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On the thinness and proper thinness of a graph

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SOBRE LA THINNESS Y THINNESS PROPIA DE UN GRAFO

Los grafos con *thinness* acotada fueron definidos en [48] como una generalización de los grafos de intervalos, con el propósito de desarrollar una heurística para el problema de asignación de frecuencias en redes GSM. En esta tesis introducimos el concepto de *thinness propia*, tal que los grafos con thinness propia acotada generalizan a los grafos de intervalos propios. Estudiamos la complejidad computacional de problemas relacionados al reconocimiento de grafos con thinness y thinness propia acotada por k, demostrando que algunos son **NP**-completos y otros polinomiales; aunque los problemas de reconocimiento siguen abiertos incluso para $k = 2$. El caso $k = 1$ corresponde a los grafos de intervalos y de intervalos propios, respectivamente, y por lo tanto se reconocen en tiempo polinomial.

Describimos el comportamiento de la thinness y thinness propia bajo las operaciones de grafos unión, suma y producto cartesiano. También estudiamos la relación entre ambos parámetros con otros de la literatura como cutwidth, linear MIM-width, y anidamiento de intervalos, que complementan a resultados previos sobre boxicidad y pathwidth. Finalmente describimos una amplia familia de problemas que pueden resolverse con técnicas de programación dinámica en tiempo polinomial en grafos con thinness acotada, dada cierta representación, generalizando a la familia LIST MATRIX PARTITION definida en [28], y luego para grafos con thinness propia acotada la extendemos para incluir los problemas de dominación definidos en [2] y sus versiones pesadas.

Palabras clave: Grafo, Intervalos, Thinness, Complejidad, Algoritmo.

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CONTENTS

1. INTRODUCTION

1.1 Historical context

It is widely believed that **NP**-complete problems cannot be solved in polynomial time, therefore much research has been done on the complexity of subproblems of **NP**-complete problems.

The study of restricted graph classes is a classical field in graph theory. Some of these classes are interesting in the context of graph algorithms because many classical problems that are **NP**-complete for general graphs are tractable for them. In particular, interval graphs were introduced by Hajos in 1957 [35], and are extensively used to model temporal relationships and linear restrictions. They have applications in diverse areas like molecular biology [4], ecology [16], archaeology [42], psychology [56, 17], information retrieval [31], operations research [53], artificial intelligence [1], circuit design [69], medical diagnosis [51], and bioinformatics [70].

A somewhat newer field is the study of structural graph *width* parameters. It started in the 1980s with the introduction of algorithms based on the treewidth [7], which generalized algorithms for trees and series-parallel graphs. Soon after, the clique-width was defined, which generalized algorithms for cographs and distance-hereditary graphs. Such abstractions made it possible to rapidly find polynomial time algorithms for a number of classical problems in quite large graph classes, by explaining observed similarities in previous algorithms using the same underlying *divide and conquer* techniques. A rather extreme case is Courcelle's theorem [21], which states that any graph property definable in monadic second-order logic can be decided in linear time for graphs of bounded treewidth.

Another relevant field emerged in the 1990s, called Parameterized Complexity, in a series of articles by Downey and Fellows [25, 26]. The theory is an attempt to get a better theoretical understanding of the source of the computational hardness in a given problem, in order to deal with it in practice when the "source of hardness" parameter is small. The basic complexity class is **XP**, for *slicewise polynomial time*, which contains the parameterized problems for which the parameterization is *meaningful*: a problem with parameter k belongs to **XP** if it can be solved in time $n^{O(f(k))}$ where f is a computable function depending only on k. But the most interesting class is **FPT**, for *fixed-parameter tractable*, which is the subset of **XP** of problems solvable in time $f(k) \cdot n^{O(1)}$.

In the following decades, **FPT** algorithms were found for many classical problems with respect to width parameters, and for them the quest shifted to finding algorithms with better f or smaller exponent of n [22]. But for some problems, like LIST COLORING parameterized by treewidth [29], there are hardness results which exclude them from being in **FPT** under reasonable assumptions. Still some problems, of course, remain **NP**-complete when restricted to graphs with such parameters bounded.

Another field related to this thesis is the study of the behavior of graphs parameters under graph operations. A famous example of this was the Hedetniemi conjecture [36], which stated that the chromatic number of the tensor product of two graphs equals the minimum of their individual chromatic numbers. It remained open for more than fifty years and was shown to hold for various classes, but it was finally disproven recently [63].

Graphs with bounded thinness were defined in [48] as a generalization of interval graphs. A graph $G = (V, E)$ is *k-thin* if there exist an ordering $v_1, \dots, v_n$ of V and a partition of V into k classes $(V^1, \dots, V^k)$ such that, for each triple (r, s, t) with $r < s < t$, if v_r, v_s belong to the same class and $v_t v_r \in E$, then $v_t v_s \in E$. The minimum k such that G is k-thin is called the *thinness* of G. The thinness is unbounded on the class of all graphs, and interval graphs are exactly the 1-thin graphs. When the k-thin representation of a graph is given, for a constant value k, some **NP**-complete problems as MAXIMUM WEIGHTED INDEPENDENT SET and BOUNDED COLORING with fixed number of colors can be solved in polynomial time [48, 11]. These algorithms were respectively applied for improving heuristics of two real-world problems: the FREQUENCY ASSIGNMENT PROBLEM in GSM networks [48], and the DOUBLE TRAVELING SALESMAN PROBLEM WITH MULTIPLE STACKS [11].

1.2 Current work

In this thesis we introduce the concept of *proper thinness*, such that graphs with bounded proper thinness generalize proper interval graphs: graphs that are proper 1-thin are exactly proper interval graphs (see Section 2.1 for a definition). We study the complexity of problems related to the computation of these parameters; describe the behavior of the thinness and proper thinness under three graph operations; and relate thinness and proper thinness to other graph invariants.

Finally, we describe a wide family of problems that can be solved by dynamic programming techniques in polynomial time for graphs with bounded thinness, when the k-thin representation of the graph is given, generalizing for example list matrix partition problems with bounded size matrix [28], and enlarge this family of problems for graphs with bounded proper thinness, including domination-type problems in the literature (e.g. classified in [2]) and their weighted versions, such as MINIMUM WEIGHTED INDEPENDENT DOMINATING SET, MINIMUM WEIGHTED TOTAL DOMINATING SET, MINIMUM PERFECT DOMINATING SET and MINIMUM WEIGHTED EFFICIENT DOMINATING SET.

The novel results presented in this work were previously published in [10]. Except when explicitly attributed, all results can be assumed to be original, excluding those clearly recognizable to be part of the mathematical folklore. The organization of the thesis is the following:

In Section 2.1 we state the main definitions and present some basic results on thinness. In Section 2.2, we study some problems related to the recognition of k-thin graphs and proper k-thin graphs. We analyze the computational complexity of finding a suitable vertex partition when a vertex ordering is given, and, conversely, finding a vertex ordering when a vertex partition is given. In Section 2.3 we describe the behavior of the thinness and proper thinness under three graph operations: union, join, and Cartesian product. The first two results allow us to fully characterize k-thin graphs by forbidden induced subgraphs within the class of cographs. The third result is used to show the polynomiality of the t-rainbow domination problem for fixed t on graphs with bounded thinness.

In Section 3.1 we survey the relation of thinness and other width parameters in graphs. In Section 3.2 we relate the proper thinness of interval graphs to other interval graph invariants, as interval count and chains of nested intervals.

In Section 4.1 we describe a wide family of problems that can be solved in polynomial time for graphs with bounded thinness, when the representation is given. In Section 4.3 we extend that family to include dominating-like problems that can be solved in polynomial time for graphs with bounded proper thinness.

1.3 Notation and terminology

All graphs in this work are finite, undirected, and have no loops or multiple edges. For all graph-theoretic notions and notation not defined here, we refer to West [68]. Let G be a graph. Denote by $V(G)$ its vertex set, by $E(G)$ its edge set, by $\overline{G}$ its complement, by $N(v)$ the neighborhood of a vertex v in G, by $N[v]$ the closed neighborhood $N(v) \cup \{v\}$, and by $\overline{N}(v)$ the non-neighbors of v. If $X \subseteq V(G)$, denote by $N(X)$ the set of vertices not in X having at least one neighbor in X.

Denote by $G[W]$ the subgraph of G induced by $W \subseteq V(G)$, and by $G-W$ or $G \setminus W$ the graph $G[V(G) \setminus W]$. A subgraph H (not necessarily induced) of G is a *spanning subgraph* if $V(H) = V(G)$.

Denote the size of a set S by $|S|$. A *clique* (resp. *independent set*) is a set of pairwise adjacent (resp. nonadjacent) vertices. We use *maximum* to mean maximum-sized, whereas *maximal* means inclusion-wise maximal. The use of *minimum* and *minimal* is analogous.

Denote by K_n the graph induced by a clique of size n. A *claw* is the graph isomorphic to $K_{1,3}$. Let H be a graph and t a natural number. The disjoint union of t copies of the graph H is denoted by tH.

For a positive integer r, the $(r \times r)$-*grid* is the graph whose vertex set is $\{(i,j) : 1 \leq i,j \leq r\}$ and whose edge set is $\{(i,j)(k,l) : |i-k| + |j-l| = 1, \text{ where } 1 \leq i,j,k,l \leq r\}$.

A *dominating set* in a graph is a set of vertices such that each vertex outside the set has at least one neighbor in the set.

A *coloring* of a graph is an assignment of colors to its vertices such that any two adjacent vertices are assigned different colors. The smallest number t such that G admits a coloring with t colors (a t-*coloring*) is called the *chromatic number* of G and is denoted by $\chi(G)$. A coloring defines a partition of the vertices of the graph into independent sets, called *color classes*. List variations of the vertex coloring problem can be found in the literature. For a survey on that kind of related problems, see [65]. In the *list-coloring problem*, every vertex v comes equipped with a list of permitted colors $L(v)$ for it.

For a symmetric matrix M over $0, 1, *$, the M-*partition problem* seeks a partition of the vertices of the input graph into independent sets, cliques, or arbitrary sets, with certain pairs of sets being required to have no edges, or to have all edges joining them, as encoded in the matrix M: $M_{ii} = 1$ means the i-th set is a clique, while $M_{ii} = 0$ means the i-th set is an independent set; for $i \neq j$, $M_{ij} = 1$ means every vertex of the i-th set is adjacent to every vertex of the j-th set, while $M_{ij} = 0$ means there are no edges from the i-th set to the j-th set. Moreover, the vertices of the input graph can be equipped with lists, restricting the parts to which a vertex can be placed. In that case the problem is know as a *list matrix partition problem.* Such (list) matrix partition problems generalize (list) colorings and (list) homomorphisms [28].

When discussing about algorithms and data structures, we denote by n the number of vertices of the input graph G.

Given a graph G, a weight function w on $V(G)$, and a subset $S \subseteq V(G)$, the weight of S, denoted by $w(S)$ is defined as $\sum_{v \in S} w(v)$.

A class of graphs is *hereditary* when if a graph G is in the class, then every induced subgraph of G is in the same class.

A graph is a *cograph* if it contains no induced path of length four.

A graph $G(V, E)$ is a *comparability graph* if there exists an ordering $v_1, \ldots, v_n$ of V such that, for each triple (r, s, t) with $r < s < t$, if $v_r v_s$ and $v_s v_t$ are edges of G, then so is $v_r v_t$. Such an ordering is a *comparability ordering.* A graph is a *co-comparability graph* if its complement is a comparability graph.

2. THINNESS

2.1 Definitions and basic results

A graph $G = (V, E)$ is an *interval graph* if each vertex $v \in V$ can be associated to a closed interval $I_v = [l_v, r_v]$ of the real line, such that two distinct vertices $u, v \in V$ are adjacent if and only if $I_u \cap I_v \neq \emptyset$. The family $\{I_v\}_{v \in V}$ is an *interval representation* of G. An undirected graph G is a *proper interval graph* if there is an interval representation of G in which no interval properly contains another. In the same way, an undirected graph G is a *unit interval graph* if there is an interval representation of G in which all the intervals have the same length.

In 1969, Roberts [57] proved that the classes of proper interval graphs, unit interval graphs, and interval graphs with no claw as induced subgraph coincide.

The right-end ordering of the vertices of an interval graph satisfies the following property: for each triple (r, s, t) with $r < s < t$, if $v_t v_r \in E$, then $v_t v_s \in E$. In other words, the neighbors of vertex t with index less than t are $t-1, t-2, \ldots, t-d$ for some $d \geq 0$. Moreover, a graph G is an interval graph if and only if there exists an ordering of its vertices satisfying the property above [55, 52].

Proof. Suppose an interval graph has intervals $I_r < I_s < I_t$ ordered by the right-end. If $I_t \cap I_r \neq \emptyset$, then $I_t \cap I_s \neq \emptyset$ too, because $l(I_t) \leq r(I_r) \leq r(I_s)$, which implies the property.

If ordering $v_1 < \ldots < v_n$ satisfies the property, let $m(i)$ be the minimum j such that $v_j \in N[v_i]$. Then let $I_i = [m(i), i]$. □

Let us repeat and extend the definition of k-thinness given in the introduction. A graph $G = (V, E)$ is *k-thin* if there exist an ordering $v_1, \ldots, v_n$ of V and a partition of V into k classes such that, for each triple (r, s, t) with $r < s < t$, if v_r, v_s belong to the same class and $v_t v_r \in E$, then $v_t v_s \in E$. An ordering and a partition satisfying those properties are said to be *consistent*. The minimum k such that G is k-thin is called the *thinness* of G and denoted by $\text{thin}(G)$.

The *thinness* of a graph was introduced by Mannino, Oriolo, Ricci, and Chandran in 2007 [48]. Graphs with *bounded thinness* (thinness bounded by a constant value) are a generalization of interval graphs, which are exactly the graphs of thinness 1, and capture some of their algorithmic properties.

Let $\overline{tK_2}$ be the complement of a matching of size t.

Theorem 1: *[48]* For every $t \geq 1$, $\text{thin}(\overline{tK_2}) = t$.

Proof. Let $V = \{x_1, \ldots, x_t, y_1, \ldots, y_t\}$, where $(x_i, y_i) \notin E$. Define the partition $V^i = \{x_i, y_i\}$, and observe that any ordering is consistent with it. So the graph is t-thin.

Now suppose that there exists an ordering and a consistent partition in $t-1$ classes. Denote by $f(V^i)$ the first element of V^i in the ordering. Clearly there exists at least one pair $\{x_j, y_j\}$ such that $\cup_i f(V^i) \cap \{x_j, y_j\} = \emptyset$. Assume w.l.o.g. that such pair is (x_1, y_1) and $x_1 < y_1$.

Let V^j be the class of x_1. Then y_1 is adjacent to $f(V^j)$ but not to x_1. But $f(V^j) < x_i < y_i$, which is absurd. □

The right-end ordering of the vertices of a proper interval graph satisfies the following property: for each triple (r, s, t) with $r < s < t$, if $v_t v_r \in E$, then $v_t v_s \in E$ and $v_r v_s \in E$. In other words, the neighbors of vertex t with index less than t are $t-1, t-2, \ldots, t-d$, and those with index greater than t are $t+1, t+2, \ldots, t+d'$. Moreover, G is a proper interval graph if and only if there exists an ordering of its vertices satisfying the property above [24, 47].

We define the concept of *proper thinness* of graphs as follows.

A graph $G = (V, E)$ is *proper k-thin* if there exist an ordering $v_1, \ldots, v_n$ of V and a partition of V into k classes $(V^1, \ldots, V^k)$ such that, for each triple (r, s, t) with $r < s < t$, if v_r, v_s belong to the same class and $v_t v_r \in E$, then $v_t v_s \in E$ and if v_s, v_t belong to the same class and $v_r v_t \in E$, then $v_r v_s \in E$. Equivalently, G is proper k-thin if both $v_1, \ldots, v_n$ and $v_n, \ldots, v_1$ are consistent with the partition. In this case, the partition and the ordering $v_1, \ldots, v_n$ are said to be *strongly consistent*, and the minimum k such that G is proper k-thin is called the *proper thinness* of G and denoted by $\mathrm{pthin}(G)$.

Since k-thin graphs are defined as a generalization of interval graphs, proper k-thin graphs arise naturally as a generalization of proper interval graphs. And from the definition it can be seen that a graph is proper 1-thin if and only if it is a proper interval graph. Moreover, the proper thinness of the class of interval graphs is unbounded (See Proposition 17).

2.2 Algorithmic aspects

We will deal in this section with some questions related to the recognition problem of (proper) k-thin graphs. The recognition problem itself is open so far for both classes, but we will show that, given a vertex ordering of a graph, we can find in polynomial time a partition into a minimum number of classes which is (strongly) consistent with the ordering. On the other hand, we will show that given a graph and a vertex partition, it is **NP**-complete to decide if there exists an ordering of the vertices of the graph which is (strongly) consistent with the partition.

Theorem 2: Given a graph G and an ordering $<$ of its vertices, one can find in polynomial time graphs $G_<$ and $\tilde{G}_<$ with the following properties:

(1) $V(G_<) = V(\tilde{G}_<) = V(G)$;

(2) the chromatic number of $G_<$ (resp. $\tilde{G}_<$) is equal to the minimum integer k such that there is a partition of $V(G)$ into k sets that is consistent (resp. strongly consistent) with the order $<$, and the color classes of a valid coloring of $G_<$ (resp. $\tilde{G}_<$) form a partition consistent (resp. strongly consistent) with $<$;

(3) $G_<$ and $\tilde{G}_<$ are co-comparability graphs.

In particular, the minimum integer k as in (2) and a partition into k vertex sets can be computed in polynomial time. Moreover, if G is a co-comparability graph and $<$ a comparability ordering of $\overline{G}$, then $G_<$ and $\tilde{G}_<$ are spanning subgraphs of G.

Proof. Let G be a graph and $<$ an ordering of its vertices. We will build a graph $G_<$ such that $V(G_<) = V(G)$, and $v < w$ are adjacent in $G_<$ if and only if they cannot belong to the same class of a partition which is consistent with $<$. By definition of consistency, this happens if and only if there is a vertex z in G such that $v < w < z$, z is adjacent to v and nonadjacent to w. So define $E(G_<)$ such that for $v < w$, $vw \in E(G_<)$ if and only if there is a vertex z in G such that $v < w < z$, $zv \in E(G)$ and $zw \notin E(G)$.

We build $\tilde{G}_<$ in a similar way. In this case, for $v < w$, $vw \in E(\tilde{G}_<)$ if and only if either there is a vertex z in G such that $v < w < z$, $zv \in E(G)$ and $zw \notin E(G)$ or there is a vertex x in G such that $x < v < w$, $xw \in E(G)$ and $xv \notin E(G)$.

Let us see that $<$ is a comparability ordering both for $\overline{G_<}$ and $\overline{\tilde{G}_<}$. Suppose on the contrary that there is a triple $r < s < t$ in $V(G)$ such that rs, st are edges of $\overline{G_<}$ (resp. $\overline{\tilde{G}_<}$) and rt is not an edge of $\overline{G_<}$ (resp. $\overline{\tilde{G}_<}$). By definition of $G_<$ (resp. $\tilde{G}_<$), there is a vertex z such that $r < s < t < z$, $zr \in E(G)$ and $zt \notin E(G)$ (resp. either there is a vertex z such that $r < s < t < z$, $zr \in E(G)$ and $zt \notin E(G)$, or there is a vertex x in G such that $x < r < s < t$, $xt \in E(G)$ and $xr \notin E(G)$). If $zs \notin E(G)$, then rs is an edge of $G_<$ (resp. $\tilde{G}_<$), a contradiction. If $zs \in E(G)$, then st is an edge of $G_<$ (resp. $\tilde{G}_<$), a contradiction as well. The case of x for $\tilde{G}_<$ is symmetric, if $xs \notin E(G)$, then st is an edge of $\tilde{G}_<$, a contradiction. If $xs \in E(G)$, then rs is an edge of $\tilde{G}_<$, a contradiction as well. So $G_<$ and $\tilde{G}_<$ are co-comparability graphs, being $<$ a comparability ordering for $\overline{G_<}$ and $\overline{\tilde{G}_<}$, respectively.

As we have defined $G_<$ (resp. $\tilde{G}_<$) such that $V(G_<) = V(\tilde{G}_<) = V(G)$, and $v < w$ are adjacent in $G_<$ (resp. $\tilde{G}_<$) if and only if they cannot belong to the same class of a partition which is consistent (resp. strongly consistent) with $<$, it follows that there is a one-to-one correspondence between partitions of $V(G)$ consistent (resp. strongly consistent) with $<$ and colorings of $G_<$ (resp. $\tilde{G}_<$). In particular, the minimum k such that there is a partition of $V(G)$ into k sets that is consistent (resp. strongly consistent) with $<$ is the chromatic number of $G_<$ (resp. $\tilde{G}_<$). An optimum coloring of $G_<$ (resp. $\tilde{G}_<$) can be computed in polynomial time [32].

To complete the proof of the theorem, suppose now that G is a co-comparability graph and $<$ is a comparability ordering for $\overline{G}$. Let $v < w$ adjacent in $G_<$ (resp. $\tilde{G}_<$). By definition, there is a vertex z in G such that $v < w < z$, $vz \in E(G)$ and $wz \notin E(G)$ (resp. either there is a vertex z in G such that $v < w < z$, $vz \in E(G)$ and $wz \notin E(G)$, or there is a vertex x in G such that $x < v < w$, $xw \in E(G)$ and $xv \notin E(G)$). If $vw \notin E(G)$, being $\overline{G}$ a comparability graph, $vz \notin E(G)$, a contradiction. So $vw \in E(G)$. This proves that $G_<$ is a spanning subgraph of G. The case of x for $\tilde{G}_<$ is symmetric, if $vw \notin E(G)$, being $\overline{G}$ a comparability graph, $xw \notin E(G)$, a contradiction. So in any case $vw \in E(G)$. This proves that $\tilde{G}_<$ is a spanning subgraph of G as well. □

A direct consequence of this result is the following, that was already proved in [11] for the case of thinness.

Corollary 3: If G is a co-comparability graph, $\text{thin}(G) \leq \text{pthin}(G) \leq \chi(G)$. Moreover, any vertex partition given by a coloring of G and any comparability ordering for its complement are strongly consistent.

As already observed in [11], the bound $\text{thin}(G) \leq \text{pthin}(G) \leq \chi(G)$ for co-comparability graphs can be arbitrarily bad: for example, if G is a clique of size n, then $\text{thin}(G) = \text{pthin}(G) = 1$ and $\chi(G) = n$. However, it holds with equality for graphs $\overline{tK_2}$, because $\text{thin}(\overline{tK_2}) = \text{pthin}(\overline{tK_2}) = \chi(\overline{tK_2}) = t$ (Theorem 1 and Corollary 3).

We implemented the algorithms of Theorem 2 in order to classify all graphs of up to 8 vertices with respect to their thinness and proper thinness. The code and classification is available in [23].

In contrast with Theorem 2, if a partition is given, it is **NP**-complete to decide the existence of a (strongly) consistent ordering.

(Strongly) Consistent ordering with a given partition
Instance: A graph $G = (V, E)$ and a partition of V into non-empty subsets.
Question: Does there exist a total order $<$ of V (strongly) consistent with the given partition?

The proof is based on a reduction from the following problem, which is known to be **NP**-complete [34].

Non-Betweenness
Instance: A finite set A and a collection S of ordered triples of distinct elements of A.
Question: Does there exist a total order $<$ of A such that for each $(x, y, z) \in S$, it is never the case that $x < y < z$ or $z < y < x$ (i.e. y is *not between* x and z)?

We start with an easy lemma.

Lemma 4: Let G be a graph, $<$ an ordering of $V(G)$ and $V_1, \ldots, V_k$ a partition of $V(G)$ that is consistent with $<$. Let $\{x_i, y_i\} \subseteq V_i$, for $i = 1, 2$, such that x_1x_2 and y_1y_2 are the only edges between $\{x_1, y_1\}$ and $\{x_2, y_2\}$. Then $x_1 < y_1$ if and only if $x_2 < y_2$.

Proof. By symmetry, let us assume that y_1 is the biggest vertex according to $<$. Again by symmetry, to prove the lemma it is enough to prove that $x_2 < y_2$. By definition of consistency, since x_2 and y_2 are in the same class and y_1 is adjacent to y_2 but not to x_2, it is not possible that $y_2 < x_2 < y_1$. □

Theorem 5: The problem (Strongly) Consistent ordering with a given partition is **NP**-complete.

Proof. First note that (Strongly) Consistent ordering with a given partition is in **NP**, by using the total order of V as the certificate.

Now let us prove its **NP**-hardness. Given an instance (A, S) of Non-Betweenness, build a graph $G = (V, E)$ and a partition $V_0, V_1, \ldots V_{|S|}$ of V as follows.

Fix an ordering of the triples in S. Vertices of V_0 are in one-to-one correspondence with elements of A. For $i = 1, \ldots, |S|$, V_i has 3 vertices, and they are in a one-to-one correspondence with the elements of the i-th triple in S. Let us call a^i the element of V_i that corresponds to $a \in A$, for $i = 0, \ldots, |S|$.

Define the edges of G as follows: for each triple $(x, y, z) \in S$, let V_i be its corresponding set. The only edge in the subgraph induced by $\{x^i, y^i, z^i\}$ is x^iz^i. The remaining edges of G are all the possible edges between vertices associated to the same $a \in A$.

Suppose first there is an ordering $<$ consistent with the partition $\{V_0, \ldots, V_{|S|}\}$. By Lemma 4, for each $1 \leq i \leq |S|$, the relative order of the vertices x^i, y^i, z^i is the same as the relative order of the vertices x^0, y^0, z^0. By definition of consistency and since the only edge in the subgraph induced by $\{x^i, y^i, z^i\}$ is x^iz^i, y^i is not between x^i and z^i in that order. So the order of the vertices in V_0 gives a positive answer to the instance (A, S) of Non-Betweenness.

Suppose now that there is a valid order $<$ for the instance (A, S) of Non-Betweenness. We can extend $<$ to $V(G)$ by making consecutive all the copies in $V(G)$ of an element of A. Now, let $p < q < r$ be three vertices of G such that p, q belong to the same class V_i and $rp \in E(G)$. Since V_0 is an independent set and the triples in S satisfy the non-betweenness condition, r is not in V_i. So r and p correspond to the same element a of A, and since there is at most one copy of an element of A in each V_i, q does not correspond to a copy of a. But this contradicts the fact that all the vertices of G that correspond to a same element of A are consecutive. So the situation cannot arise, and the extended order is consistent with the partition. The case in which q, r belong to the same class V_i is identical, and indeed the extended order is strongly consistent with the partition. □

The computational complexity of the decision of existence of a (strongly) consistent ordering when the number of sets in the partition is fixed is still open. So is the computational complexity of deciding if a graph is (proper) k-thin, even for fixed $k \geq 2$. In the case of proper thinness, the problem is open even within the class of interval graphs.

2.3 Thinness and graph operations

In this section we analyze the behavior of the thinness and proper thinness under different graph operations, namely union, join, and Cartesian product. The first two results allow us to fully characterize k-thin graphs by forbidden induced subgraphs within the class of cographs. The third result is used to solve in polynomial time the t-rainbow domination problem for fixed t on graphs with bounded thinness.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and the *join* of G_1 and G_2 is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$ (i.e., $\overline{G_1 \vee G_2} = \overline{G_1} \cup \overline{G_2}$).

Theorem 6: Let G_1 and G_2 be graphs. Then $\text{thin}(G_1 \cup G_2) = \max\{\text{thin}(G_1), \text{thin}(G_2)\}$ and $\text{pthin}(G_1 \cup G_2) = \max\{\text{pthin}(G_1), \text{pthin}(G_2)\}$.

Proof. Since both G_1 and G_2 are induced subgraphs of $G_1 \cup G_2$, then $\text{thin}(G_1 \cup G_2) \geq \max\{\text{thin}(G_1), \text{thin}(G_2)\}$ and the same holds for the proper thinness.

Let G_1 and G_2 be two graphs with thinness (resp. proper thinness) t_1 and t_2, respectively. Let $v_1, \ldots, v_{n_1}$ and $(V_1^1, \ldots, V_1^{t_1})$ be an ordering and a partition of $V(G_1)$ which are consistent (resp. strongly consistent). Let $w_1, \ldots, w_{n_2}$ and $(V_2^1, \ldots, V_2^{t_2})$ be an ordering and a partition of $V(G_2)$ which are consistent (resp. strongly consistent). Suppose without loss of generality that $t_1 \leq t_2$. For $G = G_1 \cup G_2$, define a partition $V^1, \ldots, V^{t_2}$ such that $V^i = V_1^i \cup V_2^i$ for $i = 1, \ldots, t_1$ and $V^i = V_2^i$ for $i = t_1 + 1, \ldots, t_2$, and define $v_1, \ldots, v_{n_1}, w_1, \ldots, w_{n_2}$ as an ordering of the vertices. By definition of union of graphs, if three ordered vertices according to the order defined in $V(G_1 \cup G_2)$ are such that the first and the third are adjacent, either the three vertices belong to $V(G_1)$ or the three vertices belong to $V(G_2)$. Since the order and the partition restricted to each of G_1 and G_2 are the original ones, the properties required for consistency (resp. strong consistency) are satisfied. □

Theorem 7: Let G_1 and G_2 be graphs. Then $\mathrm{thin}(G_1 \vee G_2) \leq \mathrm{thin}(G_1) + \mathrm{thin}(G_2)$ and $\mathrm{pthin}(G_1 \vee G_2) \leq \mathrm{pthin}(G_1) + \mathrm{pthin}(G_2)$. Moreover, if G_2 is complete, then $\mathrm{thin}(G_1 \vee G_2) = \mathrm{thin}(G_1)$.

Proof. Let G_1 and G_2 be two graphs with thinness (resp. proper thinness) t_1 and t_2, respectively. Let $v_1, \ldots, v_{n_1}$ and $(V_1^1, \ldots, V_1^{t_1})$ be an ordering and a partition of $V(G_1)$ which are consistent (resp. strongly consistent). Let $w_1, \ldots, w_{n_2}$ and $(V_2^1, \ldots, V_2^{t_2})$ be an ordering and a partition of $V(G_2)$ which are consistent (resp. strongly consistent). For $G = G_1 \vee G_2$, define a partition with $t_1 + t_2$ sets as the union of the two partitions, and $v_1, \ldots, v_{n_1}, w_1, \ldots, w_{n_2}$ as an ordering of the vertices.

Let x, y, z be three vertices of $V(G)$ such that $x < y < z$, $xz \in E(G)$, and x and y are in the same class of the partition of $V(G)$. Then, in particular, x and y both belong either to $V(G_1)$ or to $V(G_2)$. If z belongs to the same graph, then $yz \in E(G)$ because the ordering and partition restricted to each of G_1 and G_2 are consistent. Otherwise, z is also adjacent to y by the definition of join.

We have proved that the defined partition and ordering are consistent, and thus that $\mathrm{thin}(G_1 \vee G_2) \leq \mathrm{thin}(G_1) + \mathrm{thin}(G_2)$. The proof of the strong consistency, given the strong consistency of the partition and ordering of each of G_1 and G_2, is symmetric and implies $\mathrm{pthin}(G_1 \vee G_2) \leq \mathrm{pthin}(G_1) + \mathrm{pthin}(G_2)$.

Suppose now that G_2 is complete (in particular, $t_2 = 1$). Since G_1 is an induced subgraph of $G_1 \vee G_2$, then $\mathrm{thin}(G_1 \vee G_2) \geq \mathrm{thin}(G_1)$. For $G = G_1 \vee G_2$, define a partition $V^1, \ldots, V^{t_1}$ such that $V^1 = V_1^1 \cup V_2^1$ and $V^i = V_1^i$ for $i = 2, \ldots, t_1$, and define $v_1, \ldots, v_{n_1}, w_1, \ldots, w_{n_2}$ as an ordering of the vertices.

Let x, y, z be three vertices of $V(G)$ such that $x < y < z$, $xz \in E(G)$, and x and y are in the same class of the partition of $V(G)$. If z belongs to $V(G_2)$, then z is also adjacent to y, because it is adjacent to every vertex in $G - z$. If z belongs to $V(G_1)$, then x, y, and z, belong to $V(G_1)$ due to the definition of the order of the vertices, and thus $yz \in E(G)$ because the ordering and partition restricted to G_1 are consistent. This proves $\mathrm{thin}(G_1 \vee G_2) \leq \mathrm{thin}(G_1)$, and therefore $\mathrm{thin}(G_1 \vee G_2) = \mathrm{thin}(G_1)$. □

The following lemma shows a way of obtaining graphs with high thinness, using the join operator.

Lemma 8: If G is not complete, then $\mathrm{thin}(G \vee 2K_1) = \mathrm{thin}(G) + 1$.

Proof. By Theorem 7, $\mathrm{thin}(G \vee 2K_1) \leq \mathrm{thin}(G) + \mathrm{thin}(2K_1) = \mathrm{thin}(G) + 1$. On the other hand, as $G \vee 2K_1$ contains G as induced subgraph, $\mathrm{thin}(G \vee 2K_1) \geq \mathrm{thin}(G)$.

First notice that if $\mathrm{thin}(G) = 1$ but G is not complete, then $G \vee 2K_1$ contains C_4 as induced subgraph, so it is not an interval graph, and $\mathrm{thin}(G \vee 2K_1) \geq 2$, as claimed.

Suppose then that $\mathrm{thin}(G) = k > 1$ and $\mathrm{thin}(G \vee 2K_1) = k$ as well, and let $<$ be an ordering of the vertices of $G \vee 2K_1$ consistent with a partition $V^1, \ldots, V^k$. Let v, v' be the vertices of the graph $2K_1$, and suppose $v < v'$. Without loss of generality we may assume $v \in V^k$. As $\mathrm{thin}(G) = k$, $V^k \cap V(G) \neq \emptyset$. Since $v' > v$, v' is nonadjacent to v, and v' is adjacent to all the vertices in $V^k \cap V(G)$, v has to be the smallest vertex in V^k. Let $z \in V^k \cap V(G)$ and suppose there is a vertex $x > z$ in $V(G)$. As x is adjacent to v', it is adjacent to z as well. So, we can define a new order $<'$ on $V(G \vee 2K_1)$ that preserves the order $<$ in $V^1 \cup V^{k-1} \cup \{v\}$ and such that the vertices of $V^k - \{v\}$ are the largest. By the observations above, this order $<'$ is still consistent with the partition $V^1, \ldots, V^k$. But it is also consistent with the partition $V^{1'}, \ldots, V^{k'}$ in which $V^{1'} = V^1 \cup V^k - \{v\}$, $V^{i'} = V^i$ for $1 < i < k$, and $V^{k'} = \{v\}$. This implies that $\text{thin}(G) < k$, a contradiction that completes the proof of the theorem. □

Cographs were defined in [18], where it was shown that they are exactly the graphs with no induced path of length four. Cographs admit a full decomposition theorem. Let the *trivial* graph be the one with one vertex only.

Proposition 9: *[18]* Every cograph that is not trivial is either the union or the join of two smaller cographs.

We will use this structural property along with the theorems about thinness of union and join of graphs to prove the following.

Theorem 10: Let G be a cograph and $t \geq 1$. Then G has thinness at most t if and only if G contains no $\overline{(t+1)K_2}$ as induced subgraph.

Proof. The only if part holds by Theorem 1, because the class of k-thin graph is hereditary for every k.

We will prove the if part by induction on the number of vertices of the cograph G. If G is a trivial graph, then $\text{thin}(G) = 1$ and the theorem holds. If G is not trivial, by Proposition 9, it is either union or join of two smaller cographs G_1 and G_2, with thinness t_1 and t_2, respectively.

Suppose first $G = G_1 \cup G_2$. By Theorem 6, $\text{thin}(G) = \max\{t_1, t_2\}$. If t_1 (resp. t_2) is greater than one, then by inductive hypothesis G_1 (resp. G_2) contains $\overline{t_1K_2}$ (resp. $\overline{t_2K_2}$) as induced subgraph, and so does G.

Suppose now that $G = G_1 \vee G_2$. If one of them is complete (suppose without loss of generality G_2), then, by Theorem 7, $\text{thin}(G) = t_1$. If t_1 is greater than one, then by inductive hypothesis G_1 contains $\overline{t_1K_2}$ as induced subgraph, and so does G. If none of them is complete, then, by that fact and the inductive hypothesis, G_1 contains $\overline{t_1K_2}$ and G_2 contains $\overline{t_2K_2}$ as induced subgraph. As $\overline{t_1K_2} \vee \overline{t_2K_2} = \overline{(t_1+t_2)K_2}$, G contains $\overline{(t_1+t_2)K_2}$ as induced subgraph, thus $\text{thin}(G) \geq t1 + t2$ (Theorem 1). By Theorem 7, $\text{thin}(G) \leq t1+t2$, and therefore $\text{thin}(G) = t1+t2$. This finishes the proof of the theorem. □

A characterization by minimal forbidden induced subgraphs for k-thin graphs, $k \geq 2$, is open.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The Cartesian product $G_1 \square G_2$ is a graph whose vertex set is the Cartesian product $V_1 \times V_2$, and such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \square G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

Theorem 11: Let G_1 and G_2 be graphs. Then $\text{thin}(G_1 \square G_2) \leq \text{thin}(G_1)|V(G_2)|$ and $\text{pthin}(G_1 \square G_2) \leq \text{pthin}(G_1)|V(G_2)|$.

Proof. Let $G_1 = (V_1, E_1)$ be a k-thin (resp. proper k-thin) graph, and let $v_1, \dots, v_{n_1}$ and $(V_1^1, \dots, V_1^k)$ be an ordering and a partition of V_1 which are consistent (resp. strongly consistent). Let $G_2 = (V_2, E_2)$, $n_2 = |V_2|$, and $w_1, \dots, w_{n_2}$ an arbitrary ordering of V_2. Consider $V_1 \times V_2$ lexicographically ordered with respect to the orderings of V_1 and V_2 above. Consider now the partition $\{V^{i,j}\}_{1 \leq i \leq k,\ 1 \leq j \leq n_2}$ such that $V^{i,j} = \{(v, w_j) : v \in V_1^i\}$ for each $1 \leq i \leq k$, $1 \leq j \leq n_2$. We will show that this ordering and partition of $V_1 \times V_2$ are consistent (resp. strongly consistent). Let $(v_p, w_i), (v_q, w_j), (v_r, w_\ell)$ be three vertices appearing in that ordering in $V_1 \times V_2$.

Case 1: $p = q = r$. In this case, the three vertices are in different classes, so no restriction has to be satisfied.

Case 2: $p = q < r$. In this case, (v_p, w_i) and (v_q, w_j) are in different classes. So suppose G_1 is proper k-thin and $(v_q, w_j), (v_r, w_\ell)$ belong to the same class, i.e., $j = \ell$. Vertices (v_p, w_i) and (v_r, w_ℓ) are adjacent in $G_1 \Box G_2$ if and only if $i = \ell$ and $v_p v_r \in E_1$. But then $(v_p, w_i) = (v_q, w_j)$, a contradiction.

Case 3: $p < q = r$. In this case, (v_q, w_j) and (v_r, w_ℓ) are in different classes. So suppose G_1 is k-thin and $(v_p, w_i), (v_q, w_j)$ belong to the same class, i.e., $i = j$. Vertices (v_p, w_i) and (v_r, w_ℓ) are adjacent in $G_1 \Box G_2$ if and only if $i = \ell$ and $v_p v_r \in E_1$. But then $(v_r, w_\ell) = (v_q, w_j)$, a contradiction.

Case 4: $p < q < r$. Suppose first G_1 is k-thin (resp. proper k-thin) and $(v_p, w_i), (v_q, w_j)$ belong to the same class, i.e., $i = j$ and v_p, v_q belong to the same class in G_1. Vertices (v_p, w_i) and (v_r, w_ℓ) are adjacent in $G_1 \Box G_2$ if and only if $i = \ell$ and $v_p v_r \in E_1$. But then $j = \ell$ and since the ordering and the partition are consistent (resp. strongly consistent) in G_1, $v_r v_q \in E_1$ and so (v_r, w_ℓ) and (v_q, w_j) are adjacent in $G_1 \Box G_2$. Now suppose that G_1 is proper k-thin and $(v_q, w_j), (v_r, w_\ell)$ belong to the same class, i.e., $j = \ell$. Vertices (v_p, w_i) and (v_r, w_ℓ) are adjacent in $G_1 \Box G_2$ if and only if $i = \ell$ and $v_p v_r \in E_1$. But then $i = j$ and since the ordering and the partition are strongly consistent in G_1, $v_p v_q \in E_1$ and so (v_p, w_i) and (v_q, w_j) are adjacent in $G_1 \Box G_2$. □

Corollary 12: If G is (proper) k-thin then $G \Box K_t$ is (proper) kt-thin. In particular, if G is a (proper) interval graph then $G \Box K_t$ is (proper) t-thin.

For a graph $G(V, E)$ and an integer t, we say that f is a *t-rainbow dominating function* if it assigns to each vertex $v \in V$ a subset of $\{1, \dots, t\}$ such that $\cup_{u \in N(v)} f(u) = \{1, \dots, t\}$ for all v with $f(v) = \emptyset$. Consider the following generalization of the dominating set problem.

t-RAINBOW DOMINATION PROBLEM
Instance: A graph $G = (V, E)$.
Find: a t-rainbow dominating function that minimizes $\sum_{v \in V} |f(v)|$.

The t-RAINBOW DOMINATION PROBLEM is equivalent to MINIMUM DOMINATING SET of $G \Box K_t$ [13]. As a consequence of Corollary 12 and the last remark in Section 3.1, it can be solved in polynomial time on graphs with bounded thinness for fixed values of t. This generalizes the polynomiality for interval graphs, recently proved by Hon, Kloks, Liu, and Wang in [38] (the algorithm for $t = 2$ is claimed in [37]). The problem for proper interval graphs was stated as an open question by Brešar and Kraner Šumenjak in [13].

The behavior of thinness and proper thinness under many of the graph products defined in the literature was later studied in [12].

3. OTHER PARAMETERS

3.1 Thinness and other width parameters

Many width parameters are defined in the literature. In this section we compile the results relating the thinness with some of them, namely pathwidth [59], treewidth [5, 60], clique-width [19], cutwidth [45], MIM-width [66], and boxicity [58].

In [48] it was proved that the thinness of a graph is at most the pathwidth plus one, and that the gap may be high, since the pathwidth of a complete graph with r vertices is $r-1$, while its thinness is 1.

Proof. If $(X_1, \ldots, X_t)$ is a path decomposition of width $k-1$, let's see we can construct a k-partition with a consistent ordering.

First, we say that a path decomposition of width $k-1$ is *smooth* if $|X_i| = k$ for all i and each pair of adjacent bags differ in exactly one vertex. Any path decomposition can be converted to a smooth one, preserving the width, see [8].

So we can assume that the path decomposition is smooth. The ordering $v_1, \ldots, v_n$ will be the same as the ordering in which the vertices appear in the path decomposition (the ones from X_1 can be in any relative order between them). Let's define $X(i)$ as the bag with the minimum index in which v_i appears.

To create the classes of the k-partition, for each i we put all vertices of X_i in distinct classes (at first we distribute the ones from X_1, then the one of $X_2 - X_1$ goes to the class used by the one of $X_1 - X_2$, and so on).

For each i, all the neighbors of v_i that are before it in the ordering are also in the bag $X(i)$, by definition of path decomposition. Therefore, all of them are in different classes.

Now consider the class in which one of those neighbors v_j is. We'll see that there isn't another vertex v_k in the class such that $j < k < i$. If such a vertex exists, then v_j and v_k couldn't share a bag, but since $v_j \in X(i)$ then $v_j \in X(k)$ by definition, which is absurd. $\square$

On the other hand, in [15] it was proved that the boxicity is a lower bound for the thinness of a graph, and it was pointed out that the difference can be large, as an $(r \times r)$-grid has boxicity 2 and thinness $\Theta(r)$.

Proof. Given a k-thin representation of G, we have to give an interval graph representation $I_1, \ldots, I_k$ of G.

Define I_h as follows:

- $I_h[V^h] = G[V^h]$
- $I_h[V \setminus V^h]$ is a clique
- if $u \in V \setminus V^h$, let r be the minimum such that $(u, v^h_r) \in E$. Then let $(u, v^h_i) \in E_h$ for all $i \geq r$.

$\square$

The *vertex isoperimetric peak* of a graph G, denoted as $b_v(G)$, is defined as $\max_s \min_{X \subset V, |X|=s} |N(X)|$. The thinness of the grid was estimated by using the following result, that was also used in [6] to give a lower bound of the thinness of a complete binary tree. We will use it as well to estimate the thinness of complete m-ary trees.

Lemma 13: *[15]* For every graph G, $\text{thin}(G) \geq b_v(G)/\Delta(G)$.

Proof. First we need the following lemma:

If for a graph G there exists an integer s such that every $X \subset V$ of size s satisfies $|\delta_{\text{out}}(X)| \geq k$, then $\text{thin}(G) \geq k/\Delta(G)$.

Here $\delta_{\text{out}}(X)$ is the *outer boundary* of X, the set of vertices in $V - X$ with at least one neighbor in X.

To prove the lemma:

If $\text{thin}(G) = t$, take a partition $V_1, \ldots, V_t$ and a consistent ordering $v_1, \ldots, v_n$. Let S be the set containing the last s vertices of the ordering.

For any i, let x be the lowest vertex in $V_i \cap \delta_{\text{out}}(S)$. Then it is adjacent to a vertex of S, say y. Now, this y must be adjacent to all of $V_i \cap \delta_{\text{out}}(S)$, by the definition of consistent ordering. This implies that $|V_i \cap \delta_{\text{out}}(S)| \leq \Delta$.

Therefore,

$$|\delta_{\text{out}}(S)| = \sum_i |V_i \cap \delta_{\text{out}}(S)| \leq t\Delta$$

and so there are $t \geq |\delta_{\text{out}}(S)|/\Delta \geq k/\Delta$ classes. □

Interval graphs have thinness 1 and unbounded clique-width [33], while cographs have clique-width 2 [20] and unbounded thinness, because $\overline{tK_2}$ is a cograph for every t, so the parameters are not comparable.

Complete graphs have high treewidth and thinness 1, and trees instead have treewidth 1 but we have the following result.

Theorem 14: For every fixed value m, the thinness of the complete m-ary tree on n vertices is $\Theta(\log n)$.

Proof. In [67] it was proved that the vertex isoperimetric peak of the complete m-ary tree of height h is $\Theta(h)$. On the other hand, it was proved in [27, 62] that the pathwidth of the complete m-ary tree of height h is $\Theta(h)$. As the thinness of a graph is upper bounded by the pathwidth plus one [48] and using Lemma 13, it follows that the thinness of the complete m-ary tree of height h is $\Theta(h)$, and this proves the theorem. □

The *cutwidth* of a graph G, denoted as $\text{cutw}(G)$, is the smallest integer k such that the vertices of G can be arranged in a linear layout $v_1, \ldots, v_n$ in such a way that for every $i = 1, \ldots, n-1$, there are at most k edges with one endpoint in $\{v_1, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$.

Theorem 15: For every graph G, $\text{thin}(G) \leq \text{cutw}(G) + 1$. Moreover, a linear layout realizing the cutwidth leads to a consistent partition into at most $\text{cutw}(G) + 1$ classes.

Proof. Let G be a graph of cutwidth k, and let $v_1, \ldots, v_n$ such that for every $i = 1, \ldots, n-1$, there are at most k edges with one endpoint in $\{v_1, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$. Let $G_<$ be the graph defined as in Theorem 2 for the order $v_1, \ldots, v_n$. Since $G_<$ is a co-comparability graph, its chromatic number equals the size of a maximum clique of it [49]. Suppose that $G_<$ has a clique H of size $k+2$, and let v_i be the vertex of higher index in H. By definition of $G_<$, for each $i' < i$ such that $v_{i'} \in H$, there exists $j > i$ such that v_j is adjacent to $v_{i'}$ and not adjacent to v_i. So, there are at least $k+1$ edges with one endpoint in $\{v_1, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$, a contradiction. □

The gap may be high, as for example on cliques.

The *linear MIM-width* of a graph G, denoted as $\mathrm{lmimw}(G)$, is the smallest integer k such that the vertices of G can be arranged in a linear layout $v_1, \ldots, v_n$ in such a way that for every $i = 1, \ldots, n-1$, the size of a maximum induced matching in the bipartite graph formed by the edges of G with an endpoint in $\{v_1, \ldots, v_i\}$ and the other one in $\{v_{i+1}, \ldots, v_n\}$ is at most k. This is the linear version of a parameter called MIM-width [66], that is a lower bound for the linear MIM-width.

Theorem 16: For every graph G, $\mathrm{lmimw}(G) \leq \mathrm{thin}(G)$. Moreover, a linear ordering $v_1, \ldots, v_n$ realizing the thinness, satisfies that the size of a maximum induced matching in the bipartite graph formed by the edges of G with an endpoint in $\{v_1, \ldots, v_i\}$ and the other one in $\{v_{i+1}, \ldots, v_n\}$ is at most $\mathrm{thin}(G)$.

Proof. Let $k = \mathrm{thin}(G)$ and consider a k-thin representation of G, with ordering $<$ of $V(G)$, namely $v_1 < \cdots < v_n$, and a partition of $V(G)$ into k classes. Let $1 \leq i \leq n-1$ and let M be a maximum induced matching in the bipartite graph formed by the edges of G with an endpoint in $\{v_1, \ldots, v_i\}$ and the other one in $\{v_{i+1}, \ldots, v_n\}$. Suppose $v_r v_t$ and $v_s v_q$ belong to M, with $r < s \leq i$, $t, q \geq i+1$. If v_r and v_s belong to the same class of the partition, by definition of k-thin representation, $v_s v_t$ is also an edge, a contradiction with the fact that M is an induced matching. So, $|M| \leq k$, thus $\mathrm{lmimw}(G) \leq \mathrm{thin}(G)$. □

As a corollary, given a graph G provided with a k-thin representation, a wide family of problems known as Locally Checkable Vertex Subset and Vertex Partitioning problems can be solved in $n^{O(k)}$ time [66], as this holds for MIM-width k and a suitable ordering. This family of problems is not comparable (inclusion-wise) with the one in Section 4.1, but encompasses maximum weighted independent set and minimum weighted dominating set.

3.2 Interval graphs with high proper thinness

In this section we first show that proper thinness of the class of interval graphs is unbounded. Then we relate the proper thinness of interval graphs to other interval graphs invariants, like interval count. A family of interval graphs with arbitrarily large proper thinness is the following.

Let $h \geq 1$, and define *claw$_h$* as the graph obtained from the complete ternary tree of height h by adding all the edges between a vertex of the tree and its ancestors. It is easy to see that claw_h is an interval graph for every $h \geq 1$ (an interval representation of claw_3 can be seen in Figure 3.1). The graph claw_1 is the claw.

Proposition 17: *[61]* For any $h \geq 1$, $\mathrm{pthin}(\mathrm{claw}_h) = h+1$.

Proof. Let $h \geq 1$. We will label the vertices of $G = \mathrm{claw}_h$ as v^i_j such that $0 \leq i \leq h$, $1 \leq j \leq 3^i$, v^0_1 is the root of the ternary tree, and for each $0 \leq i \leq h-1$, $1 \leq j \leq 3^i$, the

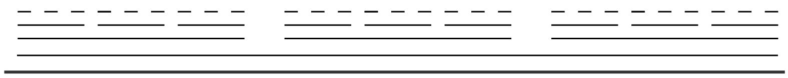

Fig. 3.1: An interval representation of claw_3.

children of v_j^i are v_{3j-2}^{i+1}, v_{3j-1}^{i+1}, and v_{3j}^{i+1}. Let us consider an ordering $<$ and a partition of $V(G)$ that are strongly consistent. Without loss of generality, by symmetry, we may assume $v_2^i < v_1^i < v_3^i$ for every $i \geq 1$.

Let us show now that for every $0 \leq i' < i \leq h$, v_1^i and $v_1^{i'}$ cannot be in the same class of the partition. Otherwise, if $v_1^i < v_1^{i'}$ then the fact of $v_2^i < v_1^i < v_1^{i'}$, $v_2^i v_1^{i'} \in E(G)$ and $v_2^i v_1^i \notin V(G)$ contradicts the definition of strong consistency, and if $v_1^{i'} < v_1^i$ then the fact of $v_1^{i'} < v_1^i < v_3^i$, $v_3^i v_1^{i'} \in E(G)$ and $v_3^i v_1^i \notin V(G)$ contradicts the definition of strong consistency.

So, $v_1^0, \ldots, v_1^h$ are all in different classes of the partition and $\text{pthin}(\text{claw}_h) \geq h+1$. On the other hand, a partition of the vertices according to its height in the tree, and a postorder of the vertices of the tree are strongly consistent. Thus $\text{pthin}(\text{claw}_h) = h+1$. □

This example is also a classical example of a graph with high interval count and high length of a chain of nested intervals. We will relate the proper thinness of interval graphs to these two interval graphs invariants.

The *interval count* of an interval graph G is the minimum number of different interval sizes needed in an interval representation of G (see for example [14, 44]). Graphs with interval count at most k are also known as *k-length interval graphs.*

A *k-nested interval graph* is an interval graph admitting an interval representation in which there are no chains of $k+1$ intervals nested in each other [43]. It is easy to see that k-nested interval graphs are a superclass of k-length interval graphs. We have also the following property.

Proposition 18: *[50]* Every k-nested interval graph is proper k-thin.

Proof. Let G be a k-nested interval graph and consider an interval representation of G with no chains of $k+1$ intervals nested in each other. It is a known result that we may assume that all the interval endpoints are distinct. We label each interval by the length of the longest chain of nested intervals ending in it, and these labels define the partition of the vertices into classes, that are at most k. Now, we order the vertices according to their intervals by the right endpoint (left to right). That order is consistent with the partition in which the only class contains all vertices of G, so, in particular, it is consistent with every other partition refining it. Let us see that the consistency is strong. Let $r < s < t$ such that s and t are in the same class of the partition. Let I_r, I_s, I_t their corresponding intervals. By definition of the classes, $I_s \not\subseteq I_t$, otherwise the length of the longest chain of nested intervals ending in I_s would be strictly greater than the one for I_t. As the right endpoint of I_t is greater than the one of I_s, it follows that the left endpoint of I_t is also greater than the one of I_s. Thus, if I_r intersects I_t, it intersects I_s as well. So, the ordering and the partition are strongly consistent and G is proper k-thin. □

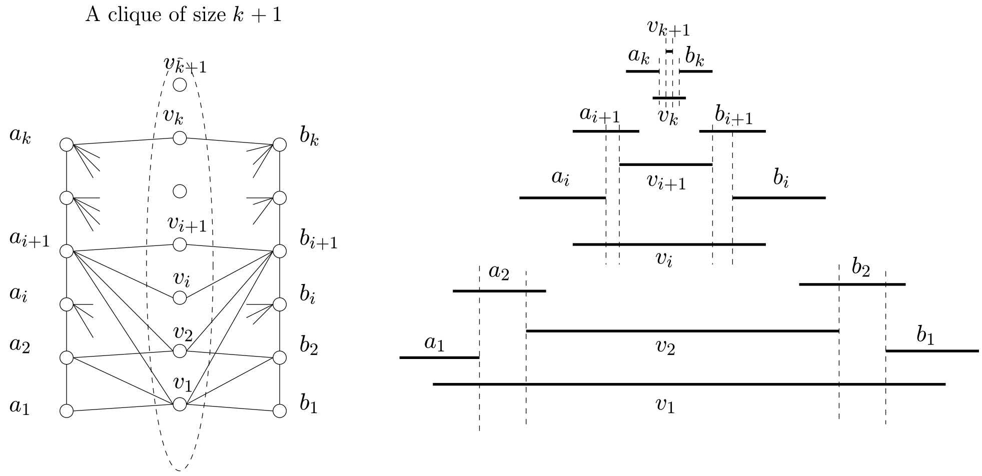

Fig. 3.2: A sketch of graph G_k and an interval representation of it.

Graphs with interval count one are known as unit interval graphs, while 1-nested interval graph are equivalent to proper interval graphs. In [57] it is shown that unit interval graphs are equivalent to proper interval graphs. So the classes proper 1-thin, 1-length interval and 1-nested interval are equivalent. We will see that for higher numbers the equivalence does not necessarily hold.

Indeed, in [30, Theorem 5, p. 177], Fishburn shows that, for every $k \geq 2$, there are 2-nested interval graphs that are not k-length interval.

We will describe a family of graphs that show that, for every $k \geq 3$, there are proper 3-thin graphs that are not k-nested interval.

Let $k \geq 1$. Let G_k with $3k+1$ vertices is defined as follows. Its vertex-set is $V_k = A_k \cup B_k \cup W_k$, where $A_k = \{a_1, \ldots, a_k\}$, $B_k = \{b_1, \ldots, b_k\}$ and $W_k = \{v_1, \ldots, v_k, v_{k+1}\}$. The subgraph induced by W_k is a clique with $k+1$ vertices; a_1 (resp., b_1) is adjacent to v_1. Then, for any $1 < i \leq k$, a_i (resp., b_i) is adjacent to a_{i-1} (resp., to b_{i-1}), and to v_j for any $j \geq i$. See Figure 3.2 for a sketch of G_k and an interval representation of it.

The graph G_1 is the claw, which is not proper interval. For higher values of k, we have the following property.

Proposition 19: *[50]* For any $k \leq 2$, G_k is proper 3-thin, but in every interval representation of it, if I_j is the interval corresponding to v_j, it holds $I_{k+1} \subseteq I_k \subseteq \cdots \subseteq I_1$.

Proof. Consider the ordering $a_1, \ldots, a_k, b_1, \ldots, b_k, v_1, \ldots, v_k, v_{k+1}$, and the three classes A_k, B_k and W_k. It is easy to see that they are strongly consistent.

Let $1 \leq i \leq k-1$. Notice that $a_i a_{i+1} v_{i+1} b_{i+1} b_i$ induce a path of length five on G_k. In every interval representation of it, the interval I_{i+1} is between the intervals corresponding to a_i and b_i and disjoint to them. As the five vertices are adjacent to v_i, it follows that the $I_{i+1} \subseteq I_i$. Finally, by the shape of interval representations of a path of length five, each of the intervals corresponding to a_k and b_k contains an endpoint of I_k. As v_{k+1} is neither adjacent to a_k nor to b_k, $I_{k+1} \subseteq I_k$. □

The following characterization was proved for k-nested interval graphs.

Lemma 20: *[43]* An interval graph is k-nested interval if and only if it has an interval representation which can be partitioned into k proper interval representations.

This lemma and the family of graphs G_k show that even if the vertices of a proper k-thin graph can be partitioned into k sets of vertices each of them inducing a proper interval graph, it is not always the case that it has an interval representation which can be partitioned into k proper interval representations.

4. PROBLEMS ON GRAPHS OF BOUNDED THINNESS

4.1 Solving combinatorial optimization problems on graphs with bounded thinness

Since a k-thin graph G does not contain $\overline{(k+1)K_2}$ as induced subgraph (Theorem 1), it has at most $|V(G)|^{2k}$ maximal cliques [54]. In particular, the MAXIMUM WEIGHTED CLIQUE problem can be solved in polynomial time on graphs with bounded thinness, by simple enumeration of the maximal cliques of the graph [64].

The MAXIMUM WEIGHTED INDEPENDENT SET problem can be solved in polynomial time on graphs with bounded thinness, when an ordering and a partition that are consistent are given [48]. In the same hypothesis, the CAPACITATED COLORING problem (in which there is an upper bound α_j on the number of vertices of color j) can be solved in polynomial time, if the number of colors s is fixed [11]. As a byproduct, in the same paper it is shown that the capacitated coloring can be solved in polynomial time for co-comparability graphs, if the number of colors s is fixed, in contrast with the case in which the bounds α_j are all equal to a fixed number h, that is **NP**-complete, even for two subclasses of co-comparability graphs: permutation graphs (for $h \geq 6$) [46] and interval graphs (for $h \geq 4$) [9]. The hardness on interval graphs implies the hardness for graphs of bounded thinness, since interval graphs are the graphs with thinness 1.

Both algorithms, the one for MAXIMUM WEIGHTED INDEPENDENT SET and the one for CAPACITATED COLORING with fixed number of colors, are based on dynamic programming. One of the main results in this work is a generalization of these algorithmic results. We describe now a generic problem that can be solved for graphs with bounded thinness, given the representation. We call it the LINEAR WEIGHTED LIST MATRIX PARTITION problem, and is defined as follows:
Instance:

- A graph $G = (V, E)$.
- A family of arbitrary nonnegative weights $w_1, \dots, w_t$ on V.
- A family of nonnegative weights $b_1, \dots, b_p$ on V bounded by a fixed polynomial in n (p fixed, $q(n)$ the bound for the weights).
- Each vertex v has a list $L(v)$ of combinations of the sets $S_1, \dots, S_r$ to which it can belong (that may include the empty combination).
- An $r \times r$ symmetric matrix M over $0, 1, *$, stating the adjacency conditions on the sets S_j, such that for $1 \leq i < j \leq r$, $M_{ii} = 1$ means S_i is a clique, $M_{ii} = 0$ means S_i is an independent set, $M_{ij} = 1$ means all the edges joining S_i and S_j have to be present, $M_{ij} = 0$ means there are no edges from S_i to S_j.
- A family of restrictions on the weight of the intersection and of the union of some families of sets. Such restrictions can be expressed as
  - $0 \leq l_{iJ\cap} \leq b_i(\bigcap_{j \in J} S_j) \leq u_{iJ\cap}$, such that $1 \leq i \leq p$, $J \subseteq \{1, \dots, r\}$.

– $0 \leq l_{iJ\cup} \leq b_i(\bigcup_{j\in J} S_j) \leq u_{iJ\cup}$, such that $1 \leq i \leq p$, $J \subseteq \{1, \ldots, r\}$.

Notice that some of these restrictions can be of cardinality, if the corresponding weight function b_i is constant.

- A set of constants c_{ij}, where $1 \leq i \leq t$ and $1 \leq j \leq r$.

Question: find sets $S_1, \ldots, S_r$ (r fixed, not necessarily disjoint), $S_j \subseteq V$ for $1 \leq j \leq r$, satisfying the restrictions, in order to maximize $\sum_{1,j} c_{ij} w_i(S_j)$.

The family of problems that can be modeled within this framework includes weighted variations of LIST MATRIX PARTITION problems with matrices of bounded size, which in turn generalize COLORING, LIST COLORING, LIST HOMOMORPHISM, EQUITABLE COLORING with different objective functions, all for fixed number of colors (or graph size in the case of homomorphism), CLIQUE COVER with fixed number of cliques, MAXIMUM WEIGHTED INDEPENDENT SET, and other graph partition problems. It models also SUM-COLORING and its more general version OPTIMUM COST CHROMATIC PARTITION problem [40] for fixed number of colors, but it does not include dominating-like problems.

Now we will provide a polynomial time algorithm for this problem, assuming we are given a k-thin representation of G, with ordering $<$ of V, namely $v_1 < \cdots < v_n$, and partition of V into k classes $V^1, \ldots, V^k$.

We will solve it as a shortest or longest path problem (according to minimization or maximization of the objective function) in an auxiliary acyclic digraph $D = (X, A)$ whose nodes correspond to *states* and whose arcs are weighted and labeled. The total weight of the path is the value of the objective function in the solution that can be built by using the arc labels. We will used the term "nodes" for the digraph D in order to avoid confusion with the vertices of the graph G.

A state is a tuple, containing:

- A number $1 \leq s \leq |V_G|$ indicating that we are considering the subgraph G_s of G, induced by $v_1, \ldots, v_s$.
- Nonnegative parameters $l_{iJ\cap}, u_{iJ\cap}, l_{iJ\cup}, u_{iJ\cup}$, for $1 \leq i \leq p$, $J \subseteq \{1, \ldots, r\}$; they are at most $2^{r+2}p$, and each of them may take a nonnegative value at most $nq(n)$, which is an upper bound for $b_i(V)$, for every $1 \leq i \leq p$.
- A family of nonnegative parameters $\{\alpha_{ij}\}_{1\leq i\leq k; 1\leq j\leq r}$, meaning that we cannot pick for S_j a vertex of the first α_{ij} vertices of the set V^i of the partition; there are kr such parameters and each of them may take a nonnegative value at most $n - 1$.
- A family of nonnegative parameters $\{\beta_{ij}\}_{1\leq i\leq k; 1\leq j\leq r}$, meaning that we cannot pick for S_j a vertex on the last β_{ij} vertices of the set V^i of the partition; there are kr such parameters and each of them may take a nonnegative value at most $n - 1$.

The total number of states is then at most $n^{2kr+1}(nq(n))^{2^{r+2}p}$, that is polynomial in n, since k, r, and p are constant and $q(n)$ is polynomial in n.

The digraph D will have nodes that correspond to possible states, organized in layers $X_0, X_1, \ldots, X_n$ such that X_0 contains only one node x_0, and the layer X_s contains the states whose first parameter is s. The layer X_n contains also only one node, corresponding to the state $(n, \{l_{iJ\cap}\}, \{u_{iJ\cap}\}, \{l_{iJ\cup}\}, \{u_{iJ\cup}\}, \{\alpha_{ij}\}, \{\beta_{ij}\})$, where the parameters $\{l_{iJ\cap}\}, \{u_{iJ\cap}\}, \{l_{iJ\cup}\}, \{u_{iJ\cup}\}$ are the ones in the original formulation of the problem and $\alpha_{ij} = \beta_{ij} = 0$ for every $1 \leq i \leq k$, $1 \leq j \leq r$.

All arcs of A have the form (u, w) with $u \in X_s$ and $w \in X_{s+1}$, for some $0 \leq s \leq n-1$.

We associate with each node of X a suitable problem, in the same framework, whose parameters correspond to the parameters in the state, but with additional constraints associated with the parameters $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$.

We will define the arcs in such a way that a node is reachable from the node in the layer X_0 if and only if the associated problem has a solution. The length of the path will be the weight of the solution, and the set of arc labels will encode the solution. Let us describe the arcs of the digraph.

Let w be a node with parameters $(1, \{l_{iJ\cap}\}, \{u_{iJ\cap}\}, \{l_{iJ\cup}\}, \{u_{iJ\cup}\}, \{\alpha_{ij}\}, \{\beta_{ij}\})$.

Let $1 \leq \ell \leq k$ such that $v_1 \in V^\ell$. For each $\tilde{J} \in L(v_1)$ (in particular $\tilde{J} \subseteq \{1, \ldots, r\}$), such that:

1.1 For each $j \in \tilde{J}$, $\beta_{\ell j} = \alpha_{\ell j} = 0$.

1.2 For each $J \subseteq \tilde{J}$, $l_{iJ\cap} \leq b_i(v_1) \leq u_{iJ\cap}$.

1.3 For each $J \not\subseteq \tilde{J}$, $l_{iJ\cap} = 0$.

1.4 For each J such that $J \cap \tilde{J} \neq \emptyset$, $l_{iJ\cup} \leq b_i(v_1) \leq u_{iJ\cup}$.

1.5 For each J such that $J \cap \tilde{J} = \emptyset$, $l_{iJ\cup} = 0$.

We add an arc from x_0 to w, labeled by $\tilde{J}$ and of weight $\sum_{1 \leq i \leq t; j \in \tilde{J}} c_{ij} w_i(v_1)$. If no $\tilde{J}$ satisfies conditions 1.1–1.5, no arc ending in w is added. If more than one arc $x_0 w$ was added, we can keep only the one with maximum (resp. minimum) weight if we are solving a maximization (resp. minimization) problem.

Note that if we add the arc $x_0 w$ labeled by $\tilde{J}$, then the solution $S_j = \{v_1\}$ for $j \in \tilde{J}$, $S_j = \emptyset$ for $j \notin \tilde{J}$ has weight $\sum_{1 \leq i \leq t; j \in \tilde{J}} c_{ij} w_i(v_1)$ and satisfies the state described by w: condition 1.1 says that v_1 (the first and last vertex of V^ℓ in G_1) is allowed to be picked for every set S_j for $j \in \tilde{J}$; conditions 1.2–1.5 say that the assignment does not violate weight constraints.

Let w be a node with parameters $(s, \{l_{iJ\cap}\}, \{u_{iJ\cap}\}, \{l_{iJ\cup}\}, \{u_{iJ\cup}\}, \{\alpha_{ij}\}, \{\beta_{ij}\})$, $1 < s \leq n$.

Let $1 \leq \ell \leq k$ such that $v_s \in V^\ell$. For each $\tilde{J} \in L(v_s)$, such that:

s.1 For each $j \in \tilde{J}$, $\beta_{\ell j} = 0$.

s.2 For each $j \in \tilde{J}$, $\alpha_{\ell j} < |V^\ell \cap \{v_1, \ldots, v_s\}|$.

s.3 For each $J \subseteq \tilde{J}$, $b_i(v_s) \leq u_{iJ\cap}$.

s.4 For each J such that $J \cap \tilde{J} \neq \emptyset$, $b_i(v_s) \leq u_{iJ\cup}$.

We add an arc from u to w, labeled by $\tilde{J}$ and of weight $\sum_{1 \leq i \leq t; j \in \tilde{J}} c_{ij} w_i(v_s)$, where u has parameters $(s-1, \{l'_{iJ\cap}\}, \{u'_{iJ\cap}\}, \{l'_{iJ\cup}\}, \{u'_{iJ\cup}\}, \{\alpha'_{ij}\}, \{\beta'_{ij}\})$, such that:

$s'.1$ Let $1 \leq j \leq r$. If there exists $j' \in \tilde{J}$ such that $M_{jj'} = 0$, then $\beta'_{\ell j} = \max\{\beta_{\ell j} - 1, |N(v_s) \cap V^\ell \cap \{1, \dots, s-1\}|\}$, and for $1 \leq i \leq k$, $i \neq \ell$, $\beta'_{ij} = \max\{\beta_{ij}, |N(v_s) \cap V^i \cap \{1, \dots, s-1\}|\}$. Otherwise, $\beta'_{\ell j} = \max\{0, \beta_{\ell j} - 1\}$, and for $1 \leq i \leq k$, $i \neq \ell$, $\beta'_{ij} = \beta_{ij}$.

$s'.2$ Let $1 \leq j \leq r$. If there exists $j' \in \tilde{J}$ such that $M_{jj'} = 1$, then $\alpha'_{\ell j} = \max\{\min\{|V^\ell \cap \{1, \dots, s-1\}|, \alpha_{\ell j}\}, |\overline{N}(v_s) \cap V^\ell \cap \{1, \dots, s-1\}|\}$, and for $1 \leq i \leq k$, $i \neq \ell$, $\alpha'_{ij} = \max\{\alpha_{ij}, |\overline{N}(v_s) \cap V^i \cap \{1, \dots, s-1\}|\}$. Otherwise, $\alpha'_{\ell j} = \min\{|V^\ell \cap \{1, \dots, s-1\}|, \alpha_{\ell j}\}$, and for $1 \leq i \leq k$, $i \neq \ell$, $\alpha'_{ij} = \alpha_{ij}$.

$s'.3$ For each $J \subseteq \tilde{J}$, $l'_{iJ\cap} = \max\{0, l_{iJ\cap} - b_i(v_s)\}$ and $u'_{iJ\cap} = u_{iJ\cap} - b_i(v_s)$.

$s'.4$ For each $J \not\subseteq \tilde{J}$, $l'_{iJ\cap} = l_{iJ\cap}$ and $u'_{iJ\cap} = u_{iJ\cap}$.

$s'.5$ For each J such that $J \cap \tilde{J} \neq \emptyset$, $l'_{iJ\cup} = \max\{0, l_{iJ\cup} - b_i(v_s)\}$ and $u'_{iJ\cup} = u_{iJ\cup} - b_i(v_s)$.

$s'.6$ For each J such that $J \cup \tilde{J} = \emptyset$, $l'_{iJ\cup} = l_{iJ\cup}$ and $u'_{iJ\cup} = u_{iJ\cup}$.

If no $\tilde{J}$ satisfies conditions $s.1$–$s.4$, no arc ending in w is added. If more than one arc from the same vertex u to w was added, we can keep only the one with maximum (resp. minimum) weight if we are solving a maximization (resp. minimization) problem.

That is, if an arc is added, the arc corresponds to the choice of adding the vertex v_s to the sets $\{S_j\}_{j \in \tilde{J}}$, the conditions required imply that the choice is valid for w in the case that the state described by u admits a solution, the label of the arc keeps track of the choice made, and the cost corresponds to the weight that the choice adds to the objective function.

Note that if we add the arc uw labeled by $\tilde{J}$, then for a solution $\{S'_j\}_{1 \leq j \leq r}$ for G_{s-1} satisfying the state described by u, then the solution $\{S_j\}_{1 \leq j \leq r}$ for G_s such that $S_j = S'_j \cup \{v_s\}$ for $j \in \tilde{J}$, $S_j = S'_j$ for $j \notin \tilde{J}$ satisfies the state described by w. Conditions $s.1$ and $s.2$ say that v_s (the last vertex of V^ℓ in G_s) is allowed to be picked for every set S_j for $j \in \tilde{J}$. Condition $s'.1$ ensures on one hand that the conditions imposed by the parameters $\{\beta_{ij}\}$ in w are satisfied by the solution of u, and, on the other hand, that if $j' \in \tilde{J}$ and $1 \leq j \leq r$ are such that $M_{jj'} = 0$ then no neighbor of v_s belongs to S'_j, as required. Similarly, condition $s'.2$ ensures on one hand that the conditions imposed by the parameters $\{\alpha_{ij}\}$ in w are satisfied also by the solution of u, and, on the other hand, that if $j' \in \tilde{J}$ and $1 \leq j \leq r$ are such that $M_{jj'} = 1$ then all vertices in S'_j are adjacent to v_s, as required. These conditions strongly use that the order and the partition are consistent. Finally, conditions $s.3$–$s.4$, and $s'.3$–$s'.6$ ensure that the solution does not violate weight constraints.

Moreover, the difference of weight of the solution $\{S_j\}_{1 \leq j \leq r}$ with respect to $\{S'_j\}_{1 \leq j \leq r}$ is exactly $\sum_{1 \leq i \leq t; j \in \tilde{J}} c_{ij} w_i(v_s)$.

In that way, a directed path in the digraph corresponds to an assignment of vertices of the graph to lists of sets and its weight is the value of the objective function for the corresponding assignment.

The digraph has a polynomial number of nodes and can be built in polynomial time. Since it is acyclic, both the longest path and shortest path can be computed in linear time in the size of the digraph by topological sorting.

Remark 1: The thinness is not preserved by the complement operation of graphs (see for instance Theorem 1). However, for every fixed k, the same problem can be solved for the complement $\overline{G}$ of a k-thin graph G, simply by swapping ones and zeroes in the restriction matrix M.

4.2 Hardness results for relaxed problems

In the previous section, LINEAR WEIGHTED LIST MATRIX PARTITION (LWLMP) was defined as an optimization problem. In this section we will only refer to the decision version of the problem in order to show hardness results.

After we published our algorithm for the LWLMP problem assuming a k-thin representation is given, subsequent work [3] showed that the conditions for the algorithm telling that some parameters have to be bounded are, in some sense, necessary: the relaxation of any of these conditions make the problem **NP**-hard for interval graphs.

The same paper also proved that the LWLMP problem is $\mathbf{W}[1]$-hard with respect to r, which means that there is no **FPT** algorithm under the standard complexity assumption that $\mathbf{FPT} \neq \mathbf{W}[1]$. Note that our algorithm has time complexity $n^{O(2^r)}$ for interval graphs, and we now know that we cannot hope for a $f(r) \cdot n^{O(1)}$ algorithm, where f is any function depending only on r.

We will proceed to describe these results, which depend on the WEIGHTED LOCALLY BOUNDED LIST COLORING (WLBLC) problem, defined as follows:
Instance:

- A graph $G = (V, E)$ with vertex weights w.
- Integers q and k.
- A partition $V^1, \ldots, V^q$ of V.
- A list of qk integer bounds W_{ij} with $\sum_j W_{ij} = \sum_{v \in V^i} w(v)$ for all i.
- Each vertex has a list $L(v) \subseteq \{1, \ldots, k\}$.

Question: decide whether there exists a k-coloring of G, so that the sum of weights of vertices with color j in V^i equals W_{ij} for all (i, j), and the color of v belongs to $L(v)$ for all v.

This problem is **NP**-complete, since it is trivially in **NP** and generalizes the 3-COLORING problem. But note that the LWLMP problem is a generalization of this one (relaxing the bounds on some of the parameters), because of the following mapping:

- The strategy is to have $r = qk$ sets, where S_{ij} will contain the vertices in partition i and color class j.
- Matrix M consists of 0 for entries of the same color class, and the rest of entries are $*$.
- If $v \in V^i$, $L(v) = \{a_1, \ldots, a_d\}$ is mapped to $\{a_{i1}, \ldots, a_{id}\}$.
- Restrictions: the S_{ij} form a partition of V; and the weighted cardinality of S_{ij} is W_{ij}. This implies that we only need $p = 2$, where b_1 is the constant function 1, and $b_2 = w$.

The paper proves that the WLBLC problem remains **NP**-complete under a number of constraints, which in turn imply similar statements for relaxed versions of the LWLMP problem.

Theorem 21 (1 of [3]): WLBLC is weakly **NP**-complete in edgeless graphs, even if $k = 1$, $q = 1$ and $|L(v)| = 2$ for all v.

Corollary 22: If b_i in LWLMP is not polynomially bounded in n, the problem is weakly **NP**-complete in edgeless graphs, even if $r = 1$, $p = 2$, $|L(v)| = 2$ for all v and without matrix M and a linear function.

Theorem 23 (2 of [3]): WLBLC is **NP**-complete in edgeless graphs, even if $q = 1$, without L, and w is polynomially bounded in n.

Corollary 24: If r in LWLMP is not fixed, the problem is **NP**-complete in edgeless graphs, even if $p = 2$, and without M and a linear function.

Theorem 25 (3 of [3]): WLBLC is **NP**-complete in star forests, even if $q = 1$ and without w.

Corollary 26: If r in LWLMP is not fixed, the problem is **NP**-complete in star forests, even if $p = 1$, b constant, and without L, M and a linear function.

Theorem 27 (4 of [3]): WLBLC is **NP**-complete in linear forests, even if $q = 1$, $|L(v)| = 2$ for all v, and without w.

Corollary 28: If r in LWLMP is not fixed, the problem is **NP**-complete in linear forests, even if $p = 1$, b constant, and $|L(v)| = 2$ for all v, and without M and a linear function.

Theorem 29 (5 of [3]): WLBLC is **NP**-complete in star forests, even if $k = 2$, $|L(v)| = 2$ for all v, and without w.

Corollary 30: If k in LWLMP is not fixed, the problem is **NP**-complete in star forests, even if $p = 1$, b constant, and $|L(v)| = 2$ for all v, and without M and a linear function.

Theorem 31 (6 of [3]): WLBLC is **NP**-complete in linear forests, even if $k = 2$, $|L(v)| = 2$ for all v, and without w.

Corollary 32: If k in LWLMP is not fixed, the problem is **NP**-complete in linear forests, even if $p = 1$, b constant, and $|L(v)| = 2$ for all v, and without M and a linear function.

Note that edgeless graphs, linear forests and star forests are interval graphs (indeed, edgeless graphs and linear forests are also proper interval graphs), so these corollaries allow us to conclude that several possible relaxations of our problem are **NP**-complete for graphs with thinness 1, and some of them for graphs with proper thinness 1.

The paper also proves that WLBLC is **W**[1]-hard with respect to k in edgeless graphs, with $q = 1$ and without list coloring. So that result also holds for our problem, with respect to r and restricted to interval graphs.

4.3 Extending the family of problems solvable on graphs with bounded proper thinness

We start by the following observation: in a proper k-thin representation of a graph G, with ordering $<$ of V, namely $v_1 < \cdots < v_n$, and partition of V into k classes $V^1, \ldots, V^k$, for each pair of vertices $v_s < v_r$ that are in the same class, $N[v_s] \cap \{v_1, \ldots, v_s\} \supseteq N[v_r] \cap \{v_1, \ldots, v_s\}$. This allows us to handle other kinds of restrictions as for example domination type constraints.

Namely, if we are considering the subgraph G_s of G induced by $\{v_1, \ldots, v_s\}$ but we "keep in mind" that we still need to dominate some of the vertices in $\{v_{s+1}, \ldots, v_n\}$ with vertices of G_s, we can summarize these conditions into at most k of them (each imposed by vertices of $\{v_{s+1}, \ldots, v_n\}$ in each partition class).

For graphs with bounded proper thinness k, when the proper k-thin representation of the graph is given, we can add now to the instance (with respect to Section 4.1) this kind of restrictions:

- $l_{ij(N)} \leq |S_i \cap N(v)| \leq u_{ij(N)} \quad \forall v \in S_j$, such that $l_{ij(N)} \in \{0, 1\}$ and $u_{ij(N)} \in \{1, \infty\}$ (it can be $i = j$), $1 \leq i, j \leq r$.
- $l_{ij[N]} \leq |S_i \cap N[v]| \leq u_{ij[N]} \quad \forall v \in S_j$, such that $l_{ij[N]} \in \{0, 1\}$ and $u_{ij[N]} \in \{1, \infty\}$ (it can be $i = j$), $1 \leq i, j \leq r$.

In this way the framework includes domination-type problems in the literature and their weighted versions, such as MINIMUM WEIGHTED INDEPENDENT DOMINATING SET, MINIMUM WEIGHTED TOTAL DOMINATING SET, MINIMUM PERFECT DOMINATING SET and MINIMUM WEIGHTED EFFICIENT DOMINATING SET, and b-COLORING [39] with fixed number of colors.

We will keep the notation of Section 4.1 and describe how to modify the algorithm in order to take into account the new restrictions. Now the vertex order and the partition of G are strongly consistent.

Each state now will be augmented with some new parameters:

- a family of nonnegative parameters $\{\gamma_{ij}\}_{1 \leq i \leq k; 1 \leq j \leq r}$, meaning that the last γ_{ij} vertices of V^i have already a neighbor in S_j (of index higher than them); there are kr such parameters and each of them may take a nonnegative value at most $n - 1$.
- a family of nonnegative parameters $\{\gamma^2_{ij}\}_{1 \leq i \leq k; 1 \leq j \leq r}$, meaning that the last γ^2_{ij} vertices of V^i have already two neighbors in S_j (of index higher than them); there are kr such parameters and each of them may take a nonnegative value at most $n - 1$.
- a family of nonnegative parameters $\{\lambda_{ijc}\}_{1 \leq i, c \leq k; 1 \leq j \leq r}$, meaning that, for each value $1 \leq c \leq k$, S_j has to contain at least one vertex in the set that is the union over $1 \leq i \leq k$ of the last λ_{ijc} vertices of V^i (if the union is empty, this means no restriction associated with (c, S_j)); there are $k^2 r$ such parameters and each of them may take a nonnegative value at most $n - 1$.

The total number of states is then multiplied by at most n^{k^2r+2kr}, that keeps it polynomial in n, since k and r are constant.

The value of all these parameters in the only node of the layer X_n of the digraph is zero.

Now the problems associated with the nodes of X will have the additional constraints associated with the new restrictions and the parameters $\{\gamma_{ij}\}$, $\{\gamma^2_{ij}\}$, and $\{\lambda_{ijc}\}$.

Let us describe the additional conditions for the arcs of the digraph, whose labels and weights are still the same as in Section 4.1.

Let w be a node with parameters $(1, \ldots, \{\gamma_{ij}\}, \{\gamma^2_{ij}\}, \{\lambda_{ijc}\})$.

Let $1 \leq \ell \leq k$ such that $v_1 \in V^\ell$. For each $\tilde{J} \in L(v_1)$ (in particular $\tilde{J} \subseteq \{1, \ldots, r\}$) satisfying 1.1–1.5, and such that:

1.6 For each $1 \leq i \leq r$, $j \in \tilde{J}$, such that $l_{ij(N)} = 1$, $\gamma_{\ell i} > 0$.

1.7 For each $i \notin \tilde{J}$, $j \in \tilde{J}$, such that $l_{ij[N]} = 1$, $\gamma_{\ell i} > 0$.

1.8 For each $1 \leq i \leq r$, $j \in \tilde{J}$, such that $u_{ij(N)} = 1$ or $u_{ij[N]} = 1$, $\gamma^2_{\ell i} = 0$.

1.9 For each $i, j \in \tilde{J}$, such that $u_{ij[N]} = 1$, $\gamma_{\ell i} = 0$.

1.10 For each $j \notin \tilde{J}$ and for each $1 \leq c \leq k$, $\lambda_{\ell jc} = 0$.

We add an arc from x_0 to w, labeled by $\tilde{J}$ and of weight $\sum_{1 \leq i \leq t; j \in \tilde{J}} c_{ij} w_i(v_1)$. If no $\tilde{J}$ satisfies conditions 1.1–1.10, no arc ending in w is added. If more than one arc x_0w was added, we can keep only the one with maximum (resp. minimum) weight if we are solving a maximization (resp. minimization) problem.

Note that if we add the arc x_0w labeled by $\tilde{J}$, then the solution $S_j = \{v_1\}$ for $j \in \tilde{J}$, $S_j = \emptyset$ for $j \notin \tilde{J}$ has weight $\sum_{1 \leq i \leq t; j \in \tilde{J}} c_{ij} w_i(v_1)$ and satisfies the state described by w: conditions 1.1–1.5 ensure the properties required in Section 4.1; conditions 1.6–1.9 ensure the validity of the two new families of restrictions about lower and upper bounds of neighbors of vertices of one set in other set, and condition 1.10 ensures that the restrictions imposed by the parameters $\{\lambda_{ijc}\}$ are satisfied.

Let w be a node with parameters $(s, \{l_{iJ\cap}\}, \{u_{iJ\cap}\}, \{l_{iJ\cup}\}, \{u_{iJ\cup}\}, \{\alpha_{ij}\}, \{\beta_{ij}\}, \{\gamma_{ij}\}, \{\gamma^2_{ij}\}, \{\lambda_{ijc}\})$, $1 < s \leq n$.

Let $1 \leq \ell \leq k$ such that $v_s \in V^\ell$. For each $\tilde{J} \in L(v_s)$ satisfying s.1–s.4, and such that:

s.5 For each $1 \leq i \leq r$, $j \in \tilde{J}$, such that $u_{ij(N)} = 1$ or $u_{ij[N]} = 1$, $\gamma^2_{\ell i} = 0$.

s.6 For each $i, j \in \tilde{J}$, such that $u_{ij[N]} = 1$, $\gamma_{\ell i} = 0$.

s.7 For each $j \notin \tilde{J}$ and for each $1 \leq c \leq k$, either $\lambda_{\ell jc} = 0$, or $\lambda_{\ell jc} > 1$, or there exists $1 \leq i \leq k$, $i \neq \ell$, such that $\lambda_{ijc} > 0$ (i.e., the union over $1 \leq i \leq k$ of the last λ_{ijc} vertices of V^i is not $\{v_s\}$).

s.8 For each $1 \leq i \leq r$ such that $\gamma_{\ell i} = 0$ and there exists $j \in \tilde{J}$ such that $l_{ij(N)} = 1$, $N(v_s) \cap \{1, \ldots, s-1\} \neq \emptyset$.

$s.9$ For each $i \notin \tilde{J}$ such that $\gamma_{\ell i} = 0$ and there exists $j \in \tilde{J}$ such that $l_{ij[N]} = 1$, $N(v_s) \cap \{1, \dots, s-1\} \neq \emptyset$.

Let $\{\lambda^0_{ijc}\}_{1 \leq i,c \leq k; 1 \leq j \leq r}$ be defined this way: for every $j \in \tilde{J}$ and every $1 \leq c \leq k$ such that $\lambda_{\ell jc} > 0$, let $\lambda^0_{ijc} = 0$ for every $1 \leq i \leq k$; for every $j \in \tilde{J}$ and every $1 \leq c \leq k$ such that $\lambda_{\ell jc} = 0$, let $\lambda^0_{ijc} = \lambda_{ijc}$ for every $1 \leq i \leq k$; for every $j \notin \tilde{J}$ and every $1 \leq c \leq k$, let $\lambda^0_{\ell jc} = \max\{0, \lambda_{\ell jc} - 1\}$ and let $\lambda^0_{ijc} = \lambda_{ijc}$ for every $1 \leq i \leq k$, $i \neq \ell$.

Let $\{\lambda^1_{jc}\}_{1 \leq c \leq k; 1 \leq j \leq r}$ be defined as $\lambda^1_{jc} = 0$ if $\lambda^0_{ijc} = 0$ for every $1 \leq i \leq k$, $\lambda^1_{jc} = 1$ otherwise.

We add an arc from u to w, labeled by $\tilde{J}$ and of weight $\sum_{1 \leq i \leq t; j \in \tilde{J}} c_{ij} w_i(v_s)$, where u has parameters $(s-1, \{l'_{iJ\cap}\}, \{u'_{iJ\cap}\}, \{l'_{iJ\cup}\}, \{u'_{iJ\cup}\}, \{\alpha'_{ij}\}, \{\beta'_{ij}\}, \{\gamma'_{ij}\}, \{{\gamma^2}'_{ij}\}, \{\lambda'_{ijc}\})$, satisfies conditions $s'.2$–$s'.6$, and:

$s'.7$ For each $1 \leq i \leq r$ such that $\gamma_{\ell i} = 0$ and there exists $j \in \tilde{J}$ such that $l_{ij(N)} = 1$, if $\lambda^1_{i\ell} = 0$, then $\lambda'_{j'i\ell} = |N(v_s) \cap V^{j'} \cap \{1, \dots, s-1\}|$ for each $1 \leq j' \leq k$; otherwise, $\lambda'_{j'i\ell} = \lambda^0_{j'i\ell}$ for every $1 \leq j' \leq k$ (recall that, by the observations above about proper thinness, $\lambda^0_{j'i\ell} = \min\{\lambda^0_{j'i\ell}, |N(v_s) \cap V^{j'} \cap \{1, \dots, s-1\}|\}$).

$s'.8$ For each $i \notin \tilde{J}$ such that $\gamma_{\ell i} = 0$ and there exists $j \in \tilde{J}$ such that $l_{ij[N]} = 1$, if $\lambda^1_{i\ell} = 0$, then $\lambda'_{j'i\ell} = |N(v_s) \cap V^{j'} \cap \{1, \dots, s-1\}|$ for each $1 \leq j' \leq k$; otherwise, $\lambda'_{j'i\ell} = \lambda^0_{j'i\ell}$ for every $1 \leq j' \leq k$.

$s'.9$ For each i, j, c not comprised in conditions $s'.7$ and $s'.8$, $\lambda'_{ijc} = \lambda^0_{ijc}$.

$s'.10$ Let $1 \leq j \leq r$. If there exists $j' \in \tilde{J}$ satisfying at least one of the following:

- $M_{jj'} = 0$
- ($u_{jj'(N)} = 1$ or $u_{jj'[N]} = 1$) and $\gamma_{\ell j} > 0$
- $j \in \tilde{J}$ and $u_{jj'[N]} = 1$

then, $\beta'_{\ell j} = \max\{\beta_{\ell j} - 1, |N(v_s) \cap V^\ell \cap \{1, \dots, s-1\}|\}$, and for $1 \leq i \leq k$, $i \neq \ell$, $\beta'_{ij} = \max\{\beta_{ij}, |N(v_s) \cap V^i \cap \{1, \dots, s-1\}|\}$. Otherwise, $\beta'_{\ell j} = \max\{0, \beta_{\ell j} - 1\}$, and for $1 \leq i \leq k$, $i \neq \ell$, $\beta'_{ij} = \beta_{ij}$.

$s'.11$ For each $j \in \tilde{J}$: if $|N(v_s) \cap V^\ell \cap \{1, \dots, s-1\}| \geq \gamma_{\ell j} - 1$, then $\gamma'_{\ell j} = |N(v_s) \cap V^\ell \cap \{1, \dots, s-1\}|$ and ${\gamma^2}'_{\ell j} = \max\{0, \gamma_{\ell j} - 1\}$; otherwise, $\gamma'_{\ell j} = \max\{0, \gamma_{\ell j} - 1\}$ and ${\gamma^2}'_{\ell j} = \max\{\gamma^2_{\ell j} - 1, |N(v_s) \cap V^\ell \cap \{1, \dots, s-1\}|\}$.

$s'.12$ For each $j \in \tilde{J}$, $1 \leq i \leq k$, $i \neq \ell$: if $|N(v_s) \cap V^i \cap \{1, \dots, s-1\}| \geq \gamma_{ij}$, then $\gamma'_{ij} = |N(v_s) \cap V^i \cap \{1, \dots, s-1\}|$ and ${\gamma^2}'_{ij} = \gamma_{ij}$; otherwise, $\gamma'_{ij} = \gamma_{ij}$ and ${\gamma^2}'_{ij} = \max\{\gamma^2_{ij}, |N(v_s) \cap V^i \cap \{1, \dots, s-1\}|\}$.

If no $\tilde{J}$ satisfies conditions $s.1$–$s.9$, no arc ending in w is added. If more than one arc from the same vertex u to w was added, we can keep only the one with maximum (resp. minimum) weight if we are solving a maximization (resp. minimization) problem.

That is, if an arc is added, the arc corresponds to the choice of adding the vertex v_s to the sets $\{S_j\}_{j \in \tilde{J}}$, the conditions required imply that the choice is valid for w in the case that the state described by u admits a solution, the label of the arc keeps track of the choice made, and the cost corresponds to the weight that the choice adds to the objective function.

Note that if we add the arc uw labeled by $\tilde{J}$, then for a solution $\{S'_j\}_{1\leq j\leq r}$ for G_{s-1} satisfying the state described by u, then the solution $\{S_j\}_{1\leq j\leq r}$ for G_s such that $S_j = S'_j \cup \{v_s\}$ for $j \in \tilde{J}$, $S_j = S'_j$ for $j \notin \tilde{J}$ satisfies the state described by w.

Condition $s'.10$ ensures on one hand that the conditions imposed by the parameters $\{\beta_{ij}, u_{ij(N)}, u_{ij[N]}\}$ in w are satisfied by the solution of u, and, on the other hand, that if $j' \in \tilde{J}$ and $1 \leq j \leq r$ are such that $M_{jj'} = 0$ then no neighbor of v_s belongs to S'_j, as required. Conditions $s'.7$–$s'.9$ together with $s.7$–$s.9$ define parameters $\{\lambda'_{ijc}\}$ in u in order to guarantee in w both the conditions imposed by the lower bounds $\{l_{ij(N)}, l_{ij[N]}\}$ and those imposed by the parameters $\{\lambda_{ijc}\}$. Finally, conditions $s'.11$ and $s'.12$ properly update the definition of parameters $\{\gamma'_{ij}, {\gamma^{2}}'_{ij}\}$ according to the choice $\tilde{J}$ for v_s. Conditions $s'.2$–$s'.6$ were analyzed above in Section 4.1.

As in that case, the difference of weight of the solution $\{S_j\}_{1\leq j\leq r}$ with respect to $\{S'_j\}_{1\leq j\leq r}$ is exactly $\sum_{1\leq i\leq t; j\in\tilde{J}} c_{ij} w_i(v_s)$.

In that way, a directed path in the digraph corresponds to an assignment of vertices of the graph to lists of sets and its weight is the value of the objective function for the corresponding assignment.

The digraph has a polynomial number of nodes and can be built in polynomial time. Since it is acyclic, both the longest path and shortest path can be computed in linear time in the size of the digraph by topological sorting.

5. CONCLUSIONS AND OPEN PROBLEMS

We described a wide family of combinatorial optimization problems that can be solved in polynomial time on classes of bounded thinness and bounded proper thinness. We think that some restrictions can be further generalized (especially the domination type ones), with more involved sets of parameters and transition rules. We tried to keep it as simpler as possible, yet including many of the classical combinatorial optimization problems in the literature.

We also proved a number of theoretical results, some of them related to the recognition problem for the classes, others relating the concept of thinness and proper thinness to other known graph parameters, and analyzing their behavior under the graph operations union, join, and Cartesian product (this last result has been extended to other graph products in a subsequent work).

Some open problems are the following.

- Characterize (proper) k-thin graphs by minimal forbidden induced subgraphs (or at least within some graph class, we did it for thinness in cographs).
- Find sufficient conditions, for instance a family subgraphs to forbid as induced subgraphs, for a graph to be (proper) k-thin, even if these graphs are not necessarily forbidden induced subgraphs for (proper) k-thin graphs. These kind of results have been obtained for MIM-width in [41].
- Study the behavior of thinness under other graph operators like, for example, graph powers and the clique graph.
- What is the complexity of computing the (proper) thinness of a graph? Or deciding if it is at most k for some fixed values k?
- Can we develop some randomized algorithm to test just a subset of vertex orderings and obtain with high probability an approximation of the (proper) thinness?
- Given a partition of the vertex set into a *fixed* number k of classes, what is the complexity of deciding if there is a (strongly) consistent order for the vertices w.r.t. that partition (and finding it)? (We have proved that for an arbitrary number of classes the problems are NP-complete, and we have solved in polynomial time the symmetric problem, i.e., given the ordering, find a minimum (strongly) consistent partition.)

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