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Self–similar asymptotics in non–symmetrical convergent viscous gravity currents

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Abstract. We investigate the evolution of the ridge produced by the non–symmetrical convergent motion of two substrates over which an initially uniform layer of a Newtonian liquid rests. The lack of symmetry of the flow arises because the substrates move with different velocities. We focus on the self–similar regimes that occur in this process. For short times, within the linear regime, the height and the width increase as $t^{1/2}$ and the profile is symmetric, independently of degree of asymmetry of the motion of the substrates. In the self–similar regime for large time, the height and the width of the ridge follow the same power laws as in the symmetric case, but the profiles are asymmetric.

1. Introduction

As in the companion paper [1] we investigate here the physics involved in the scaling laws for the evolution of the orogenic belts previously obtained by means of dimensional analysis [2]. The basic aspects of the process of mountain building and of the models developed to study it are discussed in the introduction of [1] and we shall not repeat them here. Here we extend our recent work [3] to include the effects of an asymmetric motion of the substrate. To this purpose we investigate a simple model that consists of a uniform layer of a Newtonian liquid resting over a horizontal substrate divided in two parts, that for $t > 0$ are pushed one against the other with different velocities. These asymmetric motions drag the liquid and produce a ridge as shown schematically in figure 1. Here we give the details of the evolution of the current that, as in the symmetric case, has two self–similar regimes that occur in different space–time domains and whose scaling laws we investigate.

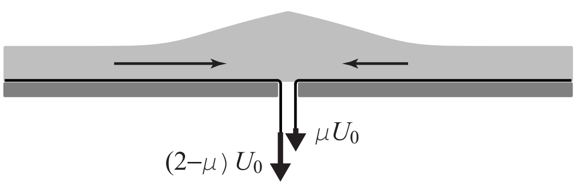

Figure 1. Formation of a ridge due to the asymmetric convergent motion of the substrates.

2. Basic equations

We consider a uniform liquid layer with thickness H_0 that rests on a rigid horizontal surface. At $T = 0$ the left half of the substrate starts to move with velocity $(2-\mu)U_0 > 0$ and the right half with velocity $-\mu U_0 < 0$, where $0 \le \mu \le 1$ (so that the left substrate has a velocity that is larger or equal than that of the right substrate). Then $\mu = 0$ corresponds to the fully asymmetric case where the right substrate remains at rest, and $\mu = 1$ correspond to the symmetric case[1] when both substrates move with equal speed U_0. The total mass inflow is $2U_0H_0$ for any μ.

Since we consider only $0 \le \mu \le 1$ the mass flux coming from the left is larger then the mass flux coming from the right. We assume that the flow is slow and dominated by viscosity so that we can apply the lubrication approximation [4] and we neglect surface effects. Let $H \equiv H(X, T)$ be the thickness of the liquid layer and $U \equiv U(X, T)$ the vertically averaged horizontal velocity. We define the dimensionless variables u, h, x, t by means of

$$U = U_0 u, \quad H = H_0 h,$$

$$X = \frac{g}{3\nu}\frac{H_0^3}{U_0}x, \quad T = \frac{g}{3\nu}\frac{H_0^3}{U_0^2}t,$$

where g is the acceleration of gravity and ν is the kinematic viscosity. Then the evolution is governed by the following equations:

$$\begin{aligned} h_t &= (\mu-2)h_x + \left(h^3h_x\right)_x, && \text{for } x<0 \\ h_t &= \mu h_x + \left(h^3h_x\right)_x, && \text{for } x>0 \end{aligned} \tag{1}$$

The boundary condition at $x \to \pm\infty$ is $h(\pm\infty, t) = 1$. At $x = 0$ the thickness h is continuous, but $\partial h/\partial x$ has a discontinuity given by

$$\left.\frac{\partial h}{\partial x}\right|_{x\to 0^-} - \left.\frac{\partial h}{\partial x}\right|_{x\to 0^+} = \frac{2}{h(0,t)^2}. \tag{2}$$

This condition implies that at $x = 0$ there is a mass flux form left to right, which did not happen in the symmetric case. Then for a given μ one must solve the equations (1) with the initial condition $h(x, 0) = 1$, and subject to the matching condition (2) in $x = 0$. Notice that that now the conservation of mass takes the form

$$\int_{-\infty}^{\infty} h_t dx = 2.$$

It should be mentioned that Buck and Sokoutis [5] performed sandbox experiment to produce in the laboratory a convergent current of this kind, with only the left substrate moving ($\mu = 0$). They also derived an evolution equation for the totally asymmetric case that is equivalent to equation(1) and presented a solution for the linear regime (see below) that however is incorrect.

3. Numerical solutions

The problem (1–2) does not have closed form solutions so that it must be solved numerically. Some results are shown in figure 2. Although the current is not symmetric the peak of the ridge remains at $x = 0$ for all t. In addition the profile is initially symmetric for all μ, but the asymmetry appears later and grows with t. As in the symmetric case, $h(x)$ has an inflexion point in each side, that moves towards larger $|x|$ as t increases, tending to approach the leading part of the ridge, where h is close to 1. It can be also noticed that the aspect ratio (heigth/width) of the ridge diminishes with time. Qualitatively similar results are found for all values of μ.

[1] This is the case considered in [3]; in this reference some preliminary results for the asymmetric case were also reported but notice that a different notation was employed. The formulae given there can be obtained from those given in the present paper by the substitution $U_0 \to U_0/(2-\mu)$ and $\mu \to (2-\mu)/\mu$.

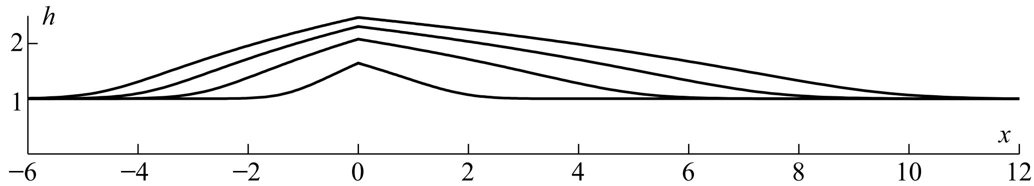

Figure 2. Numerical profiles with $\mu = 0.1$ and for $t = 0.6$, 2.4, 4.2, 6.

4. Behaviour for $t \ll 1$

For small t, when $h \approx 1$, we can write $h = 1 + z$ with $z \ll 1$ and linearize equations (1), that reduce to

$$\begin{aligned} z_t &= -(2-\mu) z_x + z_{xx}, \qquad &\text{for } x < 0 \\ z_t &= \mu z_x + z_{xx}, \qquad &\text{for } x > 0 \end{aligned}$$

At $x = 0$ the continuity of z must be satisfied and the matching condition takes the form

$$\left.\frac{\partial z}{\partial x}\right|_{x\to 0^-} - \left.\frac{\partial z}{\partial x}\right|_{x\to 0^+} = 2.$$

We propose now a self–similar solution with the form

$$z = t^{1/2} f_1(\psi) \quad \text{for } \psi < 0$$

$$z = t^{1/2} f_2(\psi) \quad \text{for } \psi > 0,$$

where $\psi = x/2t^{1/2}$. Here the exponents of t in front of $f_{1,2}$ and in the definition of ψ are all equal to 1/2 to ensure that the linearized matching condition at $x = 0$ and the mass conservation do not depend on t. This implies that in this regime the height and the width of the ridge increase as $t^{1/2}$ regardless of μ. However μ appears in the equations for $f_{1,2}$:

$$f_1'' + 2\left[\psi - t^{1/2}(2-\mu)\right] f_1' - 2 f_1 = 0 \quad \text{for } \psi < 0,$$

$$f_2'' + 2\left[\psi + t^{1/2}\mu\right] f_2' - 2 f_2 = 0 \quad \text{for } \psi > 0.$$

For t very small ($t \ll \min[x/2\mu, x/2(2-\mu)]$) we neglect the second terms in both brackets to obtain

$$f_1'' + 2\psi f_1' - 2 f_1 = 0 \quad \text{for } \psi < 0,$$

$$f_2'' + 2\psi f_2' - 2 f_2 = 0 \quad \text{for } \psi > 0.$$

The f functions must vanish at infinity. At $\psi = 0$ we require $f_1(0) = f_2(0)$ and $f_1'(0) - f_2'(0) = 4$. We finally obtain the following solution

$$f_1 = 2\frac{e^{-\psi^2}}{\sqrt{\pi}} + 2\psi\left[\mathrm{erf}(\psi) + 1\right], \qquad \text{if } \psi < 0,$$

$$f_2 = 2\frac{e^{-\psi^2}}{\sqrt{\pi}} + 2\psi\left[\mathrm{erf}(\psi) - 1\right], \qquad \text{if } \psi > 0,$$

where $\mathrm{erf}(\psi)$ is the error function. From this it can be observed that in this regime μ does not appear in the solution, that is symmetric despite that the motion of the substrates is not symmetric.

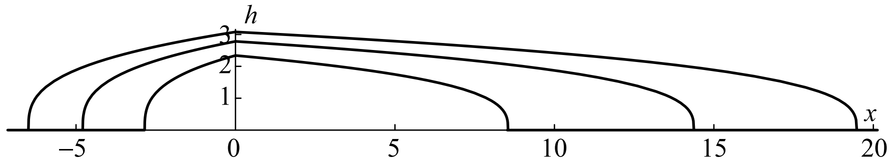

Figure 3. Self–similar solution (3–4) with μ =0.5 and t =10, 20 and 30.

5. Behavior for $t \gg 1$

For very large times the solution approaches a different self–similar regime. In this limit we shall assume first that $\mu \neq 0$, because the totally asymmetric case $\mu = 0$ must be studied separately. For $\mu \neq 0$ we seek a solution of the form

$$h = t^{1/4} F_1(\xi) \quad \text{for } \xi < 0,$$

$$h = t^{1/4} F_2(\xi) \quad \text{for } \xi > 0,$$

where $\xi = x/t^{3/4}$. This choice of the exponents of t is necessary in order to ensure that the matching condition as well as the conservation of mass are independent of t. With these hypothesis, assuming $t \gg 1$ and consequently retaining only the leading terms, and then integrating once the resulting equations we obtain the following equations for F_1 and F_2

$$-(2-\mu)F_1 + F_1^3 F_1' = \quad A_l \quad \text{if} \quad \xi < 0,$$

$$\mu F_2 + F_2^3 F_2' = \quad A_r \quad \text{if} \quad \xi > 0,$$

where A_l and A_r are integration constants. The conditions at $\xi = 0$ are now

$$F_1(0) = F_2(0) \equiv F_w, \quad F_1'(0) - F_2'(0) = \frac{2}{F_w^2}.$$

To satisfy the last condition it is necessary that $A_l = A_r \equiv A$. Notice however that it can be shown that the solution thus obtained does not satisfy the conditions at $\xi \to \pm\infty$. Following our previous work [3] we assume $A = 0$, and then the self–similar solution for $t \gg 1$ that satisfies mass conservation is

$$h = \begin{cases} 0 & \text{for} \quad -\infty < x < x_l, \\ h_w(1 - \frac{x}{x_l})^{1/3} & \text{for} \quad x_l < x < 0, \\ h_w(1 - \frac{x}{x_r})^{1/3} & \text{for} \quad 0 < x < x_r, \\ 0 & \text{for} \quad x_r < x < \infty, \end{cases} \tag{3}$$

where

$$h_w = \left[4\mu\,(2-\mu)\,t\right]^{1/4}, \quad x_l = -\frac{h_w^3}{3(2-\mu)}, \quad x_r = \frac{h_w^3}{3\mu}. \tag{4}$$

Since $h \gg 1$ in the present regime, $x_{l,r}$ can be called the "fronts" of the current at the left and the right, respectively. From (3–4) we see that the right front is farther from $x = 0$ than the left front, as can be expected since the flow from the left is larger than that from the right (see figure 3). From (4) we have $|x_r/x_l| = (2-\mu)/\mu$, the fraction of the total mass that accumulates for $x > 0$ is constant. The width of the ridge $x_r - x_l$ decreases with μ, while its height as well as its aspect ratio increase (see figure 4). The powers of t in the expressions of the width and the height of the profile agree with those of the scaling laws obtained in [2].

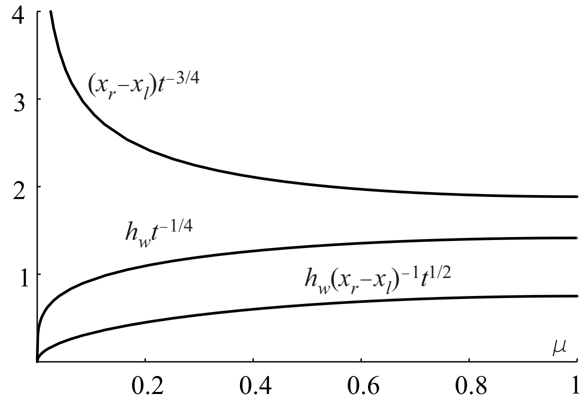

Figure 4. Dependence of the width, height and aspect ratio with $\mu > 0$.

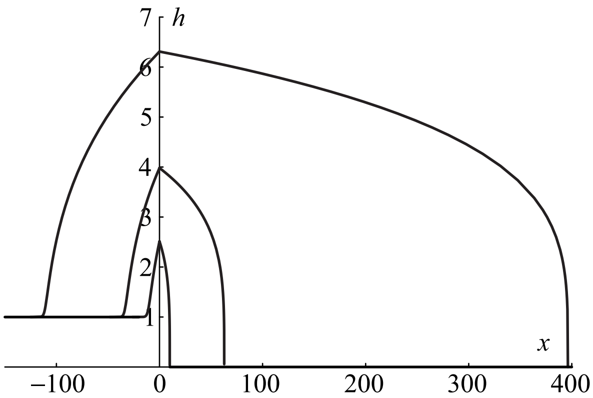

Figure 5. Solution (5–6) in the totaly asymmetric case for t =10, 100 and 1000.

Clearly the previous treatment fails for $\mu = 0$, so that this case must be considered separately. We proceed as before, but now we cannot set $A = 0$ because this choice leads to a solution independent of x for $x > 0$. Then we take $A = -t^{-1/4}$ so that $h(x \to -\infty) = 1$. It can be shown that this choice allows to obtain an approximate solution for $\mu = 0$ and $t \gg 1$ of the form

$$\begin{aligned} x &= \tfrac{(h^3 - h_w^3)}{3} + \tfrac{(h^2 - h_w^2)}{2} + h - h_w - \log\left(\tfrac{h_w - 1}{h-1}\right) \quad \text{for} \quad -\infty < x < 0, \\ h &= \begin{cases} h_w(1 - \frac{x}{x_r})^{1/4} & \text{for} \quad 0 < x < x_r, \\ 0 & \text{for} \quad x_r < x < \infty, \end{cases} \end{aligned} \tag{5}$$

where

$$h_w = (10t)^{1/5}, \quad x_r = \frac{h_w^4}{4}. \tag{6}$$

Notice that in the totaly asymmetric case the height of the ridge growths as $t^{1/5}$ and that $x_r \propto t^{4/5}$. On the other hand if we define $x_l < 0$ as the place where $h = \text{const.} \approx 1$, it can be shown that $x_l \propto t^{3/5}$. Then the fraction of the total mass of the ridge that remains at $x < 0$ decreases as $t^{-1/5}$. In the figure 5 we show this solution at different times. This totally asymmetric approximate solution is not self–similar. However the part of the solution corresponding to $x > 0$ is self–similar. The part corresponding to $x < 0$ is actually quasi–self–

similar [3; 6; 7] and for very large t tends to self–similarity, but of a different kind as that for $x > 0$, since $x_l \propto t^{3/5}$ and $x_r \propto t^{4/5}$.

6. Conclusions

The main results we have obtained are that the lack of symmetry of the flow does not modify the scaling laws that govern the growth of the ridge, that are the same as those obtained in the symmetric case [3]. The maximum height of the ridge remains at $x = 0$ for all t, regardless of the asymmetry. In the linear self–similar regime (small t) the ridge is symmetric around $x = 0$, its shape does not depend on μ, and its height and its width scale as $t^{1/2}$. This solution is given by f_1 and f_2. For larger t the ridge becomes asymmetric and the profile for $x < 0$ is narrower and steeper than for $x > 0$. For very large t and $0 < \mu \leq 1$ a self–similar regime occurs given by (3) and (4). The profile of the ridge is asymmetric, but its height and its width scale as $t^{1/4}$ and $t^{3/4}$, respectively, as in the symmetric case. The case $\mu = 0$ is different. As t increases, the portion of the ridge in $x > 0$ becomes more dominant and its height and its width scale as $t^{1/5}$ and $t^{4/5}$, respectively.

Acknowledgments

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