AN H-SYSTEM FOR A REVOLUTION SURFACE WITHOUT BOUNDARY

P. AMSTER, P. DE NÁPOLI, AND M. C. MARIANI

Received 20 November 2003; Accepted 25 April 2005

We study the existence of solutions an H-system for a revolution surface without boundary for H depending on the radius f. Under suitable conditions we prove that the existence of a solution is equivalent to the solvability of a scalar equation $N(a) = L/\sqrt{2}$, where $N : \mathcal{A} \subset \mathbb{R}^+ \to \mathbb{R}$ is a function depending on H. Moreover, using the method of upper and lower solutions we prove existence results for some particular examples. In particular, applying a diagonal argument we prove the existence of unbounded surfaces with prescribed H.

1. Introduction

The prescribed mean curvature equation for a vector function $X : \overline{\Omega} \to \mathbb{R}^3$ is given by the following nonlinear system of partial differential equations:

$$\triangle X = 2H(X)X_u \wedge X_v \quad (u,v) \in \Omega. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain, $\wedge$ denotes the exterior product in $\mathbb{R}^3$ and $H : \mathbb{R}^3 \to \mathbb{R}$ is a given function. It is well known that if X is isothermal, namely

$$|X_u| - |X_v| = X_u X_v = 0 \tag{1.2}$$

then H is the mean curvature of the surface parameterized by X (see, e.g., [8]). Equation (1.1) is also known in the literature as an H-system.

The parametric Plateau and Dirichlet problems for (1.1) have been extensively studied by different authors (see [3–5, 8–10]). Nonparametric and more general quasilinear equations are considered in [1, 2, 6, 7].

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2006, Article ID 93163, Pages 1–10
DOI 10.1155/AAA/2006/93163

We will consider the particular case of a revolution surface

$$X(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big) \tag{1.3}$$

with $f,g \in C^2(I)\cap C(\bar{I})$ such that $f>0$ over the open interval $I\subset\mathbb{R}$. Then (1.1) reads

$$\begin{aligned} f''-f &= -2H(f,g)fg' \quad \text{in } I \\ g'' &= 2H(f,g)ff' \quad \text{in } I, \end{aligned} \tag{1.4}$$

where $H:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is given.

It is easy to see that any solution of (1.4) verifies the equality

$$(f')^2+(g')^2 = f^2+c. \tag{1.5}$$

Hence, the isothermal condition (1.2) holds if and only if $c=0$.

We will study (1.4) for a compact surface without boundary. Without loss of generality we may assume that $I=(0,L)$, and hence the problem reads

$$\begin{gathered} f''-f = -2H(f,g)fg' \quad \text{in } I \\ g'' = 2H(f,g)ff' \quad \text{in } I \\ f(0)=f(L)=0, \quad f>0 \text{ in } I \\ g'(0)=g'(L)=0. \end{gathered} \tag{1.6}$$

In particular, when H depends only on the radius f, from the equality

$$g'' = 2H(f)ff', \qquad g'(0)=0, \tag{1.7}$$

we easily reduce problem (1.6) to a single equation: indeed, if $\widetilde{H}(t)=\int_0^t sH(s)ds$, it holds that $g'(t)=2\widetilde{H}(f(t))$, and $g'(L)=2\widetilde{H}(f(L))$. Thus, solving (1.6) is equivalent to obtain a positive solution of the problem

$$\begin{gathered} f''-f = -4H(f)f\widetilde{H}(f) \quad \text{in } (0,+\infty) \\ f(0)=f(L)=0 \end{gathered} \tag{1.8}$$

with $H:\mathbb{R}^+\to\mathbb{R}$. We remark that if $\widetilde{H}>0$ then $g'>0$, and if f is a positive solution of (1.8) the parametrization X given in (1.3) defines a regular revolution surface. For example, this holds when H is positive.

We will also consider the case $L=+\infty$, namely the problem

$$\begin{gathered} f''-f = -4H(f)f\widetilde{H}(f) \quad \text{in } I \\ f(0)=0, \qquad f(+\infty)=r, \end{gathered} \tag{1.9}$$

where $r > 0$ is a constant. Note that if f is a positive solution of (1.9) then $g'(+\infty) = 2\widetilde{H}(r)$. Thus, if $\widetilde{H}(r) > 0$ it follows that the surface parameterized by X is unbounded in the direction $z \to +\infty$ of the upper halfspace $\mathbb{R}^2 \times \mathbb{R}^+$.

The paper is organized as follows. In Section 2 we prove that under suitable conditions the existence of a positive solution of (1.8) is equivalent to the solvability of the scalar equation $N(a) = L/\sqrt{2}$, where N is defined by

$$N(a) = \int_0^a \frac{dz}{\sqrt{\phi(a) - \phi(z)}} \tag{1.10}$$

with

$$\phi(u) := 2\widetilde{H}^2(u) - \frac{u^2}{2}. \tag{1.11}$$

Moreover, we prove existence and uniqueness of solutions for some particular examples.

In Section 3 we apply the method of upper and lower solutions and a diagonal argument in order to prove the existence of solutions of problem (1.9).

2. A scalar equation for (1.8)

In this section we study the existence of positive solutions of (1.8). Let us first note that if ϕ is defined as in (1.11), the problem may be written as

$$\begin{gathered} f'' + \phi'(f) = 0 \quad \text{in } I \\ f(0) = f(L) = 0 \end{gathered} \tag{2.1}$$

Then we have the following theorem.

Theorem 2.1. *Let $a \in \mathcal{A}$, where*

$$\mathcal{A} = \{a \in \mathbb{R}^+ : \phi(a) > \phi(u) \text{ for } 0 < u < a\}, \tag{2.2}$$

and let N be defined by (1.10).

Then (2.1) admits at most one positive solution f with $a = \|f\|_{C([0,1])}$. Furthermore, (2.1) admits a positive solution f with $a = \|f\|_{C([0,1])}$ if and only if $N(a) = L/\sqrt{2}$.

Proof. Let f be a positive solution of (2.1) with $a = \|f\|_{C([0,1])}$, and fix $x_0 \in (0,L)$ such that $f(x_0) = a$. Multiplying the equation by f' it follows by integration that

$$E := (f')^2 + 2\phi(f) = 2\phi(a). \tag{2.3}$$

Note that if $f'(x) = 0$ for some $x \in (0,L)$ then $\phi(f(x)) = \phi(a)$, and hence $f(x) = a$. We conclude that x_0 is the only critical point of f. Thus,

$$\begin{aligned} f' &= \sqrt{2(\phi(a) - \phi(f))} \quad 0 < x < x_0, \\ f' &= -\sqrt{2(\phi(a) - \phi(f))} \quad x_0 < x < L. \end{aligned} \tag{2.4}$$

This implies that

$$\begin{aligned}x_0 - x &= \int_x^{x_0} \frac{f'}{\sqrt{2(\phi(a)-\phi(f))}} = \int_{f(x)}^{a} \frac{dz}{\sqrt{2(\phi(a)-\phi(z))}} \quad \text{for } 0 < x \le x_0,\\ x - x_0 &= -\int_{x_0}^{x} \frac{f'}{\sqrt{2(\phi(a)-\phi(f))}} = \int_{f(x)}^{a} \frac{dz}{\sqrt{2(\phi(a)-\phi(z))}} \quad \text{for } x_0 \le x < L.\end{aligned} \tag{2.5}$$

In particular, $x_0 = L - x_0$, and then $x_0 = L/2$. Furthermore, for $x = 0$ we obtain

$$\frac{L}{2} = \int_0^a \frac{dz}{\sqrt{2(\phi(a)-\phi(z))}} = \frac{N(a)}{\sqrt{2}}. \tag{2.6}$$

Conversely, if $N(a) = L/\sqrt{2}$ for some $a \in \mathcal{A}$, define f implicitly by

$$\int_{f(x)}^{a} \frac{dz}{\sqrt{\phi(a)-\phi(z)}} = \sqrt{2}\left(x - \frac{L}{2}\right) \quad \text{for } x \ge \frac{L}{2} \tag{2.7}$$

and extend it by symmetry for $x < L/2$. It is immediate to verify that f is a positive solution of problem (2.1). Moreover, from the above computations it is clear that if $\tilde{f}$ is a positive solution with $\|\tilde{f}\|_{C([0,1])} = a$, then $\tilde{f} = f$. □

Remark 2.2. The proof of existence of a solution in the previous theorem holds for any $a \in \mathrm{Dom}(N) \subset \mathbb{R}^+$ such that $N(a) = L/\sqrt{2}$.

Remark 2.3. If H is bounded in a neighborhood of 0, then $\phi'(0^+) < 0$ and hence $0 \notin \overline{\mathcal{A}}$.

Example 2.4. As an application, we may consider $H : \mathbb{R}^+ \to \mathbb{R}^+$ given by $H(u) = cu^\sigma$ for some $\sigma > -2$ and $c \ne 0$. In this case $\tilde{H}(u) = (c/(\sigma+2))u^{\sigma+2}$, and $\phi(u) = (2c^2/(\sigma+2)^2)u^{2\sigma+4} - u^2/2$. For $\sigma > -1$ a simple computation shows that $\mathcal{A} = [\alpha, +\infty)$, with $\alpha = ((\sigma+2)/2|c|)^{1/(\sigma+1)}$. Moreover, N is strictly non-increasing, with

$$\lim_{a\to\alpha^+} N(a) = +\infty \qquad \lim_{a\to+\infty} N(a) = 0. \tag{2.8}$$

On the other hand, if $-2 < \sigma < -1$, it holds that $\mathcal{A} = (0, \alpha]$, with $\alpha = ((\sigma+2)/4c^2)^{1/(2\sigma+2)}$. Moreover, N is strictly non-decreasing, with

$$\lim_{a\to 0} N(a) = 0 \qquad \lim_{a\to\alpha^-} N(a) = +\infty. \tag{2.9}$$

Thus, in both cases it follows that the problem admits a unique solution. The case $\sigma = -1$

corresponds to the well known linear problem $-f'' = (4c^2 - 1)f$. Here

$$\mathcal{A} = \begin{cases} \varnothing & \text{if } 4c^2 \le 1 \\ \mathbb{R}^+ & \text{if } 4c^2 > 1. \end{cases} \tag{2.10}$$

Moreover, if $4c^2 > 1$ then

$$N(a) \equiv \frac{\pi}{\sqrt{2(4c^2 - 1)}}, \tag{2.11}$$

and hence $N(a) = L/\sqrt{2}$ if and only if $4c^2 - 1 = (\pi/L)^2$.

3. Upper and lower solutions and unbounded revolution surfaces

In this section we apply the method of upper and lower solutions in order to solve a nonhomogeneus Dirichlet problem associated to (1.4). In particular, applying a diagonal argument we prove the existence of solutions of (1.9).

We recall that $(\alpha,\beta) \in (C^2([0,+\infty)))^2$ is an ordered couple of a lower and an upper solution of the problem if $\alpha \le \beta$ and

$$\begin{gathered} \alpha'' - \alpha \ge -4H(\alpha)\alpha\widetilde{H}(\alpha) \quad \text{in } (0,+\infty) \\ \alpha(0) \le 0, \qquad \alpha(+\infty) \le r, \\ \beta'' - \beta \le -4H(\beta)\beta\widetilde{H}(\beta) \quad \text{in } (0,+\infty) \\ \beta(0) \ge 0, \qquad \beta(+\infty) \ge r. \end{gathered} \tag{3.1}$$

For simplicity we will assume that H is continuously differentiable.

Remark 3.1. If f is a solution of (1.9), then $f''(+\infty) = r - 4r\widetilde{H}(r)H(r)$. As $f(+\infty) < \infty$, it follows that

$$4\widetilde{H}(r)H(r) = 1. \tag{3.2}$$

In particular, if (3.2) holds we may take $\beta \equiv r$ as an upper solution.

Theorem 3.2. *Let (α,β) be an ordered couple of a lower and an upper solution of (1.9), let $N > 0$ and let c_N be any constant with $\alpha(N) \le c_N \le \beta(N)$. Then the Dirichlet problem*

$$\begin{gathered} f'' - f = -4H(f)f\widetilde{H}(f) \quad \text{in } (0,N) \\ f(0) = 0, \qquad f(N) = c_N \end{gathered} \tag{3.3}$$

admits at least one solution f with $\alpha|_{[0,N]} \le f \le \beta|_{[0,N]}$.

Proof. Fix a constant $\lambda \ge -1$ such that

$$\lambda \ge -2(\widetilde{H}^2)''(u) \tag{3.4}$$

for any $u \in \mathbb{R}$ such that

$$\inf_{[0,N]} \alpha \leq u \leq \sup_{[0,N]} \beta. \tag{3.5}$$

This choice of λ implies that the function $\xi(x) := -4H(x)x\widetilde{H}(x) - \lambda x$ is non-increasing.

We will construct a sequence $\{f_n\}$ given recursively by $f_0 = \alpha$ and f_{n+1} the unique solution of the linear problem

$$\begin{gathered} f_{n+1}'' - (1+\lambda) f_{n+1} = -4H(f_n) f_n \widetilde{H}(f_n) - \lambda f_n \quad \text{in } (0,N) \\ f_{n+1}(0) = 0, \qquad f_{n+1}(N) = c_N. \end{gathered} \tag{3.6}$$

We claim that $\{f_n\}$ is non-decreasing, with $\alpha \leq f_n \leq \beta$. Indeed, as

$$\begin{gathered} f_1'' - (1+\lambda) f_1 = -4H(\alpha)\alpha\widetilde{H}(\alpha) - \lambda\alpha \leq \alpha'' - (1+\lambda)\alpha, \\ f_1(0) \geq \alpha(0), \qquad f_1(N) \geq \alpha(N) \end{gathered} \tag{3.7}$$

by the comparison principle we deduce that $f_1 \geq \alpha$. Now assume that $f_n \geq f_{n-1}$ then

$$\begin{aligned} f_{n+1}'' - (1+\lambda) f_{n+1} &= -4H(f_n) f_n \widetilde{H}(f_n) - \lambda f_n \\ &\leq -4H(f_{n-1}) f_{n-1} \widetilde{H}(f_{n-1}) - \lambda f_{n-1} \\ &= f_n'' - (1+\lambda) f_n \end{aligned} \tag{3.8}$$

and we deduce that $f_{n+1} \geq f_n$.

On the other hand, $f_0 = \alpha \leq \beta$, and if $f_n \leq \beta$ we have that

$$f_{n+1}'' - (1+\lambda) f_{n+1} = -4H(f_n) f_n \widetilde{H}(f_n) - \lambda f_n \geq -4H(\beta)\beta\widetilde{H}(\beta) - \lambda\beta \leq \beta'' - (1+\lambda)\beta. \tag{3.9}$$

As

$$f_{n+1}(0) \leq \beta(0), \qquad f_{n+1}(N) \leq \beta(N), \tag{3.10}$$

using again the comparison principle we deduce that $f_{n+1} \leq \beta$.

It follows that $\{f_n\}$ converges pointwise to some function f. By the standard a priori bounds and using the fact that $\alpha \leq f_n \leq \beta$ for each n we have that

$$||f_n||_{H^2} \leq c_0 + c_1 ||4H(f_{n-1}) f_{n-1} \widetilde{H}(f_{n-1}) - \lambda f_{n-1}||_{L^2} \leq C \tag{3.11}$$

for some constant C. Thus, if we suppose that $f_n \nrightarrow f$ uniformly, taking a subsequence we may assume that $\| f_n - f \|_{C([0,N])} \geq \varepsilon$ for some $\varepsilon > 0$. By the Sobolev imbedding $H^2(0,N) \hookrightarrow C^1([0,N])$, taking a subsequence we may assume that f_n converges to some function $g \neq f$

for the C^1-norm, a contradiction. Hence $f_n \to f$ uniformly, and $f_n'' \to f - 4H(f)f\widetilde{H}(f)$. It follows that f is a solution of the problem. □

Remark 3.3. In the previous proof, it is easy to see that the convergence is more accurate for smaller values of λ. Indeed, if $\overline{\lambda} \geq \lambda$, with λ as before, the corresponding sequence $\{\overline{f}_n\}$ given recursively by $\overline{f}_0 = \alpha$ and $\overline{f}_{n+1}$ the unique solution of the linear problem

$$\begin{gathered}\overline{f}''_{n+1} - (1+\overline{\lambda})\overline{f}_{n+1} = -4H(\overline{f}_n)\overline{f}_n\widetilde{H}(\overline{f}_n) - \overline{\lambda}\overline{f}_n \quad \text{in } (0,N)\\ \overline{f}_{n+1}(0) = 0, \qquad \overline{f}_{n+1}(N) = c_N\end{gathered} \tag{3.12}$$

is non-decreasing and converges to a solution of the problem. We claim that $\overline{f}_n \leq f_n$ for every n: indeed, this is trivial for $n = 0$, and if the claim is true for n we have that

$$\begin{aligned} f''_{n+1} - (1+\lambda)f_{n+1} &= -4H(f_n)f_n\widetilde{H}(f_n) - \lambda f_n \leq -4H(\overline{f}_n)\overline{f}_n\widetilde{H}(\overline{f}_n) - \lambda\overline{f}_n\\ &= \overline{f}''_{n+1} - (1+\overline{\lambda})\overline{f}_{n+1} + (\overline{\lambda} - \lambda)\overline{f}_n\\ &= \overline{f}''_{n+1} - (1+\lambda)\overline{f}_{n+1} + (\overline{\lambda} - \lambda)(\overline{f}_n - \overline{f}_{n+1}). \end{aligned} \tag{3.13}$$

Using the inductive hypothesis and the fact that $\{\overline{f}_k\}$ is nondecreasing, it follows that $f_{n+1} \geq \overline{f}_{n+1}$.

To conclude this remark, note that $\{f_n\}$ and $\{\overline{f}_n\}$ converge to the same solution. Indeed, it suffices to replace β by

$$\overline{\beta} = \lim_{n\to\infty} \overline{f}_n \tag{3.14}$$

in the proof of Theorem 3.2. As $\overline{\beta} \leq \beta$, the definition of $\{f_n\}$ coincides with the previous one, and $f_n \leq \overline{\beta}$ for every n.

Theorem 3.4. *Let (α,β) be an ordered couple of a lower and an upper solution of (1.9) with $\alpha(+\infty) = \beta(+\infty) = r$. Then (1.9) admits a solution f with $\alpha \leq f \leq \beta$.*

Proof. For any $N \in \mathbb{N}$, by the previous theorem we may choose a solution f_N of (3.3) with $c_N = (\alpha(N)+\beta(N))/2$ such that $\alpha|_{[0,N]} \leq f_N \leq \beta|_{[0,N]}$. Moreover, if $\varphi_N(x) = (f_N(M)/M)x$, there exist constants c_M, $\overline{c}_M$ independent of N such that

$$||f_N - \varphi_N||_{H^2(0,M)} \leq \overline{c}_M||H(f_N)f_N\widetilde{H}(f_N)||_{L^2(0,M)} \leq c_M \tag{3.15}$$

for any $N \geq M$. For $M = 1$ we may take a subsequence, still denoted $\{f_N\}$, which converges uniformly in $[0,1]$ to some function f^1. Repeating the procedure we may assume that f_N converges uniformly in $[0,M]$ to a function f^M. Then $f : [0,+\infty) \to [0,+\infty)$ given by $f(x) = f^N(x)$ if $x \leq N$ solves (1.9). Indeed, it is clear that f is well defined, and that $f(0)=0$, $f(+\infty)=r$. Moreover, as f_N'' converges uniformly in $[0,M]$ to $f-4H(f)f\widetilde{H}(f)$, for any test function $\xi\in C_0^\infty(0,M)$ we obtain that

$$\int_0^M (f-4H(f)f\widetilde{H}(f))\xi=\lim_{N\to\infty}\int_0^M f_N''\xi=\lim_{N\to\infty}\int_0^M f_N\xi''=\int_0^M f\xi'', \tag{3.16}$$

and the proof follows. □

Example 3.5. Assume that (3.2) holds, and that $(\widetilde{H}^2)''\le 0$ on $[0,r]$. Then (α,β) given by

$$\alpha(x)=r(1-e^{-x}),\quad \beta\equiv r \tag{3.17}$$

is an ordered couple of a lower and an upper solution of (1.9). Indeed,

$$\alpha''-\alpha=-r=-2(\widetilde{H}^2)'(r)\ge -2(\widetilde{H}^2)'(\alpha) \tag{3.18}$$

since $0\le\alpha\le r$. From the previous theorem we deduce that (1.9) admits at least one positive solution between α and β.

3.1. Some numerical experiments. The method described in the proof of Theorem 3.2 can be implemented as a numerical method to compute the solution in an effective way. At each step of the iterative procedure, we have to solve a linear differential equation, with Dirichlet boundary conditions.

With that purpose, we use the standard finite difference method: We split the interval $[0,N]$ into k small sub-intervals of length $h=k/N$, and we denote by f_n^i the approximate value of $f_n(x_i)$. Then, we approximate the linear problem (3.6) by the linear system of equations

$$\frac{f_{n+1}^{i+1}-2f_{n+1}^{i}+f_{n+1}^{i-1}}{h^2}-(1+\lambda)f_{n+1}^i=-4H(f_n^i)f_n^i\widetilde{H}(f_n^i)-\lambda f_n^i\quad (1\le i\le k-1) \tag{3.19}$$

subject to the boundary conditions

$$f_{n+1}^0=0,\qquad f_{n+1}^k=c. \tag{3.20}$$

Let us recall that the energy introduced in (2.3) is constant, for any solution of the problem, therefore we can use the discrete quantity

$$E^i(h)=\left(\frac{f_n^i-f_n^{i-1}}{h}\right)^2+2\phi(f_n^i) \tag{3.21}$$

as a test for the accuracy of the method. We stop the iteration when this quantity is close enough to a constant, for the desired precision ε_0, that is, when

$$\left|\frac{E^i(h)-E^{i-1}(h)}{h}\right|<\varepsilon_0\quad \forall i. \tag{3.22}$$

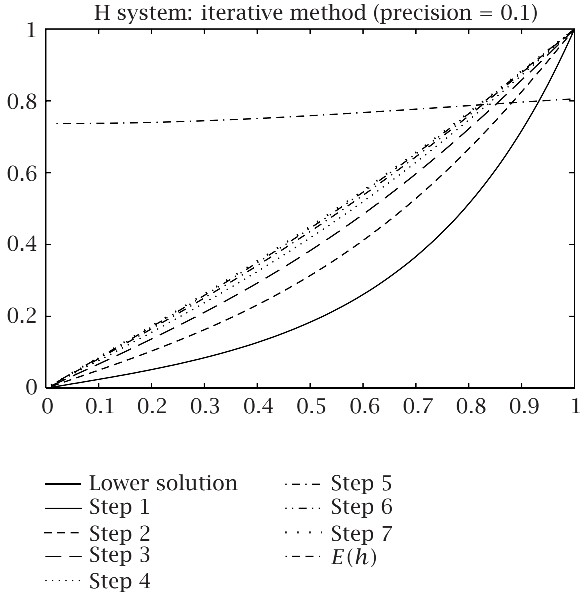

Figure 3.1. H system: iterative method (precision = 0.1).

We have implemented this numerical scheme using GNU Octave for different choices of H. In Figure 3.1, we present the case $H(x) = x$, $N = 1$, $\lambda = 10$ and $\varepsilon_0 = 0.1$.

Acknowledgment

The first and second authors are partially supported by Fundación Antorchas. The last author is partially supported by ADVANCE-NSF.

References

[1] P. Amster, M. M. Cassinelli, and M. C. Mariani, *Solutions to general quasilinear elliptic second order problems*, Nonlinear Studies **7** (2000), no. 2, 283–289.

[2] ______, *Solutions to quasilinear equations by an iterative method*, Bulletin of the Belgian Mathematical Society. Simon Stevin **7** (2000), no. 3, 435–441.

[3] P. Amster and M. C. Mariani, *Two iterative schemes for an H-system*, JIPAM. Journal of Inequalities in Pure and Applied Mathematics **6** (2005), no. 1, 7, Article 5.

[4] P. Amster, M. C. Mariani, and D. F. Rial, *Existence and uniqueness of H-system's solutions with Dirichlet conditions*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods **42** (2000), no. 4, 673–677.

[5] H. Brezis and J.-M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Communications on Pure and Applied Mathematics **37** (1984), no. 2, 149–187.

[6] A. Capietto, J. Mawhin, and F. Zanolin, *Boundary value problems for forced superlinear second order ordinary differential equations*, Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar, Vol. 12 (Paris, 1991–1993), Pitman Res. Notes Math. Ser., vol. 302, Longman Sci. Tech., Harlow, 1994, pp. 55–64.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren der mathematischen Wissenschaften, vol. 224, Springer, Berlin, 1983.
[8] S. Hildebrandt, *On the Plateau problem for surfaces of constant mean curvature*, Communications on Pure and Applied Mathematics **23** (1970), 97–114.
[9] M. Struwe, *Plateau's Problem and the Calculus of Variations*, Mathematical Notes, vol. 35, Princeton University Press, New Jersey, 1988.
[10] G. F. Wang, *The Dirichlet problem for the equation of prescribed mean curvature*, Annales de l'Institut Henri Poincaré. Analyse Non Linéaire **9** (1992), no. 6, 643–655.

P. Amster: FCEyN, Departamento de Matemática, Universidad de Buenos Aires,
Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina;
Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina
E-mail address: pamster@dm.uba.ar

P. De Nápoli: FCEyN, Departamento de Matemática, Universidad de Buenos Aires,
Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina;
Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina
E-mail address: pdenapo@dm.uba.ar

M. C. Mariani: Department of Mathematical Sciences, New Mexico State University Las Cruces,
NM 88003-8001, USA
E-mail address: mmariani@nmsu.edu