Informe año sabático

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Resumen

El presente informe resume las tareas desarrolladas durante el medio ao sabático desarrollado en el período mencionado. Durante el mismo visté la New York University at Shanghai (cuatro meses) y Stanford University (15 días).

1. Visitas académicas

1. New York Univesity at Shanghai, 1 de Septiembre de 2016 al 22 de Diciembre de 2016.
2. Stanford University, 6 de Enero de 2017 a 21 de Enero de 2017.

2. Trabajo realizado

Durante la visita a NYU Shanghai terminamos la redacción de los artículos [13, 14]. Ambos han sido enviados para ser considerados para publicación en revistas internacionales con referato. Asimismo, recibimos un referato positivo del artículo [16] y trabajamos en las correcciones solicitadas por el referee que demandaron mucho más trabajo del esperado. El artículo será publicado en Stochastic Processes and their Applications.

Paralelamente, trabajamos junto con Vladas Sidoravicius (NYU-SH), Ruojun Huang (Stanford) y Amir Dembo (Stanford) en un nuevo proyecto denominado *Lattice-free growth models* [8] que describimos en la siguiente sección. Este trabajo fue lo que motivó tambin mi visita a Stanford en el mes de Enero de 2017.

El tercer proyecto desarrollado en este período se denomina *Tumor growth Markov models* [15]. Está siendo realizado en colaboración con Krishnamurti Ravishankar (New York State University), quien visitó NYU-SH durante el mismo período.

Por último, trabajamos junto con Matthieu Jonckheere y Julián Martínez en el proyecto *Hydrodynamic limit and selection principle for a branching-selection particle system and the F-KPP equation* [12] que se encuentra en la etapa final de redacción.

3. Otros trabajos publicados

En este período también se publicaron los artículos [1, 10]

4. Proyectos

4.1. Dinmica de poblaciones con seleccin natural

La dinmica de poblaciones ha sido objeto de innumerables estudios desde el punto de vista matemtico. Entre ellos, una gran parte se ha dedicado al estudio de modelos que contemplan el fenmeno de seleccin natural: el proceso gradual por el cual rasgos biolgicos hereditarios se vuelven ms o menos comunes en una poblacin como una funcin del efecto de rasgos heredados en el xito reproductivo diferencial de los organismos que interactan con su entorno. Este mecanismo es clave para entender la evolucin.

Con la irrupcin de la gentica de poblaciones y la sntesis evolutiva moderna en los aos treinta del siglo pasado, el fenmeno de la seleccin natural pas a ser ampliamente aceptado y junto con ello, los modelos matemticos comenzaron a emerger. Tal es el caso de la renombrada ecuacin de F-KPP, introducida por Fisher en el ao 1937 e independientemente por Kolmogorov, Petrovskii y Piskunov en el mismo ao [11, 18]. La ecuacin se lee

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + rf(v), \quad x \in \mathbb{R},\, t > 0, \\ v(0,x) &= v_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

Asumamos para simplificar que f es de la forma $f(s) = s^2 - s$ y nos restringimos a datos iniciales v_0 que son funciones de distribucin de medidas de probabilidad en $\mathbb{R}$.
Ms de dos mil artculos en revistas de primer nivel se han dedicado a estudiar distintos aspectos de esta ecuacin. Tanto Fisher por un lado como Kolmogorov, Petrovskii y Piskunov por el otro, probaron que esta ecuacin admite un nmero infinito de soluciones de tipo onda viajera $v(t,x) = w_c(x - ct)$ que viajan a velocidad c. Este hecho es de alguna forma inesperado desde el punto de vista del modelado ya que uno esperara una nica onda viajera a la que converge la solucin independiente del dato inicial con el que se comienza. En palabras de Fisher [11, p. 359]

> "Common sense would, I think, lead us to believe that, though the velocity of advance might be temporarily enhanced by this method, yet ultimately, the velocity of advance would adjust itself so as to be the same irrespective of the initial conditions. If this is so, this equation must omit some essential element of the problem, and it is indeed clear that while a coefficient of diffusion may represent the biological conditions adequately in places where large numbers of individuals of both types are available, it cannot do so at the extreme front and back of the advancing wave, where the numbers of the mutant and the parent gene respectively are small, and where their distribution must be largely sporadic".

Fisher propuso una forma de superar esta dificultad, relacionada con la representacin probabilstica dada mas adelante por McKean [21], basada en Movimientos Brownianos que ramifican (BBM). El principio general detrs de esto es que los efectos microscpicos deben ser tenidos en cuenta para describir correctamente el fenmeno fsico.
Con el mismo punto de vista, Brunet, Derrida y sus coautores [4, 5, 6, 7] comenzaron en los aos noventa un estudio meticuloso del efecto del ruido microscpico en la propagacin de frentes conjeturando cierta universalidad de este fenmeno y sosteniendo esta conjetura con argumentos heursticos y simulaciones numricas. Esto dio lugar a una enorme cantidad de trabajos que estudian el cambio del comportamiento del frente de propagacin cuando los efectos microscpicos son tenidos en cuenta. Tanto con argumentos heursticos y numricos [4, 5, 6, 7, 17] como as tambin con pruebas rigurosas [2, 3, 9, 19, 20].
En este contexto, estudiamos una serie de sistemas de partculas que caen en esta clase de universalidad. Estudiamos sus propiedades y probamos que efectivamente pertenecen a esta clase, denominada *clase de universalidad de la ecuacin* F-KPP. Todos ellos son sistemas con N partculas en donde se presenta un mecanismo de ramificacin, uno de mutacin y uno de seleccin. A continuacin mencionamos dos de ellos.

Supervivencia del ms apto Consideramos N partculas que se mueven de acuerdo a Movimientos Brownianos independientes. Cada una de ellas est asociada a un Proceso de Poisson de parmetro uno. A diferencia del modelo anterior, en los momentos en que suena un reloj de Poisson, la partcula involucrada elige a otra de las $N-1$ partculas restantes y la menor de ellas dos toma la posicin de la mayor de ambas. Ciertamente este es un mecanismo de ramificacin-seleccin y se lo interpreta como el que el ms apto se ramifica y el menos apto muere. Para este modelo probamos que la medida empírica converge, cuando $N \to \infty$ a una ecuacin macroscpica lmite que en este caso viene dada por la F-KPP (1). Es un desafo probar que lo mismo ocurre con procesos de Lévy ms generales. También probamos que para cada N fijo, la nube de partículas se mueve asintóticamente (cuando $t \to \infty$) a una velocidad v_N que es determinística y sólo depende de N. Más aún, estas velocidades son crecientes y $v_N \nearrow \sqrt{2}$, la velocidad minimal de la F-KPP.

Modelos de crecimiento de tumores En la misma lnea, junto con Krishnamurti Ravishankar (NYU-Shanghai) trabajamos en el siguiente modelo. Denotamos con $(\eta_t(x),\, x \in \mathbb{N})$ al nmero de partculas en el sitio x a tiempo t.
Sea $Q = (q(x,y),\, x, y \in \mathbb{N})$ una matriz de tasas de un proceso de Markov a puro salto en $\mathbb{N}$. $q(x,0) = -(q(x,x) + \sum_x q(x,y))$ es la tasa de absorcin desde el estado x.

1. Las partculas evolucionan en forma independiente segn la matriz de tasas Q
2. Las partculas son absorbidas en un estado que llamamos 0 a tasa $\sum_x q(x,0)$.
3. Se crean partculas en un estado, que llamamos 1, a tasa $\sum_x \beta(x)\eta_t(x)$.

Un caso de particular inters es cuando Q corresponde a un proceso de nacimiento y muerte. Este modelo ha sido utilizado para modelar el crecimiento de tumores bajo tratamiento. Nos interesa tambin particularmente el caso $\beta(x) = mx^\alpha$ for $0 < \alpha \leq 1$ por cuestiones de modelado.
En [22] se prueba que bajo estas condiciones el proceso es supercrtico si y slo si

$$A_0 := \int_0^\infty \gamma_1(t)\, dt > 1.$$

Donde

$$\gamma_1(t) = \sum_x \beta(x)\mathbb{E}(1\{X_t^1 = x\})$$

es la tasa de creacin media. Ms precisamente,

1. $A_0 < 1 \Rightarrow \mathbb{E}\eta_t(x) \to 0$ para todo $x \in \mathbb{N}$.
2. $A_0 = 1 \Rightarrow \mathbb{E}\eta_t(x)$ converge a una constante.
3. $A_0 > 1 \Rightarrow \mathbb{E}\eta_t(x)$ crece exponencialmente.

Para este proceso esperamos probar

Caso supercrtico $A_0 > 1$

1. Existe $\sigma > 0$ tal que $|\eta_t|e^{-\sigma t} \to c > 0$ cuando $t \to \infty$.
2. $\eta_t(\cdot)/|\eta_t| \to \nu$ donde ν es una medida de probabilidad en $\mathbb{N}$ que resulta ser la distribucin cuasi-estacionaria minimal de Q.

Caso subcrtico $A_0 < 1$

1. Extincin casi segura, es decir $\mathbb{P}(|\eta_t| > 0 \,\text{para todo}\, t \geq 0) = 0$.
2. Existencia de medida cuasi estacionaria, es decir, existe una medida de probabilidad μ en $\mathbb{N}^\mathbb{N}$ tal que $\mathbb{P}_\mu(\eta_t \in A \,|\, |\eta_t| \neq 0) = \mu(A)$ para todo A boreliano de $\mathbb{N}^\mathbb{N}$.

El caso crtico $A_0 = 1$ es, como de costumbre, el ms dificil, pero esperamos poder decir algo tambin en esta situacin.

4.2. Modelos de crecimiento geométrico

Dado un parámetro $\varepsilon > 0$, una función de densidad $F(\gamma, \cdot)$ y un núcleo $g \colon S^{d-1} \to \mathbb{R}_{\geq 0}$ simétrico y con $\|g\|_{L^1(S^{d-1})} = 1$ consideramos el siguiente proceso de Markov en el espacio de dominios estrellados con frontera parametrizada por γ. Dado γ_t se espera un tiempo exponencial τ de media ε y luego se sortea un punto ξ en la esfera de acuerdo a la densidad $F(\gamma_t, \theta)d\theta$. El nuevo estado viene dado por

$$\gamma_{t+\tau}(\theta) = \gamma_t(\theta) + \sqrt{\varepsilon} g(\frac{\theta - \xi}{\sqrt{\varepsilon}} y_{\gamma_t}).$$

El parámetro y_γ es elegido para garantizar que el volumen medio añadido es ε. Dada la libertad para elegir el punto ξ, este modelo incluye una gran cantidad de fenómenos distintos. Nos interesan especialmente alguno de ellos como ser cuando $F(\gamma, \cdot)$ es (i) la densidad de la medida armónica de γ, (ii) inversamente proporcional a la distancia al origen, (iii) independiente de γ, (iv) una función monótona de la curvatura.
Para estos procesos estudiamos el comportamiento de γ_t cuando $t \to \infty$ y cuando $\to 0$. Para el primer caso, para una clase muy amplia de densidades F podemos probar que el proceso es ergódico y por ende converge en distribución cuando $t \to \infty$. Estamos trabajando en una prueba del límite hidrodinámico para t fijo. La ecuación límite viene dada por

$$\frac{d}{dt}\gamma_t = \frac{1}{y_{\gamma_t}} F(\gamma_t, \cdot).$$

Observar que no depende de g. Se puede ver que para una familia amplia de densidades F las soluciones de esta ecuación convergen a una bola cuando $t \to \infty$. Esto da una clase de universalidad para este tipo de modelos. Este trabajo esta siendo llevado a cabo en colaboración con A. Dembo, R. Huang y V. Sidoravicius [8].

Referencias

[1] A. Asselah, P. A. Ferrari, P. Groisman, and M. Jonckheere. Fleming-Viot selects the minimal quasi-stationary distribution: the Galton-Watson case. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(2):647–668, 2016.

[2] J. Bérard and J. Gouéré. Brunet-Derrida behavior of branching-selection particle systems on the line. *Comm. Math. Phys.*, 298(2):323–342, 2010.

[3] M. Bieniek, K. Burdzy, and S. Pal. Extinction of Fleming-Viot-type particle systems with strong drift. *Electron. J. Probab.*, 17(11):1–15, 2011.

[4] E. Brunet and B. Derrida. Shift in the velocity of a front due to a cutoff. *Phys. Rev. E (3)*, 56(3, part A):2597–2604, 1997.

[5] É. Brunet and B. Derrida. Effect of microscopic noise on front propagation. *J. Statist. Phys.*, 103(1-2):269–282, 2001.

[6] E. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Noisy traveling waves: effect of selection on genealogies. *Europhys. Lett.*, 76(1):1–7, 2006.

[7] É. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Effect of selection on ancestry: an exactly soluble case and its phenomenological generalization. *Phys. Rev. E (3)*, 76(4):041104, 20, 2007.

[8] A. Dembo, P. Groisman, R. Huang, and V. Sidoravicius. Lattice-free growth models. *In preparation*, 2017.

[9] R. Durrett and D. Remenik. Brunet-derrida particle systems, free boundary problems and Wiener-Hopf equations. *Ann. Probab.*, 39(6):2043–2078, 2011.

[10] R. Fernández, P. Groisman, and S. Saglietti. Stability of gas measures under perturbations and discretizations. *Rev. Math. Phys.*, 28(10):1650022, 46, 2016.

[11] R. Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7:353–369, 1937.

[12] P. Groisman, M. Jonchkheere, and J. Martínez. Hydrodynamic limit and selection principle for a branching-selection particle system and the f-kpp equation. *In preparation*, 2017.

[13] P. Groisman and M. Jonckheere. Front propagation and quasi-stationary distributions for one-dimensional Lévy processes. *arXiv preprint arXiv:1609.09338*, 2016.

[14] P. Groisman and M. Jonckheere. Front propagation and quasi-stationary distributions: two faces of the same coin. *Preprint*, 2016.

[15] P. Groisman and R. Krishnamurti. Tumor growth Markov models. *In preparation*, 2017.

[16] P. Groisman, S. Saglietti, and N. Saintier. Metastability for small random perturbations of a pde with blow-up. *arXiv preprint arXiv:1501.01724*, 2015.

[17] D. A. Kessler and H. Levine. Fluctuation-induced diffusive instabilities. *Nature*, 394:556–558, 1998.

[18] A. Kolmogorov, I. Petrovsky, and N. Piscounov. Etude de l'equation de la diffusion avec croissance de la quantite de matiere et son application a un probleme biologique. *Bull. Univ. Etat Moscou*, A 1:1(25), 1937.

[19] P. Maillard. Branching brownian motion with selection. *arXiv preprint arXiv:1210.3500*, 2012.

[20] P. Maillard. Branching Brownian motion with selection of the N right-most particles: An approximate model, Feb. 2012.

[21] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323–331, 1975.

[22] L. Triolo. Space structures and different scales for many-component biosystems. *Markov Process. Related Fields*, 11(2):389–404, 2005.

Front propagation and quasi-stationary distributions: two faces of the same coin

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Dedicated to Chuck Newman on the occasion of his 70th birthday.

Abstract

We analyze the connection between front propagation and quasi-stationary distributions in translation invariant one-dimensional Markov processes. We describe the link between them through the microscopic models known as Branching Brownian Motion with selection and Fleming-Viot.

1 Introduction

A selection mechanism in front propagation can be thought of as follows: a certain phenomenology is described through an equation that admits an infinite number of traveling-wave solutions, but there is only one which has a physical meaning, the one with minimal velocity. Under mild assumptions on initial conditions, the solution converges to this minimal-velocity traveling wave. The most remarkable example of this fact is the celebrated F-KPP equation (for Fisher, Kolmogorov-Petrovskii-Piskunov)

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + rg(v), \quad x \in \mathbb{R},\, t > 0, \\ v(0,x) &= v_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

Assume for simplicity that g has the form $g(s) = s^2 - s$, but this can be generalized up to some extent. We also restrict ourselves to initial data v_0 that are distribution functions of probability measures in $\mathbb{R}$. The equation was introduced in 1937 [15, 24] as a model for the evolution of a genetic trait and since then, has been widely studied.

Both Fisher and Kolmogorov, Petrovskii and Piskunov proved independently that this equation admits an infinite number of traveling wave solutions (TW) of the form $v(t,x) =$

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$w_c(x - ct)$ that travel at velocity c. This fact is somehow unexpected from the modeling point of view.

Fisher proposed a way to overcome this difficulty, related to the probabilistic representation given later on by McKean [29], weaving links between solutions to (1) and Branching Brownian Motion. The general principle behind is that microscopic effects should be taken into account to properly describe the physical phenomena. With a similar point of view in mind, Brunet, Derrida and coauthors [8, 9, 10, 11] started in the nineties a study of the effect of microscopic noise in front propagation for equation (1) and related models, which resulted in a huge number of works that study the change in the behavior of the front when microscopic effects are taken into account. These works include both numerical and heuristic arguments [8, 9, 10, 11, 22] as well as rigorous proofs [4, 5, 13, 26]. Before that, Bramson et al. [7] gave the first rigorous proof of a microscopic model for (1) that has a unique velocity for every initial condition. They also prove that these velocities converge in the macroscopic scale to the minimum velocity of (1), and call this fact a *microscopic selection principle*, as opposed to the macroscopic selection principle stated above, that holds for solutions of the hydrodynamic equation.

Consider a Markov process $X = (X_t,\, t \geq 0)$, killed at some state or region that we call 0, defined on certain filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The absorption time is defined by $\tau = \inf\{t > 0\colon X_t \in 0\}$. The conditioned evolution at time t is defined by

$$\mu_t^\gamma(\cdot) := \mathbb{P}_\gamma(X_t \in \cdot | \tau > t).$$

Here γ denotes the initial distribution of the process. A probability measure ν is said to be a quasi-stationary distribution (QSD) if $\mu_t^\nu = \nu$ for all $t \geq 0$.

The *Yaglom limit* is a probability measure ν defined by

$$\nu := \lim_{t\to\infty} \mu_t^{\delta_x},$$

if it exists and does not depend on x. It is known that if the Yaglom limit exists, then it is a QSD. A general principle is that the Yaglom limit *selects* the minimal QSD, i.e. the Yaglom limit is the QSD with minimal mean absorption time. This fact has been proved for a wide class of processes that include birth and death process, Galton-Watson processes [31], random walks [21] and Brownian Motion [27] among others.

It is a typical situation that there is an infinite number of quasi-stationary distributions, but the *Yaglom limit* (the limit of the conditioned evolution of the process started from a deterministic initial condition) selects the minimal one, i.e. the one with minimal expected time of absorption.

This description reveals that similar phenomena occur in both contexts (TW and QSD). The purpose of this note is to show why and how are they related. We first explain the link through the example of Brownian Motion and then we show how to extend this relation to more general Lévy processes. This article has essentially no proofs. In the companion paper [20] we give rigorous proofs for the existence of a precise bijection between TW and QSD in the context of one-dimensional Lévy processes.

2 Macroscopic models

We elaborate on the two macroscopic models we study: front propagation and QSD.

2.1 Front propagation in the F-KPP

Since the seminal papers [15, 24], equation (1) has received a huge amount of attention for several reasons. Among them, it is one of the simplest models explaining several phenomena that are expected to be universal. For instance, it admits a continuum of traveling wave solutions that can be parametrized by their velocity c. More precisely, for each $c \in [\sqrt{2r}, +\infty)$ there exists a function $w_c \colon \mathbb{R} \to [0,1]$ such that

$$v(t,x) = w_c(x - ct)$$

is a solution to (1). For $c < \sqrt{2r}$, there is no traveling wave solution, [1, 24]. Hence $c^* = \sqrt{2r}$ represents the minimal velocity and w_{c^*} the minimal traveling wave. Moreover, if v_0 verifies for some $0 < b < \sqrt{2r}$

$$\lim_{x \to \infty} e^{bx}(1 - v_0(x)) = a > 0,$$

then

$$\lim_{t \to \infty} v(t, x + ct) = w_c(x), \quad \text{for } c = r/b + \frac{1}{2}b, \tag{2}$$

see [29, 30]. If the initial measure has compact support (or fast enough decay at infinity), the solution converges to the minimal traveling wave and the domain of attraction and velocity of each traveling wave is determined by the tail of the initial distribution. A smooth traveling wave solution of (1) that travels at velocity c is a solution to

$$\frac{1}{2}w'' + cw + r(w^2 - w) = 0. \tag{3}$$

2.2 Quasi-stationary distributions

For Markov chains in finite state spaces, the existence and uniqueness of QSDs as well as the convergence of the conditioned evolution to this unique QSD for every initial measure follows from Perron-Frobenius theory. The situation is more delicate for unbounded spaces as there can be zero, one, or an infinite number of QSD. Among those distributions, the *minimal* QSD is the one that minimizes $\mathbb{E}_\nu(\tau)$. Here $\mathbb{E}$ denotes expectation respect to $\mathbb{P}$.

The presence of an infinite number of quasi-stationary distributions might be anomalous from the modeling point of view, in the sense that no physical nor biological meaning has been attributed to them. The reason for their presence here and in the front propagation context is similar: when studying for instance population or gene dynamics through the conditioned evolution of a Markov process, we implicitly consider an infinite population and microscopic effects are lost.

So, as Fisher suggests, in order to avoid the undesirable infinite number of QSD, we should take into account microscopic effects. A natural way to do this is by means of interacting particle systems. We discuss this in Section 3.

Brownian Motion with drift Quasi-stationary ditributions for Brownian Motion with constant drift towards the origin are studied in [27, 28]. We briefly review here some of the results of these papers and refer to them for the details.

For $c > 0$ we consider a one-dimensional Brownian Motion $X = (X_t)_{t\geq 0}$ with drift $-c$ defined by $X_t = B_t - ct$. Here $(B_t)_{t\geq 0}$ is a one dimensional Wiener process defined in the standard Wiener space. We use $\mathbb{P}_x$ for the probability defined in this space such that $(B_t)_{t\geq 0}$ is Brownian Motion started at x and $\mathbb{E}_x$ for expectation respect to $\mathbb{P}_x$. Define the hitting time of zero, when the process is started at $x > 0$ by $\tau_x(c) = \inf\{t > 0 \colon X_t = 0\}$ and denote with P_t^c the sub-Markovian semigroup defined by

$$P_t^c f(x) = \mathbb{E}_x(f(X_t)1_{\{\tau_x(c)>t\}}). \tag{4}$$

In this case, differentiating (4) and after some manipulation it can be seen that the conditioned evolution μP_t has a density $u(t,\cdot)$ for every $t > 0$ and verifies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}(t,x) + c\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\frac{\partial u}{\partial x}(t,0)u(t,x), \quad t > 0, x > 0, \\ u(t,0) &= u(t,+\infty) = 0, \quad t > 0, \end{aligned} \tag{5}$$

It is easy to check that if ν is a QSD, the hitting time of zero, started with ν is an exponential variable of parameter r and hence ν is a QSD if and only if there exists $r > 0$ such that

$$\nu P_t^c = e^{-rt}\nu, \quad \text{for any} \quad t > 0. \tag{6}$$

Differentiating (4) and using the semigroup property and (6) we get that ν is a QSD if and only if

$$\int (\frac{1}{2}f'' - cf')\, d\nu = -r\int f\, d\nu, \quad \text{for all } f \in C_0^\infty(\mathbb{R}_+). \tag{7}$$

Integrating by parts we get that the density w of ν must verify

$$\frac{1}{2}w'' + cw' + rw = 0. \tag{8}$$

Solutions to this equation with initial condition $w(0) = 0$ are given by

$$w(x) = \begin{cases} me^{-cx}\sin(\sqrt{2r - c^2}x) & r > \frac{c^2}{2}, \\ mxe^{-cx} & r = \frac{c^2}{2}, \\ me^{-cx}\sinh(\sqrt{c^2 - 2r}x) & r < \frac{c^2}{2}. \end{cases}$$

Here m is a normalizing constant. Observe that w defines an integrable density function if and only if $0 < r \leq c^2/2$ (or equivalently, $c \geq \sqrt{2r}$). One can thus parametrize the set of QSDs by their eigenvalues r, $\{\nu_r \colon 0 < r \leq c^2/2\}$. For each r, the distribution function of ν_r, $v(x) = \int_0^x w(y)\, dy$ is a monotone solution of (8) with boundary conditions

$$v(0) = 0, \qquad v(+\infty) = 1. \tag{9}$$

3 Particle systems

In this section we introduce two particle systems. The first one is known as Branching Brownian Motion (BBM) with selection of the N right-most particles (N-BBM). As a consequence of the link between BBM and F-KPP that we describe below, this process can be thought of as a microscopic version of F-KPP. The second one is called Fleming-Viot and was introduced by Burdzy, Ingemar, Holyst and March [12], in the context of Brownian Motion in a d-dimensional bounded domain. It is a slight variation of the original one introduced by Fleming and Viot [16]. The first interpretation of this process as a microscopic version of a conditioned evolution is due to Ferrari and Marić [14].

3.1 BBM and F-KPP equation

One-dimensional supercritical Branching Brownian Motion is a well-understood object. Particles diffuse following standard Brownian Motion started at the origin and branch at rate 1 into two particles. As already underlined, its connection with the F-KPP equation and traveling waves was pointed out by McKean in the seminal paper [29]. Denote with N_t the number of particles alive at time $t \geq 0$ and $\xi_t(1) \leq \cdots \leq \xi_t(N_t)$ the position of the particles enumerated from left to right. McKean's representation formula states that if $0 \leq v_0(x) \leq 1$ and we start the process with one particle at 0 (i.e. $N(0) = 1$, $\xi_0(1) = 0$), then

$$v(t,x) := \mathbb{E}\left(\prod_{i=1}^{N_t} v_0(\xi_t(i) + x)\right)$$

is the solution of (1). Of special interest is the case where the initial condition is the Heaviside function $v_0 = \mathbf{1}\{[0, +\infty)\}$ since in this case

$$v(t,x) = \mathbb{P}(\xi_t(1) + x > 0) = \mathbb{P}(\xi_t(N_t) < x).$$

This identity as well as various martingales obtained as functionals of this process have been widely exploited to obtain the precise behavior of solutions of (1), using analytic as well as probabilistic tools.

3.2 N-BBM and Durrett-Remenik equation

Consider now a variant of BBM where the N right-most particles are selected. In other words, each time a particle branches, the left-most one is killed, keeping the total number of particles constant.

This process was introduced by Brunet and Derrida [8, 9] as part of a family of models of branching-selection particle systems to study the effect of microscopic noise in front propagation.

Durrett and Remenik [13] considered a slightly different process in the Brunet-Derrida class: N-BRW. The system starts with N particles. Each particle gives rise to a child at rate one. The position of the child of a particle at $x \in \mathbb{R}$ is $x + y$, where y is chosen according to a probability distribution with density ρ, which is assumed symmetric and with finite expectation. After each birth, the $N+1$ particles are sorted and the left-most one is deleted, in order to keep always N particles. They prove that if at time zero the particles are distributed according to independent variables with distribution $u_0(x)dx$, then the empirical measure of this system converges to a deterministic probability measure ν_t for every t, which is absolutely continuous with density $u(t,\cdot)$, a solution of the following free-boundary problem

Find (γ, u) *such that*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \int_{-\infty}^{\infty} u(t,y)\rho(x-y)\,dy \quad \forall x > \gamma(t), \\ \int_{\gamma(t)}^{\infty} u(t,y)\,dy &= 1, \quad u(t,x) = 0, \quad \forall x \leq \gamma(t), \\ u(0,x) &= u_0(x). \end{aligned} \tag{10}$$

They also find all the traveling wave solutions for this equation. Just as for the BBM, there exists a minimal velocity $c^* \in \mathbb{R}$ such that for $c \geq c^*$ there is a unique traveling wave solution with speed c and no traveling wave solution with speed c for $c < c^*$. The value c^* and the behavior at infinity of the traveling waves can be computed explicitly in terms of the Laplace transform of the random walk. In Section 4 we show that these traveling waves correspond to QSDs of drifted random walks.

It follows from renewal arguments that for each N, the process seen from the left-most particle is ergodic, which in turn implies the existence of a velocity v_N at which the empirical measure travels for each N. Durrett and Remenik prove that these velocities are increasing and converge to c^* as N goes to infinity.

We can interpret this fact as a *weak selection principle*: the microscopic system has a unique velocity for each N (as opposed to the limiting equation) and the velocities converge to the minimal velocity of the macroscopic equation. The word "weak" here refers to the fact that only convergence of the velocities is proved, but not convergence of the empirical measures in equilibrium.

In view of these results, the same theorem is expected to hold for a N-BBM that branches at rate r. In this case the limiting equation is conjectured to be given by

Find (γ, u) *such that*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}(t,x) + ru(t,x) \quad \forall x > \gamma(t), \\ \int_{\gamma(t)}^{\infty} u(t,y)\,dy &= 1, \quad u(t,x) = 0, \quad \forall x \leq \gamma(t), \\ u(0,x) &= u_0(x). \end{aligned} \tag{11}$$

The empirical measures in equilibrium are expected to converge to the minimal traveling wave.

Traveling waves Let us look at the traveling wave solutions $u(t,x) = w(x - ct)$ of (11). Plugging in the equation we see that they must verify

$$\frac{1}{2}w'' + cw' + rw = 0, \quad w(0) = 0, \quad \int_0^\infty w(y)\, dy = 1, \tag{12}$$

which is exactly (8). Note nevertheless that in (12) the parameter r is part of the data of the problem (the branching rate) and c is part of the unknown (the velocity), while in (8) the situation is reversed: c is data (the drift) and r unknown (the absorption rate under the QSD). However, we have the following relation

$$\begin{array}{c}c \text{ is a minimal velocity for } r\\ \text{in (12)}\end{array} \iff \begin{array}{c}r \text{ is a maximal absorption rate for } c\\ \text{in (8)}\end{array}$$

Observe also that $1/r$ is the mean absorption time for the QSD associated to r and hence, if r is maximal, the associated QSD is minimal. So the minimal QSD for Brownian Motion in $\mathbb{R}_+$ and the minimal velocity traveling wave of (11) are one and the same. They are given by

$$u_{c^*(r)}(x) = u_{r^*(c)} = 2r^* x e^{-\sqrt{2r^*}} = (c^*)^2 x e^{-c^* x},$$

which is the one with fastest decay at infinity.

Again, the distribution function v of u is a monotone solution to the same problem but with boundary conditions given by $v(0) = 0$, $v(+\infty) = 1$.

3.3 Fleming-Viot and QSD

The Fleming-Viot process can be thought of as a microscopic version of conditioned evolutions. Its dynamics are built with a continuous time Markov process $X = (X_t,\, t \geq 0)$ taking values in a metric space $\Lambda \cup \{0\}$, that we call the *driving process.* We assume that 0 is absorbing in the sense that

$$\mathbb{P}_{\delta_0}(X_t = 0) = 1, \qquad \forall t \geq 0.$$

As before, we use τ for the absorption time

$$\tau = \inf\{t > 0 \colon X_t \notin \Lambda\},$$

and P_t for the sub-Markovian semigroup defined by

$$P_t f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}\{\tau > t\}).$$

For a given $N \geq 2$, the Fleming-Viot process is an interacting particle system with N particles. We use $\xi_t = (\xi_t(1), \ldots, \xi_t(N)) \in \Lambda^N$ to denote the state of the process, $\xi_t(i)$ denotes the position of particle i at time t. Each particle evolves according to X and independently of the others unless it hits 0, at which time, it chooses one of the $N - 1$ particles in Λ uniformly and takes its position. The genuine definition of this process is not obvious and in fact is not true in general. It can be easily constructed for processes with bounded jumps to 0, but is much more delicate for diffusions in bounded domains [6, 17] and it does not hold for diffusions with a strong drift close to the boundary of Λ.

Here we are also interested in the empirical measure of the process

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_t(i)}. \tag{13}$$

Its evolution mimics the conditioned evolution: the mass lost from Λ is redistributed in Λ proportionally to the mass at each state. Hence, as N goes to infinity, we expect to have a deterministic limit given by the conditioned evolution of the driving process X, i.e.

$$\mu_t^N(A) \to \mathbb{P}(X_t \in A | \tau > t) \qquad (N \to \infty).$$

This is proved in [32] by the martingale method in great generality. See also [18] for a proof based on sub and super-solutions for PDEs and correlations inequalities. A much more subtle question is the ergodicity of the process for fixed N and the behavior of these invariant measures as $N \to \infty$. As a general principle it is expected that

Conjecture 3.1. *If the driving process X has a Yaglom limit ν, then the Fleming-Viot process driven by X is ergodic, with (unique) invariant measure λ^N and the empirical measures* (13) *distributed according to λ^N converge to ν.*

We refer to [18] for an extended discussion on this issue. This conjecture has been proved to be true for subcritical Galton-Watson processes, where a continuum of QSDs arises [2] and also for certain birth and death processes [33].

We have again here a *microscopic selection principle*: whereas there exists an infinite number of QSDs, when microscopic effects are taken into account (through the dynamics of the Fleming-Viot process), there is a unique stationary distribution for the empirical measure, which selects asymptotically the minimal QSD of the macroscopic model.

When the driving process is a one dimensional Brownian Motion with drift $-c$ towards the origin as in Section 2.2, the proof of the whole picture remains open, but the ergodicity of FV for fixed N has been recently proved [3, 23].

So, from [32, Theorem 2.1] we have that for every $t > 0$, μ_t^N converges as $N \to \infty$ to a measure μ_t with density $u(t, \cdot)$ satisfying (5). The open problem is to prove a similar statement in equilibrium. Observe that u is a stationary solution of (5) if and only if it solves (8) for some $r > 0$. Hence, although equations (11) and (5) are pretty different, stationary solutions to (5) coincide with traveling waves of (11).

3.4 Choose the fittest and F-KPP equation

We introduce now the last microscopic model that is useful to explain the relation between all these phenomena. In this subsection $\xi_t = (\xi_t(1), \ldots, \xi_t(N))$ will denote the state of the $N-$ particle system that we describe below. In this model particles perform independent standard Brownian Motion. Additionally each particle has a Poisson process with rate 1/2 and when this process rings, the particle chooses uniformly among the other particles to form a pair. Among this pair of particles, the particle with the smaller position adopts the position of the other one.

We consider again the empirical measure of this process as in (13) and denote with $v^N(x,t)$ the cumulative distribution function

$$v^N(x,t) := \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{\xi_t^i \le x\} \tag{14}$$

It is proved in [19] that if there is a distribution function v_0 such that $v_N(\cdot,0) \to v_0$ uniformly, in probability, then for all $t > 0$

$$\lim_{N\to\infty} \|v^N(\cdot,t) - v(\cdot,t)\|_\infty = 0, \qquad \text{in probability.} \tag{15}$$

Here v is the solution of the F-KPP equation (1). Moreover, for each N it is easily seen that the process is ergodic. This implies the existence of an asympotic velocity

$$v_N = \lim_{t\to\infty} \frac{\xi_t(i)}{t},$$

independent of i. It is also proved in [19] that, as $N \to \infty$, $v_N \nearrow \sqrt{2r}$, the minimal velocity of (1).

Summing up

1. The link between N-BBM and Fleming Viot, in the Brownian Motion case is clear. Both processes evolve according to N independent Brownian Motions and branch into two particles. At branching times, the left-most particle is eliminated (selection) to keep the population size constant. The difference is that while N-BBM branches at a constant rate Nr, Fleming-Viot branches each time a particle hits 0. This explains why in the limiting equation for N-BBM the branching rate is data and the velocity is determined by the system while in the hydrodynamic equation for Fleming-Viot the velocity is data and the branching rate is determined by the system.

2. The empirical measure of N-BBM is expected to converge in finite time intervals to the solution of (11). This is supported by the results of [13] where the same result is proved for random walks.

3. The empirical measure of Fleming-Viot driven by Brownian Motion converges in finite time intervals to the solution of (5).

4. Both N-BBM seen from the left-most particle and FV are ergodic and their empirical measure in equilibrium is expected to converge to the deterministic measure given by the minimal solution of (12).

   Note though that while for N-BBM r is data and minimality refers to c, for Fleming-Viot c is data and minimality refers to $1/r$ (microscopic selection principle).

5. $u(t,x) = w(x-ct)$ is a traveling wave solution of (11) if and only if w is the density of a QSD for Brownian Motion with drift $-c$ and eigenvalue $-r$.

6. c is minimal for r (in (12)) if and only if $1/r$ is minimal for c. So, we can talk of a "minimal solution of (12)", which is both a minimal QSD and a minimal velocity traveling wave.

7. The microscopic selection principle is conjectured to hold in both cases, with the same limit, but a rigorous proof is still unavailable.

4 Traveling waves and QSD for Lévy processes

Let $X = (X_t, t \geq 0)$ be a Lévy process with values in $\mathbb{R}$, defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, Laplace exponent $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\mathbb{E}(e^{\theta X_t}) = e^{\psi(\theta)t}$ and generator $\mathcal{L}$. This centered Lévy process plays the role of Brownian Motion in the previous sections. Now, for $c > 0$ we consider the *drifted process* X^c given by $X_t^c = X_t - ct$. It is immediate to see that the Laplace exponent of X^c is given by $\psi_c(\theta) = \psi(\theta) - c\theta$, that C_0^2 is contained in the domain of the generator $\mathcal{L}_c$ of X^c, and that $\mathcal{L}_c f = \mathcal{L}f - cf'$. Recall that the forward Kolmogorov equation for X is given by

$$\frac{d}{dt}\mathbb{E}_x(f(X_t)) = \mathcal{L}f(x),$$

while the forward Kolmogorov (or Fokker-Plank) equation for the density u (which exists since $\sigma > 0$) is given by

$$\frac{d}{dt}u(t,x) = \mathcal{L}^* u(t,\cdot)(x).$$

Here $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. As in the Brownian case, we consider

- A branching Lévy process (BLP) $(\bar{\xi}_t(1), \ldots, \bar{\xi}_t(N_t))$ driven by $\mathcal{L}^*$.
- A branching Lévy process with selection of the N rightmost particles (N-BLP), driven by $\mathcal{L}$.
- A Fleming-Viot process driven by $\mathcal{L}_c$ (FV).

We focus on the last two processes. For a detailed account on BLP, we refer to [25]. Let us just mention that the F-KPP equation can be generalized in this context to:

$$\begin{aligned} &\frac{\partial v}{\partial t} = \mathcal{L}^* v + rg(v), \quad x \in \mathbb{R},\ t > 0, \\ &v(0,x) = v_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{16}$$

A characterization of the traveling waves as well as sufficient conditions of existence are then provided in [20, 25].

For N-BLP we expect (but a proof is lacking) that the empirical measure converges to a deterministic measure whose density is the solution of the generalized Durrett-Remenik equation

Find (γ, u) *such that*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \mathcal{L}^* u(t,x) + r u(t,x), \quad x > \gamma(t), \\ \int_{\gamma(t)}^{\infty} u(t,y)\,dy &= 1, \quad u(t,x) = 0, \quad x \leq \gamma(t), \\ u(0,x) &= u_0(x), \qquad x \geq 0. \end{aligned} \tag{17}$$

Existence and uniqueness of solutions to this problem have to be examined.

We show below the existence of traveling wave solutions for this equation under mild conditions on $\mathcal{L}$ based on the existence of QSDs.

Concerning FV, it is known [32] that the empirical measure converges to the deterministic process given by the conditioned evolution of the process, which has a density for all times and verifies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \mathcal{L}^* u(t,x) + c\frac{\partial u}{\partial x}(t,x) - u(t,x)\int_0^{\infty} \mathcal{L}^* u(t,y)dy \quad t>0, x>0, \\ u(t,0) &= u(t,+\infty) = 0, \quad t > 0, \end{aligned} \tag{18}$$

Proposition 4.1. *The following statements are equivalent:*

- *The probability measure ν with density w is a QSD for X^c with eigenvalue $-r$,*
- *$u(t,x) = w(x-ct)$ is a traveling wave solution with speed c for the free-boundary problem* (17)*, with parameter r.*

Proof. Denote $\langle f, g\rangle = \int f(x)g(x)dx$. A QSD ν for X^c with eigenvalue $-r$ is a solution of the equation

$$\langle \nu\mathcal{L}_c + r\nu, f\rangle = 0, \forall f \in C_0^2.$$

Using that $\mathcal{L}_c^* f = \mathcal{L}^* f + cf'$ and writing that ν has density w, we obtain that

$$\mathcal{L}^* w + cw' + rw = 0,$$

which in turn is clearly equivalent to w being a traveling wave solution with speed c for (17). ∎

In the companion paper [20] we prove that the picture that we described for the Brownian Motion case holds in more generality. Our result states that, under mild conditions, for given $r, c > 0$ we have that there exists a QSD ν_r for X^c with eigenvalue $-r$ if and only if there is a traveling wave w_c for (16) that travels at velocity c and moreover, ν_r is minimal for c if and only if w_c is minimal for r. Jointly with Proposition 4.1 this also proves that the existence of a traveling wave for (16) is also equivalent to the existence of a traveling wave for the Durrett-Remenik equation (17).

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References

[1] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978.

[2] Amine Asselah, Pablo A. Ferrari, Pablo Groisman, and Matthieu Jonckheere. Fleming–Viot selects the minimal quasi-stationary distribution: The Galton–Watson case. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(2):647–668, 2016.

[3] Amine Asselah and Marie-Noémie Thai. A note on the rightmost particle in a fleming-viot process. *arXiv preprint arXiv:1212.4168*, 2012.

[4] J. Bérard and J.B. Gouéré. Brunet-Derrida behavior of branching-selection particle systems on the line. *Comm. Math. Phys.*, 298(2):323–342, 2010.

[5] Julien Berestycki, Nathanaël Berestycki, and Jason Schweinsberg. The genealogy of branching Brownian motion with absorption. *Ann. Probab.*, 41(2):527–618, 2013.

[6] Mariusz Bieniek, Krzysztof Burdzy, and Sam Finch. Non-extinction of a Fleming-Viot particle model. *Probab. Theory Related Fields*, 153(1-2):293–332, 2012.

[7] M. Bramson, P. Calderoni, A. De Masi, P. Ferrari, J. Lebowitz, and R. H. Schonmann. Microscopic selection principle for a diffusion-reaction equation. *J. Statist. Phys.*, 45(5-6):905–920, 1986.

[8] E. Brunet and B. Derrida. Shift in the velocity of a front due to a cutoff. *Phys. Rev. E (3)*, 56(3, part A):2597–2604, 1997.

[9] É. Brunet and B. Derrida. Effect of microscopic noise on front propagation. *J. Statist. Phys.*, 103(1-2):269–282, 2001.

[10] É. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Noisy traveling waves: effect of selection on genealogies. *Europhys. Lett.*, 76(1):1–7, 2006.

[11] É. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Effect of selection on ancestry: an exactly soluble case and its phenomenological generalization. *Phys. Rev. E (3)*, 76(4):041104, 20, 2007.

[12] K. Burdzy, R. Holyst, D. Ingerman, and P. March. Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of Laplacian eigenfunctions. *J. Phys. A, Math. Gen.*, 29(11):2633–2642, 1996.

[13] R. Durrett and D. Remenik. Brunet-derrida particle systems, free boundary problems and Wiener-Hopf equations. *Ann. Probab.*, 39(6):2043–2078, 2011.

[14] P.A. Ferrari and N. Marić. Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, 12:no. 24, 684–702 (electronic), 2007.

[15] RA Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7:353–369, 1937.

[16] W.H. Fleming and M. Viot. Some measure-valued Markov processes in population genetics theory. *Indiana Univ. Math. J.*, 28(5):817–843, 1979.

[17] I. Grigorescu and M. Kang. Immortal particle for a catalytic branching process. *Probability Theory and Related Fields*, pages 1–29, 2011.

[18] P. Groisman and M. Jonckheere. Simulation of quasi-stationary distributions on countable spaces. *Markov Process. Related Fields*, 19(3):521–542, 2013.

[19] P. Groisman, M. Jonckheere, and Martínez J. Another branching-selection particle system associated to the F-KPP equation. In preparation.

[20] Pablo Groisman and Matthieu Jonckheere. A note on front propagation and quasi-stationary distributions for one-dimensional Lévy processes. *arXiv preprint arXiv:1609.09338*, 2013.

[21] Donald L. Iglehart. Random walks with negative drift conditioned to stay positive. *J. Appl. Probability*, 11:742–751, 1974.

[22] D. A. Kessler and H. Levine. Fluctuation-induced diffusive instabilities. *Nature*, 394:556–558, 1998.

[23] M. Kolb and A. Wübker. Geometric ergodicity of a Fleming-Viot interaction particle process with constant drift. *Personal communication.*

[24] A. Kolmogorov, I. Petrovsky, and N. Piscounov. Etude de l'equation de la diffusion avec croissance de la quantite de matiere et son application a un probleme biologique. *Bull. Univ. Etat Moscou*, A 1:1(25), 1937.

[25] A. E. Kyprianou. A note on branching Lévy processes. *Stochastic Process. Appl.*, 82(1):1–14, 1999.

[26] Pascal Maillard. Branching brownian motion with selection. *arXiv preprint arXiv:1210.3500*, 2012.

[27] S. Martinez, P. Picco, and J. San Martin. Domain of attraction of quasi-stationary distributions for the Brownian motion with drift. *Adv. in Appl. Probab.*, 30(2):385–408, 1998.

[28] S. Martínez and J. San Martín. Quasi-stationary distributions for a Brownian motion with drift and associated limit laws. *J. Appl. Probab.*, 31(4):911–920, 1994.

[29] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323–331, 1975.

[30] H. P. McKean. A correction to: "Application of Brownian motion to the equation of Kolmogorov-Petrovskiĭ-Piskonov" (Comm. Pure Appl. Math. **28** (1975), no. 3, 323–331). *Comm. Pure Appl. Math.*, 29(5):553–554, 1976.

[31] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probability*, 3:403–434, 1966.

[32] Denis Villemonais. Interacting particle systems and Yaglom limit approximation of diffusions with unbounded drift. *Electron. J. Probab.*, 16:no. 61, 1663–1692, 2011.

[33] Denis Villemonais. Minimal quasi-stationary distribution approximation for a birth and death process. *Electron. J. Probab.*, 20:no. 30, 18, 2015.

Front propagation and quasi-stationary distributions for one-dimensional Lévy processes

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Abstract

We jointly investigate the existence of quasi-stationary distributions for one dimensional Lévy processes and the existence of traveling waves for the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation associated with the same motion. Using probabilistic ideas developed by S. Harris [13], we show that the existence of a traveling wave for the F-KPP equation associated with a centered Lévy processes that branches at rate r and travels at velocity c is equivalent to the existence of a quasi-stationary distribution for a Lévy process with the same movement but drifted by $-c$ and killed at zero, with mean absorption time $1/r$. This also extends the known existence conditions in both contexts. As it is discussed in [12], this is not just a coincidence but the consequence of a relation between these two phenomena.

Keywords: quasi-stationary distributions, traveling waves, branching random walk, branching Lévy proceses.
MSC 2010: 60J68, 60J80, 60G51.

1 Introduction

Let $\mathcal{L}$ be the generator of a centered one-dimensional Lévy process (precise definitions and assumptions are be given below) and consider the (generalized) F-KPP equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{L}^* u + r(u^2 - u), \quad x \in \mathbb{R},\ t > 0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

Here $\mathcal{L}^*$ denotes the adjoint of $\mathcal{L}$. Both Fisher and Kolmogorov, Petrovskii and Piskunov considered this equation for $\mathcal{L} = \frac{d^2}{dx^2}$ and proved independently that in this case this equation admits traveling wave solutions of the form $u(t,x) = w_c(x - ct)$ that travel at velocity c for every $c \geq \sqrt{2r}$, [11, 15].

It is well known [5, 6, 18, 24] that a large class of equations describing the propagation of a front into an unstable region have properties similar to (1). These equations admit traveling-wave solutions for any velocity c larger than a minimal velocity c^* and the front moves with this minimal velocity c^* for any initial data with "light enough" tails.

For the Brownian case $\mathcal{L} = \frac{d^2}{dx^2}$ we have $c^* = \sqrt{2r}$ and for more general $\mathcal{L}$ the minimal velocity can be computed in terms of the Legendre transform of the process (see Theorem 1.1 below). This was essentially done by Kyprianou [16] using the seminal McKean's representation

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[20] for the solutions of (1). We complete this characterization in this note to arrive to our main theorem.

The theory of quasi-stationary distributions has its own counterpart. It is a typical situation that there is an infinite number of quasi-stationary distributions while the *Yaglom limit* (the limit of the conditioned evolution of the process started from a deterministic initial condition) selects the minimal one, i.e. the one with minimal expected time of absorption [7, 10, 23].

To be more precise, consider a Lévy process $(X_t - ct)_{t\geq 0}$ with generator $\mathcal{L} - c\frac{d}{dx}$ killed at the origin defined in certain filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with expectation denoted by $\mathbb{E}$. The absorption time is defined by $\tau = \inf\{t > 0 \colon X_t - ct = 0\}$. The conditioned evolution at time t is defined by

$$\mu_t^\gamma(\cdot) := \mathbb{P}^\gamma(X_t - ct \in \cdot | \tau > t).$$

Here γ denotes the initial distribution of the process and $\mathbb{P}^\gamma(\cdot) = \mathbb{P}(\cdot | X_0 \sim \gamma)$. A probability measure ν is said to be a quasi-stationary distribution (QSD) if $\mu_t^\nu = \nu$ for all $t \geq 0$.

The *Yaglom limit* is a probability measure ν defined by

$$\nu := \lim_{t\to\infty} \mu_t^{\delta_x},$$

if the limit exists and does not depend on x. It is known that if the Yaglom limit exists, then it is a QSD. A general principle is that the Yaglom limit *selects* the minimal QSD, i.e. the Yaglom limit is the QSD with minimal mean absorption time. This fact has been proved for a wide class of processes that include birth and death process, subcritical Galton-Watson processes, drifted random walks and Brownian motion among others, but the conjecture is still pen for a much wider class of processes.

In the last decades, a great deal of attention has been given to establish on the one hand conditions for the existence of quasi-stationary measures of Lévy processes (see for instance [17, 19]) and on the other hand to the existence of traveling waves for (1) [16]. The purpose of this note is to show that given parameters $r, c > 0$, the existence of a traveling wave for (1) with velocity c is equivalent to the existence of a QSD ν for $\mathcal{L} - c\frac{d}{dx}$ with expected absorption time $\mathbb{E}_\nu(\tau) = 1/r$. Moreover, minimal velocity TWs are in a one-to-one correspondence with minimal absorption time QSDs with the same parameters. Note that when dealing with traveling-waves the branching rate r is an input while the velocity c is chosen by the system, while when dealing with QSDs the velocity c is the input and r is chosen by the system.

Although our proof consists in showing that the conditions for the existence of TW and QSD coincide, in a companion paper [12] we show that these is not just a coincidence but that the two phenomena are essentially two faces of the same coin.

All in all, our main result reads.

Theorem 1.1. *Under assumption* **A** *(stated below), the following are equivalent:*

1. *There exists a non trivial traveling wave for* (1) *with velocity c, i.e. a solution to*

$$\mathcal{L}^* w + cw' + rw(w-1) = 0. \tag{2}$$

2. *There exists an (absolutely continuous) QSD for $\mathcal{L} - c\frac{d}{dx}$ with expected absorption time $1/r$, i.e. a solution to,*

$$\mathcal{L}^* v + cv' + rv = 0. \tag{3}$$

3. *$r \leq \Gamma(c)$, where Γ is the Legendre transform of the Laplace exponent of $\mathcal{L}$.*

4. *A branching Lévy process driven by $\mathcal{L} - c\frac{d}{dx}$, absorbed in 0 gets almost surely extinct.*

Moreover, c is a minimal velocity for $(\mathcal{L}^*, r)$ if and only if $1/r$ is a minimal mean absorption time for $\mathcal{L} - c\frac{d}{dx}$.

Remark 1.2. *In* (2) *the domain is $\mathbb{R}$ and the boundary conditions are $w(+\infty) = 1 - w(-\infty) = 1$, while in* (3) *the domain is $(0, +\infty)$ and also $v \geq 0$, $v(0) = 0$, $\int v = 1$ is imposed.*

2 Preliminaries

Let $X = (X_t)_{t\geq 0}$ be a Lévy process with values in $\mathbb{R}$, defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and Laplace exponent $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathbb{E}(e^{\theta X_t}) = e^{\psi(\theta)t},$$

such that

$$\psi(\theta) = b\theta + \sigma^2\frac{\theta^2}{2} + g(\theta),$$

where $b \in \mathbb{R}$, $\sigma > 0$ (which ensures that X is non-lattice) and g is defined in terms of the jump measure Π supported in $\mathbb{R}\setminus\{0\}$ by

$$g(\theta) = \int_{\mathbb{R}} (e^{\theta x} - 1 - \theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx), \qquad \int_{\mathbb{R}} (1\wedge x^2)\Pi(dx) < \infty.$$

L t $\theta^\star_+ = \sup\{\theta\colon |\psi(\theta)| < \infty\}$, $\theta^\star_- = \inf\{\theta\colon |\psi(\theta)| < \infty\}$ and recall that ψ is strictly convex in $(\theta^\star_-, \theta^\star_+)$ and by monotonicity $\psi(\theta^\star_\pm) = \psi(\theta^\star\mp)$ and $\psi'(\theta^\star_\pm) = \psi'(\theta^\star\mp)$ are well defined as well as the derivative at zero $\psi'(0) = \mathbb{E}(X_1)$, that we assume to be zero. We also assume that $\theta^\star_\pm > 0$. The generator of X applied to a function $f \in C_0^2$, the class of compactly supported functions with continuous second derivatives, gives

$$\mathcal{L}f(x) = \frac{1}{2}\sigma^2 f''(x) + bf'(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - yf'(x)\mathbf{1}\{|y|\leq 1\})\Pi(dy).$$

The adjoint of $\mathcal{L}$ is also well defined in C_0^2 and has the form

$$\mathcal{L}^* f(x) = \frac{1}{2}\sigma^2 f''(x) - bf'(x) + \int_{\mathbb{R}} (f(x-y) - f(x) + yf'(x)\mathbf{1}\{|y|\leq 1\})\Pi(dy).$$

It is immediate to see that the Laplace exponent of $(X_t - ct)_{t\geq 0}$ is given by $\psi_c(\theta) = \psi(\theta) - c\theta$ for $\theta \in [\theta^\star_-, \theta^\star_+]$ and that C_0^2 is contained in the domain of the generator $\mathcal{L} - c\frac{d}{dx}$. We denote by Γ the Legendre transform of ψ, i.e.,

$$\Gamma(\alpha) = \sup_{\theta\in\mathbb{R}} \alpha\theta - \psi(\theta).$$

Similarly we will denote $\bar{\Gamma}$ the Legendre transform of the Laplace exponent of the dual process $(-X_t)_{t\geq 0}$,

$$\bar{\Gamma}(\alpha) = \sup_{\theta\in\mathbb{R}} \alpha\theta - \psi(-\theta).$$

Observe that since $\sigma > 0$, Γ as well as $\bar{\Gamma}$ are defined in $\mathbb{R}$. To summarize, hereafter we assume

(A) $\sigma > 0$, $\theta^\star_\pm > 0$ and $\mathbb{E}(X_1) = 0$.

Recall that the backward Kolmogorov equation for X is given by

$$\frac{d}{dt}\mathbb{E}^x(f(X_t)) = \mathcal{L}f(x),$$

while the forward Kolmogorov (or Fokker-Plank) equation for the density u (which exists since $\sigma > 0$) is given by

$$\frac{d}{dt}u(t,x) = \mathcal{L}^* u(t,\cdot)(x).$$

We will consider on the one hand Lévy processes with generator $\mathcal{L}$ (or $\mathcal{L}^*$) that evolve in $\mathbb{R}$ and on the other hand Lévy processes with generator $\mathcal{L} - c\frac{d}{dx}$, killed at zero. A probability measure in $\mathbb{R}_+$ with density v is a QSD for the process $(X_t - ct)_{t\geq 0}$ killed at 0, if and only if, v is a positive solution of (3).

We will need the following.

Lemma 2.1 (Girsanov theorem for Lévy processes). *Let $M_t^\theta := \exp(\theta X_t - \psi_c(\theta)t)$ and the measure $\tilde{\mathbb{Q}}$ be defined by*

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = M_t^\theta, \qquad t \in [0, +\infty). \tag{4}$$

Then $(M_t^\theta)_{t\geq 0}$ is a martingale and under $\tilde{\mathbb{Q}}$, $(X_t)_{t\geq 0}$ is a Lévy process with drift $\mathbb{E}_{\tilde{\mathbb{Q}}}(X_1) = \psi_c'(\theta) = \psi'(\theta) - c$, variance σ^2, and jump measure $e^{\theta x} d\pi(x)$.

2.1 Some useful results on branching Lévy processes

Consider a continuous time branching process with binary branching at rate $r > 0$. Each individual performs independent Lévy processes with generator $\mathcal{L}$ started at the position of his ancestor at her birth-time. Details on the construction of this process can be found in [16]. Call N_t the number of individuals in the process at time t and $(\zeta_t^i, 1 \leq i \leq t)$ the positions of the individuals that are alive at time t. We call $Z_t = (\zeta_t^1, \ldots, \zeta_t^{N_t})$ and $Z = (Z_t)_{t\geq 0}$ a branching Lévy process (BLP) driven by $\mathcal{L}$. For some results, we need to consider BLP killed at some barrier $x \in \mathbb{R}$, the extension of the definition to this situation is straightforward.

The following proposition is proved in [1, 2]. See also [4, Theorem 4.17] for an alternative proof with spines and a setting closer to ours.

Proposition 2.2. *Let Z be a BLP driven by $\mathcal{L}$ and R_t the position of the maximum of Z_t. Then*

$$\lim_{t\to\infty} \frac{R_t}{t} = \Gamma^{-1}(r).$$

By means of this proposition we obtain the following partial extension of Theorem 1 in [3].

Proposition 2.3. *Let $\mathring{Z}$ be a BLP driven by $\mathcal{L} - c\frac{d}{dx}$ started at $x > 0$ and killed at the origin.*

(i) If $r \leq \Gamma(c)$, then $\mathring{Z}$ gets extinct with probability 1.

(ii) If $r > \Gamma(c)$, then for any interval $A \subset \mathbb{R}^+$, $\mathbb{P}(\sum_{i=1}^{N_t} \mathbf{1}_{\{\mathring{\zeta}_t^i \in A\}} \to \infty) > 0$.

Proof. Observe that $\mathring{Z}$ can be constructed straightforward with the trajectories of a non-absorbed process driven by the same generator. We just need to delete all the paths that touched the negative semi-axes at some time. In the case $r < \Gamma(c)$, we can directly use the previous proposition to see that the maximum of the non-absorbed branching process satisfies $\frac{R_t}{t} \to \Gamma^{-1}(r) - c < 0$ which implies that R_t is almost surely negative after some finite time. This in turn implies extinction of $\mathring{Z}$. For the critical case, we need to slightly refine the arguments given in [4].

Consider the branching Lévy process Z driven by $\mathcal{L}$ (without killing at 0) defined in the same filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and define the martingale

$$Z_t^\theta = \sum_{i=1}^{N_t} \exp(\theta\zeta_t^i - (\psi_c(\theta) + r)t),$$

as well as the change of measure,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t^\theta.$$

On some suitably augmented filtration $\tilde{\mathcal{F}}_t \supset \mathcal{F}_t$, the new process can be seen as a branching process with a spine $(S_t)_{t\geq 0}$ which branches at rate $2r$ and follows a motion given by the change of measure (4), i.e., a Lévy process with drift $\psi_c'(\theta) = \psi'(\theta) - c$, variance σ^2, and jump measure $e^{\theta x} d\pi(x)$. The other particles follow the usual process X. See [4] for details on this construction.

Since we assumed $r = \Gamma(c)$, we can define θ_c such that $\psi_c(\theta_c) = -\Gamma(c)$ and so $\psi_c'(\theta_c) = 0$. From now on we choose $\theta = \theta_c$ in the change of measure and hence, the spine (S_t) is centered.

As a consequence, it is recurrent (as a non trivial Lévy process). It follows that $\limsup_t S_t = \infty$. Now bounding $Z_t^{\theta_c}$ by the contribution of the spine, we have

$$\limsup Z_t^{\theta_c} \geq \limsup \exp(\theta_c S_t - (\psi_c(\theta_c) + r)t) = \exp(\theta_c S_t).$$

Since $1/Z^{\theta_c}$ is a positive super-martingale (under $\mathbb{Q}$), it converges $\mathbb{Q}-$almost surely and so does Z^{θ_c}. Hence,

$$\lim_{t\to\infty} Z_t^{\theta_c} = \infty, \quad \mathbb{Q} - \text{a.s.}$$

Observe that if $B \in \mathcal{F}_\infty$ we have

$$\mathbb{Q}(B) = \int_B \limsup_{t\to\infty} Z_t^\theta \, d\mathbb{P} + \mathbb{Q}(B \cap \{\limsup_{t\to\infty} Z_t^\theta = \infty\}).$$

It then follows that if $\lim Z_t^{\theta_c} = \infty$, under $\mathbb{Q}$, then $\lim Z_t^{\theta_c} = 0$, under $\mathbb{P}$. Finally, let

$$R_t := \max_{1\leq i\leq N_t} \zeta_t^i - ct,$$

and observe that $\exp(\theta_c R_t) \leq Z_t^{\theta_c}$, which implies that $\exp(\theta_c R_t)$ tends to 0 $\mathbb{P}-$a.s. and hence R_t tends to $-\infty$. As before, this implies extinction of $\mathring{Z}$.

To prove *(ii)*, denote $\mathring{Z}_t(A) := \sum_{i=1}^{N_t} \mathbf{1}_{\{\mathring{\zeta}_t^i \in A\}}$. We use the many-to-one lemma to get

$$\mathbb{E}(\mathring{Z}_t(A)) = e^{rt}\mathbb{P}(X_t - ct \in A, \min_{0\leq s\leq t} X_s - cs \geq 0). \tag{5}$$

To compute the last probability we can discretize the time variable and consider the random walk $S_n^\delta = X_{n\delta} - c\delta n$. Following [22, Theorem 4] and [14, Theorem 2.1] we obtain that the decay parameter for the process $(X_t - ct)$ killed at zero is given by $\Gamma(c)$ and hence for every $r > \Gamma(c)$ the r.h.s of (5) grows to infinity. So, for any $x > 0$ we can choose t^* large enough to guarantee $\mathbb{E}^x(\mathring{Z}_{t^*}(A)) > 1$. Let $x = \inf A$. We can assume $x > 0$ without loss of generality. Consider the (discrete time) Galton-Watson process with offspring distribution $\mathring{Z}_{t^*}(A)$, started with one individual at x. This process at time n bounds from below $\mathring{Z}_{nt^*}(A)$ and since it is supercritical we have that $\mathring{Z}_{nt^*}(A)$ grows exponentially fast as $n \to \infty$ with positive probability. Now,

$$\mathbb{P}\Big(\mathring{Z}_s(A) \leq \frac{\mathring{Z}_{nt^*}(A)}{2} \text{ for some } nt^* \leq s \leq (n+1)t^* \Big| \mathring{Z}_{nt^*}(A)\Big) \leq$$
$$\mathbb{P}^x(X_s - cs \leq 0 \text{ for some } 0 \leq s \leq t^*)^{\mathring{Z}_{nt^*}(A)/2}$$

and the conditional Borel-Cantelli lemma [9, p. 207] implies the result. ∎

3 Quasi-stationary distributions and traveling waves

In this section we prove the equivalence between existence of traveling waves and quasi-stationary distributions. The proof boils down to show that both are equivalent to the absorption of a BLP driven by $\mathcal{L} - c\frac{d}{dx}$ and killed at the origin.

3.1 Existence of Quasi-stationary disributions

We first deal with the quasi-stationary distributions.

Proposition 3.1. *The following are equivalent*

1. *There exists a QSD for $\mathcal{L} - c\frac{d}{dx}$ killed at 0 with mean absorption time $1/r$.*
2. $r \leq \Gamma(c)$.

Remark 3.2. *The existence of a QSD for $r = \Gamma(c)$ has been established in [17] under stronger assumptions on the Lévy process.*

Proof. 1) $\Longrightarrow$ 2) (Non-existence). Assume there exists a non-trivial QSD ν and suppose $r > \Gamma(c)$. Since $\sigma > 0$, there necessarily exists a density v being the Radon-Nikodym derivative of ν with respect to the Lebesgue measure on $\mathbb{R}_+$. Note that $v(0) = 0$ and on $\mathbb{R}_+$ we have

$$\mathcal{L}^* v + cv' + rv = 0.$$

Let $\mathring{\hat{Z}} = (\mathring{\zeta}_t^1, \ldots, \mathring{\zeta}_t^{N_t})$ be a branching Lévy process driven by $\mathcal{L}^* + c\frac{d}{dx}$ killed at 0 and started at $x > 0$. The process

$$M_t = \sum_{i=1}^{N_t} v(\mathring{\zeta}_i(t)),$$

is a martingale. On the other hand, for every $A \subset \mathbb{R}^+$,

$$\mathbb{E}^x(M_t) \geq (\inf_A v)\mathbb{E}^x \sum_{i=1}^{N_t} \mathbf{1}_{\{\mathring{\zeta}_t^i \in A\}} = (\inf_A v) e^{rt} \mathbb{P}^x(-X_t + ct \in A, \min_{0 \leq s \leq t} -X_s + cs \geq 0). \quad (6)$$

We want to show that the r.h.s in (6) goes to infinity. Observe that if we take $A = \mathbb{R}^+$ we know the asymptotic behavior of the probability on the r.h.s of (6), but since $\inf_{R^+} v = 0$ this is useless. So we need to choose a smaller A. Irreducibility implies that $\inf_A v > 0$ for every $A \subset \mathbb{R}_+$ bounded and at a positive distance from the origin. We are going to choose $A = [\frac{1}{n}, n]$ for an adequate $n > 0$. Consider the process $X^n = (X_t^n)_{t \geq 0}$ with generator $\mathcal{L} - c\frac{d}{dx}$ killed at $\frac{1}{n}$ and n and call $p_n(x, t, B) = \mathbb{P}^x(X_t^n \in B)$ the transition semigroup and λ_n its decay parameter ([22, Theorem 6]) such that for every interval B

$$-\lim_{t \to \infty} \frac{1}{t} \log p_n(x, t, B) = \lambda_n.$$

We use p_∞, λ_∞, etc. when we deal with the process in $\mathbb{R}^+$ killed at the origin. We will show that $\lambda_n \searrow \lambda_\infty = \Gamma(c)$ and hence, since $r > \Gamma(c)$ we can choose n such that $r - \lambda_n > 0$ and the r.h.s of (6) goes to infinity. A contradiction to the fact that M_t is a martingale. Here we are using the fact that the exit problem from $[\frac{1}{n}, n]$ for a process with generator $\mathcal{L}^* + c\frac{d}{dx}$ started at x is equivalent to the exit problem from the same interval for a process with generator $\mathcal{L} - c\frac{d}{dx}$ started at $y = n - x + \frac{1}{n}$.

Since (λ_n) is decreasing in n, we only need to show $\lim \lambda_n \leq \lambda_\infty$. By means of time-discretization, using the splitting technique (which allows us to assume that X_t^n has an atom) and the subadditive ergodic theorem [22, Section 4], it can be shown that there exists a sequence of times $t_k \nearrow \infty$, $\varepsilon > 0$ and a constant $c > 0$, both depending on x and ε but not on n such that

$$-\frac{1}{t_k} \log p_n(y, t_k, (y - \varepsilon, y + \varepsilon)) + \frac{c}{t_k} \geq \lambda_n.$$

For fixed $t_k < \infty$ we can take $n \to \infty$ to obtain

$$-(1/t_k) \log p_\infty(y, t_k, B) + \frac{c}{t_k} \geq \lim_{n \to \infty} \lambda_n.$$

Now we let $k \to \infty$ to get $\lambda_\infty \geq \lim_n \lambda_n$. The fact that $\lambda_\infty = \Gamma(c)$ was already shown in the course of the proof of Proposition 2.3.

2) $\Longrightarrow$ 1) (Existence). As before, note that ν is a QSD with density v if and only if

$$\int f(\mathcal{L}^* v + cv' + rv) = 0, \quad (7)$$

for all $f \in \mathcal{D}$ where $\mathcal{D}$ is a subset of the domain of the generator with killing, i.e. the original generator but with domain composed of functions vanishing at 0, with the property that for every measurable set $A \subset \mathbb{R}_*^+$, there exists a sequence f_n in $\mathcal{D}$, uniformly bounded and converging pointwise to 1_A. Let $\theta > 0$ and denote by $e_{-\theta}$ the function $x \mapsto e^{-\theta x}$ and $v(x) = e^{-\theta x}h(x)$. The function $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ will be determined later. Let $(X_t)_{t\geq 0}$ be a Lévy process with generator $\mathcal{L}$. We compute

$$\begin{aligned}\left(\mathcal{L}^* + c\frac{d}{dx}\right) v(x) &= \frac{d}{dt}\mathbb{E}^0 \left[e^{-\theta(x-X_t+ct)}h(x - X_t + ct)\right]_{t=0},\\ &= e^{-\theta x}\frac{d}{dt}\left[e^{\psi(\theta)t}\mathbb{E}^0\left(e^{\theta X_t-(c\theta+\psi(\theta))t}h(x-X_t+ct)\right)\right]_{t=0}\\ &= e^{-\theta x}\frac{d}{dt}\left[e^{(\psi(\theta)-c\theta)t}\tilde{\mathbb{E}}^0\left(h(x-X_t+ct)\right)\right]_{t=0}\\ &= e^{-\theta x}\left((\psi(\theta)-c\theta)h(x)+\tilde{\mathcal{L}}h(x)\right).\end{aligned}$$

here $\tilde{\mathbb{E}}$ denotes expectation under the measure $\tilde{\mathbb{Q}}$ defined by (4) and $\tilde{\mathcal{L}}$ is the generator of a Lévy process with drift $\tilde{\mathbb{E}}(\tilde{X}_1) = \tilde{\mathbb{E}}(-X_1) = c - \psi'(\theta)$, variance σ^2 and jump measure $e^{-\theta x}d\pi(-x)$ as in Lemma 2.1. Hence

$$\mathcal{L}^*v + cv' + rv = e^{-\theta x}\left((\psi(\theta) - c\theta + r)h(x) + \tilde{\mathcal{L}}h(x)\right).$$

We obtained that (7) is equivalent to the following equation

$$\int fe_{-\theta}(\tilde{\mathcal{L}}h + (r + \psi_c(\theta))h) = 0,$$

Note that since ψ is a convex function and $-\Gamma(c) \leq -r$, it is possible to choose θ such that $\psi(\theta) - c\theta = -r$. Hence (7) is equivalent to

$$\int f\tilde{\mathcal{L}}h = 0,$$

for all $f \in e_{-\theta}\mathcal{D} = \{g = e_{-\theta}u, u \in \mathcal{D}\}$. We then look for harmonic functions for the killed Lévy process $\tilde{X}$ with generator $\tilde{\mathcal{L}}$.

Define the renewal measure associated to $\tilde{X}$

$$h(x) = \mathbb{E}\int_0^\infty \mathbf{1}_{\{\tilde{H}_t\geq x\}}dt,$$

where $\tilde{H} = (\tilde{H}_t)_{t\geq 0}$ is the ladder process associated to $-\tilde{X}$.

Let θ_c be defined by $\Gamma(c) = c\theta_c - \psi(\theta_c)$. For $\theta \leq \theta_c$, the process $\tilde{X}$ does not drift to $-\infty$, since

$$\tilde{\mathbb{E}}^0(\tilde{X}_1) = c - \psi'(\theta) = -\psi_c'(\theta) \geq 0.$$

This implies that the function h is harmonic (see Lemma 1 in [8]) and since moreover $\tilde{X}_1$ has a finite mean, the renewal theorem implies that h is asymptotically equivalent to the identity and so

$$\int e_{-\theta}h < \infty.$$

Then v is the density of a QSD with absorption rate r. ∎

3.2 Existence of Traveling waves

We now present the corresponding equivalence for the case of traveling waves which was actually the inspiration for the equivalence presented previously. Let us underline that these results are already known except in the critical case $r = \Gamma(c)$, [16]. The proof is included for completeness but follows the proof of [13] who himself quote the results of Neveu [21].

Proposition 3.3 ([13, 21]). *The following are equivalent*

1. *There exists a solution to* (2).
2. $r \leq \Gamma(c)$.

Proof. 1) $\Longrightarrow$ 2) (Non-existence). Assume the existence of a non-trivial traveling wave w. This allows us to define the multiplicative positive martingale

$$M_t = \prod_{i=1}^{N_t} w(\bar{\zeta}_t^i + ct).$$

Here $\bar{Z}_t = (\bar{\zeta}_t^1, \dots, \bar{\zeta}_t^{N_t})$ is a BLP driven by $\mathcal{L}^*$ (with no killing). This martingale being positive and bounded, it converges and its mean being $w(x)$, its limit is not 0. On the other hand, since $w \leq 1$,

$$M_t \leq w(\bar{L}_t + ct),$$

where $\bar{L}_t = \min_{1 \leq i \leq N_t} \bar{\zeta}_t^i$. Remark that the minimum of a BLP driven by $\mathcal{L}^*$ has the same law as $-\max_{1 \leq i \leq \bar{N}_t} \zeta_t^i$, where $Z = ((\zeta_t^i)_{1 \leq i \leq N_t})_{t \geq 0}$ is a BLP driven by $\mathcal{L}$. Proposition 2.2 implies that if $r > \Gamma(c)$, $R_t - ct = \max_{1 \leq i \leq \bar{N}_t} \zeta_t^i - ct \to +\infty$. So $\bar{L}_t \to -\infty$ and hence M_t should have a null limit. A contradiction to the the assumption.

2) $\Longrightarrow$ 1) (Existence). Neveu's method for proving the existence of traveling waves consists in constructing a multiplicative martingale from a Galton-Watson process obtained as follows.

Consider $\mathring{Z}$ a BLP driven by $\mathcal{L} - c\frac{d}{dx}$ with killing at the origin and started with one individual at $x > 0$ as in Proposition 2.3. Since $r \leq \Gamma(c)$ the process is absorbed and then the total population size is finite a.s. We can construct this random number for every $x > 0$ using a unique BLP $\mathring{\bar{Z}}$ with generator $\mathcal{L}^* + c\frac{d}{dx}$ started with one individual at the origin and killing (freezing) at x. If we couple all the processes in this way and call $G_x < \infty$ the number of individuals of $\bar{Z}$ that have reached high x, we get that $(G_x)_{x \geq 0}$ is a continuous-time Galton-Watson process, [13, 21]. Define

$$f_x(s) = \mathbb{E}(s^{G_x}),$$

and for some fixed $s \in (0, 1)$

$$w(x) = f_x^{-1}(s).$$

Note that both quantities are strictly positive since $G_x < \infty$. For $y \geq 0$ define

$$M_y^x := w(x+y)^{G_y}.$$

It turns out that $(M_y^x)_{y \geq 0}$ is a convergent martingale. To see that, observe that the branching property gives us

$$\begin{aligned}
\mathbb{E}[M_{y'}^x | \mathcal{F}_y] &= \mathbb{E}[w(x+y')^{G_{y'}} | \mathcal{F}_y], \\
&= (f_{y'-y}(w(x+y')))^{G_y}, \\
&= (f_{y'-y}(f_{y'-y}^{-1}(w(x+y))))^{G_y} = M_y^x.
\end{aligned}$$

In addition, $(M_y^x)_{y \geq 0}$ is positive and bounded and hence, it does converge and is uniformly integrable. Following the arguments of [13], for fixed t and for all y large enough

$$G_y = \sum_{k=1}^{N_t} G_{y - \bar{\zeta}_t^i}^i,$$

where the $(G^i)_{i \geq 1}$ are independent copies of $G = (G_x)_{x \geq 0}$. Hence

$$M_y^x = \prod_{i=1}^{N_t} w(x+y)^{G_{y-\bar{\zeta}_t^i}^i} = \prod_{i=1}^{N_t} M_{y-\bar{\zeta}_t^i}^{x+\bar{\zeta}_t^i, i}.$$

and the limit of the martingale satisfies

$$M^x = \prod_{i=1}^{N_t} M^{x+\bar{\zeta}_t^i,s}.$$

Taking expectations leads to

$$w(x) = \mathbb{E} \prod_{i=1}^{N_t} w(x+\bar{\zeta}_t^i),$$

which in turn implies (see Theorem 8 in [16]) that

$$\mathcal{L}^* w + cw' + r(w^2 - w) = 0.$$

∎

3.3 Proof of Theorem 1.1

Observe that Proposition 3.3 gives us 1) $\iff$ 3) while Proposition 3.1 proves 2) $\iff$ 3). The equivlence 3) $\iff$ 4) is the content of Proposition 2.3. Finally since Γ is strictly increasing, minimality of $1/r$ (for a given c) as well as minimality of c, for a given r, reduces to

$$r = \Gamma(c).$$

References

[1] J. D. Biggins. The growth and spread of the general branching random walk. *Ann. Appl. Probab.*, 5(4):1008–1024, 1995.

[2] J. D. Biggins. How fast does a general branching random walk spread? In *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pages 19–39. Springer, New York, 1997.

[3] J. D. Biggins, Boris D. Lubachevsky, Adam Shwartz, and Alan Weiss. A branching random walk with a barrier. *Ann. Appl. Probab.*, 1(4):573–581, 1991.

[4] Sergey Bocharov. *Branching Lévy Processes with inhomogeneous breeding potentials.* PhD thesis, University of Bath, 2012.

[5] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.*, 44(285):iv+190, 1983.

[6] É. Brunet and B. Derrida. Effect of microscopic noise on front propagation. *J. Statist. Phys.*, 103(1-2):269–282, 2001.

[7] J.A. Cavender. Quasi-stationary distributions of birth-and-death processes. *Adv. Appl. Probab.*, 10(3):570–586, 1978.

[8] L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.*, 10:no. 28, 948–961, 2005.

[9] Richard Durrett. *Probability.* The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991. Theory and examples.

[10] P. A. Ferrari, S. Martinez, and P. Picco. Some properties of quasi-stationary distributions in the birth and death chains: a dynamical approach. In *Instabilities and nonequilibrium structures, III (Valparaíso, 1989)*, volume 64 of *Math. Appl.*, pages 177–187. Kluwer Acad. Publ., Dordrecht, 1991.

[11] RA Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7:353–369, 1937.

[12] Pablo Groisman and Matthieu Jonckheere. Front propagation and quasi-stationary distributions: the same selection principle? *arXiv preprint arXiv:1304.4847*, 2013.

[13] S.C. Harris. Travelling-waves for the FKPP equation via probabilistic arguments. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(3):503–517, 1999.

[14] Donald L. Iglehart. Random walks with negative drift conditioned to stay positive. *J. Appl. Probability*, 11:742–751, 1974.

[15] A. Kolmogorov, I. Petrovsky, and N. Piscounov. Etude de l'equation de la diffusion avec croissance de la quantite de matiere et son application a un probleme biologique. *Bull. Univ. Etat Moscou*, A 1:1(25), 1937.

[16] A. E. Kyprianou. A note on branching Lévy processes. *Stochastic Process. Appl.*, 82(1):1–14, 1999.

[17] A. E. Kyprianou and Z. Palmowski. Quasi-stationary distributions for Lévy processes. *Bernoulli*, 12(4):571–581, 2006.

[18] J. S. Langer. Models of pattern formation in first-order phase transitions. In *Directions in condensed matter physics*, volume 1 of *World Sci. Ser. Dir. Condensed Matter Phys.*, pages 165–186. World Sci. Publishing, Singapore, 1986.

[19] S. Martinez, P. Picco, and J. San Martin. Domain of attraction of quasi-stationary distributions for the Brownian motion with drift. *Adv. in Appl. Probab.*, 30(2):385–408, 1998.

[20] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, 28(3):323–331, 1975.

[21] J. Neveu. Multiplicative martingales for spatial branching processes. *Seminar on Stochastic Processes, Birkhauser, in: E. Cinlar, K.L. Chung, R.K. Getoor (Eds.),*, pages 223–241, 1987.

[22] Pekka Tuominen and Richard L. Tweedie. Exponential decay and ergodicity of general Markov processes and their discrete skeletons. *Adv. in Appl. Probab.*, 11(4):784–803, 1979.

[23] E.A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23(4):683–700, 1991.

[24] Wim van Saarloos. Three basic issues concerning interface dynamics in nonequilibrium pattern formation. *Physics reports*, 301(1):9–43, 1998.

METASTABILITY FOR SMALL RANDOM PERTURBATIONS OF A PDE WITH BLOW-UP

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ABSTRACT. We study random perturbations of a reaction-diffusion equation with a unique stable equilibrium and solutions that blow-up in finite time. If the strength of the perturbation $\varepsilon > 0$ is small and the initial data is in the domain of attraction of the stable equilibrium, the system exhibits metastable behavior: its time averages remain stable around this equilibrium until an abrupt and unpredictable transition occurs which leads to explosion in a finite time (but exponentially large in ε^{-2}). Moreover, for initial data in the domain of explosion we show that the explosion times converge to the one of the deterministic solution.

1. INTRODUCTION

We consider, for $\varepsilon > 0$, the stochastic process $U^{u,\varepsilon}$ which formally satisfies the stochastic partial differential equation

$$(1.1) \qquad \begin{cases} \partial_t U^{u,\varepsilon} &= \partial^2_{xx} U^{u,\varepsilon} + g(U^{u,\varepsilon}) + \varepsilon \dot{W} & t > 0\,,\, 0 < x < 1 \\ U^{u,\varepsilon}(t,0) &= U^{u,\varepsilon}(t,1) = 0 & t > 0 \\ U^{u,\varepsilon}(0,x) &= u(x) \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is given by $g(u) := u|u|^{p-1}$ for fixed $p > 1$, $\dot{W}$ is space-time white noise and u is a continuous function satisfying $u(0) = u(1) = 0$.

This process can be thought of as a random perturbation of the dynamical system U^u given by the solution of (1.1) with $\varepsilon = 0$, i.e. U^u satisfies the partial differential equation

$$(1.2) \qquad \begin{cases} \partial_t U^u &= \partial^2_{xx} U^u + g(U^u) & t > 0\,,\, 0 < x < 1 \\ U^u(t,0) &= 0 & t > 0 \\ U^u(t,1) &= 0 & t > 0 \\ U^u(0,x) &= u(x) & 0 < x < 1. \end{cases}$$

Equation (1.2) is of reaction-diffusion type, a broad class of evolution equations which naturally arise in the study of phenomena as diverse as diffusion of a fluid through a porous material, transport in a semiconductor, coupled chemical reactions with spatial diffusion, population genetics, among others. In all these cases, the equation represents an approximate model of the phenomenon and thus it is of interest to understand how its description might change if subject to small random perturbations.

An important feature of (1.2) is that it admits solutions which are only local in time and blow up in a finite time. Indeed, the system has a unique stable equilibrium, the null function $\mathbf{0}$, and a countable family of unstable equilibria, all of which are saddle points. The stable equilibrium possesses a domain of attraction $\mathcal{D}_0$ satisfying that if $u \in \mathcal{D}_0$ then the solution U^u of (1.2) with initial datum u is globally defined and converges to $\mathbf{0}$ as time tends to infinity. Similarly, each unstable equilibrium has its own stable manifold, the union of which constitutes the boundary of $\mathcal{D}_0$. Finally, for $u \in \mathcal{D}_e := \overline{\mathcal{D}_0}^c$ the system

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blows up in finite time, i.e. there exists a time $0 < \tau^u < +\infty$ such that the solution U^u is defined for all $t \in [0, \tau^u)$ but satisfies

$$\lim_{t \nearrow \tau^u} \|U^u(t, \cdot)\|_\infty = +\infty.$$

The behavior of the system is, in some aspects, similar to the double-well potential model studied in [1, 12]. Indeed, (1.2) can be reformulated as

$$\partial_t U^u = -\frac{\partial S}{\partial \varphi}(U^u)$$

where S is the potential formally given by

$$S(v) = \int_0^1 \left[\frac{1}{2}\left(\frac{dv}{dx}\right)^2 + G(v)\right],$$

where we take $G(v) := -\frac{|v|^{p+1}}{p+1}$ as opposed to the term $G(v) = \frac{\nu}{4}v^4 - \frac{\mu}{2}v^2$ appearing in the double-well potential model. In our system, instead of having two wells, each being the domain of attraction of the two stable equilibria of the system, we have only one which corresponds to $\mathcal{D}_{\mathbf{0}}$. Since our potential tends to $-\infty$ along every direction, we can imagine the second well in our case as being infinity and thus there is no return from there once the system reaches its bottom. Moreover, since the potential behaves like $-s^{p+1}$ in every direction, if the system falls into this "infinite well" it will reach its bottom (infinity) in a finite time (blow-up).

Upon adding a small noise to (1.2), one wonders if there are any qualitative differences in behavior between the deterministic system (1.2) and its stochastic perturbation (1.1). For short times both systems should behave similarly, since in this case the noise term will be typically of much smaller order than the remaining terms in the right hand side of (1.1). However, due to the independent and normally distributed increments of the perturbation, when given enough time the noise term will eventually reach sufficiently large values so as to induce a significant change of behavior in (1.1). We are interested in understanding what changes might occur in the blow-up phenomenon due to this situation and, more precisely, which are the asymptotic properties as $\varepsilon \to 0$ of the explosion time of (1.1) for the different initial data. Based on all of the considerations above, we expect the following scenario:

i. *Thermalization.* For initial data in $\mathcal{D}_{\mathbf{0}}$, the stochastic system is at first attracted towards this equilibrium. Once near it, the terms in the right hand side of (1.2) become negligible and so the process is then pushed away from the equilibrium by noise. Being away from $\mathbf{0}$, the noise becomes overpowered by the remaining terms in the right hand side of (1.1) and this allows for the previous pattern to repeat itself: a large number of attempts to escape from the equilibrium, followed by a strong attraction towards it.

ii. *Tunneling.* Eventually, after many frustrated attempts, the process succeeds in escaping $\mathcal{D}_{\mathbf{0}}$ and reaches the domain of explosion, the set of initial data for which (1.2) blows up in finite time. Since the probability of such an event is very small, we expect this escape time to be exponentially large. Furthermore, due to the large number of attempts that are necessary, we also expect this time to show little memory of the initial data.

iii. *Final excursion.* Once inside the domain of explosion, the stochastic system is forced to explode by the dominating source term g.

This type of phenomenon is known as *metastability*: the system behaves for a long time as if it were under equilibrium, but then performs an abrupt transition towards the real equilibrium (in our case, towards infinity). The former description was proved rigorously for the (infinite-dimensional) double-well potential model in [1, 12], inspired by the work in [10] for its finite-dimensional analogue. Their proofs rely heavily on large deviations estimates for $U^{u,\varepsilon}$ established in [8] for the infinite-dimensional system and in [9] for the finite-dimensional setting. In our case, we are only capable of proving the existence of local solutions of (1.1) and in fact, explosions *will* occur for $U^{u,\varepsilon}$. As a consequence, we will not be able to apply these same estimates directly, as the validity of these estimates relies on a proper control of the growth of solutions which does not hold in our setting. Localization techniques apply reasonably well to deal with the process until it escapes any fixed bounded domain but they cannot be used to say what happens from then onwards. Since we wish to focus specifically on trajectories that blow up in finite time, it is clear that a new approach is needed for this last part, one which involves a careful study of the blow-up phenomenon. Unfortunately, when dealing with perturbations of differential equations with blow-up, understanding how the behavior of the blow-up time is modified or even showing the persistence of the blow-up phenomenon itself is by no means an easy task in most cases. There are no general results addressing this matter, not even for nonrandom perturbations. This is why the usual approach to this kind of problems is to consider particular models such as ours.

The article is organized as follows. In Section 2 we give some preliminary definitions, introduce the local Freidlin-Wentzell estimates and afterwards present our main results. In Section 3 we give a detailed description of the deterministic system (1.2). Section 4 focuses on the explosion time of the stochastic system for initial data in the domain of explosion. The construction of an auxiliary domain G is performed in Section 5 and the study of the escape from G is carried out in Section 6. In Section 7 we establish metastable behavior for solutions with initial data in the domain of attraction of the stable equilibrium. Finally, we include at the end an appendix with some auxiliary results to be used throughout our analysis.

2. Definitions and results

2.1. The deterministic PDE. Our purpose in this section is to study equation (1.2). We assume that the source term $g : \mathbb{R} \to \mathbb{R}$ is given by $g(u) = u|u|^{p-1}$ for fixed $p > 1$ and also that u belongs to the space $C_D([0,1])$ of continuous functions on $[0,1]$ satisfying homogeneous Dirichlet boundary conditions, namely

$$C_D([0,1]) = \{v \in C([0,1]) : v(0) = v(1) = 0\}.$$

The space $C_D([0,1])$ is endowed with the supremum norm, i.e.

$$\|v\|_\infty = \sup_{x\in[0,1]} |v(x)|.$$

For any choice of $r > 0$ and $v \in C_D([0,1])$, we let $B_r(v)$ denote the closed ball in $C_D([0,1])$ of center v and radius r. Whenever the center is the null function $\mathbf{0}$, we simply write B_r. Equation (1.2) can be reformulated as

$$\partial_t U = -\frac{\partial S}{\partial \varphi}(U) \tag{2.1}$$

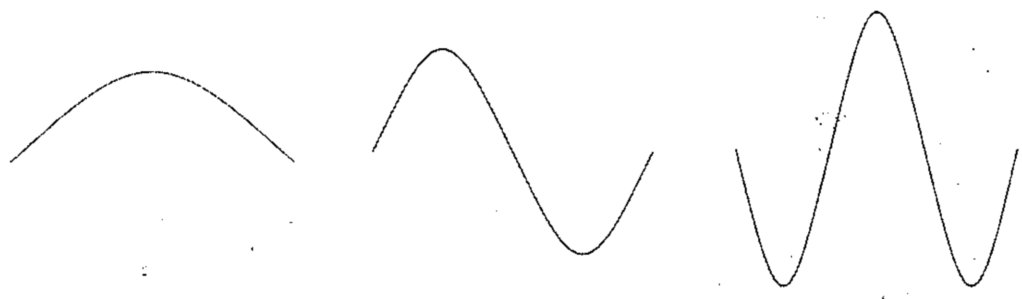

FIGURE 1. Examples of unstable equilibria: z, $z^{(2)}$ and $z^{(-3)}$.

where the *potential* S is the functional on $C_D([0,1])$ given by

$$S(v) = \begin{cases} \displaystyle\int_0^1 \left[\frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{|v|^{p+1}}{p+1}\right] & \text{if } v \in H_0^1((0,1)) \\ +\infty & \text{otherwise.} \end{cases}$$

Here $H_0^1((0,1))$ denotes the Sobolev space of square-integrable functions defined on $[0,1]$ with square-integrable weak derivative which vanish at the boundary $\{0,1\}$. Recall that $H_0^1((0,1))$ can be embedded into $C_D([0,1])$ so that the potential is indeed well-defined. We refer the reader to the appendix for a review of some of the main properties of S which shall be required throughout our work.

The formulation on (2.1) is interpreted as the validity of

$$\int_0^1 \partial_t U(t,x)\varphi(x)dx = \lim_{h\to 0} \frac{S(U+h\varphi)-S(U)}{h}$$

for any $\varphi \in C^1([0,1])$ with $\varphi(0) = \varphi(1) = 0$. It is known that for any $u \in C_D([0,1])$ there exists a unique solution U^u to equation (1.2) defined on some maximal time interval $[0,\tau^u)$ where $0 < \tau^u \leq +\infty$ is called the *explosion time* of U^u (see [17] for further details). In general, we will say that this solution belongs to the space

$$C_D([0,\tau^u)\times[0,1]) = \{v \in C([0,\tau^u)\times[0,1]) : v(\cdot,0) = v(\cdot,1) \equiv 0\}.$$

However, whenever we wish to make its initial datum u explicit we will do so by saying that the solution belongs to the space

$$C_{D_u}([0,\tau^u)\times[0,1]) = \{v \in C([0,\tau^u)\times[0,1]) : v(0,\cdot) = u \text{ and } v(\cdot,0) = v(\cdot,1) \equiv 0\}.$$

The origin $\mathbf{0} \in C_D([0,1])$ is the unique stable equilibrium of the system and it is in fact asymptotically stable. It corresponds to the unique local minimum of the potential S. There is also a family of unstable equilibria of the system corresponding to the remaining critical points of the potential S, all of which are saddle points. Among these unstable equilibria there exists only one of them which is nonnegative (see [4, p. 3] for details) which we denote by z. It can be shown that this equilibrium z is in fact strictly positive for $x \in (0,1)$, symmetric with respect to the axis $x = \frac{1}{2}$ (i.e. $z(x) = z(1-x)$ for all $x \in [0,1]$) and that is both of minimal potential and minimal norm among all the unstable equilibria. The remaining equilibria are obtained by alternating scaled copies of both z and $-z$ as Figure 1 shows. We establish this fact rigurously in Section 3.

2.2. **Definition of solution for the SPDE.** In general, equations like (1.1) do not admit strong solutions in the usual sense as they may not be globally defined but instead defined *up to an explosion time*. In the following we formalize the idea of explosion and properly define the concept of solutions of (1.1).

First, we fix a probability space $(\Omega, \mathcal{F}, P)$ on which we have defined a Brownian sheet

$$W = \{W(t,x) : (t,x) \in \mathbb{R}^+ \times [0,1]\},$$

i.e. a stochastic process satisfying the following properties:

i. W has continuous paths, i.e. $(t,x) \mapsto W(t,x)(\omega)$ is continuous for every $\omega \in \Omega$.

ii. W is a centered Gaussian process with covariance structure given by

$$\mathrm{Cov}(W(t,x), W(s,y)) = (t \wedge s)(x \wedge y)$$

for every $(t,x), (s,y) \in \mathbb{R}^+ \times [0,1]$.

Then, for every $t \geq 0$ we define

$$\mathcal{G}_t = \sigma(W(s,x) : 0 \leq s \leq t, x \in [0,1])$$

and denote its augmentation by $\mathcal{F}_t$.[1] The family $(\mathcal{F}_t)_{t\geq 0}$ constitutes a filtration on $(\Omega, \mathcal{F})$. A *solution up to an explosion time* of the equation (1.1) on $(\Omega, \mathcal{F}, P)$ with respect to the Brownian sheet W and with initial datum $u \in C_D([0,1])$ is a stochastic process $U^{u,\varepsilon} = \{U^{u,\varepsilon}(t,x) : (t,x) \in \mathbb{R}^+ \times [0,1]\}$ satisfying the following properties:

i. $U^{u,\varepsilon}(0,\cdot) \equiv u$

ii. $U^{u,\varepsilon}$ has continuous paths taking values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

iii. $U^{u,\varepsilon}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. for every $t \geq 0$ the mapping

$$(\omega, x) \mapsto U^{u,\varepsilon}(t,x)(\omega)$$

is $\mathcal{F}_t \otimes \mathcal{B}([0,1])$-measurable.

iv. If Φ denotes the fundamental solution of the heat equation on the interval $[0,1]$ with homogeneous Dirichlet boundary conditions, which is given by the formula

$$\Phi(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \left[\exp\left(-\frac{(2n+y-x)^2}{4t} \right) - \exp\left(-\frac{(2n+y+x)^2}{4t} \right) \right],$$

and for $n \in \mathbb{N}$ we define the stopping time $\tau_\varepsilon^{(n),u} := \inf\{t > 0 : \|U^{u,\varepsilon}(t,\cdot)\|_\infty \geq n\}$ then for every $n \in \mathbb{N}$ we have P-a.s.:

- $\displaystyle\int_0^1 \int_0^{t\wedge\tau_\varepsilon^{(n),u}} |\Phi(t \wedge \tau_\varepsilon^{(n),u} - s, x, y) g(U^{u,\varepsilon}(s,y))| ds dy < +\infty$ for all $t \in \mathbb{R}^+$

- $U^{u,\varepsilon}(t \wedge \tau_\varepsilon^{(n),u}, x) = I_H^{(n)}(t,x) + I_N^{(n)}(t,x)$ for all $(t,x) \in \mathbb{R}^+ \times [0,1]$, where

$$I_H^{(n)}(t,x) = \int_0^1 \Phi(t \wedge \tau_\varepsilon^{(n),u}, x, y) u(y) dy$$

and

$$I_N^{(n)}(t,x) = \int_0^{t\wedge\tau_\varepsilon^{(n),u}} \int_0^1 \Phi(t \wedge \tau_\varepsilon^{(n),u} - s, x, y) \left(g(U^{u,\varepsilon}(s,y)) dy ds + \varepsilon dW(s,y) \right). \tag{2.2}$$

[1]This means that $\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N})$ where $\mathcal{N}$ denotes the class of all P-null sets of $\mathcal{G}_\infty = \sigma(\mathcal{G}_t : t \in \mathbb{R}^+)$.

The stochastic integral in the right-hand side of (2.2) is to be understood in the sense of Walsh [20]. We call the random variable $\tau_\varepsilon^u := \lim_{n\to+\infty} \tau_\varepsilon^{(n),u}$ the *explosion time* of $U^{u,\varepsilon}$. Notice that the assumption of continuity of $U^{u,\varepsilon}$ over $\overline{\mathbb{R}}$ implies that:

- $\tau_\varepsilon^u = \inf\{t > 0 : \|U^{u,\varepsilon}(t,\cdot)\|_\infty = +\infty\}$
- $\|U^{u,\varepsilon}((\tau_\varepsilon^u)^-,\cdot)\|_\infty = \|U^{u,\varepsilon}(\tau_\varepsilon^u,\cdot)\|_\infty = +\infty$ on $\{\tau_\varepsilon^u < +\infty\}$.

We stipulate that $U^{u,\varepsilon}(t,\cdot) \equiv U^{u,\varepsilon}(\tau_\varepsilon^u,\cdot)$ for $t \geq \tau$ whenever $\tau_\varepsilon^u < +\infty$ but we do not assume that $\lim_{t\to+\infty} U^{u,\varepsilon}(t,\cdot)$ exists if $\tau_\varepsilon^u = +\infty$. Furthermore, since any initial datum $u \in C_D([0,1])$ is bounded, we always have $P(\tau_\varepsilon^u > 0) = 1$. It can be shown that there exists a (pathwise) unique solution $U^{u,\varepsilon}$ of (1.1) up to an explosion time and that it has the strong Markov property, i.e. if $\tilde{\tau}$ is a stopping time of $U^{u,\varepsilon}$ then, conditional on $\tilde{\tau} < \tau_\varepsilon^u$ and $U^{u,\varepsilon}(\tilde{\tau},\cdot) = v$, the future $\{U^{u,\varepsilon}(t+\tilde{\tau},\cdot) : 0 < t < \tau_\varepsilon^u - \tilde{\tau}\}$ is independent of the past $\{U^{u,\varepsilon}(s,\cdot) : 0 \leq s \leq \tilde{\tau}\}$ and identical in law to the solution of (1.1) with initial datum v. We refer to [13, 20] for details.

2.3. **Local Freidlin-Wentzell estimates.** One of the main tools we use in the study of solutions of (1.1) is the local large deviations principle we briefly describe next.

Given $u \in C_D([0,1])$ and $T > 0$, we consider the metric space of continuous functions

$$C_{D_u}([0,T]\times[0,1]) = \{v \in C([0,T]\times[0,1]) : v(0,\cdot) = u \text{ and } v(\cdot,0) = v(\cdot,1) \equiv 0\}$$

with the distance d_T induced by the supremum norm, i.e. for $v, w \in C_{D_u}([0,T]\times[0,1])$

$$d_T(v,w) := \sup_{(t,x)\in[0,T]\times[0,1]} |v(t,x) - w(t,x)|,$$

and define the rate function $I_T^u : C_{D_u}([0,T]\times[0,1]) \to [0,+\infty]$ by the formula

$$I_T^u(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T\int_0^1 |\partial_t\varphi - \partial_{xx}\varphi - g(\varphi)|^2 & \text{if } \varphi \in W_2^{1,2}([0,T]\times[0,1])\,,\ \varphi(0,\cdot) = u \\ +\infty & \text{otherwise.} \end{cases}$$

Here $W_2^{1,2}([0,T]\times[0,1])$ is the closure of $C^\infty([0,T]\times[0,1])$ with respect to the norm

$$\|\varphi\|_{W_2^{1,2}} = \left(\int_0^T\int_0^1 \left[|\varphi|^2 + |\partial_t\varphi|^2 + |\partial_x\varphi|^2 + |\partial_{xx}\varphi|^2\right]\right)^{\frac{1}{2}},$$

i.e. the Sobolev space of square-integrable functions defined on $[0,T]\times[0,1]$ with one square-integrable weak time derivative and two square-integrable weak space derivatives.

By following the lines of [1, 8, 19], it is possible to establish a large deviations principle for solutions of (1.1) with rate function I as given above whenever the source term g is globally Lipschitz (even though they do not work with a globally Lipschitz source, their analysis carries over to this simpler context). Unfortunately, this is not the case for us. Nonetheless, by employing localization arguments like the ones carried out in [11], one can obtain a weaker version of this principle which only holds locally, i.e. while the process remains inside any fixed bounded region. More precisely, we have the following result.

Theorem 2.1. *If for each $n \in \mathbb{N}$ and $u \in C_D([0,1])$ we define*

$$\tau^{(n),u} := \inf\{t > 0 : \|U^u(t,\cdot)\|_\infty \geq n\} \qquad \text{and} \qquad T_\varepsilon^{(n),u} := \tau_\varepsilon^{(n),u} \wedge \tau^{(n),u}$$

where $\tau_\varepsilon^{(n),u}$ is defined as in Section 2.2, then the following estimates hold:

- **Lower bound.** *For any $\delta, h > 0$ and $n \in \mathbb{N}$, there exists ε_0 such that*

$$P\left(d_{T\wedge\mathcal{T}_\varepsilon^{(n),u}}(U^{u,\varepsilon},\varphi) < \delta\right) \geq e^{-\frac{I_T^u(\varphi)+h}{\varepsilon^2}} \tag{2.3}$$

for all $0 < \varepsilon < \varepsilon_0$, $u \in C_D([0,1])$ and $\varphi \in C_{D_u}([0,T]\times[0,1])$ with $\|\varphi\|_\infty \leq n$.

- **Upper bound.** *For any $\delta > 0$ and $n \in \mathbb{N}$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\sup_{u\in C_D([0,1])} P\left(d_{T\wedge\mathcal{T}_\varepsilon^{(n),u}}(U^{u,\varepsilon},U^u) > \delta\right) \leq e^{-\frac{C}{\varepsilon^2}}, \tag{2.4}$$

for all $0 < \varepsilon < \varepsilon_0$.

The usual large deviations estimates for these type of systems usually feature a more refined version of the upper bound than the one we give here (see [1], for example). However, the estimate in (2.4) is enough for our purposes and so we do not pursue any generalizations of it here. Also, notice that both estimates are somewhat uniform in the initial datum. This uniformity is obtained as in [1] by using the fact that g is Lipschitz when restricted to bounded sets. We refer to [1, 8] for further details.

2.4. **Main results.** Our purpose is to study the asymptotic behavior as $\varepsilon \to 0$ of $U^{u,\varepsilon}$, the solution of (1.1), for the different initial data $u \in C_D([0,1])$. We now state our results. For simplicity purposes, in the following when computing probabilities of events we may drop the superscript u from the usual notation and instead make the initial datum explicit by adding it as a subscript under the probability sign. In this way, whenever we write P_u instead of P it means that in the event in question all initial data are set to u.

In many occasions throughout the sequel we will be interested in obtaining estimates which hold (in a suitable sense) uniformly in the initial condition. However, since $C_D([0,1])$ is an infinite-dimensional space, uniformity over compact sets will not be very informative, while uniformity over closed bounded sets alone will in general be too much to expect. The following definition introduces the precise class of subsets for which we will be able to obtain uniform estimates.

Definition 2.1. Given $\mathcal{D} \subseteq C_D([0,1])$, we will say that $\mathcal{K} \subseteq C_D([0,1])$ is *$\mathcal{D}$-compactifiable* if $\mathcal{K}$ is bounded and there exists $t_0 > 0$ such that $\inf_{u\in\mathcal{K}} \tau^u > t_0$ and for each $t \in (0,t_0]$ the closure of $\mathcal{K}(t) := \{U^u(t,\cdot) : u \in \mathcal{K}\}$ is a compact set contained in $\mathcal{D}$.

It is straightforward to see that any compact set $\mathcal{K}$ is $\mathcal{D}$-compactifiable for any $\mathcal{D}$ having $\mathcal{K}$ in its interior. However, due to the regularizing property of the solutions to 1.2 studied in the appendix, there exist many $\mathcal{D}$-compactifiable sets which are not compact. Indeed, in Lemma 5.3 below we will see that if $\mathcal{D} \subseteq C_D([0,1])$ is open and $\mathcal{K}$ is a compact set contained in $\mathcal{D}$ then any sufficiently small neighborhood of $\mathcal{K}$ is also $\mathcal{D}$-compactifiable.

Now, our first result deals with the continuity of the explosion time for initial data in the domain of explosion $\mathcal{D}_e$. In this case one expects the stochastic and deterministic systems both to exhibit a similar behavior for any $\varepsilon > 0$ sufficiently small, since then the noise will not be able to grow fast enough so as to overpower the quickly exploding source term g. We show this to be truly the case for $u \in \mathcal{D}_e$ such that U^u remains bounded from one side.

Theorem 2.2. *Let $\mathcal{D}_e^*$ be the set of initial data $u \in \mathcal{D}_e$ such that U^u explodes only through one side, i.e. U^u remains bounded either from below or above until its explosion time τ^u. Then given $\delta > 0$ and a $\mathcal{D}_e^*$-compactifiable set $\mathcal{K}$ there exists a constant $C > 0$ such that*

$$\sup_{u\in\mathcal{K}} P_u(|\tau_\varepsilon - \tau| > \delta) \leq e^{-\frac{C}{\varepsilon^2}}.$$

The main differences in behavior between the stochastic and deterministic systems appear for initial data in $\mathcal{D}_0$, where metastable behavior is observed. According to the characterization of metastability for stochastic processes in [3, 10], this behavior is given by two facts: the time averages of the process remain stable until an abrupt transition occurs and a different value is attained; furthermore, the time of this transition is unpredictable in the sense that, when suitably rescaled, it should have an exponential distribution. We manage to establish this description rigorously for our system whenever $1 < p < 5$. This rigorous description is contained in the remaining results.

Define the quantity $\Delta := 2(S(z) - S(\mathbf{0}))$. Our second result states that for any $u \in \mathcal{D}_0$ the asymptotic magnitude of τ_ε^u is, up to logarithmic equivalence, of order $e^{\frac{\Delta}{\varepsilon^2}}$.

Theorem 2.3. *Given $\delta > 0$ and a $\mathcal{D}_0$-compactifiable set $\mathcal{K}$, if $1 < p < 5$ then we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} \left| P_u \left(e^{\frac{\Delta - \delta}{\varepsilon^2}} < \tau_\varepsilon < e^{\frac{\Delta + \delta}{\varepsilon^2}} \right) - 1 \right| \right] = 0.$$

Theorem 2.3 suggests that, for initial data $u \in \mathcal{D}_0$, the typical route of $U^{u,\varepsilon}$ towards infinity involves passing through one of the unstable equilibria of minimal energy, $\pm z$. This seems reasonable since, as we will see in Section 5, for $1 < p < 5$ the barrier imposed by the potential S is the lowest there. The following result establishes this fact rigorously.

Theorem 2.4. *Given $\delta > 0$ and a $\mathcal{D}_0$-compactifiable set $\mathcal{K}$, if $1 < p < 5$ then we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} \left| P_u \left(\tau_\varepsilon(\mathcal{D}_0^c) < \tau_\varepsilon \,,\, U^\varepsilon(\tau_\varepsilon(\mathcal{D}_0^c), \cdot) \in B_\delta(\pm z) \right) - 1 \right| \right] = 0,$$

where $\tau_\varepsilon^u(\mathcal{D}_0^c) := \inf\{t > 0 : U^{u,\varepsilon}(t, \cdot) \notin \mathcal{D}_0\}$ and $B_\delta(\pm z) := B_\delta(z) \cup B_\delta(-z)$.

Our next result is concerned with the the asymptotic loss of memory of τ_ε^u. For $\varepsilon > 0$ define the scaling coefficient

$$\beta_\varepsilon = \inf\{t \geq 0 : P_\mathbf{0}(\tau_\varepsilon > t) \leq e^{-1}\} \tag{2.5}$$

Observe that Theorem 2.3 implies that the family $(\beta_\varepsilon)_{\varepsilon > 0}$ satisfies $\lim_{\varepsilon \to 0} \varepsilon^2 \log \beta_\varepsilon = \Delta$. This next result states that for any $u \in \mathcal{D}_0$ the normalized explosion time $\frac{\tau_\varepsilon^u}{\beta_\varepsilon}$ converges in distribution to an exponential random variable of mean one.

Theorem 2.5. *Given $\delta > 0$ and a $\mathcal{D}_0$-compactifiable set $\mathcal{K}$, if $1 < p < 5$ then we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} \left| P_u(\tau_\varepsilon > t\beta_\varepsilon) - e^{-t} \right| \right] = 0.$$

for any $t > 0$.

Finally, we show the stability of time averages of continuous functions evaluated along paths of the process starting in $\mathcal{D}_0$, i.e. they remain close to the value of the function at $\mathbf{0}$. These time averages are taken along intervals of length going to infinity and times may be taken as being almost (in a suitable scale) the explosion time. This is telling us that, up until the explosion time, the system spends most of its time in a small neighborhood of $\mathbf{0}$.

Theorem 2.6. *There exists a sequence $(R_\varepsilon)_{\varepsilon > 0}$ with $\lim_{\varepsilon \to 0} R_\varepsilon = +\infty$ and $\lim_{\varepsilon \to 0} \frac{R_\varepsilon}{\beta_\varepsilon} = 0$ such that given $\delta > 0$ for any $\mathcal{D}_0$-compactifiable set $\mathcal{K}$ we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} P_u \left(\sup_{0 \leq t \leq \tau_\varepsilon - 3R_\varepsilon} \left| \frac{1}{R_\varepsilon} \int_t^{t+R_\varepsilon} f(U^\varepsilon(s, \cdot))ds - f(\mathbf{0}) \right| > \delta \right) \right] = 0$$

for any bounded continuous function $f : C_D([0,1]) \to \mathbb{R}$.

Theorem 2.2 is proved in Section 4, the remaining results are proved in Sections 6 and 7. Perhaps the proof of Theorem 2.2 is where one finds major differences with other works in the literature dealing with similar problems, namely [10, 12]. This is due to the fact that for this part we cannot use large deviations estimates as these do. The remaining results were established in [1, 12] for the tunneling time in an infinite-dimensional double-well potential model, i.e. the time the system takes to go from one well to the bottom of the other one. Our proofs are similar to the ones found in these references, although we have the additional difficulty of dealing with solutions which are not globally defined.

3. Phase diagram of the deterministic system

In this section we review the behavior of solutions to (1.2) for the different initial data in $C_D([0,1])$. We begin by characterizing the unstable equilibria of the system.

Proposition 3.1. A function $w \in C_D([0,1])$ is an equilibrium of the system if and only if there exists $n \in \mathbb{Z}$ such that $w = z^{(n)}$, where for each $n \in \mathbb{N}$ we define $z^{(n)} \in C_D([0,1])$ by the formula

$$z^{(n)}(x) = \begin{cases} n^{\frac{2}{p-1}} z(nx - [nx]) & \text{if } [nx] \text{ is even} \\ \\ -n^{\frac{2}{p-1}} z(nx - [nx]) & \text{if } [nx] \text{ is odd} \end{cases}$$

and also define $z^{(-n)} := -z^{(n)}$ and $z^{(0)} := \mathbf{0}$. Furthermore, for each $n \in \mathbb{Z}$ we have

$$\|z^{(n)}\|_\infty = |n|^{\frac{2}{p-1}} \|z\|_\infty \qquad \text{and} \qquad S(z^{(n)}) = |n|^{2\left(\frac{p+1}{p-1}\right)} S(z). \tag{3.1}$$

Proof. Since the function z is smooth and strictly positive on $(0,1)$, it follows from (1.2) that $\partial_x z$ is decreasing in $(0,1)$ so that the limits $\partial_x z(0^+)$ and $\partial_x z(1^-)$ exist. Moreover, since z is symmetric with respect to the $x = \frac{1}{2}$ axis, we have in fact that $\partial_x z(0^+) = -\partial_x z(1^-)$. Similarly, since z vanishes in the boundary of $[0,1]$, we also have $\partial^2_{xx} z(0^+) = \partial^2_{xx} z(1^-) = 0$. From these observations, it is simple to verify that each $z^{(n)}$ is an equilibrium of the system (in particular, twice differentiable) and satisfies (3.1). Therefore, we must only check that for any equilibrium $w \in C_D([0,1]) - \{\mathbf{0}\}$ there exists $n \in \mathbb{Z} - \{0\}$ such that $w \equiv z^{(n)}$.

Thus, let $w \in C_D([0,1]) - \{\mathbf{0}\}$ be an equilibrium of (1.2) and define the sets

$$G^+ = \{x \in (0,1) : w(x) > 0\} \qquad \text{and} \qquad G^- = \{x \in (0,1) : w(x) < 0\}.$$

Since $w \neq \mathbf{0}$ at least one of these sets must be nonempty. On the other hand, if only one of them is nonempty then, since z is the unique nonnegative equilibrium different from $\mathbf{0}$, we must have either $w = z$ or $w = -z$. Therefore, we may assume that both G^+ and G^- are nonempty. Notice that since G^+ and G^- are open sets we may write them as

$$G^+ = \bigcup_{k \in \mathbb{N}} I_k^+ \qquad \text{and} \qquad G^- = \bigcup_{k \in \mathbb{N}} I_k^-$$

where the unions are disjoint and each $I_k^\pm$ is a (possibly empty) open interval.

We first show that each union must be finite. Take $k \in \mathbb{N}$ and suppose we can write $I_k^+ = (a_k, b_k)$ for some $0 \leq a_k < b_k \leq 1$. It is easy to check that $\tilde{w}_k : [0,1] \to \mathbb{R}$ given by

$$\tilde{w}_k(x) := (b_k - a_k)^{\frac{2}{p-1}} w(a_k + (b_k - a_k)x)$$

is a nonnegative equilibrium of the system different from $\mathbf{0}$ and thus it must be $\tilde{w}_k = z$. This implies that $\|w\|_\infty \geq (b_k - a_k)^{-\frac{2}{p-1}} \|\tilde{w}_k\|_\infty = (b_k - a_k)^{-\frac{2}{p-1}} \|z\|_\infty$ from where we see that an infinite number of nonempty I_k^+ would contradict the fact that w is bounded. Thus, we see that G^+ is a finite union of open intervals and by symmetry so is G^-.

The same argument also implies that for each interval $I_k^{\pm} = (a_k, b_k)$ the graph of $w|_{I_k^{\pm}}$ coincides with that of $\pm z$ but when scaled by the factor $(b_k - a_k)^{-\frac{2}{p-1}}$. More precisely, for all $x \in [0,1]$ we have

$$w(a_k + (b_k - a_k)x) = \pm(b_k - a_k)^{-\frac{2}{p-1}} z(x). \tag{3.2}$$

Now, Hopf's Lemma [7, p. 330] implies that $\partial_x z(0^+) > 0$ and $\partial_x z(1^-) < 0$. Furthermore, since z is symmetric with respect to $x = \frac{1}{2}$ we have in fact that $\partial_x z(0^+) = -\partial_x z(1^-) > 0$. In light of (3.2) and the fact that w is everywhere differentiable, the former tells us that plus and minus intervals must present themselves in alternating order, that their closures cover all of $[0,1]$ and also that their lengths are all the same. Combining this with (3.2) we conclude the proof. □

As a consequence of Proposition 3.1 we obtain the following important corollary.

Corollary 3.2. The functions $\pm z$ minimize the pontential S and the supremum norm among all the unstable equilibria of (1.2). In particular, we have $\inf_{u \in \mathcal{W}} S(u) = S(\pm z)$, where

$$\mathcal{W} := \{u \in C_D([0,1]) : \tau^u = +\infty \text{ and } \lim_{t \to +\infty} U^u(t, \cdot) = z^{(n)} \text{ for some } n \in \mathbb{Z} - \{0\}\}$$

denotes the union of all stable manifolds corresponding to the different unstable equilibria.

Proof. The first statement is clear from Proposition 3.1 while the second one is deduced from the first since the mapping $t \mapsto S(U^u(t,\cdot))$ is monotone decreasing and continuous for any $u \in H_0^1((0,1))$ (see Proposition 8.7). □

Concerning the asymptotic behavior of solutions to (1.2), the following dichotomy was proved by Cortázar and Elgueta in [4].

Proposition 3.3. Let U^u denote the solution to (1.2) with initial datum $u \in C_D([0,1])$. Then one of these two possibilities must hold:

i. $\tau^u < +\infty$ and U^u blows up as $t \nearrow \tau^u$, i.e. $\lim_{t \nearrow \tau^u} \|U^u(t,\cdot)\|_\infty = +\infty$

ii. $\tau^u = +\infty$ and U^u converges to some stationary solution $z^{(n)}$ as $t \to +\infty$.

Proposition 3.3 allows us to split the space $C_D([0,1])$ of initial data into three parts

$$C_D([0,1]) = \mathcal{D}_0 \cup \mathcal{W} \cup \mathcal{D}_e \tag{3.3}$$

where $\mathcal{D}_0$ denotes the domain of attraction of the origin $\mathbf{0}$, $\mathcal{D}_e$ is the domain of explosion of the system, i.e. the set of all initial data for which the system explodes in finite time, and $\mathcal{W}$ denotes the union of all stable manifolds associated to the unstable equilibria. It can be seen that both $\mathcal{D}_0$ and $\mathcal{D}_e$ are open sets and that $\mathcal{W}$ is the common boundary separating them. The following proposition gives a useful characterization of $\mathcal{D}_e$.

Proposition 3.4 ([17, Theorem 17.6]). The domain of explosion $\mathcal{D}_e$ satisfies

$$\mathcal{D}_e = \{u \in C_D([0,1]) : S(U^u(t,\cdot)) < 0 \text{ for some } 0 \leq t < \tau^u\}.$$

Furthermore, we have $\lim_{t \nearrow \tau^u} S(U^u(t,\cdot)) = -\infty$.

From these results one can obtain a precise description of the domains $\mathcal{D}_0$ and $\mathcal{D}_e$ in the region of nonnegative data. Cortázar and Elgueta proved the following result in [5].

Proposition 3.5.

i. Assume $u \in C_D([0,1])$ is nonnegative and such that U^u is globally defined and converges to z as $t \to +\infty$. Then for $v \in C_D([0,1])$ we have that:

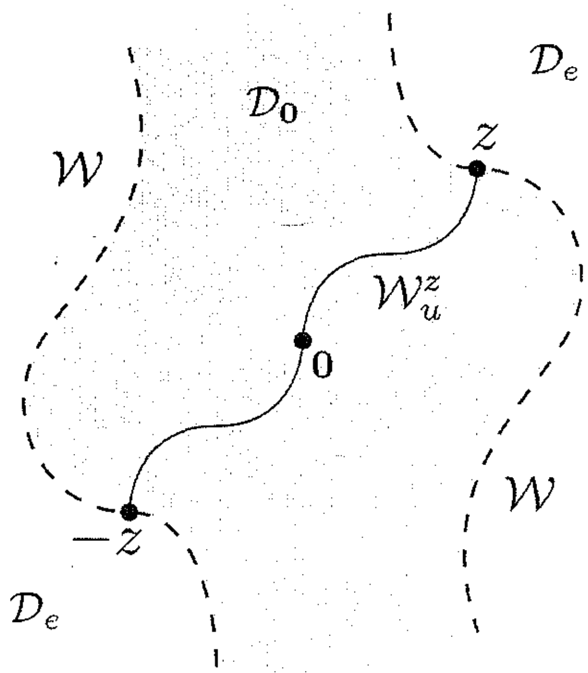

FIGURE 2. The phase diagram of equation (1.2).

- $\mathbf{0} \le v \le u \Longrightarrow U^v$ is globally defined and converges to $\mathbf{0}$ as $t \to +\infty$.
- $u \le v \Longrightarrow U^v$ explodes in finite time.

ii. For every nonnegative $u \in C_D([0,1])$ there exists $\lambda_c^u > 0$ such that for every $\lambda > 0$:

- $0 < \lambda < \lambda_c^u \Longrightarrow U^{\lambda u}$ is globally defined and converges to $\mathbf{0}$ as $t \to +\infty$.
- $\lambda = \lambda_c^u \Longrightarrow U^{\lambda u}$ is globally defined and converges to z as $t \to +\infty$.
- $\lambda > \lambda_c^u \Longrightarrow U^{\lambda u}$ explodes in finite time.

This last result yields the existence of an unstable manifold of the saddle point z which is contained in the region of nonnegative initial data and which we shall denote by $\mathcal{W}_u^z$. It is 1-dimensional, has nonempty intersection with both $\mathcal{D}_0$ and $\mathcal{D}_e$ and joins z with $\mathbf{0}$. By symmetry, a similar description also holds for the opposite unstable equilibrium $-z$. Figure 2 depicts the decomposition in (3.3) together with the unstable manifolds $\mathcal{W}_u^{\pm z}$. By exploiting the structure of the remaining unstable equilibria given by Proposition 3.1 one can verify for each of them the analogue of (ii) in Proposition 3.5. Details in [18].

4. Asymptotic behavior of τ_ε^u for $u \in \mathcal{D}_e$

In this section we investigate the continuity properties of the explosion time τ_ε^u for initial data in the domain of explosion $\mathcal{D}_e$. We show that, under suitable conditions on the initial datum $u \in \mathcal{D}_e$, the random explosion time τ_ε^u converges in probability to the deterministic explosion time τ^u as $\varepsilon \to 0$. To be more precise, let us consider the sets of initial data in $\mathcal{D}_e$ which explode only through $+\infty$ or $-\infty$, i.e.

$$\mathcal{D}_e^+ = \left\{ u \in \mathcal{D}_e : \inf_{(t,x)\in[0,\tau^u)\times[0,1]} U^u(t,x) > -\infty \right\}$$

and

$$\mathcal{D}_e^- = \left\{ u \in \mathcal{D}_e : \sup_{(t,x)\in[0,\tau^u)\times[0,1]} U^u(t,x) < +\infty \right\}.$$

Notice that $\mathcal{D}_e^+$ and $\mathcal{D}_e^-$ are disjoint and also that they satisfy the relation $\mathcal{D}_e^- = -\mathcal{D}_e^+$. Furthermore, we shall see below that $\mathcal{D}_e^+$ is an open set. Let us write $\mathcal{D}_e^* := \mathcal{D}_e^+ \cup \mathcal{D}_e^-$. The result we are to prove is the following.

Theorem 4.1. *For any $\mathcal{D}_e^*$-compactifiable set $\mathcal{K}$ and $\delta > 0$ there exists a constant $C > 0$ such that*

$$\sup_{u\in\mathcal{K}} P_u(|\tau_\varepsilon - \tau| > \delta) \leq e^{-\frac{C}{\varepsilon^2}}.$$

We split the proof of Theorem 4.1 into two parts: proving first a lower bound and then an upper bound for τ_ε. The first one is a consequence of the continuity of solutions to (1.1) with respect to ε on intervals where the deterministic solution remains bounded. The precise estimate is contained in the following proposition.

Proposition 4.1. Given any $\mathcal{D}_e$-compactifiable set $\mathcal{K}$ and $\delta > 0$, there exists a constant $C > 0$ such that

$$\sup_{u\in\mathcal{K}} P_u(\tau_\varepsilon < \tau - \delta) \leq e^{-\frac{C}{\varepsilon^2}}. \tag{4.1}$$

Proof. First, let us observe that since $\mathcal{K}$ is $\mathcal{D}_e$-compactifiable we may assume that $\tau^u > \delta$ for all $u \in \mathcal{K}$. Now, for each $u \in \mathcal{D}_e$ define the quantity

$$M_u := \sup_{0\leq t\leq \max\{0,\tau^u-\delta\}} \|U^u(t,\cdot)\|_\infty.$$

By the continuity of solutions once again we obtain that the application $u \mapsto M_u$ is both upper semicontinuous and finite on $\mathcal{D}_e$ so that, since for each $u \in \mathcal{K}$ we have

$$M_u \leq \sup_{t\in[0,t_0]} \|U^u(t,\cdot)\|_\infty + M_{U^u(t_0,\cdot)} \leq \sup_{t\in[0,t_0]} \|U^u(t,\cdot)\|_\infty + \sup_{v\in\mathcal{K}(t_0)} M_v$$

for all $t_0 < \inf_{u\in\mathcal{K}} \tau^u - \delta$, by Proposition 8.2 we conclude that $M := \sup_{u\in\mathcal{K}} M_u < +\infty$. Similarly, since $u \mapsto \tau^u$ is both continuous and finite on $\mathcal{D}_e$ (see Corollary 4.4 below for a proof of this) we also obtain that $\mathcal{T} := \sup_{u\in\mathcal{K}} \tau^u < +\infty$. Hence, for $u \in \mathcal{K}$ we get

$$\begin{aligned} P_u(\tau_\varepsilon^u < \tau^u - \delta) &\leq P\left(d_{(\tau^u-\delta)\wedge\mathcal{T}_\varepsilon^{(M_u+1),u}}(U^{u,\varepsilon}, U^u) > \frac{1}{2}\right) \\ &\leq P\left(d_{(\mathcal{T}-\delta)\wedge\mathcal{T}_\varepsilon^{(M+1),u}}(U^{u,\varepsilon}, U^u) > \frac{1}{2}\right). \end{aligned}$$

By the estimate (2.4) we conclude (4.1). □

To establish the upper bound we consider for each $u \in \mathcal{D}_e^+$ the process

$$Z^{u,\varepsilon} := U^{u,\varepsilon} - V^{\mathbf{0},\varepsilon}$$

where $U^{u,\varepsilon}$ is the solution of (1.1) with initial datum u and $V^{\mathbf{0},\varepsilon}$ is the solution of (1.1) with source term $g \equiv 0$ and initial datum $\mathbf{0}$, constructed using the same Brownian sheet in both cases. Note that $V^{\mathbf{0},0} \equiv \mathbf{0}$ and also that, since the source term 0 is globally Lipschitz, the family $(V^{(\mathbf{0},\varepsilon)})_{\varepsilon>0}$ satisfies a *global* large deviations principle, i.e. analogous to the one stated in Theorem 2.1 but with $T \wedge \mathcal{T}_\varepsilon^{(n),u}$ replaced by T everywhere. Also, observe that $Z^{u,\varepsilon}$ satisfies the random partial differential equation

$$\left\{\begin{array}{lll} \partial_t Z^{u,\varepsilon} & = \partial_{xx}^2 Z^{u,\varepsilon} + g(Z^{u,\varepsilon} + V^{\mathbf{0},\varepsilon}) & t > 0,\, 0 < x < 1 \\ Z^{u,\varepsilon}(t,0) & = Z^{u,\varepsilon}(t,1) = 0 & t > 0 \\ Z^{u,\varepsilon}(0,x) & = u(x). & \end{array}\right. \tag{4.2}$$

Furthermore, since $V^{\mathbf{0},\varepsilon}$ is globally defined and remains bounded on finite time intervals, we have that $Z^{u,\varepsilon}$ and $U^{u,\varepsilon}$ share the same explosion time. Hence, to obtain the desired upper bound on τ_ε^u we may study the behavior of $Z^{u,\varepsilon}$. The advantage of this approach is that, in general, the behavior of $Z^{u,\varepsilon}$ will be easier to understand than that of $U^{u,\varepsilon}$. Indeed, each realization of $Z^{u,\varepsilon}$ is the solution of a partial differential equation which one can handle with PDE techniques.

Now, a straightforward calculation using the mean value theorem shows that whenever $\|V^{0,\varepsilon}\|_\infty < 1$ the process $Z^{u,\varepsilon}$ satisfies the inequality

$$\partial_t Z^{u,\varepsilon} \geq \partial^2_{xx} Z^{u,\varepsilon} + g(Z^{u,\varepsilon}) - h|Z^{u,\varepsilon}|^{p-1} - h \tag{4.3}$$

where $h := p2^{p-1}\|V^{0,\varepsilon}\|_\infty > 0$. Therefore, to establish the upper bound on τ_ε^u we first consider for $h > 0$ the solution $\underline{Z}^{(h),u}$ to the equation

$$\begin{cases} \partial_t \underline{Z}^{(h),u} &= \partial^2_{xx}\underline{Z}^{(h),u} + g(\underline{Z}^{(h),u}) - h|\underline{Z}^{(h),u}|^{p-1} - h \qquad t > 0\,,\, 0 < x < 1 \\ \underline{Z}^{(h),u}(t,0) &= \underline{Z}^{(h),u}(t,1) = 0 \qquad t > 0 \\ \underline{Z}^{(h),u}(0,x) &= u(x). \end{cases} \tag{4.4}$$

and obtain a convenient upper bound for the explosion time of this new process valid for every h sufficiently small. By showing then that for h suitably small the process $\underline{Z}^{(h),u}$ explodes through $+\infty$, the fact that $Z^{u,\varepsilon}$ is a supersolution to (4.4) will yield the desired upper bound on the explosion time of $Z^{u,\varepsilon}$, provided that $\|V^{0,\varepsilon}\|_\infty$ remains small enough. For this last part is where the assumption that $u \in \mathcal{D}_e^+$ is necessary. Lemma 4.3 below contains the proper estimate on $\underline{\tau}^{(h),u}$, the explosion time of $\underline{Z}^{(h),u}$.

Definition 4.2. For $h \geq 0$ we define the potential $\underline{S}^{(h)}$ on $C_D([0,1])$ associated to (4.4) by the formula

$$\underline{S}^{(h)}(v) = \begin{cases} \displaystyle\int_0^1 \left[\frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{|v|^{p+1}}{p+1} + h\frac{g(v)}{p} + hv\right] & \text{if } v \in H_0^1((0,1)) \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that $\underline{S}^{(0)}$ coincides with our original potential S. Moreover, it is easy to check that for all $h \geq 0$ the potential $\underline{S}^{(h)}$ satisfies all properties established for S in the appendix.

Lemma 4.3. Given $\delta > 0$ there exists $M > 0$ such that:

i. For every $0 \leq h < 1$, any $u \in C_D([0,1])$ with $\underline{S}^{(h)}(u) \leq -\frac{M}{2}$ verifies $\underline{\tau}^{(h),u} < \frac{\delta}{2}$.
ii. Given $K > 0$ there exist constants $\rho_{M,K}, h_{M,K} > 0$ depending only on M and K such that any $u \in C_D([0,1])$ satisfying $S(u) \leq -M$ and $\|u\|_\infty \leq K$ verifies

$$\sup_{v \in B_{\rho_{M,K}}(u)} \underline{\tau}^{(h),v} < \delta$$

for all $0 \leq h < h_{M,K}$.

Proof. Let us take $\delta > 0$ and show first that (i) holds for an appropriate choice of M. For fixed $M > 0$ and $0 \leq h < 1$, let $u \in C_D([0,1])$ be such that $\underline{S}^{(h)}(u) \leq -\frac{M}{2}$ and consider the application $\phi^{(h),u} : [0, \underline{\tau}^{(h),u}) \to \mathbb{R}^+$ given by the formula

$$\phi^{(h),u}(t) = \int_0^1 \left(\underline{Z}^{(h),u}(t,x)\right)^2 dx.$$

It is simple to verify that $\phi^{(h),u}$ is continuous and that for any $t_0 \in (0, \underline{\tau}^{(h),u})$ it satisfies

$$\frac{d\phi^{(h),u}}{dt}(t_0) \geq -4\underline{S}^{(h)}(u^{(h)}_{t_0}) + 2\int_0^1 \left[\left(\frac{p-1}{p+1}\right)|u^{(h)}_{t_0}|^{p+1} - h\left(\frac{p+2}{p}\right)|u^{(h)}_{t_0}|^p - h|u^{(h)}_{t_0}|\right] \tag{4.5}$$

where we write $u^{(h)}_{t_0} := \underline{Z}^{(h),u}(t_0,\cdot)$ for convenience. Hölder's inequality reduces (4.5) to

(4.6)
$$\frac{d\phi^{(h),u}}{dt}(t_0) \geq -4\underline{S}^{(h)}(u^{(h)}_{t_0})+2\left[\left(\frac{p-1}{p+1}\right)\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}} - h\left(\frac{p+2}{p}\right)\|u^{(h)}_{t_0}\|^{p}_{L^{p+1}} - h\|u^{(h)}_{t_0}\|_{L^{p+1}}\right].$$

Observe that, by definition of $\underline{S}^{(h)}$ and the fact that the map $t \mapsto \underline{S}^{(h)}(u^{(h)}_t)$ is decreasing, we obtain the inequalities

$$\frac{M}{2} \leq -\underline{S}^{(h)}(u^{(h)}_{t_0}) \leq \frac{1}{p+1}\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}} + h\|u^{(h)}_{t_0}\|^{p}_{L^{p+1}} + h\|u^{(h)}_{t_0}\|_{L^{p+1}}$$

from which we deduce that by taking M sufficiently large one can force $\|u^{(h)}_{t_0}\|_{L^{p+1}}$ to be large enough so as to guarantee that

$$\left(\frac{p-1}{p+1}\right)\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}} - h\left(\frac{p+2}{p}\right)\|u^{(h)}_{t_0}\|^{p}_{L^{p+1}} - h\|u^{(h)}_{t_0}\|_{L^{p+1}} \geq \frac{1}{2}\left(\frac{p-1}{p+1}\right)\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}}$$

is satisfied for any $0 \leq h < 1$. Therefore, we see that if M sufficiently large then for all $0 \leq h < 1$ the application $\phi^{(h),u}$ satisfies

$$\frac{d\phi^{(h),u}}{dt}(t_0) \geq 2M + \left(\frac{p-1}{p+1}\right)\left(\phi^{(h),u}(t_0)\right)^{\frac{p+1}{2}} \tag{4.7}$$

for every $t_0 \in (0, \underline{\tau}^{(h),u})$, where to obtain (4.7) we have once again used Hölder's inequality and the fact that the map $t \mapsto \underline{S}^{(h)}(u^{(h)}_t)$ is decreasing. Now, it is straightforward to show that the solution y of the ordinary differential equation

$$\begin{cases} \dot{y} = 2M + \left(\frac{p-1}{p+1}\right) y^{\frac{p+1}{2}} \\ y(0) \geq 0 \end{cases}$$

explodes before time

$$T = \frac{\delta}{4} + \frac{2^{\frac{p+1}{2}}(p+1)}{(p-1)^2(M\delta)^{\frac{p-1}{2}}}.$$

Indeed, either y explodes before time $\frac{\delta}{4}$ or $\tilde{y} := y(\cdot + \frac{\delta}{4})$ satisfies

$$\begin{cases} \dot{\tilde{y}} \geq \left(\frac{p-1}{p+1}\right) \tilde{y}^{\frac{p+1}{2}} \\ \tilde{y}(0) \geq \frac{M\delta}{2}, \end{cases}$$

and $\tilde{y}$ can be seen to explode before time

$$\tilde{T} = \frac{2^{\frac{p+1}{2}}(p+1)}{(p-1)^2(M\delta)^{\frac{p-1}{2}}}$$

by performing the standard integration method. If M is taken sufficiently large then T can be made strictly smaller than $\frac{\delta}{2}$ which, by (4.7), implies that $\underline{\tau}^{(h),u} < \frac{\delta}{2}$ as desired.

Now let us show statement (ii). Given $K > 0$ let us take $M > 0$ as above and consider $u \in C_D([0,1])$ satisfying $S(u) \leq -M$ and $\|u\|_\infty \leq K$. Using Propositions 8.9 and 8.7 adapted to the system (4.4) we may find $\rho_{M,K} > 0$ sufficiently small so as to guarantee that for some small $0 < t_u < \frac{\delta}{2}$ any $v \in B_{\rho_{M,K}}(u)$ satisfies

$$\underline{S}^{(h)}(\underline{Z}^{(h),v}(t_u,\cdot)) \leq \underline{S}^{(h)}(u) + \frac{M}{4}$$

for all $0 \leq h < 1$. Notice that this is possible since the constants in Proposition 8.9 adapted to this context can be taken independent from h provided that h remains bounded. These constants still depend on $\|u\|_\infty$ though, so that the choice of $\rho_{M,K}$ will inevitably depend on both M and K. Next, let us take $0 < h_{M,K} < 1$ so as to guarantee that $\underline{S}^{(h)}(u) \leq -\frac{3M}{4}$ for every $0 \leq h < h_{M,K}$. Notice that, since $\underline{S}^{(h)}(u) \leq S(u) + h(K^p + K)$, it is possible to choose $h_{M,K}$ depending only on M and K. Thus, for any $v \in B_{\rho_{M,K}}(u)$ we obtain $\underline{S}^{(h)}(\underline{Z}^{(h),v}(t_u, \cdot)) \leq -\frac{M}{2}$ which, by the choice of M, implies that $\underline{\tau}^{(h),v} < t_u + \frac{\delta}{2} < \delta$. This concludes the proof. □

Let us observe that the system $\overline{Z}^{(0),u}$ coincides with U^u for every $u \in C_D([0,1])$. Thus, by the previous lemma we obtain the following corollary.

Corollary 4.4. The application $u \mapsto \tau^u$ is continuous on $\mathcal{D}_e$.

Proof. Given $u \in \mathcal{D}_e$ and $\delta > 0$ we show that there exists $\rho > 0$ such that for all $v \in B_\rho(u)$ we have

$$-\delta + \tau^u < \tau^v < \tau^u + \delta.$$

To see this we first notice that by Proposition 8.3 there exists $\rho_1 > 0$ such that $-\delta + \tau^u < \tau^v$ for any $v \in B_{\rho_1}(u)$. Moreover, by (i) in Lemma 4.3 we may take $M, \tilde{\rho}_2 > 0$ such that $\tau^{\tilde{v}} < \delta$ for any $\tilde{v} \in B_{\tilde{\rho}_2}(\tilde{u})$ with $\tilde{u} \in C_D([0,1])$ verifying $S(\tilde{u}) \leq -M$. For any such M, by Proposition 3.4 we may find some $0 < t_M < t^u$ such that $S(U^u(t_M, \cdot)) \leq -M$ and using Proposition 8.3 we may take $\rho_2 > 0$ such that $U^v(t_M, \cdot) \in B_{\tilde{\rho}_2}(U^u(t_M, \cdot))$ for any $v \in B_{\rho_2}(u)$. This implies that $\tau^v < t_M + \delta < \tau^u + \delta$ for all $v \in B_{\rho_2}(u)$ and thus by taking $\rho = \min\{\rho_1, \rho_2\}$ we obtain the result. □

The following two lemmas provide the necessary tools to obtain the uniformity in the upper bound claimed in Theorem 4.1.

Lemma 4.5. Given $M > 0$ and $u \in \mathcal{D}_e$ let us define the quantities

$$\mathcal{T}_M^u = \inf\{t \in [0, \tau^u) : S(U^u(t, \cdot)) < -M\} \qquad \text{and} \qquad \mathcal{R}_M^u = \sup_{0 \leq t \leq \mathcal{T}_M^u} \|U^u(t, \cdot)\|_\infty.$$

Then the applications $u \mapsto \mathcal{T}_M^u$ and $u \mapsto \mathcal{R}_M^u$ are both upper semicontinuous on $\mathcal{D}_e$.

Proof. We must see that the sets $\{\mathcal{T}_M < \alpha\}$ and $\{\mathcal{R}_M < \alpha\}$ are open in $\mathcal{D}_e$ for all $\alpha > 0$. But the fact that $\{\mathcal{T}_M < \alpha\}$ is open follows at once from Proposition 8.9 and $\{\mathcal{R}_M < \alpha\}$ is open by Proposition 8.3. □

Lemma 4.6. For each $u \in \mathcal{D}_e^+$ let us define the quantity

$$\mathcal{I}^u := \inf_{(t,x) \in [0,\tau^u) \times [0,1]} U^u(t,x).$$

Then the application $u \mapsto \mathcal{I}^u$ is lower semicontinuous on $\mathcal{D}_e^+$.

Proof. Notice that $\mathcal{I}^u \leq 0$ for any $u \in \mathcal{D}_e^+$ since $U^u(t,0) = U^u(t,1) = 0$ for all $t \in [0, \tau^u)$. Therefore, it will suffice to show that the sets $\{\alpha < \mathcal{I}\}$ are open in $\mathcal{D}_e^+$ for every $\alpha < 0$. With this purpose in mind, given $\alpha < 0$ and $u \in \mathcal{D}_e^+$ such that $\alpha < \mathcal{I}^u$, take $\beta_1, \beta_2 < 0$ such that $\alpha < \beta_1 < \beta_2 < \mathcal{I}^u$ and let y be the solution to the ordinary differential equation

$$\begin{cases} \dot{y} = -|y|^p \\ y(0) = \beta_2. \end{cases} \tag{4.8}$$

Define $t_\beta := \inf\{t \in [0, t^y_{max}) : y(t) < \beta_1\}$, where t^y_{max} denotes the explosion time of y. Notice that by the lower semicontinuity of S for any $M > 0$ we have $S(U^u(\mathcal{T}_M^u, \cdot)) \leq -M$ and thus, by Lemma 4.3, we may choose M such that

$$\sup_{v \in B_\rho(U^u(\mathcal{T}_M^u, \cdot))} \tau^v < t_\beta \tag{4.9}$$

for some small $\rho > 0$. Moreover, if $\rho < \mathcal{I}^u - \beta_2$ then every $v \in B_\rho(U^u(\mathcal{T}^u_M, \cdot))$ satisfies $\inf_{x\in[0,1]} v(x) \geq \beta_2$ so that U^v is in fact a supersolution to the equation (4.8). By (4.9) this implies that $v \in \mathcal{D}^+_e$ and $\mathcal{I}^v \geq \beta_1 > \alpha$. On the other hand, by Proposition 8.3 we may take $\delta > 0$ sufficiently small so that for every $w \in B_\delta(u)$ we have $\mathcal{T}^u_M \leq \tau^w$ and

$$\sup_{t\in[0,\mathcal{T}^u_M]} \|U^w(t,\cdot) - U^u(t,\cdot)\|_\infty < \rho.$$

Combined with the previous argument, this yields the inclusion $B_\delta(u) \subseteq \mathcal{D}^+_e \cap \{\alpha < \mathcal{I}\}$. In particular, this shows that $\{\alpha < \mathcal{I}\}$ is open and thus concludes the proof. □

Remark 4.7. The preceding proof shows, in particular, that the set $\mathcal{D}^+_e$ is open.

The conclusion of the proof of Theorem 4.1 is contained in the next proposition.

Proposition 4.8. Given any $\mathcal{D}^*_e$-compactifiable set $\mathcal{K}$ and $\delta > 0$ there exists a constant $C > 0$ such that

$$\sup_{u\in\mathcal{K}} P_u(\tau_\varepsilon > \tau + \delta) \leq e^{-\frac{C}{\varepsilon^2}}. \tag{4.10}$$

Proof. Since $\mathcal{D}^-_e = -\mathcal{D}^+_e$ and $U^{-u} = -U^u$ for $u \in C_D([0,1])$, without loss of generality we may assume that $\mathcal{K}$ is contained in $\mathcal{D}^+_e$. Let us begin by noticing that for any $M > 0$

$$\mathcal{T}_M := \sup_{u\in\mathcal{K}} \mathcal{T}^u_M < +\infty \qquad \text{and} \qquad \mathcal{R}_M := \sup_{u\in\mathcal{K}} \mathcal{R}^u_M < +\infty.$$

Indeed, by Proposition 8.2 we may choose $t_0 \geq 0$ small so as to guarantee that the orbits $\{U^u(t,\cdot) : 0 \leq t \leq t_0, u \in \mathcal{K}\}$ remain uniformly bounded and the family $\{U^u(t_0,\cdot) : u \in \mathcal{K}\}$ is contained in a compact set $\mathcal{K}' \subseteq \mathcal{D}^+_e$ at a positive distance from $\partial\mathcal{D}^+_e$. But then we have

$$\mathcal{T}_M \leq t_0 + \sup_{u\in\mathcal{K}'} \mathcal{T}^u_M \qquad \text{and} \qquad \mathcal{R}_M \leq \sup_{0\leq t\leq t_0, u\in\mathcal{K}} \|U^u(t,\cdot)\|_\infty + \sup_{u\in\mathcal{K}'} \mathcal{R}^u_M$$

and both right hand sides are finite due to Lemma 4.5 and the fact that $\mathcal{T}^u_M$ and $\mathcal{R}_M$ are both finite for each $u \in \mathcal{D}_e$ by Proposition 3.4. Similarly, by Lemma 4.6 we also have

$$\mathcal{I}_\mathcal{K} := \inf_{u\in\mathcal{K}} \mathcal{I}^u > -\infty.$$

Now, for each $u \in \mathcal{K}$ and $\varepsilon > 0$ by the Markov property we have for any $\rho \in (0,1)$

$$P_u(\tau_\varepsilon > \tau + \delta) \leq P(d_{\mathcal{T}_M \wedge \mathcal{T}^{(\mathcal{R}_M+1),u}_\varepsilon}(U^{u,\varepsilon}, U^u) > \rho) + \sup_{v\in B_\rho(U^u(\mathcal{T}^u_M,\cdot))} P_v(\tau_\varepsilon > \delta). \tag{4.11}$$

The first term on the right hand side is taken care of by (2.4) so that in order to show (4.10) it only remains to deal with the second term by choosing M and ρ appropriately. The argument given to deal with this term is similar to that of the proof of Lemma 4.6. Let y be the solution to the ordinary differential equation

$$\begin{cases} \dot{y} = -|y|^p - |y|^{p-1} - 1 \\ y(0) = \mathcal{I}_\mathcal{K} - \frac{1}{2}. \end{cases} \tag{4.12}$$

Define $t_\mathcal{I} := \inf\{t \in [0, t^y_{max}) : y(t) < \mathcal{I}_\mathcal{K} - 1\}$, where t^y_{max} denotes the explosion time of y. By Lemma 4.3, we may choose M such that

$$\sup_{v\in B_{\rho_M}(U^u(\mathcal{T}^u_M,\cdot))} \tau^{(h),v} < \min\{\delta, t_\mathcal{I}\} \tag{4.13}$$

for all $0 \leq h < h_M$, where $\rho_M > 0$ and $h_M > 0$ are suitable constants. The key observation here is that, since $\mathcal{R}_M < +\infty$, we may choose these constants so as not to depend on u but rather on M and $\mathcal{R}_M$ themselves. Moreover, if $\rho_M < \frac{1}{2}$ then every $v \in B_{\rho_M}(U^u(\mathcal{T}^u_M,\cdot))$ satisfies $\inf_{x\in[0,1]} v(x) \geq \mathcal{I}_\mathcal{K} - \frac{1}{2}$ so that $\underline{Z}^{(h),v}$ is in fact a supersolution to the equation

(4.12) for all $0 \leq h < \min\{h_M, 1\}$. By (4.13) the former implies that $\underline{Z}^{(h),v}$ explodes through $+\infty$ and that it remains bounded from below by $\mathcal{I}_{\mathcal{K}} - 1$ until its explosion time which, by (4.13), is smaller than δ. In particular, we see that if $\|V^{0,\varepsilon}\|_\infty < \min\{1, \frac{h_M}{p2^{p-1}}\}$ then $Z^{v,\varepsilon}$ explodes before $\underline{Z}^{(h),v}$ does, so that we have that $\tau_\varepsilon < \delta$ under such conditions. Hence, we conclude that

$$\sup_{v \in B_{\rho_M}(U^u(\mathcal{T}^u_M,\cdot))} P_v(\tau_\varepsilon > \delta) \leq P\left(\sup_{t\in[0,\delta]} \|V^{0,\varepsilon}(t,\cdot)\|_\infty \geq \min\left\{1, \frac{h_M}{p2^{p-1}}\right\}\right)$$

which, by the upper bound in the LDP for the family $(V^{0,\varepsilon})_{\varepsilon>0}$, gives the desired control on the second term in the right hand side of (4.11). Thus, by taking $\rho := \rho_M$ in (4.11), we obtain the result.

□

This last proposition in fact shows that for $\delta > 0$ and a given $\mathcal{D}^*_\varepsilon$-compactifiable there exist constants $M, C > 0$ such that

$$\sup_{u\in\mathcal{K}} P_u(\tau_\varepsilon > \mathcal{T}^u_M + \delta) \leq e^{-\frac{C}{\varepsilon^2}}.$$

By using the fact that $\mathcal{T}_M < +\infty$ for all $M > 0$ we obtain the following useful corollary.

Corollary 4.9. For any $\mathcal{D}^*_\varepsilon$-compactifiable set $\mathcal{K}$ and $\delta > 0$ there exist constants $\tau_{\mathcal{K}}, C > 0$ such that

$$\sup_{u\in\mathcal{K}} P_u(\tau_\varepsilon > \tau_{\mathcal{K}}) \leq e^{-\frac{C}{\varepsilon^2}}.$$

5. Construction of an auxiliary domain

To study the behavior of the explosion time for initial data in $\mathcal{D}_0$ it is convenient to introduce an auxiliary bounded domain $G \subseteq C_D([0,1])$ containing a neighborhood B_c of the stable equilibrium and such that for any initial data $u \in B_c$ the escape time from this domain is asymptotically equivalent to the explosion time. By doing so we can then reduce our original problem to a simpler one: characterizing the escape from this domain. This becomes a simpler problem because, since the escape only depends on the behavior of the system while it remains inside a bounded region, local large deviation estimates can be successfully applied to its study. This approach is not new, it was originally proposed in [10] to study the finite-dimensional double-well potential model. However, in our present setting the construction of this auxiliary domain is much more involved and, as a matter of fact, a priori it is not even clear that such a domain exists for every value of $p > 1$. The aim of this section is to construct such a domain for $1 < p < 5$. The following lemma will play a key role in this. Its proof was communicated to us by Philippe Souplet.

Lemma 5.1. If $1 < p < 5$ then the set $\{u \in \overline{\mathcal{D}_0} : S(u) \leq a\}$ is compact in $C_D([0,1])$ for any $a > 0$.

Proof. For $a > 0$ and $v \in \{u \in \overline{\mathcal{D}_0} : S(u) \leq a\}$ consider $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$\psi(t) := \int_0^1 (U^v(t,\cdot))^2.$$

A direct computation shows that for every $t_0 > 0$ the function ψ satisfies

$$\frac{d\psi(t_0)}{dt} = -4S(U^v(t,\cdot)) + 2\left(\frac{p-1}{p+1}\right)\int_0^1 |U^v(t,\cdot)|^{p+1}.$$

By Proposition 8.7 and Hölder's inequality we then obtain

$$\frac{d\psi(t_0)}{dt} \geq -4a + 2\left(\frac{p-1}{p+1}\right)(\psi(t_0))^{\frac{p+1}{2}}$$

which implies that $\psi(0) \leq B := \left[2a\left(\frac{p+1}{p-1}\right)\right]^{\frac{2}{p+1}}$ since otherwise ψ (and therefore U^v) would explode in finite time. Now, by the Gagliardo-Niremberg interpolation inequality (recall that v is absolutely continuous since $S(v) < +\infty$)

$$\|v\|_\infty^2 \leq C_{GN}\|v\|_{L^2}\|\partial_x v\|_{L^2},$$

we obtain

$$\int_0^1 |v|^{p+1} \leq \|v\|_{L^2}^2\|v\|_\infty^{p-1} \leq C_{GN}^{\frac{p-1}{2}} B^{\frac{p+3}{4}}\|\partial_x v\|_{L^2}^{\frac{p-1}{2}} \leq C_{GN}^{\frac{p-1}{2}} B^{\frac{p+3}{4}}(2a + \int_0^1 |v|^{p+1})^{\frac{p-1}{4}}$$

which for $p < 5$ implies the bound

$$\int_0^1 |v|^{p+1} \leq B' := \max\left\{2a, \left[C_{GN}^{\frac{p-1}{2}} B^{\frac{p+3}{4}} 2^{\frac{p-1}{4}}\right]^{\frac{4}{5-p}}\right\}. \tag{5.1}$$

Since $S(v) \leq a$ we see that (5.1) implies the bound $\|\partial_x v\|_{L^2} \leq \sqrt{2B'}$. Thus, we conclude

$$\|v\|_\infty \leq \left(C_{GN}^{p-1} 2BB'\right)^{\frac{1}{4}}$$

which shows that $\{u \in \overline{\mathcal{D}_0} : S(u) \leq a\}$ is bounded in $C_D([0,1])$. On the other hand, by the absolute continuity of v we have that for any $x \leq y \in [0,1]$

$$|v(y) - v(x)| \leq \int_x^y |\partial_x v| \leq \|\partial_x v\|_{L^2}\sqrt{|x-y|} \leq \sqrt{2B'|x-y|}$$

which shows $\{u \in \overline{\mathcal{D}_0} : S(u) \leq a\}$ is also equicontinuous. By the Arzela-Ascoli theorem, we conclude that $\{u \in \overline{\mathcal{D}_0} : S(u) \leq a\}$ has compact closure. Finally, since this set is also closed by the lower semicontinuity of S (see Proposition 8.8), we conclude the result. □

Remark 5.2. The proof of Lemma 5.1 is the only instance throughout our work in which the assumption $p < 5$ is used. As a matter of fact, we only require the weaker condition that there exists $\alpha > 0$ such that the set $\{u \in \overline{\mathcal{D}_0} : S(u) \leq S(z)+\alpha\}$ is compact. However, determining the validity of this condition for arbitrary $p > 1$ does not seem simple.

The following lemma will also play an important role in the construction.

Lemma 5.3. If $\mathcal{D} \subseteq C_D([0,1])$ is open then for any compact set $\mathcal{K} \subseteq \mathcal{D}$ there exists $\delta > 0$ such that the δ-neighborhood of $\mathcal{K}$

$$\mathcal{K}_\delta := \{u \in C_D([0,1]) : d(u, \mathcal{K}) \leq \delta\}$$

is $\mathcal{D}$-compactifiable.

Proof. It follows from Propositions 8.3 and 8.5 that for any $u \in \mathcal{D}$ there exists some $\delta_u > 0$ such that the ball $B_{\delta_u}(u)$ is $\mathcal{D}$-compactifiable. Now, by the compactness of $\mathcal{K}$ we can select $u_1, \ldots, u_k \in \mathcal{K}$ such that

$$\mathcal{K} \subseteq \bigcup_{i=1}^k B_{\frac{\delta_{u_i}}{2}}(u_i).$$

It is then straightforward to check that for $\delta := \min_{i=1,\ldots,k} \frac{\delta_{u_i}}{2}$ we have

$$\mathcal{K}_\delta \subseteq \bigcup_{i=1}^k B_{\delta_{u_i}}(u_i).$$

Finally, since the finite union of $\mathcal{D}$-compactifiable sets is again $\mathcal{D}$-compactifiable, it follows that $\mathcal{K}_\delta$ is $\mathcal{D}$-compactifiable since it is a subset of a $\mathcal{D}$-compactifiable set. □

Before we can carry on with the next proposition, we need to introduce some definitions.

Definition 5.4. Given $T > 0$ and $\varphi \in C_D([0,T] \times [0,1])$ we define the *rate* $I(\varphi)$ of φ by the formula

$$I(\varphi) := I_T^{\varphi(0,\cdot)}(\varphi),$$

where $I_T^{\varphi(0,\cdot)}$ is defined as in Section 2.3.

Definition 5.5. We say that a function $\varphi \in C_D([0,T]\times[0,1])$ is *regular* if both derivatives $\partial_t \varphi$ and $\partial^2_{xx}\varphi$ exist and belong to $C_D([0,T] \times [0,1])$.

Proposition 5.6. Given $T > 0$, for any $\varphi \in C_D \cap W_2^{1,2}([0,T] \times [0,1])$ such that $\partial^2_{xx}\varphi(0,\cdot)$ exists and belongs to $C_D([0,1])$ we have that

$$I(\varphi) \geq 2\left[\sup_{0\leq T'\leq T} \left(S(\varphi(T',\cdot)) - S(\varphi(0,\cdot))\right)\right]. \tag{5.2}$$

Proof. Assume first that φ is regular. Using that $(x-y)^2 = (x+y)^2 - 4xy$ for $x, y \in \mathbb{R}$, for any $0 \leq T' \leq T$ we obtain that

$$\begin{aligned} I(\varphi) &= \frac{1}{2}\int_0^T \int_0^1 |\partial_t\varphi - \partial^2_{xx}\varphi - g(\varphi)|^2 \geq \frac{1}{2}\int_0^{T'} \int_0^1 |\partial_t\varphi - \partial^2_{xx}\varphi - g(\varphi)|^2 \\ &= \frac{1}{2}\int_0^{T'} \int_0^1 \left[|\partial_t\varphi + \partial^2_{xx}\varphi + g(\varphi)|^2 - 4\left(\partial^2_{xx}\varphi + g(\varphi)\right)\partial_t\varphi\right] \\ &= \frac{1}{2}\int_0^{T'} \left[\left(\int_0^1 |\partial_t\varphi + \partial^2_{xx}\varphi + g(\varphi)|^2\right) + 4\frac{dS(\varphi(t,\cdot))}{dt}\right] \\ &\geq 2\left(S(\varphi(T',\cdot)) - S(\varphi(0,\cdot))\right). \end{aligned}$$

Taking supremum on T' yields the result in this particular case. Now, if φ is not necessarily regular then by [8, Theorem 6.9] we may take a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of regular functions converging to φ on $C_{D_{\varphi(0,\cdot)}}([0,T] \times [0,1])$ and such that $\lim_{n\to+\infty} I(\varphi_n) = I(\varphi)$ is satisfied. The result in the general case then follows from the validity of (5.2) for regular functions and the lower semicontinuity of S. □

In order to properly interpret the content of Proposition 5.6 we need to introduce the concept of *quasipotential* for our system. We do so in the following definitions.

Definition 5.7. Given $u, v \in C_D([0,1])$ a *path from u to v* is a continuous function $\varphi \in C_D([0,T] \times [0,1])$ for some $T > 0$ such that $\varphi(0,\cdot) = u$ and $\varphi(T,\cdot) = v$.

Definition 5.8. Given $u, v \in C_D([0,1])$ we define the *quasipotential* $V(u,v)$ from u to v by the formula

$$V(u,v) = \inf\{I(\varphi) : \varphi \text{ path from } u \text{ to } v\}.$$

Furthermore, given a subset $B \subseteq C_D([0,1])$ we define the quasipotential from u to B as

$$V(u,B) := \inf\{V(u,v) : v \in B\}.$$

We refer the reader to the appendix for a review of the properties of V we shall use.

In a limiting sense, made rigorous through the large deviations estimates in Section 2.3, the quasipotential $V(u,v)$ represents the energy cost for the stochastic system to travel from u to (an arbitrarily small neighborhood of) v. Notice that Lemma 5.1 implies that $\lim_{n\to+\infty} V(\mathbf{0}, \partial B_n \cap \mathcal{D}_0) = +\infty$, which says that the energy cost for the stochastic system starting from $\mathbf{0}$ to explode in a finite time while remaining inside $\mathcal{D}_0$ is infinite. Thus, should explosion occur, it would involve the system stepping outside $\mathcal{D}_0$ and crossing $\mathcal{W}$. In view of Proposition 5.6, the crossing of $\mathcal{W}$ will typical take place through $\pm z$ since the energy cost for performing such a feat is the lowest there. Therefore, if we wish the escape from G to capture the essential characteristics of the explosion phenomenon in the stochastic system (at least when starting from $\mathbf{0}$) then it is important to guarantee that this escape involves passing through (an arbitrarily small neighborhood of) $\pm z$. Not only this, but we also require that once the system escapes this domain then it explodes with overwhelming probability in a quick fashion, i.e. before some time τ^* which does not depend on ε. More precisely, we wish to consider a bounded domain $G \subseteq C_D([0,1])$ verifying the following properties:

Conditions 5.9.

i. There exists $r_0 > 0$ such that $B_{2r_0} \subseteq \mathcal{D}_0 \cap G$.
ii. There exists $c > 0$ such that $B_c \subseteq B_{r_0}$ and for all $v \in B_c$ the solution U^v to (1.2) with initial datum v is globally defined and converges to $\mathbf{0}$ without escaping B_{r_0}.
iii. There exists a closed subset $\partial^{\pm z}$ of the boundary ∂G which satisfies
   - $V(\mathbf{0}, \partial G - \partial^{\pm z}) > V(\mathbf{0}, \partial^{\pm z}) = V(\mathbf{0}, \pm z)$.
   - •• $\partial^{\pm z}$ is $\mathcal{D}^*_{\varepsilon}$-compactifiable.

In principle, we have seen that such a domain is useful to study the behavior of the explosion time whenever the initial datum of the stochastic system is (close to) the origin. Nevertheless, by the local estimate (2.4), when starting inside $\mathcal{D}_0$ the system will typically visit a small neighborhood of the origin before crossing $\mathcal{W}$ and thus such a choice of G will also be suitable to study the explosion time for arbitrary initial data in $\mathcal{D}_0$.

The construction of the domain G is done as follows. Since $\mathcal{D}_0$ is open we may choose $r_0 > 0$ such that B_{3r_0} is contained in $\mathcal{D}_0$. Furthermore, by the asymptotic stability of $\mathbf{0}$ we may choose $c > 0$ verifying (ii) in Conditions 5.9. Now, given $\zeta_1 > 0$ by Lemma 5.1 we may take $n_0 \in \mathbb{N}$ such that $n_0 > 3r_0$ and the set $\{u \in \overline{\mathcal{D}_0} : S(u) \leq S(z) + \zeta_1\}$ is contained in the interior of the ball B_{n_0-1}. We then define the pre-domain $\tilde{G}$ as

$$\tilde{G} := B_{n_0} \cap \overline{\mathcal{D}_0}. \tag{5.3}$$

Notice that since both B_{n_0} and $\overline{\mathcal{D}_0}$ are closed sets we have that

$$\partial\tilde{G} = (\mathcal{W} \cap B_{n_0}) \cup (\partial B_{n_0} \cap \mathcal{D}_0)$$

which, by the particular choice of n_0 and Proposition 8.7, implies $\min_{u \in \partial\tilde{G}} S(u) = S(z)$. By Propositions 5.6 and 8.8 we thus obtain $V(\mathbf{0}, \partial\tilde{G}) \geq \Delta$. Next, if for $u \in C_D([0,1])$ we let u^- denote the negative part of u, i.e. $u^- = \max\{-u, 0\}$, then since $z^- = \mathbf{0}$ we may find $\tilde{r}_z > 0$ such that $-u^- \in \mathcal{D}_0$ for any $u \in B_{\tilde{r}_z}(z)$. Finally, if for $r > 0$ we write $B_r(\pm z) := B_r(z) \cup B_r(-z)$ and take $r_z > 0$ such that $r_z \leq \tilde{r}_z$, $B_{2r_z}(\pm z)$ is contained in the interior of B_{n_0}, z is the unique equilibrium point of the system lying inside $B_{r_z}(z)$ and $B_{r_z}(z) \cap B_{r_z}(-z) = \emptyset$, then we define our final domain G as

$$G = \tilde{G} \cup B_{r_z}(\pm z).$$

Let us now check that this domain satisfies all the required conditions. We begin by noticing that (i) and (ii) in Conditions 5.9 are immediately satisfied by the choice of n_0.

Now, let us also observe that for any $r > 0$

$$\inf\{S(u) : u \in \partial\tilde{G} - B_r(\pm z)\} > S(z). \tag{5.4}$$

Indeed, if this were not the case then there would exist a sequence $(u_k)_{k\in\mathbb{N}} \in \partial\tilde{G} - B_{r_z}(\pm z)$ such that $\lim_{k\to+\infty} S(u_k) = S(z)$. Since $\inf_{u\in\partial B_{n_0}\cap\mathcal{D}_0} S(u) > S(z)$ holds by choice of n_0, we can assume that $u_k \in \mathcal{W} - B_r(\pm z)$ and $S(u_k) \leq S(z) + \zeta_1$ for all $k \in \mathbb{N}$. Therefore, we conclude by Lemma 5.1 that there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ which converges to some limit $u_\infty \in C_D([0,1])$ as $j \to +\infty$. Since the potential S is lower semicontinuous and $\mathcal{W}$ is both closed and invariant under the deterministic flow, by Proposition 8.7 we conclude that $u_\infty = \pm z$ which contradicts the fact that $(u_{k_j})_{j\in\mathbb{N}}$ was at a positive distance from these equilibria. Hence, we obtain (5.4). In particular, this implies that $V(\mathbf{0}, \partial\tilde{G} - B_r(\pm z)) > \Delta$ holds for any choice of $r > 0$. Let us then take $\zeta_2 > 0$ such that $\Delta + \zeta_2 < V(\mathbf{0}, \partial\tilde{G} - B_{\frac{r_z}{2}}(\pm z))$ and define

$$\tilde{\partial}^z := \{u \in \partial B_{r_z}(z) \cap \overline{\mathcal{D}_e} : V(\mathbf{0}, u) \leq \Delta + \zeta_2\}.$$

The set $\tilde{\partial}^z$ is nonempty, compact and contained in $\mathcal{D}_e^+$. Indeed, one can construct for each $\alpha > 0$ a path from $\mathbf{0}$ to $B_{r_z}(z) \cap \overline{\mathcal{D}_e}$ with rate function less than $\Delta + \alpha$, which immediately implies that $\tilde{\partial}^z$ is nonempty. This path is essentially obtained by going from $\mathbf{0}$ to z by describing the orbit given by the unstable manifold $\mathcal{W}_u^z$ in reverse order, then making a linear interpolation towards $(1+h)z$ for some $h \in (0, r_z)$ sufficiently small and ultimately following the deterministic flow until it reaches $B_{r_z}(z)$ (notice that this will eventually happen since $(1+h)z \in \mathcal{D}_e$ by Proposition 3.5 and $\mathcal{D}_e$ is invariant). We refer to [18, Lemma 4.3] for details on the construction. To see that it compact, we first notice that, being a subset of $B_{r_z}(z)$, we have

$$\sup_{u\in\tilde{\partial}^z} \|u\|_\infty \leq \|z\|_\infty + r_z < +\infty \tag{5.5}$$

which shows that $\tilde{\partial}^z$ is bounded. Furthermore, since

$$\sup_{u\in\tilde{\partial}^z} S(u) \leq S(z) + \frac{\zeta_2}{2}$$

by Proposition 5.6, using (5.5) and the lower semicontinuity of V (see Proposition 8.10), one can proceed as in the proof of Lemma 5.1 to show that $\tilde{\partial}^z$ is equicontinuous and thus also compact. Finally, to check that $\tilde{\partial}^z$ is contained in $\mathcal{D}_e^+$ we first show that $\tilde{\partial}^z \cap \mathcal{W} = \emptyset$. Indeed, if there existed some $u \in \tilde{\partial}^z \cap \mathcal{W}$ then, since $B_{r_z}(z)$ is contained in B_{n_0}, we would have that $u \in \mathcal{W} \cap B_{n_0}$ and therefore, since u belongs to $\partial B_{r_z}(z)$, that $u \in \partial\tilde{G} - B_{\frac{r_z}{2}}(\pm z)$ by definition of $\tilde{\partial}^z$. This would imply the contradiction $V(\mathbf{0}, \partial\tilde{G} - B_{\frac{r_z}{2}}(\pm z)) \leq \Delta + \zeta_2$ and therefore $\tilde{\partial}^z \cap \mathcal{W}$ must be empty. In particular, we obtain that $\tilde{\partial}^z$ is contained in $\mathcal{D}_e$. To see that it is in fact contained in $\mathcal{D}_e^+$ we note that, by the comparison principle and choice of $\tilde{r}_z$, we have

$$-\infty < \inf_{(t,x)\in[0,\tau^u)\times[0,1]} U^{-u^-}(t,x) \leq \inf_{(t,x)\in[0,\tau^u)\times[0,1]} U^u(t,x)$$

for all $u \in \tilde{\partial}^z$, so that the inclusion $\tilde{\partial}^z \subseteq \mathcal{D}_e^+$ is now immediate.

It follows from all these facts about $\tilde{\partial}^z$ and Lemma 5.3 that there exists some $\delta > 0$ such that the neighborhood $\tilde{\partial}^z_\delta$ is also $\mathcal{D}_e^+$-compactifiable, so that we may define

$$\partial^z := \tilde{\partial}^z_\delta \cap \partial B_{r_z}(z)$$

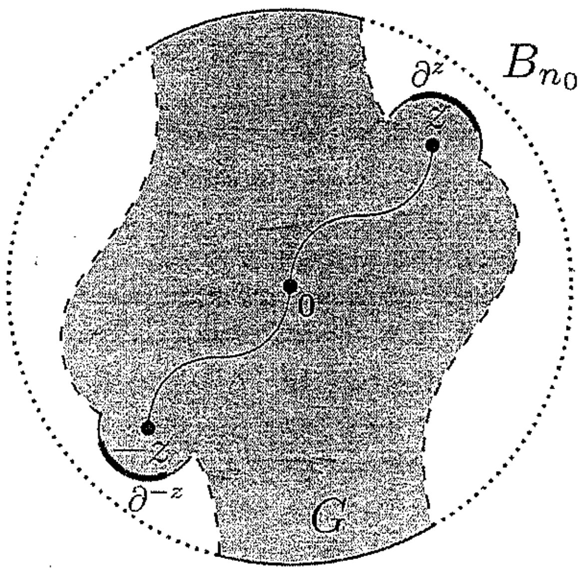

FIGURE 3. The auxiliary domain G

and set $\partial^{\pm z} := \partial^z \cup (-\partial^z)$. Since one can easily check that

$$\partial G = [\partial \tilde{G} - B_{r_z}(\pm z)] \cup [\partial B_{r_z}(\pm z) \cap \overline{\mathcal{D}_e}]$$

we conclude that $V(\mathbf{0}, \partial G - \partial^{\pm z}) \geq \Delta + \zeta_2$. On the other hand, by using Proposition 5.6 together with the existence of paths as described above, which go from $\mathbf{0}$ to $\tilde{\partial}^z$ by passing through z and have a rate function which can be made arbitrarily close to Δ, we get that $V(\mathbf{0}, \partial^z) = V(\mathbf{0}, \tilde{\partial}^z) = V(\mathbf{0}, \pm z) = \Delta$ from which one obtains

$$V(\mathbf{0}, \partial G - \partial^{\pm z}) > V(\mathbf{0}, \partial^z) = V(\mathbf{0}, \pm z).$$

Finally, since $-\partial^z$ is $\mathcal{D}_e^-$-compactifiable by the symmetry of (1.2), we conclude that $\partial^{\pm z}$ is $\mathcal{D}_e^*$-compactifiable and so condition (iii) also holds. See Figure 3.

Remark 5.10. Let us notice that, by Corollary 4.9, ($\bullet\bullet$) in Conditions 5.9 implies that there exist constants $\tau^*, C > 0$ such that

$$\sup_{u \in \partial^{\pm z}} P_u(\tau_\varepsilon > \tau^*) \leq e^{-\frac{C}{\varepsilon^2}}$$

for all $\varepsilon > 0$ sufficiently small. Since ($\bullet$) guarantees that the escape from G will typically take place through $\partial^{\pm z}$, this tells us that both τ_ε and $\tau_\varepsilon(\partial G)$ are asymptotically equivalent, so that it will suffice to study the escape from G in order to establish each of our results.

6. THE ESCAPE FROM G

The problem of escaping a bounded domain with similar characteristics to the ones detailed in Conditions 5.9 already appears in the literature. In [10, 15], the authors study the escape from a finite-dimensional domain containing a stable equilibrium and only one saddle point. Our domain G bears the additional difficulties of being infinite-dimensional and also of possibly containing other unstable equilibria besides $\pm z$. On the other hand, in [1] the author deals with an infinite-dimensional domain, but this domain has unstable equilibria only in its boundary and does not contain any of them in its interior as opposed to what happens in our current situation. Despite the fact that our domain does not quite fall into any of the cases studied before, all the results of interest in our present setting can still be obtained by combining the ideas from these previous works, eventually making some slight modifications along the way. We outline below the main results regarding the escape from the domain G and refer the reader to [18] for details on their proofs. Hereafter, $c > 0$ is taken as in Conditions 5.9. Also, for any given closed set $\Gamma \subseteq C_D([0,1])$ we write

$$\tau_\varepsilon^u(\Gamma) := \inf\{t > 0 : U^{u,\varepsilon}(t,\cdot) \in \Gamma\}.$$

Our first result deals with the asymptotic order of magnitude of the exit time from G.

Theorem 6.1. *Given $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in B_c} \left| P_u \left(e^{\frac{\Delta - \delta}{\varepsilon^2}} < \tau_\varepsilon(\partial G) < e^{\frac{\Delta + \delta}{\varepsilon^2}} \right) - 1 \right| \right] = 0.$$

The second result gives information about the typical escape routes chosen by U^ε.

Theorem 6.2. *The stochastic system verifies*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in B_c} P_u \left(U^\varepsilon(\tau_\varepsilon(\partial G), \cdot) \notin \partial^{\pm z} \right) \right] = 0.$$

Furthermore, if $\tilde{G}$ is the pre-domain constructed in Section 5, then for any $\delta > 0$

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in B_c} P_u \left(U^\varepsilon \left(\tau_\varepsilon(\partial \tilde{G}), \cdot \right) \notin B_\delta(\pm z) \right) \right] = 0. \tag{6.1}$$

The asymptotic distribution of the exit time is established in this third result.

Theorem 6.3. *For each $\varepsilon > 0$ define the normalization coefficient $\gamma_\varepsilon > 0$ by the relation*

$$P_0(\tau_\varepsilon(\partial G) > \gamma_\varepsilon) = e^{-1}.$$

Then there exists $\rho > 0$ such that for every $t \geq 0$

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in B_\rho} |P_u(\tau_\varepsilon(\partial G) > t\gamma_\varepsilon) - e^{-t}| \right] = 0. \tag{6.2}$$

Finally, the stability of time averages is shown in the fourth and last result.

Theorem 6.4. *There exists a sequence $(R_\varepsilon)_{\varepsilon > 0}$ with $\lim_{\varepsilon \to 0} R_\varepsilon = +\infty$ and $\lim_{\varepsilon \to 0} \frac{R_\varepsilon}{\gamma_\varepsilon} = 0$ such that given $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in B_c} P_u \left(\sup_{0 \leq t \leq \tau_\varepsilon(\partial G) - 3R_\varepsilon} \left| \frac{1}{R_\varepsilon} \int_t^{t+R_\varepsilon} f(U^\varepsilon(s,\cdot))ds - f(\mathbf{0}) \right| > \delta \right) \right] = 0$$

for any bounded continuous function $f : C_D([0,1]) \to \mathbb{R}$.

Remark 6.1. We would like to point out that the main technical point in the proof of Theorem 6.3 is to show that for small $\rho > 0$

$$\lim_{\varepsilon \to 0} \left[\sup_{u,v \in B_\rho} \left[\sup_{t > t_0} |P_u(\tau_\varepsilon(\partial G) > t\gamma_\varepsilon) - P_v(\tau_\varepsilon(\partial G) > t\gamma_\varepsilon)| \right] \right] = 0. \tag{6.3}$$

We do this by proceeding as in [2] with the help of the coupling of solutions with different initial data proposed in [14]. Some technical difficulties which are not present in [2] arise in the construction of the coupling due to the behavior of the source term g but, nonetheless, it is still possible to couple solutions with initial data sufficiently close to $\mathbf{0}$ so that (6.3) can be obtained. We refer to [18] for details.

7. Asymptotic behavior of τ_ε^u for $u \in \mathcal{D}_0$

We now use the analysis from Section 6 to derive our main results with respect to the metastable behavior of $U^{u,\varepsilon}$ for initial data $u \in \mathcal{D}_0$. We begin by showing that, uniformly over $\mathcal{D}_0$-compactifiable sets, for $u \in \mathcal{D}_0$ the system $U^{u,\varepsilon}$ typically visits a small neighborhood of $\mathbf{0}$ before explosion without ever exiting $\mathcal{D}_0$.

Lemma 7.1. For any $\mathcal{D}_0$-compactifiable set $\mathcal{K}$ and $\rho > 0$

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in\mathcal{K}}|P_u(\tau_\varepsilon(B_\rho) < \tau_\varepsilon) - 1|\right] = 0. \tag{7.1}$$

Proof. Note that for any $u \in \mathcal{D}_0$ the system U^u reaches the interior of $B_{\frac{\rho}{2}}$ in a finite time

$$\tau^u(B^\circ_{\frac{\rho}{2}}) := \inf\{t \geq 0 : d(U^u(t,\cdot), \mathbf{0}) < \delta\}$$

while remaining at all times inside the ball B_{r^u}, where $r^u := \sup_{t\geq 0}\|U^u(t,\cdot)\|_\infty < +\infty$. Furthermore, for any $\mathcal{D}_0$-compactifiable set $\mathcal{K}$ the quantities $\tau_{\mathcal{K},\frac{\rho}{2}} := \sup_{u\in\mathcal{K}}\tau^u(B^\circ_{\frac{\rho}{2}})$ and $r_\mathcal{K} := \sup_{u\in\mathcal{K}} r^u$ are both finite. Indeed, observe that for any $t_0 > 0$ we have that

$$\sup_{u\in\mathcal{K}}\tau^u(B^\circ_\delta) \leq t_0 + \sup_{u\in\mathcal{K}(t_0)}\tau^u(B^\circ_\delta). \tag{7.2}$$

Now, since the application $u \mapsto \tau^{u,+}(B_\delta)$ is upper semicontinuous (and finite) on $\mathcal{D}_0$ by the continuity of the solutions to (1.2) with respect to the initial datum (Proposition 8.3), it follows that the right-hand side of (7.2) is finite for $t_0 > 0$ sufficiently small since then $\mathcal{K}(t_0)$ has compact closure contained in $\mathcal{D}_0$. Similarly, to see that $r_\mathcal{K}$ is finite we note that for any $t_0 > 0$

$$r_\mathcal{K} \leq \sup_{u\in\mathcal{K},t\in[0,t_0]}\|U^u(t,\cdot)\|_\infty + \sup_{u\in\mathcal{K},t\geq t_0}\|U^u(t,\cdot)\|_\infty \tag{7.3}$$

and that by Proposition 8.2 the first term in the right-hand side of (7.3) is finite for every t_0 sufficiently small. That $\sup_{u\in\mathcal{K},t\geq t_0}\|U^u(t,\cdot)\|_\infty$ is finite then follows as before, due to the fact that the application $u \mapsto r^u$ is also both upper semicontinuous and finite on $\mathcal{D}_0$.

Finally, let us notice that if we write $\mathcal{T}_\varepsilon^\mathcal{K} := \tau_{\mathcal{K},\frac{\rho}{2}} \wedge \tau_\varepsilon^{(r_\mathcal{K}+1)}$ then for any $u \in \mathcal{K}$ we have the bound

$$P_u(\tau_\varepsilon \leq \tau_\varepsilon(B_\rho)) \leq P_u\left(\tau_\varepsilon(B_\rho) > \tau_{\mathcal{K},\frac{\rho}{2}}\right) + P_u\left(\tau_\varepsilon \leq \tau_{\mathcal{K},\frac{\rho}{2}}\right) \tag{7.4}$$

with

$$P_u\left(\tau_\varepsilon(B_\rho) > \tau_{\mathcal{K},\frac{\rho}{2}}\right) \leq P_u\left(d_{\mathcal{T}_\varepsilon^\mathcal{K}}(U^\varepsilon, U) > \min\left\{\frac{\rho}{2},\frac{1}{2}\right\}\right)$$

and

$$P_u\left(\tau_\varepsilon \leq \tau_{\mathcal{K},\frac{\rho}{2}}\right) \leq P_u\left(d_{\mathcal{T}_\varepsilon^\mathcal{K}}(U^\varepsilon, U) > \frac{1}{2}\right).$$

The uniform bounds given by (2.4) now allow us to conclude the result. □

The next step is to show that, for initial data in a small neighborhood of the origin, the explosion time and the exit time from G are asymptotically equivalent.

Lemma 7.2. If $\tau^* > 0$ is taken as in Remark 5.10 then

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in B_c} P_u(\tau_\varepsilon > \tau_\varepsilon(\partial G) + \tau^*)\right] = 0. \tag{7.5}$$

Proof. For any $u \in B_c$ the strong Markov property implies that

$$P_u(\tau_\varepsilon > \tau_\varepsilon(\partial G) + \tau^*) \le \sup_{v \in B_c} P_v\left(U^\varepsilon(\tau_\varepsilon(\partial G), \cdot) \notin \partial^{\pm z}\right) + \sup_{v \in \partial^{\pm z}} P_v(\tau_\varepsilon > \tau^*).$$

We may now conclude the result by using Theorem 6.2 and Remark 5.10. □

With these two lemmas at hand, we can now show the remaning results of Section 2.4. Indeed, Theorem 2.3 follows from Theorem 6.1 by using the strong Markov property together with Lemmas 7.1 and 7.2. Furthermore, Theorem 2.4 follows from Lemma 7.4 for $\rho = c$, where c is as in Conditions 5.9, together with (6.1) for $\delta > 0$ sufficiently small so as to guarantee that $B_\delta(\pm z)$ is contained in the interior of B_{n_0}, where n_0 is as in (5.3). Finally, Lemma 7.2 implies that $\lim_{\varepsilon \to 0} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 1$ from which, together with Lemma 7.4 and the strong Markov property, we get Theorems 2.5 and 2.6 by using Theorems 6.3 and 6.4. We leave the details to the reader, which are completely straightforward.

8. Appendix

8.1. Comparison principle.

Proposition 8.1. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz functions. For $u, v \in C([0,1])$ consider U^u and U^v the solutions of the equation

$$\partial_t U = \partial^2_{xx} U + f_1(U) + f_2(U)\dot{W}$$

with initial data u and v, respectively, and boundary conditions satisfying

$$P(U(t,\cdot)|_{\partial[0,1]} \ge V(t,\cdot)|_{\partial[0,1]} \text{ for all } t \ge 0) = 1.$$

Then, if $u \ge v$ we have that

$$P\left(U^u(t,x) \ge U^v(t,x) \text{ for all } t \ge 0, x \in [0,1]\right) = 1.$$

A proof of this result can be found on [6, p. 130]. Let us notice that by taking $f_2 \equiv 0$ one obtains a comparison principle for deterministic partial differen ial equations.

8.2. Growth and regularity estimates.

Proposition 8.2. Given a bounded set $B \subseteq C_D([0,1])$ there exists $t_B > 0$ such that $\tau^u > t_B$ for every $u \in B$ and

$$\sup_{u \in B, t \in [0,t_B]} \|U^u(t,\cdot)\|_\infty < +\infty.$$

Proof. For each $n \in \mathbb{N}$ let us consider a globally Lipschitz map g_n such that $g_n|_{B_n} \equiv g|_{B_n}$, and let $U^{(n),u}$ be the solution of (1.2) with g replaced by g_n. Notice that, by uniqueness of the solution of (1.2), for all $u \in C_D([0,1])$ we have that

$$U^u(t,x) \equiv U^{(n),u}(t,x)$$

for all $x \in [0,1]$ and $t \le \tau^{(n),u}$, where

$$\tau^{(n),u} := \inf\left\{t \ge 0 : \|U^{(n),u}(t,\cdot)\|_\infty \ge n\right\}$$

and also that, since g_n is globally Lipschitz, $U^{(n),u,\varepsilon}$ is globally defined. Now, if y_n^+, y_n^- respectively denote the solutions to the equations

$$\begin{cases} \dot{y}_n^{\pm} = g_n(y_n^{\pm}) \\ y_n^{\pm}(0) = \pm \sup_{u \in B} \|u\|_\infty \end{cases}$$

then by the continuity of y_n^+ and y_n^- there exists $t_{n,B} > 0$ sufficiently small so that

$$\sup_{t\in[0,t_{n,B}]} y_n^+(t) \leq \sup_{u\in B} \|u\|_\infty + 1 \qquad \text{and} \qquad \inf_{t\in[0,t_{n,B}]} y_n^-(t) \geq -\left(\sup_{u\in B}\|u\|_\infty + 1\right).$$

Hence, by the comparison principle we conclude that

$$\sup_{u\in B, t\in[0,t_{n,B}]} \|U^{(n),u}(t,\cdot)\|_\infty \leq \sup_{u\in B}\|u\|_\infty + 1.$$

In particular, if we take $n > \sup_{u\in B}\|u\|_\infty + 1$ then for any $u \in B$ we have $\tau^{(n),u} > t_{n,B}$ and therefore that

$$\sup_{u\in B, t\in[0,t_{n,B}]} \|U^u(t,\cdot)\|_\infty < +\infty.$$

since U^u coincides with $U^{(n),u}$ until $\tau^{(n),u}$. From this the result immediately follows. □

Proposition 8.3. The following local and pointwise growth estimates hold:

i. Given a bounded set $B \subseteq C_D([0,1])$ there exist $C_B, t_B > 0$ such that
   - $\tau^u > t_B$ for any $u \in B$
   - For any pair $u, v \in B$ and $t \in [0, t_B]$
   $$\|U^u(t,\cdot) - U^v(t,\cdot)\|_\infty \leq e^{C_B t}\|u - v\|_\infty.$$

ii. Given $u \in C_D([0,1])$ and $t \in [0,\tau^u)$ there exist $C_{u,t}, \delta_{u,t} > 0$ such that
   - $\tau^v > t$ for any $v \in B_{\delta_{u,t}}(u)$
   - For any $v \in B_{\delta_{u,t}}(u)$ and $s \in [0,t]$
   $$\|U^u(s,\cdot) - U^v(s,\cdot)\|_\infty \leq e^{C_{u,t}s}\|u - v\|_\infty.$$

Proof. These are standard continuity estimates with respect to the initial datum which can be found, for example, in [16]. □

Proposition 8.4. If $u \in C_D([0,1])$ then $\partial^2_{xx}U^u$ exists for any $t \in (0,\tau^u)$. Furthermore, for any bounded set $B \subseteq C_D([0,1])$ there exists a time $t_B > 0$ such that

- $\tau^u > t_B$ for any $u \in B$
- For any $t \in (0, t_B)$ we have $\sup_{u\in B}\left[\max\{\|\partial_x U^u(t,\cdot)\|_\infty, \|\partial^2_{xx}U^u(t,\cdot)\|_\infty\}\right] < +\infty$.

Proof. One can obtain this result by following the analysis in the proof of [1, Lemma A.1]. □

Proposition 8.5. For any bounded set $B \subseteq C_D([0,1])$ there exists $t_B > 0$ such that

- $\tau^u > t_B$ for any $u \in B$
- For any $t \in (0, t_B)$ there exist positive constants R_t, N_t such that for every $u \in B$ the function $U^u(t,\cdot)$ belongs to the compact set

$$\gamma_{R_t,N_t} = \{v \in C_D([0,1]) : \|v\|_\infty \leq R_t\,,\, |v(x) - v(y)| \leq N_t|x-y| \text{ for all } x,y \in [0,1]\}.$$

Proof. This is direct consequence of Propositions 8.2-8.4 and the mean value theorem. □

Proposition 8.6. The following local and pointwise growth estimates hold:

i. Given a bounded set $B \subseteq C_D([0,1])$ there exists $t_B > 0$ such that
   - $\tau^u > t_B$ for any $u \in B$
   - For any $t \in (0, t_B)$ there exists $C_{t,B} > 0$ such that for all $u, v \in B$
   $$\|\partial_x U^u(t,\cdot) - \partial_x U^v(t,\cdot)\|_\infty \leq C_{t,B}\|u - v\|_\infty.$$

ii. Given $u \in C_D([0,1])$ and $t \in (0,\tau^u)$ there exist $C_{u,t}, \delta_{u,t} > 0$ such that
   - $\tau^v > t$ for any $v \in B_{\delta_{u,t}}(u)$

- For any $v \in B_{\delta_{u,t}}(u)$

$$\|\partial_x U^u(t,\cdot) - \partial_x U^v(t,\cdot)\|_\infty \leq C_{u,t}\|u - v\|_\infty.$$

Proof. These estimates also follow from the analysis in the proof of [1, Lemma A.1]. □

8.3. Properties of the potential S.

Proposition 8.7. The mapping $t \mapsto S(U^u(t,\cdot))$ is monotone decreasing and continuous for any $u \in H_0^1((0,1))$.

Proof. An easy computation shows that

$$\frac{d}{dt}S(U^u(t,\cdot)) = -\int_0^1 (\partial_t U^u(t,x))^2\,dx \leq 0$$

from which the result follows. Details can be found in [17, Lemma 17.5]. □

Proposition 8.8. The potential S is lower semicontinuous.

Proof. Let $(v_k)_{k\in\mathbb{N}} \subseteq C_D([0,1])$ be a sequence converging to some limit $v_\infty \in C_D([0,1])$. We must check that

$$S(v_\infty) \leq \liminf_{k\to+\infty} S(v_k). \tag{8.1}$$

Notice that since $(v_k)_{k\in\mathbb{N}}$ is convergent in the supremum norm we have, in particular, that

$$\sup_{k\in\mathbb{N}} \|v_k\|_{L^{p+1}} < +\infty \tag{8.2}$$

and therefore that $\liminf_{k\to+\infty} S(v_k) > -\infty$. Hence, by passing to an subsequence if necessary, we may assume that the limit in (8.1) exists and is finite so that, in particular, we have that $(S(v_k))_{k\in\mathbb{N}}$ is bounded. This implies that each v_k is absolutely continuous and, furthermore, that the sequence $(v_k)_{k\in\mathbb{N}}$ is bounded in $H_0^1((0,1))$ by (8.2). Therefore, there exists some subsequence $(v_{k_j})_{j\in\mathbb{N}}$ which is weakly convergent in $H_0^1((0,1))$ and also strongly convergent in $L^2([0,1])$ to some limit v_∞^*. Notice that since $(v_k)_{k\in\mathbb{N}}$ converges in the supremum norm to v_∞, it also converges in L^q for every $q \geq 1$. In particular, we have that $v_\infty^* = v_\infty$ and thus, by the lower semicontinuity of the H_0^1-norm with respect to the weak topology, we conclude that

$$\|\partial_x v_\infty\|_{L^2} \leq \liminf_{j\to+\infty} \|\partial_x v_{k_j}\|_{L^2}.$$

Finally, since $(v_k)_{k\in\mathbb{N}}$ converges to v_∞ in L^{p+1} and we have $S(u) = \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}$ for all $u \in H_0^1$, we obtain (8.1). □

Proposition 8.9. Given $u \in C_D([0,1])$ and $t \in (0,\tau^u)$ there exist constants $C_{u,t}, \delta_{u,t} > 0$ such that

- $\tau^v > t$ for any $v \in B_{\delta_{u,t}}(u)$
- For any $v \in B_{\delta_{u,t}}(u)$ one has

$$\|S(U^u(t,\cdot)) - S(U^v(t,\cdot))\|_\infty \leq C_{u,t}\|u - v\|_\infty.$$

Proof. This is a direct consequence of Propositions 8.6 and 8.3. □

8.4. Properties of the quasipotential V.

Proposition 8.10. The mapping $u \mapsto V(\mathbf{0}, u)$ is lower semicontinuous on $C_D([0,1])$.

Proof. Let $(u_k)_{k\in\mathbb{N}} \subseteq C_D([0,1])$ be a sequence converging to some limit $u_\infty \in C_D([0,1])$. We must check that

$$V(\mathbf{0}, u_\infty) \leq \liminf_{k\to+\infty} V(\mathbf{0}, v_k). \tag{8.3}$$

If $S(u_\infty) = +\infty$ then by Proposition 5.6 we see that $V(\mathbf{0}, u_\infty) = +\infty$ and thus by the lower semicontinuity of S we conclude that $\lim_{v\to u} V(\mathbf{0}, v) = +\infty$ which establishes (8.3) in this particular case. Now, if $S(u_\infty) < +\infty$ then, by the lower semicontinuity of S and the continuity in time of the solutions to (1.2), given $\delta > 0$ there exists $t_0 > 0$ sufficiently small such that $S(U^{u_\infty}(t_0,\cdot)) > S(u_\infty) - \frac{\delta}{2}$. Moreover, by Proposition 8.3 we may even assume that t_0 is such that

$$\|U^{u_k}(t_0,\cdot) - U^{u_\infty}(t_0,\cdot)\|_\infty \leq 2\|u_k - u_\infty\|_\infty$$

for any $k \in \mathbb{N}$ sufficiently large. Thus, given k sufficiently large and a path φ_k from $\mathbf{0}$ to u_k we construct a path $\varphi_{k,\infty}$ from $\mathbf{0}$ to u_∞ by the following steps:

i. We start from $\mathbf{0}$ and follow φ_k until we reach u_k.
ii. From u_k we follow the deterministic flow U^{u_k} until time t_0.
iii. We then join $U^{u_k}(t_0,\cdot)$ and $U^{u_\infty}(t_0,\cdot)$ by a linear interpolation of speed one.
iv. From $U^{u_\infty}(t_0,\cdot)$ we follow the reverse deterministic flow until we reach u_∞.

By the considerations made in the proof of [18, Lemma 4.3] it is not difficult to see that there exists $C > 0$ such that for any $k \in \mathbb{N}$ sufficiently large we have

$$I(\varphi_{k,\infty}) \leq I(\varphi_k) + C\|u_k - u_\infty\|_\infty + \delta$$

so that we ultimately obtain

$$V(\mathbf{0}, u_\infty) \leq \liminf_{k\to+\infty} V(\mathbf{0}, u_k) + \delta.$$

Since $\delta > 0$ can be taken arbitrarily small we conclude (8.3). □

Proposition 8.11. For any $u, v \in C_D([0,1])$ the map $t \mapsto V(u, U^v(t,\cdot))$ is decreasing.

Proof. Given $0 \leq s < t$ and a path φ from u to $U^v(s,\cdot)$ we may extend φ to a path $\tilde{\varphi}$ from u to $U^v(t,\cdot)$ simply by following the deterministic flow afterwards. It follows that

$$V(u, U^v(t,\cdot)) \leq I(\tilde{\varphi}) = I(\varphi)$$

which, by taking infimum over all paths from u to $U^v(s,\cdot)$, yields the desired monotonicity. □

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References

[1] Stella Brassesco. Some results on small random perturbations of an infinite-dimensional dynamical system. *Stochastic Process. Appl.*, 38(1):33–53, 1991.

[2] Stella Brassesco. Unpredictability of an exit time. *Stochastic Process. Appl.*, 63(1):55–65, 1996.

[3] Marzio Cassandro, Antonio Galves, Enzo Olivieri, and Maria Eulália Vares. Metastable behavior of stochastic dynamics: a pathwise approach. *J. Statist. Phys.*, 35(5-6):603–634, 1984.

[4] Carmen Cortázar and Manuel Elgueta. Large time behaviour of solutions of a nonlinear reaction-diffusion equation. *Houston J. Math.*, 13(4):487–497, 1987.

[5] Carmen Cortázar and Manuel Elgueta. Unstability of the steady solution of a nonlinear reaction-diffusion equation. *Houston J. Math.*, 17(2):149–155, 1991.

[6] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[7] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[8] William G. Faris and Giovanni Jona-Lasinio. Large fluctuations for a nonlinear heat equation with noise. *J. Phys. A*, 15(10):3025–3055, 1982.

[9] Mark I. Freidlin and Alexander D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, third edition, 2012. Translated from the 1979 Russian original by Joseph Szücs.

[10] Antonio Galves, Enzo Olivieri, and Maria Eulália Vares. Metastability for a class of dynamical systems subject to small random perturbations. *Ann. Probab.*, 15(4):1288–1305, 1987.

[11] Pablo Groisman and Santiago Saglietti. Small random perturbations of a dynamical system with blow-up. *J. Math. Anal. Appl.*, 385(1):150–166, 2012.

[12] Fabio Martinelli, Enzo Olivieri, and Elisabetta Scoppola. Small random perturbations of finite- and infinite-dimensional dynamical systems: unpredictability of exit times. *J. Statist. Phys.*, 55(3-4):477–504, 1989.

[13] Henry P. McKean. *Stochastic integrals*. AMS Chelsea Publishing, Providence, RI, 2005. Reprint of the 1969 edition, with errata.

[14] Carl Mueller. Coupling and invariant measures for the heat equation with noise. *Ann. Probab.*, 21(4):2189–2199, 1993.

[15] Enzo Olivieri and Maria Eulália Vares. *Large deviations and metastability*, volume 100 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[16] C. V. Pao. *Nonlinear parabolic and elliptic equations*. Plenum Press, New York, 1992.

[17] Pavol Quittner and Philippe Souplet. *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[18] Santiago Saglietti. *Metastability for a PDE with blow-up and the FFG dynamics in diluted models*. PhD thesis, arXiv:1501.01731.

[19] Richard B. Sowers. Large deviations for a reaction-diffusion equation with non-Gaussian perturbations. *Ann. Probab.*, 20(1):504–537, 1992.

[20] John B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.

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EXACTAS UBA

Ref.: Exp. (FCEN) Nº 505.663/15

Ciudad de Buenos Aires, 10 ABR 2017

VISTO lo dispuesto en el artículo 50º del Estatuto Universitario que instituye el Año Sabático para profesores regulares de la Universidad,

CONSIDERANDO:

Que por Resolución CD Nº 467/16 se solicitó al Consejo Superior se autorice al Dr. Pablo José Groisman Profesor Regular Adjunto con dedicación exclusiva del Departamento de Matemática a hacer uso del Año Sabático,

Que por Resolución CS Nº 4782/16 se aprobó dicha solicitud otorgando licencia entre el 1 de agosto de 2016 y hasta el 28 de febrero de 2017,

Que en cumplimiento con el Art. 12º de la Resolución CS Nº 4518/93, el Dr. Pablo José Groisman presentó su informe de actividades,

Que es necesario cumplir con lo establecido por los Art. 13º y 14º de la citada resolución,

Lo aconsejado por la Comisión de Enseñanza, Programas y Planes de Estudio,

Lo actuado por este cuerpo en la sesión realizada en el día de la fecha,

En uso de las atribuciones que le confiere el art. 113º del Estatuto Universitario,

EL CONSEJO DIRECTIVO DE LA FACULTAD DE CIENCIAS EXACTAS Y NATURALES RESUELVE:

Artículo 1º: Aprobar el informe correspondiente a las actividades desempeñadas por el Dr. Pablo José Groisman durante su Año Sabático.

Artículo 2º: : Enviar un ejemplar del informe a la Biblioteca de esta Facultad.

Artículo 3º: Regístrese, notifíquese a quienes corresponda, elévese al Consejo Superior y cumplido, archívese.

RESOLUCIÓN CD Nº 0571

JORGE ZILBER
SECRETARIO ACADEMICO ADJUNTO

Dr. JUAN CARLOS REBOREDA
DECANO