha \wedge \beta)$.

(a) We can do this by showing $\neg(\alpha \wedge \beta) \rightarrow \neg\alpha <_{\mathbf{K}} \neg(\alpha \wedge \beta) \rightarrow \alpha$, or equivalently, $\beta \vee \neg\alpha <_{\mathbf{K}} \alpha$. Suppose for *reductio* that this is not the case. Then $\alpha \leq_{\mathbf{K}} \beta \vee \neg\alpha$. Since $\alpha \leq_{\mathbf{K}} \alpha$, (**EE3**) yields $\alpha \leq_{\mathbf{K}} \alpha \wedge (\beta \vee \neg\alpha)$, hence $\alpha \leq_{\mathbf{K}} \alpha \wedge \beta$, so that by (**EE1**) $\alpha \leq_{\mathbf{K}} \beta$, contrary to the conditions.

(b) Suppose to the contrary that $\beta <_{\mathbf{K}} \alpha$ and $\beta \in \mathbf{K}{*}\neg(\alpha \wedge \beta)$. There are two cases according to $(*_{EBR})$: **(b1)** $\vdash \alpha \wedge \beta$. Then $\vdash \beta$, hence by (**EE5**) $\alpha \leq_{\mathbf{K}} \beta$, contrary to the conditions. **(b2)** $\neg(\alpha \wedge \beta) \rightarrow \neg\beta <_{\mathbf{K}} \neg(\alpha \wedge \beta) \rightarrow \beta$, or equivalently by **Property 2.5.37** to $\alpha \wedge \beta <_{\mathbf{K}} \beta$, from which it follows by (**EE1**) that

$\alpha \wedge \beta <_{\mathbf{K}} \alpha$. We arrive to a contradiction according to (**EE3**). This conclude the proof. ■

Proof of Theorem 2.5.45.

(**EE1**) Let $\alpha \leq_{\mathbf{K}} \beta$, $\beta \leq_{\mathbf{K}} \gamma$ and $\alpha \in \mathbf{K}*\neg(\alpha \wedge \gamma)$. We need to prove $\gamma \in \mathbf{K}*\neg(\alpha \wedge \gamma)$.

(a) $\alpha \in \mathbf{K}*\neg(\alpha \wedge \beta)$. Then by $(C \leq_*)$, $\beta \in \mathbf{K}*\neg(\alpha \wedge \beta)$. Then by *closure* $\alpha \wedge \beta \in \mathbf{K}*\neg(\alpha \wedge \beta)$. It follows by *consistency* and *success* that $\vdash \alpha \wedge \beta$, then $\vdash \beta$. *Closure* yields $\beta \in \mathbf{K}*\neg(\beta \wedge \gamma)$. Then by $(C \leq_*)$, $\gamma \in \mathbf{K}*\neg(\beta \wedge \gamma)$. Then by *closure* $\beta \wedge \gamma \in \mathbf{K}*\neg(\beta \wedge \gamma)$. It follows by *consistency* and *success* that $\vdash \beta \wedge \gamma$, then $\vdash \gamma$. Hence by *closure* $\gamma \in \mathbf{K}*\neg(\alpha \wedge \gamma)$.

(b) $\alpha \notin \mathbf{K}*\neg(\alpha \wedge \beta)$. Let $\gamma \notin \mathbf{K}*\neg(\alpha \wedge \gamma)$. Then by *closure* $\nvdash \gamma$, and it follows that $\nvdash \beta \wedge \gamma$, then by *success* and *consistency* $\beta \wedge \gamma \notin \mathbf{K}*\neg(\beta \wedge \gamma)$. Since $\beta \leq_{\mathbf{K}} \gamma$ and $(C \leq_*)$, $\beta \notin \mathbf{K}*\neg(\beta \wedge \gamma)$. We will arrive to a contradiction by proving **(b1)** $\alpha \in \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$ and **(b2)** $\alpha \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$:

(b1) Since $\vdash \neg(\alpha \wedge \beta \wedge \gamma) \leftrightarrow \neg((\alpha \wedge \gamma) \vee (\alpha \wedge \neg\beta))$, it follows by *disjunctive overlap* that $\mathbf{K}*\neg(\alpha \wedge \gamma) \cap \mathbf{K}*(\alpha \wedge \neg\beta) \subseteq \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$. By hypothesis $\alpha \in \mathbf{K}*\neg(\alpha \wedge \gamma)$ and by *closure* and *suceess* $\alpha \in \mathbf{K}*(\alpha \wedge \neg\beta)$; hence $\alpha \in \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$.

(b2) Due to the hypothesis condition $\alpha \notin \mathbf{K}*\neg(\alpha \wedge \beta)$ it enough to prove that $\mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma) \subseteq \mathbf{K}*\neg(\alpha \wedge \beta)$. Due to *disjunctive inclusion* and $\vdash \neg(\alpha \wedge \beta \wedge \gamma) \leftrightarrow$

$(\neg(\alpha \wedge \beta) \vee \neg\gamma)$ it suffices to prove that $\alpha \wedge \beta \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$. Since $\alpha \notin \mathbf{K}*\neg(\alpha \wedge \beta)$, by *closure* $\nvdash \alpha$ and consequently $\nvdash (\alpha \wedge \beta \wedge \gamma)$; then by *consistency* $(\alpha \wedge \beta \wedge \gamma) \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$. Then by *closure* either $(\alpha \wedge \beta) \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$ or $\gamma \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$. In the first case we already have what we need. In the second case it follows by *closure* that $(\beta \wedge \gamma) \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$; then by *disjunctive inclusion* and *extensionality* $\mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma) \subseteq \mathbf{K}*\neg(\beta \wedge \gamma)$. Since $\beta \notin \mathbf{K}*\neg(\beta \wedge \gamma)$, $\beta \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$, hence by *closure* $(\alpha \wedge \beta) \notin \mathbf{K}*\neg(\alpha \wedge \beta \wedge \gamma)$ that concludes the proof.

(**EE2**) Let $\vdash \alpha \to \beta$, and $\alpha \in \mathbf{K}*\neg(\alpha \wedge \beta)$. Then by *closure* $\beta \in \mathbf{K}*\neg(\alpha \wedge \beta)$; hence by $(C \leq_*)$ $\alpha \leq_{\mathbf{K}} \beta$.

(**EE3**) We have three subcases:

(a) $\alpha \notin \mathbf{K}*\neg(\alpha \wedge \beta)$. Then by *extensionality* $\alpha \notin \mathbf{K}*\neg(\alpha \wedge (\alpha \wedge \beta))$, hence by $(C \leq_*)$ $\alpha \leq_{\mathbf{K}} (\alpha \wedge \beta)$.

(b) $\beta \notin \mathbf{K}*\neg(\alpha \wedge \beta)$. In the same way as in **(a)**, $\beta \leq_{\mathbf{K}} (\alpha \wedge \beta)$.

(c) $\alpha \in \mathbf{K}*\neg(\alpha \wedge \beta)$ and $\beta \in \mathbf{K}*\neg(\alpha \wedge \beta)$. Then by *closure*, $(\alpha \wedge \beta) \in \mathbf{K}*\neg(\alpha \wedge \beta)$. Hence by $(C \leq_*)$, $\alpha \leq_{\mathbf{K}} (\alpha \wedge \beta)$ and $\beta \leq_{\mathbf{K}} (\alpha \wedge \beta)$.

(**EE4**) From left to right, let $\alpha \notin \mathbf{K}$. Then for all β by *vacuity* $\mathbf{K}*\neg(\alpha \wedge \beta) = \mathbf{K}+\neg(\alpha \wedge \beta)$. Suppose that $\alpha \in \mathbf{K}*\neg(\alpha \wedge \beta)$. Then $(\neg(\alpha \wedge \beta) \to \alpha) \in \mathbf{K}$, and since $\mathbf{K}$ is logically closed, $\alpha \in \mathbf{K}$. Contradiction, then for all β $\alpha \notin \mathbf{K}*\neg(\alpha \wedge \beta)$; hence by $(C \leq_*)$ for all β, $\alpha \leq_{\mathbf{K}} \beta$.

For the other direction let $\alpha \leq_{\mathbf{K}} \beta$ for all β; then in particular $\alpha \leq_{\mathbf{K}} \neg\alpha$. Then by $(C \leq_*)$ if $\alpha \in \mathbf{K}*\neg(\alpha \wedge \neg\alpha)$ then $\neg\alpha \in \mathbf{K}*\neg(\alpha \wedge \neg\alpha)$. By *vacuity*, since $\mathbf{K}$ is consistent, $\mathbf{K}*\neg(\alpha \wedge \neg\alpha) = \mathbf{K}$. Then if $\alpha \in \mathbf{K}$, then $\neg\alpha \in \mathbf{K}$. Hence $\alpha \notin \mathbf{K}$.

(**EE5**) Let $\beta \leq_{\mathbf{K}} \alpha$ for all β. Then, in particular $\top \leq_{\mathbf{K}} \alpha$. Then by $(C \leq_*)$ if $\top \in \mathbf{K}*\neg(\alpha \wedge \top)$ then $\alpha \in \mathbf{K}*\neg(\alpha \wedge \top)$. Then by *closure* $\alpha \in \mathbf{K}*\neg(\alpha \wedge \top)$ that is equivalent by *extensionality* to $\alpha \in \mathbf{K}*\neg\alpha$. Hence by *success* and *consistency* $\vdash \alpha$.

($*_{EBR}$) For left to right direction, let $\beta \in \mathbf{K}*\alpha$ and $\nvdash \neg\alpha$, then by *closure* $(\alpha \rightarrow \beta) \in \mathbf{K}*\alpha$ and by *consistency* and *success* $(\alpha \rightarrow \neg\beta) \notin \mathbf{K}*\alpha$. Then by *extensionality* $(\alpha \rightarrow \beta) \notin \mathbf{K}*((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta))$ and $(\alpha \rightarrow \neg\beta) \in \mathbf{K}*((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta))$. Hence by $(C \leq_*)$, $(\alpha \rightarrow \neg\beta) \leq_{\mathbf{K}} (\alpha \rightarrow \beta)$ and $(\alpha \rightarrow \beta) \nleq_{\mathbf{K}} (\alpha \rightarrow \neg\beta)$ and consequently $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$. For the other direction if $\vdash \neg\alpha$, then by *closure* and *success* it follows that $\beta \in \mathbf{K}*\alpha$ for all β. Let $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$. Then by $(C \leq_*)$ and *extensionality* $(\alpha \rightarrow \beta) \in \mathbf{K}*\alpha$; hence by *closure* and *success* $\beta \in \mathbf{K}*\alpha$. ∎

Part II

Contraction Without Recovery

Chapter 3

Recovery and Minimal Change

3.1 Five faces of Recovery

In **Chapter 2** we showed that one of the basic principles of the AGM theory [AGM85] is that belief changes should take place with minimal loss of previous beliefs. In the opinion of the AGM trio, the postulate of recovery guarantees minimal loss of contents in the contraction process[1]. However, several authors have criticized the *recovery* postulate. Recent works [Han98a, Mak97, FR97, RP, Lev97] show that the recovery postulate is one of the most controversial issues in belief revision. In the first section of the present chapter describes recovery from five angles or models in which

[1] "*When contracting K with respect to A, the loss of information should be as small as possible. The recovery postulate* $(K-5)$ [recovery] *is one way of guaranteeing this.*" [Gär88, p.65].

"*Clearly $A \div x$ will have to be "fairly big" as a subset of A in order to satisfy this.* [recovery]" [Mak85, p.352].

"*...we are thus free to appreciate the considerations in its favour, notably its appeal as a sign of "minimality" of the change made to the belief set,...*" [Mak97, p.478].

is possible to define AGM contraction: *Postulates, partial meet functions, epistemic entrenchment, safe/kernel contraction* and *spheres systems*. It also shows how the intuitions or non-intuitions that surround recovery appear or disappear in each of them. We will deal only with belief sets and not with belief bases. In the second section we present several withdrawal functions. The analysis of the first section is based in:

[•] EDUARDO FERMÉ. Five faces of recovery. In H.Rott and M-A Williams, editors, *Frontiers in Belief Revision*. Kluwer Academic Publisher, 1999. to appear

New theorems for the second parts appear in:

[•] EDUARDO FERMÉ AND RICARDO RODRIGUEZ. A brief note about the Rott contraction. *Logic Journal of the IGPL*, 6(6):835–842, 1998.

3.1.1 Recovery and minimal change in the AGM axiomatic approach

The debate on *recovery* has focused mainly in the axiomatic presentation. Consequently, this is the best known of the fives faces of *recovery*.

One possible first approach to the minimal loss of previous beliefs is the following: a sentence β must be discarded in the contraction of $\mathbf{K}$ by α, only if it presence in the contracted set would lead to α being inferred. The following postulate reflected this reasoning:

- **Fullness:** [AGM85] If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}-\alpha$ then $\nvdash \alpha$ and $\beta \to \alpha \in \mathbf{K}-\alpha$

We showed in **Subsection 2.5.1.1** that *fullness* provokes too large subsets and is therefore a non-intuitive property (see in particular **Example 2.5.7**).

The AGM theory proposes as a rule of minimality the postulate of *recovery*, according to which it is enough to add (by expansion) the contracted sentence to recover totally the original theory.

●**Recovery:** $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + \alpha$

3.1.1 [Gär88] Fullness implies Recovery

The following example shows a situation where *recovery* is reasonable:

> **Example 3.1.2** [Hanss] I believed that I had my latchkey on me (α). Then I felt in my left pocket, where I usually keep it, and did not find it. I lost my belief in α (but without starting to believe in $\neg\alpha$ instead). Half a second later, I found the key, and regained my belief in α.

Hansson showed that *recovery* gives rise to unintuitive results as we can see in the following examples:

> **Example 3.1.3** [Han93a] "I believe that "Cleopatra had a son" (α) and that "Cleopatra had a daughter" (β), and thus also "Cleopatra had a child' ($\alpha \vee \beta$, briefly δ). Then I receive information that makes me give up my belief in δ, and contract my belief set accordingly, forming $\mathbf{K}-\delta$. Soon afterwards, I learn from a reliable source that "Cleopatra had a child". It seems perfectly reasonable for me to then add δ (i.e. $\alpha \vee \beta$) to my set of beliefs without also reintroducing either α or β."

In this example, the problem appears because when we contract $\mathbf{K}$ by $\alpha \vee \beta$, by *recovery* the sentence $(\alpha \vee \beta) \rightarrow (\alpha \wedge \beta)$ must remain in the contracted set.

Example 3.1.4 [Han93a][2] "I previously entertained the two beliefs, "*x is divisible by 2*" (α) and "*x is divisible by 6*" (β). When I received new information that induced me to give up the first of these beliefs (α), the second (β) had to go as well (since α would otherwise follow from β).

I then received new information that made me accept the belief "*x is divisible by 8.*" (ϵ). Since α follows from ϵ, $(\mathbf{K}-\alpha)+\alpha$ is a subset of $(\mathbf{K}-\alpha)+\epsilon$, then by *recovery* I obtain "*x is divisible by 24*" (δ), contrary to intuition."

In the above example we showed that retaining the sentence $\mu = \alpha \rightarrow \beta$ in the contraction of $\mathbf{K}$ by α gives rise to unintuitive results. Therefore μ must be removed in the process of contraction by α. Due to *recovery*, AGM contraction cannot eliminate μ. However not all the $\alpha \rightarrow \beta$ sentences are undesirable.

Makinson [Mak97, p. 478] noted that *"as soon as contraction makes use of the notion "y is believed only because of x', we run into counterexamples to* recovery". He argued that this is only because of the use of a justificatory structure that is not represented in the belief set and that, without this structure, *recovery* can be accepted; or, in Makinson's words, it can be accepted in a "naked" theory. In [Han98a], Hansson replied that *"Actual human beliefs always have such a justificatory structure (...). It is difficult if not impossible to find examples about which we can have intuitions, and in which the belief set is not associated with a justificatory structure that guides our intuitions. Against this background, it is not surprising that, as Makinson says,* recovery *"appears to be free of intuitive counterexamples" (...). It also seems to be free of confirming examples of*

[2]'We use here the modified version introduced in [FR97]

the kind."

Niederée [Nie91] found several unintuitive properties that follow from *recovery*:

Observation 3.1.5 [Nie91] Let $\mathbf{K}$ be a belief set and $\alpha \in \mathbf{K}$. Then, regardless of whether or not β is in $\mathbf{K}$, *recovery* together with *closure* implies that:

1. $\alpha \rightarrow \beta \in \mathbf{K}-(\alpha \vee \beta)$,
2. $\alpha \in (\mathbf{K}-(\alpha \vee \beta)) + \beta$ and
3. $\neg\beta \in (\mathbf{K}-(\alpha \vee \beta)) + \neg\alpha$

In **Chapter 4** we will propose a selection mechanism to determine which $\alpha \rightarrow \beta$ sentences must be discarded (this mechanism represents the justificatory structure referred by Makinson). The result is a contraction function without *recovery* (semi contraction).

Another possible approach, proposed by Hansson, is to relax the fullness condition in the following sense: If a sentence β was removed in the contraction of $\mathbf{K}$ by α, then β contributes to the fact that α will be deduced in $\mathbf{K}$:

●**Core-retainment:**[Han91b] If $\beta \in K$ and $\beta \notin \mathbf{K}-\alpha$ then there is some set $\mathbf{H}$ such that $\mathbf{H} \subseteq \mathbf{K}$ and $\alpha \notin Cn(\mathbf{H})$ but $\alpha \in Cn(\mathbf{K} \cup \{\beta\})$.

This *intuitive way* leads again to *recovery*, since core-retainment is equivalent to *recovery* in presence of the other basic AGM postulates [Han91b][3].

[3] However this equivalence disappears in belief bases.

For the last proposal we quote a comment by Makinson with respect to saturability [Mak85, page 357]: "in general, neither $\alpha \vee \beta$ nor $\alpha \vee \neg\beta$ should be retained in the process of eliminating α from $\mathbf{K}$, unless there is "some reason" in $\mathbf{K}$ for their continued presence". The proposal is to extend this idea in the sense that *in general, no β should be retained in the process to eliminating α from $\mathbf{K}$, unless there is "some reason" in $\mathbf{K}$ for their continued presence*. This condition was explored by Fuhrmann [Fuh91] and gives rise to the filtering condition:

> "If β has been retracted from a base B in order to bar derivations of α from B, then the contraction of Cn(B) by α should not contain any sentences which were in Cn(B) "just because" β was in Cn(B)."

The filtering condition is a different notion of minimal change from that of recovery, since $\alpha \rightarrow \beta$ maybe in $\mathbf{K}$ "just because" β is in $\mathbf{K}$.

3.1.2 Recovery in Partial Meet Contraction

In the original development of partial meet contraction [AM81, AM82], the first approach was to select only one element of $\mathbf{K}\perp\alpha$ (the set of maximally inclusive subsets of $\mathbf{K}$ that do not imply α). This approach, called originally Choice and later Maxichoice produces constructions that are "too large"; in fact it corresponds to the basic AGM postulates plus fullness. In the other extreme, we can consider a function that returns only the propositions that are common to all the elements of $\mathbf{K}\perp\alpha$; but this function, full meet contraction, returns only $Cn(\{\neg\alpha\})\cap\mathbf{K}$ when $\alpha \in \mathbf{K}$ and $\nvdash \alpha$. Full meet contraction also satisfies the basic AGM postulates. Partial meet contraction takes the common elements of a selected subset of $\mathbf{K}\perp\alpha$ and corresponds exactly to the basic AGM postulates (see **Observation 2.5.15**). If the selection is transitively relational between the members of $\mathbf{K}\perp\alpha$, then it corresponds to the basic and supplementary postulates.

Full meet contraction is the *lower bound* for contractions satisfying recovery, in the sense that if a contraction satisfies recovery, then it must be a superset of full meet contraction. In a superficial reading, recovery is reasonable since it appears in all possible combinations of $\mathbf{K}\perp\alpha$. However the following question remains: If recovery is defended as the postulate that guarantees minimal change: how it is possible to have both *too large* (maxi-choice) and *too small* (full meet) contractions that both satisfy recovery? Due to this possibility, it appears that recovery does not guarantee minimal change.

On the other hand, we can note that the elements of $\mathbf{K}\perp\alpha$ become saturated (i.e., become maximal consistent subsets of the language) by adding $\neg\alpha$ [AM81]. In [Lev91, pp. 134], Levi argued that not only the elements of $\mathbf{K}\perp\alpha$ guarantee minimal loss but all the saturatable sets; and presented an alternative contraction, based on a selection among all the saturatable subsets of $\mathbf{K}$ with respect to α instead of $\mathbf{K}\perp\alpha$. Hansson and Olsson [HO95] proved that this contraction corresponds exactly to a contraction function that satisfies *failure* (If $\vdash \alpha$, then $\mathbf{K}-\alpha = \mathbf{K}$) and all the AGM postulates except *recovery*. We explain Levi contraction in **Subsection 3.2.1**.

3.1.3 Recovery in Epistemic Entrenchment

Epistemic entrenchment [Gär88, GM88] is the AGM presentation where *recovery* appears most unintuitive. In **Subsection 2.5.2** we showed that AGM contraction postulates and epistemic entrenchment are connected by the following equivalences :

$(C \leq)$ $\alpha \leq_{\mathbf{K}} \beta$ if and only if $\alpha \notin \mathbf{K}{-}(\alpha \wedge \beta)$ or $\vdash (\alpha \wedge \beta)$.

Gärdenfors' entrenchment-based contraction:

$(-_G)$ $\beta \in \mathbf{K}{-}\alpha$ if and only if $\beta \in \mathbf{K}$ and, either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$.

The crucial clause of this definition is $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$. It does not seem possible to intuitively justify this relation unless one accepts the recovery postulate.[4] In fact Gärdenfors admitted "The comparison is somewhat counterintuitive" [Gär92]. Rott [Rot91b] proposed a more intuitive definition, later called *Rott Contraction* [RP]:

$(-_{\mathrm{R}})$ $\beta \in \mathbf{K}{-}\alpha$ if and only if $\beta \in \mathbf{K}$ and, either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} \beta$.

Rott proved that Rott contraction satisfies all the AGM postulates except recovery [Rot91b]. This construction was later independently axiomatized in [Pag96], [FR98] and [RP]. In subsection 3.2.2 we introduce Rott contraction and reproduce the characterization from [FR98]. Rott [Rot91b] proved that for all α, $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$. Hansson [Hanss] proved that Rott contraction satisfies the implausible postulate of expulsiveness[5].

Lindström and Rabinowicz have proposed [LR91, pp. 115]:

[4] "*Perhaps the best way of motivate this condition* $[(-_G)]$ ... *(Note that this argument does not stand completely on its own feet, since it presumes* $(C \leq)$ *and the validity of several of the basic postulates for contraction including most conspicuously* $(K-5)$ [recovery]"[GM88, pp. 89-90]

[5] If $\nvdash \alpha$ and $\nvdash \beta$, then either $\alpha \notin \mathbf{K}{-}\beta$ or $\beta \notin \mathbf{K}{-}\alpha$.

> *"One would like to say that the truth lies somewhere in between the two extremes: the original proposal* [in this case equivalent with Rott contraction]*and Grove's definition* [in this case equivalent with Gärdenfors' entrenchment-based contraction]*seem to give us the* lower *and* upper *limit for contraction."*

This condition was called *Lindström's and Rabinowicz's interpolation thesis* [Rot95a]. According to this thesis, a *reasonable* entrenchment-based contraction operation should lie between Rott's and Gärdenfors' operators; and consequently, it should not, in general, satisfy recovery.

3.1.4 Recovery in safe/kernel contraction

We saw that *recovery* semms to be indefensible in the epistemic entrenchment approach. On the other hand, *recovery* appears as a logical consequence in safe/kernel contraction. Basically safe/kernel contraction [AM85, Han94a][6] are based on the identification of the minimal subsets of $\mathbf{K}$ that imply α (i.e., the minimal proofs of α) and elimination in the contraction of at least one sentence from each such subset (see **subsection 2.5.3**). We first note that *recovery* can be expressed as:

- If $\beta \in \mathbf{K}$, then $\alpha \rightarrow \beta \in \mathbf{K}{-}\alpha$

[6] The theorems providing connections between safe contraction and transitively relational partial meet contraction are due to Alchourrón and Makinson [AM86] and Rott [Rot92a]. The connections between kernel contraction and partial meet contraction can be found in [Han94a].

It is trivial to show that sentences of the form $\alpha \to \beta$ are never used in a minimal proof to demonstrate α. Consequently *recovery* is not only guaranteed in safe/kernel contraction, but it also appears impossible to define any kind of kernel contraction (on belief sets) without it.

3.1.5 Recovery in Grove's spheres system

In the possible worlds approach or Grove's spheres system [Gro88] (See **Section 2.6**), we will make a difference between the model corresponding to partial meet contraction and that corresponding to transitively relational partial meet contraction. We show pictures of both in **Figure 12** and **Figure 13**.

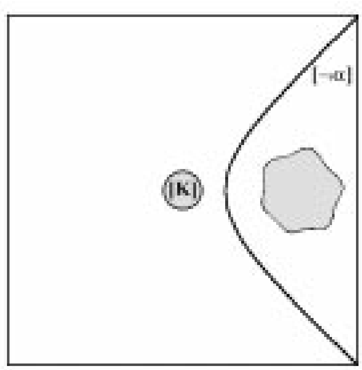


Figure 12 Partial Meet Contraction.

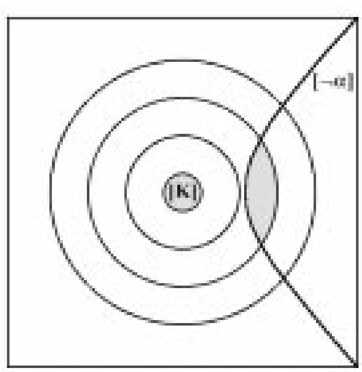


Figure 13 Transitively Relational Partial Meet Contraction.

In the first case, a partial meet contraction of **K** by α consists in adding to the **K**-worlds a subset of $\neg\alpha$-worlds. If we translate recovery to the possible worlds context, it means that only $\neg\alpha$-worlds are added in the contraction by α. In this context, recovery appears intuitive. What could the reason to be add α-worlds (that verify and enforce the belief in α) in the process of discarding α? It is not an easy question to answer for critics of recovery.

In the second case, the situation is quite different. We can see that the contraction function extends the actual set $||\mathbf{K}||$ of possible worlds to reach the $\neg\alpha$-worlds. Why not extend it to encompass all the possible α-worlds that are at least as close as the nearest $\neg\alpha$-worlds? In this analysis, recovery appears implausible. The proposed contraction corresponds exactly to Rott contraction [RP] as introduced in section 3.1.3.

3.1.6 Conclusions

We exhibited the faces of *recovery* in five presentations of the AGM theory. In doing this we found one presentation in which *recovery* is definitively implausible (epistemic entrenchment) and one on which it is a natural and unavoidable condition (safe/kernel contraction). In the other three presentations it is possible to enrich the polemic with different points of view.

In the postulates approach we showed that *recovery* can provoke unintuitive results; but there are countless cases where its application is correct. Even if we have good reasons not to accept *recovery*, it should not be completely rejected. This disctiction allow us to present in **Chapter 4** a variant of AGM contraction in which *recovery* is satisfied in some cases but not in others. In the next section we present the two most important withdrawal functions in the literature. Levi contraction [Lev91] and *Rott Contraction.*

3.2 Some withdrawal functions

3.2.1 Levi Contraction

In **Lemma 2.5.5** we show that the elements of $\mathbf{K}\perp\alpha$ become saturated (i.e., become maximal consistent subsets of the language) when add $\neg\alpha$. In [Lev91, pp. 134], Levi argues that not only the elements of $\mathbf{K}\perp\alpha$ guarantee minimal loss but all the saturatable sets, and that *measures of information* should be replaced by *measures of informational value*. He presents an alternative contraction, which is quite similar to *partial meet AGM contraction* but is based on a selection among all the saturatable subsets of $\mathbf{K}$ with respect to α.

Definition 3.2.1 Let $\mathbf{K}$ be a belief set and α a sentence. Then the *saturatable* set $S(\mathbf{K}, \alpha)$ is the set such that for all $\mathbf{H}$, $\mathbf{H} \in S(\mathbf{K}, \alpha)$ if and only if:

$$\begin{cases} \mathbf{H} \subseteq \mathbf{K} \\ \mathbf{H} = Cn(\mathbf{H}) \\ \mathbf{H}+\neg\alpha \text{ is a maximal consistent subset of the language.} \end{cases}$$

The *partial meet Levi contraction* is as defined in follows:

Definition 3.2.2 [HO95] Let $\mathbf{K}$ be a set of sentences, α a sentence and γ a selection function for $\mathbf{K}$. The *partial meet Levi contraction* of $\mathbf{K}$ that is generated by γ is the operation $\sim_\gamma$ such that for all sentences α: $\mathbf{K} \sim_\gamma \alpha = \cap\gamma(S(\mathbf{K}, \alpha))$.

It is easy to show that $\mathbf{K}\perp\alpha \subseteq S(\mathbf{K}, \alpha)$, hence every *partial meet AGM contraction function* is a *partial meet Levi contraction function*.

Hansson and Olsson [HO95] obtained a representation theorem for the *partial meet Levi contraction*:

THEOREM 3.2.3 [HO95] Let **K** be a set of sentences. An operator $\sim$ on **K** is a *partial meet Levi contraction* if and only if $\sim$ satisfies *closure, inclusion, vacuity, success, extensionality,* and *failure.*

●**Failure [FH94]:** If $\vdash \alpha$, then $\mathbf{K}-\alpha = \mathbf{K}$.

Failure was introduced in [FH94]. It is a direct consequence of *inclusion* and *recovery.* We return to it in **Section 4.2**.

3.2.2 Rott contraction

In **Subsection 2.5.2** we showed that Gärdenfors' entrenchment-based contraction is defined using $x <_{\mathbf{K}} (x \vee y)$. Hans Rott [Rot91b] has remarked that the comparison is not intuitive, and proposed the following alternative definition of a contraction operation from an entrenchment ordering:

Rott's entrenchment-based contraction

($-_{\mathrm{R}}$) $\beta \in \mathbf{K}-\alpha$ if and only if $\beta \in \mathbf{K}$ and, either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} \beta$.

Rott also provided the following result:

Observation 3.2.4 Let $\leq_{\mathbf{K}}$ be a standard entrenchment ordering on a consistent belief set **K**. Furthermore let $-_{\mathrm{R}}$ be the Rott entrenchment-based contraction on **K** defined by condition ($-_{\mathrm{R}}$) from $\leq_{\mathbf{K}}$. Then $-_{\mathrm{R}}$ satisfies all the AGM postulates except *recovery.*

Rott also proved that for all α, $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$.

In order to characterize the Rott contraction; we need the following postulates[7]:

- **Converse Conjunctive Inclusion:** If $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\beta$ then $\beta \notin \mathbf{K}-\alpha$ or $\vdash \alpha$ or $\vdash \beta$

- **Failure [FH94]:** If $\vdash \alpha$, then $\mathbf{K}-\alpha = \mathbf{K}$.

- **Strong Inclusion:** If $\alpha \notin \mathbf{K}-\beta$ then $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$.

THEOREM 3.2.5

1. Let $\leq_{\mathbf{K}}$ be a standard entrenchment ordering on a consistent belief set $\mathbf{K}$. Furthermore, let $-_{\mathrm{R}}$ be Rott's entrenchment-based contraction on $\mathbf{K}$, defined from $\leq_{\mathbf{K}}$ by condition $(-_{\mathrm{R}})$. Then $-_{\mathrm{R}}$ satisfies *closure, inclusion, success, extensionality, conjunctive overlap, failure, strong inclusion* and *converse conjunctive inclusion*, and $(C \leq)$ also holds.

2. Let $-$ be an operation on a consistent belief set $\mathbf{K}$ that satisfies *closure, inclusion, success, extensionality, conjunctive overlap, failure, strong inclusion* and *converse conjunctive inclusion*. Furthermore let $\leq_{\mathbf{K}}$ be the relation that is defined from $-$ by $(C \leq)$. Then $\leq_{\mathbf{K}}$ satisfies the standard entrenchment postulates, and $(-_{\mathrm{R}})$ also holds.

[7]**Note:** Another axiomatic characterization was independently discovered by Hans Rott and Maurice Pagnucco [RP]. Here we reproduce only the characterization of our own.

Rott and Pagnucco [RP] demonstrated that *converse conjunctive inclusion* follows from the other postulates and can be discarded from the list of axioms. We presented here the theorem as it appeared in [FR98].

Rott and Pagnucco [RP] proved close relations between Rott contraction and AGM contraction: Let $-$ be an AGM contraction. Then the corresponding Rott contraction $-_{\mathrm{R}}$ can be defined as follows,

$$\mathbf{K} -_{\mathrm{R}} \alpha = \begin{cases} \{\beta : \beta \in \mathbf{K}-(\alpha \wedge \beta)\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise.} \end{cases}$$

Due to the Harper identity, it is easy to define $-_{\mathrm{R}}$ in terms of an AGM revision. Let $*$ be an AGM contraction function:

$$\mathbf{K} -_{\mathrm{R}} \alpha = \begin{cases} \{\beta : \beta \in \mathbf{K} \cap \mathbf{K}{*}\neg(\alpha \wedge \beta)\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise.} \end{cases}$$

For the other direction, let $-$ be a Rott contraction. Then the corresponding AGM contraction $-$ can be defined as follows,

$$\mathbf{K} - \alpha = \begin{cases} \mathbf{K} \cap (\mathbf{K} -_{\mathrm{R}} \neg\alpha) + \alpha & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise.} \end{cases}$$

Other interesting postulates that Rott contraction satisfies are:

- **Expulsiveness [Hanss]:** If $\nvdash \alpha$ and $\nvdash \beta$, then either $\alpha \notin \mathbf{K}-\beta$ or $\beta \notin \mathbf{K}-\alpha$.

- **Linear Hierarchical Ordering:** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$ or $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$.

- **Linearity:** $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$ or $\mathbf{K}-\alpha \subseteq \mathbf{K}-\beta$.

Levi [Lev97] presented an equivalent construction, *mild contraction* but the motivations that inspired Levi are quite different. The arguments of Levi exceed the background contents of this work; consequently we refer the reader to [Lev97].

3.3 Proofs of Chapter 3

Proof of Theorem 3.2.5

Part 1 *Closure, inclusion, success, extensionality* and *conjunctive overlap* follow from **Observation 3.2.4**.

Converse conjunctive inclusion Let $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\beta$. We have two cases: 1. $\beta \in \mathbf{K}-\beta$: It follows from *success* that $\vdash \beta$. 2. $\beta \notin \mathbf{K}-\beta$: then $\beta \notin \mathbf{K}-(\alpha \wedge \beta)$. It follows from $(-_{\mathrm{R}})$ that $\beta \notin \mathbf{K}$ or ($\beta \leq_{\mathbf{K}} \alpha \wedge \beta$ and $\nvdash \alpha$). We have two subcases 2.1. $\beta \notin \mathbf{K}$: then by *inclusion* (see observation 3.2.4) $\beta \notin \mathbf{K}-\alpha$. 2.2. $\beta \in \mathbf{K}$: then $\beta \leq_{\mathbf{K}} \alpha \wedge \beta$. By (**EE2**) $\alpha \wedge \beta \leq_{\mathbf{K}} \alpha$ and by (**EE1**) $\beta \leq_{\mathbf{K}} \alpha$ hence by $(-_{\mathrm{R}})$ $\beta \notin \mathbf{K}-\alpha$ or $\vdash \alpha$.

Failure It follows trivially, since if $\vdash \alpha$, then by $(-_{\mathrm{R}})$ $\beta \in \mathbf{K}-\alpha$ if and only if $\beta \in \mathbf{K}$.

Strong inclusion Let $\alpha \notin \mathbf{K}-\beta$. By $(-_{\mathrm{R}})$ $\alpha \notin \mathbf{K}$ or ($\nvdash \beta$ and $\alpha \leq_{\mathbf{K}} \beta$). We have two subcases: 1. $\alpha \notin \mathbf{K}$: then $\mathbf{K}-\alpha = \mathbf{K}$ (since by observation 3.2.4, $-$ satisfies *vacuity*); then $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$ (since by observation 3.2.4, $-$ satisfies *inclusion*). 2. $\alpha \in \mathbf{K}$: then ($\nvdash \beta$ and $\alpha \leq_{\mathbf{K}} \beta$) Let $\delta \in \mathbf{K}-\beta$, then (by Rott contraction) $\delta \in \mathbf{K}$ and

$\beta <_{\mathbf{K}} \delta$. By (**EE1**) $\alpha <_{\mathbf{K}} \delta$ then by $(-_{\mathrm{R}})$ $\delta \in \mathbf{K}{-}\alpha$ hence $\mathbf{K}{-}\beta \subseteq \mathbf{K}{-}\alpha$.

$(C \leq)$ For one direction let $\alpha \leq_{\mathbf{K}} \beta$ and $\alpha \in \mathbf{K}{-}(\alpha \wedge \beta)$. We need to prove $\vdash \alpha \wedge \beta$. By $(-_R)$ we have: $\alpha \in \mathbf{K}{-}(\alpha \wedge \beta)$ if and only if $\alpha \in \mathbf{K}$ and either $\vdash \alpha \wedge \beta$ or $\alpha \wedge \beta <_{\mathbf{K}} \alpha$. Therefore: $\vdash \alpha \wedge \beta$ or $\alpha \wedge \beta <_{\mathbf{K}} \alpha$. Let $\alpha \wedge \beta <_{\mathbf{K}} \alpha$: then, by (**EE3**), $\beta \leq_{\mathbf{K}} \alpha \wedge \beta$; and since $\alpha \leq_{\mathbf{K}} \beta$, we have by (**EE1**) that $\alpha \leq_{\mathbf{K}} (\alpha \wedge \beta)$, contradiction, hence $\alpha \wedge \beta$.

For the second direction we have two subcases: **1.** $\alpha \notin \mathbf{K}{-}(\alpha \wedge \beta)$**:** Then for $(-_R)$, $\alpha \notin \mathbf{K}$ or $\nvdash \beta$ and $\alpha \leq_{\mathbf{K}} \alpha \wedge \beta$. If $\alpha \notin \mathbf{K}$, then $\alpha \leq_{\mathbf{K}} \beta$ follows (by (**EE4**)). If $\alpha \leq_{\mathbf{K}} \alpha \wedge \beta$, by (**EE1**), (since by (**EE2**) $\alpha \wedge \beta \leq_{\mathbf{K}} \beta$), $\alpha \leq_{\mathbf{K}} \beta$. **2.** $\vdash \alpha \wedge \beta$**:** Then $\vdash \beta$, hence by (**EE2**), $\alpha \leq_{\mathbf{K}} \beta$. This complete the proof.

Part 2 (**EE2**) – (**EE5**) are proved by Gärdenfors and Makinson from *closure, inclusion, success, extensionality, failure* and $(C \leq)$ in [GM88], pp. 93-94.

(**EE1**) We demonstrate by *reductio ad absurdum.* Let $\alpha \leq_{\mathbf{K}} \beta$, $\beta \leq_{\mathbf{K}} \delta$ and $\alpha \not\leq_{\mathbf{K}} \delta$. It follows by $(C \leq)$ that: (a) either $\vdash \alpha \wedge \beta$ or $\alpha \notin \mathbf{K}{-}(\alpha \wedge \beta)$; (b) either $\vdash \beta \wedge \delta$ or $\beta \notin \mathbf{K}{-}(\beta \wedge \delta)$; and (c) $\nvdash \alpha \wedge \delta$ and $\alpha \in \mathbf{K}{-}(\alpha \wedge \delta)$.

1. Let $\vdash \alpha \wedge \beta$: then $\vdash \alpha$ and $\vdash \beta$. By *closure* $\beta \in \mathbf{K}{-}(\beta \wedge \delta)$, then by condition (b) $\vdash \beta \wedge \delta$, so $\vdash \delta$, and $\vdash \alpha \wedge \delta$; contradiction.

2. Let $\vdash \beta \wedge \delta$: By *closure* $\delta \in \mathbf{K}{-}(\alpha \wedge \delta)$, then by condition (c) and *closure* $\alpha \wedge \delta \in \mathbf{K}-(\alpha \wedge \delta)$; hence by *success* $\vdash \alpha \wedge \delta$; absurd.

By 1. and 2. (a), (b) and (c) are reduced to $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$, $\beta \notin \mathbf{K}-(\beta \wedge \delta)$, $\nvdash \alpha \wedge \delta$ and $\alpha \in \mathbf{K}-(\alpha \wedge \delta)$.

3. By *strong inclusion* $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$, then by *converse conjunctive inclusion* we have $\vdash \alpha$ or $\vdash \beta$ or $\alpha \notin \mathbf{K}-\beta$. But since $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$ and $\beta \notin \mathbf{K}-(\beta \wedge \delta)$, by *closure* $\nvdash \alpha$ and $\nvdash \beta$. Then $\alpha \notin \mathbf{K}-\beta$ and by *strong inclusion* we have $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$.

4. $\beta \notin \mathbf{K}-(\beta \wedge \beta)$ implies by *strong inclusion* that $\mathbf{K}-(\beta \wedge \delta) \subseteq \mathbf{K}-\beta$, then by *converse conjunctive inclusion* we have $\vdash \beta$ or $\vdash \delta$ or $\beta \notin \mathbf{K}-\delta$; by *closure* $\nvdash \beta$ and $\nvdash \delta$ (since by *success* and *closure* $\delta \notin \mathbf{K}-(\alpha \wedge \delta)$). Then $\beta \notin \mathbf{K}-\delta$ and by *strong inclusion* we have $\mathbf{K}-\delta \subseteq \mathbf{K}-\beta$.

5. It follows from *success* that $\delta \notin \mathbf{K}-\delta$, so by *closure* $(\alpha \wedge \delta) \notin \mathbf{K}-\delta$; and since $\delta \notin \mathbf{K}-(\alpha \wedge \delta)$ we obtain by *strong inclusion* that $\mathbf{K}-\delta = \mathbf{K}-(\alpha \wedge \delta)$. So $\alpha \in \mathbf{K}-(\alpha \wedge \delta) = \mathbf{K}-\delta \subseteq \mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$. Hence by *success* $\vdash \alpha$; contradiction.

$-_{\mathrm{R}}$ ($\Rightarrow$) Let $\beta \in \mathbf{K}-\alpha$ and $\nvdash \alpha$. It follows by *inclusion* that $\beta \in \mathbf{K}$. We have two cases: 1. $\vdash \beta$: By *closure* $\beta \in \mathbf{K}-(\alpha \wedge \beta)$, then by *success* $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$; hence, by $(C \leq)$, $\alpha \leq_{\mathbf{K}} \beta$. By *success* and *closure* $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$; then, by $(C \leq)$, $\alpha \leq_{\mathbf{K}} \beta$. Hence $\alpha <_{\mathbf{K}} \beta$. 2. $\nvdash \beta$: By *converse conjunctive inclusion* $\mathbf{K}-(\alpha \wedge \beta) \nsubseteq \mathbf{K}-\beta$, then by *strong inclusion* $\beta \in \mathbf{K}-(\alpha \wedge \beta)$; from *success* and *closure* we have that $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$; hence, by $(C \leq)$, $\alpha \leq_{\mathbf{K}} \beta$. By *success* and *closure* $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$; then, by $(C \leq)$, $\alpha \leq_{\mathbf{K}} \beta$. Hence $\alpha <_{\mathbf{K}} \beta$.

$(\Leftarrow)$ 1. Let $\beta \in \mathbf{K}$ and $\vdash \alpha$. By *failure* $\mathbf{K}-\alpha = \mathbf{K}$ then $\beta \in \mathbf{K}-\alpha$.

2. Let $\beta \in \mathbf{K}$ and $\alpha <_{\mathbf{K}} \beta$. By $(C \leq)$ $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$, then by *strong inclusion* $\mathbf{K}-(\alpha \wedge \beta)$. For *reductio ad absurdum* let $\beta \notin \mathbf{K}-\alpha$; then $\beta \notin \mathbf{K}-(\alpha \wedge \beta)$ then by $(C \leq)$, $\beta \leq_{\mathbf{K}} \alpha$. Contradiction. ■

Chapter 4

The Semi-Contraction Functions

4.1 Introduction

In **Chapter 3** we analysed the controversial postulate of *Recovery* and presented some operations that violated it (withdrawals) from the literature. In this chapter we present a withdrawal function, called *Semi-Contraction*, that attempts to satisfy *minimal loss of information* and *minimal loss of informational value.*

We propose: (1) An axiomatic characterization of semi-contraction. (2) An alternative construction for semi-contraction based on *semi-saturatable sets*, inspired by Levi's *saturatable sets.* (3) A special kind of semi-contraction that satisfies the *Lindström and Rabinowicz interpolation thesis* [LR91]. (4) A modified version of the Harper identity, that allows, with the Levi identity, the correspondence one to one between semi-contraction and AGM revision.

The major parts of the results of this chapter appeared in:

[•] Eduardo Fermé. On the logic of theory change: Contraction without recovery. *Journal of Logic, Language and Information*, 7:127–137, 1998.

[•] Eduardo Fermé and Ricardo Rodríguez. Semi-contraction: Axioms and construction. 1997. (manuscript).

4.2 Axioms for a sensible withdrawal

Examples 3.1.3 and **3.1.4** showed that in AGM contraction the recovery postulate can give rise to non-intuitive results, but we also saw that there are cases in which *recovery* should hold. Our purpose is to define axioms for a sensible *withdrawal function*, that preserves the *principle of minimal loss of information* but removes the sentences that provoke these non-intuitive results. In this context *closure, inclusion, vacuity, success* and *extensionality* must hold.

However, finding counterexamples of *recovery* does not mean that recovery must be eliminated completely. There are many cases where *recovery* is a desired property. We must find a new postulate that preserves recovery in certain cases but allows us to eliminate the "$\alpha \to \beta$" sentences that provoke unintuitive results. In the last case, we also want to retain the possibility of recovering the original belief set.

If when contracting by α we eliminate sentences of the form $\alpha \to \beta$, we cannot recover the original set of sentences by simple adding simply. To re-obtain the whole original set of beliefs we must reintroduce not only α but also all the $\alpha \to \beta$ sentences lost in the contraction, i.e., this should happen when adding: $\alpha \wedge (\alpha \to \beta_1) \wedge \ldots (\alpha \to \beta_n)$, which is equivalent to:

$\alpha \wedge \beta_1 \wedge \ldots \beta_n$. Consequently, we delegate the task of recovering the whole set to a sentence $\beta = \alpha \wedge \beta_1 \wedge \ldots \beta_n$. We formalize this idea in the following postulate:

●**Proxy Recovery:** If $\mathbf{K} \neq \mathbf{K}-\alpha$ then there exists some $\beta \in \mathbf{K}$ such that $\mathbf{K}-\alpha \not\vdash \beta$ and $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + \beta$.

Proxy recovery is a weaker version of *recovery*. When *recovery* is satisfied, *proxy recovery* holds taking $\beta = \alpha$. The converse of the last formula of this postulate follows from inclusion.

In the limiting case in which the sentence to be removed is a tautology (which is impossible to remove) *recovery* and *inclusion* guarantee that the result of this contraction is the original belief set **K**. If we reject *recovery* we must explicitly add this intuitive condition:

●**Failure [FH94]:** If $\vdash \alpha$, then $\mathbf{K}-\alpha = \mathbf{K}$.

Definition 4.2.1 Let **K** be a belief. An operator $-$ on **K** is a *sensible withdrawal* function if and only if it satisfies *closure, inclusion, vacuity, success, extensionality, failure*, and *proxy recovery.*

Note that when the language is finite, every withdrawal function satisfies *proxy recovery*, and then all Levi contractions are semi-contractions and conversely (just let $\beta : Cn(\beta) = \mathbf{K}$).

4.3 Construction of Semi-Contraction

In **Example 3.1.3** and **Example 3.1.4** we show that retaining the sentence $\mu = \alpha \to \beta$ in the contraction of $\mathbf{K}$ by α can provoke unintuitive results. Therefore μ must be removed in the process of contraction by α. Due to *recovery*, AGM contraction does not eliminate μ.

However, not all the $\alpha \to \beta$ sentences are undesirable. Makinson [Mak97, pp 478] noted that "as soon as contraction makes use of the notion *y is believed only because of x*, we run into counterexamples to recovery". He argues that this is because we make use of a justificatory structure that is not represented in the belief set (see **Chapter 3**). Consequently we need a selection mechanism to determine which $\alpha \to \beta$ sentences must be discarded. The semi-contraction function does just this, through the combined used of a unique AGM contraction and a selection function:

Definition 4.3.1 Let $\mathbf{A}$ be a set of sentences. A semi-selection function for $\mathbf{A}$ is a function Sel such that:

1. If $\mathbf{A}$ is non-empty, then $Sel(\mathbf{A}) \in \mathbf{A}$
2. If $\mathbf{A}$ is empty, then $Sel(\mathbf{A}) = \top$.

Definition 4.3.2 A function $\overline{}_s : \mathcal{K} \times \mathcal{L} \to \mathcal{L}$ is a **semi-contraction** function if and only if there is a *partial meet AGM contraction function* $-$ and a smi-selection function Sel such that for all $\mathbf{K} \in \mathcal{K}$ and $\alpha \in \mathcal{L}$:

$$\mathbf{K} \overline{}_s \alpha = (\mathbf{K} - \alpha) \cap (\mathbf{K} - (\alpha \to Sel(\mathbf{K} \setminus \mathbf{K} - \alpha)))$$

Sel selects an element of $(\mathbf{K} \setminus \mathbf{K}{-}\alpha)$; this is equivalent to selecting some finite subset of $(\mathbf{K} \setminus \mathbf{K}{-}\alpha)$, as we see in the following property:

4.3.3 If $\beta_1 \in \mathbf{K}\backslash\mathbf{K}{-}\alpha$ and $\beta_2 \in \mathbf{K}\backslash\mathbf{K}{-}\alpha$ then $\beta_1 \wedge \beta_2 \in \mathbf{K}\backslash\mathbf{K}{-}\alpha$

The intuitions that guide the axioms for sensible withdrawals are the same as those that inspire semi-contraction, as we can see in the following lemma:

Lemma 4.3.4

Every *semi-contraction function* defined as in **Definition 4.3.2** satisfies *closure, inclusion, vacuity, success, extensionality, failure* and *proxy recovery.*

This lemma and the axiomatic characterization of Levi contraction [HO95] imply that semi-contraction is a special case of withdrawal; more general than AGM contraction, but less general than Levi contraction. Formally:

Observation 4.3.5

1. Every *semi-contraction function* defined as in **Definition 4.3.2** is a *partial meet Levi contraction function.*
2. Every *partial meet AGM contraction function* is a *semi-contraction function* defined as in **Definition 4.3.2**.

One interesting point is the relation between the semi-contraction and *recovery*:

Definition 4.3.6 Let $\mathbf{K}$ be a belief set, $-$ a contraction function for $\mathbf{K}$ and α a sentence. $-$ satisfies *α-recovery* if and only if $\mathbf{K} \subseteq (\mathbf{K}{-}\alpha) + \alpha$

Observation 4.3.7 Every *semi-contraction function* satisfies α-*recovery* if and only if $\vdash \alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)$.

Clearly a semi-contraction function satisfies *recovery* if and only if it satisfies α-*recovery* for all α. Moreover if a semi-contraction satisfies α-*recovery*, then $\mathbf{K}\,\bar{_s}\,\alpha = \mathbf{K}-\alpha$.

4.4 Semi-Saturatable Contraction

We have shown that semi-contraction functions are situated between Levi and AGM contractions. In this section our purpose is to find an alternative construction in terms of the remainder sets and Levi's saturatable sets. Since semi-contraction is equivalent to the intersection of the same AGM contraction applied to α and $\alpha \rightarrow \beta$, respectively, an obvious approach is:

$$\mathbf{K}\,\bar{_s}\,\alpha = \cap\gamma(\mathbf{K}\bot\alpha) \bigcap \cap\gamma(\mathbf{K}\bot(\alpha \rightarrow \beta))$$

Since in semi-contraction $\beta \in \mathbf{K} \setminus \mathbf{K}-\alpha$, we also need to add the constraint that $\exists \mathbf{H} \in \cap\gamma(\mathbf{K}\bot\alpha)$: $\beta \notin \mathbf{H}$. This constraint and the use of two different remainder sets encourage us to find a simple selection function over a unique set.

Since the semi-contractions are withdrawals, $S(\mathbf{K}, \alpha)$ appears as a candidate, but again, the selection function must be constrained to select at least one $\mathbf{H}$ such that $\beta \notin \mathbf{H}$. This condition is given by the set $S(\mathbf{K}, (\alpha \vee \beta))$. However, there remains the constraint that we want to recover the whole set $\mathbf{H}$ by adding $\alpha \wedge \beta$. Consequently we add this constraint and define the *semi-saturatable* sets for α and β as subsets of $S(\mathbf{K}, (\alpha \vee \beta))$ as follows:

Definition 4.4.1 Let $\mathbf{K}$ be a belief set and α, β sentences. Then the *semi-saturatable* set $SS(\mathbf{K},\alpha,\beta)$ is the set such that $\mathbf{H} \in SS(\mathbf{K},\alpha,\beta)$ if and only if:

$$\begin{cases} \mathbf{H} \subseteq \mathbf{K} \\ \mathbf{H} = Cn(\mathbf{H}) \\ \mathbf{H}{+}(\neg\alpha \wedge \neg\beta) \text{ is a maximal consistent subset of the language.} \\ \mathbf{K} \subseteq \mathbf{H}{+}(\alpha \wedge \beta) \end{cases}$$

The following observations formalize the relationship between the elements of $SS(\mathbf{K},\alpha,\beta)$ and $S(\mathbf{K},\alpha \vee \beta)$ and also relate them to $\mathbf{K} \perp (\alpha \vee \beta)$:

Observation 4.4.2 If $\alpha \vee \beta \in \mathbf{K}$, then $\mathbf{K}\perp(\alpha \vee \beta) \subseteq SS(\mathbf{K},\alpha,\beta)$.

Observation 4.4.3 $SS(\mathbf{K},\alpha,\beta) \subseteq S(\mathbf{K},\alpha \vee \beta)$.

Similarly to the construction of partial meet AGM and Levi contraction, we now build contraction functions by means of a selection function over the semi-saturatable set $SS(\mathbf{K},\alpha,\beta)$:

Definition 4.4.4 Let $\mathbf{K}$ be a belief set. A *selection function* for $\mathbf{K}$ is a function γ such that for all sentences α:
(1) If $SS(\mathbf{K},\alpha,\beta)$ is non-empty, then $\gamma(SS(\mathbf{K},\alpha,\beta))$ is a non-empty subset of $SS(\mathbf{K},\alpha,\beta)$.
(2) If $SS(\mathbf{K},\alpha,\beta)$ is empty, then $\gamma(SS(\mathbf{K},\alpha,\beta)) = \mathbf{K}$.

Definition 4.4.5 Let $\mathbf{K}$ be a belief set. An operation $\bar{s}$ on $\mathbf{K}$ is a *semi-saturatable contraction* if and only if there is a selection function γ for $\mathbf{K}$ defined as in **Definition 4.4.4**, such that for all sentences α: $\mathbf{K}\bar{s}\alpha = \cap\gamma(SS(\mathbf{K}, \alpha, \beta))$, where $\beta = f(\mathbf{K}, \alpha)$ for a function $f : \mathbf{K} \times \mathcal{L} \to \mathcal{L}$.

Clearly, the role of f is the same as the role of Sel in semi-contraction, i.e., $Sel(\mathbf{K} \setminus \mathbf{K}-\alpha) = f(\mathbf{K}, \alpha)$. The next lemma shows the relationship between *semi-saturatable contraction* and semi-contraction:

Lemma 4.4.6 Let $\mathbf{K}$ be a belief set and $\sim$ a semi-saturatable contraction function for $\mathbf{K}$. Then $\sim$ is a *semi-contraction function*, i.e., there exists a *partial meet AGM contraction function* $-$ such that $\mathbf{K}\sim\alpha = \mathbf{K}-\alpha \cap \mathbf{K}-(\alpha \to \beta)$, $\beta \in \mathbf{K} \setminus \mathbf{K}-\alpha$.

Finally, we relate the axioms for a *sensible withdrawal* with the construction by means of semi-saturatable sets:

Lemma 4.4.7 Let $\mathbf{K}$ be a belief set and $\sim$ a sensible withdrawal for $\mathbf{K}$. Then there is a selection function γ on $\mathbf{K}$ such that $\mathbf{K}\sim\alpha = \cap\gamma(SS(\mathbf{K}, \alpha, \beta))$, where $\beta = f(\mathbf{K}, \alpha)$ for a function $f : \mathbf{K} \times \mathcal{L} \to \mathcal{L}$.

4.5 Characterizations of Semi-Contraction

Based on **Lemmas 4.3.4, 4.4.6** and **4.4.7** we can characterize semi-contraction functions as follows:

THEOREM 4.5.1 Let $\mathbf{K}$ be a belief set and $\underset{s}{-}$ an operator on $\mathbf{K}$. Then the following conditions are equivalent:

1) $\underset{s}{-}$ is a semi-contraction function as defined in **Definition 4.3.2**, i.e., there is a *partial meet AGM contraction function* $-$ and a semi-selection function Sel such that for all α, $\mathbf{K}\underset{s}{-}\alpha = \mathbf{K}-\alpha \cap \mathbf{K}-(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$.

2) $\underset{s}{-}$ is a *sensible withdrawal* as defined in **Definition 4.2.1**, i.e., it satisfies *closure, inclusion, vacuity, success, extensionality, failure* and *proxy recovery.*

3) $\underset{s}{-}$ is a *semi-saturatable contraction* function as defined in **Definition 4.4.1**, i.e., there is a selection function γ on $\mathbf{K}$ such that $\mathbf{K}\underset{s}{-}\alpha = \cap\gamma(SS(\mathbf{K}, \alpha, \beta))$, where $\beta = f(\mathbf{K}, \alpha)$ for a function $f : \mathbf{K} \times \mathcal{L} \rightarrow \mathcal{L}$.

4.6 Epistemic Entrenchment for Semi-Contraction

In **Subsection 2.5.2** we recalled the relations between *transitively relational partial meet AGM contraction* function and epistemic entrenchment. Since semi-contraction is defined using a unique *partial meet AGM contraction*, if the latter is transitively relational then it is easy to construct a semi-contraction function based on an epistemic entrenchment relation and $(C \leq)$.

For the first contraction, $\mathbf{K}-\alpha$, the condition is the same as $(-_{\mathrm{G}})$.

For the second contraction, $\mathbf{K}-(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$, we use $(-_{\mathrm{G}})$ again, using $(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$ instead of α; i.e., $\beta \in \mathbf{K}-(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$ if and only if $\beta \in \mathbf{K}$ and, either $\vdash (\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))$ or $(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)) <_{\mathbf{K}} ((\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)) \vee \beta)$.

The next step is to define $(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)$ in terms of an entrenchment ordering: $\mathbf{K} \setminus \mathbf{K}-\alpha = \{\epsilon \mid \epsilon \in \mathbf{K}$ and $\nvdash \alpha$ and $(\alpha \vee \epsilon) \leq_{\mathbf{K}} \alpha\}$

We combine all the above conditions and obtain the following definition:

$(-_{\mathrm{S}})$ $\beta \in \mathbf{K} \underset{s}{-} \alpha$ if and only if $\beta \in \mathbf{K}$ and; either $\vdash \alpha$ or $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$ and; either $\vdash (\alpha \to Sel(\mathbf{H}))$ or $(\alpha \to Sel(\mathbf{H})) <_{\mathbf{K}} ((\alpha \to Sel(\mathbf{H})) \vee \beta)$, where $\mathbf{H} = \{\epsilon \mid \epsilon \in \mathbf{K}$ and $\nvdash \alpha$ and $(\alpha \vee \epsilon) \leq_{\mathbf{K}} \alpha\}$.

Due to the construction of $(-_{\mathrm{S}})$, we can relate this to semi-contraction:

Observation 4.6.1 Let $\leq_{\mathbf{K}}$ be a standard epistemic entrenchment ordering on a consistent belief set $\mathbf{K}$. Furthermore, let $\underset{s}{-}$ be an entrenchment-contraction on $\mathbf{K}$ based on $\leq_{\mathbf{K}}$ defined via condition $(-_{\mathrm{S}})$. Then $\underset{s}{-}$ is a semi-contraction function, and $(C \leq)$ also holds.

Observation 4.6.2 Let $\sim$ be a semi-contraction function on the consistent belief set $\mathbf{K}$ and $-$ its associate *partial meet AGM contraction* such that $-$ is also transitively relational. Furthermore, let $\leq_{\mathbf{K}}$ be the relation that is derived from $-$ through $(C \leq)$. Then $\leq_{\mathbf{K}}$ satisfies the standard entrenchment postulates and $(-_{\mathrm{S}})$ also holds.

4.7 Construction of Interpolated Semi-Contraction

We saw in **Section 3.1.3** that according to the *Lindström and Rabinowicz interpolation thesis*, a reasonable contraction function must be situated between *partial meet AGM contraction* and *severe withdrawal.* We show in this section what additional restrictions on $\overline{s}$ are needed to obtain an interpolated semi-contraction function; i.e., such that for all α, $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} \overline{s} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$

We will introduce the basic ideas informally. We will assume an epistemic entrenchment ordering $\leq_{\mathbf{K}}$ for $\mathbf{K}$ and the partial meet AGM contraction and severe withdrawal $-_{\mathrm{G}}$ and $-_{\mathrm{R}}$ based on $\leq_{\mathbf{K}}$. $\overline{s}$ is the semi-contraction based on $-_{\mathrm{G}}$, and Sel its associated selection function.

It is trivial that $\mathbf{K} \overline{s} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$. For the other condition, $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} \overline{s} \alpha$, we must show $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha \cap \mathbf{K} -_{\mathrm{G}} (\alpha \rightarrow \beta)$ for $\beta = Sel(\mathbf{K} \setminus \mathbf{K} -_{\mathrm{G}} \alpha)$. $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$ so we only have to prove that $\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} (\alpha \rightarrow \beta)$. This condition holds if $\vdash \alpha \rightarrow \beta$ or $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{R}} \alpha$. When $\nvdash \alpha \rightarrow \beta$, then $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{R}} \alpha$ if and only if $\alpha \rightarrow \beta \leq_{\mathbf{K}} \alpha$. By means of ($C \leq$), we write it as follows: $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{G}} ((\alpha \rightarrow \beta) \wedge \alpha)$, or equivalently $\alpha \rightarrow \beta \notin \mathbf{K} -_{\mathrm{G}} (\alpha \wedge \beta)$.

We can formalize the above explanation in the following theorem:

THEOREM 4.7.1 Let $\mathbf{K}$ be a belief set, $\leq_{\mathbf{K}}$ an epistemic entrenchment ordering for $\mathbf{K}$, $-_{\mathrm{R}}$ the *severe withdrawal*, and $-_{\mathrm{G}}$ the *partial meet AGM contraction function* associated with the epistemic entrenchment ordering $\leq_{\mathbf{K}}$. Let $\overline{s}$ be the associated semi-contraction of $-_{\mathrm{G}}$, and Sel its selection function. If

$\beta = Sel(\mathbf{K} \setminus \mathbf{K} -_{\mathrm{G}} \alpha)$ satisfies $\alpha \to \beta \notin \mathbf{K} -_{\mathrm{G}} (\alpha \wedge \beta)$, then:
$\mathbf{K} -_{\mathrm{R}} \alpha \subseteq \mathbf{K} \overline{_s} \alpha \subseteq \mathbf{K} -_{\mathrm{G}} \alpha$ for all α.

The converse of this theorem is not true, since there are contraction functions that satisfy the *interpolation thesis* but they are not semi-contractions. An example can be found in the proof section of the chapter.

4.8 Semi-Contraction and Grove's spheres system

In the possible worlds approach, there are three possible ways to construct semi-contraction function. The first one is when the associated AGM contraction satisfies only the basic AGM postulates, or equivalently, is an AGM partial meet contraction. The second one is when the associated AGM contraction satisfies all basic and supplementary AGM postulates, or equivalently, is a transitively relational AGM partial meet contraction. The third is the special case of interpoled semi-contraction. These approaches are illustrated in the following three figures [1]:

[1]Note that in the third figure, the semi-contraction is situated between the correspondent Severe and AGM contraction. This property does not hold in the second figure.

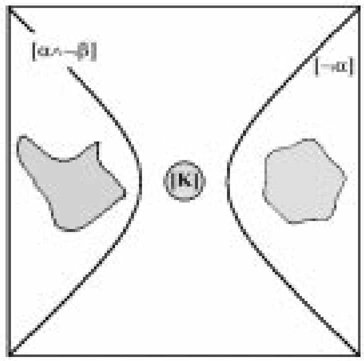


Figure 14 Semi-Contraction in possible world

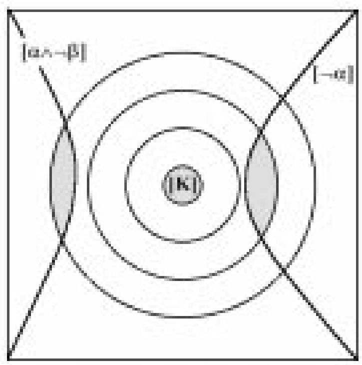


Figure 15 Semi-Contraction in spheres system

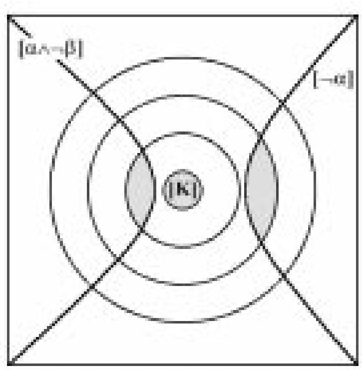


Figure 16 Interpoled Semi-Contraction

4.9 Semi-Contraction and Revision

As the Levi and Harper functions for the AGM model, we can define new functions that relate $\underset{s}{-}$ and $*$. Since $\underset{s}{-}$ is a withdrawal, we can use the Levi identity (see **Definition 2.4.16**) to define $*$ in terms of a semi-contraction:

$$\mathbf{K}_{\mathbb{R}(\underset{s}{-})}\alpha = (\mathbf{K}\underset{s}{-}\neg\alpha) + \alpha$$

Observation 4.9.1 $\mathbb{R}(\underset{s}{-})$ satisfies *closure, success, inclusion, vacuity, consistency*, and *extensionality.*

We can prove the following relation between $\underset{s}{-}$ and it associate AGM contraction $-$:

Observation 4.9.2 Let $\mathbf{K}$ be a belief set, $\underset{s}{-}$ a semi-contraction and $-$ it associated AGM contraction. Then $\mathbb{R}(\underset{s}{-}) = \mathbb{R}(-)$.

However, the Harper identity returns an AGM contraction function. Consequently, we must define a new identity to define a semi-contraction in terms of revision:

Definition 4.9.3 [Mak87] Let $\mathbf{K}$ be a belief set and Sel a semi-selection function as defined in **Definition 4.3.1**. Then $\mathbb{C}_s()$ (semi-Harper) is the function such that for every operator $*$ for $\mathbf{K}$, $\mathbb{C}()$ is the operator for $\mathbf{K}$ such that for all α:

$$\mathbf{K}_{\mathbb{C}_s(*)}\alpha = \mathbf{K} \cap (\mathbf{K}*\neg\alpha) \cap (\mathbf{K}*\neg(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha)))$$

THEOREM 4.9.4 Let $\mathbf{K}$ be a theory and $*$ an operator for $\mathbf{K}$ that satisfies the revision postulates *closure, sucess, inclusion, vacuity, consistency*, and extensionality. Then $\mathbb{C}_s()$ is an operator for $\mathbf{K}$ that satisfies *closure, inclusion, vacuity, success, extensionality, failure* and *proxy recovery.*

As in the AGM model, we can obtained a one to one correspondence between $\mathbb{R}()$ and $\mathbb{C}_s()$:

THEOREM 4.9.5 Let $\mathbf{K}$ be a theory, $-$ an operator for $\mathbf{K}$ that satisfies the contraction postulates *closure, inclusion, vacuity, success, extensionality* and *recovery*; and $\overline{}_s$ its associated semi-contraction. Then $\mathbb{C}_s(\mathbb{R}(\overline{}_s)) = \overline{}_s$.

THEOREM 4.9.6 Let $\mathbf{K}$ be a theory and $*$ an operator for $\mathbf{K}$ that satisfies the revision postulates *closure, success, inclusion, vacuity, consistency*, and *extensionality.* Then $\mathbb{R}(\mathbb{C}_s(*)) = *$.

4.10 Proofs of Chapter 4

4.10.1 Lemmas

The following lemmas will be use in the demonstrations.

Lemma 4.10.1 Let $\mathbf{K}$ be a belief set and $-$ an operator on $\mathbf{K}$ that satisfies *success, vacuity* and *failure.* Then $-$ satisfies *proxy* *recovery* if and only if it satisfies:

- **Weak Recovery:** If $\mathbf{K} \neq \mathbf{K}-\alpha$ then there exists some β such that $\mathbf{K} \vdash \beta$, $\mathbf{K}-\alpha \nvdash (\alpha \vee \beta)$ but $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + (\alpha \wedge \beta)$.

Proof of Lemma 4.10.1 *Weak recovery* to *proxy recovery* is trivial. For the converse, let δ be a sentence that satisfies the *proxy recovery* conditions and let $\beta = \alpha \wedge \delta$. It is trivial to prove that β satisfies *weak recovery*. ■

Lemma 4.10.2 Let $\mathbf{K}$ be a belief set. Let $\alpha \in \mathbf{K}$, and $\nvdash \alpha$. Then $\mathbf{K} \perp (\alpha \vee \beta) \subseteq \mathbf{K} \perp \alpha$.

Proof of Lemma 4.10.2 If $\vdash \beta$, then the proof is trivial. For the principal case, let $\nvdash \beta$ and $\mathbf{H} \in \mathbf{K} \perp (\alpha \vee \beta)$. Then $\mathbf{H} = Cn(\mathbf{H})$ and $\mathbf{H} \nvdash \alpha$. We must prove that $\mathbf{H}$ is a maximal subset of $\mathbf{K}$ that does not imply α.
Let $\mathbf{H}'$ be such that $\mathbf{H} \subset \mathbf{H}' \subseteq \mathbf{K}$. Then there exists some $\delta \in \mathbf{H}'$ such that $\delta \notin \mathbf{H}$. Since $\mathbf{H} \in \mathbf{K} \perp (\alpha \vee \beta)$, $\delta \rightarrow (\alpha \vee \beta) \in \mathbf{H}$ and $(\alpha \vee \beta) \rightarrow \alpha \in \mathbf{H}$. Thus $\delta \rightarrow \alpha \in \mathbf{H}$, so that $\mathbf{H}' \vdash \alpha$. Hence $\mathbf{H} \in \mathbf{K} \perp \alpha$. ■

Lemma 4.10.3 Let $\mathbf{B}$ be a belief set. If $\mathbf{B} \in SS(\mathbf{K}, \alpha, \beta)$, then $\exists!$ belief set $\mathbf{H}$ such that $\mathbf{B} = \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$, where:

$$\mathbf{H} \in \mathbf{K} \perp (\alpha \vee \beta)$$
$$\Delta = \{\mathbf{I} \in \mathbf{K} \perp (\alpha \vee \neg\beta) \mid \mathbf{B} \subseteq \mathbf{I}\}$$

$$\Pi = \{\mathbf{J} \in \mathbf{K} \perp (\neg\alpha \vee \beta) \mid \mathbf{B} \subseteq \mathbf{J}\}$$

$$\Delta^{\sim} = \begin{cases} \cap\Delta \text{ if } \Delta \neq \emptyset \\ \\ \mathbf{B} \text{ otherwise} \end{cases}$$

$$\Pi^{\sim} = \begin{cases} \cap\Pi \text{ if } \Pi \neq \emptyset \\ \\ \mathbf{B} \text{ otherwise} \end{cases}$$

Proof of Lemma 4.10.3 We must to prove **(a)** that **H** exists; **(b)** that **H** is unique and finally **(c)** that $\mathbf{B} = \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$.

(a) By definition of SS $\mathbf{B} \subseteq \mathbf{K}$ and $(\alpha \vee \beta) \notin \mathbf{B}$. Then by **Property 2.5.4** $\exists \mathbf{H} \in \mathbf{K}\perp(\alpha \vee \beta)$.

(b) To prove that **H** is unique suppose for *reductio ad absurdum* that $\exists \mathbf{H}'$, $\mathbf{H}' \neq \mathbf{H}$, $\mathbf{H}' \in \mathbf{K}\perp(\alpha \vee \beta)$, $\mathbf{B} \subseteq \mathbf{H}'$. Since $\mathbf{H}' \neq \mathbf{H}$ and both are maximal subsets of **K**, then $\exists \delta \in \mathbf{H}'$ such that $\delta \notin \mathbf{H}$. We have two subcases:

(b1) $\mathbf{H}{+}(\neg\alpha \wedge \neg\beta) = \mathbf{H}'{+}(\neg\alpha \wedge \neg\beta)$, then by Cn deduction $(\neg\alpha\wedge\neg\beta) \rightarrow \delta \in \mathbf{H}$ i.e., $(\alpha\vee\beta\vee\delta) \in \mathbf{H}$ and since $(\neg\delta\vee\alpha\vee\beta) \in \mathbf{H}$ then $(\alpha \vee \beta) \in \mathbf{H}$. Absurd.

(b2) $\mathbf{H}{+}(\neg\alpha \wedge \neg\beta) \neq \mathbf{H}'{+}(\neg\alpha \wedge \neg\beta)$ then $\mathbf{B}{+}(\neg\alpha \wedge \neg\beta) \subseteq \mathbf{H}{+}(\neg\alpha \wedge \neg\beta) \cap\ \mathbf{H}'{+}(\neg\alpha \wedge \neg\beta)$, hence $\mathbf{B}{+}(\neg\alpha \wedge \neg\beta)$ is not maximal subset contradicting that $\mathbf{B} \in SS(\mathbf{K}, \alpha, \beta)$. Absurd.

(c) It is trivial that $\mathbf{B} \subseteq \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$. For the other inclusion suppose that $\mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim} \not\subseteq \mathbf{B}$, then there exists some $\delta \in \mathbf{H} \cap \Delta^{\sim} \cap \Pi^{\sim}$ such that $\delta \notin \mathbf{B}$. Since $\delta \notin \mathbf{B}$, then (by **Property 2.5.4**) $\exists \mathbf{H}' \in \mathbf{K}\perp(\alpha \vee \beta \vee \delta), \mathbf{B} \subseteq \mathbf{H}'$. By **Lemma 4.10.2** $\mathbf{H}' \in \mathbf{K}\perp(\alpha \vee \beta)$, then by part **(b)** $\mathbf{H} = \mathbf{H}'$, which is absurd since $\delta \in \mathbf{H}$ and $\delta \notin \mathbf{H}'$. ■

4.10.2 Proofs

Proof of Lemma 4.3.4 *Closure, inclusion, success*, and *failure* are trivial. *Extensionality* follows since $\mathbf{K} \setminus \mathbf{K}{-}\alpha = \mathbf{K} \setminus \mathbf{K}{-}\beta$ by $-$ *extensionality.*

Vacuity: Let $\alpha \notin \mathbf{K}$; then $\mathbf{K}{-}\alpha = \mathbf{K}$, hence $Sel(\mathbf{K} \setminus \mathbf{K}{-}\alpha) = \top$, then $\mathbf{K}\,\bar{\overline{s}}\,\alpha = \mathbf{K}{-}\alpha \cap \mathbf{K}{-}\top = \mathbf{K}$.

Proxy recovery: Let $\mathbf{K}$ be a belief set, $\bar{\overline{s}}$ a semi-contraction function for $\mathbf{K}$; $-$ its associated *partial meet AGM contraction function* and β such that $\mathbf{K}\,\bar{\overline{s}}\,\alpha = \mathbf{K}{-}\alpha \cap \mathbf{K}{-}(\alpha \rightarrow \beta)$, $\beta \in Sel(\mathbf{K} \setminus \mathbf{K}{-}\alpha)$. Let $\mathbf{K} \neq \mathbf{K}\,\bar{\overline{s}}\,\alpha$ and $\delta = \alpha \wedge \beta$. Since $\mathbf{K} \neq \mathbf{K}\,\bar{\overline{s}}\,\alpha$ it follows that $\alpha \in \mathbf{K}$ and $\beta \in \mathbf{K}$, from which it follows that $\delta \in \mathbf{K}$. We need to show **(a)** that $\delta \notin \mathbf{K}{-}\alpha$ and **(b)** that $\mathbf{K} \subseteq (\mathbf{K}\,\bar{\overline{s}}\,\alpha) + \delta$.

(a) It follows by the definition of semi-contraction that $\mathbf{K} \neq \mathbf{K}{-}\alpha$ and that $\mathbf{K} \setminus \mathbf{K}{-}\alpha \neq \emptyset$; then $\beta \in \mathbf{K} \setminus \mathbf{K}{-}\alpha$, hence $\delta \in \mathbf{K} \setminus \mathbf{K}{-}\alpha$.

(b) $(\mathbf{K}{-}\alpha \cap \mathbf{K}{-}(\alpha \rightarrow \beta)) + (\alpha \wedge \beta)$

$= (\mathbf{K}{-}\alpha) + (\alpha \wedge \beta) \cap (\mathbf{K}{-}(\alpha \rightarrow \beta)) + (\alpha \wedge \beta)$

$= (\mathbf{K}{-}\alpha) + (\alpha \wedge \beta) \cap (\mathbf{K}{-}(\alpha \rightarrow \beta)) + ((\alpha \rightarrow \beta) \wedge \alpha)$

$= ((\mathbf{K}{-}\alpha) + \alpha) + \beta \cap ((\mathbf{K}{-}(\alpha \rightarrow \beta)) + (\alpha \rightarrow \beta)) + \alpha$

$=$ (by *recovery* and *inclusion*) $\mathbf{K}{+}\beta \cap \mathbf{K}{+}\alpha$

$= \mathbf{K}$ (since α and β are in $\mathbf{K}$) . ■

Proof of Observation 4.4.2 Let $(\alpha \vee \beta) \in \mathbf{K}$. If $\vdash \alpha \vee \beta$, then $\mathbf{K}\perp(\alpha \vee \beta) = \emptyset$ and we are finished. For $\nvdash \alpha \vee \beta$, let $\mathbf{H} \in \mathbf{K}\perp(\alpha \vee \beta)$. To prove that $\mathbf{H}$ is in $SS(\mathbf{K}, \alpha, \beta)$ we need to prove
(a) that $\mathbf{H} \subseteq \mathbf{K}$ and $\mathbf{H} = Cn(\mathbf{H})$: Thisfollows trivially from the definition of $\mathbf{K}\perp\alpha \vee \beta$;
(b) that $\mathbf{H}+(\neg\alpha \wedge \neg\beta)$ is a maximal consistent subset of the language; this follows from **Property 2.5.5**, since $(\alpha \vee \beta) \in \mathbf{K}$ and $(\alpha \vee \beta) \notin \mathbf{H}$; and finally
(c) that $\mathbf{K} \subseteq \mathbf{H}+(\alpha \wedge \beta)$ which follows from $\mathbf{K} \subseteq \mathbf{H}+(\alpha \vee \beta)$[2] and $\mathbf{H}+(\alpha \vee \beta) \subseteq \mathbf{H}+(\alpha \wedge \beta)$. ■

Proof of Observation 4.4.3 The demonstration is trivial, since the conditions for $S(\mathbf{K}, (\alpha \vee \beta))$ are the first three conditions for $SS(\mathbf{K}, \alpha, \beta)$. ■

Proof of Lemma 4.4.6: In **Lemma 4.10.3**, we show that for all $\mathbf{B}_i \in \gamma(SS(\mathbf{K}, \alpha, \beta))$, $\mathbf{B}_i$ can be expressed as $\mathbf{B}_i = \mathbf{H}_i \cap \Delta_i^{\sim} \cap \Pi_i^{\sim}$; then $\bigcap \gamma(SS(\mathbf{K}, \alpha, \beta)) = \bigcap_i \mathbf{H}_i \cap \Delta_i^{\sim} \cap \Pi_i^{\sim}$; where each $\mathbf{H}_i \cap \Delta_i^{\sim} \subseteq \mathbf{K}\perp\alpha$ and each $\Pi_i^{\sim} \subseteq \mathbf{K}\perp\neg\alpha \vee \beta$.

We can construct a *partial meet AGM contraction function* using a selection function that take the elements of $\mathbf{H}_i \cap \Delta_i^{\sim}$ to construct $\mathbf{K}-\alpha$ and $\Pi_i^{\sim}$ to construct $\mathbf{K}-(\neg\alpha \vee \beta)$. Let γ_2 be an arbitrary selection function and γ_1 a selection function such that:

[2] Since each member of the remainder set satisfies *recovery*, see [AM81].

$$\gamma_1(\mathbf{W}) = \begin{cases} \{\mathbf{M} \mid \mathbf{M} = \mathbf{H} \text{ or } \mathbf{M} \in \Delta_i^{\sim}\} & \text{if } \mathbf{W} = \mathbf{K}\bot\alpha \\ \{\mathbf{M} \mid \mathbf{M} \in \Pi_i^{\sim}\} & \text{if } \mathbf{W} = \mathbf{K}\bot(\neg\alpha \vee \beta) \\ \gamma_2(\mathbf{W}) & \text{otherwise.} \end{cases}$$

Clearly $\cap\gamma_1$ is a *partial meet AGM contraction* and it follows that $\bigcap\gamma(SS(\mathbf{K}, \alpha, \beta)) = \bigcap\gamma_1(\mathbf{K}\bot\alpha) \cap \bigcap\gamma_1(\mathbf{K}\bot(\neg\alpha \vee \beta))$, that concludes the proof. ■

Proof of Lemma 4.4.7 If $\vdash \alpha$ or $\alpha \notin \mathbf{K}$, then it is trivial. Let $\nvdash \alpha$ and $\alpha \in \mathbf{K}$. Due to *proxy recovery* and **Lemma 4.10.1** there exists some β such that $\beta \in \mathbf{K}$ and $(\alpha \vee \beta) \notin \mathbf{K}{\sim}\alpha$. By *inclusion*, $\mathbf{K} = \mathbf{K}{\sim}\alpha + (\alpha \wedge \beta)$.

Let $\Upsilon = \{\mathbf{U} \in \mathbf{K}\bot(\alpha \vee \beta) \mid \mathbf{K}{\sim}\alpha \subseteq \mathbf{U}\}$

Let $\Delta = \{\mathbf{I} \in \mathbf{K}\bot(\alpha \vee \neg\beta) \mid \mathbf{K}{\sim}\alpha \subseteq \mathbf{I}\}$

Let $\Pi = \{\mathbf{J} \in \mathbf{K}\bot(\neg\alpha \vee \beta) \mid \mathbf{K}{\sim}\alpha \subseteq \mathbf{J}\}$

Let $\Delta^{\sim} = \begin{cases} \cap\Delta \text{ if } \Delta \neq \emptyset \\ \mathbf{K}{\sim}\alpha \text{ otherwise} \end{cases}$

Let $\Pi^{\sim} = \begin{cases} \cap\Pi \text{ if } \Pi \neq \emptyset \\ \mathbf{K}{\sim}\alpha \text{ otherwise} \end{cases}$

We must prove **(a)** that $\Upsilon \neq \emptyset$ and **(b)** that $\mathbf{K}{\sim}\alpha = \cap\mathbf{M}$, where $\mathbf{M} = \{\mathbf{M_i} : \mathbf{M}_i \in SS(\mathbf{K}, \alpha, \beta)\}$.

(a) $(\alpha \vee \beta) \notin \mathbf{K}{\sim}\alpha$ and by *inclusion* $\mathbf{K}{\sim}\alpha \subseteq \mathbf{K}$, then by **Lemma 2.5.4** there exists some $\mathbf{U}$ such that $\mathbf{K}{\sim}\alpha \subseteq \mathbf{U}$ and $\mathbf{U} \in \mathbf{K}\perp(\alpha \vee \beta)$.

(b) Let $\mathbf{M_i} = \mathbf{U_i} \cap \Delta^{\sim} \cap \Pi^{\sim}, \mathbf{U_i} \in \Upsilon$. It follows trivially that $\mathbf{M}_i = Cn(\mathbf{M_i})$, $\mathbf{M_i} \subseteq \mathbf{K}$ and $\mathbf{K} \subseteq \mathbf{M_i}{+}(\alpha \wedge \beta)$. $\mathbf{M_i}{+}(\neg\alpha \wedge \neg\beta) = \mathbf{U_i}{+}(\neg\alpha \wedge \neg\beta) \cap \boldsymbol{\Delta}^{\sim}{+}(\neg\alpha \wedge \neg\beta) \cap \boldsymbol{\Pi}^{\sim}{+}(\neg\alpha \wedge \neg\beta)$. Since $\Delta^{\sim}$ and $\Pi^{\sim}$ both satisfy *recovery*, $\boldsymbol{\Delta}^{\sim}{+}(\neg\alpha \wedge \neg\beta) = \mathbf{K}_{\perp}$ and $\boldsymbol{\Pi}^{\sim}{+}(\neg\alpha \wedge \neg\beta) = \mathbf{K}_{\perp}$, then $\mathbf{M_i}{+}(\neg\alpha \wedge \neg\beta) = \mathbf{U_i}{+}(\neg\alpha \wedge \neg\beta)$ that is a maximal consistent subset of the language. Hence $\mathbf{M}_i \in SS(\mathbf{K}, \alpha, \beta)$.

Finally, we must prove that $\mathbf{K}{\sim}\alpha = \cap\mathbf{M}$, where $\mathbf{M} = \{\mathbf{M_i} \mid \mathbf{M_i} = \mathbf{U_i} \cap \Delta^{\sim} \cap \Pi^{\sim}, \mathbf{U_i} \in \Upsilon\}$. It follows trivially that $\mathbf{K}{\sim}\alpha \subseteq \cap\mathbf{M}$. To prove that $\cap\mathbf{M} \subseteq \mathbf{K}{\sim}\alpha$ let $\delta \in \cap\mathbf{M}, \delta \notin \mathbf{K}{\sim}\alpha$, then $\delta \in \mathbf{M_i}$, $\forall \mathbf{M_i} \in \Upsilon$. Since $\delta \notin \mathbf{K}{\sim}\alpha$, by **Lemma 2.5.4** $\exists \mathbf{H} \in \mathbf{K}\perp(\alpha \vee \beta \vee \delta)$. By **Lemma 4.10.2**, $\mathbf{H} \in \mathbf{K}\perp(\alpha \vee \beta)$, so that $\mathbf{K}{\sim}\alpha \subseteq \mathbf{H}$, and consequently $\mathbf{H} \in \Upsilon$, then $\delta \notin \cap\mathbf{M_i}$ and $\delta \notin \cap\mathbf{M}$. Absurd. ■

Proof of Theorem 4.5.1

(1) implies (2): Follows from Lemma 4.3.4.

(2) implies (3): Follows from Lemma 4.4.7.

(3) implies (1): Follows from Lemma 4.4.6. ■

Example of Section 4.7 Let $\mathcal{L}$ be the closure under truth-functional operations of $\{\alpha, \beta\}$, and let $\mathbf{K} = Cn(\{\alpha \wedge \beta\})$. We will construct $\leq_{\mathbf{K}}$ explicitly. Due to (**EE2**) it is sufficient to order the sixteen formulae in the following ordering:

$$
\begin{array}{ccccccccc}
\neg\alpha \wedge \beta \\
\alpha \wedge \neg\beta \\
\neg\alpha \wedge \neg\beta & & \alpha \wedge \beta \\
\neg\alpha & & \beta & & \alpha \\
\neg\beta & <_{\mathbf{K}} & \alpha \leftrightarrow \beta & <_{\mathbf{K}} & \alpha \vee \neg\beta & <_{\mathbf{K}} & \alpha \vee \beta & <_{\mathbf{K}} & \top \\
\alpha \not\leftrightarrow \beta & & \neg\alpha \vee \beta \\
\neg\alpha \vee \neg\beta \\
\bot
\end{array}
$$

Let $-_{\mathrm{G}}$ and $-_{\mathrm{R}}$ be the AGM contraction and severe withdrawal based on $\leq_{\mathbf{K}}$ defined via $(-_{\mathrm{G}})$ and $(-_{\mathrm{R}})$ respectively. By definition of $-_{\mathrm{G}}$, we have:

$$\mathbf{K}-_{\mathrm{G}}(\alpha \wedge \beta) = \mathbf{K}-_{\mathrm{G}}(\alpha \leftrightarrow \beta) = \mathbf{K}-_{\mathrm{G}}(\beta) = \mathbf{K}-_{\mathrm{G}}(\neg\alpha \vee \beta) = Cn(\{\alpha\})$$

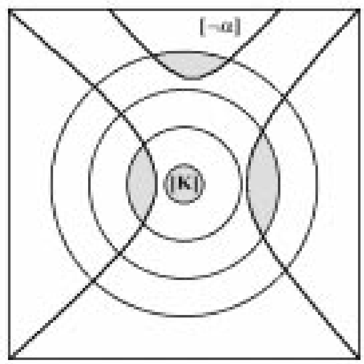
$$\mathbf{K}-_{\mathrm{G}}(\alpha) = \mathbf{K}-_{\mathrm{G}}(\alpha \vee \neg\beta) = Cn(\{\beta\})$$

$$\mathbf{K}-_{\mathrm{G}}(\alpha \vee \beta) = Cn(\{\alpha \leftrightarrow \beta\})$$

Otherwise $\mathbf{K}-_{\mathrm{G}}(x) = \mathbf{K}$.

Trivially, $-_{\mathrm{R}}$ satisfies the interpolation thesis. For $\alpha \vee \beta$, $\mathbf{K}-_{\mathrm{R}}(\alpha \vee \beta) = Cn(\emptyset)$ and it is easy to show that there is no δ such that $Cn(\{\alpha \leftrightarrow \beta\}) \cap \mathbf{K}-_{\mathrm{G}}((\alpha \vee \beta) \to \delta) = Cn(\emptyset)$. Hence $-_{\mathrm{R}}$ is not a semi-contraction. ■

Note: The following figure in the Grove's sphere-system illustrates the example. If the contraction intersects more than two spheres (as in the figure), we can not express it as a semi-contraction, since each AGM function can intersect only one sphere. This is the case of the example.



Proof of Observation 4.9.2

$\mathbf{K}_{\mathbb{R}(\overline{s})}\alpha = (\mathbf{K}\,\overline{s}\,\neg\alpha) + \alpha$; by **Definition 4.3.2** $= ((\mathbf{K}-\neg\alpha) \cap (\mathbf{K}-(\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha))) + \alpha$; by **Property 2.4.6** $= ((\mathbf{K}-\neg\alpha) + \alpha) \cap ((\mathbf{K}-(\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha))) + \alpha)$; since α is equivalent to $(\alpha \wedge (\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha)))$ then $= ((\mathbf{K}-\neg\alpha)+\alpha) \cap ((\mathbf{K}-(\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha)))+(\alpha \wedge (\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha)))) = ((\mathbf{K}-\neg\alpha)+\alpha) \cap (((\mathbf{K}-(\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha)))+(\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha))) + \alpha$; then by *recovery and inclusion*, since $(\neg\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha)) \in \mathbf{K} = ((\mathbf{K}-\neg\alpha) + \alpha) \cap \mathbf{K} + \alpha = (\mathbf{K}-\neg\alpha) + \alpha$, hence by Definition of Levi $= \mathbf{K}_{\mathbb{R}(-)}\alpha$. ■

Proof of Theorem 4.9.4 By definition of semi-Harper $\mathbf{K}_{\mathbb{C}_s(*)}\alpha = \mathbf{K} \cap (\mathbf{K}*\neg\alpha) \cap (\mathbf{K}*\neg(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha)))$; or equivalently $(\mathbf{K} \cap (\mathbf{K}*\neg\alpha)) \cap (\mathbf{K} \cap (\mathbf{K}*\neg(\alpha \to Sel(\mathbf{K} \setminus \mathbf{K}*\neg\alpha))))$; then by Harper identity $= (\mathbf{K}-\alpha) \cap (-K(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)))$, hence $\mathbb{C}_s(*)$ is a semi-contraction, hence, by **Observation 4.3.4** it satisfies *closure, inclusion, vacuity, success, extensionality, failure* and *proxy recovery.* ■

Proof of Theorem 4.9.5 By Definition of $\mathbb{C}_s()$ we have $\mathbf{K}_{\mathbb{C}_s(\mathbb{R}(\overline{s}))}\alpha = \mathbf{K} \cap (\mathbf{K}_{\mathbb{R}(\overline{s})}\neg\alpha) \cap (\mathbf{K}_{\mathbb{R}(\overline{s})}\neg(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K} \cap \mathbf{K}_{\mathbb{R}(\overline{s})}\neg\alpha)))$; applying the Levi identity and since $\mathbb{R}(-) = \mathbb{R}(\overline{s})$ we obtain $= \mathbf{K} \cap ((\mathbf{K}-\alpha) + \neg\alpha) \cap ((\mathbf{K}-(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))) + \neg(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))) = (\mathbf{K} \cap ((\mathbf{K}-\alpha) + \neg\alpha)) \cap (\mathbf{K} \cap ((\mathbf{K}-(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))) + \neg(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha)))) =$ (by **Theorem 2.4.24**) $= (\mathbf{K}-\alpha) \cap (\mathbf{K}-(\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K}-\alpha))) =$ (by definition of semi contraction) $= \mathbf{K}_{\overline{s}}\alpha$. ■

Proof of Theorem 4.9.6 By the Levi identity $\mathbf{K}_{\mathbb{R}(\mathbb{C}_s(*))}\alpha = (\mathbf{K}_{\mathbb{C}_s(*)}\neg\alpha) + \alpha$; applying the semi-Harper identity we obtain $\mathbf{K}_{\mathbb{R}(\mathbb{C}_s(*))}\alpha = (\mathbf{K} \cap (\mathbf{K}*\alpha) \cap (\mathbf{K}*\neg(\neg\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K} \cap (\mathbf{K}*\alpha))))) + \alpha$ and since $\neg\alpha \in \mathbf{K}*\neg(\neg\alpha \rightarrow Sel(\mathbf{K} \setminus \mathbf{K} \cap (\mathbf{K}*\alpha)))$, by distributing the expansion we obtain $\mathbf{K}_{\mathbb{R}(\mathbb{C}_s(*))}\alpha = (\mathbf{K}+\alpha) \cap (\mathbf{K}*\alpha) \cap \mathbf{K}_\perp$, hence by *inclusion* $= \mathbf{K}*\alpha$. ■

Part III

Revision Without Success

Chapter 5

Revision, New Information and Success

In this chapter we present different models of revision functions and their relationship with the new information. In the AGM model, revision functions always accept the new information. These is an unrealistic feature since we tend to reject the new information if (1) it contradicts and (2) it is not better than the information that has already been accepted. Arguing essentially against the success postulate of AGM revision, Isaac Levi [Lev96, p.6] wrote:

> "Sometimes, however, the input information h is called into question, and the background information is retained. Sometimes new inputs and background information are both questioned"

In the last of the quoted sentences, Levi indicated two different possibilities. Maybe the new information is fully rejected, maybe it is partially accepted. This distinction encouraged several authors to define different kinds of revision functions, where the new information is not always accepted. We will present a taxonomy of revision functions based on two different parameters: The process of revision (due imostly to Hansson [Han98d]) and the result of the revision function. In this taxonomy we include two different non-prioritized belief revision functions of our own: *Selective revision* and *Credibility-limited revision*, that are developed in detail in **Chapter 6** and **Chapter 7**.

Part of the material of this chapter appeared in:

[•] EDUARDO FERMÉ. Technical note: Irrevocable belief revision and epistemic entrenchment. 1998. (manuscript).

[•] EDUARDO FERMÉ AND SVEN OVE HANSSON. Selective revision. *Studia Logica*, 1998. In press.

[•] SVEN OVE HANSSON, EDUARDO FERMÉ, JOHN CANTWELL, AND MARCELO FALAPPA. Credibility-limited revision. 1998. (manuscript).

5.1 A Quick Survey of Revision Functions

5.1.1 AGM Revision

AGM revision [AGM85] consists basically in the incorporation of new information, preserving consistency (unless the new information is inconsistent in itself). Consequently revision eliminates (if possible) the sentences that contradict the new beliefs. Let $\mathbf{K}$ be a belief set. $\mathbf{K}*\alpha$ denotes the revision from $\mathbf{K}$ by a sentence α. AGM revision was analyzed in detail in **Chapter 2**.

5.1.2 Updating

Belief revision is understood as the process of changing the belief of an agent in a static world. Katsuno and Mendelzon [KM92] presented a model where the agent modifies his beliefs according to a change in the world, called "updating". In the syntactic presentation, their model of updating requires a strong restriction of the language of **K**: The language is propositional with a finite number of elementary propositions. Consequently, any belief set can be represented as the consequence of a proposition. Updating is characterized by the following set of axioms, where ψ is a sentence that represents the beliefs of the agent and $\psi \diamond \mu$ denotes the result of updating ψ by μ:

(U1) $\psi \diamond \mu \vdash \mu$.

(U2) If $\psi \vdash \mu$, then $\psi \diamond \mu \equiv \psi$.

(U3) If $\psi \nvdash \bot$ and $\mu \nvdash \bot$, then $\psi \diamond \mu \nvdash \bot$.

(U4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \diamond \mu_1 \equiv \psi_2 \diamond \mu_2$.

(U5) $(\psi \diamond \mu_1) \wedge \mu_2$ implies $\psi \diamond (\mu_1 \wedge \mu_2)$.

(U6) If $\psi_1 \diamond \mu_1 \vdash \mu_2$ and $\psi_2 \diamond \mu_2 \vdash \mu_1$, then $\psi_1 \diamond \mu_1 \equiv \psi_2 \diamond \mu_2$.

(U7) If ψ is complete, then $(\psi \diamond \mu_1) \wedge (\psi \diamond \mu_2)$ implies $\psi \diamond (\mu \vee \phi)$.

(U8) $(\psi_1 \vee \psi_2 \diamond \mu) \equiv (\psi_1 \diamond \mu) \vee (\psi_2 \diamond \mu)$.

Katsuno and Mendelzon also provide a "semantic" account of updating in terms of possible worlds, where the worlds of $\psi \diamond \mu$ are the nearest μ-worlds of each $\psi-$world. Since updating is based on a fixed order around each world, it is iterable. On the other hand, the following undesiderable property is derived[1]:

[1] With respect to this property, Katsuno and Mendelzon said [KM92, p.190]: "We can never repair an inconsistent theory using update, because update specifies a change in the

5.1.1 If an updating operator satisfies **(U2)** and $\psi \vdash \bot$, then $\psi \diamond \mu \vdash \bot$ for all μ.

i.e., if the belief set is inconsistent, then all the possible updates are inconsistent too.

5.1.3 Screened Revision

Makinson [Makss] proposed a simple model of non-prioritized belief revision. He defined a special set A of sentences that are immune to revision. The result of revising by sentences that violates $\mathbf{K} \cap A$ is identical to the original belief set. If the input sentence is compatible with $\mathbf{K} \cap A$, then the belief set is revised essentially in the AGM way. Formally, the screened revision for a belief set $\mathbf{K}$ is based on a function $\#_A$, with respect to the set A, defined as follows:

$$\mathbf{K}\#_A\alpha = \begin{cases} \mathbf{K}*\alpha & \text{If } \alpha \text{ is consistent with } \mathbf{K} \cap A. \\ \mathbf{K} & \text{Otherwise} \end{cases}$$

where $*$ is essentially an AGM revision function with the additional constraint that for all α, $\mathbf{K} \cap A \subseteq \mathbf{K}*\alpha$.

Hansson [Han98d] presented a more general approach, called *generalized screened revision*:

$$\mathbf{K}\#_{f(\alpha)}\alpha = \begin{cases} \mathbf{K}*\alpha & \text{If } \alpha \text{ is consistent with } \mathbf{K} \cap f(\alpha). \\ \mathbf{K} & \text{Otherwise} \end{cases}$$

where $*$ is a (modified) AGM revision function such that for all α, $\mathbf{K} \cap f(\alpha) \subseteq$

world. If there is no set of worlds that fits our current description, we have no way of recording the change in the world."

$\mathbf{K}{*}\alpha$. $f : \mathcal{L} \rightarrow \mathcal{PP}_{\mathcal{L}}$. Different properties can be added to f. Makinson [Makss] proposed, for example $f(\alpha) = \{\beta : \alpha < \beta\}$, where $<$ is a binary relation on the language.

5.1.4 Credibility-Limited Revision

In a recent work [HFCF98], we proposed a generalization of screened revision. We define a set $\mathcal{C}$ that represents all the possible credible sentences of the language, and the following function:

$$\mathbf{K}?\alpha = \begin{cases} \mathbf{K}{*}\alpha & \text{If } \alpha \in \mathcal{C} \\ \mathbf{K} & \text{Otherwise} \end{cases}$$

where $*$ is an AGM revision function and ? is the credibility-limited revision induced by $*$ and $\mathcal{C}$.

For further details of credibility-limited revision see **Chapter 7**.

5.1.5 External Revision

For belief bases, Hansson [Han93b] proposed an alternative to the Levi identity:

$$\mathbf{K} * \alpha = (\mathbf{K}{+}\alpha) - \neg\alpha$$

This identity is not plausible for belief sets, since the first step, $\mathbf{K}{+}\alpha$, typically involves an inconsistency, and therefore leads to the loss of all distinctions (since there is only one inconsistent belief set). Consequently, the following implausible property holds:

- If $\mathbf{K} \vdash \neg\alpha$ and $\mathbf{H} \vdash \neg\alpha$, then $\mathbf{K}{*}\alpha = \mathbf{H}{*}\alpha$.

However, external revision for belief sets proposes an interesting starting point for iterable functions and is closely related to the Areces and Becher *iterable AGM functions* [AB99].

On the other hand it is possible to distinguish between different inconsistent belief bases, as we will see in the following example:

Example 5.1.2 Let $A = \{\neg\alpha, \beta\}$ and $B = \{\neg\alpha, \delta\}$. Then a possible external revision for A and B by α is:

$$A * \alpha = \{\beta, \alpha\}$$
$$B * \alpha = \{\delta, \alpha\}$$

Hansson provided the following representation theorem:

THEOREM 5.1.3 The operator $*$ for a set of sentences A is an operator of external revision if and only if it satisfies:

- If $\nvdash \neg\alpha$ then $A * \alpha \nvdash \bot$. ([Base] Consistency)
- $A * \alpha \subseteq A \cup \alpha$. ([Base] Inclusion)
- If $\beta \in A$ and $\beta \notin A * \alpha$, then there exists some A' such that $A * \alpha \subseteq A' \subseteq A \cup \alpha$, $A' \nvdash \bot$, but $A' \cup \{\alpha\} \vdash \bot$. ([Base] Relevance)
- $\alpha \in A * \alpha$. ([Base] Success)
- If α and β are elements of A and it holds for all $A' \subseteq A$ that $A' \cup \alpha$ is inconsistent if and only if $A' \cup \beta$ is inconsistent, then $A \cap A * \alpha = A \cap A * \beta$.([Base] Weak Uniformity)
- $(A \cup \alpha) * \alpha = A * \alpha$. ([Base] Pre-Expansion)

For a detailed study of belief base dynamics see [Han91a, Hanss]. It is possible to define the following two variants of external revision for belief sets:

(a) $\mathbf{K} * \alpha = Cn((\mathbf{K} \cup \{\alpha\}) - \neg\alpha)$

(b) $\mathbf{K} * \alpha = Cn(\mathbf{K} \cup Cn(\alpha) - \neg\alpha)$

where $\mathbf{K}$ is a belief set and $-$ a contraction operator for belief bases. The properties and behaviour of these variants are not explored and constitute open problems.

5.1.6 Semi-Revision

Hansson proposed a modification of external revision: Instead of contracting by the negation of the sentence, external revision consolidated the expanded belief base; i.e., it made the expanded belief base consistent. One way to doing this is to contract by $\perp$ (falsum). In symbols:

$$\mathbf{K}?\alpha = (\mathbf{K}+\alpha) - \perp$$

The main difference between semi-revision and external revision is that the input sentence may be discarded in the consolidation process. Consequently this is a non-prioritized belief revision model.

On the other hand, the main difference from other non prioritized belief revision models is that the consolidation process may discard both α and $\neg\alpha$. We say that this model violates the "Non-Indifference principle"[2] Hansson provided the following representation theorem:

[2] $\alpha \in \mathbf{K}?\alpha$ or $\neg\alpha \in \mathbf{K}?\alpha$.

THEOREM 5.1.4 The operator ? for a set A of sentences is semi-revision operator if and only if it satisfies *[Base] inclusion, [Base] Relevance, [Base] Pre-Expansion* and :

- $A?\alpha \nvdash \perp$. ([Base] Strong Consistency)
- If $\alpha, \beta \in A$, then $A?\alpha = A?\beta$. ([Base] Internal Exchange)

There are many ways of modifying semi-contraction and making it applicable to belief sets:

(a) $\mathbf{K}?\alpha = Cn((\mathbf{K} \cup \{\alpha\}) - \perp)$

(b) $\mathbf{K}?\alpha = Cn(\mathbf{K} \cup Cn(\alpha) - \perp)$

(c) $\mathbf{K}?\alpha = \begin{cases} Cn((\mathbf{K} \cup \{\alpha\}) - \perp) + \alpha & \text{If } \neg\alpha \notin Cn((\mathbf{K} \cup \{\alpha\}) - \perp) \\ Cn((\mathbf{K} \cup \{\alpha\}) - \perp) & \text{Otherwise} \end{cases}$

where $\mathbf{K}$ is a belief set and $-$ is a contraction operator for belief bases. The **(b)** model allows partial acceptance of the new information. The **(c)** model is a semi-revision without "Indifference": when the new sentence α and $\neg\alpha$ both disappear in the consolidation process, the new information is added. As in external revision, the properties and behaviour of these variants are not explored and constitute open problems.

Fuhrmann [Fuh97] extended semi-revision to inputs that are belief bases or belief sets:

$$A \circ B = (A \cup B) - \perp.$$

where $\circ$ is called *merge operation*. He provided the following representation theorem:

THEOREM 5.1.5 $\circ$ is an operation of partial meet merge if and only if it satisfies:

- $A \circ B \nvdash \bot$. (Strong Consistency)
- $A \circ B \subseteq A \cup B$. (Inclusion)
- If $\alpha \in A \cup B$ and $\alpha \notin A \circ B$, then there exists some E such that $A \circ B \subseteq E \subseteq A \cup B$, $E \nvdash \bot$, but $E \cup \{\alpha\} \vdash \bot$. (Relevance)
- If $(A \cup B) = (A' \cup B')$, then $A \circ B = A' \circ B'$. (Congruence)

Olsson proposed a coherent version of semi-revision, where coherence takes the place of consistency [Ols97]. Hansson and Wassermann proposed an operation that regains consistency only in a local part of the belief base, the part that is relevant for α and $\neg\alpha$ [HW98].

5.1.7 Selective Revision

In [FHss] we proposed a selective revision operator ?, that is defined by the equality $\mathbf{K}?\alpha = \mathbf{K}{*}f(\alpha)$, where $*$ is an AGM revision operator and f a function, typically with the property $\vdash \alpha \rightarrow f(\alpha)$. Selective revision is developed in **Chapter 6**.

Other revision functions related to selective revision are models of revision among different belief sets. There are different approaches based on distance between worlds. The first one was proposed by Lin [Lin96]. It is a generalization for multiple belief sets of the revision function proposed by Dalal in [Dal88], where distance is measured numerically. The other approaches were proposed independently by Rabinowicz [Rab95] and Schlechta [Schss]. Here the distance is based on the relation "w_1 is closer to w_2 than is w_3". The selective version of credibility-limited revision is presently under development.

5.1.8 Irrevocable Belief Revision

Krister Segerberg [Seg97] proposed a special kind of revision, where the new information receives an *irrevocable* status, i.e. the same status as tautologies. Revising by the negation of an irrevocable sentence produces the inconsistent belief set. Irrevocable sentences are represented by a second belief set $\mathbf{V}$. In [Seg98] the following axiomatic characterization is given:

Definition 5.1.6 [Seg98] Let $\mathbf{V}$, $\mathbf{K}$ be belief sets in $\mathcal{L}$. A pair $(\mathbf{V}, \mathbf{K})$ is called a *complex* if and only if $\mathbf{V} \subseteq \mathbf{K}$.

Definition 5.1.7 [Seg98] Let $(\mathbf{V}, \mathbf{K})$ be a complex in a language $\mathcal{L}$. $*$: $(\mathcal{K}, \mathcal{K}) \rightarrow (\mathcal{K}, \mathcal{K})$ is an *irrevocable revision*, where for all α, $(\mathbf{V},\ \mathbf{K}) * \alpha = (\mathbf{V}_\alpha, \mathbf{K}_\alpha)$ if and only if it satisfies:

($*$1) $(\mathbf{V},\ \mathbf{K}) * \alpha$ is always a complex.

($*$2) $\mathbf{K}_\alpha \vdash \alpha$.

($*$b) $\mathbf{V}_\alpha = Cn(\mathbf{V} \cup \{\alpha\})$.

($*$3) If $\mathbf{K} \nvdash \neg\alpha$, then $\mathbf{K}_\alpha = Cn(\mathbf{K} \cup \{\alpha\})$.

($*$4) If $\mathbf{V} \nvdash \neg\alpha$ then $\mathbf{K}_\alpha \neq \mathbf{K}_\perp$.

($*$5) If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}_\alpha = \mathbf{K}_\beta$.

($*$df) $\mathbf{K}_{(\alpha\vee\beta)} = \mathbf{K}_\alpha$ or $\mathbf{K}_{(\alpha\vee\beta)} = \mathbf{K}_\beta$ or $\mathbf{K}_{(\alpha\vee\beta)} = \mathbf{K}_\alpha \cap \mathbf{K}_\beta$.

($*$c) $((\mathbf{V},\ \mathbf{K}) * \alpha) * \beta = (\mathbf{V},\ \mathbf{K}) * (\alpha \wedge \beta)$.

In [Fer98c] we defined irrevocable belief revision in terms of *epistemic entrenchment*.

Definition 5.1.8 Let $\leq$ be an entrenchment relation on a belief set $\mathbf{K}$ that satisfies transitivity, dominance, conjunctiveness, minimality and:

- **(EE5i) i-maximality:** $\alpha \in \mathbf{V}$ if and only if $\top \leq \alpha$.

$*_\leq$ is an *irrevocable entrenchment based revision* if and only if $(\mathbf{V}, \mathbf{K}) *_\leq \alpha = (\mathbf{V}_\alpha, \mathbf{K}_\alpha)$ where

- $\beta \in \mathbf{V}_\alpha$ if and only if $Cn(\mathbf{V} \cup \{\alpha\})$.
- $\beta \in \mathbf{K}_\alpha$ if and only if $\neg\alpha \in \mathbf{V}$ or $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$.
- $\beta \leq_\alpha \gamma$ if and only if $\gamma \in Cn(\mathbf{V} \cup \{\alpha\})$ or, $\alpha \rightarrow \beta \leq \alpha \rightarrow \gamma$.

THEOREM 5.1.9 Let $(\mathbf{V}, \mathbf{K})$ be a complex, $\mathbf{K} \neq \mathbf{K}_\perp$. and $*$ an operation on $(\mathbf{V}, \mathbf{K})$. Then the following conditions are equivalent:

1. $*$ is an *irrevocable revision* defined as in **Definition 5.1.7**
2. $*$ is an *irrevocable entrenchment based revision* defined as in **Definition 5.1.8**.

5.2 Taxonomy based on how to revise

We can classify the different models of belief revision according to the process that the revision function involves. The origin of this taxonomy is due to Hansson [Han98d]:

Integrated Revision: It consists in revising (or updating) the belief set in one single step.

Examples of this model: AGM revision, Updating.

Decision + Revision: It consists in a first step where it is decided if the input α is fully accepted, partially accepted or rejected and, in a second step, if α is not rejected, in which the belief set is revised by α or by the chosen part of α.

Examples of this model: Screened Revision, Selective Revision.

Integrated Choice: It consists in choosing among the originally believed sentences and the input α in one integrated step.

Examples of this model: Credibility-limited revision based on Epistemic Entrenchment, Possible world approach of Credibility-limited revision, Schlechta and Rabinowicz revision.

Contraction + Expansion: In these models revising consists in first contracting the belief state and then expanding the new state by the new belief.

Examples of this model: AGM revision, internal revision of belief bases.

Expansion + Contraction: It consists in adding the new sentence to the corpus of belief and then contracting by the negation of the input sentence.

Example of this model: External revision.

Expansion + Consolidation: It consists in adding a new sentence to the corpus of belief and then regaining consistency.

Example of this model: Semi-Contraction.

5.3 Taxonomy based in the output of the revision

Non-Indifference Revision Functions **5.3.1** $\alpha \in \mathbf{K}?\alpha$ or $\neg\alpha \in \mathbf{K}?\alpha$.

Models that satisfy this property: AGM, Updating, Screened Revision, Credibility-limited revision, Irrevocable Belief Revision, Internal Revision.

All **5.3.2** $\alpha \in \mathbf{K}?\alpha$.

Models that satisfy this property: AGM revision, internal revision, updating.

All or Nothing **5.3.3** $\alpha \in \mathbf{K}?\alpha$ or $\mathbf{K}?\alpha = \mathbf{K}$.

Models that satisfy this property: Screened revision, Credibility-limited revision.

All or Inconsistency **5.3.4** $\alpha \in \mathbf{K}?\alpha$ or $\mathbf{K}?\alpha = \mathbf{K}_{\perp}$.

Models that satisfy this property: Irrevocable belief revision.

Proxy success **5.3.5** There is some β such that $\beta, \mathbf{K}{*}\alpha \vdash \beta, \vdash \alpha \rightarrow \beta$ such that $\mathbf{K}{*}\alpha = \mathbf{K}{*}\beta$.

Models that satisfy this property: Selective revision, Schlechta and Rabinowicz revision, Lin Revision.

All or Less **5.3.6** $\alpha \in \mathbf{K}?\alpha$ or $\mathbf{K}?\alpha \subseteq \mathbf{K}$.

Models that satisfy this property: Semi-revision.

The relationship between the different properties can be seen in the following picture:

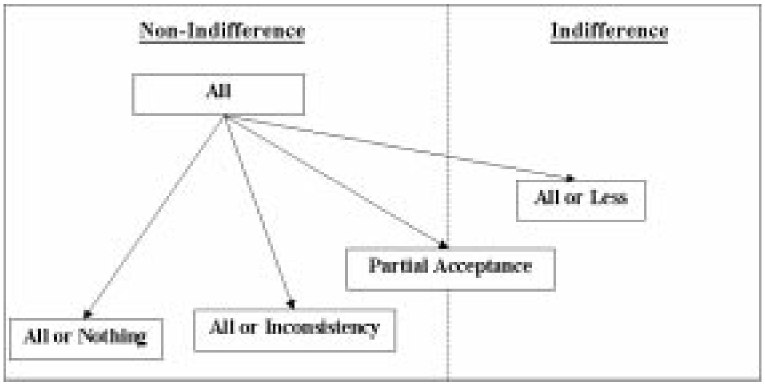


Figure 17

5.4 Proofs of Chapter 5

5.4.1 Lemmas

Lemma 5.4.1 Let $\leq_{\mathbf{K}}$ be an entrenchment ordering on a belief set $\mathbf{K}$ that satisfies (**EE1**) – (**EE4**) and (**EEi5**). Then $(\mathbf{V}, \mathbf{K})$ is a complex.

Proof: We must prove that: **(a)** $\mathbf{V}$ is a belief set:
Let $\delta \in Cn(\mathbf{V})$, we must prove that $\delta \in \mathbf{V}$. By compactness of the underlying logic there is a finite subset $\{\beta_1, ..., \beta_n\} \subseteq \mathbf{V}$, such that $\{\beta_1, ..., \beta_n\} \vdash \delta$.

Part 1. We first show that $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{V}$. For this purpose we are going to show that if $\beta_1 \in \mathbf{V}$ and $\beta_2 \in \mathbf{V}$ then $\beta_1 \wedge \beta_2 \in \mathbf{V}$. The rest follows by iteration of the same procedure. It follows from (**EE2**) that $\beta 1 \leq \beta_1 \wedge \beta_2$ or $\beta 2 \leq \beta_1 \wedge \beta_2$; then by (**EE1**) and (**EE5i**), $\beta_1 \wedge \beta_2 \in \mathbf{V}$.

Part 2. By repeated use of **Part 1**, we know that $\{\beta_1 \wedge ... \wedge \beta_n\} \in \mathbf{V}$. Since $\beta_1 \wedge ... \wedge \beta_n \vdash \delta$, by (**EE2**) $\beta_1 \wedge ... \wedge \beta_n \leq \delta$, hence by (**EE1**) and (**EEi5**), $\delta \in \mathbf{V}$.

(b) $\mathbf{V} \subseteq \mathbf{K}$: Let $\beta \in \mathbf{V}$, then by (**EEi5**) $\top \leq \beta$. If $\beta \leq \gamma$ for all γ it follows from (**EE1**) that $\top \leq \gamma$ for all γ, and since $\top \in \mathbf{K}$, then by (**EE4**), $\mathbf{K} = \mathbf{K}_\perp$. Hence $\beta \in \mathbf{K}$. If $\beta \leq \gamma$ for all γ is not satisfied, by (**EE4**), $\beta \in \mathbf{K}$. ■

Lemma 5.4.2 Let $\leq$ be an entrenchment relation on a belief set $\mathbf{K}$ that satisfies (**EE1**) – (**EE4**) and (**EEi5**). Let $*_\leq$ be defined as in **Definition 5.1.8**. Then:

(a) $(\mathbf{V}_\alpha, \mathbf{K}_\alpha)$ is a complex .

b [Rot91b] $\leq_\alpha$ satisfies (**EE1**) – (**EE4**)

c $\leq_\alpha$ satisfies (**EEi5**).

Proof:

(a) $(\mathbf{V}_\alpha, \mathbf{K}_\alpha)$ **is a complex:** We must prove the following cases:

$\mathbf{K}_\alpha$ **is a belief set:** This proof is quite similar to the proof of **Lemma 5.4.1.a**. Let $\delta \in Cn(\mathbf{K}_\alpha)$. By compactness of the underlying logic there is a finite subset $\{\beta_1, ..., \beta_n\} \subseteq \mathbf{K}_\alpha$, such that $\{\beta_1, ..., \beta_n\} \vdash \delta$. If $\alpha \vdash \perp$, then it follows trivially from the definition of $\mathbf{K}_\alpha$ that $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K}_\alpha$ and $\delta \in \mathbf{K}_\alpha$. Let $\alpha \not\vdash \perp$. Then:

Part 1. We show first that $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K}_\alpha$. For this purpose we are going to show that if $\beta_1 \in \mathbf{K}_\alpha$ and $\beta_2 \in \mathbf{K}_\alpha$ then $\beta_1 \wedge \beta_2 \in \mathbf{K}_\alpha$. The rest follows by iteration of the same procedure. It follows from $\beta_1 \in \mathbf{K}_\alpha$

by the definition of $\mathbf{K}_\alpha$ that $(\alpha \to \neg\beta_1) < (\alpha \to \beta_1)$. Then by **Property 2.5.36**, $\neg\alpha < (\alpha \to \beta_1)$. Then it follows from $\beta_2 \in \mathbf{K}_\alpha$ that $\neg\alpha < (\alpha \to \beta_2)$. By (**EE3**), either $(\alpha \to \beta_1) \leq ((\alpha \to \beta_1) \wedge (\alpha \to \beta_2))$ or $(\alpha \to \beta_1) \leq ((\alpha \to \beta_1) \wedge (\alpha \to \beta_2))$. In the same way, by **Property 2.5.37**, either $(\alpha \to \beta_1) \leq (\alpha \to (\beta_1 \wedge \beta_2))$ or $(\alpha \to \beta_2) \leq (\alpha \to (\beta_1 \wedge \beta_2))$. In the first case, we use (**EE1**) and $\neg\alpha < (\alpha \to \beta_1)$ to obtain $\neg\alpha < (\alpha \to (\beta_1 \wedge \beta_2))$ and in the second we use $\neg\alpha < (\alpha \to \beta_2)$ to obtain the same result. It follows that $\beta_1 \wedge \beta_2 \in \mathbf{K}_\alpha$.

Part 2. By repeated use of **Part 1.**, we know that $\{\beta_1 \wedge ... \wedge \beta_n\} \in \mathbf{K}_\alpha$. Let $\vdash \beta \leftrightarrow \beta_1 \wedge ... \wedge \beta_n$. We also have $\vdash \beta \to \delta$, then by the definition of $\mathbf{K}_\alpha$ $(\alpha \to \neg\beta) < (\alpha \to \beta)$. Since $\vdash (\alpha \to \beta) \to (\alpha \to \delta)$ and $\vdash (\alpha \to \neg\delta) \to (\alpha \to \neg\beta)$, (**EE2**) yields $(\alpha \to \beta) \leq (\alpha \to \delta)$ and $(\alpha \to \neg\delta) \leq (\alpha \to \neg\beta)$. We can apply (**EE1**) to $(\alpha \to \neg\delta) \leq (\alpha \to \neg\beta)$, $(\alpha \to \neg\beta) < (\alpha \to \beta)$ and $(\alpha \to \beta) \leq (\alpha \to \delta)$ to obtain $(\alpha \to \neg\delta) < (\alpha \to \delta)$. Hence by the definition of $\mathbf{K}_\alpha$, $\delta \in \mathbf{K}_\alpha$.

$\mathbf{V}_\alpha \subseteq \mathbf{K}_\alpha$: Let $\beta \in \mathbf{V}_\alpha$. by the definition of $\mathbf{V}_\alpha$, $\alpha \to \beta \in \mathbf{V}$. If $\alpha \to \neg\beta \in \mathbf{V}$, then $\neg\alpha \in \mathbf{V}$, hence $\mathbf{V}_\alpha \subseteq \mathbf{K}_\alpha = \mathbf{K}_\perp$. Let $\alpha \to \neg\beta \notin \mathbf{V}$, then $\alpha \to \neg\beta < \alpha \to \beta$, hence $\beta \in \mathbf{K}_\alpha$ from which it follows that $\mathbf{V}_\alpha \subseteq \mathbf{K}_\alpha$.

(b) (**EE1**) – (**EE4**) See [Rot91b].

(c) (**EEi5**) If $\beta \in Cn(\mathbf{V} \cup \{\alpha\})$ then $\alpha \to \beta \in \mathbf{V}$, then by (**EEi5**), for all γ $\alpha \to \alpha \leq \alpha \to \beta$, hence $\gamma \leq_\alpha \beta$ for all γ. If $\gamma \leq_\alpha \beta$ for all γ, then in particular $\alpha \leq_\alpha \beta$, $\alpha \to \alpha \leq \alpha \to \beta$.

It follows by (**EE1**) and (**EE2**) that, then $\alpha \rightarrow \beta \in \mathbf{V}$, hence $\beta \in Cn(\mathbf{V} \cup \{\alpha\})$. ■

5.4.2 Proofs

Proof of Theorem 5.1.9

(1.) to (2.)

(**EE1**) Let $\alpha \leq \beta$, $\beta \leq \gamma$ and $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\gamma)}$. We need prove $\gamma \in \mathbf{K}_{\neg(\alpha\wedge\gamma)}$.

(a) $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$: Then by $(C \leq_*)$, $\beta \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. By **(∗1)** $\alpha \wedge \beta \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. It follows by **(∗1)** that $\mathbf{K} \neq \mathbf{K}_{\perp}$ and by **(∗4)** $\alpha \wedge \beta \in \mathbf{V}$, then $\beta \in V$. **(∗1)** and **(∗b)** yield $\beta \in \mathbf{K}_{\neg(\beta\wedge\gamma)}$. Then by $(C \leq_*)$, $\gamma \in \mathbf{K}_{\neg(\beta\wedge\gamma)}$ and by the same reasoning we arrive at $\gamma \in \mathbf{V}$. Hence by **(∗1)** and **(∗b)**, $\gamma \in \mathbf{K}_{\neg(\alpha\wedge\gamma)}$.

(b) $\alpha \notin \mathbf{K}_{\neg(\alpha\wedge\beta)}$. Let $\gamma \notin \mathbf{K}_{\neg(\alpha\wedge\gamma)}$. Then by **(∗1)** and **(∗b)**, $\gamma \notin \mathbf{V}$, and it follows that $\beta \wedge \gamma \notin \mathbf{V}$, then by **(∗2)** and **(∗4)**, $\beta \wedge \gamma \notin \mathbf{K}_{\neg(\beta\wedge\gamma)}$. Since $\beta \leq \gamma$ and $(C \leq_*)$, $\beta \notin \mathbf{K}_{\neg(\beta\wedge\gamma)}$. We will arrive at an absurd by proving **(b1)** $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$ and **(b2)** $\alpha \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$:

(b1) Since $\vdash \neg(\alpha \wedge \beta \wedge \gamma) \leftrightarrow (\neg(\alpha \wedge \gamma) \vee (\alpha \wedge \neg\beta))$, it follows by **(∗df)** and **(∗5)** that $\mathbf{K}_{\neg(\alpha\wedge\gamma)} \cap \mathbf{K}_{(\alpha\wedge\neg\beta)} \subseteq \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$. By hypothesis $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\gamma)}$ and by **(∗1)** and **(∗2)** $\alpha \in \mathbf{K}_{(\alpha\wedge\neg\beta)}$; hence $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$.

(b2) Due to the hypothesis condition $\alpha \notin \mathbf{K}_{\neg(\alpha\wedge\beta)}$ it is enough to prove that $\mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)} \subseteq \mathbf{K}_{\neg(\alpha\wedge\beta)}$. Due to **(∗df)**, **(∗5)** and $\vdash \neg(\alpha \wedge \beta \wedge \gamma) \leftrightarrow (\neg(\alpha \wedge \beta) \vee \neg\gamma)$

it enough to prove that $\alpha \wedge \beta \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$. Since $\alpha \notin \mathbf{K}_{\neg(\alpha\wedge\beta)}$, then by **(∗1)** and **(∗b)** $\alpha \notin \mathbf{V}$ and consequently $(\alpha \wedge \beta \wedge \gamma) \notin \mathbf{V}$; then by **(∗4)** $(\alpha \wedge \beta \wedge \gamma) \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$. Then by **(∗1)** either $(\alpha \wedge \beta) \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$ or $\gamma \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$. In the first case we already have what we need. In the second case, it follows from **(∗1)** that $(\beta \wedge \gamma) \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$; then by **(∗df)** and **(∗5)** $\mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)} \subseteq \mathbf{K}_{\neg(\beta\wedge\gamma)}$ and since $\beta \notin \mathbf{K}_{\neg(\beta\wedge\gamma)}$, $\beta \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$, hence by **(∗1)** $(\alpha\wedge\beta) \notin \mathbf{K}_{\neg(\alpha\wedge\beta\wedge\gamma)}$ that concludes the proof.

(**EE2**) Let $\vdash \alpha \rightarrow \beta$, and $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. Then by **(∗1)** $\beta \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$; hence by $(C \leq_*)$ $\alpha \leq \beta$.

(**EE3**) We have three subcases:

(a) $\alpha \notin \mathbf{K}_{\neg(\alpha\wedge\beta)}$. Then by **(∗5)** $\alpha \notin \mathbf{K}_{\neg(\alpha\wedge(\alpha\wedge\beta))}$, hence by $(C \leq_*)$ $\alpha \leq (\alpha \wedge \beta)$.

(b) $\beta \notin \mathbf{K}_{\neg(\alpha\wedge\beta)}$. In the same way as **(a)**, $\beta \leq (\alpha \wedge \beta)$.

(c) $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$ and $\beta \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. Then by **(∗1)**, $(\alpha \wedge \beta) \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. Hence by $(C \leq_*)$, $\alpha \leq (\alpha\wedge\beta)$ and $\beta \leq (\alpha\wedge\beta)$.

(**EE4**) From left to right, let $\alpha \notin \mathbf{K}$. Then for all β by **(∗3)** $\mathbf{K}_{\neg(\alpha\wedge\beta)} = Cn(\mathbf{K} \cup \{\neg(\alpha \wedge \beta)\}$. Suppose that $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. Then $(\neg(\alpha \wedge \beta) \rightarrow \alpha) \in \mathbf{K}$, and since $\mathbf{K}$ is logically closed, $\alpha \in \mathbf{K}$. Contradiction, then for all β $\alpha \notin \mathbf{K}_{\neg(\alpha\wedge\beta)}$; hence by $(C \leq_*)$ for all β, $\alpha \leq \beta$. For the other direction let $\alpha \leq \beta$ for all β; then in particular $\alpha \leq \neg\alpha$. Then by $(C \leq_*)$ if $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\neg\alpha)}$ then $\neg\alpha \in \mathbf{K}_{\neg(\alpha\wedge\neg\alpha)}$. By **(∗3)**, since $\mathbf{K}$ is consistent, $\mathbf{K}_{\neg(\alpha\wedge\neg\alpha)} = \mathbf{K}$. Then if $\alpha \in \mathbf{K}$, then $\neg\alpha \in \mathbf{K}$.

Hence $\alpha \notin \mathbf{K}$.

(**EEi5**) For one direction, let $\beta \leq \alpha$ for all β. Then, in particular, $\top \leq \alpha$. Then by $(C \leq_*)$ "if $\top \in \mathbf{K}_{\neg(\alpha\wedge\top)}$ then $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\top)}$". Then by **(∗1)** $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\top)}$ that is equivalent by **(∗5)** to $\alpha \in \mathbf{K}_{\neg\alpha}$. Hence by **(∗2)** and **(∗4)** $\alpha \in \mathbf{V}$.

For the other direction, let $\alpha \in \mathbf{V}$, then by **(∗b)**, $\alpha \in \mathbf{V}_{\alpha\wedge\beta}$ for all β. Then by **(∗1)**, $\alpha \in \mathbf{K}_{\alpha\wedge\beta}$ for all β, hence by $(C \leq_*)$, $\beta \leq \alpha$ for all β.

$*$ **is an irrevocable entrenchment-based revision** We must prove that $\mathbf{V}_\alpha$, $\mathbf{K}_\alpha$ and $\leq_\alpha$ are as in **Definition 5.1.8**.

$\mathbf{V}_\alpha$**:** It follows directly from postulate **(∗b)**.

$\mathbf{K}_\alpha$**:** For the left to right direction, let $\beta \in \mathbf{K}_\alpha$ and $\neg\alpha \notin \mathbf{V}$, then by **(∗1)** $(\alpha \to \beta) \in \mathbf{K}_\alpha$ and by **(∗4)** and **(∗2)** $(\alpha \to \neg\beta) \notin \mathbf{K}_\alpha$. Then by **(∗5)** $(\alpha \to \beta) \in \mathbf{K}_{((\alpha\to\beta)\wedge(\alpha\to\neg\beta))}$ and $(\alpha \to \neg\beta) \notin \mathbf{K}_{((\alpha\to\beta)\wedge(\alpha\to\neg\beta))}$. Hence by $(C \leq_*)$, $(\alpha \to \neg\beta) \leq (\alpha \to \beta)$ and $(\alpha \to \beta) \not\leq (\alpha \to \neg\beta)$; i.e., $(\alpha \to \neg\beta) < (\alpha \to \beta)$. For the other direction if $\neg\alpha \in \mathbf{V}$. It follows by **(∗b)** that $\neg\alpha \in \mathbf{V}_\alpha$, and by **(∗1)** that $\neg\alpha \in \mathbf{K}_\alpha$, hence by **(∗2)**, $\beta \in \mathbf{K}_\alpha = \mathbf{K}_\perp$. Let $(\alpha \to \neg\beta) < (\alpha \to \beta)$. Then by $(C \leq_*)$ and **(∗5)** $(\alpha \to \beta) \in \mathbf{K}_\alpha$; hence by **(∗1)** and **(∗2)** $\beta \in \mathbf{K}_\alpha$.

$\leq_\alpha$**:** For one direction, let $\beta \leq_\alpha \gamma$. It follows by $(C \leq_*)$ that "if $\beta \in \mathbf{K}_{\alpha\ \neg(\beta\wedge\gamma)}$, then $\gamma \in \mathbf{K}_{\alpha\ \neg(\beta\wedge\gamma)}$". Then by **(∗c)** "if $\beta \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$, then $\gamma \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$". Due to **(∗1)** and **(∗2)**, it follows that "$\beta \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$ if and only if $\alpha \to \beta \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$" and "$\gamma \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$ if and only if $\alpha \to \gamma \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$". Then "if $\alpha \to \beta \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$

then $\alpha \to \gamma \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$". Since $\vdash (\alpha \wedge \neg(\beta \wedge \gamma)) \leftrightarrow (\neg((\alpha \to \beta) \wedge (\alpha \to \gamma)))$, hence by **(∗5)** and $(C \leq_*)$, we conclude that $\alpha \to \beta \leq \alpha \to \gamma$.

For the other direction, if $\gamma \in Cn(\mathbf{V} \cup \{\alpha\})$. Then by **(∗b)**, $\gamma \in \mathbf{V}_\alpha$. By **(∗b)** and **(∗1)**, $\gamma \in \mathbf{K}_{\alpha\ \neg(\beta\wedge\gamma)}$ for all β, hence by $(C \leq_*)$, $\beta \leq_\alpha \gamma$. If $\alpha \to \beta \leq \alpha \to \gamma$ then by $(C \leq_*)$ and **(∗5)**, "if $\alpha \to \beta \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$ then $\alpha \to \gamma \in \mathbf{K}_{\alpha\wedge\neg(\beta\wedge\gamma)}$". It follows by **(∗c)** that "if $\alpha \to \beta \in \mathbf{K}_{\alpha\ \neg(\beta\wedge\gamma)}$ then $\alpha \to \gamma \in \mathbf{K}_{\alpha\ \neg(\beta\wedge\gamma)}$". Then by **(∗1)** and **(∗2)**, "if $\beta \in \mathbf{K}_{\alpha\ \neg(\beta\wedge\gamma)}$ then $\gamma \in \mathbf{K}_{\alpha\ \neg(\beta\wedge\gamma)}$"; hence by $(C \leq_*)$, $\beta \leq_\alpha \gamma$. ∎

(2.) to (1.)

(∗1) See **Lemma 5.4.2.a**.

(∗2) If $\neg\alpha \in \mathbf{V}$, then it follows trivially from the definition of $\mathbf{K}_\alpha$ that $\alpha \in \mathbf{K}_\alpha$. Let $\neg\alpha \notin \mathbf{V}$. By (**EEi5**) $\neg\alpha < (\neg\alpha \vee \alpha)$ or equivalently by **Property 2.5.37** $(\alpha \to \neg\alpha) < (\alpha \to \alpha)$. Hence by the definition of $\mathbf{K}_\alpha$, $\alpha \in \mathbf{K}_\alpha$.

(∗b) It follows trivially from the definition of $\mathbf{V}_\alpha$.

(∗3) Let $\neg\alpha \notin \mathbf{K}$. We must prove that $\mathbf{K}_\alpha = Cn(\mathbf{K} \cup \{\alpha\})$. We will prove this identity by double inclusion. For the first direction let $\beta \in \mathbf{K}_\alpha$. We want to show that $\beta \in Cn(\mathbf{K} \cup \{\alpha\})$, which can be done by showing that $\alpha \to \beta \in \mathbf{K}$. by the definition of $\mathbf{K}_\alpha$, since $\beta \in \mathbf{K}_\alpha$, $(\alpha \to \neg\beta) < (\alpha \to \beta)$; hence by (**EE4**), $(\alpha \to \beta) \in \mathbf{K}$. For the other direction, let $\beta \in Cn(\mathbf{K} \cup \{\alpha\})$.

Then $\alpha \to \beta \in \mathbf{K}$. Due to $\neg\alpha \notin \mathbf{K}$, $Cn(\mathbf{K} \cup \{\alpha\}) \neq \mathbf{K}_\perp$, then $\neg\beta \notin Cn(\mathbf{K} \cup \{\alpha\})$, then $\alpha \to \neg\beta \notin \mathbf{K}$; and it follows by (**EE4**) that $(\alpha \to \neg\beta) < (\alpha \to \beta)$. Hence, by the definition of $\mathbf{K}_\alpha$, $\beta \in \mathbf{K}_\alpha$.

(∗4) Let $\mathbf{V} \nvdash \neg\alpha$ and assume that $\mathbf{K} = \mathbf{K}_\perp$. Then by the definition of $\mathbf{K}_\alpha$, $(\alpha \to \neg\perp) < (\alpha \to \perp)$. By **Property 2.5.37** $\top < \perp$. Contradiction by (**EE2**). Hence $\mathbf{K} \neq \mathbf{K}_\perp$.

(∗5) Let $\vdash \alpha \leftrightarrow \beta$. If $\neg\alpha \in \mathbf{V}$ then $\beta \in \mathbf{V}$, hence by the definition of $\mathbf{K}_\alpha$ and $\mathbf{K}_\beta$, $\mathbf{K}_\alpha = \mathbf{K}_\beta = \mathbf{K}_\perp$. By **Property 2.5.37** it follows for all γ that $(\alpha \to \neg\gamma) < (\alpha \to \gamma)$ if and only if $(\beta \to \neg\gamma) < (\beta \to \gamma)$; hence $\mathbf{K}_\alpha = \mathbf{K}_\beta$.

(∗df) If $\vdash \alpha$, then $\vdash (\alpha \vee \beta) \leftrightarrow \beta$ and the rest follows from the previous proof of **(∗5)**. Equivalently if $\vdash \beta$. Let $\nvdash \alpha$ and $\nvdash \beta$. We have three subcases[3]:

(a) $\neg\alpha < \neg\beta$. It follows from $\neg\alpha < \neg\beta$ and (**EE3**) that $\neg\alpha =_\leq (\neg\alpha \wedge \neg\beta)$. Then $\neg\alpha \notin \mathbf{V}$. We will prove that $\mathbf{K}_{(\alpha\vee\beta)} = \mathbf{K}_\alpha$. For one direction let $\delta \in \mathbf{K}_\alpha$. It follows from the definition of $\mathbf{K}_\alpha$ that $(\alpha \to \neg\delta) < (\alpha \to \delta)$. Then by **Property 2.5.36**, $\neg\alpha < (\alpha \to \delta)$. Since (**EE2**) yields $\neg\beta < (\beta \to \delta)$, we use (**EE1**) to obtain both $(\neg\alpha \wedge \neg\beta) < (\alpha \to \delta)$ and $(\neg\alpha \wedge \neg\beta) < (\beta \to \delta)$. (**EE2**) and (**EE3**) yield $(\neg\alpha \wedge \neg\beta) < ((\alpha \vee \beta) \to \delta)$. Hence $((\alpha \vee \beta) \to \neg\delta) < ((\alpha \vee \beta) \to \delta)$ from which it follows that $\delta \in \mathbf{K}_{(\alpha\vee\beta)}$.

For the other direction, let $\delta \in \mathbf{K}_{(\alpha\vee\beta)}$. It follows by

[3]We write $\alpha =_\leq \beta$ if and only if $\alpha \leq \beta$ and $\beta \leq \alpha$.

$\neg\alpha =_{\leq} (\neg\alpha \wedge \neg\beta)$ that $\not\vdash (\neg\alpha \wedge \neg\beta)$; then by the definition of $\mathbf{K}_\alpha$, $(\neg\alpha \wedge \neg\beta) < ((\alpha \vee \beta) \rightarrow \delta)$. By (**EE2**) $((\alpha \vee \beta) \rightarrow \delta) \leq (\alpha \rightarrow \delta)$. (**EE1**) yields $\neg\alpha < (\alpha \rightarrow \delta)$, hence $\alpha \rightarrow \neg\delta < \alpha \rightarrow \delta$ from which it follows that $\delta \in \mathbf{K}_\alpha$.

(b) $\neg\beta < \neg\alpha$: Similar to case **(a)**; $\mathbf{K}_{(\alpha\vee\beta)} = \mathbf{K}_\beta$.

(c) $\neg\alpha =_{\leq} \neg\beta$. Then $\neg\alpha =_{\leq} \neg\beta =_{\leq} (\neg\alpha\wedge\neg\beta)$ that implies that $\neg\alpha \in \mathbf{V}$ if and only if $\neg\beta \in \mathbf{V}$ if and only if $(\neg\alpha \wedge \neg\beta) \in \mathbf{V}$, hence if $\neg\alpha \in \mathbf{V}$, then $\mathbf{K}_{(\alpha\vee\beta)} = \mathbf{K}_\alpha = \mathbf{K}_\beta = \mathbf{K}_\perp$. Let $\neg\alpha \notin \mathbf{V}$. Then: $\delta \in \mathbf{K}_\alpha \cap \mathbf{K}_\beta$ if and only if (by the definition of $\mathbf{K}_\alpha$ and $\mathbf{K}_\beta$), $\neg\alpha < (\alpha \rightarrow \delta)$ and $\neg\beta < (\alpha \rightarrow \delta)$ if and only if (by (**EE1**)) $(\neg\alpha \wedge \neg\beta) < (\alpha \rightarrow \delta)$ and $(\neg\alpha \wedge \neg\beta) < (\alpha \rightarrow \delta)$ if and only if (by (**EE2**) and by (**EE3**)) $(\neg\alpha\wedge\neg\beta) < ((\alpha\vee\beta) \rightarrow \delta)$ if and only if (by the definition of $\mathbf{K}_{(\alpha\vee\beta)}$) $\delta \in \mathbf{K}_{(\alpha\vee\beta)}$.

(∗c) Let $((\mathbf{V}, \mathbf{K}) * \alpha) * \beta = (\mathbf{V}', \mathbf{K}')$. We will use double inclusion to prove this: For the first direction, let $\gamma \in \mathbf{K}'$. We have two cases:

(a) $\neg\beta \in \mathbf{V}_\alpha$. Then $\neg\beta \in Cn(\mathbf{V} \cup \{\alpha\})$, from which it follows that $\neg\alpha\vee\neg\beta \in \mathbf{V}$, then by the definition of $\mathbf{K}_{(\alpha\wedge\beta)}$, $\mathbf{K}_{(\alpha\wedge\beta)} = \mathbf{K}_\perp$, hence $\mathbf{K}' \subseteq \mathbf{K}_{(\alpha\wedge\beta)}$.

(b) $\neg\beta \notin \mathbf{V}_\alpha$. It follows by the definition of $\mathbf{K}'$ that $(\beta \rightarrow \neg\gamma) <_\alpha (\beta \rightarrow \gamma)$. Then by the definition of $\leq_\alpha$, $(\alpha \rightarrow (\beta \rightarrow \neg\gamma)) < (\alpha \rightarrow (\beta \rightarrow \gamma))$. By **Property 2.5.37** this is equivalent to $((\alpha \wedge \beta) \rightarrow \neg\gamma) <_\alpha ((\alpha \wedge \beta) \rightarrow \gamma)$; then by the definition of $\mathbf{K}_{(\alpha\wedge\beta)}$, $\gamma \in \mathbf{K}_{(\alpha\wedge\beta)}$; hence $\mathbf{K}' \subseteq \mathbf{K}_{(\alpha\wedge\beta)}$.

For the second direction let $\gamma \in \mathbf{K}_{(\alpha\wedge\beta)}$. We have two cases:

(c) $\neg(\alpha\wedge\beta) \in \mathbf{V}$. Then it follows that $\neg\beta \in \mathbf{V}_\alpha = Cn(\mathbf{V}\cup\{\alpha\})$, then by the definition of $\mathbf{K}'$, $\mathbf{K}' = \mathbf{K}_\perp$, hence $\mathbf{K}_{(\alpha\wedge\beta)} \subseteq \mathbf{K}'$.

(d) $\neg(\alpha \wedge \beta) \notin \mathbf{V}$. It follows by the definition of $\mathbf{K}_{(\alpha\wedge\beta)}$ that $((\alpha\wedge\beta) \to \neg\gamma) < ((\alpha\wedge\beta) \to \gamma)$, then by **Property 2.5.37**, $(\alpha \to (\beta \to \neg\gamma)) < (\alpha \to (\beta \to \gamma))$, then by the definition of $<_\alpha$, $\beta \to \neg\gamma <_\alpha \beta \to \gamma$. Hence $\gamma \in \mathbf{K}'$ and $\mathbf{K}_{(\alpha\wedge\beta)} \subseteq \mathbf{K}'$.

$(C \leq_*)$ Let $\alpha \leq \beta$ and $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. There are two subcases according to the definition of $\mathbf{K}_{\neg(\alpha\wedge\beta)}$: **(a)** $\neg(\alpha \wedge \beta) \in \mathbf{V}$: Hence $\beta \in \mathbf{K}_{\neg(\alpha\wedge\beta)} = \mathbf{K}_\perp$. (b) $\neg(\alpha \wedge \beta) \notin \mathbf{V}$: Then $(\neg(\alpha \wedge \beta) \to \neg\alpha) < (\neg(\alpha \wedge \beta) \to \alpha)$, then by **Property 2.5.37**, $(\beta \vee \neg\alpha) < \alpha$. By (**EE2**), $\beta \leq (\beta \vee \neg\alpha)$, it then follows by (**EE1**) that $\beta < \alpha$. We obtain a contradiction, hence the second case is not possible.

The other direction can be proved by showing that **(a)** if $\beta < \alpha$, then $\alpha \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$ and **(b)** if $\beta <_{\mathbf{K}} \alpha$, then $\beta \notin \mathbf{K}{*}\neg(\alpha \wedge \beta)$.

(a) We can to this by showing $\neg(\alpha \wedge \beta) \to \neg\alpha < \neg(\alpha \wedge \beta) \to \alpha$, or equivalently, $\beta \vee \neg\alpha < \alpha$. Suppose for *reductio* that this is not the case. Then $\alpha \leq \beta \vee \neg\alpha$. Since $\alpha \leq \alpha$, (**EE3**) yields $\alpha \leq \alpha \wedge (\beta \vee \neg\alpha)$, hence $\alpha \leq \alpha \wedge \beta$, so that by (**EE1**) $\alpha \leq \beta$, contrary to the conditions.

(b) Suppose to the contrary that $\beta < \alpha$ and $\beta \in \mathbf{K}_{\neg(\alpha\wedge\beta)}$. There are two cases: **(b1)** $\alpha \wedge \beta \in \mathbf{V}$. Then $\beta \in \mathbf{V}$, hence by (**EEi5**) $\alpha \leq \beta$, contrary to the conditions. **(b2)** $\neg(\alpha\wedge\beta) \to \neg\beta < \neg(\alpha\wedge\beta) \to \beta$, or equivalently by **Property 2.5.37**, to $\alpha\vee\neg\beta < \beta$, which is impossible since by (**EE2**), $\alpha \leq \alpha\vee\neg\beta$. This conclude the proof. ■

Chapter 6

Selective Revision

6.1 Introduction

In standard accounts of belief revision [AGM85, KM92, Rot93], the new information is always accepted. In **Chapter 5** we introduced several models of belief revision have been developed that allow for two options: either the new information is fully accepted or it is completely rejected. In this chapter we introduce a model that also allows for a third possibility: to accept parts of the new information and reject the rest of it.

The following example illustrates the practical relevance of this third option:

Example 6.1.1 One day when you return back from work, your son tells you, as soon as you see him: "A dinosaur has broken grandma's vase in the living-room". You probably accept one part of the information, namely that the vase has been broken, while rejecting the part of it that refers to a dinosaur.

In this chapter we introduce a selective revision operator $\circ$ that is defined by the equality $\mathbf{K}\circ\alpha = \mathbf{K}*f(\alpha)$, where $*$ is an AGM revision operator and f a function, typically with the property $\vdash \alpha \rightarrow f(\alpha)$.

The major parts of the result of this chapter appeared in:

[•] EDUARDO FERMÉ AND SVEN OVE HANSSON. Selective revision. *Studia Logica*, 1998. In press.

6.2 Postulates for Selective Revision

Four of the six AGM postulates are equally plausible for selective revision as for standard revision. These are *closure, inclusion, consistency*, and *extensionality*. The *vacuity* postulate is more debatable. It has been questioned even in a non-selective framework [KM92], and the reasons to do so are stronger in a framework for selective revision. It may be argued that even if the input sentence α does not logically contradict $\mathbf{K}$, there may be non-logical reasons for not accepting it completely, so that *vacuity* should not hold.

The *success* postulate should clearly not hold for selective revision, but it is of interest to investigate weakened versions of it. The following postulate, introduced in [Han97], ensures that an input is accepted if it is consistent with the original belief set:

- **Weak success [Han97]:** If $\mathbf{K} \nvdash \neg\alpha$, then $\mathbf{K}\circ\alpha \vdash \alpha$.

Weak success follows logically from *vacuity*.

Another way to weaken *success* is to require that revision should take the form of accepting and fully incorporating some part of the input information. That part then acts as a proxy for the input:

●**Proxy success:** There is a sentence β, such that $\mathbf{K}\circ\alpha \vdash \beta$, $\vdash \alpha \rightarrow \beta$, and $\mathbf{K}\circ\alpha = \mathbf{K}\circ\beta$

There is an obvious way to weaken this postulate:

●**Weak proxy success:** There is a sentence β, such that $\mathbf{K}\circ\alpha \vdash \beta$ and $\mathbf{K}\circ\alpha = \mathbf{K}\circ\beta$.

Proxy success and its weak variant are unusual among belief change postulates due to their existential nature. (On the use of existential conditions, see [vB91, p.16])

The following postulate captures the intuition that previous beliefs are given up only if this is required to avoid inconsistency.

●**Consistent expansion:** If $\mathbf{K} \not\subseteq \mathbf{K}\circ\alpha$ then $\mathbf{K} \cup (\mathbf{K}\circ\alpha) \vdash \bot$.

This postulate is a direct consequence of *success* and *vacuity*:

Observation 6.2.1 If $\circ$ satisfies *vacuity* and *success* then it satisfies *consistent expansion.*

Observation 6.2.2 *Consistent expansion* does not follow from *closure, inclusion, vacuity, consistency*, and *extensionality.*

As was pointed to us by an anonymous referee for our paper [FHss], *consistent expansion* is also a weakening of:

●**Tenacity:** If $\beta \in \mathbf{K} \setminus (\mathbf{K}\circ\alpha)$ then $\mathbf{K}\circ\alpha \vdash \neg\beta$.

that has been used as a characteristic postulate of maxichoice revision in AGM theory [AM82] [Gär88, pp.58-59] [Hanss].

6.3 Constructing Selective Revision

In this section we provide a constructive model for selective revision that makes use of the power of the AGM apparatus. We also provide the corresponding representation theorems.

Definition 6.3.1 Let $\mathbf{K}$ be a belief set, $*$ a *partial meet revision* for $\mathbf{K}$ and f a function from $\mathcal{L}$ to $\mathcal{L}$. The *selective revision* $\circ$, based on $*$ and f, is the operation such that for all sentences α:

$$\mathbf{K}{\circ}\alpha = \mathbf{K}{*}f(\alpha)$$

f is the *transformation function* on which $\circ$ is based.

The following is a list of properties that the transformation function may satisfy:

implication $\vdash \alpha \rightarrow f(\alpha)$.

weak implication If $\mathbf{K} \nvdash \neg\alpha$, then $\vdash \alpha \rightarrow f(\alpha)$.

idempotence $\vdash f(f(\alpha)) \leftrightarrow f(\alpha)$.

internalized negation $\vdash \neg f(\alpha) \rightarrow f(\neg\alpha)$.

externalized negation $\vdash f(\neg\alpha) \rightarrow \neg f(\alpha)$.

monotony If $\vdash \alpha \rightarrow \beta$ then $\vdash f(\alpha) \rightarrow f(\beta)$.

extensionality If $\vdash \alpha \leftrightarrow \beta$ then $\vdash f(\alpha) \leftrightarrow f(\beta)$.

consistency preservation If $\nvdash \neg\alpha$, then $\nvdash \neg f(\alpha)$.

consistency $\nvdash \neg f(\alpha)$.

disjunctive distribution $\vdash f(\alpha \vee \beta) \leftrightarrow f(\alpha) \vee f(\beta)$.

disjunctive factoring Either $\vdash f(\alpha \vee \beta) \leftrightarrow f(\alpha)$ or $\vdash f(\alpha \vee \beta) \leftrightarrow f(\beta)$ or $\vdash f(\alpha \vee \beta) \leftrightarrow f(\alpha) \vee f(\beta)$.

conjunctive distribution $\vdash f(\alpha \wedge \beta) \leftrightarrow f(\alpha) \wedge f(\beta)$.

maximality $\vdash f(\alpha) \leftrightarrow \alpha$.

weak maximality If $\mathbf{K} \nvdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow \alpha$.

disjunctive maximality Either $\vdash f(\alpha) \leftrightarrow \alpha$ or $\vdash f(\neg\alpha) \leftrightarrow \neg\alpha$.

Some interrelations among properties of the transformation function are listed in the following observation:

Observation 6.3.2

1. If f satisfies implication, then it satisfies internalized negation.
2. If f satisfies implication and externalized negation then it satisfies maximality.
3. If f satisfies extensionality and disjunctive distribution then it satisfies monotony.
4. If f satisfies weak maximality with respect to $\mathbf{K}$ and $\mathbf{K} \neq \mathbf{K}_{\perp}$, then f satisfies disjunctive maximality.
5. If $\mathbf{K} \nvdash \perp$ and $\mathbf{K} \notin \mathcal{L} \perp\perp$ (i.e., $\mathbf{K}$ is consistent but not a maximal consistent subset of the language) then f cannot satisfy

simultaneously monotony, consistency, and weak maximality with respect to $\mathbf{K}$.

The following two observations show how these properties of the transformation function give rise to properties of the selective revision function.

Observation 6.3.3 Let $\mathbf{K}$ be a belief set in a language $\mathcal{L}$, $*$ a revision operator for $\mathbf{K}$ that satisfies the six basic AGM postulates, and f a transformation function. Let $\circ$ be the selective revision function on $\mathbf{K}$ generated from $*$ and f. Then:

1. $\circ$ satisfies *closure* and *consistent expansion*.
2. If f satisfies extensionality then $\circ$ satisfies *extensionality*.
3. If f satisfies weak implication then $\circ$ satisfies *inclusion*.
4. If f satisfies weak maximality then $\circ$ satisfies *inclusion* and *vacuity*.
5. If f satisfies consistency preservation then $\circ$ satisfies *consistency*.
6. If f satisfies maximality then $\circ$ satisfies *success*.
7. If f satisfies implication then $\circ$ satisfies *consistency*.
8. If f satisfies idempotence, then $\circ$ satisfies *weak proxy success*.
9. If f satisfies idempotence and implication, then $\circ$ satisfies *proxy success*.

In the limiting case when f satisfies maximality, $\circ$ is a partial meet revision function.

It should be noted that *weak implication* and *weak maximality* differ from the rest of the listed properties of f by referring to a belief set $\mathbf{K}$. It is no surprise that these properties can be used to obtain the postulates of *inclusion* and *vacuity*, the only basic AGM revision postulates that refer to the belief set $\mathbf{K}$.

Observation 6.3.4 Let $\mathbf{K}$ be a belief set in a language $\mathcal{L}$, $*$ a revision operator for $\mathbf{K}$ that satisfies the eight basic and supplementary AGM postulates, and f a transformation function. Let $\circ$ be the selective revision function on $\mathbf{K}$ generated from $*$ and f. Then:

1. If f satisfies implication and conjunctive distribution, then $\circ$ satisfies *superexpansion*.
2. If f satisfies disjunctive distribution, then $\circ$ satisfies *disjunctive overlap*.
3. If f satisfies disjunctive factoring, then $\circ$ satisfies *disjunctive factoring*.
4. If f satisfies implication and disjunctive distribution, then $\circ$ satisfies *disjunctive inclusion*.

The following representation theorems have been obtained for three classes of selective revision functions.

THEOREM 6.3.5 Let $\mathcal{L}$ be a finite language, $\mathbf{K}$ a belief set in $\mathcal{L}$ and $\circ$ an operator on $\mathbf{K}$. Then the following conditions are equivalent:

1. $\circ$ satisfies *closure, inclusion, vacuity, consistency, extensionality*, and *consistent expansion.*
2. There exists a revision function $*$ for $\mathbf{K}$ that satisfies the six basic AGM postulates, and a transformation function f that satisfies extensionality, consistency preservation, and weak maximality, such that $\mathbf{K}\circ\alpha = \mathbf{K}*f(\alpha)$ for all α.

THEOREM 6.3.6 Let $\mathcal{L}$ be a finite language, $\mathbf{K}$ a belief set in $\mathcal{L}$ and $\circ$ an operator on $\mathbf{K}$. Then the following conditions are equivalent:

1. $\circ$ satisfies *closure, inclusion, vacuity, consistency, extensionality, consistent expansion*, and *weak proxy success.*

2. There exists a revision function $*$ for $\mathbf{K}$ that satisfies the six basic AGM postulates, and a transformation function f that satisfies extensionality, consistency preservation, weak maximality, and idempotence, such that $\mathbf{K}\circ\alpha = \mathbf{K}*f(\alpha)$ for all α.

THEOREM 6.3.7 Let $\mathcal{L}$ be a finite language, $\mathbf{K}$ a belief set in $\mathcal{L}$ and $\circ$ an operator on $\mathbf{K}$. Then the following two conditions are equivalent:

1. $\circ$ satisfies *closure, inclusion, vacuity, consistency, extensionality, consistent expansion*, and *proxy success.*

2. There exists a revision function $*$ for $\mathbf{K}$ that satisfies the six basic AGM postulates, and a transformation function f that satisfies extensionality, consistency preservation, weak maximality, idempotence, and implication, such that $\mathbf{K}\circ\alpha = \mathbf{K}*f(\alpha)$ for all α.

The 1. to 2. direction of **Theorem 6.3.5** can be proved with a construction such that, in the principal case when α is inconsistent with $\mathbf{K}$, $f(\alpha)$ is taken to be equivalent with the whole of $\mathbf{K}\circ\alpha$ and $\mathbf{K}*\alpha$ is taken to be equivalent with α itself. Refinements of this construction can be used for **Theorems 6.3.6** and **6.3.7**. For details, the reader is referred to the **Section 6.4**.

The postulates for selective revision referred to in **Theorems 6.3.5-6.3.7** are, in addition to five of the six AGM postulates: *consistent expansion* that follows form the AGM postulates *vacuity* and *success*; and *weak proxy success* and *proxy success* that both follow from *success*. Hence, these operations of selective revision are weakened variants of AGM revision. The representation theorems indicate that these constructions provide a fairly faithful extension of the AGM framework to allow for less than total acceptance of new information.

6.4 Proofs of Chapter 6

Proof of Observation 6.2.1. Let $\mathbf{K} \not\subseteq \mathbf{K}\circ\alpha$. It follows from *vacuity* that $\mathbf{K} \vdash \neg\alpha$ and from *success* that $\mathbf{K}\circ\alpha \vdash \alpha$. Hence, $\mathbf{K} \cup (\mathbf{K}*\alpha) \vdash \bot$.

Proof of Observation 6.2.2. [The idea for this proof was provided by David Makinson.]

$$\text{Let } \mathbf{K}{*}\alpha = \begin{cases} \mathbf{K}{+}\alpha & \text{if } \mathbf{K} \nvdash \neg\alpha \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

It is trivial to prove that $*$ satisfies *closure, inclusion, vacuity, consistency*, and *extensionality*. However it does not satisfies *consistent expansion*: Let α be such that $\nvdash \alpha$ and $\nvdash \neg\alpha$ and let $\mathbf{K} = Cn(\{\neg\alpha\})$. Then $\mathbf{K}{*}\alpha = Cn(\emptyset)$, so $\mathbf{K} \not\subseteq \mathbf{K}{*}\alpha$ but $\mathbf{K} \cup (\mathbf{K}{*}\alpha) = Cn(\{\neg\alpha\}) \cup Cn(\emptyset) \nvdash \bot$.

Proof of Observation 6.3.2.

1. It follows from implication that $\vdash \alpha \rightarrow f(\alpha)$ and $\vdash \neg\alpha \rightarrow f(\neg\alpha)$. From this it follows truth-functionally that $\vdash \neg f(\alpha) \rightarrow f(\neg\alpha)$.

2. From implication we obtain $\vdash \neg\alpha \rightarrow f(\neg\alpha)$ and from externalized negation $\vdash f(\neg\alpha) \rightarrow \neg f(\alpha)$. Hence $\vdash \neg\alpha \rightarrow \neg f(\alpha)$ or equivalently $\vdash f(\alpha) \rightarrow \alpha$. Since implication also yields $\vdash \alpha \rightarrow f(\alpha)$ we can conclude that $\vdash f(\alpha) \leftrightarrow \alpha$.

3. Let $\vdash \alpha \rightarrow \beta$. Then $\vdash \beta \leftrightarrow \alpha \vee \beta$, and it follows from extensionality that $\vdash f(\alpha \vee \beta) \leftrightarrow f(\beta)$. We can combine this with $\vdash f(\alpha) \rightarrow f(\alpha \vee \beta)$, that follows from disjunctive distribution, and obtain $\vdash f(\alpha) \rightarrow f(\beta)$.

4. It follows from $\mathbf{K} \neq \mathbf{K}_\perp$ that $\mathbf{K} \nvdash \alpha$ or $\mathbf{K} \nvdash \neg\alpha$, so by weak maximality $\vdash f(\alpha) \leftrightarrow \alpha$ or $\vdash f(\neg\alpha) \leftrightarrow \neg\alpha$.

5. Since $\mathbf{K} \in \mathcal{L} \perp\perp$, there is some β such that $\mathbf{K} \nvdash \beta$ and $\mathbf{K} \nvdash \neg\beta$. It follows from monotony that $\vdash f(\beta \wedge \neg\beta) \to f(\beta)$ and $\vdash f(\beta \wedge \neg\beta) \to f(\neg\beta)$, and from weak maximality that $\vdash f(\beta) \leftrightarrow \beta$ and $\vdash f(\neg\beta) \leftrightarrow \neg\beta$. Hence $\vdash f(\beta \wedge \neg\beta) \to (\beta \wedge \neg\beta)$, which contradicts consistency.

Proof of Observation 6.3.3.

1. Trivial, since by Observations 2.5.19 and 6.2.1 $*$ satisfies *closure* and *consistent expansion*.

2. Let $\vdash \alpha \leftrightarrow \beta$. Then, by f-extensionality $\vdash f(\alpha) \leftrightarrow f(\beta)$, and by $*$-*extensionality* $\mathbf{K}{*}f(\alpha) = \mathbf{K}{*}f(\beta)$, or equivalently $\mathbf{K}{\circ}\alpha = \mathbf{K}{\circ}\beta$.

3. We prove by cases:

(a) $\mathbf{K} \vdash \neg\alpha$

then $\mathbf{K}{+}\alpha = \mathbf{K}_{\perp}$, so that $\mathbf{K}{\circ}\alpha \subseteq \mathbf{K}{+}\alpha$

(b) $\mathbf{K} \nvdash \neg\alpha$. Then $\mathbf{K}{\circ}\alpha = \mathbf{K}{*}f(\alpha)$

$\mathbf{K}{*}f(\alpha) \subseteq \mathbf{K}{+}f(\alpha)$ ($*$-*inclusion*)

$\mathbf{K}{+}f(\alpha) \subseteq \mathbf{K}{+}\alpha$ (weak implication)

hence $\mathbf{K}{\circ}\alpha \subseteq \mathbf{K}{+}\alpha$.

4. *Inclusion* follows from part **3** since weak maximallity implies weak implication. For *vacuity*, suppose that $\mathbf{K} \nvdash \neg\alpha$. Then by weak maximality, $\vdash \alpha \leftrightarrow f(\alpha)$ so that $\mathbf{K}{\circ}\alpha = \mathbf{K}{*}\alpha$, and by $*$-*vacuity* $\mathbf{K}{+}\alpha \subseteq \mathbf{K}{\circ}\alpha$.

5. Suppose that $\nvdash \neg\alpha$. Then by consistency preservation $\nvdash \neg f(\alpha)$, hence by $*$-*consistency* $\mathbf{K}{*}f(\alpha) \nvdash \perp$, hence $\mathbf{K}{\circ}\alpha \nvdash \perp$.

6. Trivial, since by definition $*$ satisfies *success* and by maximality $\mathbf{K}\circ\alpha = \mathbf{K}*\alpha$.

7. From part 5, since implication implies consistency preservation.

8. By Definition 6.3.1 and idempotence $\mathbf{K}\circ\alpha = \mathbf{K}*f(\alpha) = \mathbf{K}*f(f(\alpha)) = \mathbf{K}\circ f(\alpha)$. Since $\mathbf{K}*f(\alpha) \vdash f(\alpha)$ we therefore have $\mathbf{K}\circ\alpha = \mathbf{K}\circ f(\alpha)$ and $\mathbf{K}\circ\alpha \vdash f(\alpha)$, which is sufficient to prove that $\circ$ satisfies *proxy success*.

9. This follows from the proof of part 8 since f satisfies implication.

Proof of Observation 6.3.4.

1. $\mathbf{K}\circ(\alpha \wedge \beta) = \mathbf{K}*f(\alpha \wedge \beta)$ (definition of $\circ$)
$= \mathbf{K}*(f(\alpha) \wedge f(\beta))$ (conjunctive distribution and $*$-*extensionality*)
$\subseteq (\mathbf{K}*f(\alpha)) + f(\beta)$ (since $*$ satisfies *superexpansion*)
$\subseteq (\mathbf{K}*f(\alpha)) + \beta$ (implication)
$= (\mathbf{K}\circ\alpha) + \beta$.

2. $(\mathbf{K}\circ\alpha) \cap (\mathbf{K}\circ\beta) = (\mathbf{K}*f(\alpha)) \cap (\mathbf{K}*f(\beta))$
$\subseteq \mathbf{K}*(f(\alpha) \vee f(\beta))$ (*disjuntive overlap* for $*$)
$= \mathbf{K}*f(\alpha \vee \beta)$ (disjunctive distribution and $*$-*extensionality*)
$=\mathbf{K}\circ(\alpha \vee \beta)$.

3. By f-disjunctive factoring and $*$-*extensionality*

$$\mathbf{K}{*}f(\alpha \vee \beta) = \begin{cases} \mathbf{K}{*}f(\alpha), \text{ or} \\ \mathbf{K}{*}f(\beta), \text{ or} \\ \mathbf{K}{*}(f(\alpha) \vee f(\beta)) \end{cases}$$

By observation 2.4.15, $*$ satisfies *disjunctive factoring*,

$$\text{thus } \mathbf{K}{*}(f(\alpha) \vee f(\beta)) = \begin{cases} \mathbf{K}{*}f(\alpha), \text{ or} \\ \mathbf{K}{*}f(\beta), \text{ or} \\ \mathbf{K}{*}f(\alpha) \cap \mathbf{K}{*}f(\beta) \end{cases}$$

4. Let $\mathbf{K}{\circ}(\alpha \vee \beta) \nvdash \neg\alpha$. Then:

$\mathbf{K}{\circ}(\alpha \vee \beta) \nvdash \neg f(\alpha)$(implication)

then $\mathbf{K}{*}(f(\alpha) \vee f(\beta)) \nvdash \neg f(\alpha)$ (disjunctive distribution, $*$-*extensionality*)

$\mathbf{K}{*}(f(\alpha) \vee f(\beta)) \subseteq \mathbf{K}{*}f(\alpha)$ ($*$-*disjunctive inclusion*)

$\mathbf{K}{*}f(\alpha \vee \beta) \subseteq \mathbf{K}{*}f(\alpha)$(disjunctive distribution)

$\mathbf{K}{\circ}(\alpha \vee \beta) \subseteq \mathbf{K}{\circ}\alpha$(definition of $\circ$).

Proof of Theorem 6.3.5. **1. implies 2.:** We first define f and $*$: Let e be any function such that for any two sentences α and β if $\vdash \alpha \leftrightarrow \beta$ then $e(\alpha) = e(\beta)$ and $\vdash \alpha \leftrightarrow e(\alpha)$.

$$f(\alpha) = \begin{cases} e(\alpha) & \text{if } \mathbf{K} \nvdash \neg\alpha \\ e(\&(\mathbf{K}{\circ}\alpha)) & \text{otherwise} \end{cases}$$

$$\mathbf{K}{*}\beta = \begin{cases} \mathbf{K}{+}\beta & \text{if } \mathbf{K} \nvdash \neg\beta \\ Cn(\{\beta\}) & \text{otherwise} \end{cases}$$

We need to show **(a)** that f satisfies the properties, **(b)** that $*$ is a partial meet revision and **(c)** that $\mathbf{K}\circ\alpha = \mathbf{K}*f(\alpha)$ for all α.

(a) It follows directly that f satisfies extensionality (since $\circ$ satisfies *extensionality*) and weak maximality. To show that it satisfies consistency preservation we need to consider the two clauses of the definition of f. First, if $\mathbf{K} \nvdash \neg\alpha$, then α is consistent. Secondly, since $\circ$ satisfies *consistency*, $\&(\mathbf{K}\circ\alpha)$ is consistent if $\nvdash \neg\alpha$.

(b) To show that $*$ is a partial meet revision, we need to prove that it satisfies the six AGM postulates. It follows directly from the definition that *closure, success, inclusion, vacuity*, and *extensionality* are satisfied. To show that it satisfies *consistency*, let $\nvdash \neg\beta$. If $\mathbf{K} \nvdash \neg\beta$ then it follows from our definition of $*$ that $\mathbf{K}*\beta = \mathbf{K}+\beta$, and hence $\mathbf{K}*\beta$ is consistent. If $\mathbf{K} \vdash \neg\beta$, then $\mathbf{K}*\beta = Cn(\{\beta\})$, and since $\nvdash \neg\beta$, $\mathbf{K}*\beta$ is consistent.

(c) Finally, we need to prove that $\mathbf{K}\circ\alpha = \mathbf{K}*f(\alpha)$. There are two major cases, according to whether or not $\mathbf{K}$ implies $\neg\alpha$:

(c1) If $\mathbf{K} \nvdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow \alpha$ so that $\mathbf{K} \nvdash \neg f(\alpha)$. Hence $\mathbf{K}*f(\alpha) = \mathbf{K}+f(\alpha) = \mathbf{K}+\alpha$ and since $\circ$ satisfies vacuity $\mathbf{K}\circ\alpha = \mathbf{K}+\alpha$.

(c2) If $\mathbf{K} \vdash \neg\alpha$, then $f(\alpha) = e(\&(\mathbf{K}\circ\alpha))$. We have two subcases. First, if $\mathbf{K} \vdash \neg f(\alpha)$ then $\mathbf{K}*f(\alpha) = Cn(\{f(\alpha)\}) = \mathbf{K}\circ\alpha$. Seccondly if $\mathbf{K} \nvdash \neg f(\alpha)$ or equivalently $\mathbf{K} \nvdash \neg e(\&(\mathbf{K}\circ\alpha))$, then we can use *consistent expansion* to obtain $\mathbf{K} \subseteq \mathbf{K}\circ\alpha$, hence $\mathbf{K}*f(\alpha) = \mathbf{K}+f(\alpha) = \mathbf{K}+e(\&(\mathbf{K}\circ\alpha)) = \mathbf{K}\circ\alpha$.

2. implies 1.: This direction of the proof follows from **Observation 6.3.3**.

Proof of Theorem 6.3.6. 1. implies 2.: We first define f and $*$:

$$f(\alpha) = \begin{cases} \alpha & \text{if } \mathbf{K} \nvdash \neg\alpha \\ r(\alpha) & \text{otherwise, where } r \text{ is a function such that for} \\ & \text{all } \alpha \text{ and } \alpha',\ r(\alpha) = r(\alpha'),\ \mathbf{K}{\circ}\alpha \vdash r(\alpha), \text{ and} \\ & \mathbf{K}{\circ}\alpha = \mathbf{K}{\circ}r(\alpha). \end{cases}$$

This definition is possible since $\circ$ satisfies *weak proxy success.*

$$\mathbf{K}{*}\beta = \begin{cases} \mathbf{K}{\circ}\beta & \text{if } \mathbf{K}{\circ}\beta \vdash \beta \\ \mathbf{K} *' \beta & \text{otherwise, where } *' \text{ is any operation that satisfies} \\ & \quad \text{the six basic AGM postulates.} \end{cases}$$

We need to show **(a)** that f satisfies the properties, **(b)** that $*$ is a partial meet revision and **(c)** that $\mathbf{K}{\circ}\alpha = \mathbf{K}{*}f(\alpha)$ for all α.

(a) That f satisfies weak maximality follows directly from the definition of f. To show that f satisfies consistency preservation let $\nvdash \neg\alpha$: If $\mathbf{K} \nvdash \neg\alpha$, then $f(\alpha) = \alpha$ is consistent. If $\mathbf{K} \vdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow r(\alpha)$ and $\mathbf{K}{\circ}\alpha \vdash r(\alpha)$. Since α is consistent so is $\mathbf{K}{\circ}\alpha$, thus $r(\alpha)$ is consistent, hence $f(\alpha)$ is consistent. To show that f satisfies extensionality, let $\vdash \alpha \leftrightarrow \gamma$: If $\mathbf{K} \nvdash \neg\alpha$, then $\mathbf{K} \nvdash \neg\gamma$, and we have $f(\alpha) = \alpha$ and $f(\gamma) = \gamma$, hence $\vdash f(\alpha) \leftrightarrow f(\gamma)$. If $\mathbf{K} \vdash \neg\alpha$, then $f(\alpha) = r(\alpha)$. Since $\mathbf{K} \vdash \neg\gamma$ we also have $f(\gamma) = r(\gamma)$.

By $\circ$-*extensionality* $\mathbf{K}\circ\alpha = \mathbf{K}\circ\gamma$, from which follows that $r(\alpha) = r(\gamma)$, hence $f(\alpha) = f(\gamma)$. Finally we show that f satisfies idempotence. If $\mathbf{K} \nvdash \neg\alpha$ then $f(f(\alpha)) = f(\alpha)$ follows directly. Let $\mathbf{K} \vdash \neg\alpha$ then $f(\alpha) = r(\alpha)$. If $\mathbf{K} \nvdash \neg r(\alpha)$, then $f(f(\alpha)) = r(\alpha)$. If $\mathbf{K} \nvdash \neg r(\alpha)$, then $f(f(\alpha)) = r(r(\alpha))$. By definition $\mathbf{K}\circ r(r(\alpha)) = \mathbf{K}\circ r(a)$, from which it follows that $r(\alpha) = r(r(\alpha))$. Hence $f(f(\alpha)) = r(\alpha) = f(\alpha)$.

(b) That $*$ satisfies *closure, inclusion, vacuity, extensionality* and *consistency* is trivial, since $\circ$ and $*'$ both satisfy these five postulates. That $*$ satisfies *success* also follows directly from the definition.

(c) We need to prove that $\mathbf{K}\circ\alpha = \mathbf{K}{*}f(\alpha)$. If $\mathbf{K} \nvdash \neg\alpha$, then $f(\alpha) = \alpha$ and $\mathbf{K}\circ f(\alpha) = \mathbf{K}\circ\alpha$ follows directly. By $\circ$-*vacuity* $\mathbf{K}\circ f(\alpha) \vdash f(\alpha)$. By the definition of $*$, $\mathbf{K}{*}f(\alpha) = \mathbf{K}\circ f(\alpha)$. Hence $\mathbf{K}{*}f(\alpha) = \mathbf{K}\circ\alpha$.

If $\mathbf{K} \vdash \neg\alpha$, then it follows from the definitions of f and r, and $\circ$-*extensionality* that $\mathbf{K}\circ\alpha \vdash f(\alpha)$ and $\mathbf{K}\circ f(\alpha) = \mathbf{K}\circ\alpha$.

Hence $\mathbf{K}{*}f(\alpha) = \mathbf{K}\circ\alpha$.

2. implies 1.: This part of the proof follows from **Observation 6.3.3**.

Proof of Theorem 6.3.7. This proof is quite similar to that of Theorem 6.3.6. To show **1. implies 2.**, we define f to be a function such that for all α, $\mathbf{K}\circ\alpha \vdash f(\alpha)$, $\vdash \alpha \to f(\alpha)$, and $\mathbf{K}\circ\alpha = \mathbf{K}\circ f(\alpha)$ and that if $\vdash \alpha \leftrightarrow \alpha'$, then $f(\alpha) = f(\alpha')$. The existence of such function follows from *proxy success*. The proof that f satisfies extensionality, consistency preservation, weak maximality, and idempotence are essentially the same, and the implication property follows trivially. To show that **2. implies 1.** we only have to add a proof of proxy success. This follows from Observation 6.3.3.

Part IV

Credibility-Limited Functions

Chapter 7

Credibility-Limited Revision

In **Chapter 5** we presented models of non-prioritized belief revision, and in **Chapter 6**, we introduced Selective Revision. In this chapter we introduce Credibility-Limited revision, a revision model where the new information must to reach our limit of credibility to be accepted. The following example can be used to illustrate our intuitions:

Example 7.0.1

1. Marco tells me: “Today I have lunch with my father”. I believe him.
2. Elías tells me: “Today I have lunch with the King Gustav”. I don’t believe him.

In the first item, we are disposed to accept the new information, but in the second case, our reaction is to reject it. The reason is that in the second case, the new belief exceed our *Credibility Limit* of tolerance to new information. The fact that King Gustav has lunch with Elías is “too distant” from our corpus of beliefs. We will introduce five types of construction of Credibility-Limited revision. In **Section 7.1** postulates for credibility-limited revision are proposed. In **Section 7.2**, five types for constructions for credibility-limited revision are proposed, and in **Section 7.3** they are axiomatically characterized.

The material of this chapter was appeared in:

[•] Sven Ove Hansson, Eduardo Fermé, John Cantwell, and Marcelo Falappa. Credibility-limited revision. 1998. (manuscript).

7.1 Postulates and their interrelations

In **Section 2.4** we presented the AGM postulate. Our general approach will be to give up the *success* postulate while retaining as much as possible of the other AGM postulates. The following are useful weakenings of the *success* postulate:

- **Relative success:** $\alpha \in \mathbf{K}\circ\alpha$ or $\mathbf{K}\circ\alpha = \mathbf{K}$.
- **Disjunctive success:** $\alpha \in \mathbf{K}\circ\alpha$ or $\neg\alpha \in \mathbf{K}\circ\alpha$.
- **Strict improvement:** If $\alpha \in \mathbf{K}\circ\alpha$ and $\alpha \rightarrow \beta$, then $\beta \in \mathbf{K}\circ\beta$.
- **Regularity:** If $\beta \in \mathbf{K}\circ\alpha$ then $\beta \in \mathbf{K}\circ\beta$.
- **Strong regularity:** If $\neg\beta \notin \mathbf{K}\circ\alpha$ then $\beta \in \mathbf{K}\circ\beta$.

Intuitively, we may call a sentence α credible, relative to a belief set $\mathbf{K}$ and a revision operator $\circ$ for $\mathbf{K}$ if and only if $\alpha \in \mathbf{K}\circ\alpha$. Under this interpretation, *strict improvement* says that credibility is preserved under logical weakening, *regularity* that the resulting new belief state contains only credible sentences and *strong regularity* that it contains all sentences with incredible negations.

The *consistency* postulate of AGM requires $\mathbf{K}\circ\alpha$ to be consistent only when α is consistent. In credibility-limited belief revision, *success is relaxed*, and it is therefore natural to consider the following stronger consistency postulate:

- **Strong consistency:** $\mathbf{K}\circ\alpha \neq \mathbf{K}_\perp$.

We will also have use for the following consistency postulates:

- **Consistency preservation [Makss]:** If $\mathbf{K} \neq \mathbf{K}_\perp$ then $\mathbf{K}\circ\alpha \neq \mathbf{K}_\perp$.

- **Weak consistency preservation [KM92]:** If $\mathbf{K} \neq \mathbf{K}_\perp$ and $\nvdash \neg\alpha$, then $\mathbf{K}\circ\alpha \neq \mathbf{K}_\perp$.

The following two postulate will also turn out to be useful:

- **Disjunctive constancy:** If $\mathbf{K}\circ\alpha = \mathbf{K}\circ\beta = \mathbf{K}$ then $\mathbf{K}\circ(\alpha \vee \beta) = \mathbf{K}$.

- **Consistent expansion [FHss]:** If $\mathbf{K} \nsubseteq \mathbf{K}\circ\alpha$ then $\mathbf{K} \cup (\mathbf{K}\circ\alpha) \vdash \perp$.

Disjunctive Constancy follows from *vacuity, success*, and *consistency. Consistent expansion* follows from *vacuity* and *relative success.*

Subexpansion is a fairly plausible property for conventional (prioritized) belief revision, but it is much less so for non-prioritized revision. This can be seen from examples such that $\beta \notin \mathbf{K}$, $\neg\beta \notin \mathbf{K}$, $\alpha \notin \mathbf{K}\circ\alpha$, and $\alpha \wedge \beta \notin \mathbf{K}\circ(\alpha \wedge \beta)$. (For instance, let α be denote that there is a living dinosaur in Australia and β that there is a living tree in Australia that existed at the time of the dinosaurs.) Such a pattern cannot simultaneously *subexpansion*, *relative success*, and *closure.* (From $\alpha \notin \mathbf{K}\circ\alpha$ and *relative* *success* follows $\mathbf{K}\circ\alpha = \mathbf{K}$. Since $\neg\beta \notin \mathbf{K}$ we then have $\neg\beta \notin \mathbf{K}\circ\alpha$, and due to *closure*, $\mathbf{K}\circ\alpha \nvdash \neg\beta$, so that by *subexpansion* $\beta \in \mathbf{K}\circ(\alpha \wedge \beta)$. Since $\beta \notin \mathbf{K}$ it follows from *relative success* that $\alpha \wedge \beta \in \mathbf{K} \circ (\alpha \wedge \beta)$, contrary to the conditions.) This problem can be avoided if we replace subexpansion by the following variant, that is equivalent with subexpansion whenever success holds.

●**Guarded subexpansion:** If $\alpha \in \mathbf{K}\circ\alpha$ and $\mathbf{K}\circ\alpha \nvdash \neg\beta$, then $(\mathbf{K}\circ\alpha)+\beta \subseteq \mathbf{K}\circ(\alpha \wedge \beta)$.

The above-mentioned relationships in AGM theory between supplementary postulates and their equivalents (*subexpansion, superexpansion, disjunctive overlap, disjunctive inclusion*, and *disjunctive factoring*) depend on the *success* postulate. They can, however, be reconstructed without that postulate, provided that subexpansion is replaced by *guarded subexpansion*.

Observation 7.1.1 Let $\circ$ be an operation on a belief set $\mathbf{K}$.

1. If $\circ$ satisfies *closure, extensionality, superexpansion, disjunctive success* and *strict improvement* then it satisfies *disjunctive overlap.*

2. If $\circ$ satisfies *closure, vacuity, extensionality, disjunctive success, strict improvement* and *disjunctive overlap* then it satisfies *superexpansion.*

3. If $\circ$ satisfies *extensionality, disjunctive success, strict improvement* and *guarded subexpansion* then it satisfies *disjunctive inclusion.*

4. If $\circ$ satisfies *closure, vacuity, extensionality, disjunctive success* and *disjunctive inclusion* then it satisfies *guarded subexpansion.*

5. If $\circ$ satisfies *disjunctive factoring* then it satisfies *guarded subexpansion.*

6. If $\circ$ satisfies *closure, vacuity, extensionality, disjunctive success* and *disjunctive factoring* then it satisfies *guarded subexpansion.*

7. If $\circ$ satisfies *closure, vacuity, consistency, extensionality, disjunctive success, strict improvement, disjunctive overlap* and *disjunctive inclusion* then it satisfies *disjunctive factoring.*

Corollary 7.1.2 Let $\circ$ be an operation on a belief set $\mathbf{K}$ that satisfies *closure, vacuity, consistency, extensionality, strict improvement* and *relative success* . Then it satisfies:

1. *Superexpansion* if and only if it satisfies *disjunctive overlap.*

2. *Guarded subexpansion* if and only if it satisfies *disjunctive inclusion.*

3. *Disjunctive factoring* if and only if satisfies both *superexpansion* and *guarded subexpansion*

7.2 Constructions

In this section, we are going to introduce five constructions credibility-limited revision on belief sets. The first of these is the most general one. Its basic assumption is simply that some inputs are accepted, whereas others are not. Those that are accepted form the set $\mathcal{C}$ of credible sentences.

Definition 7.2.1 Let $\mathbf{K}$ be a logically closed set of sentences. The operation $\circ$ on $\mathbf{K}$ is a credibility-limited revision on $\mathbf{K}$ if and only if there is an AGM revision $*$ on $\mathbf{K}$ (satisfying the six basic postulates) and a set $\mathcal{C}$ of sentences such that for all sentences α:

$$\mathbf{K}\circ\alpha = \begin{cases} \mathbf{K}\circ\alpha & \text{if } \alpha \in \mathcal{C} \\ \mathbf{K} & \text{otherwise} \end{cases}$$

Note that if $\mathcal{C} \subseteq \mathbf{K} \neq \mathbf{K}_{\perp}$, then $\mathbf{K}\circ\alpha = \mathbf{K}$ for all α.

The following are some plausible conditions on $\mathcal{C}$:

●**Closure under Logical Equivalence:**

If $\vdash \alpha \leftrightarrow \beta$ and $\alpha \in \mathcal{C}$, then $\beta \in \mathcal{C}$.

●**Single sentence closure:** If $\alpha \in \mathcal{C}$, then $Cn(\{\alpha\}) \subseteq \mathcal{C}$.

●**Disjunctive completeness:** If $\alpha \vee \beta \in \mathcal{C}$, then either $\alpha \in \mathcal{C}$ or $\beta \in \mathcal{C}$.

●**Negation completeness:** $\alpha \in \mathcal{C}$ or $\neg\alpha \in \mathcal{C}$.

●**Element consistency:** If $\alpha \in \mathcal{C}$, then $\alpha \nvdash \perp$.

●**Expansive credibility:** If $\mathbf{K} \nvdash \alpha$, then $\neg\alpha \in \mathcal{C}$.

●**Outcome credibility:** If $\alpha \in \mathcal{C}$, then $\mathbf{K}\circ\alpha \subseteq \mathcal{C}$.

The generalization of single sentence closure to full logical closure ($Cn(\mathcal{C}) \subseteq \mathcal{C}$) is patently unreasonable; each of α and β may be credible without $\alpha \wedge \beta$ being so (an obvious example is to let $\beta = \neg\alpha$).

Observation 7.2.2 1. Single sentence closure implies closure under logical equivalence.

2. If C satisfies single sentence consequence and $\bot \in \mathcal{C}$, then $\mathbf{K}\circ\alpha = \mathbf{K}$ for all α.

3. If $\mathcal{C} \neq \emptyset$ and $\mathcal{C}$ satisfies single sentence closure and disjunctive completeness, then it satisfies negation completeness.

4. Dijunctive completeness does not follows from negation completeness and single sentence closure.

Our second construction is a modified version of David Makinson's screened revision [Makss]. Makinson made use of a set A of potential core beliefs that are immune to revision. The belief set $\mathbf{K}$ should be revised by the input sentence α if α is consistent with the set $A \cap \mathbf{K}$ of actual core beliefs, otherwise not. In our version, we have replaced $A \cap \mathbf{K}$ by a set A of core beliefs. For expository convenience we will present this construction as a special case of **Definition 7.2.1**, with the set A of core beliefs as the determinant of whether or not a sentence α is a member of the set $\mathcal{C}$ of credible sentences:

Definition 7.2.3 Let $\circ$ be a credibility-limited revision operator for $\mathbf{K}$, based on $*$ and $\mathcal{C}$. Then it is:

1. an operator of *core beliefs revision* if and only if there is a set $A \subseteq \mathcal{L}$ such that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$.

2. an operator of *consistent core beliefs revision* if and only if there is a consistent set $A \subseteq \mathcal{L}$ such that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$.

3. an operator of *endorsed core beliefs revision* if and only if there is a set $A \subseteq \mathbf{K}$ such that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$.

If $\mathbf{K}$ is consistent, then all endorsed core beliefs revisions are also consistent core beliefs revisions.

Our third construction is a modification of epistemic entrenchment. In **Subsection 2.5.2** we saw that is possible to define entrenchment-based revision from an entrenchment ordering [LR91, Rot91a]:

($*_{EBR}1$)**:** $\beta \in \mathbf{K}*\alpha$ if and only if either $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$ or $\alpha \vdash \bot$.

Given the standard properties of the entrenchment relation, this is equivalent with:

($*_{EBR}2$)**:** $\beta \in \mathbf{K}*\alpha$ if and only if either $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$ or $\neg\alpha$ is maximally entrenched.

To construct non-prioritized entrenchment-based revision, we can make use of (**EE1**) – (**EE4**) but give up (**EE5**)) (maximality). Furthermore, we can use the following variant of ($*_{EBR}2$):

Definition 7.2.4 $\circ$ is an entrenchment-based non-prioritized revision operator based on $\leq$ if and only if:

($\circ_{EBR}$)**:** $\beta \in \mathbf{K}*\alpha$ if and only if either $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}}$ $(\alpha \rightarrow \beta)$ or $\beta \in \mathbf{K}$ and $\neg\alpha$ is maximally entrenched.

The added condition$\beta \in \mathbf{K}$ is needed to assure that *(strong) consistency* is given priority over *success*.

Our fourth construction makes use of the one-to-one correspondence that persists between propositions (sets of possible worlds) and belief sets. In a propositional approach, operations of belief change are performed on the set $\|\mathbf{K}\|$ of possible worlds. Indeed, the standard AGM revision operator (partial meet revision) of $\mathbf{K}$ by α corresponds to the selection of a subset of $\|\alpha\|$ that is non-empty if $\|\alpha\|$ is non-empty and equal to $\|\mathbf{K}\| \cap \|\alpha\|$ if $\|\mathbf{K}\| \cap \|\alpha\|$ is non-empty. [Hanss, Gro88]. We propose to distinguish between credible and incredible worlds, and to require that the latter never be included in an outcome proposition. Again, it is convenient to introduce the new construction as a special case of credibility-limited revision.

Definition 7.2.5 Let $\circ$ be a credibility-limited revision operator for the belief set $\mathbf{K}$, based on $\mathcal{C}$. Then $\circ$ is:

1. an operator of credible worlds revision if and only if there is a set $\mathcal{W}_{\mathcal{C}}$ of possible worlds such that: $\alpha \in \mathcal{C}$ if and only if there is some $w \in \mathcal{W}_{\mathcal{C}}$ such that $\alpha \in w$.

2. an operator of non-empty credible worlds revision if and only if this holds for a set $\mathcal{W}_{\mathcal{C}} \neq \emptyset$ of possible worlds.

3. an operator of endorsed credible worlds revision if and only if this holds for a set $\mathcal{W}_{\mathcal{C}}$ such that $\|\mathbf{K}\| \subseteq \mathcal{W}_{\mathcal{C}}$.

If $\mathbf{K}$ is consistent, then all endorsed credible worlds revisions are non-empty credible worlds revisions. Two plausible additional conditions should be mentioned that relate $\mathcal{W}_{\mathcal{C}}$ to the outcome of the operation:

- **Outcome credibility :** $\|\mathbf{K}\circ\alpha\| \cap \mathcal{W}_{\mathcal{C}} \neq \emptyset$.
- **Strong outcome credibility:** $\|\mathbf{K}\circ\alpha\| \subseteq \mathcal{W}_{\mathcal{C}}$.

Our fifth and last model is a variant of the previous one. Grove's sphere-based operations make use of the simple intuition that the outcome of revising $\|\mathbf{K}\|$ by $\|\alpha\|$ consists of those elements of $\|\alpha\|$ that are as close as possible to $\|\mathbf{K}\|$. For that purpose, $\|\mathbf{K}\|$ can be thought of as surrounded by a system of concentric spheres [Gro88]. Each sphere represents a degree of closeness or similarity to $\|\mathbf{K}\|$. The outcome of revising $\|\mathbf{K}\|$ by $\|\alpha\|$ should be the intersection of $\|\alpha\|$ with the narrowest sphere around $\|\mathbf{K}\|$ that has a non-empty intersection with $\|\alpha\|$. The equivalence of this construction with the full set of (basic and supplementary) AGM postulates is a standard result in AGM theory. [Gär88].

Our modification consists in relaxing the standard requirements on sphere systems, so that not all possible worlds are elements of any sphere; i.e., by relaxing \$6 of **Definition 2.6.15**:

Definition 7.2.6 \$ is a *credibility-limited system of spheres around* $Th(\cap\$)$ if and only if it satisfies conditions \$1 – \$5 of **Definition 2.6.15**.

Definition 7.2.7 Let \$ be a credibility-limited system of spheres around **K**. The operator $\circ$ is a credibility-limited revision operator for \$ if and only if it satisfies:

$$\mathbf{K}\circ\alpha = \begin{cases} \bigcap Th(\|\alpha\| \cap \mathbf{S}_\alpha) & \text{if } \|\alpha\| \cap \mathbf{S}_\alpha \neq \emptyset \\ \mathbf{K} & \text{otherwise} \end{cases}$$

7.3 Representation theorems

This section reports a series of representation results through which the postulates of **Section 7.1** and the constructions of **Section 7.2** are closely knit together. **Theorem 7.3.1** provides the starting-point, characterizing essentially those credibility-limited revisions that are available within an extensional framework. **Theorem 7.3.3** exhibits some one-to-one correspondences between additional revision postulates and additional properties of the set $\mathcal{C}$ of credible sentences. **Theorems 7.3.4-7.3.9** provide us with a series of axiomatically characterized constructions of increasing strength. The major results of this section are summarized in **Figure 18**.

THEOREM 7.3.1 Let $\mathbf{K}$ be a consistent and logically closed set and $\circ$ an operation on $\mathbf{K}$. Then the following three conditions are equivalent:

1. $\circ$ satisfies *closure, relative success, inclusion, weak consistency preservation, consistent expansion*, and *extensionality*.

2. There is an AGM revision operator $*$ for $\mathbf{K}$ and a set $\mathcal{C} \subseteq \mathcal{L}$ that is closed under logical equivalence, and such that $\circ$ is the credibility-limited revision induced by $*$ and $\mathcal{C}$.

3 There is an AGM revision operator $*$ for $\mathbf{K}$ and a set $\mathcal{C} \subseteq \mathcal{L}$ that satisfies $\mathbf{K} \subseteq \mathcal{C}$ and is closed under logical equivalence, and such that $\circ$ is the credibility-limited revision induced by $*$ and $\mathcal{C}$.

It follows from **Theorem 7.3.1** that the condition $\mathbf{K} \subseteq \mathcal{C}$ has no effects on the properties of the operator $\circ$. The reason for this should be clear from the following observation:

Observation 7.3.2 Let $\mathbf{K}$ be a consistent and logically closed set of sentences and $*$ an AGM revision on $\mathbf{K}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two sets of sentences. Let $\circ_1$ be the credibility-limited revision based on $\mathcal{C}_1$ and $*$, and $\circ_2$ that based on $\mathcal{C}_2$ and $*$. Then:

If $\mathcal{C}_1 \setminus \mathbf{K} = \mathcal{C}_2 \setminus \mathbf{K}$, then $\mathbf{K} \circ_1 \alpha = \mathbf{K} \circ_2 \alpha$ for all α.

THEOREM 7.3.3 Let $\mathbf{K}$ be a consistent and logically closed set and $\circ$ an operation on $\mathbf{K}$. Then the following pairs of conditions are equivalent:

1. $\circ$ satisfies *closure, relative success, inclusion, weak consistency preservation, consistent expansion, extensionality* and

(a) Strict improvement: If $\alpha \in \mathbf{K}\circ\alpha$ and $\alpha \to \beta$, then $\beta \in \mathbf{K}\circ\beta$.

(b) Disjunctive constancy: If $\mathbf{K}\circ\alpha = \mathbf{K}\circ\beta = \mathbf{K}$ then $\mathbf{K}\circ(\alpha \vee \beta) = \mathbf{K}$.

(c) Disjunctive success: $\alpha \in \mathbf{K}\circ\alpha$ or $\neg\alpha \in \mathbf{K}\circ\alpha$.

(d) Strong consistency: $\mathbf{K}\circ\alpha \neq \mathbf{K}_\perp$.

(e) Vacuity: If $\mathbf{K} \nvdash \neg\alpha$, then $\mathbf{K}+\alpha \subseteq \mathbf{K}\circ\alpha$.

2. There is an AGM revision operator $*$ for $\mathbf{K}$ and a set $\mathcal{C} \subseteq \mathcal{L}$ that is closed under logical equivalence and satisfies

(a) Single sentence closure: If $\alpha \in \mathcal{C}$, then $Cn(\{\alpha\}) \subseteq \mathcal{C}$.

(b) Disjunctive completeness: If $\alpha \vee \beta \in \mathcal{C}$, then either $\alpha \in \mathcal{C}$ or $\beta \in \mathcal{C}$.

(c) Negation completeness: $\alpha \in \mathcal{C}$ or $\neg\alpha \in \mathcal{C}$.

(d) Element consistency: If $\alpha \in \mathcal{C}$, then $\alpha \nvdash \perp$.

(e) Expansive credibility: If $\mathbf{K} \nvdash \alpha$, then $\neg\alpha \in \mathcal{C}$.

THEOREM 7.3.4 Let $\mathbf{K}$ be a consistent and logically closed set and $\circ$ an operation on $\mathbf{K}$. Then the following four conditions are equivalent:

0. $\circ$ satisfies *closure, relative success, inclusion, strong consistency, consistent expansion, extensionality, strict improvement*, and *disjunctive constancy.*

1. There is an AGM revision operator $*$ for $\mathbf{K}$ and a set $\mathcal{C} \subseteq \mathcal{L}$ that is closed under logical equivalence, and satisfies single sentence closure, disjunctive completeness and element consistency, and such that $\circ$ is the credibility-limited revision induced by $*$ and $\mathcal{C}$.

2 It is a core beliefs revision.

3 It is a credible worlds revision.

(*Weak consistency preservation* could be redundantly added to the list of postulates in this theorem, since it follows from *strong consistency*.)

THEOREM 7.3.5 Let $\mathbf{K}$ be a consistent and logically closed set and $\circ$ an operation on $\mathbf{K}$. Then the following four conditions are equivalent:

0. $\circ$ satisfies *closure, relative success, inclusion, strong consistency, consistent expansion, extensionality, strict improvement, disjunctive constancy*, and *disjunctive success*.

1. There is an AGM revision operator $*$ for $\mathbf{K}$ and a set $\mathcal{C} \subseteq \mathcal{L}$ that is closed under logical equivalence, and satisfies single sentence closure, disjunctive completeness, element consistency and negation completeness, and such that $\circ$ is the credibility-limited revision induced by $*$ and $\mathcal{C}$.

2 It is a consistent core beliefs revision.

3 It is a non-empty credible worlds revision.

THEOREM 7.3.6 Let $\mathbf{K}$ be a consistent and logically closed set and $\circ$ an operation on $\mathbf{K}$. Then the following three conditions are equivalent:

0 $\circ$ satisfies *closure, relative success, inclusion, strong consistency, extensionality, strict improvement, disjunctive constancy* and *vacuity*.

1 There is an AGM revision operator for **K** and a set $\mathcal{C} \subseteq \mathcal{L}$ that is closed under logical equivalence, and satisfies single sentence closure, disjunctive completeness, element consistency, and expansive credibility, and such that $\circ$ is the credibility-limited revision induced by $*$ and $\mathcal{C}$.

2 It is an endorsed core beliefs revision.

3 It is an endorsed credible worlds revision.

(*Consistent expansion* and *disjunctive success* could, redundantly, have been added to the list of postulates in this theorem. The former follows from *relative success* and *vacuity*, and the latter from *vacuity* alone.)

THEOREM 7.3.7 Let **K** be a consistent and logically closed set and $\circ$ an endorsed credible worlds revision on **K**. Then the following two conditions are equivalent:

1. $\circ$ satisfies *strong regularity*.

2. $\circ$ is an endorsed core beliefs revision that satisfies strong outcome credibility.

THEOREM 7.3.8 Let **K** be a consistent and logically closed set and $\circ$ an endorsed credible worlds revision on **K**. Then the following three conditions are equivalent:

1. $\circ$ satisfies *regularity*.

2. $\circ$ is an endorsed credible worlds revision satisfying outcome credibility.

3. $\circ$ satisfies: If $\alpha \in \mathcal{C}$, then $\mathbf{K} \circ \alpha \subseteq \mathcal{C}$.

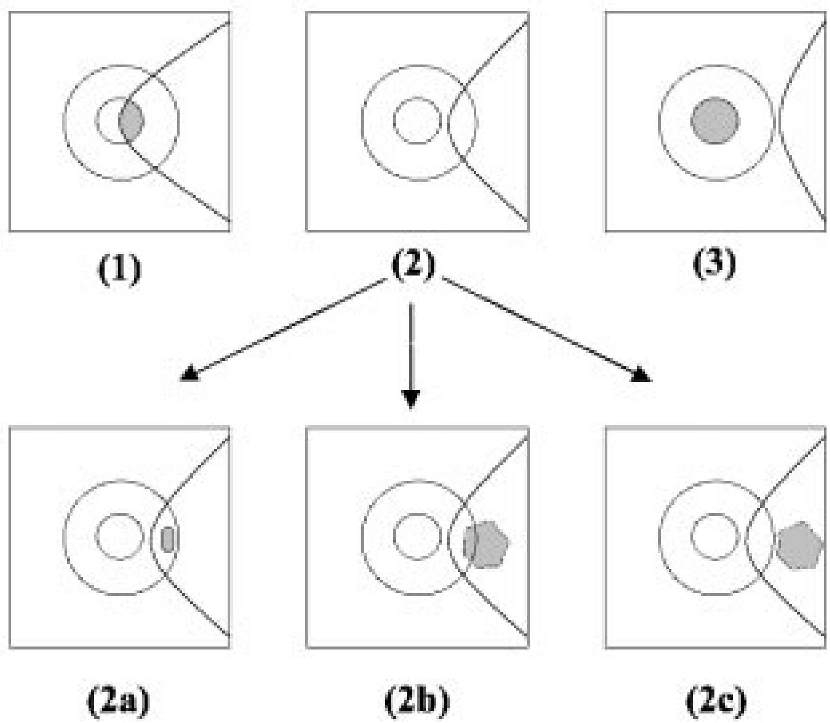


Figure 18

Theorems 7.3.6, **7.3.7** and **7.3.8** are diagrammatically summarized in **Figure 19**. In endorsed credible worlds revision, the set of credible worlds is a superset of the set $\|\mathbf{K}\|$ of worlds compatible with the belief set. If $\|\alpha\|$ intersects with $\|\mathbf{K}\|$, then the outcome of revision is equal to the belief set corresponding to $\|\mathbf{K}\| \cap \|\alpha\|$, see **(1)** in the figure. If $\|\alpha\|$ does not intersect with $\mathcal{W}_{\mathcal{C}}$, as in **(3)**, then the outcome is $\|\mathbf{K}\|$. In the intermediate case, when $\|\alpha\|$ intersects with $\mathcal{W}_{\mathcal{C}}$ but not with $\|\mathbf{K}\|$, the outcome may be a proposition that either **(2a)** consists only of credible worlds, **(2b)** consists in part of credible and in part of incredible worlds, or **(2c)** consists only of incredible worlds. A good case can be made that **(2c)**, and perhaps also **(2b)**, should be excluded. **Regularity** corresponds exactly to the exclusion of case **(2c)** and *strong regularity* to the exclusion of both cases **(2b)** and case **(2c)**.

THEOREM 7.3.9 Let **K** be a consistent and logically closed set and $\circ$ an operator on **K**. Then the following three conditions are equivalent:

0. $\circ$ satisfies *closure, relative success, inclusion, strong consistency, extensionality, strict improvement, vacuity, strong regularity*, and *disjunctive factoring.*

1. $\circ$ is an entrenchment-based non-prioritized revision in the sense of **Definition 7.2.4** based on an entrenchment relation $\leq$ on **K** that satisfies properties (**EE1**) – (**EE4**).

2. $\circ$ is a sphere-based revision operator around **K** in the sense of **Definition clss**.

(Disjunctive constancy can redundantly be added to the list of postulates in this theorem, since it follows from disjunctive factoring.)

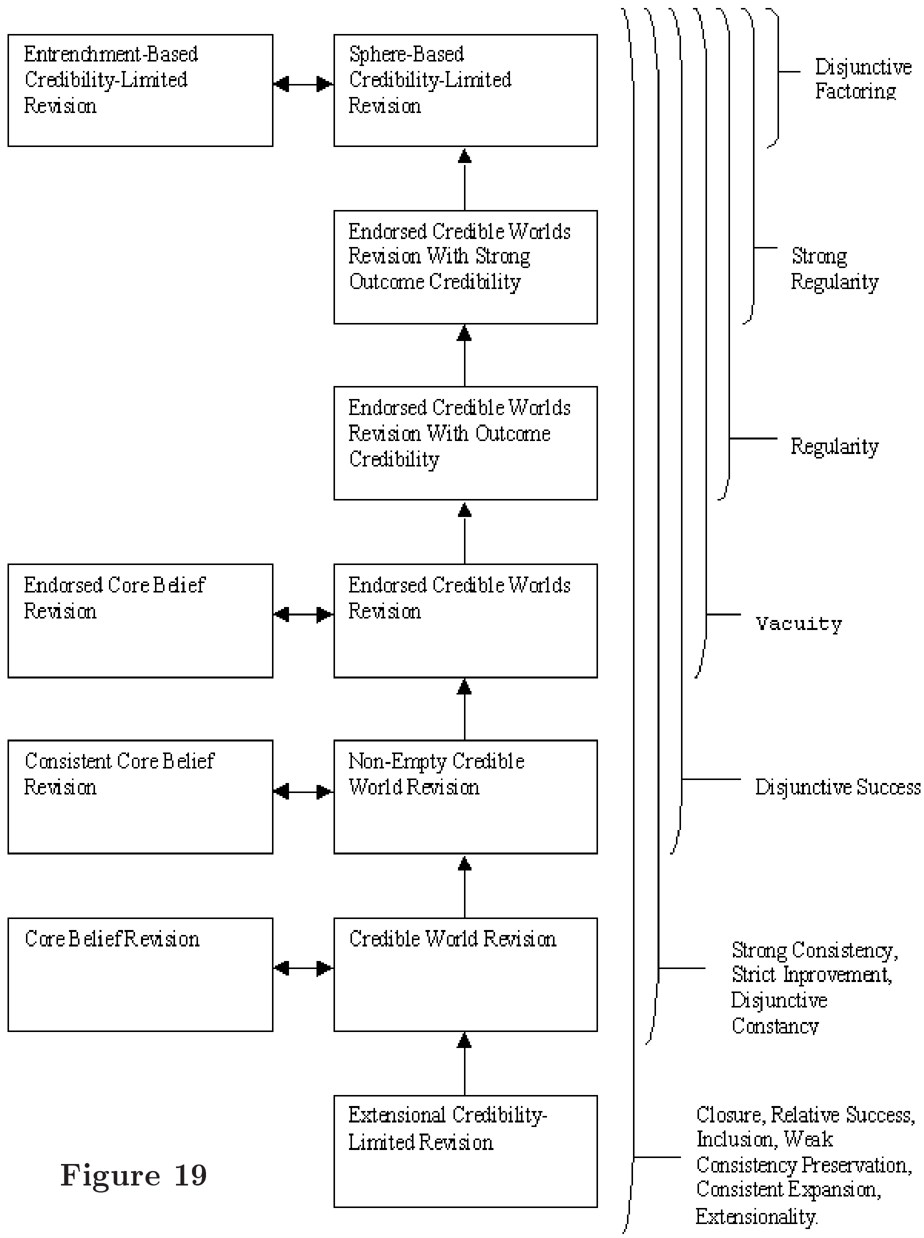

Entrenchment-Based Credibility-Limited Revision
Sphere-Based Credibility-Limited Revision
Endorsed Credible Worlds Revision With Strong Outcome Credibility
Endorsed Credible Worlds Revision With Outcome Credibility
Endorsed Core Belief Revision
Endorsed Credible Worlds Revision
Consistent Core Belief Revision
Non-Empty Credible World Revision
Core Belief Revision
Credible World Revision
Extensional Credibility-Limited Revision
Disjunctive Factoring
Strong Regularity
Regularity
Vacuity
Disjunctive Success
Strong Consistency, Strict Improvement, Disjunctive Constancy
Closure, Relative Success, Inclusion, Weak Consistency Preservation, Consistent Expansion, Extensionality.

Figure 19

7.4 Proofs of the Chapter

7.4.1 Lemmas

Lemma 7.4.1 Let $\mathbf{K}$ be a consistent and logically closed set and $\circ$ an endorsed credible worlds revision on $\mathbf{K}$, based on the set $\mathcal{W}_\mathcal{C}$ of credible worlds. Then $\circ$ is also an endorsed credible worlds revision on $\mathbf{K}$, based on the set $\|Th(\mathcal{W}_\mathcal{C})\|$.

Proof: There is a w such that $\alpha \in w \in \mathcal{W}_\mathcal{C}$
iff there is a $w \in \mathcal{W}_\mathcal{C}$ such that $\neg\alpha \notin w$
iff $\neg\alpha \notin Th(\mathcal{W}_\mathcal{C})$
iff there is a $w \in \|Th(\mathcal{W}_\mathcal{C})\|$ such that $\neg\alpha \notin w$
iff there is a $w \in \|Th(\mathcal{W}_\mathcal{C})\|$ such that $\alpha \in w$
iff there is a w such that $\alpha \in w \in \|Th(\mathcal{W}_\mathcal{C})\|$. ■

Lemma 7.4.2 Let $\leq$ be a relation on $\mathcal{L}$ that satisfies (**EE1**), (**EE2**) and (**EE3**). Then $\|\{\delta \mid \alpha \leq \delta\}\| \subseteq \|\beta\|$ if and only if $\alpha \leq \beta$.

Proof: For one direction, let $\|\{\delta \mid \alpha \leq \delta\}\| \subseteq \|\beta\|$. Then by compactness there are $\delta_1, ...\delta_n$ such that for each such δ_i, $\alpha \leq \delta_i$, and furthermore $\|\{\delta_1, ...\delta_n\}\| \subseteq \|\beta\|$. It follows from (**EE3**) that $\alpha \leq \delta_1 \wedge ... \wedge \delta_n$, from (**EE2**) that $\delta_1 \wedge ... \wedge \delta_n \leq \beta$, and then from (**EE1**) that $\alpha \leq \beta$. The other direction is trivial.

Lemma 7.4.3 Let $\circ$ satisfy *vacuity, relative success, strict improvement* and *strong consistency*. Then it satisfies: *If* $\alpha \wedge \beta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$ then $\beta \in \mathbf{K}\circ\neg\beta$.

Proof: Let $\beta \notin \mathbf{K}\circ\neg\beta$. It follows from *vacuity* and *relative success* that $\neg\beta \in \mathbf{K}\circ\neg\beta$. By *strict improvement*, $\neg\alpha \vee \neg\beta \in \mathbf{K} \circ (\neg\alpha \vee \neg\beta)$. Hence, according to *strong consistency*, $\alpha \wedge \beta \notin \mathbf{K} \circ (\neg\alpha \vee \neg\beta)$.

Lemma 7.4.4 Let $\mathbf{K}$ be a consistent belief set, and let $\circ$ satisfy *closure, vacuity, relative success, extensionality, disjunctive inclusion*, and *strong consistency*. Then it satisfies: *If* $\neg\beta \in \mathbf{K} \circ \beta$, *then* $\neg\beta \in \mathbf{K}\circ(\alpha \vee \beta)$.

Proof: Suppose for *reductio* that $\neg\beta \in \mathbf{K}\circ\beta$ and $\neg\beta \notin \mathbf{K}\circ(\alpha \vee \beta)$. Then by *closure* and *extensionality*, $\neg(\alpha \vee \beta) \notin \mathbf{K}\circ(\alpha \vee \beta)$. Due to *vacuity* and *relative success*, $(\alpha \vee \beta) \in \mathbf{K}\circ(\alpha \vee \beta)$. By *disjunctive inclusion* and *closure*, it follows from $\neg\beta \notin \mathbf{K}\circ(\alpha \vee \beta)$ that $\mathbf{K}\circ(\alpha \vee \beta) \subseteq \mathbf{K}\circ\beta$. By *strong consistency* it follows from $\neg\beta \in \mathbf{K}\circ\beta$ that $\beta \notin \mathbf{K}\circ\beta$, hence *relative success* yields $\mathbf{K}\circ\beta = \mathbf{K}$. It follows from this and $\mathbf{K}\circ(\alpha \vee \beta) \subseteq \mathbf{K}\circ\beta$ that $\mathbf{K}\circ(\alpha \vee \beta) \subseteq \mathbf{K}$.
Now suppose that $\mathbf{K} \not\subseteq \mathbf{K}\circ(\alpha \vee \beta)$. It then follows from *vacuity* that $\neg(\alpha \vee \beta) \in \mathbf{K}$ and from *relative success* that $\alpha \vee \beta \in \mathbf{K}\circ(\alpha \vee \beta)$. Since $\mathbf{K}\circ(\alpha \vee \beta) \subseteq \mathbf{K}$, it follows that $\mathbf{K}$ is inconsistent, contrary to the conditions. We can conclude that $\mathbf{K} \subseteq \mathbf{K}\circ(\alpha \vee \beta)$. Since we already have $\mathbf{K}\circ(\alpha \vee \beta) \subseteq \mathbf{K}$, it follows that $\mathbf{K}\circ(\alpha \vee \beta) = \mathbf{K}$. Since by supposition, $\beta \notin \mathbf{K}\circ(\alpha \vee \beta)$, it follows that $\neg\beta \notin \mathbf{K}$, contrary to our assumption $\neg\beta \in \mathbf{K}\circ\beta$ and $\mathbf{K}\circ\beta = \mathbf{K}$ that was shown above. This contradiction concludes the proof. ∎

Lemma 7.4.5 Let $\circ$ satisfy *vacuity*. Then it satisfies: *If* $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \beta)$, *then* $\alpha \in \mathbf{K}$.

Proof: Let $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \beta)$. It is sufficient to show that if $\alpha \wedge \beta \notin \mathbf{K}$, then $\alpha \in \mathbf{K}$. Let $\alpha \wedge \beta \notin \mathbf{K}$. Then *vacuity* yields $\mathbf{K} \circ \neg(\alpha \wedge \beta) = \mathbf{K} + \neg(\alpha \wedge \beta)$, hence $\alpha \in \mathbf{K} + \neg(\alpha \wedge \beta)$, hence by deduction $\neg(\alpha \wedge \beta) \rightarrow \alpha \in \mathbf{K}$, hence $\alpha \in \mathbf{K}$. ∎

Lemma 7.4.6 (Modified from [Canssb]) Let D be a non-empty subset of $\mathcal{P}(\mathcal{L})$ such that (1) for all X in D, $X = \|Th(X)\|$, and (2) for all elements X and Y of D, either $X \subseteq Y$ or $Y \subseteq X$. Furthermore, let $\alpha \in \mathcal{L}$. Then: If $\cap D \subseteq \|\alpha\|$ then there is some element X of D such that $X \subseteq \|\alpha\|$.

Proof: Suppose to the contrary that $\cap D \subseteq \|\alpha\|$ and that $Y \not\subseteq \|\alpha\|$ for all $Y \in D$.

Let $S = \cup\{Th(Y) \mid Y \in D\}$ Then $\alpha \notin S$.

Next, let $Y \in D$. We are going to show that $\|S\| \subseteq Y$. Let $u \in \|S\|$. Then it holds for all $\beta \in S$ that $u \in \|\beta\|$. But since $Th(Y) \subseteq S$ it holds for all $\beta \in Th(Y)$ that $u \in \|\beta\|$. Hence, $u \in Y$. We have shown that for all $Y \in D$, $\|S\| \subseteq Y$. Hence, $\|S\| \subseteq \cap D$.

Next, we are going to show that $\|S\| \not\subseteq \|\alpha\|$. Suppose to the contrary that $\|S\| \subseteq \|\alpha\|$. Then it follows by compactness that there are $\delta_1, ..., \delta_n \in S$ such that $\|\delta_1\| \cap ... \cap \|\delta_n\| \subseteq \|\alpha\|$. Hence there are $Y_1, ..., Y_n \in D$ such that for each δ_k with $1 \leq k \leq n$, $\delta_k \in Th(Y_k)$. But $Th(Y_1), ..., Th(Y_n)$ form an inclusion chain so there must be some Y_i such that $\delta_1, ..., \delta_n \in Th(Y_i)$. Then $\delta_1 \wedge ... \wedge \delta_n \in Th(Y_i)$, i.e., $Y_i \subseteq \|\delta_1\| \cap ... \cap \|\delta_n\|$. Hence $Y_i \subseteq \|\alpha\|$, contrary to our assumption. We can conclude that $\|S\| \not\subseteq \|\alpha\|$. Since $\|S\| \subseteq \cap D$, it follows that $\cap D \not\subseteq \|\alpha\|$, contrary to our assumption. ■.

7.4.2 Proofs

Proof of Observation 7.1.1.

1. See A.8.13.

2. See A.8.15.

3. See A.8.16.

4. See A.8.18.

5. Trivial from set theory.

6. See A.8.19.

7. See A.8.23.

Proof of Observation 7.2.2.

1. Trivial.

2. Trivial.

3. Single sentence consequence and $\mathcal{C} \neq \emptyset$ implies that $\alpha \vee \neg\alpha \in \mathcal{C}$, to this we can apply disjunctive completeness.

4. Let p and q be the only atomic sentences. Let $\mathcal{C} = Cn(\{\neg p\}) \cup Cn(\{\neg q\}) \cup Cn(\{\neg(p \leftrightarrow q)\})$. It is obvious that single sentence closure holds. That negation completeness hold is easy to show by chwcking through the 16 sentences contained in the language. To see that disjunctive completeness does not hold. Note that $p \vee q \in \mathcal{C}$, whereas $p \notin \mathcal{C}$ and $q \notin \mathcal{C}$.

Proof of Theorem 7.3.1

(2)-to-(1) It follows directly from the construction that *closure, relative success, inclusion, weak consistency preservation*, and *extensionality* are satisfied. For *consistent expansion*, let $\mathbf{K} \nsubseteq \mathbf{K}\circ\alpha$. Then $\mathbf{K}\circ\alpha = \mathbf{K}*\alpha$ and $\alpha \in \mathcal{C}$. It follows from the *vacuity* and *success* postulates satisfied by $*$ that $*$ satisfies *consistent expansion.*

(1)-to(3) Let $*$ be the operation such that

(i) if $\alpha \in \mathbf{K}\circ\alpha$, then $\mathbf{K}*\alpha = \mathbf{K}\circ\alpha$

(ii) if $\alpha \notin \mathbf{K}\circ\alpha$, then $\mathbf{K}*\alpha = \mathbf{K} *' \alpha$ for some AGM revision operator $*'$.

Furthermore, let $\mathcal{C} = \{\alpha \mid \alpha \in \mathbf{K}\circ\alpha\}$. We need to show: **(A1)** that $\mathcal{C}$ is closed under logical equivalence, **(A2)** that $\mathbf{K} \subseteq \mathcal{C}$, **(B)** that $*$ is an AGM revision operator, and **(C)** that $\circ$ is induced by $*$ and $\mathcal{C}$.

Part A1 To show that $\mathcal{C}$ is closed under logical equivalence, let $\alpha \in \mathcal{C}$ and let $\vdash \alpha \leftrightarrow \beta$. Then $\alpha \in \mathbf{K}\circ\alpha$. It follows from $\circ-$*closure* that $\beta \in \mathbf{K}\circ\alpha$ and from $\circ-$*extensionality* that $\mathbf{K}\circ\alpha = \mathbf{K}\circ\beta$. Then $\beta \in \mathbf{K}\circ\beta$, hence $\beta \in \mathcal{C}$.

Part A2 Let $\alpha \in \mathbf{K}$. It follows from *relative success* that $\alpha \in \mathbf{K}\circ\alpha$, hence $\alpha \in \mathcal{C}$.

Part B We can do this by showing that $*$ satisfies the six basic AGM postulates:

$*-$*extensionality*: Let $\vdash \alpha \leftrightarrow \beta$ and $\alpha \in \mathbf{K}\circ\alpha$. There are two cases. First case, $\alpha \in \mathbf{K}{\circ}\alpha$: Then $\mathbf{K}{\circ}\alpha = \mathbf{K}{*}\alpha$. It follows from $\circ-$*closure* that $\beta \in \mathbf{K}{\circ}\alpha$ and from $\circ-$*extensionality* that $\mathbf{K}{\circ}\alpha = \mathbf{K}{\circ}\beta$. Hence $\beta \in \mathbf{K}{\circ}\beta$, hence $\mathbf{K}{*}\beta = \mathbf{K}{\circ}\beta$, hence $\mathbf{K}{*}\beta = \mathbf{K}{*}\alpha$.

Second case, $\alpha \notin \mathbf{K}{\circ}\alpha$: Then we have $\beta \notin \mathbf{K}{\circ}\beta$ from $\circ-$*closure*. It follows from clause (ii) of the definition of $*$ that $\mathbf{K}{*}\alpha = \mathbf{K} *' \alpha$ and $\mathbf{K}{*}\beta = \mathbf{K} *' \beta$. Due to $*'-$*extensionality* we have $\mathbf{K} *' \alpha = \mathbf{K} *' \beta$ and hence $\mathbf{K}{*}\alpha = \mathbf{K}{*}\beta$.

$*-$*closure* Follows in case (i) from $\circ-$*closure* and in case (ii) from $*'-$*closure*.

$*-$*success*: In case (i), $\alpha \in \mathbf{K}{\circ}\alpha$ and $\mathbf{K}{*}\alpha = \mathbf{K}{\circ}\alpha$. In case (ii), $\alpha \in \mathbf{K} *' \alpha$ and $\mathbf{K}{*}\alpha = \mathbf{K} *' \alpha$.

$*-$*inclusion*: In case (i), we can use $\circ-$*inclusion* and in case (ii) $*'-$*inclusion*.

$*-$*consistency*: Let α be a consistent sentence. We have to prove that $\mathbf{K}{*}\alpha$ is consistent. In clause (i) of the definition of $*$, it follows from $\circ-$*weak consistency preservation* that $\mathbf{K}{\circ}\alpha$ is consistent and hence $\mathbf{K}{*}\alpha$ is consistent. In clause (ii), we can use $*'-$*consistency* to obtain the desired result.

$*-$*vacuity*: Let $\neg\alpha \notin \mathbf{K}$. There are two cases.

Case 1, $\alpha \in \mathbf{K}{\circ}\alpha$: Suppose that $\mathbf{K} \not\subseteq \mathbf{K}{\circ}\alpha$. Then it follows from $\circ-$*consistent expansion* that $\mathbf{K} \cup (\mathbf{K}{\circ}\alpha)$ is inconsistent, then according to $\circ-$*inclusion* so is $\mathbf{K} \cup \mathbf{K} + \alpha$, hence $\mathbf{K} \vdash \neg\alpha$, contrary to the conditions. Hence $\mathbf{K} \subseteq \mathbf{K}{\circ}\alpha$. It follows by $\circ-$*closure* from $\alpha \in \mathbf{K}{\circ}\alpha$

and $\mathbf{K} \subseteq \mathbf{K}{\circ}\alpha$ that $\mathbf{K} + \alpha \subseteq \mathbf{K}{\circ}\alpha$. It follows from $\alpha \in \mathbf{K}{\circ}\alpha$ that $\mathbf{K}{*}\alpha = \mathbf{K}{\circ}\alpha$, hence $\mathbf{K} + \alpha \subseteq \mathbf{K}{*}\alpha$.

Case 2, $\alpha \notin \mathbf{K}{\circ}\alpha$: We have $\mathbf{K}{*}\alpha = \mathbf{K} *' \alpha$, and we can use $*'-$*vacuity*.

Part C There are two cases. (1) If $\alpha \in \mathcal{C}$, then $\mathbf{K}{\circ}\alpha = \mathbf{K}{*}\alpha$. (2) If $\alpha \notin \mathcal{C}$, then $\alpha \notin \mathbf{K}{\circ}\alpha$. It follows from $\circ-$*relative success* that $\mathbf{K}{\circ}\alpha = \mathbf{K}$.

(3)-to-(2) : Obvious. ■

Proof of Observation 7.3.2: Let $\mathcal{C}_1 \setminus \mathbf{K} = \mathcal{C}_2 \setminus \mathbf{K}$. Case 1, $\alpha \in \mathbf{K}$: Then $\mathbf{K}{*}\alpha = \mathbf{K}$ and consequently $\mathbf{K} \circ_1 \alpha = \mathbf{K}$ and $\mathbf{K} \circ_2 \alpha = \mathbf{K}$. Case 2, $\alpha \notin \mathbf{K}$: Then, since $\mathcal{C}_1 \setminus \mathbf{K} = \mathcal{C}_2 \setminus \mathbf{K}$, we have two subcases: either (2a) $\alpha \in \mathcal{C}_1$ and $\alpha \in \mathcal{C}_2$, or (2b) $\alpha \notin \mathcal{C}_1$ and $\alpha \notin \mathcal{C}_2$. In both subcases, $\mathbf{K} \circ 1\alpha = \mathbf{K} \circ 2\alpha$ follows directly.■

Proof of Theorem 7.3.3

part (a) Construction to postulates: Due to **Theorem 7.3.1** it only remains to show that single sentence closure implies *strict improvement*.

Let $\alpha \in \mathbf{K}{\circ}\alpha$ and $\vdash \alpha \to \beta$.

Case 1, $\alpha \in \mathcal{C}$: By single sentence closure $\beta \in \mathcal{C}$, hence $\beta \in \mathbf{K}{\circ}\beta$.

Case 2, $\alpha \notin \mathcal{C}$: Then $\mathbf{K}{\circ}\alpha = \mathbf{K}$. Furthermore, since $\vdash \alpha \to \beta$ and $\mathbf{K}$ is logically closed, we have $\beta \in \mathbf{K}$.

Case 2a, $\beta \in \mathcal{C}$: Then $\mathbf{K}{\circ}\beta = \mathbf{K}{*}\beta$, and $*-$*success* yields $\beta \in \mathbf{K}{*}\beta$.

Case 2b, $\beta \notin \mathcal{C}$: Then $\mathbf{K}{\circ}\beta = \mathbf{K}$, and since $\beta \in \mathbf{K}$ we have $\beta \in \mathbf{K}{\circ}\beta$.

Postulates to construction: The following addition will be made to the proof of the corresponding part of **Theorem 7.3.1** (including the construction $\mathcal{C} = \{\alpha \mid \alpha \in \mathbf{K}{\circ}\alpha\}$ introduced there): We need to show that single sentence closure holds. Excluding the trivial direction, let $\alpha \in \mathcal{C}$ and $\vdash \alpha \rightarrow \beta$. It follows from $\alpha \in \mathcal{C}$ that $\alpha \in \mathbf{K}{\circ}\alpha$, and *strict improvement* yields $\beta \in \mathbf{K}{\circ}\beta$, hence $\beta \in \mathcal{C}$.

Part (b) Construction-to-postulates: Let the conditions given in (2) be satisfied. Due to the **Theorem 7.3.1** we only have to show that *disjunctive constancy* is satisfied. Let $\mathbf{K} = \mathbf{K}{\circ}\alpha = \mathbf{K}{\circ}\beta$. There are three cases. (a) $\alpha \in \mathcal{C}$: Then $\mathbf{K}{*}\alpha = \mathbf{K}{\circ}\alpha$ and it follows from $\mathbf{K} = \mathbf{K}{\circ}\alpha$ that $\alpha \in \mathbf{K}$, hence $\alpha \vee \beta \in \mathbf{K}$, from which follows $\mathbf{K}{\circ}\alpha \vee \beta = \mathbf{K}$. (b) $\beta \in \mathcal{C}$: Proved analogously. (c) $\alpha \notin \mathcal{C}$ and $\beta \notin \mathcal{C}$. It follows from disjunctive completeness that $\alpha \vee \beta \notin \mathcal{C}$, hence $\mathbf{K}{\circ}\alpha \vee \beta = \mathbf{K}$.

Postulates-to-construction: Let $*$ be the operation such that:

(i) if $\alpha \in \mathbf{K}{\circ}\alpha$, then $\mathbf{K}{*}\alpha = \mathbf{K}{\circ}\alpha$

(ii) if $\alpha \notin \mathbf{K}{\circ}\alpha$, then $\mathbf{K}{*}\alpha = \mathbf{K} *' \alpha$ for some AGM revision operator $*'$.

Furthermore, let $\mathcal{C} = \{\alpha \mid \mathbf{K} \neq \mathbf{K}{\circ}\alpha\}$. We need to show: (A1) that $\mathcal{C}$ is closed under logical equivalence, (A2) that $\mathcal{C}$ satisfies disjunctive completeness, (B) that $*$ is an AGM revision operator, and (C) that $\circ$ is induced by $*$ and $\mathcal{C}$.

Part A1: To show that $\mathcal{C}$ is closed under logical equivalence, let $\alpha \in \mathcal{C}$ and let $\vdash \alpha \leftrightarrow \beta$. It follows from $\circ-$*extensionality* that $\mathbf{K}\circ\alpha = \mathbf{K}\circ\beta$, hence $\beta \in \mathcal{C}$.

Part A2: To show that $\mathcal{C}$ satisfies disjunctive completeness, let $\alpha \notin \mathcal{C}$ and $\beta \notin \mathcal{C}$. Our task is to show that $\alpha \vee \beta \notin \mathcal{C}$. It follows from $\alpha \notin \mathcal{C}$ and $\beta \notin \mathcal{C}$ that $\mathbf{K} = \mathbf{K}\circ\alpha = \mathbf{K}\circ\beta$. Since *disjunctive constancy* holds, we then have $\mathbf{K} = \mathbf{K}\circ\alpha \vee \beta$, from which follows that $\alpha \vee \beta \notin \mathcal{C}$.

Part B: That $*$ satisfies the six basic AGM postulates can be shown exactly as in the proof of **Theorem 7.3.1**, since the definition of $*$ is the same.

Part C: There are two cases. (1) If $\alpha \in \mathcal{C}$, then $\mathbf{K} \neq \mathbf{K}\circ\alpha$. It follows from *relative success* that $\alpha \in \mathbf{K}\circ\alpha$. Hence, according to our definition of $*$, $\mathbf{K}\circ\alpha = \mathbf{K}*\alpha$. (2) If $\alpha \notin \mathcal{C}$, we have $\mathbf{K}\circ\alpha = \mathbf{K}$ directly from the definition of $\mathcal{C}$.

Part (c) Construction to postulates: Due to **Theorem 7.3.1** it only remains to be shown that *disjunctive success* holds if negation completeness is satisfied. It follows from negation completeness that for all α, either $\mathbf{K}\circ\alpha = \mathbf{K}*\alpha$ or $\mathbf{K} \circ \neg\alpha = \mathbf{K} * \neg\alpha$. Due to $*-$*success*, it follows from this that $\circ$ satisfies *disjunctive success*.

Postulates to construction. The following addition will be made to the proof of the corresponding part of **Theorem 7.3.1**: We need to show that negation completeness is satisfied. According to *disjunctive success*, either $\alpha \in \mathbf{K}\circ\alpha$ or $\neg\alpha \in \mathbf{K}\circ\neg\alpha$. In the first case, $\alpha \in \mathcal{C} = \{\alpha \mid \alpha \in \mathbf{K}\circ\alpha\}$, and in the second case $\neg\alpha \in \mathcal{C}$ follows in the same way.

Part (d) Construction-to-postulates: In addition to the corresponding part of **Theorem 7.3.1**, it is sufficient to show that if *strong consistency* does not hold, then it is does not either hold that if $\alpha \in \mathcal{C}$, then $\alpha \nvdash \bot$. Suppose that *strong consistency* does not hold. Then there is some α such that $\mathbf{K}\circ\alpha \vdash \bot$. Since $\mathbf{K}$ is consistent, $\mathbf{K}\circ\alpha \neq \mathbf{K}$, hence $\mathbf{K}\circ\alpha = \mathbf{K}*\alpha$. Due to $*-$*consistency*, $\alpha \vdash \bot$. From $\mathbf{K} \neq \mathbf{K}\circ\alpha = \mathbf{K}*\alpha$ follows $\alpha \in \mathcal{C}$.

Postulates to construction: The following addition will be made to the proof of the corresponding part of **Theorem 7.3.1**: We need to show that if $\alpha \in \mathcal{C}$, then $\alpha \nvdash \bot$. For *reductio*, suppose to the contrary that $\alpha \in \mathcal{C}$ and $\alpha \vdash \bot$. It follows from $\alpha \in \mathcal{C}$ that $\mathbf{K}\circ\alpha = \mathbf{K}*\alpha$, due to $*-$*success* $\mathbf{K}*\alpha \vdash \bot$, hence $\mathbf{K}\circ\alpha \vdash \bot$, contrary to *strong consistency*.

Part (e) Construction-to-postulates: In addition to the corresponding part of **Theorem 7.3.1**, it is sufficient to show that if the construction satisfies expansive credibility, then *vacuity* holds. Let expansive credibility hold. In order to prove *vacuity*, let $\neg\alpha \notin \mathbf{K}$. Then it follows from expansive credibility that $\alpha \in \mathcal{C}$, hence by the definition of $\circ$, $\mathbf{K}\circ\alpha = \mathbf{K}*\alpha$. By $*-$*vacuity*, $\mathbf{K}*\alpha = \mathbf{K} + \alpha$.

Postulates to construction: The following addition will be made to the proof of the corresponding part of **Theorem 7.3.1**: We need to show that $\mathcal{C}$ satisfies expansive credibility. For that purpose, let $\mathbf{K} \nvdash \alpha$. Then it follows from vacuity that $\neg\alpha \in \mathbf{K} \circ \neg\alpha$, hence by the definition of $\mathcal{C}$, $\neg\alpha \in \mathcal{C}$. ■

Proof of Theorem 7.3.4

(0)-to-(1) and (1)-to-(0) Directly from **Theorems 7.3.1** and **7.3.3**.

(1)-to-(2) Let the three conditions hold. Let $A = \{\alpha \mid \neg\alpha \notin \mathcal{C}\}$. It is sufficient to show (1) that $\alpha \in \mathcal{C}$ iff $\neg\alpha \notin A$, and (2) $A = Cn(A)$.

(1): $\alpha \in \mathcal{C}$ iff $\neg(\alpha \notin \mathcal{C})$

iff $\neg(\neg\alpha \in \{\neg\alpha \mid \alpha \notin \mathcal{C}\})$

iff $\neg(\neg\alpha \in A)$

iff $\neg\alpha \notin A$.

(2): By element consistency, $\perp \notin \mathcal{C}$, so that $\top \in A$ and hence $A \neq \emptyset$.

In order to prove that $A = Cn(A)$, let $\alpha \in Cn(A)$. We assume compactness. Since A is non-empty, there are $\beta_1, ...\beta_n \in A$ such that $\{\beta_1, ...\beta_n\} \vdash \alpha$. We need to show that $\alpha \in A$.

It follows from $\beta_1, ...\beta_n \in A$ that $\neg\beta_1, ...\neg\beta_n \notin \mathcal{C}$. It follows from repeated use of disjunctive completeness that $\neg\beta_1 \vee ... \vee \neg\beta_n \notin \mathcal{C}$.

Suppose that $\neg\alpha \in \mathcal{C}$. Then, since $\neg\alpha \vdash \neg\beta_1 \vee ... \vee \neg\beta_n$, single sentence closure yields $\neg\beta_1 \vee ... \vee \neg\beta_n \in \mathcal{C}$, contrary to what was just shown. We may conclude that $\neg\alpha \notin \mathcal{C}$, hence $\alpha \in A$. This finishes the proof.

(2)-to-(1) We need to show that all core beliefs revisions satisfy the three postulates. Let the operator be a core beliefs revision, i.e., let there be some A such that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$. To show that single sentence closure is satisfied, let $\alpha \in \mathcal{C}$ and $\vdash \alpha \rightarrow \beta$. Then $\alpha \in \mathcal{C}$ yields $A \nvdash \neg\alpha$, and $\vdash \alpha \rightarrow \beta$

yields $\vdash \neg\beta \rightarrow \neg\alpha$. Hence $A \nvdash \neg\beta$, so that $\beta \in \mathcal{C}$.

To show that disjunctive completeness is satisfied, let $\alpha \vee \beta \in \mathcal{C}$. It follows from the definition of core beliefs revisions that $A \nvdash \neg(\alpha \vee \beta)$, hence $A \nvdash \neg\alpha \wedge \neg\beta$, hence either $A \nvdash \neg\alpha$ or $A \nvdash \neg\beta$. In the former case, $\alpha \in \mathcal{C}$, in the latter $\beta \in \mathcal{C}$.

To show that element consistency is satisfied, let $\alpha \vdash \bot$. Then $A \nvdash \neg\alpha$, hence $\alpha \notin \mathcal{C}$.

(2)-to-(3) Let the operator be a core beliefs revision. Let $\mathcal{W}_{\mathcal{C}} = \|A\|$. We then have:

$\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$

iff there is some $w \in \mathcal{W}_{\mathcal{C}}$ such that $w \nvdash \neg\alpha$

iff there is some $w \in \mathcal{W}_{\mathcal{C}}$ such that $\alpha \in w$.

(3)-to-(2) Let the operator be a credible worlds revision. Let $A = Th(\mathcal{W}_{\mathcal{C}})$. Then:

$\alpha \in \mathcal{C}$ iff there is some $w \in \mathcal{W}_{\mathcal{C}}$ such that $\alpha \in w$

iff there is some $w \in \mathcal{W}_{\mathcal{C}}$ such that $w \nvdash \neg\alpha$

iff $Th(\mathcal{W}_{\mathcal{C}}) \nvdash \neg\alpha$

iff $A \nvdash \neg\alpha$.

Proof of Theorem 7.3.5

(0)-to-(1) and (1)-to-(0) Directly from **Theorems 7.3.4** and **7.3.3, part C**.

(1)-to-(2) and (2)-to-(1) For all α, $\alpha \in \mathcal{C}$ or $\neg\alpha \in \mathcal{C}$

iff for all α, $A \nvdash \neg\alpha$ or $A \nvdash \alpha$

iff $A \nvdash \bot$.

(2)-to-(3) According to the defining characteristic of consistent core beliefs revisions, there is a consistent set $A \subseteq \mathcal{L}$ such that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$. Let $\mathcal{W}_\mathcal{C} = \|A\|$. It follows from the proof of **Theorem 7.3.4** that $\alpha \in \mathcal{C}$ iff there is some $w \in \mathcal{W}_\mathcal{C}$ such that $\alpha \in w$. That $\mathcal{W}_\mathcal{C}$ is non-empty follows directly since A is consistent.

(3)-to-(2) According to the defining characteristic of non-empty credible worlds revisions, there is a non-empty set $\mathcal{W}_\mathcal{C}$ of possible worlds such that $\alpha \in \mathcal{C}$ iff there is some $w \in \mathcal{W}_\mathcal{C}$ such that $\alpha \in w$. Let $A = \cap\mathcal{W}_\mathcal{C}$. It follows from the proof of **Theorem 7.3.4** that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$. That A is consistent follows directly since is $\mathcal{W}_\mathcal{C}$ non-empty.

Proof of Theorem 7.3.6

(0)-to-(1) From **Theorems 7.3.5 7.3.3, part E**. Two postulates have been deleted from the list: *Weak consistency preservation* (redundant, since it follows from *strong consistency*) and *consistent expansion* (redundant, since it follows from *vacuity* and *relative success* imply *consistent expansion*).

(1)-(to)-(0) Theorems 7.3.5 7.3.3, part e.

(1)-to-(2) Let $\mathcal{C}$ satisfy the listed properties. It follows from the corresponding proof of **Theorem 7.3.4** that $\circ$ is a core beliefs revision with respect to the set $A = \{\alpha \mid \neg\alpha \notin \mathcal{C}\}$. In order to show that this is also an endorsed core beliefs revision, let $\alpha \in A$. It follows that $\neg\alpha \notin \mathcal{C}$, and according to expansive credibility, $\mathbf{K} \vdash \alpha$. Since $\mathbf{K}$ is logically closed, this is sufficient to show that $\alpha \in A$, and hence this is a core beliefs revision.

(2)-to-(1) Let the operator be an endorsed core beliefs revision. It follows from the corresponding proof of **Theorem 7.3.4** that $\mathcal{C}$ satisfies single sentence closure, disjunctive completeness, and element consistency. To show that it satisfies expansive credibility, let $\mathbf{K} \nvdash \alpha$. Since $A \subseteq \mathbf{K}$, we then have $A \nvdash \alpha$, or equivalently $\neg\alpha \notin \mathcal{C}$.

(2)-to-(3) According to the defining characteristic of core beliefs revisions, there is a set $A \subseteq \mathbf{K}$ such that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$. Let $\mathcal{W}_\mathcal{C} = \|A\|$. It follows from the proof of **Theorem 7.3.4** that $\alpha \in \mathcal{C}$ iff there is some $w \in \mathcal{W}_\mathcal{C}$ such that $\alpha \in w$. It follows from $A \subseteq \mathbf{K}$ that $\|\mathbf{K}\| \subseteq \|A\|$, hence $\|\mathbf{K}\| \subseteq \mathcal{W}_\mathcal{C}$.

(3)-to-(2) Since this is an endorsed credible worlds revision, there is a non-empty set $\mathcal{W}_\mathcal{C}$ of possible worlds such $\|\mathbf{K}\| \subseteq \mathcal{W}_\mathcal{C}$ and that $\alpha \in \mathcal{C}$ iff there is some $w \in \mathcal{W}_\mathcal{C}$ such that $\alpha \in w$. Let $A = Th(\mathcal{W}_\mathcal{C})$. It follows from the proof of **Theorem 7.3.4** that $\alpha \in \mathcal{C}$ iff $A \nvdash \neg\alpha$. From $\|\mathbf{K}\| \subseteq \mathcal{W}_\mathcal{C}$ follows $A = Th(\mathcal{W}_\mathcal{C}) \subseteq Th(\|\mathbf{K}\|) = \mathbf{K}$ is consistent follows directly since is $\mathcal{W}_\mathcal{C}$ non-empty. ■

Proof of Theorem 7.3.7

(1)-to-(2) We will use the same construction as in **Theorems 7.3.1** and **7.3.6**, but with the further specification that $*'$ (the revision operator for residual cases) is defined so that $\mathbf{K} *' \alpha = Cn(\alpha)$ for all α. (It can easily be checked that this is an AGM operator.) It remains to be shown that outcome credibility holds, i.e., that $\|\mathbf{K}\circ\alpha\| \cap \mathcal{W}_{\mathcal{C}} \neq \emptyset$.

Let $\beta \notin \mathcal{C}$. Then, $\mathbf{K}\circ\beta = \mathbf{K}$. It follows from $\mathbf{K} \subseteq \mathcal{C}$ that $\beta \notin \mathbf{K}$, hence $\beta \notin \mathbf{K}\circ\beta$, hence by *regularity* $\beta \notin \mathbf{K}\circ\alpha$. Hence $\mathbf{K}\circ\alpha \subseteq \mathcal{C}$, hence $\|\mathbf{K}\circ\alpha\| \cap \mathcal{W}_{\mathcal{C}} \neq \emptyset$.

(2)-to-(3) Let (2) be satisfied, and let $\alpha \in \mathcal{C}$. Then $\mathbf{K}\circ\alpha = \mathbf{K}{*}\alpha$, hence $\|\mathbf{K}{*}\alpha\| \cap \mathcal{W}_{\mathcal{C}} \neq \emptyset$, hence there is some $w \in \mathcal{W}_{\mathcal{C}}$ such that $w \in \|\mathbf{K}{*}\alpha\|$, i.e., $\mathbf{K}{*}\alpha \subseteq w \subseteq \mathcal{W}_{\mathcal{C}}$. Hence, according to the definition of $\mathcal{W}_{\mathcal{C}}$, it holds for all $\beta \in \mathbf{K}{*}\alpha$ that $\beta \in \mathcal{C}$, hence $\mathbf{K}{*}\alpha \subseteq \mathcal{C}$.

(3) to (1) Let (3) be satisfied and let $\beta \in \mathbf{K}\circ\alpha$. There are two cases.

Case 1, $\alpha \notin \mathcal{C}$. Then $\mathbf{K}\circ\alpha = \mathbf{K}$. and $\beta \in \mathbf{K}$. Since this is an endorsed core beliefs revision, it is also an endorsed credible worlds revision (see **Theorem 7.3.6**), i.e., $\|\mathbf{K}\| \subseteq \mathcal{W}_{\mathcal{C}}$. Since $\mathbf{K}$ is consistent, it follows that $\|\mathbf{K}\| \cap \mathcal{W}_{\mathcal{C}} \neq \emptyset$ and since $\|\mathbf{K}\| \subseteq \|\beta\|$ it follows from this that $\|\beta\| \cap \mathcal{W}_{\mathcal{C}} \neq \emptyset$ or equivalently $\beta \in \mathcal{C}$, from which follows $\beta \in \mathbf{K}\circ\beta$.

Case 2, $\alpha \in \mathcal{C}$. Then $\mathbf{K}\circ\alpha = \mathbf{K}{*}\alpha$. It follows from (3) that $\mathbf{K}{*}\alpha \subseteq \mathcal{C}$, hence $\mathbf{K}\circ\alpha \subseteq \mathcal{C}$. Since $\beta \in \mathbf{K}\circ\alpha$ it follows that $\alpha \in \mathcal{C}$, hence $\beta \in \mathbf{K}\circ\beta$. ■

Proof of Theorem 7.3.7

(1) to (2) Let *strong regularity* be satisfied. It follows from **Lemma 7.4.1** that we can, without loss of generality, assume that $\circ$ is an endorsed credible worlds revision based on a set $\mathcal{W}_{\mathcal{C}}$ of credible worlds such that $\mathcal{W}_{\mathcal{C}} = \|Th(\mathcal{W}_{\mathcal{C}})\|$. Let $\beta \in Th(\mathcal{W}_{\mathcal{C}})$. Then $\|Th(\mathcal{W}_{\mathcal{C}})\| \subseteq \|\beta\|$, hence

$\mathcal{W}_\mathcal{C} \subseteq \|\beta\|$, hence $\mathcal{W}_\mathcal{C} \cap \|\neg\beta\| = \emptyset$, hence by the definition of $\mathcal{W}_\mathcal{C}$, $\neg\beta \notin \mathbf{K} \circ \neg\beta$. Applying *strong regularity* to this we obtain $\beta \in \mathbf{K}\circ\alpha$. Hence we have proved that $Th(\mathcal{W}_\mathcal{C}) \subseteq \mathbf{K}\circ\alpha$. From this follows $\|\mathbf{K}\circ\alpha\| \subseteq \mathcal{W}_\mathcal{C}$.

(2) to (1) Let (2) be satisfied, and let $\neg\beta \notin \mathbf{K}\circ\alpha$. From $\neg\beta \notin \mathbf{K}\circ\alpha$ follows $\|\mathbf{K}\circ\alpha\| \not\subseteq \|\neg\beta\|$, and then from strong outcome credibility ($\|\mathbf{K}\circ\alpha\| \subseteq \mathcal{W}_\mathcal{C}$) that $\mathcal{W}_\mathcal{C} \not\subseteq \|\neg\beta\|$, hence $\mathcal{W}_\mathcal{C} \cap \|\beta\| \neq \emptyset$, equivalently $\beta \in \mathcal{C}$, from which follows $\beta \in \mathbf{K}\circ\beta$.■

Proof of Theorem 7.3.9

(0)-to-(1) We assume that the postulates given in (0) hold for a given operator $\circ$, and we let $\leq$ be defined as follows:

$\alpha \leq \beta$ iff: If $\alpha \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$, then $\beta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$.

Our task is to show that (**EE1**) – (**EE4**) and that $\circ$ is entrenchment-based on $\leq$ in the sense of **Definition 7.2.4**.

(**EE1**): Let $\alpha \leq \beta$, $\beta \leq \delta$ and $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \delta)$. We need to prove that $\delta \in \mathbf{K} \circ \neg(\alpha \wedge \delta)$. There are two cases.

Case 1, $\alpha \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$: Since $\alpha \leq \beta$ we then have $\beta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$. By *closure*, $\alpha \wedge \beta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$. **Lemma 7.4.3** yields $\beta \in \mathbf{K}\circ\neg\beta$ and **Lemma 7.4.4** yields $\beta \in \mathbf{K}\circ\neg(\beta \wedge \delta)$. Since $\beta \leq \delta$, it follows that $\delta \in \mathbf{K}\circ\neg(\beta \wedge \delta)$. By *closure*, $\beta \wedge \delta \in \mathbf{K}\circ\neg(\beta \wedge \delta)$. **Lemma 7.4.3** then yields $\delta \in \mathbf{K} \circ \neg\delta$, and **Lemma 7.4.4** yields $\delta \in \mathbf{K} \circ \neg(\alpha \wedge \delta)$. This concludes the proof of Case 1.

Case 2, $\alpha \notin \mathbf{K}\circ\neg(\alpha \wedge \beta)$: It then follows from **Lemma 7.4.4** that $\alpha \notin \mathbf{K} \circ \neg\alpha$. We are going to assume for *re-*

ductio that $\delta \notin \mathbf{K}\circ(\alpha \wedge \delta)$. It then follows directly by **Lemma 7.4.4** that $\delta \notin \mathbf{K} \circ \neg\delta$. **Lemma 7.4.3** then yields $\beta \wedge \delta \notin \mathbf{K}\circ\neg(\beta \wedge \delta)$. Since $\beta \leq \delta$, we can conclude from this (using *closure*) that $\beta \notin \mathbf{K}\circ\neg(\beta \wedge \delta)$. We are going to show (1) that $\alpha \in \mathbf{K}\circ\neg(\alpha \wedge \beta \wedge \delta)$ and (2) that $\alpha \notin \mathbf{K}\circ\neg(\alpha \wedge \beta \wedge \delta)$.

Ad 1: Since $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \delta)$, **Lemma 7.4.5** yields $\alpha \in \mathbf{K}$. Hence by *relative success* and *closure*, $\alpha \in \mathbf{K} \circ (\alpha \wedge \neg\beta)$. By assumption, $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \delta)$. Due to *disjunctive overlap* (that holds according to **Observation 7.1.1, part 5**), $\mathbf{K} \circ \neg(\alpha \wedge \delta) \cap \mathbf{K} \circ (\alpha \wedge \neg\beta) \subseteq \mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$, hence $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$.

Ad 2: It follows from *disjunctive factoring* and *extensionality* that $\mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$ is equal to one of $\mathbf{K}\circ\neg(\alpha \wedge \beta) \cap \mathbf{K} \circ \neg\delta$, $\mathbf{K}\circ\neg(\alpha \wedge \beta)$ and $\mathbf{K} \circ \neg\delta$. Since, as we have just seen, $\alpha \notin \mathbf{K}\circ\neg(\alpha \wedge \beta)$ and $\delta \notin \mathbf{K} \circ \neg\delta$, it follows that either $\alpha \notin \mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$ or $\delta \notin \mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$. In the former case we are done. In the latter case, we also have $\beta \wedge \delta \notin \mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$, and it follows by disjunctive inclusion (that holds according to parts 6 and 3 of **Observation 7.1.1**) and *extensionality* that $\mathbf{K}\circ\neg(\alpha\wedge\beta\wedge\delta) \subseteq \mathbf{K}\circ\neg(\beta \wedge \delta)$. Since $\beta \notin \mathbf{K}\circ\neg(\beta \wedge \delta)$ as shown above, it follows that $\beta \notin \mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$, hence due to *closure* and *extensionality* $\alpha \wedge \beta \notin \mathbf{K} \circ \neg(\alpha \wedge \beta \vee \neg\delta)$. It follows from *disjunctive inclusion* that $\mathbf{K} \circ \neg(\alpha \wedge \beta \vee \neg\delta) \subseteq \mathbf{K} \circ \neg(\alpha \wedge \beta)$. *Extensionality* yields $\mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta) \subseteq \mathbf{K} \circ \neg(\alpha \wedge \beta)$. Since in this case $\alpha \notin \mathbf{K}\circ\neg(\alpha \wedge \beta)$, it follows that $\alpha \notin \mathbf{K} \circ \neg(\alpha \wedge \beta \wedge \delta)$. This is the contradiction we needed.

(**EE2**): Let $\vdash \alpha \to \beta$ and $\alpha \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$. Then by *closure*, $\beta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$.

(**EE3**) : There are three cases:

(1) $\alpha \notin \mathbf{K}\circ\neg(\alpha \wedge \beta)$: Then by *extensionality*, $\alpha \notin \mathbf{K} \circ \neg(\alpha \wedge (\alpha \wedge \beta))$, and by the definition of $\leq$ follows $\alpha \leq \alpha \wedge \beta$.

(2) $\beta \notin \mathbf{K}\circ\neg(\alpha \wedge \beta)$: Then $\beta \leq \alpha \wedge \beta$ follows in the same way.

(3) $\alpha \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$ and $\beta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$: Then by *closure*, $\alpha \wedge \beta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$, and it follows from the definition of $\leq$ that $\alpha \leq \alpha \wedge \beta$ and $\beta \leq \alpha \wedge \beta$.

(**EE4**): For one direction, let $\alpha \notin \mathbf{K}$. Then by *closure*, $\alpha\wedge\beta \notin \mathbf{K}$. It follows by *vacuity* that for all β, $\mathbf{K}\circ\neg(\alpha\wedge\beta) = \mathbf{K} + \neg(\alpha \wedge \beta)$. Now suppose that $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \beta)$. Then $\alpha \in \mathbf{K} + \neg(\alpha \wedge \beta)$, thus $\neg(\alpha \wedge \beta) \to \alpha \in \mathbf{K}$. Since $\mathbf{K}$ is logically closed, it follows that $\alpha \in \mathbf{K}$, contrary to our conditions. Hence $\alpha \notin \mathbf{K} \circ \neg(\alpha\wedge\beta)$. It follows from the definition of $\leq$ that $\alpha \leq \beta$. This holds for all β, which finishes this direction of the proof of (**EE4**).

For the other direction, let $\alpha \leq \beta$ for all β. Then in particular, $\alpha \leq \neg\alpha$, i.e., according to the definition of $\leq$, if $\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \neg\alpha)$, then $\neg\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \neg\alpha)$. Since $\mathbf{K}$ is consistent it follows from *vacuity* that $\mathbf{K} \circ \neg(\alpha \wedge \neg\alpha) = \mathbf{K}$. Hence, if $\alpha \in \mathbf{K}$ then $\neg\alpha \in \mathbf{K}$. Since $\mathbf{K}$ is consistent, it follows that $\alpha \notin \mathbf{K}$. This finishes this part of the proof.

$\circ$ **is entrenchment-based with respect to** $\leq$: We are going to show that $\beta \in \mathbf{K}\circ\alpha$ iff either (1) $\alpha \to \neg\beta < \alpha \to \beta$

or (2) $\neg\alpha$ is maximal and $\beta \in \mathbf{K}$.

For one direction, let $\beta \in \mathbf{K}\circ\alpha$. There are two cases.

First case, $\neg\alpha \in \mathbf{K}\circ\alpha$. Then it follows from *strong consistency* and *relative success* that $\mathbf{K}\circ\alpha = \mathbf{K}$, hence $\beta \in \mathbf{K}$. Let δ be any sentence. It follows from **Lemma 7.4.4**, since $\neg\alpha \in \mathbf{K}\circ\alpha$, that $\neg\alpha \in \mathbf{K} \circ \neg(\alpha \wedge \delta)$, hence by the definition of $\leq$, $\delta \leq \neg\alpha$, hence $\neg\alpha$ is maximal.

Second case, $\neg\alpha \notin \mathbf{K}\circ\alpha$. By *closure* and $\beta \in \mathbf{K}\circ\alpha$, we have $\alpha \to \beta \in \mathbf{K}\circ\alpha$. Hence, by *closure* $\alpha \to \neg\beta \notin \mathbf{K}\circ\alpha$. By *extensionality*, $\alpha \to \beta \in \mathbf{K} \circ ((\alpha \to \beta) \wedge (\alpha \to \neg\beta))$ and $\alpha \to \neg\beta \notin \mathbf{K} \circ ((\alpha \to \beta) \wedge (\alpha \to \neg\beta))$. The definition of $\leq$ yields $\alpha \to \neg\beta < \alpha \to \beta$.

For the other direction, there are again two cases.

First case, $\alpha \to \neg\beta < \alpha \to \beta$: Then, since $\alpha \to \beta \leq \alpha \to \neg\beta$ does not hold, it follows by *extensionality* from the definition of $\leq$ that $\alpha \to \beta \in \mathbf{K}\circ\alpha$ and $\alpha \to \neg\beta \notin \mathbf{K}\circ\alpha$. By *closure*, $\neg\alpha \notin \mathbf{K}\circ\alpha$. It then follows from *vacuity* and *relative success* that $\alpha \in \mathbf{K}\circ\alpha$, hence by *closure* $\beta \in \mathbf{K}\circ\alpha$.

Second case, $\neg\alpha$ is maximal and $\beta \in \mathbf{K}$: Then $\top \leq \neg\alpha$, hence (according to the definition of $\leq$) $\neg\alpha \in \mathbf{K}\circ\neg(\neg\alpha\wedge\top)$, by *extensionality* $\neg\alpha \in \mathbf{K}\circ\alpha$. By *relative success* and *strong consistency*, $\mathbf{K}\circ\alpha = \mathbf{K}$, hence $\beta \in \mathbf{K}\circ\alpha$.

(1)-to-(0) Let $\leq$ be an entrenchment relation satisfying (**EE1**)-(**EE4**) with respect to $\mathbf{K}$, and let $\circ$ be the operator that is based on $\leq$ in the manner of **Definition 7.2.4**. We need to show that the listed postulates hold.

Closure: Let $\varepsilon \in Cn(\mathbf{K}\circ\alpha)$. Then there is, by compactness of the underlying logic , a finite subset $\{\beta_1, ..., \beta_n\}$ of $\mathbf{K}{\circ}\alpha$ such that $\{\beta_1, ..., \beta_n\} \vdash \varepsilon$.

Part 1: We are first going to show that $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K}{\circ}\alpha$. For this purpose we are going to show that if $\beta_1 \in \mathbf{K}{\circ}\alpha$ and $\beta_2 \in \mathbf{K}{\circ}\alpha$, then $\beta_1 \wedge \beta_2 \in \mathbf{K}{\circ}\alpha$. The rest follows by iteration of the same procedure.

It follows from $\beta_1 \in \mathbf{K}{\circ}\alpha$, by the above definition of our entrenchment-based revision $\circ$, that either $(\alpha \rightarrow \neg\beta_1) < (\alpha \rightarrow \beta_1)$ or $\beta_1 \in \mathbf{K}$ and there is no δ such that $\neg\alpha < \delta$.

Case 1, $\beta_1 \in \mathbf{K}$ and there is no δ such that $\neg\alpha < \delta$: Then it does not hold that $\neg\alpha < \alpha \rightarrow \beta_2$. Hence, since $\beta_2 \in \mathbf{K}{\circ}\alpha$ we have (according to the same definition of $\circ$) $\beta_2 \in \mathbf{K}$. Since $\mathbf{K}$ is logically closed, we may conclude that $\beta_1 \wedge \beta_2 \in \mathbf{K}$. We also know that there is no δ such that $\neg\alpha < \delta$, and hence we may conclude that $\beta_1 \wedge \beta_2 \in \mathbf{K}{\circ}\alpha$.

Case 2, $\alpha \rightarrow \neg\beta_1 < \alpha \rightarrow \beta_1$: Equivalently, $\neg\alpha < \alpha \rightarrow \beta_1$. Then it follows from $\beta_2 \in \mathbf{K}{\circ}\alpha$ that $\neg\alpha < \alpha \rightarrow \beta_2$.

By (**EE3**), either $(\alpha \rightarrow \beta_1) \leq (\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2)$ or $(\alpha \rightarrow \beta_2) \leq (\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2)$. Equivalently (by **Property 2.5.37**), either $(\alpha \rightarrow \beta_1) \leq (\alpha \rightarrow (\beta_1 \wedge \beta_2))$ or $(\alpha \rightarrow \beta_2) \leq (\alpha \rightarrow (\beta_1 \wedge \beta_2))$. In the first case, we use (**EE1**) and $\neg\alpha < \alpha \rightarrow \beta_1$ to obtain $\neg\alpha < (\alpha \rightarrow \beta_1 \wedge \beta_2)$ and in the second case we use $\neg\alpha < \alpha \rightarrow \beta_2$ to obtain the same result. It follows that $\beta_1 \wedge \beta_2 \in \mathbf{K}{\circ}\alpha$.

Part 2: By repeated use of part 1, we know that $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K}{\circ}\alpha$. Let $\beta \leftrightarrow \beta_1 \wedge ... \wedge \beta_n$. We also have $\vdash \beta \rightarrow \varepsilon$. There are two cases.

Case 1, $\beta \in \mathbf{K}$ and there is no δ such that $\neg\alpha < \delta$. Then $\varepsilon \in \mathbf{K}$ follows from $\beta \in \mathbf{K}$, and the definition of $\circ$ yields $\varepsilon \in \mathbf{K}\circ\alpha$.

Case 2, $(\alpha \to \neg\beta) < (\alpha \to \beta)$. Since $\vdash (\alpha \to \beta) \to (\alpha \to \varepsilon)$, (**EE2**) yields $(\alpha \to \beta) \leq (\alpha \to \varepsilon)$. Since $\vdash (\alpha \to \neg\varepsilon) \to (\alpha \to \neg\beta)$, (**EE2**) yields $(\alpha \to \neg\varepsilon) \leq (\alpha \to \neg\beta)$. We can apply (**EE1**) to $(\alpha \to \neg\varepsilon) \leq (\alpha \to \neg\beta)$, $(\alpha \to \neg\beta) < (\alpha \to \beta)$, and $(\alpha \to \beta) \leq (\alpha \to \varepsilon)$, and obtain $(\alpha \to \neg\varepsilon) < (\alpha \to \varepsilon)$ from which follows $\varepsilon \in \mathbf{K}\circ\alpha$.

Relative success: Let $\alpha \notin \mathbf{K}\circ\alpha$. We have to show that $\mathbf{K}\circ\alpha = \mathbf{K}$.

By the definition of $\circ$ it follows from $\alpha \notin \mathbf{K}\circ\alpha$ that $\alpha \to \neg\alpha \not< \alpha \to \alpha$. By **Property 2.5.37**, $\neg\alpha \not< \top$. Hence $\neg\alpha$ is maximally entrenched. It follows from (**EE2**) that shows that $\neg\alpha \leq \alpha \to \beta$ and $\neg\alpha \leq \alpha \to \neg\beta$, so that $\alpha \to \beta$ and $\alpha \to \neg\beta$ are both maximally entrenched, thus both equally entrenched. Hence for all β, $\alpha \to \neg\beta \not< \alpha \to \beta$, hence by the definition of $\circ$, $\mathbf{K}\circ\alpha = \mathbf{K}$.

Inclusion: Let $\beta \in \mathbf{K}\circ\alpha$. We want to show that $\beta \in \mathbf{K}+\alpha$, which can be done by showing that $\alpha \to \beta \in \mathbf{K}$.

First case, $\alpha \to \neg\beta < \alpha \to \beta$: Then by (**EE4**), $\alpha \to \beta \in \mathbf{K}$.

Second case, $\beta \in \mathbf{K}$: Then $\alpha \to \beta \in \mathbf{K}$ follows from the closure of $\mathbf{K}$.

Strong consistency: Suppose to the contrary that $\bot \in \mathbf{K}\circ\alpha$. Since $\mathbf{K}$ is consistent, $\bot \notin \mathbf{K}$, and it follows by the definition of $\circ$ from $\bot \in \mathbf{K}\circ\alpha$ that $\alpha \to \top < \alpha \to \bot$. Then by **Property 2.5.37**, $\top < \neg\alpha$, which is impossible due to

(**EE2**).

Extensionality: Let $\vdash \alpha \leftrightarrow \alpha'$. It follows from (**EE2**) that $\alpha = \alpha'$. In the same way, it follows for all β that $\alpha \to \neg\beta = \alpha' \to \neg\beta$ and $\alpha \to \beta = \alpha' \to \beta$. By substitution into the definition of $\circ$ we obtain $\mathbf{K}\circ\alpha = \mathbf{K}\circ\alpha'$.

Strict improvement: Let $\alpha \in \mathbf{K}\circ\alpha$ and $\vdash \alpha \to \beta$. We need to prove that $\beta \in \mathbf{K}\circ\beta$. There are two cases according to the definition of $\circ$.

Case 1, $\alpha \in \mathbf{K}$ and there is no δ such that $\neg\alpha < \delta$. There are two subcases according to whether or not $\neg\beta$ is maximal.

Case 1a, $\neg\beta$ is maximal: Since we also have $\beta \in \mathbf{K}$, it follows directly from the definition of $\circ$ that $\beta \in \mathbf{K}\circ\beta$.

Case 1b, $\neg\beta$ is not maximal: Then $\neg\beta < \top$, by **Property 2.5.37** $\beta \to \neg\beta < \beta \to \beta$, hence $\beta \in \mathbf{K}\circ\beta$.

Case 2, $\alpha \to \neg\alpha < \alpha \to \alpha$. Then equivalently, $\neg\alpha < \top$. By (**EE2**), $\neg\beta \leq \neg\alpha$, hence $\neg\beta < \top$, by **Property 2.5.37** $\beta \to \neg\beta < \beta \to \beta$, hence $\beta \in \mathbf{K}\circ\beta$.

Vacuity: Let $\neg\alpha \notin \mathbf{K}$. We have to show that $\mathbf{K}+\alpha \subseteq \mathbf{K}\circ\alpha$. Let $\beta \in \mathbf{K} + \alpha$. Then $\alpha \to \beta \in \mathbf{K}$, and since $\neg\alpha \notin \mathbf{K}$, $\mathbf{K}$ is consistent. Therefore, $\neg\beta \notin \mathbf{K} + \alpha$, hence $\alpha \to \neg\beta \notin \mathbf{K}$, hence by (**EE4**) $\alpha \to \neg\beta < \alpha \to \beta$, then $\beta \in \mathbf{K}\circ\alpha$.

Strong regularity: Let $\neg\beta \notin \mathbf{K}\circ\alpha$. We have to prove that $\beta \in \mathbf{K}\circ\beta$. We are first going to prove that $\neg\beta < \top$. Suppose to the contrary that $\top \leq \neg\beta$. It follows from $\neg\beta \notin \mathbf{K}\circ\alpha$, according to the definition of $\circ$, that $\neg((\alpha \to \beta < (\alpha \to \neg\beta))$, hence by **Property 2.5.36**, $\neg(\neg\alpha < (\alpha \to \neg\beta))$, hence $(\alpha \to \neg\beta \leq \neg\alpha$. From this, $\top \leq \neg\beta$ and $\neg\beta \leq (\alpha \to \neg\beta)$

(that follows by (**EE2**)), we obtain $\top \leq \neg\alpha$. Hence, due **Property 2.5.39**, there is no δ such that $\neg\alpha < \delta$. From this and $\neg\beta \notin \mathbf{K}\circ\alpha$ it follows, due to the second clause of the definition of $\circ$, that $\neg\beta \notin \mathbf{K}$.

However, it also follows from the consistency of $\mathbf{K}$ that there must be some ϕ such that $\phi < \top$. Hence by (**EE1**), $\phi < \neg\beta$, hence by (**EE4**) $\neg\beta \in \mathbf{K}$. We can conclude from this contradiction that $\neg\beta < \top$.

Property 2.5.37, yields $(\beta \to \neg\beta) < (\beta \to \beta)$. According to the first clause of the definition of $\circ$, $\beta \in \mathbf{K}\circ\beta$.

Disjunctive factoring: There are three cases:

Case 1, $\alpha \notin \mathbf{K}\circ\alpha$ and $\beta \notin \mathbf{K}\circ\beta$. From $\alpha \notin \mathbf{K}\circ\alpha$ follows, via the definition of $\circ$, that $\alpha \to \neg\alpha \not< \alpha \to \alpha$, hence by **Property 2.5.20** and **Property 2.5.37** $\top \leq \neg\alpha$, hence by (**EE2**) $\neg\alpha$ is maximal. It follows from **Definition 7.2.4** that $\alpha \notin \mathbf{K}$. Hence, according to the same definition, for all δ, $\delta \in \mathbf{K}\circ\alpha$ iff $\delta \in \mathbf{K}$, hence $\mathbf{K}\circ\alpha = \mathbf{K}$. It follows in the same way that $\mathbf{K}\circ\beta = \mathbf{K}$. We need to show that $\mathbf{K}\circ(\alpha \vee \beta) = \mathbf{K}$.

Case 1a, let both $\neg\alpha$ and $\neg\beta$ be maximally entrenched. Then it follows by (**EE3**) that $\neg\alpha \wedge \neg\beta$ is maximally entrenched. Since $\neg\alpha \wedge \neg\beta$ is maximally entrenched, it follows from (**EE2**) that so is $(\alpha \vee \beta) \to \neg\delta$ for all δ. It then follows from **Definition 7.2.4** that $\mathbf{K}\circ(\alpha \vee \beta) = \mathbf{K}$.

Case 1b, $\neg\alpha$ is not maximally entrenched. We then have $\neg\alpha < \top$, hence $\neg\alpha < \alpha \to \alpha$, hence by the definition of $\circ$, $\alpha \in \mathbf{K}\circ\alpha = \mathbf{K}$. Since $\mathbf{K}$ is logically closed it follows that $\alpha \vee \beta \in \mathbf{K}$. Since $\mathbf{K}$ is consistent, *vacuity* yields

$\mathbf{K}\circ(\alpha \vee \beta) = \mathbf{K} + (\alpha \vee \beta) = \mathbf{K}$.

Case 2, $\alpha \notin \mathbf{K}\circ\alpha$ and $\beta \in \mathbf{K}\circ\beta$. We are first going to show that it is not the case that $\beta \in \mathbf{K}$ and $\neg\beta$ is maximally entrenched. Suppose to the contrary that this is the case. Since $\mathbf{K}$ is consistent, it then follows from (**EE4**) that, $\neg\beta$ is not minimal. Then according to (**EE4**), $\neg\beta \in \mathbf{K}$, contrary to the consistency of $\mathbf{K}$. We may conclude from this contradiction that it is not the case that $\beta \in \mathbf{K}$ and $\neg\beta$ is maximal.

Hence, since $\beta \in \mathbf{K}\circ\beta$, we can conclude from the definition of $\circ$ that $\beta \to \neg\beta < \beta \to \beta$, hence by **Property 2.5.37** $\neg\beta < \top$. By (**EE2**), $\neg\alpha \wedge \neg\beta \leq \neg\beta$, hence by (**EE1**) $\neg\alpha \wedge \neg\beta < \top$. Since $\top \leq \neg\alpha$, we have by (**EE3**) and (**EE2**) that $\neg\alpha \wedge \neg\beta = \neg\beta$. By **Property 2.5.37**, $\neg(\alpha \vee \beta = \neg\beta$. Let δ be any sentence. By (**EE2**), since $\neg\alpha$ is maximal, so is $\alpha \to \delta$. Hence by (**EE3**) and (**EE2**), $(\alpha \to \delta) \wedge (\beta \to \delta) = (\beta \to \delta)$. By **Property 2.5.37**, $(\alpha \vee \beta \to \delta) = (\beta \to \delta)$. Hence for all δ, $\neg\beta < \beta \to \delta$ if and only if $\neg\alpha\vee\beta < (\alpha\vee\beta) \to \delta$. Since neither $\neg\alpha \wedge \neg\beta$ nor $\neg\beta$ is maximal, it follows from the definition of $\circ$ that for all δ, $\delta \in \mathbf{K}\circ(\alpha \vee \beta)$ iff $\delta \in \mathbf{K}\circ\beta$.

Case 3, $\alpha \in \mathbf{K}\circ\alpha$ and $\beta \in \mathbf{K}\circ\beta$: Using the symmetry of this case, we have two subcases.

Case 3a, $\neg\alpha < \neg\beta$: For one direction, let $\delta \in \mathbf{K}\circ\alpha$. Then, since $\neg\alpha$ is not maximal, according to the definition of $\circ$ we have $\neg\alpha < \alpha \to \delta$. It also follows from $\neg\alpha < \neg\beta$, by **Property 2.5.35** that $\neg\alpha = \neg\alpha \wedge \neg\beta$. Since (**EE2**) yields $\neg\beta \leq \beta \to \delta$, we can use (**EE1**) to obtain both

$\neg\alpha \wedge \neg\beta < \beta \rightarrow \delta$ and $\neg\alpha \wedge \neg\beta < \alpha \rightarrow \delta$. (**EE3**) yields $\neg\alpha \wedge \neg\beta < (\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)$, hence by **Property 2.5.37** $\neg\alpha \wedge \neg\beta < (\alpha \vee \beta \rightarrow \delta)$, hence $\delta \in \mathbf{K}\circ(\alpha \vee \beta)$.

For the other direction, let $\delta \in \mathbf{K}\circ(\alpha \vee \beta)$. It follows from $\neg\alpha < \neg\beta$ that $\neg\alpha \wedge \neg\beta = \neg\alpha$, hence $\neg\alpha \wedge \neg\beta$ is not maximal, hence it follows from $\delta \in \mathbf{K}\circ(\alpha \vee \beta)$ that $\neg\alpha \wedge \neg\beta < (\alpha \vee \beta) \rightarrow \delta$. By (**EE2**), $(\alpha \vee \beta) \rightarrow \delta \leq \alpha \rightarrow \delta$. (**EE1**) yields $\neg\alpha < \alpha \rightarrow \delta$, hence $\delta \in \mathbf{K}\circ\alpha$.

Case 3b, $\neg\alpha = \neg\beta$. Then $\neg\alpha = \neg\beta = \neg\alpha \wedge \neg\beta$. For one direction, let $\delta \in \mathbf{K}\circ\alpha \cap \mathbf{K}\circ\beta$. Then $\neg\alpha < \alpha \rightarrow \delta$ and $\neg\beta < \beta \rightarrow \delta$. Then by (**EE2**) and (**EE1**) $\neg\alpha \wedge \neg\beta < \alpha \rightarrow \delta$ and $\neg\alpha \wedge \neg\beta < \beta \rightarrow \delta$. (**EE3**) and (**EE2**) yield $\neg\alpha \wedge \neg\beta < (\alpha \vee \beta) \rightarrow \delta$. Hence $\delta \in \mathbf{K}\circ(\alpha \vee \beta)$.

For the other direction, let $\delta \in \mathbf{K}\circ(\alpha \vee \beta)$. Then we have $\neg\alpha \wedge \neg\beta < (\alpha \vee \beta) \rightarrow \delta$. We already know that $\neg\beta = \neg\alpha \wedge \neg\beta$, and (**EE2**) yields $(\alpha \vee \beta) \rightarrow \delta \leq \beta \rightarrow \delta$. Using (**EE1**) to combine this, we obtain $\neg\beta < \beta \rightarrow \delta$, hence $\delta \in \mathbf{K}\circ\beta$.

(1)-to-(2) Let $\leq$ be a relation on $\mathcal{L}$ that satisfies (**EE1**)$-$(**EE4**) with respect to the consistent belief set **K**. Furthermore, let $\$_{\leq}$ be the set such that $X \in \$_{\leq}$ iff it satisfies the following four conditions:

($\$_{\leq}1$) $\|\mathbf{K}\| \subseteq X$.

($\$_{\leq}2$) $X = \cap\{\|\alpha\| \mid X \subseteq \|\alpha\|\}$

($\$_{\leq}3$) $X \subseteq \mathcal{P}(\mathcal{L} \perp\perp$.

($\$_{\leq}4$) For all α, $\beta \in \mathcal{L}$, if $X \subseteq \|\alpha\|$ and $\alpha \leq \beta$, then $X \subseteq \|\beta\|$.

Let $S_\alpha = \cap G \in \$_\leq \mid G \cap \|\alpha\| \neq \emptyset$, and let $Th(\cap \$_\leq) = \mathbf{K}$. We need to prove that $\$_\leq$ is a sphere system around $\mathbf{K}$, i.e., that it satisfies $\$1 - \$5$, and that the revision operator based on it, for all inputs, yields the same outcome as the entrenchment-based revision operator based on $\leq$ in the manner of **Definition 7.2.4**. (Note in what follows that $\|\{\delta \mid \alpha \leq \delta\}\| = \cap\{\|\delta\| \mid \alpha \leq \delta\}$.)

Intermediate result A: If $\bot < \alpha$, then $\|\{\delta \mid \alpha \leq \delta\|\} \in \$_\leq$.

Proof: We need to show that $\|\{\delta \mid \alpha \leq \delta\}\|$ satisfies conditions $(\$_\leq 1) - (\$_\leq 4)$. $(\$_\leq 1)$: Let $u \in \|\mathbf{K}\|$ and let $\alpha \leq \delta$. Since $\bot < \alpha$, (**EE1**) yields $\bot < \delta$, so that $\delta \in \mathbf{K}$, hence $u \in \|\delta\|$. We can conclude that $u \in \|\{\delta \mid \alpha \leq \delta\|\}$.

$(\$_\leq 2)$: It follows by set theory that $\|\{\delta \mid \alpha \leq \delta\}\| \subseteq \cap\{\|\varepsilon\| \mid \|\{\delta \mid \alpha \leq \delta\}\| \subseteq \|\varepsilon\|\}$. For the other direction, let $u \notin \|\{\delta \mid \alpha \leq \delta\}\|$. Then there is some δ such that $\alpha \leq \delta$, $u \notin \|\delta\|$ and $\|\{\delta \mid \alpha \leq \delta\|\} \subseteq \|\delta\|$. Then $u \notin \cap\{\|\varepsilon\| \mid \|\{\delta \mid \alpha \leq \delta\}\| \subseteq \|\varepsilon\|\}$.

$(\$_\leq 3)$: It follows directly from the definition.

$(\$_\leq 4)$:, let $\|\{\delta \mid \alpha \leq \delta\}\| \subseteq \|\delta\|$ and $\delta \leq \varepsilon$. According to **Lemma 7.4.2**, $\alpha \leq \delta$, by (**EE1**) $\alpha \leq \varepsilon$, hence by **Lemma 7.4.2**, $\|\{\delta \mid \alpha \leq \delta\|\} \subseteq \|\varepsilon\|$.

\$4 holds: In order to show that $\cup \$_\leq \in \$_\leq$, we have to show that $\cup \$_\leq$ satisfies conditions $\$_\leq 1 - \$_\leq 4$.

$\$_\leq 1$: It follows from $(\$_\leq 1)$ that for all $X \in \$_\leq$, $\|\mathbf{K}\| \subseteq X$. Hence, $\|\mathbf{K}\| \subseteq \cup \$_\leq$.

$\$_\leq 3$: It follows directly from the definition.

$\$_{\leq}2$: We need to show that $\cup\$_{\leq} = \cap\{\|\alpha\| \mid \cup\$_{\leq} \subseteq \alpha\}$. The left-to-right inclusion is obvious. For the right-to-left inclusion, we first need to show that $\cup\$_{\leq} = \cap\{\|\alpha\| \mid \top \leq \alpha\}$.
Let $X \in \$_{\leq}$. Furthermore, let $\top \leq \alpha$. It follows from $(\$_{\leq}3)$ that $X \subseteq \|T\|$, and hence from $(\$_{\leq}4)$ that $X \subseteq \|\alpha\|$. Hence, $X \subseteq \cap\{\|\alpha\| \mid \top \leq \alpha\}$ Since **K** is consistent, $\bot < \top$, hence due to the intermediate result A, $\cap\{\|\alpha\| \mid \top \leq \alpha\} \in \$_{\leq}$. Combining these two results, we obtain $\cup\$_{\leq} = \cap\{\|\alpha\| \mid \top \leq \alpha\}$.
It follows from **Lemma 7.4.2** that $\top \leq \alpha$ iff $\cap\{\|\alpha\| \mid \top \leq \alpha\} \subseteq \|\alpha\|$. Hence $\cap\{\|\alpha\| \mid \top \leq \alpha\} = \cap\{\|\alpha\} \mid \cap\{\|\alpha\| \mid \top \leq \alpha\} \subseteq \|\alpha\|\| = \cap\{\|\alpha\| \mid \cup\$_{\leq} \subseteq \|\alpha\|\} = \cup\$_{\leq}$.
$\$_{\leq}4$: Let $\cup\$_{\leq} \subseteq \|\alpha\|$ and $\alpha \leq \beta$. Then it holds for each $X \in \$_{\leq}$ that $X \subseteq \|\alpha\|$, hence, according to $(\$_{\leq}4)$, $X \subseteq \|\beta\|$. Since this holds for all $X \in \$_{\leq}$, we have $\cup\$_{\leq} \subseteq \|\beta\|$ as desired.

\$1 holds: This follows trivially from \$4.

bf \$3 holds: Let $X, Y \in \$_{\leq}$ and $X \not\subseteq Y$. Then according to $(\$_{\leq}2)$ there is some $\|\alpha\|$ such that $Y \subseteq \|\alpha\|$ and $X \not\subseteq \|\alpha\|$. In order to show that $Y \subseteq X$ is it sufficient, according to $(\$_{\leq}2)$, to show that if $X \subseteq \|\beta\|$, then $Y \subseteq \|\beta\|$. Let $X \subseteq \|\beta\|$. Then since $X \not\subseteq \|\alpha\|$ we can conclude from $(\$_{\leq}4)$ that $\beta \leq \alpha$ does not hold, hence by the **Property 2.5.20** of the entrenchment relation (**Property 2.5.20**), $\alpha \leq \beta$. Since $Y \subseteq \|\alpha\|$ we can again apply $(\$_{\leq}4)$, and obtain $Y \subseteq \|\beta\|$ as desired.

Intermediate result B: Let D be a non-empty subset of

$\$_{\leq}$. Then $\cap D \in \$_{\leq}$.

Proof: We have to show that $\cap D$ satisfies the conditions $(\$_{\leq 1}) - (\$_{\leq}4)$.

$(\$_{\leq}1)$ Since $\|\mathbf{K}\|$ is a subset of each element of D, it is a subset of $\cap D$.

$(\$_{\leq}2)$ It follows directly that $\cap D \subseteq \cap\{\|\delta\| \mid \cap D \subseteq \|\delta\|\}$. In order to show that $\cap\{\|\delta\| \mid \cap D \subseteq \|\delta\|\} \subseteq \cap D$, let $u \notin \cap D$. Then there is some $X \in D$ such that $u \notin X$. Then there must be some δ such that $X \subseteq \|\delta\|$ and $u \notin \|\delta\|$. It follows that $\cap D \subseteq \|\delta\|$. Hence $u \notin \cap\{\|\delta\| \mid \cap D \subseteq \|\delta\|\}$.

$(\$_{\leq}3)$ Since each element of D is a subset of $\mathcal{P}(\mathcal{L} \perp\!\perp$, so is $\cap D$.

$(\$_{\leq}4)$ Let $\cap D \subseteq \|\alpha\|$ and $\alpha \leq \beta$. Since \$3 holds, it follows from **Lemma 7.4.6** that there is some X such that $X \in D$ and $X \subseteq \|\alpha\|$. Then $X \subseteq \|\beta\|$, hence $\cap D \subseteq \|\beta\|$.

\$2 holds: This follows directly from intermediate result B.

\$5 holds: Let $\|\alpha\| \cap (\cup\$_{\leq}) \neq \emptyset$. We can conclude from \$4 that was proved above that $\{G \in \$_{\leq} \mid G \cap \|\alpha\|\} \neq \emptyset$. It follows from intermediate result B that $\cap\{G \in \$_{\leq} \mid G \cap \|\alpha\| \neq \emptyset\} \in \$_{\leq}$, and by the definition for this part of the proof, $S_\alpha = \cap\{G \in \$_{\leq} \mid G \cap \|\alpha\| \neq \emptyset\}$. Hence $S_\alpha \in \$_{\leq}$. Suppose for *reductio* that $S_\alpha \cap \|\alpha\| = \emptyset$. Then $S_\alpha \subseteq \|\neg\alpha\|$, i.e., $\cap\{G \in \$_{\leq} \mid G \cap \|\alpha\| \neq \emptyset\} \subseteq \|\neg\alpha\|$. It follows from **Lemma 7.4.6** that there is some $X \in S_\alpha$ such that $X \subseteq \|\neg\alpha\|$, contrary to the definition of S_α .

Proof that $Th(\cap\$_{\leq}) = \mathbf{K}$: For one direction, let $\alpha \in Th(\cap\$_{\leq})$. Then $\cap\$_{\leq} \subseteq \|\alpha\|$. Suppose that $\cap\$_{\leq} \subseteq \|\perp\|$.

Then, according to \$2 that was shown above, $\emptyset \in \$_{\leq}$, hence according to $(\$_{\leq}1)$, $\|\mathbf{K}\| \subseteq \emptyset$, contrary to the consistency of $\mathbf{K}$. Hence $\cap\$_{\leq} \not\subseteq \|\bot\|$. According to \$2, $\cap\$_{\leq} \in \$_{\leq}$. Hence we can use $(\$_{\leq}4)$ to conclude from $\cap\$_{\leq} \subseteq \|\alpha\|$ and $\cap\$_{\leq} \not\subseteq \|\bot\|$ that $\alpha \not\leq_{\bot}$, hence by the **Property 2.5.20** of $\leq$ $\bot < \alpha$, hence $\alpha \in \{\beta \mid \bot < \beta\}$. It follows from (**EE4**) that $\mathbf{K} = \{\beta \mid \bot < \beta\}$. Hence $\alpha \in \mathbf{K}$, which is sufficient for this direction of the proof.

For the other direction, let $\alpha \in \mathbf{K}$. We are going to show $\|\mathbf{K}\| \in \$_{\leq}$, from which follows $\cap\$_{\leq} \subseteq \|\mathbf{K}\|$ and hence $\mathbf{K} \subseteq Th(\cap\$_{\leq})$ as desired. We need to show that $(\$_{\leq}1) - (\$_{\leq}4)$ are satisfied. The proofs of $(\$_{\leq}1) - (\$_{\leq}3)$ are trivial. For $(\$_{\leq}4)$, let $\|\mathbf{K}\| \subseteq \|\alpha\|$ and $\alpha \leq \beta$. Then $\alpha \in \mathbf{K}$, by (**EE4**) $\bot < \alpha$, hence by (**EE1**) $\bot < \beta$, then $\beta \in \mathbf{K}$, hence $\|\mathbf{K}\| \subseteq \|\beta\|$.

Intermediate result C: If $\neg\alpha < \top$, then $S_\alpha = \cap\{\|\beta\| \mid \neg\alpha < \beta\}$.

Proof: One direction: Let $u \in S_\alpha$ and suppose for *contradictio* that $u \notin \cap\{\|\beta\| \mid \neg\alpha < \beta\}$. Then there is some β such that $\neg\alpha < \beta$ and $u \notin \|\beta\|$. It follows from intermediate result A that $\cap\{\|\delta\| \mid \beta \leq \delta\} \in \$_{\leq}$. Since $\beta \leq \beta$, we have $\cap\{\|\delta\| \mid \beta \leq \delta\} \subseteq \|\beta\|$. By **Lemma 7.4.2**, $\cap\{\|\delta\| \mid \beta \leq \delta\} \not\subseteq \|\neg\alpha\|$.

It follows from $u \in S_\alpha$ and $u \notin \|\beta\|$ that $S_\alpha \not\subseteq \|\beta\|$. We already know that either $\cap\{\|\delta\| \mid \beta \leq \delta\} \subseteq S_\alpha$ or $S_\alpha \subseteq \cap\{\|\delta\| \mid \beta \leq \delta\}$. Since $\cap\{\|\delta\| \mid \beta \leq \delta\} \not\subseteq \|\neg\alpha\|$ and $\cap\{\|\delta\| \mid \beta \leq \delta\} \subseteq \|\beta\|, S_\alpha \not\subseteq \cap\{\|\delta\| \mid \beta \leq \delta\}$. Hence

$\cap\{\|\delta\| \mid \beta \leq \delta\} \subset S_\alpha$. Since S_α is the smallest sphere that is not a subset of $\|\neg\alpha\|$, $\cap\{\|\delta\| \mid \beta \leq \delta\} \subseteq \|\neg\alpha\|$, contrary to what was shown above.

Other direction: Let $u \in \cap\{\|\beta\| \mid \neg\alpha < \beta\}$. Then $u \in \|\beta\|$ for all β such that $\neg\alpha < \beta$. Suppose for *contradictio* that $u \notin S_\alpha$. Then there is some β such that $S_\alpha \subseteq \|\beta\|$ and $u \notin \|\beta\|$. It follows that $\neg\alpha < \beta$ does not hold, hence that $\beta \leq \neg\alpha$. It follows from $S_\alpha \subseteq \|\beta\|$ and $\beta \leq \neg\alpha$ that $S_\alpha \subseteq \|\neg\alpha\|$. This contradicts the definition of S_α.

Intermediate result D: $\alpha < \top$ iff $\cup\$_\leq \not\subseteq \|\alpha\|$.

Proof: As was shown in the proof of \$4, $\cup\$_\leq = \cap\{\|\beta\| \mid \top \leq \beta\}$. Hence, according to **Lemma 7.4.2**, $\alpha < \top$ iff $\cup\$_\leq \not\subseteq \|\alpha\|$.

Revision-equivalence holds: Let $\circ_\leq$ be the credibility-limited revision operator based on $\leq$ in the manner of **Definition 7.2.4**, i.e.,

$$\mathbf{K} \circ_\leq \alpha = \begin{cases} \{\beta \mid \neg\alpha < \alpha \to \beta & \text{if } \neg\alpha < \top \\ \mathbf{K} & \text{otherwise} \end{cases}$$

Let $\circ_\$$ be the operator based on $\$_\leq$ in the manner for **Definition 7.2.7**, i.e., let

$$\mathbf{K} \circ_\$ \leq \alpha = \begin{cases} Th(\|\alpha\| \cap S_\alpha) \text{ if } \|\alpha\| \cap (\cup\$_\leq) \neq \emptyset \\ \mathbf{K} \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

We need to show that for all $\alpha, \mathbf{K} \circ_\leq \alpha = \mathbf{K} \circ \$_\leq \alpha$.

For one direction, let $\beta \in \mathbf{K} \circ_\leq \alpha$. First case, $\neg\alpha < \top$:

Then it follows from the definition of $\circ_\leq$ that $\neg\alpha < \alpha \to \beta$. From intermediate result D follows $\cup\$_\leq \not\subseteq \|\neg\alpha\|$, so that $\|\alpha\| \cap (\cup\$_\leq) \neq \emptyset$. Since $\neg\alpha < \alpha \to \beta$, intermediate result C yields $S_\alpha \subseteq \|\alpha \to \beta\|$. Hence $S_\alpha \subseteq \|\neg\alpha\| \cup \|\beta\|$, so that $S_\alpha \cap \|\alpha\| \subseteq \|\beta\|$, and thus $\beta \in Th(\|\alpha\| \cap S_\alpha)$.

Second case, $\neg\alpha < \top$ does not hold. Then $\mathbf{K} \circ_\leq \alpha = \mathbf{K}$. By intermediate result D, $\cup\$_\leq \subseteq \|\neg\alpha\|$, hence $\|\alpha\| \cap (\cup\$_\leq) = \emptyset$, hence we are in the second clause so that $\mathbf{K} \circ \$_\leq \alpha = \mathbf{K}$.

For the other direction, let $\beta \in \mathbf{K} \circ \$_\leq \alpha$. First case, $\|\alpha\| \cap (\cup\$_\leq) \neq \emptyset$ and $\beta \in Th(\|\alpha\| \cap S_\alpha)$. It follows from intermediate result D that $\neg\alpha < \top$. It follows from $\beta \in Th(\|\alpha\| \cap S_\alpha)$ that $(\|\alpha\| \cap S_\alpha) \subseteq \|\beta\|$, hence $S_\alpha \subseteq \|\alpha \to \beta\|$. Intermediate result C yields $\cap\{\|\delta\| \mid \neg\alpha < \delta\} \subseteq \|\alpha \to \beta\|$, hence by compactness there are $\beta_1, ...\beta_n$ such that for each of these β_i, $\neg\alpha < \beta_i$ and that $\{\beta_1, ...\beta_n\} \vdash \alpha \to \beta$. By **Property 2.5.23**, $\neg\alpha < (\beta_1 \wedge ... \wedge \beta_n)$. By (**EE2**), $(\beta_1 \wedge ... \wedge \beta_n) \leq \alpha \to \beta$. By (**EE1**), $\neg\alpha < \alpha \to \beta$. Hence, $\beta \in \mathbf{K} \circ_\leq \alpha$.

The second case is just the reverse of the second case of the first direction of the proof.

(2)-to-(1) Given the sphere system \$ around $\mathbf{K}$, we define the following entrenchment relation for $\mathbf{K}$:

$\alpha \leq \beta$ iff it holds for all $S \in \$$ that if $S \subseteq \|\alpha\|$ then $S \subseteq \|\beta\|$.

We need to prove that $\leq$ satisfies (**EE1**) – (**EE4**) and that the entrenchment-based operator $\circ_\leq$ that it gives rise to is identical with the sphere-based operator $\circ_\$$ that is based on \$.

(**EE1**): Trivial.

(**EE2**): If $\alpha \vdash \beta$, then $\|\alpha\| \subseteq \|\beta\|$, the rest is trivial.

(**EE3**): Let it be the case that not $\alpha \leq (\alpha \wedge \beta)$. Then there is some $S \in \$$ such that $S \subseteq \|\alpha\|$ and $S \not\subseteq \|\alpha \wedge \beta\|$. Hence $S \not\subseteq \|\beta\|$. Let $S' \subseteq \|\beta\|$. Clearly, either $S \subseteq S'$ or $S' \subseteq S$. But if $S \subseteq S'$, then we would have $S \subseteq \|\beta\|$, contrary to the conditions. Hence $S' \subseteq S$. Since $S \subseteq \|\alpha\|$ we then also have $S' \subseteq \|\alpha\|$, hence $S' \subseteq \|\alpha\| \cap \|\beta\|$, or equivalently $S' \subseteq \|\alpha \wedge \beta\|$.

(**EE4**): Since **K** is consistent, we have to show that then $\alpha \leq \bot$ iff $\alpha \notin \mathbf{K}$. (By (**EE2**), this is equivalent with the formulation used in the definition.)

For one direction, let $\alpha \leq \bot$.Then, since $\|\mathbf{K}\|$ is a sphere, we can use the definition of $\leq$ to obtain:

If $\|\mathbf{K}\| \subseteq \|\alpha\|$, then $\|\mathbf{K}\| \subseteq \|\bot\|$.

Equivalently, if $\alpha \in \mathbf{K}$, then $\bot \in \mathbf{K}$. Since **K** is consistent, $\alpha \notin \mathbf{K}$.

For the other direction, let $\alpha \notin \mathbf{K}$. Then $\cap\$ \not\subseteq \|\alpha\|$. Then for all $S \in \$$, $S \not\subseteq \|\alpha\|$. Hence for all $S \in \$$, it follows vacuously that if $S \subseteq \|\alpha\|$, then $S \subseteq \|\bot\|$.

Identity of the revision operators: First direction: Let $\beta \in \mathbf{K} \circ_{\$} \alpha$. We have two cases.

First case, $\cup\$ \not\subseteq \|\neg\alpha\|$: Then $\beta \in Th(S_\alpha \cap \|\alpha\|)$, hence $S_\alpha \cap \|\alpha\| \subseteq \|\beta\|$, so that $S_\alpha \subseteq \|\neg\alpha \vee \beta\|$. By definition, $S_\alpha \not\subseteq \|\neg\alpha\|$.

It follows from (**EE2**) that $\neg\alpha \leq \neg\alpha \vee \beta$. From $S_\alpha \subseteq \|\neg\alpha \vee \beta\|$ and $S_\alpha \not\subseteq \|\neg\alpha\|$ follows that $\neg\alpha \vee \beta \leq \neg\alpha$ does not hold, hence $\neg\alpha < \neg\alpha \vee \beta$. It follows from the definition of $\circ_{\leq}$ that $\beta \in \mathbf{K} \circ_{\leq} \alpha$.

Second case, $\cup\$ \subseteq \|\neg\alpha\|$: Then $\mathbf{K} \circ_{\$} \alpha = \mathbf{K}$ by the definition of $\circ_{\$}$, hence $\beta \in \mathbf{K}$. Next, let S be any sphere such that $S \subseteq \|\top\|$. Then $S \subseteq \cup\$ \subseteq \|\neg\alpha\|$. Hence $\top \leq \neg\alpha$.

Second direction: Let $\beta \in \mathbf{K} \circ_{\leq} \alpha$. According to the definition of $\circ_{\leq}$ there are two cases:

First case, $\neg\alpha < \alpha \rightarrow \beta$: Rewriting this condition, using the definition above of $\leq$, it holds (a) that for all $G \in \$$, if $G \subseteq \|\neg\alpha\|$ then $G \subseteq \|\neg\alpha \vee \beta\|$, and (b) that there is some $G' \in \$$ such that $G' \subseteq \|\neg\alpha \vee \beta\|$ and $G' \not\subseteq \|\neg\alpha\|$.

It follows from $G' \not\subseteq \|\neg\alpha\|$ that $G' \cap \|\alpha\| \neq \emptyset$, so that $S_\alpha \subseteq G'$. Hence $S_\alpha \subseteq \|\neg\alpha \vee \beta\| = \|\neg\alpha\| \cup \|\beta\|$, hence $S_\alpha \cap \|\alpha\| \subseteq \|\beta\|$, from which follows that $\beta \in Th(S_\alpha \cap \|\alpha\|)$. We also know from $G' \not\subseteq \|\neg\alpha\|$ that $G' \cap \|\alpha\| \neq \emptyset$, hence $(\cup\$) \cap \|\alpha\| \neq \emptyset$. It follows from this and $\beta \in Th(S_\alpha \cap \|\alpha\|)$ that $\beta \in \mathbf{K} \circ_{\$} \alpha$.

Second case, $\top \leq \neg\alpha$ and $\beta \in \mathbf{K}$. Let $G \in \$$. Then it follows from $\top \leq \neg\alpha$ and the definition of $\leq$ that if $G \subseteq \|\top\|$ then $G \subseteq \|\neg\alpha\|$. Since $G \subseteq \|\top\|$ is true for all $G \in \$$, it follows that $G \subseteq \|\neg\alpha\|$ for all $G \in \$$, hence $\cup\$ \subseteq \|\neg\alpha\|$, so that $\cup\$ \cap \|\alpha\| = \emptyset$.

It follows by (**EE4**) from $\beta \in \mathbf{K}$, i.e., (by **Property 2.5.38**) that $\perp < \beta$ and by the definition of $\leq$, that there is some $G \in \$$ such that $G \subseteq \|\beta\|$. Hence $\cap\$ \subseteq \|\beta\|$, so that $\beta \in Th(\cap\$) = \mathbf{K}$. It follows from this and $\cup\$ \cap \|\alpha\| = \emptyset$ that $\beta \in \mathbf{K} \circ_{\$} \alpha$. ■

Chapter 8

Shielded Contraction

8.1 Introduction

In **Chapter 5** we show that one of the major recent developments in the theory of belief change is the construction of models of non-prioritized belief change in which the *success* postulate is not satisfied.

In **Chapter 3** we analized the controversial postulate of *recovery*. In this chapter, we venture to question another of the basic Gärdenfors postulates for contraction, *success* (If $\nvdash \alpha$, then $\mathbf{K}-\alpha \nvdash \alpha$). The success postulate can be interpreted as requiring that all non-tautological beliefs are retractible. As was observed by Rott [Rot92b, p.54], this is not a fully realistic requirement, since actual doxastic agents are known to have beliefs (of a non-logical nature) that nothing can bring them to give up. It should be of interest, therefore, to develop models of contraction in which some non-tautological beliefs may be shielded from contraction; in short: models of shielded contraction.

In this chapter we propose three models of shielded contraction. These models are axiomatically characterized. Furthermore, a close connection to credibility-limited revision operators is shown to hold, expressible in terms of the Harper identity and a modified version of the Levi Identity.

In **Section 8.2**, postulates for shielded contraction are proposed. In **Section 8.3**, three types of constructions are proposed, and in **Section 8.4** they are axiomatically characterized. In **Section 8.5**, connections with credibility-limited revision are explored.

The results of this chapter appeared in:

[•] Shielded contraction. In H.Rott and M-A Williams, editors, *Frontiers in Belief Revision*. Kluwer Academic Publisher, 1999. to appear.

8.2 Postulates for Shielded Contraction

In **Section 2.4** we analyze the AGM postulates. We are going to focus on the effects of giving up the success postulate, and will therefore not question any of the other postulates here.

The following weakened version of success is a plausible property for shielded contraction:

●**Persistence:** If $\mathbf{K} \div \beta \vdash \beta$, then $\mathbf{K} \div \alpha \vdash \beta$.

According to persistence, if a sentence β is at all removable from the belief set, then it is removed in contraction by itself.

If an input sentence cannot be removed, then the original belief set should be retained:

●**Relative Success:** [Rot92b] $\mathbf{K} \div \alpha = \mathbf{K}$ or $\mathbf{K} \div \alpha \nvdash \alpha$.

A sentence α such that $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha \vdash \alpha$ can be called an *irretractible* sentence. Given relative success, a sentence $\alpha \in \mathbf{K}$ is irretractible if and only if $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha = \mathbf{K}$. Shielded contraction differs from classical contraction in that non-tautological sentences may be irretractible.

If two sentences α and β are both irretractible, then we should expect their conjunction to also be irretractible:

- **Conjunctive Constancy:** If $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha = \mathbf{K} \mathbin{-\!\!\circ\!\!-} \beta = \mathbf{K}$ then $\mathbf{K} \mathbin{-\!\!\circ\!\!-} (\alpha \wedge \beta) = \mathbf{K}$.

It can also be reasonably expected that the logical consequence of an irretractible belief should itself be irretractible:

- **Success Propagation:** If $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \beta \vdash \beta$ and $\vdash \beta \rightarrow \alpha$ then $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha \vdash \alpha$.

Success propagation can be shown to follow from *persistence*:

Observation 8.2.1 If an operator $\mathbin{-\!\!\circ\!\!-}$ on the belief set $\mathbf{K}$ satisfies *persistence*, then it satisfies *success propagation*.

The new postulates introduced in this section can be shown to all follow from the AGM contraction postulates (including *success*):

Observation 8.2.2 Let $\mathbf{K}$ be a belief set and $-$ an operator on $\mathbf{K}$ that satisfies *closure, inclusion, vacuity, success* and *recovery*. Then $-$ satisfies *relative success, persistence* and *conjunctive constancy*.

Observation 8.2.3 Let $\mathbf{K}$ be a belief set and $\mathbin{-\!\!\circ\!\!-}$ an operator on $\mathbf{K}$ that satisfies *inclusion* and *conjunctive overlap*. Then it satisfies *conjunctive constancy*.

8.3 Three constructions of shielded contraction

In this section, we are going to introduce three alternative constructions of shielded contraction. They are all derived through minor adjustments of well-known constructions of conventional contraction.

A fairly obvious method to construct shielded contraction is to divide the language into two parts, the retractible and the irretractible sentences, and apply a conventional contraction operator to the retractible sentences. This can be done as follows:

> **Definition 8.3.1** Let $\mathbf{K}$ be a belief set, $-$ an AGM contraction operator on $\mathbf{K}$ and $\mathcal{R}$ a subset of $\mathcal{L}$ (the set of retractible sentences). Then $\div$ is the *shielded AGM contraction* induced by $-$ and $\mathcal{R}$ if and only if:

$$\mathbf{K} \div \alpha = \begin{cases} \mathbf{K} - \alpha & \text{if } \alpha \in \mathcal{R} \\ \mathbf{K} & \text{otherwise} \end{cases}$$

This construction can be further specified by adding requirements on the structure of $\mathcal{R}$.

Since a conjunctive sentence $\alpha \wedge \beta$ can be removed from a belief set only if at least one of its conjuncts α and β is removed, we should expect that $\alpha \wedge \beta$ cannot be retractible without either α or β being so.

- **Conjunctive Completeness:** If $\alpha \wedge \beta \in \mathcal{R}$ then $\alpha \in \mathcal{R}$ or $\beta \in \mathcal{R}$.

The irretractible sentences are such that they cannot be removed from $\mathbf{K}$, no matter what contraction we perform, hence:

●**Non-Retractability Preservation:** $\mathcal{L} \setminus \mathcal{R} \subseteq \mathbf{K} \div \alpha$.

Next, suppose that α is irretractible, $\vdash \alpha \rightarrow \beta$ and β is retractible. Then $\beta \notin \mathbf{K} \div \beta$, hence $\alpha \notin \mathbf{K} \div \beta$, so that α can be retracted after all. This implausible combination is precluded by the following condition:

●**Non-Retractability Propagation:** If $\alpha \notin \mathcal{R}$, then $Cn(\{\alpha\}) \cap \mathcal{R} = \emptyset$.

A quite different approach to the construction of shielded contraction is to base it on an entrenchment relation. In **Section 2.5** we presented the standard entrenchment postulates (transitivity, dominance, conjunctiveness, minimality, and maximality).

Gärdenfors's maximality property (**EE5**) says that only tautologies can be maximally entrenched. This is exactly the condition that we want to relax: in shielded contraction non-tautologies may be maximally entrenched. Therefore, as was noted by Rott [Rot92b, p.54], an obvious way to modify entrenchment-based contraction for our purposes is to just withdraw this property (**EE5**) from the definition. In addition, however, a minor modification of the ($-_G$) condition is necessary: The clause "either $\vdash \alpha$ or" has the purpose of ensuring that $\mathbf{K} \div \alpha = \mathbf{K}$ whenever α is maximally entrenched, i.e., a tautology. It would make no difference to replace $\vdash \alpha$ here by "$\beta \leq_{\mathbf{K}} \alpha$ for all β" or, equivalenty "$\alpha \not<_{\mathbf{K}} \top$". However, when (**EE5**) has been removed, this replacement is mandatory, since tautologies and maximally entrenched sentences no longer coincide. We therefore arrive at the following definition:

Definition 8.3.2 Let $\mathbf{K}$ be a belief set and $\leq_{\mathbf{K}}$ a relation satisfying (**EE1**) – (**EE4**) with respect to $\mathbf{K}$. Then $\div$ is the *shielded entrenchment-based contraction* based on $\leq_{\mathbf{K}}$ if and only if:

$$\mathbf{K} \ominus \alpha = \begin{cases} \{\beta \in \mathbf{K} \colon \alpha <_{\mathbf{K}} (\alpha \vee \beta)\} & \text{if } \alpha <_{\mathbf{K}} \top \\ \mathbf{K} & \text{otherwise} \end{cases}$$

This definition gives rise to the following simple "backwards" connection between $\ominus$ and $\leq_{\mathbf{K}}$.

Observation 8.3.3 Let $\mathbf{K}$ be a consistent belief set and $\leq_{\mathbf{K}}$ a relation satisfying (**EE1**) – (**EE4**) with respect to $\mathbf{K}$. Let $\ominus$ be the shielded entrenchment-based contraction induced by $\leq_{\mathbf{K}}$. Then:

$(C \leq)$ $\alpha \leq_{\mathbf{K}} \beta$ if and only if: If $\alpha \in \mathbf{K} \ominus (\alpha \wedge \beta)$ then $\beta \in \mathbf{K} \ominus (\alpha \wedge \beta)$.

Our third construction is based on possible world models of belief change (see **Section 2.6**). The construction of possible worlds can be adopted to shielded contraction by shielding off a set $\Im$ of inaccessible worlds:

Definition 8.3.4 Let M be a proposition. A *shielded propositional selection function* for M is a function f such that, for some set $\Im \subseteq \mathcal{L} \perp\!\!\perp$ (the set of inaccessible worlds), it holds for all sentences α that:

(I) $f(\|\alpha\|) \subseteq \|\alpha\| \setminus \Im$

(II) If $\|\alpha\| \setminus \Im \neq \emptyset$ then $f(\|\alpha\|) \neq \emptyset$.

(III) If $(M \cap \|\alpha\|) \setminus \Im \neq \emptyset$, then $f(\|\alpha\|) = (M \cap \|\alpha\|) \setminus \Im$.

Definition 8.3.5 Let M be a proposition. An operator $\ominus$ is a *shielded propositional contraction operator* for M if and only if there is a shielded propositional selection function f for M such that for all α:

$$M \ominus \|\alpha\| = M \cup f(\|\neg\alpha\|)$$

We can adopt a sphere system to shielded contraction simply by making the outermost sphere inaccessible, i.e., by relaxing condition \$6 of **Definition 2.6.15**.

Definition 8.3.6 \$ is a *shielded system of spheres* if and only if it satisfies conditions \$1 – \$5 of **Definition 2.6.15**.

Definition 8.3.7 A shielded propositional function f for a proposition M, with the associated set $\Im$ of inaccessible worlds, is *sphere-based* if and only if there is a shielded system of spheres \$ such that:

1. $\Im = (\mathcal{L} \perp\!\perp) \setminus (\cup \$)$ and

2. for all α, if $\|\alpha\| \setminus \Im \neq \emptyset$, then $f(\|\alpha\|) = \mathbf{S}_\alpha \cap \|\alpha\|$.

A shielded propositional contraction-operator is sphere-based if and only it is based on a sphere-based propositional selection function.

8.4 Representation theorems

The following two representation theorems characterize the major constructions introduced in the previous section.

THEOREM 8.4.1 Let **K** be a consistent belief set and $\div$ an operation on **K**. Then the following conditions are equivalent:

1 $\div$ satisfies *closure, inclusion, vacuity, extensionality, recovery, relative success, success propagation*, and *conjunctive constancy*.

2 $\div$ is the operator of shielded partial meet contraction induced by a partial meet contraction operator for $\mathbf{K}$ and a set $\mathcal{R} \subseteq \mathcal{L}$ that satisfies *non-retractability propagation* and *conjunctive completeness*.

3 $\div$ is the operator of shielded partial meet contraction induced by a partial meet contraction operator for $\mathbf{K}$ and a set $\mathcal{R} \subseteq \mathcal{L}$ such that $\mathcal{L} \setminus \mathbf{K} \subseteq \mathcal{R}$ and that $\mathcal{R}$ satisfies *non-retractability propagation* and *conjunctive completeness*.

THEOREM 8.4.2 Let $\mathbf{K}$ be a consistent belief set and $\div$ an operation on $\mathbf{K}$. Then the following three conditions are equivalent:

1 $\div$ satisfies *closure, inclusion, vacuity, extensionality, recovery, relative success, persistence, conjunctive inclusion,* and *conjuntive overlap.*

2 $\div$ is the operator of shielded partial meet contraction induced by a transitively relational partial meet contraction operator for $\mathbf{K}$ and a set $\mathcal{R} \subseteq \mathcal{L}$ that satisfies *non-retractability propagation, conjunctive completeness,* and *non-retractability preservation.*

3 $\div$ is the operator of shielded partial meet contraction induced by a transitively relational partial meet contraction operator for $\mathbf{K}$ and a set $\mathcal{R} \subseteq \mathcal{L}$ such that $\mathcal{L} \setminus \mathbf{K} \subseteq \mathcal{R}$ and that $\mathcal{R}$ satisfies *non-retractability propagation, conjunctive completeness,* and *non-retractability preservation.*

4 $\mathbin{-\!\!\circ\!\!-}$ is a shielded entrenchment-based contraction.

5 There exists a shielded sphere-based contraction $\ominus$ on $\|\mathbf{K}\|$ such that $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha = \cap\|\mathbf{K} \ominus \alpha\|$ for all α.

(Note that due to **Observations 8.2.1** and **8.2.3**, *success propagation* and *conjunctive constancy* could redundantly be added the above list of postulates).

8.5 Generalized Levi and Harper identities

In this section we are going to search alternative identities like Levi and Harper identities to relate shielded contraction and credibility-limited revision.

A revision operator obtained through the Levi identity always satisfies the *success* postulate for revision. We should expect a revision operator that corresponds to shielded contraction not to satisfy that postulate. One way to achieve this is to modify the definition so that it satisfies *consistency preservation* (If $\mathbf{K} \nvdash \bot$, then $\mathbf{K} \circ \alpha \nvdash \bot$), which is a fairly plausible condition that contradicts *success*:

Definition 8.5.1 Consistency-preserving Levi identity:

$$\mathbf{K} \circ \alpha = \begin{cases} (\mathbf{K} \mathbin{-\!\!\circ\!\!-} \neg\alpha) + \alpha & \text{if } \mathbf{K} \mathbin{-\!\!\circ\!\!-} \neg\alpha \nvdash \neg\alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

For our present purposes, the consistency-preserving version of the Levi identity turned out to be the more useful one. With this definition, the postulates of shielded contraction studied above give rise, in the corresponding revision operator, to properties that have been used to characterize operators of non-prioritized revision (see **Chapters 5–7**).

Observation 8.5.2 Let **K** be a belief set. Let $\circ$ be defined via the consistency-preserving Levi identity from **K** and $\ominus$. Then:

1 $\circ$ satisfies *closure, consistency preservation* and *relative success.*

2 If $\ominus$ satisfies *inclusion*, then $\circ$ satisfies *inclusion.*

3 If $\ominus$ satisfies *inclusion* and *vacuity*, then $\circ$ satisfies *vacuity.*

4 If $\ominus$ satisfies *extensionality*, then $\circ$ satisfies *extensionality.*

5 If $\ominus$ satisfies *inclusion* and *persistence*, then $\circ$ satisfies *strong regularity.*

6 If $\ominus$ satisfies *relative success, vacuity, extensionality*, and *conjunctive constancy*, then $\circ$ satisfies *disjunctive constancy.*

7 If $\ominus$ satisfies *inclusion* and *success propagation*, then $\circ$ satisfies *strict improvement.*

8 If $\ominus$ satisfies *extensionality, relative success*, and *conjunctive inclusion*, then $\circ$ satisfies *guarded subexpansion.*

9 If $\ominus$ satisfies *closure, inclusion, extensionality, recovery, persistence, relative success*, and *conjunctive overlap*, then $\circ$ satisfies *superexpansion.*

On the other hand, the Harper identity can be used in its original form for shielded contraction. The postulates used for non-prioritized revision give rise, via the Harper identity, to postulates that were shown above to hold for shielded contraction:

Observation 8.5.3 Let $\mathbf{K}$ be a belief set. Let $\ominus$ be defined via the Harper identity from $\mathbf{K}$ and $\circ$. Then:

1 $\ominus$ satisfies *inclusion.*

2 If $\circ$ satisfies *closure*, then $\ominus$ satisfies *closure.*

3 If $\circ$ satisfies *vacuity*, then $\ominus$ satisfies *vacuity.*

4 If $\circ$ satisfies *extensionality*, then $\ominus$ satisfies *extensionality.*

5 If $\circ$ satisfies *closure* and *relative suceess*, then $\ominus$ satisfies *recovery.*

6 If $\mathbf{K} \nvdash \bot$ and $\circ$ satisfies *closure, consistency preservation*, and *strong regularity*, then $\ominus$ satisfies *persistence.*

7 If $\circ$ satisfies *closure, consistency preservation*, and *relative success*, then $\ominus$ satisfies *relative success.*

8 If $\circ$ satisfies *vacuity, consistency preservation, extensionality, relative success*, and *disjunctive constancy*, then $\circ$ satisfies *conjunctive constancy.*

9 If $\mathbf{K} \nvdash \bot$ and $\circ$ satisfies *strict improvement, relative success*, and *consistency preservation*, then $\ominus$ satisfies *success propagation.*

10 If $\circ$ satisfies *closure, extensionality, relative success*, and *superexpansion*, then $\ominus$ satisfies *conjunctive overlap.*

11 If $\circ$ satisfies *vacuity, strong regularity, strict improvement*, and *guarded subexpansion*, then $\ominus$ satisfies *conjunctive inclusion.*

As in AGM contraction and revision (see **Theorems 2.4.24** and **2.4.25**), operators of credibility-limited revision and shielded contraction are interdefinable:

> **THEOREM 8.5.4** Let $\mathbf{K}$ be a belief set and $\ominus$ an operator for $\mathbf{K}$ that satisfies the contraction postulates *closure*, *inclusion*, *recovery*, and *relative success*. Then $\mathbb{C}(\mathbb{R}(\ominus)) = \ominus$.

> **THEOREM 8.5.5** Let $\mathbf{K}$ be a belief set and $\circ$ an operator for $\mathbf{K}$ that satisfies the revision postulates *closure*, *vacuity*, *relative success*, and *consistency preservation*. Then $\mathbb{R}(\mathbb{C}(\circ)) = \circ$.

In summary, we have managed to relax the success postulate for contraction while at the same time retaining two central features of the AGM model: (1) Our constructions ore axiomatically characterized, which means that they can be tested against both semantical and syntactical intuitions. (2) The interdefinability of revision and contraction via the Levi and Harper identities has been retained.

8.6 Proofs of the Chapter

> **Lemma 8.6.1** [Rot92b] Let $\mathbf{K}$ be a belief set and $-$ an operator on $\mathbf{K}$ that satisfies *closure, inclusion, success, vacuity* and *recovery*. Then: $\mathbf{K}{-}\alpha = \mathbf{K}$ if and only if either $\mathbf{K} \nvdash \alpha$ or $\vdash \alpha$.

> **Lemma 8.6.2** [Foo90] Let $\mathbf{K}$ be a belief set and $\leq_{\mathbf{K}}$ a relation satisfying (**EE1**) and (**EE4**) with respect to $\mathbf{K}$. If $\alpha \notin \mathbf{K}$ and $\beta \in \mathbf{K}$, then $\alpha <_{\mathbf{K}} \beta$.

Proof of Observation 8.2.1: Let $\mathbf{K}{\ominus}\beta \vdash \beta$ and $\vdash \beta \rightarrow \alpha$. It follows by *persistence* that $\mathbf{K} \ominus \alpha \vdash \beta$, and from this the desired result follows directly.■

Proof of Observation 8.2.2:

Relative success: Let $\mathbf{K} \ominus \alpha \neq \mathbf{K}$. Then by **Lemma 8.6.1**, $\nvdash \alpha$, hence by *success*, $\mathbf{K} \ominus \alpha \nvdash \alpha$.

Persistence: Let $\mathbf{K} \ominus \beta \vdash \beta$. Then by *success* $\vdash \beta$. Hence $\mathbf{K} \ominus \alpha \vdash \beta$.

Conjunctive constancy: For *reductio ad absurdum* suppose that $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta = \mathbf{K}$ and $\mathbf{K} \ominus (\alpha \wedge \beta) \neq \mathbf{K}$. It follows by **Lemma 8.6.1** from $\mathbf{K} \ominus (\alpha \wedge \beta) \neq \mathbf{K}$ that $\mathbf{K} \vdash \alpha \wedge \beta$ and $\nvdash \alpha \wedge \beta$. Then $\mathbf{K} \vdash \alpha$ and $\mathbf{K} \vdash \beta$. From this and $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta = \mathbf{K}$ follows, by **Lemma 8.6.1**, $\vdash \alpha$ and $\vdash \beta$, contrary to $\nvdash \alpha \wedge \beta$ that was just shown. ■

Proof of Observation 8.2.3: Let $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$. It follows from *inclusion* that $\mathbf{K} \ominus (\alpha \wedge \beta) \subseteq \mathbf{K}$ and from *conjunctive overlap* that $\mathbf{K} \subseteq \mathbf{K} \ominus (\alpha \wedge \beta)$.■

Proof of Observation 8.3.3: For the left to right direction, let $\alpha \leq_{\mathbf{K}} \beta$ and $\alpha \in \mathbf{K} \ominus (\alpha \wedge \beta)$. It follows by **Definition 8.3.2** that $\alpha \in \mathbf{K}$. Due to the completeness of $\leq_{\mathbf{K}}$, we have two cases: First case, $\top \leq_{\mathbf{K}} \alpha$: Then (**EE1**) yields $\top \leq_{\mathbf{K}} \beta$. It follows from (**EE1**) and (**EE3**) that $\top \leq_{\mathbf{K}} \alpha \wedge \beta$. According to **Definition 8.3.2** $\mathbf{K} \ominus (\alpha \wedge \beta) = \mathbf{K}$. Since $\mathbf{K}$ is consistent and logically closed, it follows from (**EE1**), (**EE2**) and (**EE4**) that $\bot <_{\mathbf{K}} \top$. Hence by (**EE1**) $\bot <_{\mathbf{K}} \alpha \wedge \beta$. It follows from (**EE4**) that $\alpha \wedge \beta \in \mathbf{K}$, hence $\beta \in \mathbf{K} = \mathbf{K} \ominus (\alpha \wedge \beta)$. Second case, $\alpha <_{\mathbf{K}} \top$. Then it follows from **Definition 8.3.2** that $\alpha \wedge \beta <_{\mathbf{K}} (\alpha \wedge \beta) \vee \alpha$. By **Property 2.5.37**, $\alpha \wedge \beta <_{\mathbf{K}} \alpha$. Then by (**EE1**), $\alpha \wedge \beta <_{\mathbf{K}} \beta$; and by **Property 2.5.37**, $\alpha \wedge \beta <_{\mathbf{K}} (\alpha \wedge \beta) \vee \beta$. It follows from **Definition 8.3.2** that $\beta \in \mathbf{K} \ominus (\alpha \wedge \beta)$.

For the other direction assume that *if* $\alpha \in \mathbf{K} \ominus (\alpha \wedge \beta)$, *then* $\beta \in \mathbf{K} \ominus (\alpha \wedge \beta)$. We have two cases: First case, $\alpha \in \mathbf{K} \ominus (\alpha \wedge \beta)$ and $\beta \in$

$\mathbf{K}\ominus(\alpha\wedge\beta)$: If $\alpha\wedge\beta <_{\mathbf{K}} \top$, then it follows from **Definition 8.3.2**, since $\alpha,\ \beta \in \mathbf{K}\ominus(\alpha\wedge\beta)$ that $\alpha\in\mathbf{K}$, $\beta\in\mathbf{K}$, $\alpha\wedge\beta <_{\mathbf{K}} (\alpha\wedge\beta)\vee\alpha$ and $\alpha\wedge\beta <_{\mathbf{K}} (\alpha\wedge\beta)\vee\beta$. This contradicts (**EE3**) and we can conclude that $\top \leq_{\mathbf{K}} \alpha\wedge\beta$. Then by (**EE2**) and (**EE1**), $\top\leq_{\mathbf{K}}\beta$; and since by (**EE2**) $\alpha\leq_{\mathbf{K}}\top$, by (**EE1**) we obtain $\alpha\leq_{\mathbf{K}}\beta$. Second case, $\alpha\notin\mathbf{K}\ominus(\alpha\wedge\beta)$: If $\alpha\notin\mathbf{K}$, then it follows by (**EE4**) that $\alpha\leq_{\mathbf{K}}\beta$. If $\alpha\in\mathbf{K}$, then **Definition 8.3.2** yields $(\alpha\wedge\beta)\vee\alpha\leq_{\mathbf{K}}\alpha\wedge\beta$. By **Property 2.5.37**, $\alpha\leq_{\mathbf{K}}\alpha\wedge\beta$. (**EE2**) yields $\alpha\wedge\beta\leq_{\mathbf{K}}\beta$ and we can use to (**EE1**) conclude that $\alpha\leq_{\mathbf{K}}\beta$. ■

Proof of Theorem 8.4.1:

(1) to (3): We first define $\mathcal{R}$ and $-$:

$$\mathcal{R} = \{\alpha :\ \mathbf{K}\ominus\alpha \nvdash \alpha\}.$$

$$\mathbf{K}-\alpha = \begin{cases} \text{(i) } \mathbf{K}\ominus\alpha & \text{if } \alpha\in\mathcal{R} \\ \text{(ii) } \mathbf{K}\cap Cn(\{\neg\alpha\}) & \text{otherwise} \end{cases}$$

$\mathbf{K}\cap Cn(\{\neg\alpha\})$ is *full meet contraction*, that satisfies the basic (and supplementary) AGM contraction postulates [AM82]. We have to prove: **(a)** that $\mathcal{R}$ has the listed properties, **(b)** that $-$ satisfies the basic AGM contraction postulates, and **(c)** that $\ominus$ is induced by $\mathcal{R}$ and $-$.

(a): $\mathcal{L}\setminus\mathbf{K}\subseteq\mathcal{R}$: Directly from *inclusion* and definition of $\mathcal{R}$.

Non-Retractability Propagation: Let $\alpha\notin\mathcal{R}$ and $\beta\in Cn(\{\alpha\})$. Then $\mathbf{K}\ominus\alpha\vdash\alpha$, and by *success propagation* $\mathbf{K}\ominus\beta\vdash\beta$. Hence $\beta\notin\mathcal{R}$.

Conjunctive Completeness: Let $\alpha\notin\mathcal{R}$ and $\beta\notin\mathcal{R}$. Then $\mathbf{K}\ominus\alpha\vdash\alpha$ and $\mathbf{K}\ominus\beta\vdash\beta$. By *relative success*, $\mathbf{K}\ominus\alpha=\mathbf{K}\ominus\beta=\mathbf{K}$. Then $\mathbf{K}\vdash\alpha\wedge\beta$. By *conjunctive constancy* $\mathbf{K}\ominus(\alpha\wedge\beta)=\mathbf{K}$; hence $\mathbf{K}\ominus(\alpha\wedge\beta)\vdash\alpha\wedge\beta$ and $\alpha\wedge\beta\notin\mathcal{R}$.

(b): Closure: Follows in case (i) from $\ominus$ *closure* and in case (ii) since full meet contraction satisfies *closure*.

Inclusion, **success** and **recovery** are proved in the some way as *closure*.

Vacuity: Let $\mathbf{K} \nvdash \alpha$. It follows from $\ominus$ *vacuity* that $\mathbf{K} \ominus \alpha = \mathbf{K}$. Hence $\mathbf{K} \ominus \alpha \nvdash \alpha$, hence $\alpha \in \mathcal{R}$, then by our definition of $-$, $\mathbf{K} - \alpha = \mathbf{K} \ominus \alpha$, hence $\mathbf{K} - \alpha = \mathbf{K}$.

Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. There are two cases: First case, $\mathbf{K} \ominus \alpha \nvdash \alpha$: Then $\alpha \in \mathcal{R}$, so that $\mathbf{K} - \alpha = \mathbf{K} \ominus \alpha$. By $\ominus$ *extensionality*, $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$, hence $\mathbf{K} - \alpha = \mathbf{K} \ominus \beta$. It follows from $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$, $\mathbf{K} \ominus \alpha \nvdash \alpha$ and $\vdash \alpha \leftrightarrow \beta$ that $\mathbf{K} \ominus \beta \nvdash \beta$, from which follows $\mathbf{K} - \beta = \mathbf{K} \ominus \beta$, hence $\mathbf{K} - \alpha = \mathbf{K} - \beta$. Second case, $\mathbf{K} \ominus \alpha \vdash \alpha$: Then by *extensionality* $\mathbf{K} \ominus \beta \vdash \beta$. Hence $\mathbf{K} - \alpha = \mathbf{K} \cap Cn(\{\neg\alpha\}) = \mathbf{K} \cap Cn(\{\neg\beta\}) = \mathbf{K} - \beta$.

(c): Let $\ominus'$ be the shielded contraction induced by $\mathcal{R}$ and $-$. We are going to show that for all α, $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus' \alpha$. If $\alpha \in \mathcal{R}$, then $\mathbf{K} \ominus' \alpha = \mathbf{K} - \alpha = \mathbf{K} \ominus \alpha$. If $\alpha \notin \mathcal{R}$, then it follows from the definition of shielded contraction that $\mathbf{K} \ominus' \alpha = \mathbf{K}$. Furthermore, it follows from the definition of $\mathcal{R}$ that $\mathbf{K} \ominus \alpha \vdash \alpha$, and then from $\ominus$ *relative success* that $\mathbf{K} \ominus \alpha = \mathbf{K}$.

(2) to (1): Closure, Inclusion and **Vacuity:** Trivial.

Extensionality: From non-retractability propagation and $-$ *extensionality*.

Recovery: If $\alpha \in \mathcal{R}$, then by $-$ *recovery* $\mathbf{K} \subseteq (\mathbf{K} - \alpha) + \alpha = (\mathbf{K} \ominus \alpha) + \alpha$. If $\alpha \notin \mathcal{R}$, then $\mathbf{K} \subseteq \mathbf{K} + \alpha = (\mathbf{K} \ominus \alpha) + \alpha$.

Relative success: Let $\mathbf{K} \ominus \alpha \neq \mathbf{K}$. It follows from **Definition 8.3.1** that $\mathbf{K} \ominus \alpha = \mathbf{K} - \alpha$, and from **Observation 8.2.2** that either $\mathbf{K} - \alpha = \mathbf{K}$ or $\mathbf{K} - \alpha \nvdash \alpha$.

Success Propagation: Let $\vdash \beta \rightarrow \alpha$ and $\mathbf{K} \ominus \alpha \nvdash \alpha$. We have two cases: **(a)** $\alpha \in \mathcal{R}$: Since $\nvdash \alpha$ we have $\nvdash \beta$, hence by *success* $\mathbf{K} - \beta \nvdash \beta$. Hence, since by non-retractability propagation $\beta \in \mathcal{R}$, $\mathbf{K} \ominus \beta \nvdash \beta$. **(b)** $\alpha \notin \mathcal{R}$, then

$\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha = \mathbf{K}$ and $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha \nvdash \alpha$. Hence $\mathbf{K} \nvdash \beta$, from which follows that $\nvdash \beta$ and (by *inclusion*) $\mathbf{K}-\beta \nvdash \beta$. Hence $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta \nvdash \beta$.

Conjunctive constancy: Let $\mathbf{K} = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta$. We have three cases: **(a)** $\alpha \in \mathcal{R}$. Then $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha = \mathbf{K}-\alpha$, and it follows from **Lemma 8.6.1** that either $\vdash \alpha$ or $\mathbf{K} \nvdash \alpha$. If $\vdash \alpha$, then by *extensionality*, that was just proved, $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta = \mathbf{K}$. If $\mathbf{K} \nvdash \alpha$, then $\mathbf{K} \nvdash \alpha \wedge \beta$ and due to *vacuity*, that was just proved, $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) = \mathbf{K}$. **(b)** $\beta \in \mathcal{R}$. This is similar to case (a). **(c)** $\alpha \notin \mathcal{R}$ and $\beta \notin \mathcal{R}$. Then by conjunctive completeness, $(\alpha \wedge \beta) \notin \mathcal{R}$, hence $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) = \mathbf{K}$.

(3) to (2): Trivial. ■

Proof of Theorem 8.4.2:

(1) to (3): We use the same construction as in the corresponding part of the proof of **Theorem 8.4.1**. Then we need to prove only that $-$ satisfies *conjunctive inclusion* and *conjunctive overlap*, and that $\mathcal{R}$ satisfies non-retractability preservation.

Conjunctive inclusion: Let $\mathbf{K}-(\alpha \wedge \beta) \nvdash \alpha$. We have two cases: **(a)** $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha \nvdash \alpha$: Then $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha = \mathbf{K}-\alpha$ and, since by *success propagation* $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) \nvdash \alpha \wedge \beta$, we also have $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) = \mathbf{K}-(\alpha \wedge \beta)$. Then $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) \nvdash \alpha$, hence by $\mathbin{-\!\!\!\circ\!\!\!-}$ *conjunctive inclusion* $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) \subseteq \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$, from which we obtain $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$. **(b)** $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha \vdash \alpha$: Then by *persistence* $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) \vdash \alpha$, then $\mathbf{K}-(\alpha \wedge \beta) \neq \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta)$. It follows from the construction of $-$ that $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K} \cap Cn(\{\neg\alpha \vee \neg\beta\}) \subseteq \mathbf{K} \cap Cn(\{\neg\alpha\}) = \mathbf{K}-\alpha$.

Conjunctive Overlap: There are three cases: **(a)** $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha \nvdash \alpha$, $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta \nvdash \beta$: Then by *success propagation* $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) \nvdash \alpha \wedge \beta$. It follows from the definition of $-$ that $\mathbf{K}-\alpha = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$, $\mathbf{K}-\beta = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta$ and $\mathbf{K}-(\alpha \wedge \beta) =$

$\mathbf{K}\ominus(\alpha\wedge\beta)$. By $\ominus$ *conjunctive overlap* $\mathbf{K}\ominus\alpha\cap\mathbf{K}\ominus\beta\subseteq\mathbf{K}\ominus(\alpha\wedge\beta)$, hence $\mathbf{K}-\alpha\cap\mathbf{K}-\beta=\mathbf{K}-(\alpha\wedge\beta)$. **(b)** $\mathbf{K}\ominus\alpha\vdash\alpha$, $\mathbf{K}\ominus\beta\vdash\beta$: By *relative success*, $\mathbf{K}\ominus\alpha=\mathbf{K}\ominus\beta=\mathbf{K}$. Then by $\ominus$ *conjunctive overlap* $\mathbf{K}\ominus(\alpha\wedge\beta)\vdash\alpha\wedge\beta$. Hence by the definition of $-$, $\mathbf{K}-\alpha\cap\mathbf{K}-\beta=\mathbf{K}\cap Cn(\{\neg\alpha\})\cap\mathbf{K}\cap Cn(\{\neg\beta\})=\mathbf{K}\cap Cn(\{\neg(\alpha\wedge\beta)\})=\mathbf{K}-(\alpha\wedge\beta)$. **(c)** $\mathbf{K}\ominus\alpha\vdash\alpha$, $\mathbf{K}\ominus\beta\nvdash\beta$: Then by *success propagation* $\mathbf{K}\ominus(\alpha\wedge\beta)\nvdash\alpha\wedge\beta$. By the definition of $-$, $\mathbf{K}-\alpha=\mathbf{K}\cap Cn(\{\neg\alpha\})$, $\mathbf{K}-\beta=\mathbf{K}\ominus\beta$ and $\mathbf{K}-(\alpha\wedge\beta)=\mathbf{K}\ominus(\alpha\wedge\beta)$. Due to *relative success*, $\mathbf{K}\ominus\alpha=\mathbf{K}$, so that $\mathbf{K}\cap Cn(\{\neg\alpha\})\subseteq\mathbf{K}\ominus\alpha$. Hence by $\ominus$ *conjunctive overlap* $\mathbf{K}-\alpha\cap\mathbf{K}-\beta=\mathbf{K}\cap Cn(\{\neg\alpha\})\cap\mathbf{K}\ominus\beta\subseteq\mathbf{K}\ominus\alpha\cap\mathbf{K}\ominus\beta\subseteq\mathbf{K}\ominus(\alpha\wedge\beta)=\mathbf{K}-(\alpha\wedge\beta)$.

Non-retractability preservation: Let $\beta\notin\mathcal{R}$. Then $\beta\in\mathbf{K}\ominus\beta$; hence by *persistence* $\beta\in\mathbf{K}\ominus\alpha$.

(2) to (1): Let (2) be satisfied. Due to **Theorem 8.4.1** we only need to prove that $\ominus$ satisfies *persistence, conjunctive inclusion* and *conjunctive overlap*.

Persistence: Let $\mathbf{K}\ominus\beta\vdash\beta$. First case, $\beta\in\mathcal{R}$: Then $\mathbf{K}\ominus\beta=\mathbf{K}-\beta$. It follows from *success* that $\vdash\beta$, hence $\mathbf{K}\ominus\alpha\vdash\beta$. Second case, $\beta\notin\mathcal{R}$: Then by non-retractability preservation, $\mathbf{K}\ominus\alpha\vdash\beta$.

Conjunctive inclusion: Let $\mathbf{K}\ominus(\alpha\wedge\beta)\nvdash\alpha$. We have three cases: **(a)** $\alpha\wedge\beta\in\mathcal{R}$ and $\alpha\in\mathcal{R}$: Then $\mathbf{K}\ominus(\alpha\wedge\beta)=\mathbf{K}-(\alpha\wedge\beta)$ and $\mathbf{K}\ominus\alpha=\mathbf{K}-\alpha$. Hence by $-$ *conjunctive inclusion* $\mathbf{K}\ominus(\alpha\wedge\beta)=\mathbf{K}-(\alpha\wedge\beta)\subseteq\mathbf{K}-\alpha=\mathbf{K}\ominus\alpha$. **(b)** $\alpha\wedge\beta\in\mathcal{R}$ and $\alpha\notin\mathcal{R}$: Then $\mathbf{K}\ominus(\alpha\wedge\beta)=\mathbf{K}-(\alpha\wedge\beta)$ and $\mathbf{K}\ominus\alpha=\mathbf{K}$. By *inclusion*, $\mathbf{K}\ominus(\alpha\wedge\beta)=\mathbf{K}-(\alpha\wedge\beta)\subseteq\mathbf{K}=\mathbf{K}\ominus\alpha$. **(c)**: $\alpha\wedge\beta\notin\mathcal{R}$: Then $\mathbf{K}\ominus(\alpha\wedge\beta)=\mathbf{K}$. By non-retractability propagation $\alpha\notin\mathcal{R}$, hence $\mathbf{K}\ominus\alpha=\mathbf{K}=\mathbf{K}\ominus(\alpha\wedge\beta)$.

Conjunctive overlap: There are three cases: **(a)** α, $\beta\in\mathcal{R}$: Then by non-retractability propagation, $\alpha\wedge\beta\in\mathcal{R}$. By $-$ *conjunctive overlap* and definition of $\mathbin{-\!\!\!\circ\!\!\!-}$, $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha \cap \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta = \mathbf{K}-\alpha \cap \mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta)$. **(b)** $\alpha \notin \mathcal{R}$, $\beta \notin \mathcal{R}$: Then by conjunctive completeness $\alpha \wedge \beta \notin \mathcal{R}$, and by definition of $\mathbin{-\!\!\!\circ\!\!\!-}$, $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha \cap \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) = \mathbf{K}$. **(c)** $\alpha \in \mathcal{R}$, $\beta \notin \mathcal{R}$: Then $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha = \mathbf{K}-\alpha$ and $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta = \mathbf{K}$. By non-retractability propagation, $\alpha \wedge \beta \in \mathcal{R}$, so that $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) = \mathbf{K}-(\alpha \wedge \beta)$. According to $-$ *conjunctive factoring* (that follows from the AGM basic and supplementary postulates) we have three subcases: **(c1)** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$: Hence $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha \cap \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta = \mathbf{K}-\alpha \cap \mathbf{K} = \mathbf{K}-\alpha = \mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta)$. **(c2)** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$: It follows by non-retractability preservation that $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) \vdash \beta$, then since $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) = \mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$, $\mathbf{K}-\beta \vdash \beta$. Then it follows by $-$ *success* that $\vdash \beta$ and consequently $\mathbf{K}-\beta = \mathbf{K} = \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\beta$. The rest follows by $-$ *conjunctive overlap*. **(c3)** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha \cap \mathbf{K}-\beta$: Due to non-retractability preservation $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}(\alpha \wedge \beta) \vdash \beta$, which it follows that $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha \cap \mathbf{K}-\beta \vdash \beta$. Hence $\mathbf{K}-\beta \vdash \beta$ and the rest is similar to case (c2).

(3) to (2): Trivial.

(4) to (1): Let $\leq_{\mathbf{K}}$ be a relation satisfying (**EE1**) – (**EE4**) with respect to $\mathbf{K}$. Let $\mathbin{-\!\!\!\circ\!\!\!-}$ the shielded entrenchment-based contraction induced by $\leq_{\mathbf{K}}$. We have to show that $\mathbin{-\!\!\!\circ\!\!\!-}$ satisfies the listed postulates.

Closure: Let $\varepsilon \in Cn(\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha)$. We have to show that $\varepsilon \in \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$. If $\top \leq_{\mathbf{K}} \alpha$, then $\mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha = \mathbf{K}$, and the desired result follows since $\mathbf{K}$ is a belief set. Hence we may assume that $\alpha <_{\mathbf{K}} \top$. Since the underlying logic is compact, there is a finite subset $\{\beta_1, ..., \beta_n\} \subseteq \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$, such that $\{\beta_1, ..., \beta_n\} \vdash \varepsilon$. It follows from $\beta_1 \in \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$ and $\beta_2 \in \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$ that $\alpha <_{\mathbf{K}} \alpha \vee \beta_1$ and $\alpha <_{\mathbf{K}} \alpha \vee \beta_2$; then by (**EE3**) and (**EE1**), $\alpha <_{\mathbf{K}} (\alpha \vee \beta_1) \wedge (\alpha \vee \beta_2)$; and by **Property 2.5.37** $\alpha <_{\mathbf{K}} \alpha \vee (\beta_1 \wedge \beta_2)$. Hence $\beta_1 \wedge \beta_2 \in \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$. By iteration of the some procedure we obtain $\beta_1 \wedge ... \wedge \beta_n \in \mathbf{K}\mathbin{-\!\!\!\circ\!\!\!-}\alpha$. Next, let $\vdash \beta \leftrightarrow \beta_1 \wedge ... \wedge \beta_n$. Then

$\alpha <_{\mathbf{K}} \alpha \vee \beta$. It follows from $\vdash \beta \to \varepsilon$ that $\vdash (\alpha \vee \beta) \to (\alpha \vee \varepsilon)$. (**EE2**) yields $(\alpha \vee \beta) \leq_{\mathbf{K}} (\alpha \vee \varepsilon)$. It follows by (**EE1**) that $\alpha <_{\mathbf{K}} (\alpha \vee \varepsilon)$. Hence $\varepsilon \in \mathbf{K} \div \alpha$.

Inclusion: Follows trivially from **Definition 8.3.2**.

Vacuity: Let $\alpha \notin \mathbf{K}$. It follows by **Lemma 8.6.2** that $\top \not\leq_{\mathbf{K}} \alpha$. Then by **Definition 8.3.2**, $\beta \in \mathbf{K} \div \alpha$ if and only if $\beta \in \mathbf{K}$ and $\alpha <_{\mathbf{K}} \alpha \vee \beta$. By **Lemma 8.6.2** $\alpha <_{\mathbf{K}} \alpha \vee \beta$ follows for all $\beta \in \mathbf{K}$, since $\mathbf{K}$ is a belief set. Then $\beta \in \mathbf{K} \div \alpha$ if and only if $\beta \in \mathbf{K}$; hence $\mathbf{K} \div \alpha = \mathbf{K}$.

Extensionality: Let $\vdash \delta \leftrightarrow \alpha$. Then by **Property 2.5.37**, $\alpha <_{\mathbf{K}} \top$ if and only if $\delta <_{\mathbf{K}} \top$ and $\alpha <_{\mathbf{K}} \alpha \vee \beta$ if and only if $\delta <_{\mathbf{K}} \delta \vee \beta$. Hence, by **Definition 8.3.2**, $\mathbf{K} \div \alpha = \mathbf{K} \div \delta$.

Recovery: Let $\beta \in \mathbf{K}$. We have to prove that $\alpha \to \beta \in \mathbf{K} \div \alpha$. Assume to the contrary that $\alpha \to \beta \notin \mathbf{K} \div \alpha$. Then, since $\beta \in \mathbf{K}$ it follows by **Definition 8.3.2** that $\alpha <_{\mathbf{K}} \top$ and $\alpha \vee (\alpha \to \beta) \leq_{\mathbf{K}} \alpha$. By **Property 2.5.37** $\top \leq_{\mathbf{K}} \alpha$. From this contradiction we may conclude that $\alpha \to \beta \in \mathbf{K} \div \alpha$.

Relative Success: Let $\mathbf{K} \div \alpha \neq \mathbf{K}$. Then by **Definition 8.3.2** it follows that $\alpha <_{\mathbf{K}} \top$. By (**EE2**), $\alpha \not<_{\mathbf{K}} \alpha \vee \alpha$ and then by **Definition 8.3.2** $\alpha \notin \mathbf{K} \div \alpha$.

Persistence: Let $\mathbf{K} \div \beta \vdash \beta$. Then by *closure*, $\beta \in \mathbf{K} \div \beta$. By *inclusion*, $\beta \in \mathbf{K}$. Suppose that $\beta <_{\mathbf{K}} \top$. Then it follows by **Definition 8.3.2** that $\beta <_{\mathbf{K}} \beta \vee \beta$, contrary to (**EE2**). We can conclude from this contradiction that $\top \leq_{\mathbf{K}} \beta$. According to (**EE2**) it holds for all α that $\beta \leq_{\mathbf{K}} \alpha \vee \beta$. Hence by (**EE1**),$\top \leq_{\mathbf{K}} \alpha \vee \beta$. Hence by (**EE1**), it holds for all α that if $\alpha <_{\mathbf{K}} \top$, then $\alpha <_{\mathbf{K}} \alpha \vee \beta$. Since $\beta \in \mathbf{K}$, we can conclude from **Definition 8.3.2** that $\beta \in \mathbf{K} \div \alpha$.

Conjunctive inclusion: Let $\mathbf{K} \div (\alpha \wedge \beta) \nvdash \alpha$: By *closure* $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. If $\alpha \notin \mathbf{K}$; then by the previous proof of *vacuity*, $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha = \mathbf{K}$. We can therefore assume that $\alpha \in \mathbf{K}$. It follows from **Definition 8.3.2**,

$\alpha \in \mathbf{K}$, and $\alpha \notin \mathbf{K} \ominus (\alpha \wedge \beta)$ that $\alpha \wedge \beta <_{\mathbf{K}} \top$ and $\alpha \vee (\alpha \wedge \beta) \leq_{\mathbf{K}} \alpha \wedge \beta$. By **Property 2.5.37**, $\alpha \leq_{\mathbf{K}} \alpha \wedge \beta$.

Let $\delta \in \mathbf{K} \ominus (\alpha \wedge \beta)$: Then, by **Definition 8.3.2**, $\delta \in \mathbf{K}$ and $\alpha \wedge \beta <_{\mathbf{K}} (\alpha \wedge \beta) \vee \delta$. By **Property 2.5.37** $\alpha \wedge \beta <_{\mathbf{K}} (\alpha \vee \delta) \wedge (\beta \vee \delta)$. By (**EE2**), $(\alpha \vee \delta) \wedge (\beta \vee \delta) \leq_{\mathbf{K}} \alpha \vee \delta$. Applying (**EE1**) to $\alpha \leq_{\mathbf{K}} \alpha \wedge \beta$, $\alpha \wedge \beta <_{\mathbf{K}} (\alpha \vee \delta) \wedge (\beta \vee \delta)$ and $(\alpha \vee \delta) \wedge (\beta \vee \delta) \leq_{\mathbf{K}} \alpha \vee \delta$, we obtain $\alpha <_{\mathbf{K}} \alpha \vee \delta$. Furthermore, it follows by (**EE1**) from $\alpha \leq_{\mathbf{K}} \alpha \wedge \beta$ and $\alpha \wedge \beta <_{\mathbf{K}} \top$ that $\alpha <_{\mathbf{K}} \top$. It follows by **Definition 8.3.2** from $\alpha <_{\mathbf{K}} \top$, $\delta \in \mathbf{K}$ and $\alpha <_{\mathbf{K}} \alpha \vee \delta$ that $\delta \in \mathbf{K} \ominus \alpha$. We may conclude that $\mathbf{K} \ominus (\alpha \wedge \beta) \subseteq \mathbf{K} \ominus \alpha$.

Conjunctive overlap: Let $\delta \in \mathbf{K} \ominus \alpha$ and $\delta \in \mathbf{K} \ominus \beta$: According to **Definition 8.3.2**, there are four cases: **(a)** $\top \leq_{\mathbf{K}} \alpha$ and $\top \leq_{\mathbf{K}} \beta$: Then it follows by (**EE3**) and (**EE1**) that $\top \leq_{\mathbf{K}} \alpha \wedge \beta$. It follows from **Definition 8.3.2** that $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta = \mathbf{K} \ominus (\alpha \wedge \beta) = \mathbf{K}$, hence $\delta \in \mathbf{K} \ominus (\alpha \wedge \beta)$. **(b)** $\top \not\leq_{\mathbf{K}} \alpha$ and $\top \leq_{\mathbf{K}} \beta$: Then by (**EE2**) and (**EE1**), $\top \not\leq_{\mathbf{K}} \alpha \wedge \beta$. It follows by **Definition 8.3.2** from $\delta \in \mathbf{K} \ominus \alpha$ and $\alpha <_{\mathbf{K}} \top$ that $\delta \in \mathbf{K}$ and $\alpha <_{\mathbf{K}} \alpha \vee \delta$. Then by (**EE2**) and (**EE1**), $\alpha \wedge \beta <_{\mathbf{K}} \alpha \vee \delta$. Due to (**EE2**), $\beta \leq_{\mathbf{K}} \beta \vee \delta$. It follows by (**EE1**) from this and $\top \leq_{\mathbf{K}} \beta$ that $\top \leq_{\mathbf{K}} \beta \vee \delta$. Since $\alpha \wedge \beta <_{\mathbf{K}} \top$, it follows by (**EE1**) that $\alpha \wedge \beta <_{\mathbf{K}} \beta \vee \delta$. From this and $\alpha \wedge \beta <_{\mathbf{K}} \alpha \vee \delta$ it follows by (**EE3**) and (**EE2**) that $\alpha \wedge \beta <_{\mathbf{K}} (\alpha \wedge \beta) \vee \delta$. Hence, according to **Definition 8.3.2**, $\delta \in \mathbf{K} \ominus (\alpha \wedge \beta)$. **(c)** $\top \leq_{\mathbf{K}} \alpha$ and $\top \not\leq_{\mathbf{K}} \beta$: Simmetrical with case (b). **(d)** $\top \not\leq_{\mathbf{K}} \alpha$ and $\top \not\leq_{\mathbf{K}} \beta$: Then by (**EE2**) and (**EE1**) it follows that $\top \not\leq_{\mathbf{K}} \alpha \wedge \beta$. It follows from **Definition 8.3.2** that $\delta \in \mathbf{K}$, $\alpha <_{\mathbf{K}} \alpha \vee \delta$ and $\beta <_{\mathbf{K}} \beta \vee \delta$. Then by (**EE1**) and (**EE2**), $\alpha \wedge \beta <_{\mathbf{K}} \alpha \vee \delta$ and $\alpha \wedge \beta <_{\mathbf{K}} \beta \vee \delta$. It follows from (**EE1**), (**EE2**), and (**EE3**) that $\alpha \wedge \beta <_{\mathbf{K}} (\alpha \wedge \beta) \vee \delta$; hence from **Definition 8.3.2**, $\delta \in \mathbf{K} \ominus (\alpha \wedge \beta)$.

(1) to (4): Let $\ominus$ be an operator satisfying the postulates listed in (1), and let $\leq_{\mathbf{K}}$ defined as follows:

(def $\leq_{\mathbf{K}}$) $\qquad \alpha \leq_{\mathbf{K}} \beta$ iff *if* $\alpha \in \mathbf{K}\div(\alpha \wedge \beta)$, *then* $\beta \in \mathbf{K}\div(\alpha \wedge \beta)$.

We need to show that (**EE1**) – (**EE4**) are satisfied and that $\div$ is induced by $\leq_{\mathbf{K}}$ in the sense of **Definition 8.3.2**.

(**EE1**) Let $\alpha \leq_{\mathbf{K}} \beta$ and $\beta \leq_{\mathbf{K}} \delta$. In order to prove that $\alpha \leq_{\mathbf{K}} \delta$, let $\alpha \in \mathbf{K}\div(\alpha \wedge \delta)$. We have to prove $\delta \in \mathbf{K}\div(\alpha \wedge \delta)$. There are two cases: First case, $\alpha \in \mathbf{K}\div(\alpha \wedge \beta)$: By (def $\leq_{\mathbf{K}}$), $\beta \in \mathbf{K}\div(\alpha \wedge \beta)$. Then by *closure* $\alpha \wedge \beta \in \mathbf{K}\div(\alpha \wedge \beta)$, by *success propagation*, $\beta \in \mathbf{K}\div\beta$, by *persistence*, $\beta \in \mathbf{K}\div(\beta \wedge \delta)$, and by (def $\leq_{\mathbf{K}}$), $\delta \in \mathbf{K}\div(\beta \wedge \delta)$. By *closure* $\beta \wedge \delta \in \mathbf{K}\div(\beta \wedge \delta)$, by *success propagation* $\delta \in \mathbf{K}\div\delta$, and by *persistence* $\delta \in \mathbf{K}\div(\alpha \wedge \delta)$.
Second case, $\alpha \notin \mathbf{K}\div(\alpha \wedge \beta)$: Let $\delta \notin \mathbf{K}\div(\alpha \wedge \delta)$. By *persistence* $\delta \notin \mathbf{K}\div\delta$; from which it follows by *success propagation* that $\beta \wedge \delta \notin \mathbf{K}\div(\beta \wedge \delta)$. Since $\beta \leq_{\mathbf{K}} \delta$ we can conclude from *closure* and (def $\leq_{\mathbf{K}}$) that $\beta \notin \mathbf{K}\div(\beta \wedge \delta)$. We will arrive at a contradiction by proving both that $\alpha \in \mathbf{K}\div(\alpha \wedge \beta \wedge \delta)$ and that $\alpha \notin \mathbf{K}\div(\alpha \wedge \beta \wedge \delta)$: By *conjunctive overlap* it follows that $\mathbf{K}\div(\alpha \wedge \delta) \cap \mathbf{K}\div(\neg\alpha \vee \beta) \subseteq \mathbf{K}\div(\alpha \wedge \beta \wedge \delta)$. By hypothesis $\alpha \in \mathbf{K}\div(\alpha \wedge \delta)$. Furthermore by *inclusion* $\alpha \in \mathbf{K}$ and it follows by *recovery* that $\alpha \in (\mathbf{K}\div(\neg\alpha \vee \beta)) + (\neg\alpha \vee \beta)$, so that by *closure* $\alpha \in \mathbf{K}\div(\neg\alpha \vee \beta)$. Hence, $\alpha \in \mathbf{K}\div(\alpha \wedge \beta \wedge \delta)$.

Due to the hypothesis condition $\alpha \notin \mathbf{K}\div(\alpha \wedge \beta)$ it now suffices to prove that $\mathbf{K}\div(\alpha \wedge \beta \wedge \delta) \subseteq \mathbf{K}\div(\alpha \wedge \beta)$. Due to *conjunctive inclusion* this can be done by showing that $\alpha \wedge \beta \notin \mathbf{K}\div(\alpha \wedge \beta \wedge \delta)$. It follows by *persistence* from $\alpha \notin \mathbf{K}\div(\alpha \wedge \beta)$ that $\alpha \notin \mathbf{K}\div\alpha$. Hence by *success propagation*, $(\alpha \wedge \beta \wedge \delta) \notin \mathbf{K}\div(\alpha \wedge \beta \wedge \delta)$. It follows from *closure* that either $\mathbf{K}\div(\alpha \wedge \beta \wedge \delta) \nvdash \alpha \wedge \beta$ or $\mathbf{K}\div(\alpha \wedge \beta \wedge \delta) \nvdash \delta$. In the second case, it follows by *closure* that $\mathbf{K}\ominus(\alpha\wedge\beta\wedge\delta)\nvdash\beta\wedge\delta$. By *conjunctive inclusion*, $\mathbf{K}\ominus(\alpha\wedge\beta\wedge\delta)\subseteq\mathbf{K}\ominus(\beta\wedge\delta)$ and since $\beta\notin\mathbf{K}\ominus(\beta\wedge\delta)$, $\beta\notin\mathbf{K}\ominus(\alpha\wedge\beta\wedge\delta)$, so that by *closure* $\mathbf{K}\ominus(\alpha\wedge\beta\wedge\delta)\nvdash\alpha\wedge\beta$ in this case as well. It follows by *conjunctive inclusion* from $\mathbf{K}\ominus(\alpha\wedge\beta\wedge\delta)\nvdash\alpha\wedge\beta$ that $\mathbf{K}\ominus(\alpha\wedge\beta\wedge\delta)\subseteq\mathbf{K}\ominus(\alpha\wedge\beta)$. By hypothesis, $\alpha\notin\mathbf{K}\ominus(\alpha\wedge\beta)$, so that $\alpha\notin\mathbf{K}\ominus(\alpha\wedge\beta\wedge\delta)$, contrary to what shown above. This contradiction conclude this part of the proff.

(**EE2**) Let $\vdash\alpha\rightarrow\beta$. Then if $\alpha\in\mathbf{K}\ominus(\alpha\wedge\beta)$ it follows by *closure* that $\beta\in\mathbf{K}\ominus(\alpha\wedge\beta)$; hence by (def $\leq_{\mathbf{K}}$), $\alpha\leq_{\mathbf{K}}\beta$.

(**EE3**) Let $\alpha\nleq_{\mathbf{K}}\alpha\wedge\beta$. Then by (def $\leq_{\mathbf{K}}$), $\alpha\in\mathbf{K}\ominus((\alpha\wedge\beta)\wedge\alpha)$ and $\alpha\wedge\beta\notin\mathbf{K}\ominus((\alpha\wedge\beta)\wedge\alpha)$. It follows by *closure* that $\beta\notin\mathbf{K}\ominus((\alpha\wedge\beta)\wedge\alpha)$ and by *extensionality* that $\beta\notin\mathbf{K}\ominus((\alpha\wedge\beta)\wedge\beta)$. Hence, by (def $\leq_{\mathbf{K}}$), $\beta\nleq_{\mathbf{K}}\alpha\wedge\beta$.

(**EE4**) For one direction, let $\beta\notin\mathbf{K}$. Then by *inclusion* for all α, $\beta\notin\mathbf{K}\ominus(\alpha\wedge\beta)$; hence for all α, $\beta\leq_{\mathbf{K}}\alpha$. For the other direction, let $\beta\leq_{\mathbf{K}}\alpha$ for all α. Then, in particular $\beta\leq_{\mathbf{K}}\bot$. It follows from (def $\leq_{\mathbf{K}}$) that *if* $\beta\in\mathbf{K}\ominus(\beta\wedge\bot)$, *then* $\bot\in\mathbf{K}\ominus(\beta\wedge\bot)$. It follows by *inclusion*, that $\bot\notin\mathbf{K}\ominus(\beta\wedge\bot)$, so that $\beta\notin\mathbf{K}\ominus(\beta\wedge\bot)$. By *vacuity*, $\mathbf{K}\ominus(\beta\wedge\bot)=\mathbf{K}$, so that $\beta\notin\mathbf{K}$ as desired.

$\ominus$ is based on $\leq_{\mathbf{K}}$ in the manner of Definition 8.3.2: There are two cases. First case, $\top\leq_{\mathbf{K}}\alpha$: Then it follows from (def $\leq_{\mathbf{K}}$) that *if* $\top\in\mathbf{K}\ominus(\alpha\wedge\top)$, *then* $\alpha\in\mathbf{K}\ominus(\alpha\wedge\top)$. By *extensionality* and *closure*, we obtain $\alpha\in\mathbf{K}\ominus\alpha$. By *relative success*, $\mathbf{K}\ominus\alpha=\mathbf{K}$ as desired.

Second case, $\alpha<_{\mathbf{K}}\top$: We need to show that $\beta\in\mathbf{K}\ominus\alpha$ iff $\beta\in\mathbf{K}$ and $\alpha<_{\mathbf{K}}(\alpha\vee\beta)$. Due to *inclusion*, this is trivial if $\beta\notin\mathbf{K}$. Let $\beta\in\mathbf{K}$. For one direction, let $\beta\in\mathbf{K}\ominus\alpha$. Suppose for *reductio* that $(\alpha\vee\beta)\leq_{\mathbf{K}}\alpha$, *i.e.*, that *if* $\alpha\vee\beta\in\mathbf{K}\ominus((\alpha\vee\beta)\wedge\alpha)$, *then* $\alpha\in\mathbf{K}\ominus((\alpha\vee\beta)\wedge\alpha)$. It follows by *closure* and *extensionality* from $\beta\in\mathbf{K}\ominus\alpha$ that $\alpha\vee\beta\in\mathbf{K}\ominus((\alpha\vee\beta)\wedge\alpha)$. Hence

$\alpha \in \mathbf{K}\div((\alpha \vee \beta) \wedge \alpha)$, by *extensionality* $\alpha \in \mathbf{K}\div(\alpha \wedge \top)$. By *closure*, $\top \in \mathbf{K}\div(\alpha \wedge \top)$. It follows by (def $\leq_{\mathbf{K}}$) that $\top \leq_{\mathbf{K}} \alpha$, contrary to the conditions from this case. We can conclude from this contradiction that $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$.

For the other direction, let $\alpha <_{\mathbf{K}} (\alpha \vee \beta)$. It follows from (def $\leq_{\mathbf{K}}$) that $\alpha \vee \beta \in \mathbf{K}\div(\alpha \wedge (\alpha \vee \beta))$ and then from *extensionality* that $\alpha \vee \beta \in \mathbf{K}\div\alpha$. By *recovery*, $\alpha \to \beta \in \mathbf{K}\div\alpha$. Hence, by *closure*, $\beta \in \mathbf{K}\div\alpha$.

(4) to (5): Due to part (1)-to-(2) of **Theorem 7.3.9**, we only have to prove that the contraction functions are equivalent. Let $\div_{\leq}$ be the contraction function based on $\leq_{\mathbf{K}}$ and $\div_{\$}$ the shielded sphere-based contraction defined as in part (1)-to-(2) of **Theorem 7.3.9**.

(a) For the first direction, let $\beta \in \mathbf{K} \div_{\leq} \alpha$. According to **Definition 8.3.2**, $\beta \in \mathbf{K}$ and we have two cases:

(a1) $\alpha <_{\mathbf{K}} \top$: Then $\alpha <_{\mathbf{K}} \alpha \vee \beta$. It follows from $\|\mathbf{K}\| \subseteq \|\beta\|$ and part (1)-to-(2) of **Theorem 7.3.9** that $\|\mathbf{K}_{\$}\| \subseteq \|\beta\|$. Due to $\alpha <_{\mathbf{K}} \top$ it follows by part (1)-to-(2) of **Theorem 7.3.9** that $\cup\$_{\leq} \not\subseteq \|\alpha\|$. Then $\|\neg\alpha\| \cap (\cup\$_{\leq}) \neq \emptyset$. Since $\alpha <_{\mathbf{K}} \alpha \vee \beta$, part (1)-to-(2) of **Theorem 7.3.9** yields that $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha \vee \beta\|$; then $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha\| \cup \|\beta\|$, from which it follows that $\mathbf{S}_{\neg\alpha} \cap \|\neg\alpha\| \subseteq (\|\alpha\| \cup \|\beta\|) \cap \|\neg\alpha\| = \|\beta\| \cap \|\neg\alpha\| \subseteq \|\beta\|$. Then, since $\|\mathbf{K}_{\$}\| \subseteq \|\beta\|$, it follows that $\|\mathbf{K}_{\$}\| \cup (\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha}) \subseteq \|\beta\|$; hence $\beta \in \bigcap(\|\mathbf{K}_{\$}\| \cup (\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha}))$ and consequently $\beta \in \mathbf{K} \div_{\$} \alpha$.

(a2) $\alpha \not<_{\mathbf{K}} \top$: Then $\mathbf{K} \div_{\leq} \alpha = \mathbf{K}$. By part (1)-to-(2) of **Theorem 7.3.9** it follows that $\cup\$_{\leq} \subseteq \|\alpha\|$. Then $\|\neg\alpha\| \cap (\cup\$_{\leq}) = \emptyset$, which it follows that $\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha} = \emptyset$ hence by definition of $\div_{\$}$ and part (1)-to-(2) of **Theorem 7.3.9** that $\mathbf{K} \div_{\$} \alpha = \mathbf{K}_{\$} = \mathbf{K}$.

(b) For the second direction, let $\beta \in \mathbf{K} \div_{\$} \alpha$. We have two cases, according to definition of $\div_{\$}$:

(b1) $\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha} \neq \emptyset$. Then, since $\mathbf{S}_{\neg\alpha} \subseteq (\cup\$_{\leq})$, $\|\neg\alpha\| \cap (\cup\$_{\leq}) \neq \emptyset$. Then $(\cup\$_{\leq}) \nsubseteq \|\alpha\|$, from which it follows by part (1)-to-(2) of **Theorem 7.3.9** that $\alpha <_{\mathbf{K}} \top$. Since $\beta \in \bigcap(\|\mathbf{K}_{\$}\| \cup (\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha}))$, we then have $\|\mathbf{K}_{\$}\| \subseteq \|\beta\|$ and $\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha} \subseteq \|\beta\|$. From the first part we conclude that $\beta \in \mathbf{K}$. From the second part we obtain $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha \vee \beta\|$. Suppose that $\alpha \not<_{\mathbf{K}} \alpha \vee \beta$. Then $\alpha \vee \beta \not\leq_{\mathbf{K}} \alpha$, and it follows from condition ($\$_{\leq}4$) of part (1)-to-(2) of **Theorem 7.3.9**, since $\mathbf{S}_{\neg\alpha}$ is a sphere and $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha \vee \beta\|$, that $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha\|$. This contradicts the condition for this case, and we may conclude from this contradiction that $\alpha <_{\mathbf{K}} \alpha \vee \beta$; hence $\beta \in \mathbf{K} \mathbin{-\!\circ\!-}_{\leq} \alpha$.

(b2) $\mathbf{K} \mathbin{-\!\circ\!-}_{\$} \alpha = \mathbf{K}_{\$} =$ (by part (1)-to-(2) of **Theorem 7.3.9**) $\mathbf{K}$: Then $\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha} = \emptyset$, from which it follows that $\|\neg\alpha\| \cap (\cup\$_{\leq}) = \emptyset$. Then $(\cup\$_{\leq}) \subseteq \|\alpha\|$, then by part (1)-to-(2) of **Theorem 7.3.9** $\top \leq_{\mathbf{K}} \alpha$; hence $\mathbf{K} \mathbin{-\!\circ\!-}_{\leq} \alpha = \mathbf{K}$.

(5) to (4): Let $\$$ be a sphere system and let $\leq$ be defined as in part (2)-to-(1) of **Theorem 7.3.9**. Due to part (2)-to-(1) of **Theorem 7.3.9**, we only have to prove that the contraction functions are equivalent:

(a) For the first direction, let $\beta \in \mathbf{K} \mathbin{-\!\circ\!-}_{\$} \alpha$. We have two cases:

(a1) $\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha} \neq \emptyset$. Then, since $\mathbf{S}_{\neg\alpha} \subseteq (\cup\$_{\leq})$, $\|\neg\alpha\| \cap (\cup\$_{\leq}) \neq \emptyset$, from which it follows that $\beta \in \bigcap(\|\mathbf{K}_{\$}\| \cup (\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha}))$. Then $\|\mathbf{K}_{\$}\| \subseteq \|\beta\|$ and $\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha} \subseteq \|\beta\|$. From the first part we conclude that $\beta \in \mathbf{K}$. From the second part we obtain that $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha \vee \beta\|$. By definition $\mathbf{S}_{\neg\alpha} \nsubseteq \|\alpha\|$. Let $\mathbf{G} \in \$$ and $\mathbf{G} \subseteq \|\alpha\|$: then, since $\mathbf{S}_{\neg\alpha} \nsubseteq \|\alpha\|$, it follows that $\mathbf{S}_{\neg\alpha} \nsubseteq \mathbf{G}$. Then by **($\$3$)** $\mathbf{G} \subseteq \mathbf{S}_{\neg\alpha}$. Then $\mathbf{G} \subseteq \|\alpha \vee \beta\|$, from which it follows by the definition of $\leq$ that $\alpha \leq \alpha \vee \beta$. From $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha \vee \beta\|$ and $\mathbf{S}_{\neg\alpha} \nsubseteq \|\alpha\|$ we obtain by the definition of $\leq$ that $\alpha \vee \beta \not\leq \alpha$. Hence $\alpha < \alpha \vee \beta$ and consequently $\beta \in \mathbf{K} \mathbin{-\!\circ\!-}_{\leq} \alpha$.

(a2) $\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha} = \emptyset$. Then $\cup\$ \subseteq \|\alpha\|$. Then $\mathbf{K} \mathbin{-\!\circ\!-}_{\$} \alpha = \mathbf{K}$, hence $\beta \in \mathbf{K}$. Let

$\mathbf{G}$ be a sphere such that $\mathbf{G} \subseteq \|\top\|$. Then, since $\mathbf{G} \subseteq \cup\$ \subseteq \|\alpha\|$, it follows that $\top \leq \alpha$, hence $\beta \in \mathbf{K} \div_{\leq} \alpha$.

(b) For the other direction, let $\beta \in \mathbf{K} \div_{\leq} \alpha$. Then $\beta \in \mathbf{K}$. We have two cases:

(b1) $\alpha \leq_{\top}$. Then, according to **Definition 8.3.2**, $\alpha < \alpha \vee \beta$. It follows from the definition of $\leq$ that for all $\mathbf{G} \in \$$, if $\mathbf{G} \subseteq \|\alpha\|$ then $\mathbf{G} \subseteq \|\alpha \vee \beta\|$; and that there is some $\mathbf{G}' \in \$$ such that $\mathbf{G}' \subseteq \|\alpha \vee \beta\|$ and $\mathbf{G}' \nsubseteq \|\alpha\|$. By $\mathbf{G}' \nsubseteq \|\alpha\|$ it follows that $\mathbf{G}' \cap \|\neg\alpha\| \neq \emptyset$. Then by definition of $\mathbf{S}_{\neg\alpha}$, $\mathbf{S}_{\neg\alpha} \subseteq \mathbf{G}'$. Hence $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha \vee \beta\|$. Then $\mathbf{S}_{\neg\alpha} \cap \|\neg\alpha\| \subseteq \|\beta\|$; and since $\beta \in \mathbf{K}$ it follows that $\|\mathbf{K}_{\$}\| \subseteq \beta$ and consequently that $\|\mathbf{K}_{\$}\| \cup (\mathbf{S}_{\neg\alpha} \cap \|\neg\alpha\|) \subseteq \|\beta\|$ which it follows that $\beta \in \bigcap(\|\mathbf{K}_{\$}\| \cup (\|\neg\alpha\| \cap \mathbf{S}_{\neg\alpha}))$; hence $\beta \in \mathbf{K} \div_{\$} \alpha$.

(b2) $\top \leq \alpha$. Then $\mathbf{K} \div_{\leq} \alpha = \mathbf{K}$. By definition of $\leq$ it follows that if $\mathbf{G} \subseteq \|\top\|$, then $\mathbf{G} \subseteq \|\alpha\|$. Since $\mathbf{G} \subseteq \|\top\|$ for all $\mathbf{G} \in \$$; it follows that $\mathbf{G} \subseteq \|\alpha\|$ for all $\mathbf{G} \in \$$. In particular $\mathbf{S}_{\neg\alpha} \subseteq \|\alpha\|$, then $\mathbf{S}_{\neg\alpha} \cap \|\neg\alpha\| = \emptyset$, hence by definition of $\div_{\$}$ it follows that $\mathbf{K} \div_{\$} \alpha = \mathbf{K}$. ■

Proof of Observation 8.5.2.

1: Directly from the definition of $\circ$.

2: If $\mathbf{K} \circ \alpha \neq \mathbf{K}$, then by $\div$ *inclusion* $\mathbf{K} \div \neg\alpha \subseteq \mathbf{K}$, hence $\mathbf{K} \circ \alpha = (\mathbf{K} \div \neg\alpha) + \alpha \subseteq \mathbf{K} + \alpha$.

3: Let $\mathbf{K} \nvdash \neg\alpha$. Then by $\div$ *inclusion* $\mathbf{K} \div \neg\alpha \nvdash \neg\alpha$. By $\div$ *vacuity* $\mathbf{K} \div \neg\alpha = \mathbf{K}$; hence $\mathbf{K} \circ \alpha = \mathbf{K} + \alpha$.

4: Directly from the definition of $\circ$.

5: Let $\mathbf{K} \circ \alpha \nvdash \neg\beta$. We have two cases: **(a)** $\mathbf{K} \circ \alpha = \mathbf{K}$: Then $\mathbf{K} \nvdash \neg\beta$. By $\div$ *inclusion*, $\mathbf{K} \div \neg\beta \nvdash \neg\beta$, hence $\beta \in \mathbf{K} \circ \beta = (\mathbf{K} \div \neg\beta) + \beta$. **(b)** $\mathbf{K} \circ \alpha \neq \mathbf{K}$: Then by definition $\mathbf{K} \circ \alpha = (\mathbf{K} \div \neg\alpha) + \alpha$. Then $(\mathbf{K} \div \neg\alpha) + \alpha \nvdash \neg\beta$ and consequently $\mathbf{K} \div \neg\alpha \nvdash \neg\beta$. It follows by *persistence* that $\mathbf{K} \div \neg\beta \nvdash \neg\beta$, hence $\mathbf{K}\circ\beta = (\mathbf{K}\div\neg\beta) + \beta$. Then $\beta \in \mathbf{K}\circ\beta$ follows directly.

6: Let $\mathbf{K}\circ\alpha = \mathbf{K}\circ\beta = \mathbf{K}$. There are three cases: First case, $\mathbf{K}\div\neg\alpha \nvdash \neg\alpha$: Then $\mathbf{K}\circ\alpha = (\mathbf{K}\div\neg\alpha) + \alpha = \mathbf{K}$, so that $\mathbf{K} \vdash \alpha \vee \beta$. It also follows from $\mathbf{K}\div\neg\alpha \nvdash \neg\alpha$ that $(\mathbf{K}\div\neg\alpha) + \alpha \nvdash \perp$ so that $\mathbf{K} \nvdash \perp$ and consequently $\mathbf{K} \nvdash \neg(\alpha\vee\beta)$. Then by $\div$ *vacuity*, $\mathbf{K}\div\neg(\alpha\vee\beta) = \mathbf{K}$. Hence $\mathbf{K}\circ(\alpha\vee\beta) = \mathbf{K}\div\neg(\alpha\vee\beta) + (\alpha\vee\beta) = \mathbf{K}+(\alpha\vee\beta) = \mathbf{K}$. Second case, $\mathbf{K}\div\neg\beta \nvdash \neg\beta$: Similar to the first case. Third case, $\mathbf{K}\div\neg\alpha \vdash \neg\alpha$ and $\mathbf{K}\div\neg\beta \vdash \neg\beta$: Then by $\div$ *relative success* $\mathbf{K}\div\neg\alpha = \mathbf{K}\div\neg\beta = \mathbf{K}$. By *conjunctive constancy* $\mathbf{K}\div(\neg\alpha\wedge\neg\beta) = \mathbf{K}$, hence by $\div$ *extensionality* $\mathbf{K}\div\neg(\alpha\vee\beta) = \mathbf{K}$. Since $\mathbf{K} \vdash \neg(\alpha\vee\beta)$, then $\mathbf{K}\div\neg(\alpha\vee\beta) \vdash \neg(\alpha\vee\beta)$. Hence, by definition of $\circ$, $\mathbf{K}\circ(\alpha\vee\beta) = \mathbf{K}$.

7: Let $\vdash \alpha\rightarrow\beta$ and $\alpha \in \mathbf{K}\circ\alpha$. We have two cases: First case $\mathbf{K}\div\neg\alpha \nvdash \neg\alpha$: Then by *success propagation* $\mathbf{K}\div\neg\beta \nvdash \neg\beta$, so that $\mathbf{K}\circ\beta = (\mathbf{K}\div\neg\beta) + \beta$ and hence $\beta \in \mathbf{K}\circ\beta$. Second case, $\mathbf{K}\div\neg\alpha \vdash \neg\alpha$: Then $\mathbf{K}\circ\alpha = \mathbf{K}$. Due to $\div$ *inclusion* $\neg\alpha \in \mathbf{K}$. Since $\alpha \in \mathbf{K}\circ\alpha = \mathbf{K}$, it follows that $\mathbf{K} \vdash \perp$. Then, since $\mathbf{K}\circ\beta = (\mathbf{K}\div\neg\beta) + \beta$ or $\mathbf{K}\circ\beta = \mathbf{K}$, it follows that in $\beta \in \mathbf{K}\circ\beta$.

8: Let $\mathbf{K}\circ\alpha \vdash \alpha$ and $\mathbf{K}\circ\alpha \nvdash \neg\beta$. First suppose that $\mathbf{K}\div\neg\alpha \vdash \neg\alpha$. Then $\div$ *relative success* yields $\mathbf{K}\div\neg\alpha = \mathbf{K}$; and from the definition of $\circ$ we obtain $\mathbf{K}\circ\alpha = \mathbf{K}$. Then, since $\mathbf{K} \vdash \alpha$ and $\mathbf{K} \vdash \neg\alpha$, it follows that $\mathbf{K} \vdash \perp$, contrary to the assumption that $\mathbf{K}\circ\alpha \nvdash \beta$. It follows from this contradiction that $\mathbf{K}\div\neg\alpha \nvdash \neg\alpha$, hence $\mathbf{K}\circ\alpha = (\mathbf{K}\div\neg\alpha) + \alpha$. Since $\mathbf{K}\circ\alpha \nvdash \neg\beta$, it follows that $\mathbf{K}\div\neg\alpha \nvdash \alpha\rightarrow\neg\beta$. By $\div$ *extensionality* $\mathbf{K}\div\neg\alpha = \mathbf{K}\div(\neg\alpha\wedge(\alpha\rightarrow\neg\beta))$. Then by *conjunctive inclusion* and $\div$ *extensionality*, $\mathbf{K}\div(\neg\alpha\wedge(\alpha\rightarrow\neg\beta)) \subseteq \mathbf{K}\div(\alpha\rightarrow\neg\beta) = \mathbf{K}\div\neg(\alpha\wedge\beta)$.

Since $\mathbf{K}\div\neg(\alpha\wedge\beta) \nvdash \neg(\alpha\wedge\beta)$, $\mathbf{K}\circ(\alpha\wedge\beta) = (\mathbf{K}\div\neg(\alpha\wedge\beta)) + (\alpha\wedge\beta)$. Then $(\mathbf{K}\circ\alpha) + \beta = ((\mathbf{K}\div\neg\alpha) + \alpha) + \beta = (\mathbf{K}\div\neg\alpha) + (\alpha\wedge\beta) \subseteq (\mathbf{K}\div\neg(\alpha\wedge\beta)) + (\alpha\wedge\beta) = \mathbf{K}\circ(\alpha\wedge\beta)$.

9: There are three cases: **(a)** $\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\beta)\vdash\neg(\alpha\wedge\beta)$: Then by *persistence* $\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha\vdash\neg(\alpha\wedge\beta)$. Then $((\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha)+\alpha)+\beta=\mathbf{K}_\perp=(\mathbf{K}\circ\alpha)+\beta$. Hence $\mathbf{K}\circ(\alpha\wedge\beta)\subseteq(\mathbf{K}\circ\alpha)+\beta$. **(b)** $\neg(\alpha\wedge\beta)\notin\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\beta)$: Then by *success propagation*, $\neg\alpha\notin\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha$. **(b1)** $\neg(\alpha\wedge\neg\beta)\in\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta)$: Then by *relative success*, $\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta)=\mathbf{K}$. By *conjunctive overlap* and *extensionality*, $\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\beta)\cup\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta)\subseteq\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha$, then by *inclusion* $\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\beta)\subseteq\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha$. Hence $\mathbf{K}\circ(\alpha\wedge\beta)=(\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\beta))+(\alpha\wedge\beta)\subseteq((\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha)+\alpha)+\beta=(\mathbf{K}\circ\alpha)+\beta$. **(b2)** $\neg(\alpha\wedge\neg\beta)\notin\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta)$: Let $\delta\in\mathbf{K}\circ(\alpha\wedge\beta)$. Then $(\neg\alpha\vee\neg\beta\vee\delta)\in\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\beta)$. Then by *inclusion* $(\neg\alpha\vee\neg\beta\vee\delta)\in\mathbf{K}$. It follows by *recovery* that $(\neg\alpha\vee\neg\beta\vee\delta)\in(\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta))+\neg(\alpha\wedge\neg\beta)$ and since $(\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta))+\neg(\alpha\wedge\neg\beta)\subseteq(\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta))+(\alpha\wedge\beta)$, it follows by *closure* that $(\neg\alpha\vee\neg\beta\vee\delta)\in\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg(\alpha\wedge\neg\beta)$. Then by *conjunctice overlap* $(\neg\alpha\vee\neg\beta\vee\delta)\in\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha$, hence $\delta\in((\mathbf{K}\mathbin{-\!\!\!\!\circ}\neg\alpha)+\alpha)+\beta=(\mathbf{K}\circ\alpha)+\beta$.■

Proof of Observation 8.5.3.

1: Trivial.

2: Trivial.

3: Let $\mathbf{K}\nvdash\alpha$. Then by $\circ$ *vacuity* $\mathbf{K}+\neg\alpha\subseteq\mathbf{K}\circ\neg\alpha$, hence $\mathbf{K}\cap\mathbf{K}\circ\neg\alpha=\mathbf{K}$.

4: Trivial.

5: Due to $\circ$ *relative success* either $\mathbf{K}\circ\neg\alpha=\mathbf{K}$ or $\mathbf{K}\circ\neg\alpha\vdash\neg\alpha$. If $\mathbf{K}\circ\neg\alpha=\mathbf{K}$, then $\mathbf{K}\mathbin{-\!\!\!\!\circ}\alpha=\mathbf{K}$, hence $\mathbf{K}\subseteq(\mathbf{K}\mathbin{-\!\!\!\!\circ}\alpha)+\alpha$. In the other case, when $\mathbf{K}\circ\neg\alpha\vdash\neg\alpha$, let $\beta\in\mathbf{K}$. It follows from the logical closure of $\mathbf{K}$ that $\alpha\rightarrow\beta\in\mathbf{K}$ and from $\mathbf{K}\circ\neg\alpha\vdash\neg\alpha$ that $\mathbf{K}\circ\neg\alpha\vdash\alpha\rightarrow\beta$. Due to $\circ$ *closure*, $\alpha\rightarrow\beta\in\mathbf{K}\circ\neg\alpha$. Hence $\alpha\rightarrow\beta\in(\mathbf{K}\cap(\mathbf{K}\circ\neg\alpha))$, so that $\beta\in(\mathbf{K}\cap(\mathbf{K}\circ\neg\alpha))+\alpha=(\mathbf{K}\mathbin{-\!\!\!\!\circ}\alpha)+\alpha$.

6: Let $\beta \notin \mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha$. Then it follows by Harper identity that either $\beta \notin \mathbf{K}{\circ}\neg\alpha$ or $\beta \notin \mathbf{K}$. If $\beta \notin \mathbf{K}$, then $\beta \notin \mathbf{K} \cap \mathbf{K}{\circ}\neg\beta$, hence $\beta \notin \mathbf{K} \mathbin{-\!\!\circ\!\!-} \beta$. If $\beta \notin \mathbf{K}{\circ}\neg\alpha$, then it follows from *closure* that $\mathbf{K}{\circ}\neg\alpha \nvdash \beta$, and hence by *strong regularity* that $\mathbf{K}{\circ}\neg\beta \vdash \neg\beta$. Then, since $\mathbf{K} \neq \mathbf{K}_{\perp}$, it follows by *consistency preservation* that $\beta \notin \mathbf{K}{\circ}\neg\beta$, hence $\beta \notin \mathbf{K} \mathbin{-\!\!\circ\!\!-} \beta$.

7: If $\mathbf{K}{\circ}\neg\alpha = \mathbf{K}$, then $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha = \mathbf{K}$ follows directly. If $\mathbf{K}{\circ}\neg\alpha \neq \mathbf{K}$, then it follows from $\circ$ *relative success* that $\mathbf{K}{\circ}\neg\alpha \vdash \neg\alpha$. If $\mathbf{K} = \mathbf{K}_{\perp}$, then it follows by $\circ$ *closure* that $\mathbf{K}{\circ}\neg\alpha \nvdash \perp$. If $\mathbf{K} \neq \mathbf{K}_{\perp}$, then it follows by *consistency preservation* that $\mathbf{K}{\circ}\neg\alpha \nvdash \perp$. In both cases it follows from $\mathbf{K}{\circ}\neg\alpha \vdash \alpha$ and the consistency of $\mathbf{K}{\circ}\neg\alpha$ that $\alpha \notin \mathbf{K}{\circ}\neg\alpha$, hence $\alpha \notin \mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha = \mathbf{K} \cap (\mathbf{K}{\circ}\neg\alpha)$.

8: Let $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha = \mathbf{K} \mathbin{-\!\!\circ\!\!-} \beta = \mathbf{K}$. There are three cases: First case $\alpha \notin \mathbf{K}{\circ}\neg\alpha$: Then, since $\alpha \notin \mathbf{K} \cap \mathbf{K}{\circ}\neg\alpha = -\!\!\circ\!\!- K\alpha = \mathbf{K}$, it follows that $\alpha \notin \mathbf{K}$. Then $\alpha \wedge \beta \notin \mathbf{K}$, and it follows from $\circ$ *vacuity* that $\mathbf{K} \subseteq \mathbf{K}{\circ}\neg(\alpha \wedge \beta)$, so that $\mathbf{K} \mathbin{-\!\!\circ\!\!-} (\alpha \wedge \beta) = \mathbf{K} \cap \mathbf{K}{\circ}\neg(\alpha \wedge \beta) = \mathbf{K}$. Second case, $\beta \notin \mathbf{K}{\circ}\neg\beta$: Symmetrical with the first case. Third case, $\alpha \in \mathbf{K}{\circ}\neg\alpha$ and $\beta \in \mathbf{K}{\circ}\neg\beta$: If $\neg\alpha \in \mathbf{K}{\circ}\neg\alpha$, then $\mathbf{K}{\circ}\neg\alpha = \mathbf{K}_{\perp}$, then by *consistency preservation*, $\mathbf{K} = \mathbf{K}_{\perp}$; then $\mathbf{K}{\circ}\neg\alpha = \mathbf{K}$. If $\neg\alpha \notin \mathbf{K}{\circ}\neg\alpha$, then by $\circ$ *relative success*, $\mathbf{K}{\circ}\neg\alpha = \mathbf{K}$. By the same reasoning we obtain $\mathbf{K}{\circ}\neg\beta = \mathbf{K}$. It follows by *disjunctive constancy* that $\mathbf{K}{\circ}(\neg\alpha \vee \neg\beta) = \mathbf{K}$. Then by $\circ$ *extensionality*, $\mathbf{K} \mathbin{-\!\!\circ\!\!-} (\alpha \wedge \beta) = \mathbf{K} \cap \mathbf{K}{\circ}(\neg\alpha \vee \neg\beta) = \mathbf{K}$.

9: Let $\mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha \nvdash \alpha$ and $\vdash \beta \rightarrow \alpha$. If $\alpha \notin \mathbf{K}$, then $\beta \notin \mathbf{K}$, hence $\beta \notin \mathbf{K} \cap (\mathbf{K}{\circ}\neg\beta)$. If $\alpha \in \mathbf{K}$, then it follows from $\mathbf{K} \cap (\mathbf{K}{\circ}\neg\alpha) \nvdash \alpha$ that $\mathbf{K}{\circ}\neg\alpha \nvdash \alpha$. Hence $\mathbf{K}{\circ}\neg\alpha \neq \mathbf{K}$ and it follows from $\circ$ *relative success* that $\neg\alpha \in \mathbf{K}{\circ}\neg\alpha$. By *strict improvement*, $\neg\beta \in \mathbf{K}{\circ}\neg\beta$ and by *consistency preservation*, $\mathbf{K}{\circ}\neg\beta \nvdash \beta$. Hence $\mathbf{K} \cap \mathbf{K}{\circ}\neg\beta = \mathbf{K} \mathbin{-\!\!\circ\!\!-} \beta \nvdash \beta$.

10: Let $\delta \in \mathbf{K} \mathbin{-\!\!\circ\!\!-} \alpha$ and $\delta \in \mathbf{K} \mathbin{-\!\!\circ\!\!-} \beta$. Then $\delta \in \mathbf{K} \cap \mathbf{K}{\circ}\neg\alpha$ and $\delta \in \mathbf{K} \cap \mathbf{K}{\circ}\neg\beta$. Due to $\vdash \alpha \leftrightarrow \neg(\neg(\alpha \wedge \beta) \wedge \neg\alpha)$, it follows by $\circ$ *extensionality* that

$\delta \in \mathbf{K} \cap \mathbf{K}\circ(\neg(\alpha \wedge \beta) \wedge \neg\alpha) \subseteq$ (by *superexpansion*) $\mathbf{K} \cap (\mathbf{K}\circ\neg(\alpha \wedge \beta)) + \neg\alpha$. Then $\neg\alpha \rightarrow \delta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$. By the same reasoning we obtain $\neg\beta \rightarrow \delta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$. Then by $\circ$ *closure* $\neg(\alpha \wedge \beta) \rightarrow \delta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$. If $\mathbf{K}\circ\neg(\alpha \wedge \beta) = \mathbf{K}$, then it follows, since $\delta \in \mathbf{K}$, that $\delta \in \mathbf{K} \cap \mathbf{K}\circ\neg(\alpha \wedge \beta) = \mathbf{K}\div(\alpha \wedge \beta)$. If $\mathbf{K}\circ\neg(\alpha \wedge \beta) \neq \mathbf{K}$, then it follows from $\circ$ *relative success* that $\mathbf{K}\circ\neg(\alpha \wedge \beta) \nvdash \neg(\alpha \wedge \beta)$; then by $\circ$ *closure* $\delta \in \mathbf{K}\circ\neg(\alpha \wedge \beta)$, hence $\delta \in \mathbf{K} \cap \mathbf{K}\circ\neg(\alpha \wedge \beta) = \mathbf{K}\div(\alpha \wedge \beta)$.

11: Let $\mathbf{K}\div(\alpha \wedge \beta) \nvdash \alpha$. We have two possible cases according to Harper identity: First case, $\alpha \notin \mathbf{K}$: Then by $\circ$ *vacuity*, $\mathbf{K}\div\alpha = \mathbf{K} \cap \mathbf{K}+\neg\alpha = \mathbf{K}$. Hence $\mathbf{K}\div(\alpha \wedge \beta) = \mathbf{K} \cap \mathbf{K}\circ\neg(\alpha \wedge \beta) \subseteq \mathbf{K} = \mathbf{K}\div\alpha$. Second case, $\mathbf{K}\circ\neg(\alpha \wedge \beta) \nvdash \alpha$: Then by *strong regularity* $\mathbf{K}\circ\neg\alpha \nvdash \neg\alpha$. By *strict improvement* $\mathbf{K}\circ\neg(\alpha \wedge \beta) \nvdash \neg(\alpha \wedge \beta)$. It follows from *guarded subexpansion* and $\circ$ *extensionality* that $(\mathbf{K}\circ\neg(\alpha \wedge \beta)) + \neg\alpha \subseteq \mathbf{K}\circ(\neg(\alpha \wedge \beta) \wedge \neg\alpha) = \mathbf{K}\circ\neg\alpha$. Due to $(\mathbf{K}\circ\neg(\alpha \wedge \beta)) + \neg\alpha \subseteq \mathbf{K}\circ\neg\alpha$ it follows that $\mathbf{K}\circ\neg(\alpha \wedge \beta) \subseteq \mathbf{K}\circ\neg\alpha$. Hence $\mathbf{K}\div(\alpha \wedge \beta) = \mathbf{K} \cap \mathbf{K}\circ\neg(\alpha \wedge \beta) \subseteq \mathbf{K} \cap \mathbf{K}\circ\neg\alpha = \mathbf{K}\div\alpha$. ■

Proof of Theorem 8.5.4: Let $\circ = \mathbb{R}(\div)$ and $\div_2 = \mathbb{C}()R(\div))$. Then:

$$\mathbf{K} \circ \neg\alpha = \begin{cases} (\mathbf{K} \div \alpha) + \neg\alpha & \text{if } \mathbf{K} \div \alpha \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

$$\mathbf{K} \div_2 \alpha = \mathbf{K} \cap \mathbf{K}\circ\neg\alpha$$

There are two cases: First case, $\mathbf{K} \div \alpha \vdash \alpha$: Then $\mathbf{K}\circ\neg\alpha = \mathbf{K}$. By *relative success*, $\mathbf{K} \div \alpha = \mathbf{K}$. Then $\mathbf{K} \div_2 \alpha = \mathbf{K} \cap \mathbf{K}\circ\neg\alpha = \mathbf{K} \cap \mathbf{K} = \mathbf{K} = \mathbf{K} \div \alpha$. Second case, $\mathbf{K} \div \alpha \nvdash \alpha$: Then $\mathbf{K} \div_2 \alpha = \mathbf{K} \cap ((\mathbf{K} \div \alpha) + \neg\alpha)$. For one direction, let $\delta \in \mathbf{K} \div \alpha$. Then by *inclusion*, $\delta \in \mathbf{K}$ from which it follows that $\delta \in \mathbf{K} \div_2 \alpha$; hence $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div_2 \alpha$. For the other direction, let $\delta \in \mathbf{K} \div_2 \alpha$. Then $\delta \in \mathbf{K}$ and by $\div$ *closure*, $\neg\alpha \rightarrow \delta \in \mathbf{K} \div \alpha$. By *recovery*, $\alpha \rightarrow \delta \in \mathbf{K} \div \alpha$. Then by *closure* $\delta \in \mathbf{K} \div \alpha$; hence $\mathbf{K} \div_2 \alpha \subseteq \mathbf{K} \div \alpha$. This concludes the proof. ∎

Proof of Theorem 8.5.5: Let $\div = \mathbb{C}(\circ))$ and $\circ_2 = \mathbb{R}(\mathbb{C}(\circ)))$. Then:

$$\mathbf{K} \div \neg\alpha = \mathbf{K} \cap \mathbf{K} \circ \alpha$$

$$\mathbf{K} \circ_2 \alpha = \begin{cases} (\mathbf{K} \div \neg\alpha) + \alpha & \text{if } \mathbf{K} \div \neg\alpha \nvdash \neg\alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

If $\neg\alpha \notin \mathbf{K}$, then by *vacuity* $\mathbf{K} \circ \alpha = \mathbf{K}+\alpha$, hence $\mathbf{K} \div \neg\alpha = \mathbf{K}$ and $\mathbf{K} \circ_2 \alpha = \mathbf{K}+\alpha$. For the principal case, let $\neg\alpha \in \mathbf{K}$. There are two subcases: **Case (a)**, $\mathbf{K} \circ \alpha \vdash \neg\alpha$: Then $\mathbf{K} \div \neg\alpha \vdash \neg\alpha$, hence $\mathbf{K} \circ_2 \alpha = \mathbf{K}$. If $\mathbf{K} \circ \alpha \vdash \alpha$, then $\mathbf{K} \circ \alpha = \mathbf{K}_\perp$, and by *consistency preservation*, $\mathbf{K} = \mathbf{K}_\perp$, so that $\mathbf{K} \circ \alpha = \mathbf{K} \circ_2 \alpha = \mathbf{K}_\perp$. If $\mathbf{K} \circ \alpha \nvdash \alpha$, then it follows from *relative success* that $\mathbf{K} \circ \alpha = \mathbf{K}$, and consequently $\mathbf{K} \circ \alpha = \mathbf{K} \circ_2 \alpha$ in this case as well. **Case (b)**, $\mathbf{K} \circ \alpha \nvdash \neg\alpha$: Since $\mathbf{K} \vdash \neg\alpha$, we have $\mathbf{K} \circ \alpha \neq \mathbf{K}$, and *relative success* yields $\mathbf{K} \circ \alpha \vdash \alpha$. We also have $\mathbf{K} \div \neg\alpha \nvdash \neg\alpha$, so that $\mathbf{K} \circ_2 \alpha = (\mathbf{K} \div \neg\alpha) + \alpha = (\mathbf{K} \cap \mathbf{K} \circ \alpha) + \alpha$. It follows for all δ that $\delta \in \mathbf{K} \circ_2 \alpha$ iff $\alpha \rightarrow \delta \in (\mathbf{K} \cap \mathbf{K} \circ \alpha)$, iff $\alpha \rightarrow \delta \in \mathbf{K}$ and $\alpha \rightarrow \delta \in \mathbf{K} \circ \alpha$, iff $\alpha \rightarrow \delta \in \mathbf{K} \circ \alpha$ (since $\neg\alpha \in \mathbf{K}$), iff $\delta \in \mathbf{K} \circ \alpha$ (by *closure*, since $\mathbf{K} \circ \alpha \vdash \alpha$). Hence $\mathbf{K} \circ \alpha = \mathbf{K} \circ_2 \alpha$. ∎

Part V

Appendix: Alternative Postulates

A Battery of Postulates and Their Interrelations

The pourpose of this chapter is to introdue a series of variations of the original AGM postulate and their interrelation. The literature reference given for the individual postulates refer as far has been as possible the source where the respective postulate was published.

A denotes a set of sentences and $\mathbf{H}$, $\mathbf{K}$ belief sets in a language $\mathcal{L}$ that can be finite or infinite, unless we specify that it is finite. $+$, $-$, and $*$ are operators of expansion, contraction and revision for $\mathbf{K}$ respectively. We classify the postulates in two categories:

Postulates implied by the AGM postulates In this category are included weaker versions of a particular AGM postulate (e.g. *weak success*), postulates derived from more than one AGM postulates (e.g. *consistent expansion*) and equivalent reformulations of one or more AGM postulates (e.g. *core-retaiment* or *disjunctive factoring*).[1]

Other postulates

[1] We do not present weaker versions of *closure*, since we assume in the whole work deal with belief sets. Weaker versions of *closure* and their implications can be found in [Fuh91, Han92b, Han93c, Han94b, Han96a, Rot95a, Wil94a]

Note that this categorization is only respect to belief sets. For example *core retainment* can be implied by the AGM postulates, but in belief bases can not. Part of the postulates and their interrelation was appeared in:

[•] Eduardo Fermé and Sven Ove Hansson. Selective revision. *Studia Logica*, 1998. In press.

[•] Eduardo Fermé and Sven Ove Hansson. Shielded contraction. In H.Rott and M-A Williams, editors, *Frontiers in Belief Revision*. Kluwer Academic Publisher, 1999. to appear.

[•] Eduardo Fermé and Ricardo Rodríguez. Semi-contraction: Axioms and construction. 1997. (manuscript).

[•] Eduardo Fermé and Ricardo Rodriguez. A brief note about the Rott contraction. *Logic Journal of the IGPL*, 6(6):835–842, 1998.

[•] Sven Ove Hansson, Eduardo Fermé, John Cantwell, and Marcelo Falappa. Credibility-limited revision. (manuscript), 1998.

A.7 Contraction

A.7.1 Postulates implied by the AGM postulates

- **Strict improvement:** If $\alpha \notin \mathbf{K}{-}\alpha$ and $\vdash \beta \rightarrow \alpha$ then $\beta \notin \mathbf{K}{-}\beta$.

- **Failure [FH94]:** If $\vdash \alpha$, then $\mathbf{K}{-}\alpha = \mathbf{K}$.

- **Relevance [Han89]:** If $\beta \in K$ and $\beta \notin \mathbf{K}{-}\alpha$ then there is some set A such that $\mathbf{K}{-}\alpha \subseteq A \subseteq \mathbf{K}$ and $\alpha \notin Cn(A)$ but $\alpha \in Cn(\mathbf{K} \cup \{\beta\})$.

●**Core-retainment [Han91b]:** If $\beta \in K$ and $\beta \notin \mathbf{K}-\alpha$ then there is some set A such that $A \subseteq \mathbf{K}$ and $\alpha \notin Cn(A)$ but $\alpha \in Cn(\mathbf{K} \cup \{\beta\})$.

●**Negation-retainment [Hanss]:** If $\alpha \in \mathbf{K}$, then $\alpha \in \mathbf{K}-\neg\alpha$.

●**Weak recovery [FR97]:** If $\mathbf{K} \neq \mathbf{K}-\alpha$ then there exists β such that $\mathbf{K} \vdash \beta$, $\mathbf{K}-\alpha \nvdash (\alpha \vee \beta)$ but $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + (\alpha \wedge \beta)$.

●**Proxy recovery [FR97]:** If $\mathbf{K} \neq \mathbf{K}-\alpha$ then there exists β such that $\mathbf{K} \vdash \beta$, $\mathbf{K}-\alpha \nvdash \beta$ and $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + \beta$.

●**Conjunctive constancy [FH99]:** If $\mathbf{K}-\alpha = \mathbf{K}-\beta = \mathbf{K}$ then $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}$.

●**Full Vacuity [Rot92b]:** $\mathbf{K}-\alpha = \mathbf{K}$ if and only if $\alpha \notin \mathbf{K}$ or $\vdash \alpha$.

●**relative success [Rot92b]:** Either $\mathbf{K}-\alpha = \mathbf{K}$ or $\alpha \notin \mathbf{K}-\alpha$

●**Conjunctive covering [AGM85]:** Either $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$ or $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\beta$.

●**Left conjunctive reduction [Rot92b]:** If $\beta \in \mathbf{K}-(\alpha \wedge \beta)$, then $\mathbf{K}-\alpha = \mathbf{K}-(\alpha \wedge \beta)$

●**Right conjunctive reduction [Rot92b]:** If $\beta \in \mathbf{K}-(\alpha \wedge \beta)$, then $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$.

●**Conjunctive reduction:** If $\beta \in \mathbf{K}-(\alpha \wedge \beta)$, then $\mathbf{K}-\alpha \subseteq \mathbf{K}-(\alpha \wedge \beta)$

●**Reciprocity [Rot92b]:** If $\alpha \to \beta \in \mathbf{K}-\beta$ and $\beta \to \alpha \in \mathbf{K}-\alpha$, then $\mathbf{K}-\alpha = \mathbf{K}-\beta$.

●**Conjunctive factoring [AGM85]:** $\mathbf{K}-(\alpha \wedge \beta) = \begin{cases} \mathbf{K}-\alpha, \text{ or} \\ \mathbf{K}-\beta, \text{ or} \\ \mathbf{K}-\alpha \cap \mathbf{K}-\beta \end{cases}$

The relation between this postulates and the AGM postulates can be shown in the following table, where the postulates marked with • are, together, sufficent to prove the postulate. The source of the proofs are cited. The proofs of our own can be found at the end of the appendix.

Postulates implied by the AGM contraction postulates										
	AGM postulates									
Postulate	cl.	inc.	vac.	suc.	ext.	rec.	c. inc.	c. ov.	cite	proff
Strict Improvement	•			•						A.9.1
Failure		•				•			[FH94]	
Relevance	•	•	•			•			[Han89]	
Core-Retainment	•	•	•			•			[Han91b]	
Negation-Retainment	•					•			[Hanss]	A.9.2
Weak Recovery						•				
Proxy Recovery						•				
Conjunctive Constancy		•	•			•				A.9.4
Full Vacuity		•	•			•			[Rot92b]	A.9.3

Postulates implied by the AGM contraction postulates										
	AGM postulates									
Postulate	cl.	inc.	vac.	suc.	ext.	rec.	c. inc.	c. ov.	cite	proff
relative Success		•		•		•			[Rot92b]	
Conjunctive Covering		•		•	•	•		•	[AGM85]	
Left Conjunctive Reduction	•	•				•		•	[Rot92b]	
Right Conjunctive Reduction	•	•		•	•	•	•		[Rot92b]	
Conjunctive Reduction	•	•		•	•	•	•	•	[Rot92b]	
Reciprocity	•	•		•	•	•	•	•	[Rot92b]	
Conjunctive Factoring	•	•	•	•	•	•	•	•	[AGM85]	

Other interesting relation are:

A.7.1 [FH94] *Relevance* implies *recovery.*

A.7.2 [Hanss] *Relevance* implies *core retainment.*

A.7.3 [Hanss] *Inclusion* and *core retainment* imply *failure* and *vacuity.*

A.7.4 [Hanss] *Core retainment* implies *Negation retainment.*

A.7.5 [FR97] *Weak recovery* implies *proxy recovery.*

A.7.6 [FR97] *Success, vacuity, failure* and *proxy recovery* imply *weak recovery.*

A.7.7 [FH99] *Vacuity* and *failure* imply *conjunctive constancy.*

A.7.8 [Rot92b] *Success* and *failure* imply *relative success.*

A.7.9 [Rot92b] If $-$ satisfies *Closure* and *extensionality* then $-$ satisfies *reciprocity* if and only if it satisfies *conjunctive reduction.*

A.7.2 Other Contraction postulates

●**Fullness [Gär82]:** If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}-\alpha$ then $\not\vdash \alpha$ and $\beta \to \alpha \in \mathbf{K}-\alpha$

●**Saturability [AM82]:** If $\alpha \in \mathbf{K}$, then for any $\beta \in \mathcal{L}$, either $\alpha \vee \beta \in \mathbf{K}-\alpha$ or $\alpha \vee \neg\beta \in \mathbf{K}-\alpha$

●**Primeness [AGM85]:** If $\beta \wedge \delta \in \mathbf{K}$, and $\beta \vee \delta \in \mathbf{K}-\alpha$, then $\beta \in \mathbf{K}-\alpha$ or $\delta \in \mathbf{K}-\alpha$.

●**Meet identity [AGM85]:** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha \cap \mathbf{K}-\beta$

●**Decomposition [AGM85]:** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$ or $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$.

●**Strong inclusion [Pag96, FR98]:** If $\beta \notin \mathbf{K}-\alpha$, then $\mathbf{K}-\alpha \subseteq \mathbf{K}-\beta$.

●**Linearity:** $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$ or $\mathbf{K}-\alpha \subseteq \mathbf{K}-\beta$.

●**Antitony [RP]:** If $\not\vdash \alpha$, then $\mathbf{K}-\alpha \subseteq \mathbf{K}-(\alpha \wedge \beta)$.

●**Expulsiveness [Hanss]:** If $\not\vdash \alpha$ and $\not\vdash \beta$, then either $\alpha \notin \mathbf{K}-\beta$ or $\beta \notin \mathbf{K}-\alpha$.

●**$-$ 10 [Pag96]:** If $\not\vdash \beta$ and $\beta \in \mathbf{K}-\alpha$, then $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$.

●**Converse conjunctive inclusion:** If $\mathbf{K}-(\alpha\wedge\beta) \subseteq \mathbf{K}-\beta$ then $\beta \notin \mathbf{K}-\alpha$ or $\vdash \alpha$ or $\vdash \beta$.

A.7.10 [Gär82] Closure, success and *fullness* imply *recovery.*

A.7.11 [FR98] *Strong inclusion* implies *conjunctive inclusion.*

A.7.12 [FR98] *Inclusion, failure* and *strong inclusion* imply *vacuity.*

A.7.13 [FR98] *Closure, success* and *strong inclusion* imply *expulsiveness.*

A.7.14 [FR98] *Inclusion, failure, strong inclusion* and *expulsiveness* imply *linearity.*

A.7.15 *Inclusion, failure* and *expulsiveness* imply *− 10.*

A.7.16 [FR98] *Closure, success, extensionality* and *strong inclusion* imply *linear hierarchical ordering.*

A.7.17 [RP] *Inclusion, vacuity, failure, conjunctive inclusion* and *antitony* imply *strong inclusion.*

A.7.18 [RP] *Closure, conjunctive inclusion* and *strong inclusion* imply *antitony.*

A.7.19 [RP] *Closure, conjunctive inclusion* and *strong inclusion* imply *antitony.*

A.7.20 [RP] *Closure, inclusion, success, conjunctive inclusion, failure* and *antitony* imply *− 10.*

A.7.21 [RP] *success* and *antitony* imply *converse conjunctive inclusion.*

A.8 Revision

A.8.1 Postulates implied by the AGM postulates

●**Weak proxy success :** $\exists\beta$, $\mathbf{K}*\alpha \vdash \beta$ such that $\mathbf{K}*\alpha = \mathbf{K}*\beta$.

●**Proxy success :** $\exists\beta, \mathbf{K}*\alpha \vdash \beta, \vdash \alpha \rightarrow \beta$ such that $\mathbf{K}*\alpha = \mathbf{K}*\beta$.

●**Stability [Han97]:** If $\alpha \in \mathbf{K}$ and $\mathbf{K} \neq \mathbf{K}_{\perp}$, then $\alpha \in \mathbf{K}*\alpha$.

●**Strong stability :** If $\alpha \in \mathbf{K}$ then $\alpha \in \mathbf{K}*\alpha$.

●**Relative success:** $\alpha \in \mathbf{K}*\alpha$ or $\mathbf{K}*\alpha = \mathbf{K}$.

●**disjuctive succcess:** $\alpha \in \mathbf{K}*\alpha$ or $\neg\alpha \in \mathbf{K}*\alpha$.

●**Strict improvement :** If $\alpha \in \mathbf{K}*\alpha$ and $\vdash \alpha \rightarrow \beta$, then $\beta \in \mathbf{K}*\beta$.

●**Regularity :** If $\beta \in \mathbf{K}*\alpha$ then $\beta \in \mathbf{K}*\beta$.

●**Strong regularity :** If $\neg\beta \notin \mathbf{K}*\alpha$ then $\beta \in \mathbf{K}*\beta$.

●**Weak success [Han96b]:** If $\mathbf{K} \nvdash \neg\alpha$, then $\alpha \in \mathbf{K}*\alpha$.

●**Weak success 2 [Rot95b]:** If $\nvdash \neg\alpha$, then $\alpha \in \mathbf{K}*\alpha$.

●**Preservation [Gär86]:** If $\nvdash \neg\alpha$, then $\mathbf{K} \subseteq \mathbf{K}*\alpha$.

●**Consistent expansion :** If $\mathbf{K} \nsubseteq \mathbf{K}*\alpha$ then $\mathbf{K} \cup (\mathbf{K}*\alpha) \vdash \perp$.

●**Truth impertubability:** If $\vdash \alpha$ and $\mathbf{K} \neq \mathbf{K}_{\perp}$, then $\mathbf{K}*\alpha = \mathbf{K}$

●**Weak consistency preservation [KM92]:** If $\mathbf{K} \neq \mathbf{K}_\perp$ and $\nvdash \neg\alpha$, then $\mathbf{K}*\alpha \neq \mathbf{K}_\perp$.

●**Weak consistency preservation 2 [Han96b]:** If $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K}*\alpha \neq \mathbf{K}_\perp$.

●**Disjunctive constancy :** If $\mathbf{K}*\alpha = \mathbf{K}*\beta = \mathbf{K}$ then $\mathbf{K}*(\alpha \vee \beta) = \mathbf{K}$.

●**Weak idempotence [Gär88]:** If $\alpha \in \mathbf{K}$ and $\mathbf{K} \neq \mathbf{K}_\perp$, then $\mathbf{K}*\alpha = \mathbf{K}$.

●**Disjunctive overlap [Gär88]:** $(\mathbf{K}*\alpha) \cap (\mathbf{K}*\beta) \subseteq \mathbf{K}*(\alpha \vee \beta)$.

●**Disjunctive inclusion [Gär88]:** If $\mathbf{K}*(\alpha \vee \beta) \nvdash \neg\alpha$, then $\mathbf{K}*(\alpha \vee \beta) \subseteq \mathbf{K}*\alpha$.

●**Weak disjunctive inclusion [KM92]:** $\mathbf{K}*(\alpha \vee \beta) \subseteq Cn(\mathbf{K}*\alpha \cup \mathbf{K}*\beta)$.

●**Guarded subexpansion:** If $\alpha \in \mathbf{K}*\alpha$ and $\mathbf{K}*\alpha \nvdash \neg\beta$, then $(\mathbf{K}*\alpha)+\beta \subseteq \mathbf{K}*(\alpha \wedge \beta)$.

●**Cut [MG91]:** If $\beta \in \mathbf{K}*\alpha$, then $\mathbf{K}*(\alpha \wedge \beta) \subseteq \mathbf{K}*\alpha$.

●**Cautious monotony [MG91]:** If $\beta \in \mathbf{K}*\alpha$, then $\mathbf{K}*\alpha \subseteq \mathbf{K}*(\alpha \wedge \beta)$

●**Reciprocity [Gär82]:** $\mathbf{K}*\alpha = \mathbf{K}*\beta$ if and only if $\alpha \in \mathbf{K}*\beta$ and $\beta \in \mathbf{K}*\alpha$.

●**Right reciprocity [KM92]:** If $\alpha \in \mathbf{K}*\beta$ and $\beta \in \mathbf{K}*\alpha$ then $\mathbf{K}*\alpha = \mathbf{K}*\beta$.

●**Disjunctive factoring [AGM85]:** $\mathbf{K}*(\alpha \vee \beta) = \begin{cases} \mathbf{K}*\alpha, \text{ or} \\ \mathbf{K}*\beta, \text{ or} \\ \mathbf{K}*\alpha \cap \mathbf{K}*\beta \end{cases}$

●**Disjunctive priority:** If $\mathbf{K} \nvdash \neg\alpha$ and $\mathbf{K} \vdash \neg\beta$, then $\mathbf{K}*(\alpha \vee \beta) = \mathbf{K}*\alpha$.

●**Disjunctive reduction:** If $\mathbf{K}*(\alpha \vee \beta) \vdash \neg\alpha$, then $\mathbf{K}*(\alpha \vee \beta) = \mathbf{K}*\beta$.

The relation between thess postulates and the AGM postulates are shown in the following table:

Postulates implied by the AGM revision postulates										
	AGM postulates									
Postulate	cl.	suc.	inc.	vac.	cons.	ext.	sub.	sup.	cite	proff
Weak Proxy Success		•								
Proxy Success		•								
Stability		•								
Strong Stability		•								
relative Success		•								
disjuctive succcess		•								
Strict Improvement		•								
Regularity		•								
Strong regularity		•								

Postulates implied by the AGM revision postulates

cont.

	AGM postulates									
Postulate	cl.	suc.	inc.	vac.	cons.	ext.	sub.	sup.	cite	proff
Weak Success		•								
				•						
Weak Success 2		•								
Preservation				•						
	•				•					
Consistent Expansion		•		•						A.9.5
Truth Impertubability			•	•						
Weak Consistency Preservation					•					
Weak Consistency Preservation 2	•			•						
Disjunctive Constancy		•		•	•					A.9.6
Weak Idempotence			•	•					[Gär88]	
Disjunctive Overlap	•	•				•		•	[Gär88]	
Disjunctive inclusion						•	•		[Gär88]	
Weak disjunctive inclusion		•			•	•	•		[GR93]	
Guarded subexpansion							•			
Cut								•	[MG91]	
Cautious monotony	•	•			•		•		[MG91]	
Reciprocity	•	•			•		•	•	[MG91]	

Postulates implied by the AGM revision postulates **cont.**										
	AGM postulates									
Postulate	cl.	suc.	inc.	vac.	cons.	ext.	sub.	sup.	cite	proff
Right Reciprocity	•	•			•		•	•	[MG91]	
Disjunctive factoring	•	•			•	•	•	•	[Gär88]	
Disjunctive Priority	•	•	•	•	•	•	•	•		A.9.7
Disjunctive Reduction	•	•			•		•	•		A.9.8

Other interesting relations are:

A.8.1 *relative success* implies *strong stability.*

A.8.2 *Strong stability* implies *stability.*

A.8.3 *Weak success* implies *stability.*

A.8.4 If **K** is consistent, then *weak success* implies *strong stability.*

A.8.5 *Vacuity* implies *weak success.*

A.8.6 If **K** is consistent, then *vacuity* implies *relative success.*

A.8.7 *Vacuity* implies *truth imperturbability.*

A.8.8 *Truth imperturbability* and *relative success* imply *proxy success.*

A.8.9 *Inclusion, consistent expansion* and *relative success* imply *proxy success.*

A.8.10 *Vacuity* and *strong stability* imply *relative success.*

A.8.11 *Vacuity* and *relative success* imply *consistent expansion.*

A.8.12 *Weak success* and *relative success* imply *disjuctive success.*

A.8.13 *Closure, extensionality, superexpansion, relative success* and *strict improvement* imply *disjunctive overlap.*

A.8.14 [Gär88] *Closure, success, extensionality* and *disjunctive overlap* imply *superexpansion.*

A.8.15 *Closure, vacuity, extensionality, relative success, strict improvement* and *disjunctive overlap* imply *superexpansion.*

A.8.16 *Extensionality, relative success, strict improvement* and *guarded subexpansion* imply *disjunctive inclusion.*

A.8.17 [Gär88] *Closure, success* and *disjunctive inclusion* imply *subexpansion.*

A.8.18 *Closure, vacuity, extensionality, relative success* and *disjunctive inclusion* imply *guarded subexpansion.*

A.8.19 *Closure, vacuity, extensionality, relative success* and *disjunctive factoring* imply *guarded subexpansion.*

A.8.20 [Gär88] *Disjunctive factoring* implies *disjunctive overlap.*

A.8.21 [Gär88] *Closure, extensionality, success* and *disjunctive factoring* imply *disjunctive inclusion.*

A.8.22 [Gär88] *Closure, success, consistency, extensionality, disjunctive overlap* and *disjunctive inclusion* imply *disjunctive factoring.*

A.8.23 *Closure, vacuity, consistency, extensionality, relative success, strict improvement, disjunctive overlap* and *disjunctive inclusion* imply *disjunctive factoring.*

A.8.24 [MG91] If $*$ satisfies *closure* and *success* then $*$ satisfies *reciprocity* if and only if it satisfies both *cut* and *cautious monotony.*

A.8.25 *Reciprocity* implies *right reciprocity.*

A.8.2 Other revision postulates

●**Strong consistency [Han97]:** $\mathbf{K}{*}\alpha \neq \mathbf{K}_\perp$.

●**Tenacity [Gär88]:** If $\beta \in \mathbf{K}$, then either $\beta \in \mathbf{K}{*}\alpha$ or $\neg\beta \in \mathbf{K}{*}\alpha$

●**Monotonicity [Gär88]:** If $\mathbf{H} \subseteq \mathbf{K}$, then $\mathbf{H}{*}\alpha \subseteq \mathbf{K}{*}\alpha$.

●**Consistency preservation [Makss]:** If $\mathbf{K} \neq \mathbf{K}_\perp$ then $\mathbf{K}{*}\alpha \neq \mathbf{K}_\perp$.

●**Idempotence [Gär88]:** If $\alpha \in \mathbf{K}$, then $\mathbf{K}{*}\alpha = \mathbf{K}$.

A.8.26 *Tenacity* implies *disjuctive succcess.*

A.8.27 *Idempotence* implies *weak idempotence.*

A.8.28 *Strong consistency* implies *consistency, consistency preservation* and *weak consistency preservation.*

A.8.29 *Consistency preservation* implies *weak consistency preservation.*

A.9 Proofs

Proof of property A.7.10 We have to prove that if $\beta \in \mathbf{K}$, then $\beta \in (\mathbf{K}-\alpha) + \alpha$. Let $\beta \in \mathbf{K}$. We have two subcases:

(a) $\beta \in \mathbf{K}-\alpha$, then $\beta \in (\mathbf{K}-\alpha) + \alpha$.

(b) $\beta \notin \mathbf{K}-\alpha$. By *fullness* $\nvdash \alpha$. Suppose that $\alpha \to \beta \notin \mathbf{K}-\alpha$, then by *fullness* $(\alpha \to \beta) \to \alpha \notin \mathbf{K}-\alpha$ and, since α is equivalent to $(\alpha \to \beta) \to \alpha$, by *closure* $\alpha \in \mathbf{K}-\alpha$, then by *success* $\vdash \alpha$. Absurd, then $\alpha \to \beta \in \mathbf{K}-\alpha$, hence $\beta \in (\mathbf{K}-\alpha) + \alpha$. ■

Proof of property A.7.11 Let $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$. Then by *strong inclusion* $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$. ■

Proof of property A.7.12 Let $\alpha \notin \mathbf{K}$ and $\vdash \beta$. Then by *failure* $\alpha \notin \mathbf{K}-\beta = \mathbf{K}$; by *strong inclusion* $\mathbf{K} = \mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$. Hence by *inclusion* $\mathbf{K}-\alpha = \mathbf{K}$. ■

Proof of property A.7.13 Let $\nvdash \alpha$ and $\nvdash \beta$. By *closure* and *success* $\alpha \wedge \beta \notin \mathbf{K}-\alpha$ and $\alpha \wedge \beta \notin \mathbf{K}-\beta$, then by *strong inclusion*

$\mathbf{K}-\alpha \subseteq \mathbf{K}-(\alpha \wedge \beta)$ and $\mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta)$. Let $\beta \in \mathbf{K}-\alpha$, then $\beta \in \mathbf{K}-(\alpha \wedge \beta)$, then by *success* $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$; hence $\alpha \notin \mathbf{K}-\beta$. ■

Proof of property A.7.14 If $\vdash \alpha$ then by *failure* $\mathbf{K}-\alpha = \mathbf{K}$, and by *inclusion* $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$. By the same reasoning if $\vdash \beta$ then $\mathbf{K}-\alpha \subseteq \mathbf{K}-\beta$. Let $\nvdash \alpha$ and $\nvdash \beta$, then by *expulsiveness* $\beta \notin \mathbf{K}-\alpha$ or $\alpha \notin \mathbf{K}-\beta$. Hence by *strong inclusion* $\mathbf{K}-\alpha \subseteq \mathbf{K}-\beta$ or $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$. ■

Proof of property A.7.15 Let $\nvdash \beta$ and $\beta \in \mathbf{K}-\alpha$. If $\vdash \alpha$ then by *failure* $\mathbf{K}-\alpha = \mathbf{K}$, and by *inclusion* $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$. If $\nvdash \alpha$, by *expulsiveness* $\alpha \notin \mathbf{K}-\beta$ hence by *strong inclusion* $\mathbf{K}-\beta \subseteq \mathbf{K}-\alpha$.■

Proof of property A.7.16 If $\vdash \alpha$ or $\vdash \beta$ then $\vdash (\alpha \wedge \beta) \leftrightarrow \alpha$ or $\vdash (\alpha \wedge \beta \leftrightarrow \beta)$, then by *extensionality* $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$ or $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$. Let $\nvdash \alpha$ and $\nvdash \beta$. By *closure* and *success* $\alpha \wedge \beta \notin \mathbf{K}-\alpha$ and $\alpha \wedge \beta \notin \mathbf{K}-\beta$, then by *strong inclusion* $\mathbf{K}-\alpha \subseteq \mathbf{K}-(\alpha \wedge \beta)$ and $\mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta)$. By *success* $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$ or $\beta \notin \mathbf{K}-(\alpha \wedge \beta)$. Then by *strong inclusion* $\mathbf{K}-\alpha \subseteq \mathbf{K}-(\alpha \wedge \beta)$ or $\mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta)$ Hence $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$ or $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$. ■

Proof of property A.8.8

(a) $\alpha \in \mathbf{K}{*}\alpha$: just let $\beta = \alpha$.

(b) $\alpha \notin \mathbf{K}{*}\alpha$: Due to *relative success* we then have $\mathbf{K}{*}\alpha = \mathbf{K}$ and consequently $\alpha \notin \mathbf{K}$. It follows from $\alpha \notin \mathbf{K}$ that $\mathbf{K}$ is consistent. Let $\vdash \beta$. Then $\vdash \alpha \rightarrow \beta$. Since $\mathbf{K}$ is logically closed, $\beta \in \mathbf{K} = \mathbf{K}*\alpha$. It follows from *truth imperturbability* that $\mathbf{K}*\beta = \mathbf{K}$. ■

Proof of property A.8.9

(a) $\alpha \in \mathbf{K}*\alpha$: just let $\beta = \alpha$.

(b) $\alpha \notin \mathbf{K}*\alpha$: Due to *relative success* we then have $\mathbf{K}*\alpha = \mathbf{K}$ and consequently $\alpha \notin \mathbf{K}$. It follows from $\alpha \notin \mathbf{K}$ that $\mathbf{K}$ is consistent. Let $\vdash \beta$. Then $\vdash \alpha \to \beta$. Since $\mathbf{K}$ is logically closed, $\beta \in \mathbf{K}$. It follows from *inclusion* that $\mathbf{K}*\beta \subseteq \mathbf{K}+\beta = \mathbf{K}$, then $\mathbf{K}*\beta \cup \mathbf{K} = \mathbf{K}$. By *consistent expansion* $\mathbf{K} \subseteq \mathbf{K}*\beta$. Hence $\mathbf{K} \subseteq \mathbf{K}*\beta$. ■

Proof of property A.8.10 Let $\alpha \notin \mathbf{K}*\alpha$. It follows from *strong stability* that $\alpha \notin \mathbf{K}$. By *vacuity* $\mathbf{K}*\neg\alpha = \mathbf{K}+\neg\alpha$, hence $\neg\alpha \in \mathbf{K}*\neg\alpha$. ■

Proof of property A.8.11 Let $\mathbf{K} \not\subseteq \mathbf{K}*\alpha$. It follows from *vacuity* that $\neg\alpha \in \mathbf{K}$ and from *relative success* that $\alpha \in \mathbf{K}*\alpha$. Hence $\mathbf{K} \cup \mathbf{K}*\alpha \vdash \perp$. ■

Proof of property A.8.12 Let $\alpha \notin \mathbf{K}*\alpha$. Then by *weak success* $\neg\alpha \in \mathbf{K}$. By *relative success* $\mathbf{K}*\alpha = \mathbf{K}$, hence $\neg\alpha \in \mathbf{K}*\alpha$. ■

Proof of property A.8.13 There are two cases:

(a) $\alpha \vee \beta \notin \mathbf{K}*(\alpha \vee \beta)$: It follows from *strict improvement* that $\alpha \notin \mathbf{K}*\alpha$ and $\beta \notin \mathbf{K}*\beta$. Then it follows from *relative success* that $\mathbf{K}*\alpha = \mathbf{K}*\beta = \mathbf{K}*(\alpha \vee \beta) = \mathbf{K}$.

(b) $\alpha \vee \beta \in \mathbf{K}*(\alpha \vee \beta)$: Let $\delta \in \mathbf{K}*\alpha \cap \mathbf{K}*\beta$. It follows from *extensionality* that $\delta \in \mathbf{K}*((\alpha \vee \beta) \wedge \alpha)$. *Superexpansion* yields $\delta \in (\mathbf{K}*(\alpha \vee \beta)) + \alpha$. It follows by *closure* that $\alpha \rightarrow \delta \in \mathbf{K}*(\alpha \vee \beta)$. In the same way we obtain $\beta \rightarrow \delta \in \mathbf{K}*(\alpha \vee \beta)$, then by *closure* $\delta \in \mathbf{K}*(\alpha \vee \beta)$. ■

Proof of property A.8.15

(a) $\alpha \in \mathbf{K}*\alpha$:

(a1) $\alpha \wedge \beta \in \mathbf{K}*(\alpha \wedge \beta)$ and $\alpha \wedge \neg\beta \notin \mathbf{K}*(\alpha \wedge \neg\beta)$: Since *vacuity* holds, it follows from $\alpha \wedge \neg\beta \notin \mathbf{K}*(\alpha \wedge \neg\beta)$ that $\neg\alpha \vee \beta \in \mathbf{K}$.

(a1.1) $\neg\alpha \notin \mathbf{K}$: Then by *closure* $\neg\alpha \vee \neg\beta \notin \mathbf{K}$. By *closure* $\neg(\alpha \wedge \beta) \notin \mathbf{K}$. By *vacuity* $\mathbf{K}*(\alpha \wedge \beta) = \mathbf{K}+(\alpha \wedge \beta) = (\mathbf{K}+\alpha) + \beta = (\mathbf{K}*\alpha) + \beta$.

(a1.2) $\neg\alpha \in \mathbf{K}$: Let $\epsilon \in \mathbf{K}*(\alpha \wedge \beta)$. We need to prove that $\epsilon \in (\mathbf{K}*\alpha) + \beta$. By *closure*, $\neg\alpha \vee \neg\beta \vee \epsilon \in \mathbf{K}*(\alpha \wedge \beta)$. By *relative success* $\mathbf{K}*(\alpha \wedge \neg\beta) = \mathbf{K}$. Since $\neg\alpha \in \mathbf{K}$, $\neg\alpha \vee \neg\beta \vee \epsilon \in \mathbf{K} = \mathbf{K}*(\alpha \wedge \neg\beta)$. Then by *disjunctive overlap*, $\neg\alpha \vee \neg\beta \vee \epsilon \in \mathbf{K}*((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta))$. By *extensionality* $\neg\alpha \vee \neg\beta \vee \epsilon \in \mathbf{K}*\alpha$. Then $\neg\alpha \vee \neg\beta \vee \epsilon \in (\mathbf{K}*\alpha) + \beta$. Hence by *closure* $\epsilon \in (\mathbf{K}*\alpha) + \beta$.

(a2) $\alpha \wedge \beta \notin \mathbf{K}*(\alpha \wedge \beta)$ and $\alpha \wedge \neg\beta \notin \mathbf{K}*(\alpha \wedge \neg\beta)$: It follows from *relative success* that $\mathbf{K}*(\alpha \wedge \beta) = \mathbf{K}*(\alpha \wedge \neg\beta) = \mathbf{K}$. Then by *extensionality* and *disjunctive overlap* $\mathbf{K} \subseteq \mathbf{K}*\alpha$. Hence $\mathbf{K}*(\alpha \wedge \beta) \subseteq \mathbf{K}*\alpha$, hence $\mathbf{K}*(\alpha \wedge \beta) \subseteq (\mathbf{K}*\alpha) + \beta$.

(a3) $\alpha \wedge \neg\beta \in \mathbf{K}*(\alpha \wedge \neg\beta)$: Let $\epsilon \in \mathbf{K}*(\alpha \wedge \beta)$. *Closure* yields $\epsilon \vee \neg\beta \in \mathbf{K}*(\alpha \wedge \beta)$. We can also use *closure* to obtain $\epsilon \vee$

$\neg\beta \in \mathbf{K}{*}(\alpha \wedge \neg\beta)$. Hence by *extensionality* and *disjunctive overlap*, $\epsilon \vee \neg\beta \in \mathbf{K}{*}\alpha$, hence $\epsilon \in (\mathbf{K}{*}\alpha) + \beta$.

(b) $\alpha \notin \mathbf{K}{*}\alpha$: Then *strict improvement* yields $\alpha \wedge \beta \notin \mathbf{K}{*}(\alpha \wedge \beta)$. By *relative success* $\mathbf{K}{*}\alpha = \mathbf{K}{*}(\alpha \wedge \beta) = \mathbf{K}$, hence $\mathbf{K}{*}(\alpha \wedge \beta) \subseteq (\mathbf{K}{*}\alpha) + \beta$. ■

Proof of property A.8.16 Let $\neg\alpha \notin \mathbf{K}{*}(\alpha \wedge \beta)$. There are two cases:

(a) $\alpha \vee \beta \in \mathbf{K}{*}(\alpha \vee \beta)$: It follows from *guarded subexpansion* that $(\mathbf{K}{*}(\alpha \vee \beta)) + \alpha \subseteq \mathbf{K}{*}((\alpha \vee \beta) \wedge \alpha)$. By *extensionality* $\mathbf{K}{*}((\alpha \vee \beta) \wedge \alpha) = \mathbf{K}{*}\alpha$, then $(\mathbf{K}{*}(\alpha \vee \beta)) + \alpha \subseteq \mathbf{K}{*}\alpha$, hence $\mathbf{K}{*}(\alpha \vee \beta) \subseteq \mathbf{K}{*}\alpha$.

(b) $\alpha \vee \beta \notin \mathbf{K}{*}(\alpha \vee \beta)$: *Strict improvement* yields $\alpha \notin \mathbf{K}{*}\alpha$. It follows from *relative success* that $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}$ and $\mathbf{K}{*}\alpha = \mathbf{K}$, hence $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\alpha$. ■

Proof of property A.8.18 Let $\alpha \in \mathbf{K}{*}\alpha$ and $\neg\beta \notin \mathbf{K}{*}\alpha$. By *closure* $\neg(\alpha \vee \beta) \notin \mathbf{K}{*}\alpha$. By *extensionality* $\neg(\alpha \vee \beta) \notin \mathbf{K}{*}((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta))$. Then it follows from *disjunctive inclusion* and *extensionality* that $\mathbf{K}{*}\alpha \subseteq \mathbf{K}{*}(\alpha \wedge \beta)$.

(a) $\alpha \wedge \beta \in \mathbf{K}{*}(\alpha \wedge \beta)$: It follows from from $\mathbf{K}{*}\alpha \subseteq \mathbf{K}{*}(\alpha \wedge \beta)$ and $\beta \in \mathbf{K}{*}(\alpha \wedge \beta)$ that $(\mathbf{K}{*}\alpha) + \beta \subseteq \mathbf{K}{*}(\alpha \wedge \beta)$.

(b) $\alpha \wedge \beta \notin \mathbf{K}{*}(\alpha \wedge \beta)$: We prove that this is not a possible case: *Relative success* yields $\mathbf{K}{*}(\alpha \wedge \beta) = \mathbf{K}$. Since $\mathbf{K}{*}\alpha \subseteq \mathbf{K}{*}(\alpha \wedge \beta)$, then $\mathbf{K}{*}\alpha \subseteq \mathbf{K}$, hence $\alpha \in \mathbf{K}$. Since $\mathbf{K}$ is consistent (due to $\alpha \wedge \beta \notin \mathbf{K}$), then by *vacuity*

$\mathbf{K}*\alpha = \mathbf{K}+\alpha = \mathbf{K}$. Since $\neg(\alpha \wedge \beta) \notin \mathbf{K}*\alpha$, *vacuity* yields $\mathbf{K}*(\alpha \wedge \beta) = \mathbf{K}+(\alpha \wedge \beta)$ that contradicts the hypothesis. ■

Proof of property A.8.19

Let $\alpha \in \mathbf{K}*\alpha$ and $\neg\beta \notin \mathbf{K}*\alpha$. Since $\vdash \alpha \leftrightarrow ((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta))$ and *extensionality* holds, there are three cases, according to disjunctive factoring:

(a) $\mathbf{K}*\alpha = \mathbf{K}*(\alpha \wedge \beta)$:

(a1) $\mathbf{K}*\alpha = \mathbf{K}$. Then, since $\neg\beta \notin \mathbf{K}$, and since $\alpha \in \mathbf{K}*\alpha = \mathbf{K}$, it follows that $\neg\alpha \vee \neg\beta \notin \mathbf{K}$. Then by *vacuity* $\mathbf{K}*\alpha \subseteq \mathbf{K}+(\alpha \wedge \beta) = \mathbf{K}*(\alpha \wedge \beta)$, hence $(\mathbf{K}*\alpha)+\beta \subseteq (\mathbf{K}*(\alpha \wedge \beta))+\beta = \mathbf{K}*(\alpha \wedge \beta)$.

(a2) $\mathbf{K}*\alpha \neq \mathbf{K}$. Then $\mathbf{K}*(\alpha \wedge \beta) \neq \mathbf{K}$, then by *relative success* $\alpha \wedge \beta \in \mathbf{K}*(\alpha \wedge \beta)$. Hence by *closure* $(\mathbf{K}*\alpha) + \beta = (\mathbf{K}*(\alpha \wedge \beta)) + \beta = \mathbf{K}*(\alpha \wedge \beta)$.

(b) $\mathbf{K}*\alpha = \mathbf{K}*(\alpha \wedge \neg\beta)$: Since $\neg\beta \notin \mathbf{K}*\alpha$, by *relative success* $\mathbf{K}*\alpha = \mathbf{K}*(\alpha \wedge \neg\beta) = \mathbf{K}$. Then by *vacuity* $\neg\alpha \vee \beta \in \mathbf{K} = \mathbf{K}*\alpha$. Due to $\alpha \in \mathbf{K}*\alpha$, then $\beta \in \mathbf{K}$, hence $\alpha \wedge \beta \in \mathbf{K}$. It follows that $(\mathbf{K}*\alpha) + \beta = \mathbf{K}*\alpha = \mathbf{K}*(\alpha \wedge \beta)$.

(c) $\mathbf{K}*\alpha = \mathbf{K}*(\alpha \wedge \beta) \cap \mathbf{K}*(\alpha \wedge \neg\beta)$

(c1) $\alpha \wedge \beta \in \mathbf{K}*(\alpha \wedge \beta)$: We then have $\mathbf{K}*\alpha \subseteq \mathbf{K}*(\alpha \wedge \beta)$, hence $(\mathbf{K}*\alpha) + \beta \subseteq (\mathbf{K}*(\alpha \wedge \beta)) + \beta = \mathbf{K}*(\alpha \wedge \beta)$.

(c2) $\alpha \wedge \beta \notin \mathbf{K}*(\alpha \wedge \beta)$ and $\alpha \wedge \neg\beta \notin \mathbf{K}*(\alpha \wedge \neg\beta)$: It follows from *relative success* that $\mathbf{K}*(\alpha \wedge \beta) = \mathbf{K}*(\alpha \wedge \neg\beta) = \mathbf{K}$. By *vacuity* $\neg\alpha \vee \neg\beta \in \mathbf{K}$ and $\neg\alpha \vee \beta \in \mathbf{K}$, hence $\neg\alpha \in \mathbf{K}$. This is absurd, since $\alpha \in \mathbf{K}$ and due to $\alpha \wedge \beta \notin \mathbf{K}$, $\mathbf{K}$ is consistent.

(c3) $\alpha \wedge \beta \notin \mathbf{K}{*}(\alpha \wedge \beta)$ and $\alpha \wedge \neg\beta \in \mathbf{K}{*}(\alpha \wedge \neg\beta)$: It follows from *relative success* that $\mathbf{K}{*}(\alpha \wedge \beta) = \mathbf{K}$. By *vacuity* $\neg\alpha \vee \neg\beta \in \mathbf{K}$, hence $\neg\alpha \vee \neg\beta \in \mathbf{K}{*}(\alpha \wedge \beta)$. It follows from *closure* and $\alpha \wedge \neg\beta \in \mathbf{K}{*}(\alpha \wedge \neg\beta)$ that $\neg\alpha \vee \neg\beta \in \mathbf{K}{*}(\alpha \wedge \neg\beta)$, then $\neg\alpha \vee \neg\beta \in \mathbf{K}{*}\alpha$. This is absurd, since $\alpha \in \mathbf{K}{*}\alpha$ and $\neg\beta \notin \mathbf{K}{*}\alpha$. ■

Proof of property A.8.23 There are three cases:

(a) $\neg\alpha \notin \mathbf{K}{*}(\alpha \vee \beta)$ and $\neg\beta \notin \mathbf{K}{*}(\alpha \vee \beta)$: It follows from *disjunctive overlap* that $\mathbf{K}{*}\alpha \cap \mathbf{K}{*}\beta \subseteq \mathbf{K}{*}(\alpha \vee \beta)$. Since $\neg\alpha \notin \mathbf{K}{*}(\alpha \vee \beta)$, it follows from *disjunctive inclusion* that $\mathbf{K}{*}(\alpha \vee \beta) \subseteq \mathbf{K}{*}\alpha$. In the same way we obtain $\mathbf{K}{*}(\alpha \vee \beta) \subseteq \mathbf{K}{*}\beta$, hence $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\alpha \cap \mathbf{K}{*}\beta$.

(b) $\neg\alpha \in \mathbf{K}{*}(\alpha \vee \beta)$ and $\neg\beta \in \mathbf{K}{*}(\alpha \vee \beta)$: If $\vdash \neg(\alpha \vee \beta)$ then α is equivalent to $(\alpha \vee \beta)$, hence by *extensionality* $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\alpha$. If $\nvdash \neg(\alpha \vee \beta)$, then it follows from *relative success* and *consistency* that $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}$. It follows from *strict improvement* that $\alpha \notin \mathbf{K}{*}\alpha$ and $\beta \notin \mathbf{K}{*}\beta$. By *relative success* we obtain $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\alpha = \mathbf{K}{*}\beta = \mathbf{K}$.

(c) $\neg\alpha \notin \mathbf{K}{*}(\alpha \vee \beta)$ and $\neg\beta \in \mathbf{K}{*}(\alpha \vee \beta)$: By *closure*, $\neg\alpha \wedge \neg\beta \notin \mathbf{K}{*}(\alpha \vee \beta)$. By *relative success*, either $\alpha \vee \beta \in \mathbf{K}{*}(\alpha \vee \beta)$ or $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}$. Due to *vacuity*, in both cases $\alpha \vee \beta \in \mathbf{K}{*}(\alpha \vee \beta)$. Then by *closure* $\alpha \in \mathbf{K}{*}(\alpha \vee \beta)$. By *extensionality*, $\mathbf{K}{*}\alpha = \mathbf{K}{*}((\alpha \vee \beta) \wedge (\alpha \vee \neg\beta))$. By *superexpansion* (which follows from property A.8.15), $\mathbf{K}{*}((\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)) \subseteq (\mathbf{K}{*}(\alpha \vee \beta)) + (\alpha \vee \neg\beta)$. Since $\neg\alpha \notin \mathbf{K}{*}(\alpha \vee \beta)$ then by *dis-*

junctive inclusion $\mathbf{K}*(\alpha \vee \beta) \subseteq \mathbf{K}*\alpha$. Hence $\mathbf{K}*(\alpha \vee \beta) = \mathbf{K}*\alpha$. ■

A.9.1 *Closure* and *success* imply *strict improvement.*

Proof: Let $\alpha \notin \mathbf{K}-\alpha$ and $\vdash \beta \rightarrow \alpha$. By *closure* $\nvdash \alpha$, then $\nvdash \beta$, then by success $\beta \notin \mathbf{K}-\beta$. ■

A.9.2 *Closure* and *recovery* imply *negation retainment.*

Proof: Let $\alpha \in \mathbf{K}$ and suppose that $\alpha \notin \mathbf{K}-\neg\alpha$. Then by *recovery* $\neg\alpha \rightarrow \alpha \in \mathbf{K}-\neg\alpha$. Then by *closure* $\alpha \in \mathbf{K}-\neg\alpha$. Absurd, hence $\alpha \in \mathbf{K}-\neg\alpha$.

A.9.3 *Inclusion, vacuity* and *recovery* imply *full vacuity.*

Proof:

$\Rightarrow$: Let $\mathbf{K}-\alpha = \mathbf{K}$ and $\alpha \in \mathbf{K}$. Then by success $\vdash \alpha$.

$\Leftarrow$ We have two subcases:

(a) Let $\alpha \notin \mathbf{K}$. Then by *vacuity* $\mathbf{K}-\alpha = \mathbf{K}$.

(b) Let $\vdash \alpha$. By *inclusion* $\mathbf{K}-\alpha \subseteq \mathbf{K}$ and it follows by *recovery* that $\mathbf{K} \subseteq (\mathbf{K}-\alpha) + \alpha = \mathbf{K}-\alpha$. Hence $\mathbf{K}-\alpha = \mathbf{K}$. ■

A.9.4 *Inclusion, vacuity* and *recovery* imply *conjunctive constancy.*

Proof: Let $\mathbf{K}-\alpha = \mathbf{K}-\beta = \mathbf{K}$. Suppose by reductio ad absurdum that $\mathbf{K}-(\alpha \wedge \beta) \neq \mathbf{K}$. Then by property A.9.3 $\alpha \wedge \beta \in \mathbf{K}$, $\nvdash \alpha \wedge \beta$, $\alpha \in \mathbf{K}$ and $\beta \in \mathbf{K}$. Then by property A.9.3, $\vdash \alpha$ and $\vdash \beta$, hence $\vdash \alpha \wedge \beta$. Absurd. ■

A.9.5 *Success* and *vacuity* imply *consistent expansion.*

Proof: Let $\mathbf{K} \not\subseteq \mathbf{K}{*}\alpha$. It follows from *vacuity* that $\mathbf{K} \vdash \neg\alpha$ and from *success* that $\mathbf{K}{*}\alpha \vdash \alpha$. Hence, $\mathbf{K} \cup (\mathbf{K}{*}\alpha) \vdash \bot$. ■

A.9.6 *Success, vacuity* and *consistency* imply *disjunctive constancy.*

Proof: Let $\mathbf{K} = \mathbf{K}{*}\alpha = \mathbf{K}{*}\beta$. It follows from *success* that $\alpha \notin \mathbf{K}$. Since $\mathbf{K}$ is logically closed, we then have $\alpha \vee \beta \in \mathbf{K}$. If $\mathbf{K}$ is inconsistent, then it follows from *consistency* that both α and β are inconsistent, hence so is $\alpha \vee \beta$, then by *success* $\mathbf{K}{*}(\alpha \vee \beta)$ is inconsistent, hence $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}$. If $\mathbf{K}$ is consistent, then $\neg(\alpha \vee \beta) \notin \mathbf{K}$, hence by *vacuity* $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{+}(\alpha \vee \beta) = \mathbf{K}$. ■

A.9.7 *Closure, success, inclusion, vacuity, consistency, extensionality, subexpansion* and *superexpansion* imply *disjunctive priority.*

Proof:

(a) $\vdash \neg\beta$: then $\vdash \alpha \leftrightarrow \alpha \vee \beta$ and the rest is trivial by *extensionality.*

(b) $\nvdash \neg\beta$: Suppose that $\mathbf{K} \nvdash \neg\alpha$ and $\mathbf{K} \vdash \neg\beta$. By *closure* $\mathbf{K} \nvdash (\neg\alpha \wedge \neg\beta)$; then by *inclusion* and *vacuity* $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{+}(\alpha \vee \beta)$, then $\mathbf{K}{*}(\alpha \vee \beta) \vdash \neg\beta$, by *consistency* $\mathbf{K}{*}\beta \nvdash \neg\beta$, then $\mathbf{K}{*}(\alpha \vee \beta) \neq \mathbf{K}{*}\beta$ and $\mathbf{K}{*}(\alpha \vee \beta) \neq \mathbf{K}{*}\beta \cap \mathbf{K}{*}\alpha$; hence by *disjunctive factoring* $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\alpha$. ■

A.9.8 *Closure, success, consistency, extensionality, subexpansion* and *superexpansion* imply *disjunctive reduction.*

Proof: Let $\mathbf{K}{*}(\alpha \vee \beta) \vdash \neg\alpha$. The proof proceeds by cases:

(a) $\vdash \neg\alpha \wedge \neg\beta$: then by *success* $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\beta = \mathbf{K}_{\perp}$.

(b) $\nvdash \neg\alpha \wedge \neg\beta$:

(b.1) $\vdash \neg\alpha$: by *consistency* $\mathbf{K}{*}(\alpha \vee \beta) \neq \mathbf{K}_{\perp}$, by *success* $\mathbf{K}{*}\alpha = \mathbf{K}_{\perp}$, then by *disjunctive factoring* $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\beta$ or $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\alpha \cap \mathbf{K}{*}\beta = \mathbf{K}_{\perp} \cap \mathbf{K}{*}\beta = \mathbf{K}{*}\beta$.

(b.2) $\nvdash \neg\alpha$: By *success* $\mathbf{K}{*}\alpha \vdash \alpha$; by *consistency* $\mathbf{K}{*}\alpha \nvdash \neg\alpha$; then $\mathbf{K}{*}(\alpha \vee \beta) \neq \mathbf{K}{*}\alpha$ and $\mathbf{K}{*}(\alpha \vee \beta) \neq \mathbf{K}{*}\alpha \cap \mathbf{K}{*}\beta$, since by hypothesis $\mathbf{K}{*}(\alpha \vee \beta) \vdash \neg\alpha$; hence by *disjunctive factoring* $\mathbf{K}{*}(\alpha \vee \beta) = \mathbf{K}{*}\beta$. ■

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