IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Three-point functions in superstring theory on $AdS_3 \times S^3 \times T^4$

This content has been downloaded from IOPscience. Please scroll down to see the full text.

JHEP06(2009)009

(http://iopscience.iop.org/1126-6708/2009/06/009)

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 157.92.4.6
This content was downloaded on 11/08/2015 at 17:31

Please note that terms and conditions apply.

Published by IOP Publishing for SISSA

Received: *March 23, 2009*
Revised: *May 12, 2009*
Accepted: *May 27, 2009*
Published: *June 3, 2009*

Three-point functions in superstring theory on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$

Carlos A. Cardona[a] **and Carmen A. Núñez**[a,b]

[a] *Instituto de Astronomía y Física del Espacio (CONICET-UBA), C. C. 67 - Suc. 28, 1428 Buenos Aires, Argentina*

[b] *Departamento de Física, FCEN, Universidad de Buenos Aires, Ciudad Universitaria, Pab. I, 1428 Buenos Aires, Argentina*

E-mail: cargicar@iafe.uba.ar, carmen@iafe.uba.ar

Abstract: We consider R and NS spectral flow sectors of type IIB superstring theory on $\mathrm{AdS}_3\times$ $\mathrm{S}^3\times\mathrm{T}^4$ in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. We present a derivation of the vertex operators creating spectral flow images of chiral primary states previously proposed in the literature. We compute spectral flow conserving three-point functions involving these operators on the sphere. Using the bulk-to-boundary dictionary, we compare the results with the corresponding correlators in the dual conformal field theory, the symmetric product orbifold of T^4. In the limit of small string coupling, agreement is found in all the cases considered.

Keywords: Superstrings and Heterotic Strings, AdS-CFT Correspondence

© SISSA 2009

doi:10.1088/1126-6708/2009/06/009

Contents

1 Introduction

A systematic understanding of the duality between type IIB superstring theory on $\mathrm{AdS}_3\times$ $\mathrm{S}^3\times\mathrm{T}^4$ and the $\mathcal{N}=(4,4)$ non-linear sigma model on the moduli space of Yang-Mills instantons on T^4 has been achieved along recent years, based on early work in [1]–[5].

The instanton moduli space is a deformation of the symmetric product of N copies of T^4, namely $Sym(\mathrm{T}^4)^{\mathrm{N}} \equiv (\mathrm{T}^4)^{\mathrm{N}}/S_{\mathrm{N}}$ [6] and the worldsheet of the superstring is an $\mathcal{N}=1$ SL(2,$\mathbb{R}$)$\times$ SU(2) WZNW model. In the large N limit, twisted states in $Sym(\mathrm{T}^4)$ map to single states of short strings [7, 8] described by discrete representations of SL$(2,\mathbb{R})\times$ SU(2) and their spectral flow images [9]. Agreement between the spectrum and three-point functions of unflowed chiral primary string states and the corresponding dual counterparts was found in [7, 8, 10]. Conversely, the non-trivial spectral flow sectors of the string theory have been less studied and present some unclear features, such as the apparent lack of certain string states in the spectrum of the superconformal field theory (SCFT) [11] and various technical difficulties in the computation of correlation functions. Some preliminary results were obtained in [12] where, in particular, a bulk-to-boundary dictionary for 1/2 BPS states in the flowed sectors was proposed.

The aim of this paper is to study this holographic map by exploring three-point functions in both sides of the duality. The computation of worldsheet correlators basically involves three parts reflecting the fact that the theory is a direct product of free fermions

JHEP06(2009)009

and bosonic SU(2) and SL(2,$\mathbb{R}$) WZNW models. The relevant three-point functions of the free fermions and SU(2) bosons were obtained in [12]. Here we evaluate spectral flow conserving three-point functions on the sphere involving spectral flow images of chiral primary string states in the Neveu-Schwarz (NS) and Ramond (R) sectors of the SL(2, $\mathbb{R}$) WZNW model in order to complete the construction of these amplitudes in the full string theory and compare them with the conjectured dual correlators in the symmetric orbifold of T^4 obtained in [13–15]. Our results confirm the agreement of the string amplitudes with the corresponding counterparts in the dual theory.

The paper is organized as follows. After setting the notations in the next section, in section 3 we present a derivation of the vertex operators creating spectral flow images of chiral primary string states in NS and R sectors which were proposed in [12]. In section 4 we compute the SL(2, $\mathbb{R}$) part of the spectral flow conserving three-point functions involving these chiral operators and, after adding the fermionic and SU(2) parts computed in [12], we compare our results with the conjectured corresponding correlators in the dual SCFT. Finally, section 5 contains the conclusions. In the appendix we compute the Clebsch-Gordan coefficients needed to construct vertex operators in product representations of SL(2, $\mathbb{R}$).

JHEP06(2009)009

2 Notations

In order to set the notations in this section we briefly review basic aspects of the dual theories.

2.1 Review of type IIB superstring on $AdS_3\times S^3\times T^4$

Type IIB superstring theory on $AdS_3\times S^3\times T^4$ was originally studied in [2–5, 11, 16]. It has $\widehat{SL(2)} \times \widehat{SU(2)} \times \widehat{U(1)}^4$ affine worldsheet symmetry which allows to perform explicit calculations. The $\widehat{SL(2)}$ and $\widehat{SU(2)}$ supercurrents $\psi^A+\theta J^A$ and $\chi^A+\theta K^A$, respectively, satisfy the following OPE

$$\begin{aligned}
J^A(z)J^B(w) &\sim \frac{\frac{k}{2}\eta^{AB}}{(z-w)^2}+\frac{i\epsilon^{AB}{}_C J^C(w)}{z-w}\,, & K^A(z)K^B(w) &\sim \frac{\frac{k}{2}\delta^{AB}}{(z-w)^2}+\frac{i\epsilon^{AB}{}_C K^C(w)}{z-w}\,,\\
J^A(z)\psi^B(w) &\sim \frac{i\epsilon^{AB}{}_C\psi^C(w)}{z-w}\,, & K^A(z)\chi^B(w) &\sim \frac{i\epsilon^{AB}{}_C\chi^C(w)}{z-w}\,,\\
\psi^A(z)\psi^B(w) &\sim \frac{\frac{k}{2}\eta^{AB}}{z-w}\,, & \chi^A(z)\chi^B(w) &\sim \frac{\frac{k}{2}\delta^{AB}}{z-w}\,,
\end{aligned} \tag{2.1}$$

with $A=0,1,2$, $\epsilon^{012}=1$ and $\eta^{AB}=(-,+,+)$. It is convenient to introduce new currents as

$$J^A(z)=j^A(z)+\hat{j}^A(z)\,,\qquad K^A(z)=k^A(z)+\hat{k}^A(z)\,, \tag{2.2}$$

where j^A ($\hat{j}^A$) and k^A ($\hat{k}^A$) generate $SL(2)_{k+2}$ ($SL(2)_{-2}$) and $SU(2)_{k-2}$ ($SU(2)_2$) affine algebras, respectively, with

$$\hat{j}^A(z)=-\frac{i}{k}\epsilon^A{}_{BC}\psi^B(z)\psi^C(z),\qquad \hat{k}^A(z)=-\frac{i}{k}\epsilon^A{}_{BC}\chi^B(z)\chi^C(z). \tag{2.3}$$

The $\widehat{\mathrm{U}(1)}^4$ is realized in terms of free bosonic currents $i\partial Y^i$ and free fermions $\lambda^i, i = 1,2,3,4$.

The stress tensor and supercurrent are given by

$$T(z) = \frac{\eta_{AB}}{k}\left(j^A j^B - \psi^A\partial\psi^B\right) + \frac{\delta_{AB}}{k}\left(k^A k^B - \chi^A\partial\chi^B\right) + \frac{1}{2}\left(\partial Y^i\partial Y_i - \frac{1}{2}\lambda^i\partial\lambda_i\right),$$
$$G(z) = \frac{2}{k}\left(\eta_{AB}\psi^A j^B + \frac{2i}{k}\psi^0\psi^1\psi^2\right) + \frac{2}{k}\left(\delta_{AB}\chi^A k^B - \frac{2i}{k}\chi^0\chi^1\chi^2\right) + \lambda^i\partial Y_i\,. \tag{2.4}$$

The spectrum of the theory is built from those of the SL(2,$\mathbb{R}$) and SU(2) WZNW models. The Hilbert space of the former [9] is decomposed into unitary representations of the SL(2,$\mathbb{R}$)$\times$ SL(2,$\mathbb{R}$) current algebra,[1] namely the discrete lowest- and highest-weight representations $\mathcal{D}^\pm_h\otimes\mathcal{D}^\pm_h$ with $h\in\mathbb{R}$, $\frac{1}{2}<h<\frac{k+1}{2}$ and $m=\pm h, \pm h\pm 1,\dots$, the continuous representations $\mathcal{C}^\alpha_h\otimes\mathcal{C}^\alpha_h$ with $h=\frac{1}{2}+i\mathbb{R}$, $m=\alpha+\mathbb{Z}$, $\alpha\in[0,1)$, their current algebra descendants and spectral flow images, $\widehat{\mathcal{D}}^{\pm,w}_h\otimes\widehat{\mathcal{D}}^{\pm,w}_h, \widehat{\mathcal{C}}^{\alpha,w}_h\otimes\widehat{\mathcal{C}}^{\alpha,w}_h$ with $w\in\mathbb{Z}$ and the same spin and amount of spectral flow on the left- and right-moving sectors. Primary operators of spin h and worldsheet conformal dimension $\Delta^{sl}(\Phi_h) = -\frac{h(h-1)}{k}$, satisfy

$$j^a(z)\Phi_h(x,\overline{x};w,\overline{w}) \sim \frac{D^a_x\Phi_h(x,\overline{x};w,\overline{w})}{z-w}\,,\quad a=0,\pm\,, \tag{2.5}$$

where $D^+_x=\partial_x$, $D^0_x = x\partial_x + h$ and $D^-_x = x^2\partial_x + 2hx$. Expanding in modes as

$$\Phi_h(x,\overline{x}) = \sum_{m,\overline{m}}\Phi_{h,m,\overline{m}}x^{-h+m}\overline{x}^{-h+\overline{m}}\,, \tag{2.6}$$

one can read the action of the zero modes of the currents on $\Phi_{h,m,\overline{m}}$, namely

$$j^0_0\Phi_{h,m,\overline{m}} = m\Phi_{h,m,\overline{m}}\,,\qquad j^\pm_0\Phi_{h,m,\overline{m}} = \left[m\mp(h-1)\right]\Phi_{h,m\pm1,\overline{m}},\quad (m\neq\pm h)\,, \tag{2.7}$$

and $j^-_0\Phi_{h,h,\overline{m}} = j^+_0\Phi_{h,-h,\overline{m}} = 0$.

Similarly, the primary fields of the SU(2)$_{k-2}$ WZNW model with conformal dimension $\Delta^{su}(V_j) = \frac{j(j+1)}{k}$ verify

$$k^a(z)V_j(y,\overline{y};w,\overline{w}) \sim \frac{P^a_y V_j(y,\overline{y};w,\overline{w})}{z-w}\,, \tag{2.8}$$

with $P^+_y = \partial_y$, $P^0_y = y\partial_y - j$, $P^-_y = -y^2\partial_y + 2jy$ and can be expanded in modes as

$$V_j(y,\overline{y}) = \sum_{m',\overline{m}'=-j}^{j} V_{j,m',\overline{m}'}y^{j+m'}\overline{y}^{j+\overline{m}'}\,. \tag{2.9}$$

The spin $j\in\mathbb{Z}/2$ is bounded by $0\leq j\leq\frac{k-2}{2}$ and $k^+_0V_{j,j,\overline{m}'} = k^-_0V_{j,-j,\overline{m}'} = 0$,

$$k^0_0V_{j,m',\overline{m}'} = m'V_{j,m',\overline{m}'}\,,\qquad k^\pm_0V_{j,m',\overline{m}'} = (\pm m'+1+j)V_{j,m'\pm1,\overline{m}'}\,,\quad (m'\neq\pm j)\,. \tag{2.10}$$

[1] Actually, the spectrum is built on representations of the universal cover of SL(2, $\mathbb{R}$), to which we refer simply as SL(2,$\mathbb{R}$) for short.

JHEP06(2009)009

In the fermionic sector, the fields ψ^a transform in the spin $\hat{h}=-1$ representation of the global SL(2,$\mathbb{R}$)$_{-2}$ algebra and χ^a transform in the spin $\hat{j}=1$ of the SU(2)$_2$ global algebra.

Vertex operators creating unflowed physical states in the NS sector were constructed in [16]. For short, we display only the holomorphic indices. The chiral (antichiral) primaries satisfy the condition $\mathcal{H}=\mathcal{J}$ ($\mathcal{H}=-\mathcal{J}$), $\mathcal{H}$ being the spacetime conformal dimension and $\mathcal{J}$ the SU(2) charge, which implies $h=j+1$. In the -1 picture, they are given by

$$\mathcal{W}_{h,m,m'} = e^{-\varphi}(\psi\Phi_{h,m})_{h-1,m_T}V_{j,m'}\,, \tag{2.11}$$
$$\mathcal{Y}_{h,m,,m'} = e^{-\varphi}\Phi_{h,m}(\chi V_{j,m'})_{h,m'_T}\,, \tag{2.12}$$

where $(\psi\Phi_{h,m})$ and $(\chi V_{j,m'})$ denote the product representations of J^a and K^a, respectively, $m_T=h-1,h,h+1,\dots$ and $m'_T=-h,-h+1,\dots,h$.

To study the Ramond sector one needs to construct the spin fields for ψ^a,χ^a,λ^i [16]. It is convenient to have a bosonized form of the fermions such as

$$\partial H_1=\frac{2}{k}i\psi^+\psi^-,\quad \partial H_2=\frac{2}{k}i\chi^+\chi^-,\quad \partial H_3=-\frac{2}{k}i\psi^3\chi^3,\quad \partial H_4=\lambda^1\lambda^2,\quad \partial H_5=\lambda^3\lambda^4\,. \tag{2.13}$$

The spin fields take the form $S_{[\epsilon_1,\dots,\epsilon_5]}=\exp\frac{i}{2}\sum_{i=1}^5\epsilon_iH_i$, with $\epsilon_i=\pm1$. They transform as two copies of $(\frac12,\frac12)$ under SL(2)$\times$ SU(2). GSO projection requires $\prod_{i=1}^5\epsilon_i=+1$ and BRST invariance demands $\prod_{i=1}^3\epsilon_i=-1$. Following [12] we define the spin fields associated with ψ^a,χ^a as $\tilde{S}_{[\epsilon_1,\epsilon_2,\epsilon_3]}=\exp\frac{i}{2}(\epsilon_1H_1+\epsilon_2H_2+\epsilon_3H_3)$.

Decomposing the product $(\tilde{S}\Phi_{h,m}V_{j,m'})$ into representations of the total currents J^a,K^a, the chiral vertex operators in the $-\frac12$ picture take the form

$$\mathcal{R}^{\pm}_{h,m,m'}=e^{-\frac{\varphi}{2}}\left(\tilde{S}\Phi_{h,m}V_{j,m'}\right)_{h-\frac12,m_T+\frac12;j+\frac12,m'_T+\frac12}e^{\pm i(\hat{H}_4-\hat{H}_5)}\,, \tag{2.14}$$

where H_i are redefined as $\hat{H}_i=H_i+\pi\sum_{j<i}N_j$, $N_j=i\oint\partial H_i$.

Spectral flow. The algebras (2.1) are invariant under the following spectral flow automorphisms

$$\widetilde{J}^0_n=J^0_n-\frac{k}{2}w\delta_{n,0},\quad \widetilde{J}^{\pm}_n=J^{\pm}_{n\pm w},\quad \widetilde{K}^0_n=K^0_n+\frac{k}{2}w'\delta_{n,0},\quad \widetilde{K}^{\pm}_n=K^{\pm}_{n\pm w'}\,.$$

The currents j^a, $\hat{j}^a$, k^a and $\hat{k}^a$ transform under spectral flow as

$$j^0_n=\tilde{j}^0_n+\frac{k+2}{2}w\delta_{n,0}\,,\quad j^{\pm}_n=\tilde{j}^{\pm}_{n\mp w}\,,\quad k^0_n=\tilde{k}^0_n-\frac{k-2}{2}w'\delta_{n,0}\,,\quad k^{\pm}_n=\tilde{k}^{\pm}_{n\mp w'}\,, \tag{2.15}$$
$$\hat{j}^0_n=\tilde{\hat{j}}^0_n-w\delta_{n,0},\qquad \hat{j}^{\pm}_n=\tilde{\hat{j}}^{\pm}_{n\mp w}\,,\quad \hat{k}^0_n=\tilde{\hat{k}}^0_n-w'\delta_{n,0}\,,\qquad \hat{k}^{\pm}_n=\tilde{\hat{k}}^{\pm}_{n\mp w'}\,, \tag{2.16}$$

and the modes of the Virasoro generators, $L^{sl}_n=l^{sl}_n+\hat{l}^{sl}_n, L^{su}_n=l^{su}_n+\hat{l}^{su}_n$, as

$$\begin{aligned}
\tilde{L}^{sl}_n &= L^{sl}_n+w\tilde{J}^0_n+\frac{k}{4}w^2\delta_{n,0}\,, & \tilde{L}^{su}_n &= L^{su}_n+w'\tilde{K}^0_n-\frac{k}{4}w'^2\delta_{n,0}\\
l^{sl}_n &= \tilde{l}^{sl}_n-w\tilde{j}^0_n-\frac{k+2}{4}w^2\delta_{n,0}\,, & l^{su}_n &= \tilde{l}^{su}_n-w'\tilde{k}^0_n+\frac{k-2}{4}w'^2\,,\\
\hat{l}^{sl}_n &= \tilde{\hat{l}}^{sl}_n-w\tilde{\hat{j}}^0_n+\frac{1}{2}w^2\delta_{n,0}\,, & \hat{l}^{su}_n &= \tilde{\hat{l}}^{su}_n-w'\tilde{\hat{k}}^0_n+\frac{1}{2}w'^2\delta_{n,0}\,.
\end{aligned} \tag{2.17}$$

JHEP06(2009)009

The closure of the SL(2,$\mathbb{R}$) and SU(2) algebras requires the same amount of spectral flow w (w') for j^a and $\hat{j}^a$ (k^a and $\hat{k}^a$). The spectral flow maps primaries to descendants of SU(2) and it generates new representations in SL(2,$\mathbb{R}$) [9]. For the sake of simplicity, we restrict to $w > 0$ in this section.

To construct spectral flow images of chiral primaries in generic frames, we consider the SL(2,$\mathbb{R}$) sector first. A $w = 0$ affine primary is mapped by the spectral flow to a lowest-weight state of the global algebra $\Phi^{h,w}_{H,M}$ with $H = M$ satisfying [9]

$$j^0_0 \Phi^{h,w}_{H,M} = M\Phi^{h,w}_{H,M} = \left(m + \frac{k+2}{2} w\right) \Phi^{h,w}_{H,M}\,, \tag{2.18}$$

$$l_0 \Phi^{h,w}_{H,M} = \left(-\frac{h(h-1)}{k} - wm - \frac{k+2}{4} w^2\right) \Phi^{h,w}_{H,M}\,. \tag{2.19}$$

In the fermionic SL(2,$\mathbb{R}$) sector, an interesting description of the spectral flow was presented by A. Pakman in [17]. Using (2.16) – (2.17), the fermions ψ^a in the spectral flow frame obey

$$\tilde{\jmath}^0_0 \psi^a = (a-w)\psi^a, \qquad \tilde{\jmath}^-_0 \psi^a = \tilde{\jmath}^-_w \psi^a = 0\,, \qquad \hat{l}_0 \psi^a = \left(\frac{1}{2} - wa + \frac{1}{2} w^2\right) \psi^a\,, \tag{2.20}$$

i.e. ψ^a is a lowest-weight field with angular momentum $\hat{h} = a - w$. Acting with $\hat{\jmath}^+_0$, one obtains the global representation in the w sector as $\psi^{|\hat{h}|}_{\hat{m}} \sim (\hat{\jmath}^+_0)^n \psi^a$ with $\hat{m} = -\hat{h}, \ldots, \hat{h}$ up to a normalization.

All these ingredients allow to construct the representations of J^a. We denote the fields of the product representation in the NS sector as $(\psi^{|\hat{h}|}_{\hat{m}} \Phi^{w,h}_{H,M})_{\mathcal{H},\mathcal{M}}(z,\bar{z})$, where $|H - \hat{h}| \leq \mathcal{H} \leq H + \hat{h}$, $\mathcal{M} = \mathcal{H}, \mathcal{H}{+}1, \ldots$ and their worldsheet conformal weight is given by

$$\Delta^{sl}\left[(\psi^{|\hat{h}|}_{\hat{m}} \Phi^{h,w}_{H,M})_{\mathcal{H},\mathcal{M}}\right] = -\frac{h(h-1)}{k} - w(m-a) + \frac{1}{2} - \frac{k}{4} w^2\,. \tag{2.21}$$

Repeating the analysis for SU(2), one obtains the product representation $(\chi^{\hat{j}}_{\hat{m}'} V^{j,w'}_{J,M'})_{\mathcal{J},\mathcal{M}'}$, with $|J - \hat{\jmath}| \leq \mathcal{J} \leq J + \hat{\jmath}$, $-\mathcal{J} \leq \mathcal{M}' \leq \mathcal{J}$, $J = m' - \frac{k-2}{2} w'$, $\hat{\jmath} = |a - w'|$ and worldsheet conformal weight

$$\Delta^{su}\left[(\chi^{\hat{j}}_{\hat{m}'} V^{j,w'}_{J,M'})_{\mathcal{J},\mathcal{M}'}\right] = \frac{j(j+1)}{k} - w'(m'-a) + \frac{1}{2} + \frac{k}{4} w'^2\,. \tag{2.22}$$

In order to construct chiral states, we apply the spectral flow operation on the chiral primaries (2.11) and (2.12). We notice that the physical and chiral state conditions require to simultaneously spectral flow the SL(2,$\mathbb{R}$) and SU(2) product representations and we obtain

$$\mathcal{W}^{h,w}_{\mathcal{H},\mathcal{M}} = e^{-\varphi} \left(\psi^{w+1}_{\hat{m}} \Phi^{h,w}_{H,M}\right)_{\mathcal{H},\mathcal{M}} \left(\chi^{w'}_{\hat{m}'} V^{j,w'}_{J,M'}\right)_{\mathcal{J},\mathcal{M}'}\,, \tag{2.23}$$

$$\mathcal{Y}^{h,w}_{\mathcal{H},\mathcal{M}} = e^{-\varphi} \left(\psi^{w}_{\hat{m}} \Phi^{h,w}_{H,M}\right)_{\mathcal{H},\mathcal{M}} \left(\chi^{w'+1}_{\hat{m}'} V^{j,w'}_{J,M'}\right)_{\mathcal{J},\mathcal{M}'}\,, \tag{2.24}$$

where φ is the bosonization of the β, γ ghosts, $\mathcal{M} = \mathcal{H}$ and $\mathcal{M}' = -\mathcal{J}$. For generic level k, the physical state condition $(L_0 - 1)\mathcal{W} = 0$ implies $h = j + 1$, $w = w'$ and $m'_T = -m_T$

JHEP06(2009)009

(see (2.11), (2.12)), and similarly for $\mathcal{Y}$. Analogously, $G_r\mathcal{W} = (\tilde{G}_r - w\tilde{\psi}^0_r - w\tilde{\chi}^0_r)\mathcal{W} = 0$ ($G_r\mathcal{Y} = 0$) for $r > 0$ requires $m_T = h-1$ ($m_T = h$) [12]. Finally, chirality (or antichirality) demands, for both operators $\mathcal{W}$ and $\mathcal{Y}$,

$$\mathcal{H} = m_T + \frac{k}{2}w = \pm\mathcal{J}. \tag{2.25}$$

To obtain the spectral flowed $\frac{1}{2}$ BPS operators in the Ramond sector we need the product representation $(S^{\hat{j}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w'}_{J,M'})$. The discussion about the fermions applies analogously to the spin fields, *i.e.* from the lowest-weight component of the $\hat{h} = -\hat{j} = -|w \pm \frac{1}{2}|$ spin representation, given by

$$S^{w+\frac{1}{2}}_{-w-\frac{1}{2},w+\frac{1}{2}} \equiv e^{-i(w+\frac{1}{2})(\hat{H}_1+\hat{H}_2)-\frac{i}{2}\hat{H}_3}\,, \tag{2.26}$$

one constructs the global representation acting with $\hat{j}^+_0, \hat{k}^+_0$.

The chiral fields in the w sector are [12]

$$\mathcal{R}^{\pm,h,w}_{\mathcal{H},\mathcal{M}} = e^{-\frac{\varphi}{2}}\left(S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J,M'}\right)_{\mathcal{H},\mathcal{M},\mathcal{J},\mathcal{M}'} e^{\pm\frac{i}{2}(\hat{H}_4-\hat{H}_5)}\,, \tag{2.27}$$

where $S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}$ has conformal weight $\frac{3}{8}+w^2+w$, $\hat{h} = -w-\frac{1}{2} = -\hat{j}$ and $\mathcal{H} = h-\frac{1}{2}+\frac{k}{2}w = \mathcal{J}$.

2.2 Sigma model on the symmetric product orbifold of T^4

Type IIB superstring theory on $\mathrm{AdS}_3\times$ $\mathrm{S}^3\times$ T^4 with RR background is conjectured to be dual to the infrared fixed point theory living on a D1-D5 system compactified on T^4. It is convenient to use the S-dual description [18] in terms of N_1 fundamental strings and N_5 NS5-branes. The target space of the SCFT is identified with the singular orbifold $(\mathrm{T}^4)^{\mathrm{N}_1\mathrm{N}_5}/S(\mathrm{N}_1\mathrm{N}_5)$, where $S(\mathrm{N}_1\mathrm{N}_5)$ denotes the permutation group of $\mathrm{N}_1\mathrm{N}_5$ elements. It was argued in [19] that the symmetric orbifold corresponds to the point N_5=1, N_1=N.

The chiral spectrum of the sigma model is built from that of a single copy of T^4 plus operators in the twisted sectors. Each twisted sector corresponds to one conjugacy class of $S(\mathrm{N})$, labeled by positive integer partitions of N, namely

$$\sum_{l=1}^{\mathrm{N}} l k_l = \mathrm{N}\,, \tag{2.28}$$

corresponding to permutations with k_l cycles of length l. Chiral operators describing single particle states in the string theory side correspond to single cycle twist operators [18, 20]. There is one twist field for each conjugacy class of the permutation group, and chiral operators corresponding to chiral states in the dual string theory can be constructed as a sum over the group orbit, namely

$$\mathcal{O}^{\epsilon_n,\overline{\epsilon_n}}_n = [n(\mathrm{N}-\mathrm{n})!\mathrm{N}!]^{-1/2}\sum_{h\epsilon S(\mathrm{N})}\sigma^{\epsilon_n,\overline{\epsilon_n}}_{h(1\cdots n)h^{-1}}\,, \tag{2.29}$$

where $\epsilon_n = \pm 1, a$ and $\sigma^{1_n}_{(1\cdots n)}$ is a twist field corresponding to just one single element of $S(\mathrm{N})$. The global part of the $\mathcal{N} = (4,4)$ superconformal algebra forms the supergroup

JHEP06(2009)009

SU(1,1|2)$_L\times$ SU(1,1|2)$_R$ and contains the R-symmetry group SU(2)$_L\times$ SU(2)$_R$, under which the operator $\mathcal{O}_n^{\epsilon_n,\overline{\epsilon}_{\overline{n}}}(x,\overline{x})$ is a chiral state in a unitary representation with angular momentum

$$\mathcal{H}_n = \frac{n+\epsilon_n}{2}, \quad 0\le \mathcal{H}_n \le \frac{\mathrm{N}+\epsilon_n}{2}, \quad \epsilon_n = \pm 1, \tag{2.30}$$
$$\mathcal{H}_n = \frac{n}{2}, \quad 0\le \mathcal{H}_n \le \frac{\mathrm{N}}{2}, \quad \epsilon_n = a\,, \tag{2.31}$$

and similarly for $\overline{\epsilon}_{\overline{n}}$. Two- and three-point functions on the sphere for $\epsilon_n,\overline{\epsilon}_{\overline{n}}=\pm 1$, are given by [14, 15][2]

$$\left\langle \mathcal{O}_n^{\epsilon_n,\overline{\epsilon}_{\overline{n}}}(x_1,\overline{x}_1)\mathcal{O}_{-n}^{-\epsilon_n,-\overline{\epsilon}_{\overline{n}}}(x_2,\overline{x}_2)\right\rangle = |x_{12}|^{-4\mathcal{H}_n}\,, \tag{2.32}$$
$$\left\langle \mathcal{O}_{n_1}^{\epsilon_{n_1},\overline{\epsilon}_{\overline{n}_1}}\mathcal{O}_{n_2}^{\epsilon_{n_2}\overline{\epsilon}_{\overline{n}_2}}\mathcal{O}_{n_3}^{\epsilon_{n_3}\overline{\epsilon}_{\overline{n}_3}\dagger}\right\rangle = \sqrt{\frac{n_1n_2n_3}{\mathrm{N}}}\,\delta^2\left(\sum_{i=1}^{3}\mathcal{M}_{n_i}\right)C_{n_1n_2n_3}\prod_{i<j}|x_{ij}|^{-2\mathcal{H}_{n_in_j}}, \tag{2.33}$$

where $\mathcal{H}_{n_1n_2}=\mathcal{H}_{n_1}+\mathcal{H}_{n_2}-\mathcal{H}_{n_3}$, etc., $-\mathcal{H}_n\le\mathcal{M}_n\le\mathcal{H}_n$ and the coefficients $C_{n_1n_2n_3}$ are defined in terms of the SU(2) 3j symbols as

$$C_{n_1n_2n_3} = \frac{|\epsilon_{n_1}n_1+\epsilon_{n_2}n_2+\epsilon_{n_3}n_3+1|^2}{4n_1n_2n_3} \times\left|\begin{pmatrix}\mathcal{H}_{n_1} & \mathcal{H}_{n_2} & \mathcal{H}_{n_3}\\ \mathcal{M}_{n_1} & \mathcal{M}_{n_2} & \mathcal{M}_{n_3}\end{pmatrix}\right|^2\left|\frac{\mathcal{H}_{n_1n_2}!\mathcal{H}_{n_2n_3}!\mathcal{H}_{n_3n_1}!(\sum_{i=1}^3\mathcal{H}_{n_i}+1)!}{(2\mathcal{H}_{n_1})!(2\mathcal{H}_{n_2})!(2\mathcal{H}_{n_3})!}\right|.$$

Using (2.30) and $\mathcal{M}_{n_i}=\pm\mathcal{H}_{n_i}$, the delta function in (2.33) implies $\mathcal{H}_{n_in_j}=0$ for certain i,j. Specifying $n_3=n_1+n_2-1$, the non-vanishing three-point functions are those with $(\epsilon_{n_1},\epsilon_{n_2},\epsilon_{n_3})=(-,-,-)$ and $(+,-,+)$ and similarly for $\overline{\epsilon}_{\overline{n}_i}$. In this case, the product in the second line reduces to one.

Two other correlators that will be important below have been evaluated in the particular case $n_3=n_1+n_2-1$ [13], namely (we omit the obvious coordinate dependence)

$$\left\langle \mathcal{O}_{n_1}^{a,\overline{a}}\mathcal{O}_{n_2}^{-,-}\mathcal{O}_{n_3}^{a',\overline{a}'\dagger}\right\rangle = \frac{1}{\sqrt{\mathrm{N}}}\left(\frac{n_1n_3}{n_2}\right)^{1/2}\delta^{a,a'}\delta^{\overline{a},\overline{a}'}\,, \tag{2.34}$$
$$\left\langle \mathcal{O}_{n_1}^{a,\overline{a}}\mathcal{O}_{n_2}^{a',\overline{a}'}\mathcal{O}_{n_3}^{+,+\dagger}\right\rangle = \frac{1}{\sqrt{\mathrm{N}}}\left(\frac{n_1n_2}{n_3}\right)^{1/2}\xi^{a,a'}\xi^{\overline{a},\overline{a}'}\,, \quad \xi^{a,a'}=\xi^{\overline{a},\overline{a}'}=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{2.35}$$

3 Vertex operators of chiral states

In this section we present a derivation of the vertex operators creating spectral flow images of chiral primary states. These operators were proposed in [12].

[2] Contributions from surfaces with higher genus are suppressed in the large N limit.

3.1 NS sector

The Clebsch-Gordan coefficients expanding the product representation $(\psi\Phi)$ in (2.23) and (2.24) are computed in the appendix. We find

$$\left(\psi^{|\hat{h}|}_{\hat{m}}\Phi^{h,w}_{H,M}\right)_{\mathcal{H},\mathcal{M}} = \sum_{\hat{m}=-\hat{h}}^{\hat{h}} C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}}\,\psi^{|\hat{h}|}_{\hat{m}}\Phi^{h,w}_{H,M}\,, \tag{3.1}$$

where only the holomorphic part has been written and[3]

$$C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}} = \frac{(\mathcal{M}+\mathcal{H})!}{(\hat{m}+|\hat{h}|)!(\mathcal{M}-\hat{m}+H)!}\sum_{s=0}^{\hat{m}+|\hat{h}|}(-1)^{s-|\hat{h}|}\binom{\hat{m}+|\hat{h}|}{s}\frac{(\mathcal{M}-s+|\hat{h}|+H)!}{(\mathcal{M}-s+\mathcal{H})!}$$
$$\times\frac{(|\hat{h}|-H+\mathcal{M}-s-1)!}{(\mathcal{M}-s-\mathcal{H}-1)!(\mathcal{H}+|\hat{h}|-H)!}\,. \tag{3.2}$$

This can be rewritten using the generalized hypergeometric function ${}_3F_2(a,b,c;e,f|1)$ as

$$C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}} = \frac{(-1)^{\hat{m}-|\hat{h}|}\Gamma(-H+|\hat{h}|+\mathcal{M})\Gamma(H+|\hat{h}|+\mathcal{M}+1)}{\Gamma(\mathcal{H}-H+|\hat{h}|+1)\Gamma(\mathcal{M}-\mathcal{H})\Gamma(H+\mathcal{M}-\hat{m}+1)\Gamma(|\hat{h}|+\hat{m}+1)}\times$$
$${}_3F_2(-\mathcal{H}-\mathcal{M},\mathcal{H}-\mathcal{M}+1,-|\hat{h}|-\hat{m};-H-|\hat{h}|-\mathcal{M},H-|\hat{h}|-\mathcal{M}+1;1)\,,$$

with the advantage that it can be represented in terms of the Pochhammer double-loop contour integral, possessing a unique analytic continuation in the complex plane for all its indices [21, 22]. Recall that the analogous coefficients for SU(2) are related to these ones through analytic continuation.

For our purposes, it is convenient to write the vertex operators in the $x-$basis, where the isospin can be identified with the coordinates on the boundary. This can be done using [3]:

$$e^{-xJ_0^-}\mathcal{O}(z)e^{xJ_0^-} \equiv \mathcal{O}(x,z). \tag{3.3}$$

Performing this operation on the fermion fields, one gets in the unflowed frame

$$e^{-xJ_0^-}\psi^+(z)e^{xJ_0^-} = \psi^+(x,z) \equiv \psi(x,z) \tag{3.4}$$
$$= -2x\psi^0(z)+\psi^+(z)+x^2\psi^-(z), \tag{3.5}$$

and in a generic w frame

$$e^{-xJ_0^-}\psi^{|\hat{h}|}_{\hat{m}=\hat{h}}(z)e^{xJ_0^-} \equiv \psi^{|\hat{h}|}(x,z) = \sum_{\hat{m}=-\hat{h}}^{\hat{h}}\frac{(-1)^{\hat{m}+\hat{h}}\,\Gamma(2|\hat{h}|+1)}{\Gamma(\hat{m}+|\hat{h}|+1)\Gamma(|\hat{h}|-\hat{m}+1)}\psi^{|\hat{h}|}_{\hat{m}}x^{-\hat{h}+\hat{m}}\,. \tag{3.6}$$

Inserting $H = m + \frac{k+2}{2}w$ and $\hat{h} = -w-1$ in (3.2) we get

$$C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}} = (-1)^{\hat{m}+w+1}\frac{\Gamma(2w+3)}{\Gamma(\hat{m}+w+2)\Gamma(w-\hat{m}+2)}\,, \tag{3.7}$$

[3] We found convenient to denote the coefficients $C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}}$ as $\langle H,M;\hat{h},\hat{m}|H,\hat{h};\mathcal{H},\mathcal{M}\rangle$ in the appendix.

JHEP06(2009)009

which coincide with the coefficients in (3.6). Therefore, the SL(2,$\mathbb{R}$) part of the chiral vertex (2.23) may be written as

$$\left(\psi^{w+1}_{\hat{m}}\Phi^{h,w}_{H,M}\right)_{\mathcal{H},\mathcal{M}} = \sum_{\hat{m}=-w-1}^{w+1}(-1)^{\hat{m}+w+1}\frac{\Gamma(2w+3)}{\Gamma(\hat{m}+w+2)\Gamma(w-\hat{m}+2)}\psi^{w+1}_{\hat{m}}\Phi^{h,w}_{H,M}\,. \quad (3.8)$$

Expanding in modes, it is easy to see that they may be expressed in the following factorized form

$$(\psi\Phi)^{h,w}_{\mathcal{H}}(x) \equiv \sum_{\mathcal{M}}\left(\psi^{w+1}_{\hat{m}}\Phi^{h,w}_{H,M}\right)_{\mathcal{H},\mathcal{M}}\; x^{-\mathcal{H}+\mathcal{M}} = \psi^{w+1}(x)\Phi^{h,w}_{H}(x)\,. \quad (3.9)$$

This factorization always occurs in (2.23) when H and $\hat{h}$ combine to produce a chiral state.

So far, we have restricted to the holomorphic SL(2,$\mathbb{R}$) sector, but the same analysis applies to SU(2) [22] and to their antiholomorphic parts. Putting all together, we get the following vertex operators creating spectral flow images of chiral primary states in arbitrary spectral flow frames

$$\mathcal{W}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}(x,y,\overline{x},\overline{y}) = e^{-\varphi}\;\Phi^{h,w}_{H\overline{H}}(x,\overline{x})\psi^{w+1}(x)\overline{\psi}^{w+1}(\overline{x})V^{h-1,w}_{J\overline{J}}(y,\overline{y})\chi^{w}(y)\overline{\chi}^{w}(\overline{y}), \quad (3.10)$$

$$\mathcal{Y}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}(x,y,\overline{x},\overline{y}) = e^{-\varphi}\Phi^{h,w}_{H\overline{H}}(x,\overline{x})\psi^{w}(x)\overline{\psi}^{w}(\overline{x})\chi^{w+1}(y)\overline{\chi}^{w+1}(\overline{y})V^{h-1,w}_{J,\overline{J}}(y,\overline{y})\,, \quad (3.11)$$

with $J = H - 2w, \overline{J} = \overline{H} - 2w$, $\mathcal{H} = -J-1, \overline{\mathcal{H}} = -\overline{J}-1$.

3.2 Ramond sector

The product representation needed to construct the vertex operators (2.27) in the Ramond sector can be expanded as

$$\begin{aligned}\left(S^{\hat{j}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J,M'}\right)_{\mathcal{H},\mathcal{M},\mathcal{J},\mathcal{M}'} &= \sum_{\hat{m},\hat{m}'=-\hat{h}}^{\hat{h}}\left(S^{\hat{j}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J,M'}\right)C^{(M,\hat{m},\mathcal{M}),(M',\hat{m}',\mathcal{M}')}_{(H,\hat{h},\mathcal{H}),(J,\hat{h},\mathcal{J})} \quad (3.12)\\ &\equiv \sum_{\hat{m}=-\hat{h}}^{\hat{h}}\left(S^{\hat{j}}_{\hat{m}}\Phi^{h,w}_{H,M}\right)C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}}\otimes\sum_{\hat{m}'=-\hat{h}}^{\hat{h}}\left(S^{\hat{j}}_{\hat{m}'}V^{j,w}_{J,M'}\right)C^{M',\hat{m}',\mathcal{M}'}_{J,\hat{h},\mathcal{J}},\end{aligned}$$

i.e. the SL(2) and SU(2) parts factorize. The Clebsch-Gordan coefficients $C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}}$ can be computed from (3.2) taking $H = m + \frac{k+2}{2}w$ and $\hat{h} = -w - \frac{1}{2}$. Using (3.3), it is easy to see that the triple product factorizes in the $x-$basis as

$$\begin{aligned}(S\Phi V)^{h,w}_{\mathcal{H},\mathcal{J}}(x,y) &\equiv \sum_{\mathcal{M},\mathcal{M}'}\left(S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J;M'}\right)_{\mathcal{H},\mathcal{M},\mathcal{J},\mathcal{M}'}\; x^{-\mathcal{H}+\mathcal{M}}y^{\mathcal{J}+\mathcal{M}'}\\ &= S^{w+\frac{1}{2}}(x,y)\Phi^{h,w}_{H}(x)V^{j,w}_{J}(y)\,, \quad (3.13)\end{aligned}$$

where[4]

$$\begin{aligned}S^{w+\frac{1}{2}}(x,y) \equiv &\sum_{\hat{m},\hat{m}'=-(w+\frac{1}{2})}^{w+\frac{1}{2}}\left[\frac{(-1)^{\hat{m}+w+\frac{1}{2}}\;\Gamma(2w+2)}{\Gamma\left(\hat{m}+w+\frac{3}{2}\right)\Gamma\left(w-\hat{m}+\frac{3}{2}\right)}\frac{(-1)^{\hat{m}'+w+\frac{1}{2}}\;\Gamma(2w+2)}{\Gamma\left(\hat{m}'+w+\frac{3}{2}\right)\Gamma\left(w-\hat{m}'+\frac{3}{2}\right)}\right]\\ &\times S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}\,x^{\hat{m}+w+\frac{1}{2}}y^{\hat{m}'+w+\frac{1}{2}}. \quad (3.14)\end{aligned}$$

[4]These spin fields are denoted $S^{-}_{w}(x,y)$ in [12]

JHEP06(2009)009

Taking into account the antiholomorphic part, the vertex operators creating spectral flow images of chiral primary states in the Ramond sector are given by

$$\mathcal{R}^{\pm,h,w}_{\mathcal{H}\overline{\mathcal{H}}}(x,\overline{x},y,\overline{y}) = e^{-\frac{\varphi}{2}} S^{w+\frac{1}{2}}(x,y)\overline{S}^{w+\frac{1}{2}}(\overline{x},\overline{y})\Phi^{h,w}_{H,\overline{H}}(x,\overline{x})V^{j,w}_{J,\overline{J}}(y,\overline{y})e^{\pm\frac{i}{2}(\hat{H}_4-\hat{H}_5)}e^{\pm\frac{i}{2}(\hat{\overline{H}}_4-\hat{\overline{H}}_5)} . \tag{3.15}$$

The expressions (3.10), (3.11) and (3.15) that we deduced here appeared previously in [12].

4 Three-point functions of chiral states

In this section we compute $w-$conserving three-point functions involving spectral flow images of chiral primary states. We restrict to the so called extremal correlators, satisfying $j_n = j_m + j_l$.

4.1 NS-NS-NS three-point functions

Let us start by evaluating the following amplitudes

$$\mathcal{A}_3 = g_s^{-2}\left\langle \mathcal{W}^{h_1,w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,y_1,\overline{x}_1,\overline{y}_1)\mathcal{W}^{h_2,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}(x_2,y_2,\overline{x}_2,\overline{y}_2)\mathcal{W}^{h_3,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,y_3,\overline{x}_3,\overline{y}_3)\right\rangle_{S^2} , \tag{4.1}$$

$$\mathcal{A}'_3 = g_s^{-2}\left\langle \mathcal{Y}^{h_1,w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,y_1,\overline{x}_1,\overline{y}_1)\mathcal{W}^{h_2,w_2}_{\mathcal{H}_2\overline{\mathcal{H}}_2}(x_2,y_2,\overline{x}_2,\overline{y}_2)\mathcal{Y}^{h_3,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,y_3,\overline{x}_3,\overline{y}_3)\right\rangle_{S^2} . \tag{4.2}$$

The vertices $\mathcal{W}^{h,w}_{\mathcal{H},\overline{\mathcal{H}}}$, $\mathcal{Y}^{h,w}_{\mathcal{H},\overline{\mathcal{H}}}$ were defined in the -1 ghost picture. To have total ghost number -2, as required on the sphere, we change the picture of an unflowed operator for simplicity, *i.e.* [7, 8]

$$\mathcal{W}^{(0)}_h(x,y,\overline{x},\overline{y}) = \left[\left((1-h)\hat{\jmath}(x) + j(x) + \frac{2}{k}\psi(x)\chi_a(y)P^a_y\right)\times c.c.\right]\Phi_h(x,\overline{x})V_{h-1}(y,\overline{y}) , \tag{4.3}$$

$$\mathcal{Y}^{(0)}_h(x,y,\overline{x},\overline{y}) = \left[\left(h\hat{k}(y) + k(y) + \frac{2}{k}\chi(y)\psi_a(x)D^a_x\right)\times c.c.\right]\Phi_h(x,\overline{x})V_{h-1}(y,\overline{y}) . \tag{4.4}$$

As discussed in detail below, this restriction is not strictly necessary to evaluate (4.2), but further knowledge on spectral flowed affine representations than is currently available is needed to compute (4.1) in a more general situation. In any case, we shall see that including an unflowed operator does not imply any loss of generality for correlators involving spectral flow images of chiral primary states in the SL(2, $\mathbb{R}$) sector.

Replacing (4.3) in (4.1), $\mathcal{A}_3$ explicitly reads

$$\begin{aligned}\mathcal{A}_3 = g_s^{-2}&\left\langle e^{-\varphi(z_1,\bar{z}_1)}e^{-\varphi(z_2,\bar{z}_2)}\right\rangle\left\langle V^{h_1-1,w}_{J_1,\overline{J}_1}(y_1,\overline{y}_1)V^{h_2-1,w}_{J_2,\overline{J}_2}(y_2,\overline{y}_2)V_{h_3-1}(y_3,\overline{y}_3)\right\rangle\\ &\times\langle\overline{\chi}^w(\overline{y}_1)\overline{\chi}^w(\overline{y}_2)\rangle\langle\chi^w(y_1)\chi^w(y_2)\rangle\left\langle\Phi^{h_1,w}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\psi^{w+1}(x_1)\overline{\psi}^{w+1}(\overline{x}_1)\Phi^{h_2,w}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\right.\\ &\left.\times\,\psi^{w+1}(x_2)\overline{\psi}^{w+1}(\overline{x}_2)\left\{(1-h_3)\hat{\jmath}(x_3)+j(x_3)\right\}\left\{(1-h_3)\hat{\jmath}(\overline{x}_3)+j(\overline{x}_3)\right\}\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle ,\end{aligned}$$

JHEP06(2009)009

and inserting (4.4) into (4.2) and using $\psi_a(x)D^a_x = \frac{1}{2}(\psi(x)\partial_x + h\partial_x\psi(x))$ we get

$$\begin{aligned}
\mathcal{A}'_3 &= g_s^{-2}\left\langle e^{-\varphi(z_1,\bar{z}_1)}e^{-\varphi(z_2,\bar{z}_2)}\right\rangle \left\langle V^{h_1-1,w}_{J_1,\overline{J}_1}(y_1,\overline{y}_1)V^{h_2-1,w}_{J_2,\overline{J}_2}(y_2,\overline{y}_2)V_{h_3-1}(y_3,\overline{y}_3)\right\rangle \\
&\times\Big\{\Big[\left\langle\psi^w(x_1)\psi^{w+1}(x_2)\psi(x_3)\right\rangle\partial_{x_3}\left\langle\Phi^w_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^w_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle + \\
&h_3\left\langle\psi^w(x_1)\psi^{w+1}(x_2)\partial_{x_3}\psi(x_3)\right\rangle\left\langle\Phi^w_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^w_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle\Big] \\
&\times\left\langle\chi^{w+1}(y_1)\chi^w(y_2)\chi(y_3)\right\rangle\times c.c.\Big\}\ , \qquad (4.5)
\end{aligned}$$

where $w = w_1 = w_2$.

The SU(2) and fermionic expectation values were discussed in [12]. We now compute the SL(2,$\mathbb{R}$) correlators, applying the technique developed in [23].

From the integral transform

$$\Phi^{h,w}_{H,M,\overline{H},\overline{M}} = \int d^2x x^{H-M-1}\overline{x}^{\overline{H}-\overline{M}-1}\Phi^{h,w}_{H,\overline{H}}(x,\overline{x})\ , \qquad (4.6)$$

a generic three-point function in the $x-$basis, *e.g.* (we omit the z dependence for short)

$$\left\langle\Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi^{h_3,w_3}_{H_3,\overline{H}_3}(x_3,\overline{x}_3)\right\rangle = D(H_i,\overline{H}_i)\left(x_{12}^{-H_{12}}x_{23}^{-H_{23}}x_{13}^{-H_{13}}\times c.c.\right)\ , \qquad (4.7)$$

($c.c.$ stands for the antiholomorphic dependence), can be transformed to the $m-$basis as

$$\left\langle\prod_{i=1}^{3}\Phi^{h_i,w'_i}_{H_i,M_i,\overline{H}_i,\overline{M}_i}\right\rangle = (2\pi)^2D(H_i,\overline{H}_i)W(H_i,M_i,\overline{H}_i,\overline{M}_i)\delta^2(M_1+M_2+M_3), \qquad (4.8)$$

where

$$\begin{aligned}
W\left(H_i,M_i,\overline{H}_i,\overline{M}_i\right) &= \int d^2x_1d^2x_2x_1^{H_1-M_1-1}x_2^{H_2-M_2-1}\overline{x}_1^{\overline{H}_1-\overline{M}_1-1}\overline{x}_2^{\overline{H}_2-\overline{M}_2-1}|x_{12}|^{-2H_{12}} \\
&\times|1-x_1|^{-2H_{13}}|1-x_2|^{-2H_{23}}\ . \qquad (4.9)
\end{aligned}$$

Recall that the spectral flow with $w > 0$ ($w < 0$) turns primary states of the current algebra into lowest- (highest-) weight states of a global representation with $H = M = m + \frac{k+2}{2}w$, $\overline{H} = \overline{M} = \overline{m} + \frac{k+2}{2}w$ ($H = -M = -m + \frac{k+2}{2}|w|$, $\overline{H} = -\overline{M} = -\overline{m} + \frac{k+2}{2}|w|$).[5] Therefore, we are interested in the residue of the poles at say, $H_1 = M_1, \overline{H}_1 = \overline{M}_1$ and $H_2 = -M_2, \overline{H}_2 = -\overline{M}_2$. This is obtained by taking the $x_1,\overline{x}_1 \to 0$ and $x_2,\overline{x}_2 \to \infty$ limits in the integrand of $W(H_i,\overline{H}_i,M_i,\overline{M}_i)$, which simply gives

$$\left\langle\prod_{i=1}^{3}\Phi^{h_i,w'_i}_{H_i,M_i,\overline{H}_i,\overline{M}_i}\right\rangle = (2\pi)^2V^2_{\rm conf}\delta^2(M_1+M_2+M_3)D(H_i,\overline{H}_i), \qquad (4.10)$$

where $V_{\rm conf} = \int dx^2/|x|^2$.

[5] The spectral flow labels w and w' for highest/lowest weight states of global representations in the $x-$ and $m-$basis, respectively, may be related as $w' = \frac{M}{H}w$.

On the other hand, it is well known that spectral flow preserving $n-$point functions in the $m-$basis are related to correlators involving only unflowed operators as

$$\left\langle \prod_{i=1}^{n} \Phi^{h_i,w_i}_{H_i,M_i,\overline{H}_i,\overline{M}_i}(z_i,\overline{z}_i)\right\rangle_{\sum_i w_i=0} = \prod_{j<i}(z_{ij})^{-w_jm_i-w_im_j-\frac{k+2}{2}w_iw_j} \times c.c. \times \left\langle \prod_{i=1}^{n} \Phi^{w_i=0}_{h_i,m_i,\overline{m}_i}(z_i,\overline{z}_i)\right\rangle , \qquad (4.11)$$

and three-point functions of $w=0$ primary states have the following form [24, 25]:

$$\left\langle \prod_{i=1}^{3} \Phi^{w_i=0}_{h_i,m_i,\overline{m}_i}(z_i,\overline{z}_i)\right\rangle = (2\pi)^2\delta^2\Big(\sum_i m_i\Big)W(h_i,m_i,\overline{m}_i)C(h_i)|z_{12}|^{-2\Delta_{12}}|z_{13}|^{-2\Delta_{13}}|z_{23}|^{-2\Delta_{23}}, \qquad (4.12)$$

with

$$C(h_1,h_2,h_3) = -\frac{G(1-h_1-h_2-h_3)G(-h_{12})G(-h_{13})G(-h_{23})}{2\pi^2\nu^{h_1+h_2+h_3-1}\gamma\left(\frac{k+1}{k}\right)G(-1)G(1-2h_1)G(1-2h_2)G(1-2h_3)}, \qquad (4.13)$$

where $G(h) = k^{\frac{j(k+1-h)}{2k}}\Gamma_2(-h|1,k)\Gamma_2(k+1+h|1,k)$, Γ_2 being the Barnes double gamma function and $\Delta_{12} = \Delta_1+\Delta_2-\Delta_3$, $h_{12} = h_1+h_2-h_3$, etc.

Comparing with (4.10), one finds that the three-point functions involving spectral flow images of primary operators in arbitrary $w-$sectors in the $x-$basis corresponding to $w-$preserving amplitudes in the $m-$basis are given by[6]

$$\left\langle \Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi^{h_3,w_3}_{H_3,\overline{H}_3}(x_3,\overline{x}_3)\right\rangle = \frac{1}{V^2_{\text{conf}}}W(h_i,m_i,\overline{m}_i)C(h_i)x_{12}^{-H_{12}}x_{13}^{-H_{13}}x_{23}^{-H_{23}}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}} . \qquad (4.14)$$

Recall that this result holds for operators satisfying $m_1+m_2+m_3=0$.

As discussed above, for highest/lowest weight states the function $W(h_i,m_i,\overline{m}_i)$ develops poles which cancel the factor V^{-2}_{conf}. Taking, for instance, a chiral field at $x_1,\overline{x}_1$ and an antichiral one at $x_2,\overline{x}_2$, *i.e.* $m_1=\overline{m}_1=h_1, m_2=\overline{m}_2=-h_2$, the residue of the double pole is just one, and we obtain[7]

$$\mathcal{A}^1_3 \equiv \langle\Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi^{h_3,w_3}_{H_3,\overline{H}_3}(x_3,\overline{x}_3)\rangle = C(h_i)x_{12}^{-H_{12}}x_{13}^{-H_{13}}x_{23}^{-H_{23}}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}}.$$

[6] This correlation function was directly computed in the $x-$basis in [26] in the particular case $w_1=w_2=1$, $w_3=0$ using the definition of $w=1$ vertex operators given in [23]. Here we have used a different technique which is useful to evaluate correlators involving fields in arbitrary w sectors and, specially, expectation values including currents.

[7] Normalizing the two-point functions of these operators to the identity, this result agrees with the prediction formulated in [12] when the correlator involves one unflowed state. Three flowed chiral primary operators obeying $m_1+m_2+m_3=0$ cannot meet the condition $h_3=h_1+h_2-1$ under which the prediction of [12] holds.

JHEP06(2009)009

The following expectation value is also needed to evaluate $\mathcal{A}_3$:

$$\mathcal{A}_3^2 \equiv \left\langle \Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2) j(x_3)\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle .$$

The OPE $j(x)\Phi^{h,w}_{H,\overline{H}}(x',\overline{x}')$ is only known so far for $w=1$ fields [23], namely

$$\begin{aligned} j(x',z')\Phi^{h,w=1}_{H,\overline{H}}(x,\overline{x};z,\overline{z}) &= (m-h+1)\frac{(x-x')^2}{(z-z')^2}\Phi^{h,w=1}_{H+1,\overline{H}}(x,\overline{x};z,\overline{z}) \\ &\quad + \frac{1}{z'-z}\left[2H(x-x')+(x-x')^2\partial_x\right]\Phi^{h,w=1}_{H,\overline{H}}(x,\overline{x};z,\overline{z}) \,. \end{aligned} \quad (4.15)$$

Therefore, we restrict to this case. Inserting (4.15) into $\mathcal{A}_3^2$, one gets

$$\begin{aligned} \mathcal{A}_3^2 &= (1-h_1+m_1)\frac{(x_1-x_3)^2}{(z_1-z_3)^2} < \Phi^{h_1,w=1}_{H_1+1,\overline{H}_1}(x_1,\bar{x}_1)\Phi^{h_2,w=1}_{H_2,\overline{H}_2}(x_2,\bar{x}_2)\Phi_{h_3}(x_3,\bar{x}_3) > \\ &\quad + (1-h_2-m_2)\frac{(x_2-x_3)^2}{(z_2-z_3)^2} < \Phi^{h_1,w=1}_{H_1,\overline{H}_1}(x_1,\bar{x}_1)\Phi^{w=1,h_2}_{H_2+1,\overline{H}_2}(x_2,\bar{x}_2)\Phi_{h_3}(x_3,\bar{x}_3) > \\ &\quad + \frac{1}{z_3-z_1}\left[2H_1(x_1-x_3)+(x_1-x_3)^2\partial_{x_1}\right]\mathcal{A}_3^1 \\ &\quad + \frac{1}{z_3-z_2}\left[2H_2(x_2-x_3)+(x_2-x_3)^2\partial_{x_2}\right]\mathcal{A}_3^1 \,. \end{aligned}$$

The first two terms are easily evaluated using the procedure discussed above and we get

$$\begin{aligned} &\left\langle \Phi^{h_1,w=1}_{H_1+1,\overline{H}_1}(x_1,\bar{x}_1)\Phi^{h_2,w=1}_{H_2,\overline{H}_2}(x_2,\bar{x}_2)\Phi_{h_3}(x_3,\bar{x}_3)\right\rangle = \\ &\qquad W(h_i,m_1=h_1+1,m_2=-h_2,m_3) \\ &\qquad \times V_{\rm conf}^{-2}\, C(h_i)\; x_{12}^{-H_{12}-1}x_{13}^{-H_{13}-1}x_{23}^{-H_{23}+1}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}} \,, \end{aligned} \quad (4.16)$$

where

$$W(h_i,m_1=h_1+1,m_2=-h_2,m_3) = V_{\rm conf}^2\frac{h_{13}}{m_1-h_1+1}\,, \quad (4.17)$$

and similarly,

$$\begin{aligned} &\left\langle \Phi^{h_1,w=1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w=1}_{H_2+1,\overline{H}_2}(x_2,\overline{x}_2)\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle = \\ &\qquad C(h_i)\frac{h_{23}}{1-h_2-m_2} \\ &\qquad \times x_{12}^{-H_{12}-1}x_{13}^{-H_{13}+1}x_{23}^{-H_{23}-1}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}} \,. \end{aligned} \quad (4.18)$$

Putting all together, we obtain

$$\begin{aligned} \mathcal{A}_3^2 &= (3h_3-H_1-H_2)C(h_i)\; x_{12}^{-H_{12}-1}x_{13}^{-H_{13}+1}x_{23}^{-H_{23}+1}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}} \\ &\quad \times z_{12}^{-\Delta_{12}+1}z_{13}^{-\Delta_{13}-1}z_{23}^{-\Delta_{23}-1}\overline{z}_{12}^{-\overline{\Delta}_{12}}z_{13}^{-\overline{\Delta}_{13}}\overline{z}_{23}^{-\overline{\Delta}_{23}} \,, \end{aligned} \quad (4.19)$$

and analogously for the term containing the antiholomorphic current $\overline{j}(x)$ in $\mathcal{A}_3$.

JHEP06(2009)009

To write down the final result, let us recall the fermionic and SU(2) correlators (see [12] for details).

$$< \psi^{w+1}(x_1)\psi^{w+1}(x_2) > = \frac{k}{2}\frac{x_{12}^{2(w+1)}}{z_{12}^{(w+1)^2}} \ , \tag{4.20}$$

$$\begin{aligned} < \psi^{w+1}(x_1)\psi^{w+1}(x_2)\hat{j}(x_3) > &= \sum_{i=1}^{2}\frac{1}{z_{3i}}\left[2(w+1)x_{3i} + (x_{3i})^2\partial_{x_i}\right] < \psi^{w+1}(x_1)\psi^{w+1}(x_2) > \\ &= k(w+1)\frac{x_{13}x_{23}}{x_{12}}\frac{z_{12}}{z_{13}z_{23}}\frac{x_{12}^{2(w+1)}}{z_{12}^{(w+1)^2}}\,, \end{aligned} \tag{4.21}$$

$$< \psi^{w}(x_1)\psi^{w+1}(x_2)\psi(x_3) > = k\frac{x_{12}^{2w}x_{23}^{2}z_{13}^{w}}{z_{12}^{w^2+w}z_{23}^{w+1}}\,. \tag{4.22}$$

Similar expressions are obtained for χ^w.

In the SU(2) WZNW model, normalizing the two−point functions as

$$\langle V_{j_1}(y_1,\overline{y}_1;z_1,\overline{z}_1)V_{j_2}(y_2,\overline{y}_2;z_2,\overline{z}_2) = \delta_{j_1j_2}\frac{|y_{12}|^{2j_1}}{|z_{12}|^{4\Delta_{j_1}}}\,, \tag{4.23}$$

the three−point functions are given by [27]

$$\langle V_{j_1}(y_1,\bar{y}_1;z_1,\bar{z}_1)V_{j_2}(y_2,\bar{y}_2;z_2,\bar{z}_2)V_{j_3}(y_3,\bar{y}_3;z_3,\bar{z}_3)\rangle = C'(j_1,j_2,j_3)\prod_{i<j}\frac{|y_{ij}|^{2j_{ij}}}{|z_{ij}|^{2\Delta_{ij}}}\,, \tag{4.24}$$

for $j_n \le j_m + j_l$, where

$$C'(j_1,j_2,j_3) = \sqrt{\frac{\gamma(\frac{1}{k})}{\gamma(\frac{2j_1+1}{k})\gamma(\frac{2j_2+1}{k})\gamma(\frac{2j_3+1}{k})}}\,\frac{P(j_1+j_2+j_3+1)\,P(j_{12})\,P(j_{23})\,P(j_{31})}{P(2j_1)\,P(2j_2)\,P(2j_3)}$$

with $P(j) = \prod_{m=1}^{j}\gamma(\frac{m}{k}), P(0) = 1$.

As argued in [12], the structure constants for spectral flowed chiral fields in SU(2) are also given by $C'(j_i)$ for $j_n = j_m + j_l$. Therefore, collecting all the contributions and suppressing the $x-$ and $z-$dependence for short, we get

$$\mathcal{A}_3 = g_s^{-2}\frac{k^2}{4}\left|\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+1\right|^2 C'(j_i)C(h_i)\,, \tag{4.25}$$

$$\mathcal{A'}_3 = g_s^{-2}\frac{k^2}{4}\left|\mathcal{H}_1-\mathcal{H}_2+\mathcal{H}_3-1\right|^2 C'(j_i)C(h_i)\,. \tag{4.26}$$

As shown in [7, 8],

$$C'(j_i)C(h_i) = \sqrt{B(h_1)B(h_2)B(h_3)}, \qquad B(h_i) = \frac{k}{4\pi^3}\frac{\nu^{1-2h_i}}{\gamma\left(\frac{2h_i-1}{k}\right)}\,, \qquad \nu = \pi\frac{\Gamma\left(1-\frac{1}{k}\right)}{\Gamma\left(1+\frac{1}{k}\right)}\,, \tag{4.27}$$

with $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

JHEP06(2009)009

In order to compare these results with the conjectured dual counterparts, the two-point functions must be normalized to the identity. Taking into account that in the SL(2,$\mathbb{R}$) sector they are given by [23]

$$\langle \Phi^{h,w}_{H,\overline{H}}(x_1,\overline{x}_1)\Phi^{h,w}_{H,\overline{H}}(x_2,\overline{x}_2)\rangle = g_s^{-2}(2h-1+kw)B(h)x_{12}^{-2H}\overline{x}_{12}^{-2\overline{H}}\,, \tag{4.28}$$

the normalized chiral operators are defined as:

$$\mathbb{W}^{h,w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x}) \equiv \frac{4g_s\mathcal{W}^{h,w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x})}{k^2\sqrt{B(h)(2h-1+kw)}}, \qquad \mathbb{Y}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}(x,\bar{x}) \equiv \frac{4g_s\mathcal{Y}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}(x,\bar{x})}{k^2\sqrt{B(h)(2h-1+kw)}}\,. \tag{4.29}$$

Omitting the standard dependence on the coordinates, we thus get

$$\left\langle \mathbb{W}^{h_1,w}_{\mathcal{H}_1\overline{\mathcal{H}}_1}\,\mathbb{W}^{h_2,w}_{\mathcal{H}_2\overline{\mathcal{H}}_2}\,\mathbb{W}^{(0)}_{h_3}\right\rangle = \frac{4g_s}{k^2}\frac{|\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+1|^2}{\sqrt{(2h_1-1+kw)(2h_2-1+kw)(2h_3-1)}}\,, \tag{4.30}$$

$$\left\langle \mathbb{Y}^{h_1,w}_{\mathcal{H}_1\overline{\mathcal{H}}_1}\,\mathbb{W}^{h_2,w}_{\mathcal{H}_2\overline{\mathcal{H}}_2}\,\mathbb{Y}^{(0)}_{h_3}\right\rangle = \frac{4g_s}{k^2}\frac{|\mathcal{H}_1-\mathcal{H}_2+\mathcal{H}_3-1|^2}{\sqrt{(2h_1-1+kw)(2h_2-1+kw)(2h_3-1)}}\,. \tag{4.31}$$

While (4.30) was obtained for $w=1$, (4.31) holds for arbitrary w.

These three-point functions involve one unflowed operator. We restricted to this case for simplicity. However, notice that when the three operators are spectral flow images of chiral primaries of SL(2,$\mathbb{R}$) or the unflowed operator creates a highest/lowest weight primary state, the condition $h_i = \pm m_i$ together with the requirement $m_1+m_2+m_3=0$ imply, for example, $h_2 = h_1 + h_3$. Combined with the chirality condition $j_i = h_i - 1$, this gives $j_2 = j_1 + j_3 + 1$ which violates the triangular inequality $j_2 \leq j_1 + j_3$ of the SU(2) WZNW model. Therefore the SU(2) factor gives a zero for the whole three-point function. This conclusion does not apply when the unflowed operator obeys $h_3 \neq \pm m_3$. Therefore, the results (4.30) and (4.31) hold for amplitudes containing two flowed and one unflowed chiral primary operators as long as the latter does not create a highest/lowest weight state in the SL(2, $\mathbb{R}$) sector.

Let us now compare these results with the correlators in the dual theory. The level k is identified with N_5 [18] and $g_s^2 = \frac{\mathrm{N}_5}{\mathrm{N}_1}Vol(\mathrm{T}^4)$ [2, 3, 16], so these correlation functions scale as $\mathrm{N}^{-1/2}$ in the large N limit. Recall that the chiral string states $\mathcal{W}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}$, $\mathcal{Y}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}$ have been identified with the chiral operators $\mathcal{O}^{--}_{n,\bar{n}}$, $\mathcal{O}^{++}_{n,\bar{n}}$ of the SCFT, respectively [7, 8, 12]. Moreover, the proposed identification between the quantum numbers of $\mathbb{W}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}$ and those of $\mathcal{O}^{--}_{n}(x,\bar{x})$ is the following [7, 8, 12]

$$\mathcal{H}_n = \frac{n-1}{2} = h-1+\frac{k}{2}w \qquad \Rightarrow \qquad n = 2h-1+kw\,, \tag{4.32}$$

and for $\mathbb{Y}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}$ and $\mathcal{O}^{++}_{n}(x,\bar{x})$ it is

$$\mathcal{H}_n = \frac{n+1}{2} = h+\frac{k}{2}w \qquad \Rightarrow \qquad n = 2h-1+kw\,. \tag{4.33}$$

Replacing these values of n in the boundary three-point functions (2.33), one gets at leading order

$$\left\langle \mathcal{O}^{--}_{n_1}\mathcal{O}^{--}_{n_2}\mathcal{O}^{--\dagger}_{n_3}\right\rangle = \frac{1}{\sqrt{\mathrm{N}}}\frac{|\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+1|^2}{\sqrt{(2h_1-1+kw_1)(2h_2-1+kw_2)(2h_3-1)}}\,, \tag{4.34}$$

$$\left\langle \mathcal{O}^{++}_{n_1}\mathcal{O}^{--}_{n_2}\mathcal{O}^{++\dagger}_{n_3}\right\rangle = \frac{1}{\sqrt{\mathrm{N}}}\frac{|\mathcal{H}_1-\mathcal{H}_2+\mathcal{H}_3-1|^2}{\sqrt{(2h_1-1+kw_1)(2h_2-1+kw_2)(2h_3-1)}}\,, \tag{4.35}$$

in perfect agreement with (4.30) and (4.31), respectively. Furthermore, using the bulk-to-boundary dictionary, one can verify that the boundary correlators corresponding to three spectral flow images of chiral primary operators is zero because in both cases (4.34) and (4.35) the relation $h_2 = h_1 + h_3$ implies $n_2 = n_1 + n_3$, which violates the U(1) charge by one unit.

Two other correlators can be considered in the string theory corresponding to the vanishing correlators $(\epsilon_{n_1}, \epsilon_{n_2}, \epsilon_{n_3}) = (+, +, +)$ and $(-, -, +)$ in the boundary CFT, namely $\langle\prod_{i=1}^{3}\mathcal{Y}^{h_i,w_i}_{\mathcal{H}_i,\overline{\mathcal{H}}_i}\rangle$ and $\langle\prod_{i=1}^{2}\mathcal{W}^{h_i,w_i}_{\mathcal{H}_i,\overline{\mathcal{H}}_i}\mathcal{Y}^{h_3,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}\rangle$. It is easy to see that they violate the SU(2) charge conservation in the case $j_2 = j_1 + j_3$ that we are considering, and therefore they also vanish.

4.2 R-R-NS three-point functions

The chiral states $\mathcal{R}^{\pm,h,w}_{\mathcal{H},\overline{\mathcal{H}}}$ were identified with the operators $\mathcal{O}^{a,\overline{a}}_n$ in [8, 12]. To compare the corresponding three-point functions in the dual theories, the two R-R-NS correlators needed are

$$\mathbb{A}_3 = g_s^{-2}\left\langle \mathcal{R}^{\pm,h_3,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,\overline{x}_3)\mathcal{R}^{\pm,h_2,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}(x_2,\overline{x}_2)\mathcal{W}^{h_1,w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,\overline{x}_1)\right\rangle_{S^2}\,, \tag{4.36}$$

$$\mathbb{A}'_3 = g_s^{-2}\left\langle \mathcal{Y}^{h_3,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,\overline{x}_3)\mathcal{R}^{\pm,h_2,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}(x_2,\overline{x}_2)\mathcal{R}^{\pm,h_1,w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,\overline{x}_1)\right\rangle_{S^2}\,. \tag{4.37}$$

The R vertices (3.15) were obtained in the $-\frac{1}{2}$ picture, so it is not necessary to insert a picture changing operator and we can compute this amplitude for states in arbitrary w sectors, as long as $w_n = w_m + w_l$.

The SU(2) part of the three-point functions is given by $C'(j_i)$ for $j_n = j_m + j_l$ and the fermionic contributions are the following [12][8]

$$\begin{aligned}
\langle S^{w_3+\frac{1}{2}}(x_3,y_3)S^{w_2+\frac{1}{2}}(x_2,y_2)\psi^{w_1+1}(x_1)\chi^{w_1}(y_1)\rangle &= \frac{(w_1+w_2)!}{w_1!w_2!}\,,\\
\langle \psi^{w_3}(x_3)\chi^{w_3+1}(y_3)S^{w_2+\frac{1}{2}}(x_2,y_2)S^{w_1+\frac{1}{2}}(x_1,y_1)\rangle &= \frac{(w_1+w_2)!}{w_1!w_2!}\,.
\end{aligned} \tag{4.38}$$

As shown in the previous section, the SL(2,$\mathbb{R}$) contribution is simply $C(h_i)$ for two or three flowed chiral primary states satisfying $m_1 + m_2 + m_3 = 0$. If the three operators are flowed, the SU(2) spins violate the triangular inequality and the correlator vanishes, analogously to the NS-NS-NS case. When one operator is unflowed, the factor $\frac{(w_1+w_2)!}{w_1!w_2!}$ reduces to unity and we have

$$\mathbb{A}_3 = \mathbb{A}'_3 = g_s^{-2}\sqrt{B(h_1)B(h_2)B(h_3)}\,. \tag{4.39}$$

[8] We get the inverse of the result reported in [12].

JHEP06(2009)009

Normalizing the R operators as (see [12] for details)

$$\mathbb{R}^{\pm,h,w}_{\mathcal{H},\overline{\mathcal{H}}} = \sqrt{\frac{(2h-1+kw)}{2B(h)}}\, g_s \mathcal{R}^{\pm,h,w}_{\mathcal{H},\overline{\mathcal{H}}} \,, \tag{4.40}$$

we get

$$\begin{aligned}\left\langle \mathbb{R}^{\pm,h_3,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3} \mathbb{R}^{\pm,h_2,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2} \mathbb{W}^{h_1,w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1} \right\rangle &= \left\langle \mathbb{R}^{\pm,h_3,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3} \mathbb{R}^{\pm,h_2,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2} \mathbb{Y}^{h_1,w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1} \right\rangle \\ &= \frac{2g_s}{k^2}\left[\frac{(2h_3+kw_3-1)(2h_2+kw_2-1)}{(2h_1+kw_1-1)}\right]^{1/2} \,,\end{aligned} \tag{4.41}$$

for $w_1 = 0$ or $w_2 = 0$, again in agreement with the boundary correlators (2.34) and (2.35).

5 Conclusions

We have evaluated spectral flow conserving three-point functions containing spectral flow images of chiral primary states in type IIB superstring theory on $AdS_3\times S^3\times T^4$ and showed that they agree with the corresponding correlators in the dual boundary CFT. These results provide an additional verification of the AdS_3/CFT_2 correspondence, widening similar conclusions of previous works [7, 8, 10] to the non-trivial spectral flow sectors of the theory.

The matching obtained so far reflects the cancellation of the three-point structure constant of AdS_3 against that of the S^3 factor. The non-trivial fermionic contributions reduce to unity in all the non-vanishing amplitudes that we have considered here. A definite confirmation of this duality would require extending the bulk-to-boundary dictionary to descendant states. The evaluation of three-point functions involving affine descendants and their spectral flow images is an interesting subject in its own right. Actually, the spectral flow operation maps primaries into descendants both in SU(2) and SL(2, $\mathbb{R}$) and it generates new representations of the universal cover of SL(2, $\mathbb{R}$). Understanding these new representations is crucial to solve the AdS_3 WZNW model and elucidate the physical mechanism determining the truncation of the fusion rules imposed by the spectral flow symmetry [28]. In the context of the AdS_3/CFT_2 correspondence, a better comprehension of the structure of the spectral flow sectors would contribute to achieve a systematic comprehension of the hypothesis advanced in the literature.

Acknowledgments

We would like to thank W. Baron, S. Iguri and P. Minces for useful discussions. We are specially grateful to A. Pakman and an anonymous referee for carefully reading the manuscript, for pointing out a misleading conclusion in the previous version of this paper and for many interesting comments. This work was supported in part by grants PIP-CONICET/6332 and UBACyT X161.

JHEP06(2009)009

A Clebsch-Gordan coefficients

In this appendix we compute the Clebsch-Gordan coefficients (CG) expanding the product representation $(H \otimes \hat{h})$ of the SL(2,$\mathbb{R}$) algebra. We consider the case $H \in \mathcal{D}_H^{+,w}$, $\hat{h} \in \mathcal{D}_{\hat{h}}$, where

$$\mathcal{D}_H^+ : \quad \{|H,M\rangle\,; \quad H \in \mathbb{R}, \quad M = H + n, \quad n = 1,2,3\ldots\ldots\}\,, \tag{A.1}$$

is an infinite discrete representation and

$$\mathcal{D}_{\hat{h}}^+ : \quad \{|\hat{h},\hat{m}\rangle\,; \quad -\hat{h} \leq \hat{m} \leq \hat{h}\,, \quad \hat{m} \in \mathbb{Z}\}\,, \tag{A.2}$$

is a finite representation of the SL(2,$\mathbb{R}$)$_{-2}$ algebra. We use the following normalization

$$\mathbf{j}^{\pm}|H,M >= (M \mp H)|H,M \pm 1 > \tag{A.3}$$

and similarly for $|\hat{h},\hat{m}>$. A state living in the product representation may be expanded as

$$|H \otimes \hat{h}\rangle \equiv |H,\hat{h};\mathcal{H},\mathcal{M}\rangle = \sum_{M,\hat{m}} |H,M;\hat{h},\hat{m}\rangle\langle H,M;\hat{h},\hat{m}|H,\hat{h};\mathcal{H},\mathcal{M}\rangle\delta_{\mathcal{M},M+\hat{m}}\,. \tag{A.4}$$

Applying the raising operator $\mathbf{H}^+ = \mathbf{j_1^+} + \mathbf{j_2^+}$ and equating the coefficients on both sides of (A.4), the following recursion relation is obtained

$$\begin{aligned}(\mathcal{M}-\mathcal{H})\langle\mathcal{M}+1-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}+1\rangle =&(\mathcal{M}-\hat{m}-H)\langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle \\ &+ (\hat{m}-1-\hat{h})\langle\mathcal{M}-\hat{m}+1,\hat{m}-1|\mathcal{H},\mathcal{M}\rangle\,,\end{aligned} \tag{A.5}$$

where the indices H, $\hat{h}$ have been dropped for short. A similar recursion relation is obtained applying the lowering operator $\mathbf{H}^- = \mathbf{j_1^-} + \mathbf{j_2^-}$, namely

$$\begin{aligned}&(\mathcal{M}+\mathcal{H})\langle\mathcal{M}-1-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}-1\rangle = \\ &\qquad(\mathcal{M}-\hat{m}+H)\langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle + (\hat{m}+\hat{h}+1)\langle\mathcal{M}-\hat{m}-1,\hat{m}+1|\mathcal{H},\mathcal{M}\rangle\,.\end{aligned} \tag{A.6}$$

The last term in (A.5) vanishes for $\hat{m} = -\hat{h}$, *i.e.*

$$\langle\mathcal{M}+\hat{h}+1,-\hat{h}|\mathcal{H},\mathcal{M}+1\rangle = \frac{\mathcal{M}+\hat{h}-H}{\mathcal{M}-\mathcal{H}}\langle\mathcal{M}+\hat{h},-\hat{h}|\mathcal{H},\mathcal{M}\rangle\,, \tag{A.7}$$

and for $\mathcal{M} = \mathcal{H}+1$, this reads

$$\langle\mathcal{H}+\hat{h}+2,-\hat{h}|\mathcal{H},\mathcal{H}+2\rangle = (\mathcal{H}+1+\hat{h}-H)\langle M',-\hat{h}|\mathcal{H},\mathcal{H}+1\rangle\,. \tag{A.8}$$

Then, taking successively $\mathcal{M} = \mathcal{H}+2,\ldots,\mathcal{H}+n$, one finds

$$\langle M,-\hat{h}|\mathcal{H},\mathcal{M}\rangle = \frac{(\hat{h}-H+\mathcal{M}-1)!}{(\mathcal{M}-\mathcal{H}-1)!(\mathcal{H}+\hat{h}-H)!}\langle M_1',-\hat{h}|\mathcal{H},\mathcal{H}+1\rangle. \tag{A.9}$$

JHEP06(2009)009

Defining $q(\hat{m},\mathcal{M}) \equiv \frac{(-1)^{\hat{m}}(\hat{m}+\hat{h})!(\mathcal{M}-\hat{m}+H)!}{(\mathcal{M}+\mathcal{H})!}$, (A.6) may be recast as

$$\begin{aligned} q(\hat{m}+1,\mathcal{M})\langle\mathcal{M}-(\hat{m}+1),\hat{m}+1|\mathcal{H},\mathcal{M}\rangle &= q(\hat{m},\mathcal{M})\langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle \\ &\quad -q(\hat{m},\mathcal{M}-1)\langle\mathcal{M}-1-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}-1\rangle \\ &\equiv \Delta_{\mathcal{M}}[q(\hat{m},\mathcal{M})\langle M',\hat{m}|\mathcal{H},\mathcal{M}\rangle]\,. \end{aligned} \tag{A.10}$$

Applying this successively for $\hat{m}-1,\ldots,\hat{m}-n$ and using $\Delta_x^n[f](x) = \sum_{s=0}^n (-1)^s \begin{pmatrix} n \\ s \end{pmatrix} f(x-s)$, we get

$$\begin{aligned} \langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle &= \frac{1}{q(\hat{m},\mathcal{M})}\Delta_{\mathcal{M}}^{\hat{m}+\hat{h}}[q(-\hat{h},\mathcal{M})\langle M',-\hat{h}|\mathcal{H},\mathcal{M}\rangle] \\ &= \frac{1}{q(\hat{m},\mathcal{M})}\sum_{s=0}^{\hat{m}+\hat{h}}(-1)^s\begin{pmatrix}\hat{m}+\hat{h} \\ s\end{pmatrix} q(-\hat{h},\mathcal{M}-s)\langle\mathcal{M}-s+\hat{h},-\hat{h}|\mathcal{H},\mathcal{M}-s\rangle\,. \end{aligned}$$

Substituting $q(\hat{m},\mathcal{M})$ and $\langle\mathcal{M}-s+\hat{h},-\hat{h}|\mathcal{H},\mathcal{M}-s\rangle$ in this equation, we obtain

$$\begin{aligned} \langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle =& \frac{(\mathcal{M}+\mathcal{H})!}{(\hat{m}+\hat{h})!(\mathcal{M}-\hat{m}+H)!}\sum_{s=0}^{\hat{m}+\hat{h}}(-1)^{s-\hat{h}}\begin{pmatrix}\hat{m}+\hat{h} \\ s\end{pmatrix}\frac{(\mathcal{M}-s+\hat{h}+H)!}{(\mathcal{M}-s+\mathcal{H})!} \\ &\times\frac{(\hat{h}-H+\mathcal{M}-s-1)!}{(\mathcal{M}-s-\mathcal{H}-1)!(\mathcal{H}+\hat{h}-H)!}\langle\mathcal{H}+1+\hat{h},-\hat{h}|\mathcal{H},\mathcal{H}+1\rangle\,. \end{aligned}$$

Therefore, all the CG coefficients in the expansion (A.4) are expressed in terms of just one coefficient, which can be set to one.[9] As a consistency check, we compute some known cases.

In the unflowed sector, we need to decompose the product representation with $\hat{h}=1$. In this case, there are three possible combinations of $\mathcal{H}$, according to the angular momentum selection rules, namely $\mathcal{H}=H+1$, $\mathcal{H}=H$, $\mathcal{H}=H-1$. In the first case, one gets

$$\begin{aligned} (\psi\Phi)_{H+1,M}^{\omega=0} &= \sum_{M,\hat{m}}(\Phi_{H,M}^{\omega=0}\psi^{\hat{m}})\langle M-\hat{m},\hat{m}|H+1,M\rangle \\ &= \frac{1}{2}(H-M)(1+H-M)\Phi_{H,M+1}^{\omega=0}\psi^- + (H+1-M)(1+H+M)\Phi_{H,M}^{\omega=0}\psi^3 \\ &\quad +\frac{1}{2}(H+M)(1+H+M)\Phi_{H,M-1}^{\omega=0}\psi^+\,. \end{aligned} \tag{A.11}$$

For $\mathcal{H}=H$, the following field expansion is obtained

$$(\psi\Phi)_{H,M}^{\omega=0} = (M-H)\Phi_{H,M+1}^{\omega=0}\psi^- - 2M\Phi_{H,M}^{\omega=0}\psi^3 + (H+M)\Phi_{H,M-1}^{\omega=0}\psi^+\,. \tag{A.12}$$

And finally, for $\mathcal{H}=H-1$, which satisfies the chirality condition in the unflowed sector,

$$(\psi\Phi)_{H-1,M}^{\omega=0} = \Phi_{H,M+1}^{\omega=0}\psi^- - 2\Phi_{H,M}^{\omega=0}\psi^3 + \Phi_{H,M-1}^{\omega=0}\psi^+\,, \tag{A.13}$$

in accord with the decomposition given in [2], up to the global phase factor mentioned above.

[9] Recall that the CG are determined up to a global phase factor (which is a global multiplicative factor for all remaining CG).

References

[1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [SPIRES].

[2] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS_3*, *Adv. Theor. Math. Phys.* **2** (1998) 733 [hep-th/9806194] [SPIRES].

[3] D. Kutasov and N. Seiberg, *More comments on string theory on AdS_3*, *JHEP* **04** (1999) 008 [hep-th/9903219] [SPIRES].

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [SPIRES].

[5] J.R. David, G. Mandal and S.R. Wadia, *Microscopic formulation of black holes in string theory*, *Phys. Rept.* **369** (2002) 549 [hep-th/0203048] [SPIRES].

[6] M.R. Douglas, *Branes within branes*, hep-th/9512077 [SPIRES];
C. Vafa, *Instantons on D-branes*, *Nucl. Phys.* **B 463** (1996) 435 [hep-th/9512078] [SPIRES];
E. Witten, *On the conformal field theory of the Higgs branch*, *JHEP* **07** (1997) 003 [hep-th/9707093] [SPIRES].

[7] M.R. Gaberdiel and I. Kirsch, *Worldsheet correlators in AdS_3/CFT_2*, *JHEP* **04** (2007) 050 [hep-th/0703001] [SPIRES].

[8] A. Dabholkar and A. Pakman, *Exact chiral ring of AdS_3/CFT_2*, hep-th/0703022 [SPIRES].

[9] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and SL(2,R) WZW model. I*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053] [SPIRES].

[10] A. Pakman and A. Sever, *Exact $N = 4$ correlators of AdS_3/CFT_2*, *Phys. Lett.* **B 652** (2007) 60 [arXiv:0704.3040] [SPIRES].

[11] R. Argurio, A. Giveon and A. Shomer, *Superstrings on AdS_3 and symmetric products*, *JHEP* **12** (2000) 003 [hep-th/0009242] [SPIRES].

[12] G. Giribet, A. Pakman and L. Rastelli, *Spectral flow in AdS_3/CFT_2*, *JHEP* **06** (2008) 013 [arXiv:0712.3046] [SPIRES].

[13] A. Jevicki, M. Mihailescu and S. Ramgoolam, *Gravity from CFT on $S^N(X)$: symmetries and interactions*, *Nucl. Phys.* **B 577** (2000) 47 [hep-th/9907144] [SPIRES].

[14] O. Lunin and S.D. Mathur, *Three-point functions for M(N)/S(N) orbifolds with $N = 4$ supersymmetry*, *Commun. Math. Phys.* **227** (2002) 385 [hep-th/0103169] [SPIRES].

[15] O. Lunin and S.D. Mathur, *Correlation functions for M(N)/S(N) orbifolds*, *Commun. Math. Phys.* **219** (2001) 399 [hep-th/0006196] [SPIRES].

[16] D. Kutasov, F. Larsen and R.G. Leigh, *String theory in magnetic monopole backgrounds*, *Nucl. Phys.* **B550** (1999) 183 [hep-th/9812027] [SPIRES].

[17] A. Pakman, *Unitarity of supersymmetric $SL(2,R)/\mathrm{U}(1)$ and no-ghost theorem for fermionic strings in $AdS_3 \times N$*, *JHEP* **01** (2003) 077 [hep-th/0301110] [SPIRES].

[18] J.M. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [hep-th/9804085] [SPIRES].

[19] F. Larsen and E.J. Martinec, U(1) *charges and moduli in the $D1 - D5$ system*, *JHEP* **06** (1999) 019 [hep-th/9905064] [SPIRES].

JHEP06(2009)009

[20] J. de Boer, *Six-dimensional supergravity on* $S^3 \times AdS_3$ *and 2d conformal field theory*, *Nucl. Phys.* **B 548** (1999) 139 [hep-th/9806104] [SPIRES].

[21] L. Pochhammer, *On an integral with a double circulation*, *Math. Ann.* **35** (1890) 470.

[22] W.J. Holman and L.C. Biedenharn, *Complex angular momenta and the groups SU(1,1) and* SU(2), *Ann. Phys.* **39** (1966) 1; *A general study of the Wigner coefficients of SU(1,1)*, *Ann. Phys.* **47** (1968) 205.

[23] J.M. Maldacena and H. Ooguri, *Strings in* AdS_3 *and the SL(2,R) WZW model. III: correlation functions*, *Phys. Rev.* **D 65** (2002) 106006 [hep-th/0111180] [SPIRES].

[24] J. Teschner, *On structure constants and fusion rules in the* $SL(2,C)/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 546** (1999) 390 [hep-th/9712256] [SPIRES].

[25] Y. Satoh, *Three-point functions and operator product expansion in the SL(2) conformal field theory*, *Nucl. Phys.* **B 629** (2002) 188 [hep-th/0109059] [SPIRES].

[26] P. Minces, C.A. Núñez and E. Herscovich, *Winding strings in* AdS_3, *JHEP* **06** (2006) 047 [hep-th/0512196] [SPIRES].

[27] A.B. Zamolodchikov and V.A. Fateev, *Operator algebra and correlation functions in the two-dimensional Wess-Zumino* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *chiral model*, *Sov. J. Nucl. Phys.* **43** (1986) 657 [*Yad. Fiz.* **43** (1986) 1031] [SPIRES].

[28] W.H. Baron and C.A. Núñez, *Fusion rules and four-point functions in the SL(2,R) WZNW model*, *Phys. Rev.* **D 79** (2009) 086004 [arXiv:0810.2768] [SPIRES].

JHEP06(2009)009