WEAK SOLUTIONS FOR THE p-LAPLACIAN WITH A NONLINEAR BOUNDARY CONDITION AT RESONANCE

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Abstract. In this paper we deal with the existence of weak solutions of the problem $\Delta_p u = |u|^{p-2}u + f(x,u)$ with a nonlinear boundary condition given by $|\nabla u|^{p-2}\partial u/\partial\nu = \lambda|u|^{p-2}u - h(x,u)$ on the boundary of the domain. We assume Landesman-Lazer type conditions and use variational arguments to find existence of solutions.

1. Introduction.

In this paper we find conditions that provide existence of weak solutions for the problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u - h(x,u) & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. We assume that the perturbations $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are bounded Caratheodory functions. In a variational approach, the functional associated to the problem is

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{p}\int_\Omega |u|^p - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p + \int_\Omega F(x,u) + \int_{\partial\Omega} H(x,u),$$

where F and H are primitives of f and h with respect to u respectively. Weak solutions of (1.1) are critical points of J_λ in $W^{1,p}(\Omega)$, in fact if $u \in W^{1,p}(\Omega)$ is a critical point of J_λ we have,

$$\begin{aligned} J'_\lambda(u).v = & \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv - \lambda\int_{\partial\Omega} |u|^{p-2}uv \\ & + \int_\Omega f(x,u)v + \int_{\partial\Omega} h(x,u)v = 0, \qquad \forall\, v \in W^{1,p}(\Omega). \end{aligned}$$

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Let us introduce some motivation to deal with (1.1). We will say that λ is an eigenvalue for the p-Laplacian with a nonlinear boundary condition if the problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

has non trivial solutions. The set of solutions (called eigenfunctions) for a given λ will be denoted by A_λ. Problems of the form (1.2) appears in a natural way when one considers the Sobelev trace inequality. In fact, the immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact, hence there exits a constant λ_1 such that

$$\lambda_1^{1/p} \|u\|_{L^p(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

This Sobolev trace constant λ_1 can be characterized as

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx, \qquad \int_{\partial\Omega} |u|^p = 1 \right\}, \tag{1.3}$$

and is the first eigenvalue of (1.2) in the sense that $\lambda_1 \le \lambda$ for any other eigenvalue λ. The extremals (functions where the constant is attained) are solutions of (1.2). The first eigenvalue is simple and isolated, see [17]. In [11] it is proved that there exists a sequence of eigenvalues λ_n of (1.2) such that $\lambda_n \to +\infty$ as $n \to +\infty$.

The study of the eigenvalue problem when the nonlinear term is placed in the equation, that is when one consider a quasilinear problem of the form $-\Delta_p u = \lambda |u|^{p-2}u$ with Dirichlet boundary conditions, has received considerable attention, see for example [1], [2], [13], [14], [16], etc.

Resonance problems are well known in the literature. For example, for the resonance problem for the $p-$laplacian with Dirichlet boundary conditions see [3], [4], [9] and references therein.

In problem (1.1) we have a perturbation of the eigenvalue problem (1.2) given by the two nonlinear terms $f(x,u)$, $h(x,u)$. Following ideas from [9], we prove the following result, that establishes Landesman-Lazer type conditions on the nonlinear perturbation terms in order to have existence of weak solutions for (1.1).

Theorem 1.1. *Let $f^\pm := \lim_{t\to\pm\infty} f(x,t)$, $h^\pm := \lim_{t\to\pm\infty} h(x,t)$. Assume that there exists $\bar{f} \in L^q(\Omega)$ and $\bar{h} \in L^q(\partial\Omega)$, such that $|f(x,t)| \le \overline{f}$ $\forall (x,t) \in \Omega \times \mathbb{R}$ and $|h(x,t)| \le \overline{h}$ $\forall (x,t) \in \partial\Omega \times \mathbb{R}$ (where $q = p/p-1$). Also assume that either*

$$(LL)^+_\lambda : \int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v > 0$$

$$\text{for all } v \in A_\lambda \setminus \{0\}$$

or

$$(LL)^-_\lambda : \int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v < 0$$

$$\text{for all } v \in A_\lambda \setminus \{0\},$$

then (1.1) has a weak solution.

Observe that in the case where λ is not an eigenvalue the hypotheses trivially hold.

The integral conditions (of Landesman-Lazer type) that we impose for f and h will be used to prove a Palais-Smale condition for the functional J_λ associated to the problem (1.1). Observe that these conditions involve an integral balance (with the eingenfunctions v as weights) between f and h. Hence we allow perturbations both in the equation and in the boundary condition.

Let us have a close look at the conditions for the first eigenvalue. As the first eigenvalue is isolated and simple with an eigenfuction that do not change sign in Ω (we call it ϕ_1 and assume $\phi_1 > 0$), [17], the conditions involved in Theorem 1.1 for λ_1 read as

$$(LL)^+_{\lambda_1}: \quad \int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 > 0 \quad \text{and} \quad \int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 < 0$$

or

$$(LL)^-_{\lambda_1}: \quad \int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 < 0 \quad \text{and} \quad \int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 > 0.$$

For this case, $\lambda = \lambda_1$, we will prove a more general result which improve the conditions on f and h. In [3] the resonance problem for the Dirichlet problem was analyzed using bifurcation theory. If we adapt the arguments of [3] to our situation, using bifurcation techniques to deal with (1.1), we can improve the previous result by measuring the speed and the form at which f and h approaches the limits $f^\pm$ and $h^\pm$. To this end, let us suppose that there exists α and β such that

$$\lim_{s\to+\infty}(f(x,s)-f^+(x))s^\alpha = A_\alpha(x), \quad \lim_{s\to-\infty}(f(x,s)-f^-(x))s^\beta = B_\beta(x), \ a.e, \ x\in\Omega,$$

$$\lim_{s\to+\infty}(h(x,s)-h^+(x))s^\alpha = \overline{A}_\alpha(x), \quad \lim_{s\to-\infty}(h(x,s)-h^-(x))s^\beta = \overline{B}_\beta(x), \ a.ex\in\partial\Omega.$$

The limits $A_\alpha, \overline{A}_\alpha, B_\beta$ and $\overline{B}_\beta$ are taken in pointwise sense and dominated by functions in $L^1(\Omega)$ and $L^1(\partial\Omega)$.

We consider the conditions

$$(G^+_\alpha)\begin{cases} \displaystyle\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 > 0 \quad \textit{or} \\ \displaystyle\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 = 0 \quad \textit{and} \quad \int_\Omega A_\alpha(x)\phi_1^{1-\alpha} + \int_{\partial\Omega} \overline{A}_\alpha(x)\phi_1^{1-\alpha} > 0, \end{cases}$$

and

$$(G^-_\beta)\begin{cases} \displaystyle\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 < 0 \quad \textit{or} \\ \displaystyle\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 = 0 \quad \textit{and} \quad \int_\Omega B_\beta(x)\phi_1^{1-\beta} + \int_{\partial\Omega} \overline{B}_\beta(x)\phi_1^{1-\beta} < 0, \end{cases}$$

or the conditions

$$(G^-_\beta)\begin{cases} \displaystyle\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 > 0 \quad \textit{or} \\ \displaystyle\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 = 0 \quad \textit{and} \quad \int_\Omega B_\beta(x)\phi_1^{1-\beta} + \int_{\partial\Omega} \overline{B}_\beta(x)\phi_1^{1-\beta} > 0, \end{cases}$$

and

$$(G_\alpha^-)\begin{cases} \displaystyle\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 < 0 \quad or \\ \displaystyle\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 = 0 \quad and \quad \int_\Omega A_\alpha(x)\phi_1^{1-\alpha} + \int_{\partial\Omega} \overline{A}_\alpha(x)\phi_1^{1-\alpha} < 0. \end{cases}$$

Remark that this set of conditions extend the hypothesis of Theorem 1.1.

Theorem 1.2. *Let f and h be such that there exists $\bar{f} \in L^q(\Omega)$ and $\bar{h} \in L^q(\partial\Omega)$, with $|f(x,t)| \le \overline{f}\ \forall (x,t) \in \Omega \times \mathbb{R}$ and $|h(x,t)| \le \overline{h}\ \forall (x,t) \in \partial\Omega \times \mathbb{R}$ (where $q = p/p-1$). If (G_α^+) and (G_β^-) or (G_α^-) and (G_β^+) hold then problem (1.1) with $\lambda = \lambda_1$ has at least one solution.*

We can continue with this procedure and obtain even more general conditions considering the rate of convergence to zero of $(f(x,s) - f^+(x))s^\alpha - A_\alpha(x)$, for example. We leave the details to the reader. Also it is possible to consider different rates of convergence, in this case the conditions involve signs of integrals of A_α and B_α separately.

In the case $p = 2$, we deal with a linear operator in a Hilbert space, $H^1(\Omega)$, so using the Spectral Theorem for compact self-adjoint linear operators and the Fredholm alternative, we have that when λ is not an eigenvalue we do not need any additional condition to have solutions for (1.1), and if λ is an eigenvalue, we need an orthogonality condition. However when dealing with $p \neq 2$ we have to consider the problem in $W^{1,p}(\Omega)$ (which is not Hilbert) and the results is not straightforward.

Finally, let us note that nonlinear boundary conditions have only been considered in recent years. For reference purposes, we cite previous works. For the Laplace operator with nonlinear boundary conditions see for example [7], [8], [12]. For previous work for the $p-$Laplacian with nonlinear boundary conditions of different types see [6], [11], [18] and [17]. Also, one is lead to nonlinear boundary conditions in the study of conformal deformations on Riemannian manifolds with boundary, see for example [10].

2. Proofs of the results

In this section we prove theorems 1.1 and 1.2 that provide existence of solutions of (1.1). First, let us prove Theorem 1.1. We will divide the proof in two steps. Following [9], we first prove a Palais-Smale condition for the functional J_λ using the conditions of Theorem 1.1. Then we split the proof of the theorem in two cases, first we deal with $\lambda_k < \lambda < \lambda_{k+1}$, where λ_k are the variacional eigenvalues of (1.2) this allows us to obtain some geometric structure on J_λ (see [11]), and finally we treat the case where $\lambda = \lambda_k$. In this case we obtain solutions as limit of solutions for a sequence $\lambda_n \to \lambda_k$. We will see that if there is any bifurcation from infinity in $\lambda = \lambda_k$ then the bifurcation is subcritical. This fact provides a priori bounds that allow us to pass to the limit in a sequence of solutions as $\lambda_n \to \lambda_k$.

To prove these results we will need some previous lemmas (the proofs are straightforward, see [11]).

Lemma 2.1. *Let* $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *be given by*

$$A(u).v := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv,$$

then A *is a continuous, odd,* $(p-1)$*- homogeneous and continuously invertible.*

Lemma 2.2. *Let* $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *be given by*

$$B(u).v := \int_{\partial\Omega} |u|^{p-2}uv.$$

Then B *is a continuous, odd,* $(p-1)$*- homogeneous and compact.*

Lemma 2.3. *Let* $C : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *be given by*

$$C(u).v := \int_\Omega f(x,u)v + \int_{\partial\Omega} h(x,u)v.$$

Then C *is continuous and compact and* $\|C(u)\|_{W^{1,p}(\Omega)^*} \le \|\overline{f}\|_{L^q(\Omega)} + K\|\overline{h}\|_{L^q(\partial\Omega)}$. *where* K *is the best constant for the Sobolev trace inequality* $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$.

With these lemmas we can prove the following theorem.

Theorem 2.1. *Suppose that the hypotheses of Theorem 1.1 are satisfied, then* J_λ *satisfies the Palais-Smale condition, that is, for any sequence* $\{u_n\} \subset W^{1,p}(\Omega)$ *such that* $\|J_\lambda(u_n)\|_{W^{1,p}(\Omega)} \le c$ *and* $J'_\lambda(u_n) \to 0$ *there exists* $u \in W^{1,p}(\Omega)$ *such that* $u_n \to u$ *strongly in* $W^{1,p}(\Omega)$.

Proof. Let $\{u_n\}$ be a Palais-Smale sequence. If u_n is bounded then we have that there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. Using that

$$A(u_n) - \lambda B(u_n) + C(u_n) = J'_\lambda(u_n) \to 0,$$

the compactness of B and C, and the continuity of A^{-1} we have that

$$u_n \to A^{-1}(\lambda B(u) - C(u))$$

strongly in $W^{1,p}(\Omega)$. Hence if we prove that Palais-Smale sequences are bounded, the result follows. To see this, let us argue by contradiction. Assume that u_n is a Palais-Smale sequence and that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$. Let

$$v_n := \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$$

then there exists v such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$ and $v_n \to v$ in $L^p(\partial\Omega)$. We have,

$$\frac{J'_\lambda(u_n)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} = A(v_n) - \lambda B(v_n) + \frac{C(u_n)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}}. \tag{2.1}$$

Using compactness of B, continuity of A^{-1} and the fact that

$$\frac{C(u_n)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \to 0$$

we have that $v_n \to A^{-1}(\lambda B(v))$ in $W^{1,p}(\Omega)$. Hence $v_n \to v$ in $W^{1,p}(\Omega)$ and then $A(v) - \lambda B(v) = 0$ with $\|v\|_{W^{1,p}(\Omega)} = 1$. That means that $v \in A_\lambda \setminus \{0\}$.

Observe that, for a.e $x \in \{v(x) > 0\}$ we have $u_n(x) \to +\infty$ so,

$$\lim_{n\to\infty} f(x, u_n(x))v_n(x) + h(x, u_n(x))v_n(x) = f^+(x)v(x) + h^+(x)v(x),$$

and

$$\lim_{n\to\infty} \frac{F(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} + \frac{H(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}}$$

$$= \lim_{n\to\infty} v_n(x)\frac{1}{u_n(x)}\int_0^{u_n(x)} f(t,u_n(t)) + v_n(x)\frac{1}{u_n(x)}\int_0^{u_n(x)} h(t,u_n(t))$$

$$= v(x)f^+(x) + v(x)h^+(x).$$

In a similar way we obtain that, for a.e $x \in \{x : v(x) < 0\}$ we have,

$$\lim_{n\to\infty} f(x,u_n(x))v_n(x) + h(x,u_n(x))v_n(x) = f^-(x)v(x) + h^-(x)v(x),$$

and therefore

$$\lim_{n\to\infty} \frac{F(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} + \frac{H(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} = v(x)f^-(x) + v(x)h^-(x).$$

On the other hand we have,

$$pJ_\lambda(u_n) - J'_\lambda(u_n)u_n$$

$$= p\int_\Omega F(x,u_n(x)) + p\int_{\partial\Omega} H(x,u_n(x)) - \int_\Omega f(x,u_n(x))u_n - \int_{\partial\Omega} h(x,u_n(x))u_n.$$

Then

$$p\frac{J_\lambda(u_n)}{\|u_n\|_{W^{1,p}(\Omega)}} - J'_\lambda(u_n)v_n$$

$$= p\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} + p\int_{\partial\Omega} \frac{H(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} - \int_\Omega f(x,u_n(x))v_n - \int_{\partial\Omega} h(x,u_n(x))v_n.$$

The left hand side go to 0 as $n \to \infty$. Hence

$$0 = (p-1)\left[\int_{\{v>0\cap\Omega\}} f^+v + \int_{\{v>0\cap\partial\Omega\}} h^+v + \int_{\{v<0\cap\Omega\}} f^-v + \int_{\{v<0\cap\partial\Omega\}} h^-v\right]$$

which contradicts the hypothesis on f and h in Theorem 1.1. $\square$

We now use a variacional characterization for a sequence of eigenvalues for the problem (1.2). Indeed, for solutions of (1.2) we can understand critical points of the associated energy functional

$$I(u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p,$$

under the constraint $u \in M$, where $M = \{u \in W^{1,p}(\Omega) : \|u\|_{L^p(\partial\Omega)} = 1\}$. We can find a sequence of variacional eigenvalues with the characterization,

$$\lambda_k := \inf_{A\in C_k} \sup_{u\in A} I(u),$$

where

$$C_k := \{A \subset M : \text{there exists } h : S^{k-1} \to A \text{ continuos, odd and surjective }\}.$$

To prove that these λ_k are critical values one first prove a Palais-Smale condition for the functional. Next, using a deformation argument, we prove that λ_k is an eigenvalue (see [11] for the details), but it is not known if this sequence contains all the eigenvalues.

As we mentioned before, we divide the proof in two cases, $\lambda_k < \lambda < \lambda_{k+1}$ and $\lambda = \lambda_k$.

The case $\lambda_k < \lambda < \lambda_{k+1}$.

Let $A \in\ C_k$ such that $\sup_{u\in A} I(u) = m \in (\lambda_k, \lambda)$ (here we are using the definition of λ_k). Then we have, for $u \in\ A,\ t > 0$, that

$$\begin{aligned} J_\lambda(tu) &= \frac{t^p}{p}[\|u\|^p_{W^{1,p}(\Omega)} - \lambda] + \int_\Omega F(x,tu) + \int_{\partial\Omega} H(x,tu) \\ &\leq \frac{t^p}{p}(m-\lambda) + \left|\int_\Omega F(x,tu)\right| + \left|\int_{\partial\Omega} H(x,tu)\right| \\ &\leq \frac{t^p}{p}(m-\lambda) \\ &\quad + t\left(\int_\Omega |u|^p\right)^{\frac{1}{p}}\left(\int_\Omega |\overline{f}|^q\right)^{\frac{1}{q}} + t\left(\int_{\partial\Omega} |u|^p\right)^{\frac{1}{p}}\left(\int_{\partial\Omega} |\overline{h}|^q\right)^{\frac{1}{q}} \\ &\leq \frac{t^p}{p}(m-\lambda) + t\left(m\|\overline{f}\|_{L^q(\Omega)} + \|\overline{h}\|_{L^q(\partial\Omega)}\right). \end{aligned}$$

Let

$$\xi_{k+1} = \left\{u \in\ W^{1,p}(\Omega)\ /\int_\Omega |\nabla u|^p + \int_\Omega |u|^p \geq \lambda_{k+1}\int_{\partial\Omega} |u|^p\right\}.$$

If $u \in\ \xi_{k+1}$ then,

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\left[\int_\Omega |\nabla u|^p + \int_\Omega |u|^p\right] - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p + \int_\Omega F(x,u) + \int_{\partial\Omega} H(x,u) \\ &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}\left[1 - \frac{\lambda}{\lambda_{k+1}}\right] + \int_\Omega F(x,u) + \int_{\partial\Omega} H(x,u) \\ &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}\left[1 - \frac{\lambda}{\lambda_{k+1}}\right] - \|u\|_{W^{1,p}(\Omega)}\|\overline{f}\|_{L^q(\Omega)} \\ &\quad -K\|u\|_{W^{1,p}(\Omega))}\|\overline{h}\|_{L^q(\partial\Omega)}. \end{aligned}$$

This proves the coercitivity of J_λ in ξ_{k+1}, then there exists α such that,

$$\alpha := \inf_{u\in\xi_{k+1}} J_\lambda(u).$$

On the other hand we have, for $u \in A$,

$$J_\lambda(tu) \leq \frac{t^p}{p}(m-\lambda) + t\left(m\|\overline{f}\|_{L^q(\Omega)} + \|\overline{h}\|_{L^q(\partial\Omega)}\right),$$

where $m - \lambda < 0$. Then for all $u \in\ A$, as $t \to +\infty$ $J_\lambda(tu) \to -\infty$. Hence there exists $T > 0$ such that

$$\max_{u\in A, t\geq T} J_\lambda(tu) = \gamma < \alpha. \tag{2.2}$$

Let

$$T.A := \{tu :\ u \in\ A,\ t \geq T\}$$

and

$$\chi := \{h \in\ C(B_k(0,1), W^{1,p}(\Omega)) :\ h|_{S^{k-1}} \text{is odd into } T.A\}.$$

Let us see that χ is nonempty. By the definition of C_k, there exists continuous function $h : S^{k-1} \to A$ odd and surjective. Let us define $\overline{h} : B_k \to W^{1,p}(\Omega)$ as $\overline{h}(ts) = tTh(s)\ s \in S^{k-1},\ t \in [0,1]$. Clearly $\overline{h} \in \chi$.

Next, let us prove that if $h \in \chi$ then $h(B_k) \cap \xi_{k+1} \neq \emptyset$. If there exists any $u \in h(B_k)$ such that $\int_{\partial\Omega} |u|^p = 0$ then $u \in \xi_{k+1}$.

Suppose now that $\int_{\partial\Omega} |u|^p \neq 0$ for all $u \in h(B_k)$, and let us consider

$$\widetilde{h}(x_1, ..., x_{k+1}) = \begin{cases} \pi h(x_1, ..., x_k) & x_{k+1} \geq 0 \\ \\ -\pi h(-x_1, ..., -x_k) & x_{k+1} < 0, \end{cases}$$

where

$$\pi u = \frac{u}{\|u\|_{L^p(\partial\Omega)}}.$$

Then, if $x_{k+1} \geq 0$

$$\widetilde{h}(x_1, ..., x_{k+1}) = \pi(-h(-x_1, ..., -x_k)) = -\pi h(-x_1, ..., -x_k)$$

and hence

$$\widetilde{h}(-x_1, ..., -x_{k+1}) = -\pi h(x_1, ..., x_k) = -\widetilde{h}(x_1, ..., x_{k+1}).$$

In an analogous way for $x_{k+1} < 0$, we have

$$\widetilde{h}(x_1, ..., x_{k+1}) = -\widetilde{h}(-x_1, ..., -x_{k+1}),$$

then $\widetilde{h}$ is odd. Hence $\widetilde{h}(S^k) \in C^{k+1}$. On the other hand, we have,

$$\lambda_{k+1} = \inf_{A \in C^{k+1}} \sup_{u \in A} I(u),$$

then

$$\lambda_{k+1} \leq \sup_{u \in \widetilde{h}(S^k)} I(u).$$

Hence, for some $u \in \widetilde{h}(S^k)$, that is, for some $x \in S^k$ such that $u = \widetilde{h}(x)$ we have

$$\lambda_{k+1} \leq I(u).$$

That implies that $\widetilde{h}(x) \in \xi_{k+1}$. Using the definition of $\widetilde{h}$ we obtain that $h(x) \in \xi_{k+1}$. Then $h(B_k) \cap \xi_{k+1} \neq \emptyset$.

Theorem 2.2. *Let*

$$c := \inf_{h \in \chi} \sup_{x \in B_k} J_\lambda h(x),$$

then c is a critical value for J_λ, with $c \geq \alpha$.

Proof. for all $h \in \chi$, there exists $x \in B_k$ such that $h(x) \in \xi_{k+1}$, then $J_\lambda(h(x)) \geq \alpha$. Hence

$$\sup_{x \in B_k} J_\lambda(h(x)) \geq \alpha \quad \forall\ h \in \chi.$$

Therefore, $c \geq \alpha > \gamma$, where γ is given by (2.2).

Let us argue by contradiction. Suppose that c is a regular value, then using the deformation lemma, with $\beta = c$ and $\tilde{\epsilon} < c - \gamma$, we have that there exists a deformation $\Phi(u,t)$ that verifies the usual properties (see [11]). If $u \in TA$ then,

$$J_\lambda(u) \leq \gamma < \beta - \tilde{\epsilon},$$

then by one of the properties of the deformation lemma we have $\Phi(u,t) = u$.

By the definition of c, there exists $h \in \chi$ such that,

$$\sup_{x\in B_k} J_\lambda(h(x)) \le c+\epsilon. \tag{2.3}$$

Let $\tilde h(\cdot) := \Phi(h(\cdot),1)$, if $x \in S^{k-1}$ we have that $h(x) \in TA$, then $\tilde h(x) = \Phi(h(x),1) = h(x)$ and hence $\tilde h|_{S^{k-1}} = h|_{S^{k-1}}$. We also have $\tilde h(-x) = \Phi(h(-x),1) = \Phi(h(x),1) = \tilde h(x)$. We obtain that $\tilde h \in \chi$. Using (2.3) and the deformation lemma we have

$$\sup_{x\in B_k} J_\lambda(\tilde h(x)) = \sup_{x\in B_k} J_\lambda(\Phi(h(x),1)) \le c-\epsilon,$$

a contradiction that proves that c is a critical value. □

Case $\lambda = \lambda_k$.

We will prove the result under $(LL)^+_{\lambda_k}$, the case where $(LL)^-_{\lambda_k}$ holds is completely analogous.

Lemma 2.4. *If $(LL)^+_{\lambda_k}$ is satisfied, then exists $\delta > 0$ such that $(LL)^+_\mu$ is satisfied for all $\mu \in (\lambda_k - \delta, \lambda_k + \delta)$.*

Proof. Arguing by contradiction, let us assume that there exists $\mu_n \to \lambda_k$ and corresponding eigenfunctions $\{v_n\}$, $\|v_n\|_{W^{1,p}(\Omega)} = 1$, such that,

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla w + \int_\Omega |v_n|^{p-2} v_n w = \mu_n \int_{\partial\Omega} |v_n|^{p-2} v_n \quad \forall w \in W^{1,p}(\Omega) \tag{2.4}$$

and

$$\int_{\{v_n>0\cap\Omega\}} f^+ v_n + \int_{\{v_n>0\cap\partial\Omega\}} h^+ v_n + \int_{\{v_n<0\cap\Omega\}} f^- v_n + \int_{\{v_n<0\cap\partial\Omega\}} h^- v_n \le 0, \tag{2.5}$$

for all n. Then, since $\{v_n\}$ is bounded, there exists $v \in W^{1,p}(\Omega)$ such that $v_n \to v$ in $L^p(\partial\Omega)$. Taking

$$\phi_n(w) = \mu_n \int_{\partial\Omega} |v_n|^{p-2} v_n w$$

and

$$\phi(w) = \lambda_k \int_{\partial\Omega} |v|^{p-2} v w,$$

we have that $\phi_n \to \phi$ in $(W^{1,p}(\Omega))^*$. Using the continuity of A^{-1}, we have that $v_n \to v$ in $W^{1,p}(\Omega)$. Then, taking limits in (2.4) and (2.5) we have

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla w + \int_\Omega |v|^{p-2} v w = \lambda_k \int_{\partial\Omega} |v|^{p-2} v, \quad \forall w \in W^{1,p}(\Omega),$$

and

$$\int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v \le 0.$$

Which contradicts the fact that $(LL)^+_{\lambda_k}$ is satisfied. □

Now we assume that $\lambda_{k-1} \le \lambda_k - \delta$ and let $\{\mu_n\} \subset (\lambda_k - \delta, \lambda_k)$ be an increasing sequence such that $\mu_n \to \lambda_k$. We will construct a decreasing sequence $\{c_n\}$ of critical values corresponding to J_{μ_n}, and then we will see that the sequence corresponding to the critical points $\{u_n\}$ is bounded and converge to a certain u that is a critical point for J_{λ_k}.

Lemma 2.5. *There exists a decreasing sequence of critical values, $\{c_n\}$ associated with the functional J_{μ_n}.*

Proof. Let $A \in C^{k-1}$, $T_1 > 0, \xi_k$ and χ_1 as in the first part $(\lambda_k < \lambda < \lambda_{k+1})$ such that,

$$c_1 := \inf_{h\in\chi_1} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x))$$

is a critical value for J_{μ_1}. To define c_2, let us chose the same A and ξ_k, but we take $T_2 > T_1$ that provides the correspondent χ_2. Then $T_1A \subset T_2A$, $\chi_2 \subset \chi_1$ and,

$$\inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) \ge \inf_{h\in\chi_1} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) = c_1$$

for all $h_1 \in \chi_1$. Let us define,

$$h_2(x) := \begin{cases} h_1(2x) & |x| \le \frac{1}{2}, \\ \\ h_1\left(\frac{x}{|x|}\right)[1 + 2(|x| - \frac{1}{2})T_2] & |x| > \frac{1}{2}. \end{cases}$$

For $|x| \ge 1/2$, $h_2(x) \in T_1A$ then

$$J_{\mu_1}(h_2(x)) \le \gamma < \alpha \le J_{\mu_1}(u), \qquad \forall u \in \xi_{k+1}.$$

Then there exists $y \in B_k$ such that $h(y) \in \xi_{k+1}$ and

$$J_{\mu_1}(h_2(x)) \le \gamma < \alpha \le J_{\mu_1}(h_2(y)).$$

That is, for all x with $|x| \ge 1/2$ there exists $y \in B_k$ such that $J_{\mu_1}(h_2(x)) > J_{\mu_1}(h_2(y))$. Then

$$\sup_{x\in B_{k-1}} J_{\mu_1}(h_2(x)) = \sup_{|x|\le 1/2} J_{\mu_1}(h_2(x))$$

$$= \sup_{|x|\le 1/2} J_{\mu_1}(h_1(2x)) = \sup_{x\in B_{k-1}} J_{\mu_1}(h_1(x)).$$

Hence

$$c_1 := \inf_{h\in\chi_1} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) = \inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)).$$

On the other hand we have,

$$J_{\mu_2}(u) = J_{\mu_1}(u) + \frac{1}{p}(\mu_1 - \mu_2)\int_{\partial\Omega} |u|^p \le J_{\mu_1}(u) \quad \forall u \in W^{1,p}(\Omega),$$

then

$$\inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) \ge \inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_2}(h(x)) := c_2.$$

We conclude that $c_1 \ge c_2$. Continuing with this procedure we find a sequence c_n with the desired properties. ◻

Let $\{u_n\}$ be the sequence of critical points associated with $\{c_n\}$ then

$$J'_{\mu_n}(u_n) = A(u_n) - \mu_n B(u_n) + C(u_n) = 0.$$

If $\{u_n\}$ is bounded then exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$, then $u_n \to A^{-1}(\lambda_k B(u) - C(u))$ in $W^{1,p}(\Omega)$. Hence u is a critical point for J_{λ_k} and we have proved our result.

We will next show that $\{u_n\}$ must be bounded. This means that if there exists (μ_n, u_n) solutions of (1.1) with $\mu_n \to \lambda_k$ such that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$ then the sequence μ_n verifies $\mu_n > \lambda_k$, that is the only posible bifurcation from infinity at $\lambda = \lambda_k$ is subcritical.

Lemma 2.6. *If $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$, then there exists $v \in A_{\lambda_k} \setminus \{0\}$ such that*

$$\frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \to v.$$

Proof. Let

$$v_n := \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$$

then we have that $v_n \rightharpoonup v$. Using that

$$A(v_n) - \mu_n B(v_n) - \frac{C(u_n)}{\|u_n\|^{p-1}} = 0, \tag{2.6}$$

the compactness of B and the continuity of A^{-1} we have that $v_n \to A^{-1}(\lambda_k B(v))$, then $v_n \to v$, with $\|v\|_{W^{1,p}(\Omega)} = 1$. Taking limit in (2.6) we have $A(v) = \lambda_k B(v)$, then $v \in A_{\lambda_k} \setminus \{0\}$. ∎

Making similar calculations as in the proof of Theorem 2.1, we get

$$pc_n = pJ_{\mu_n} - J'_{\mu_n}.\mu_n =$$

$$p\int_\Omega F(x,u_n) + p\int_{\partial\Omega} H(x,u_n) - \int_\Omega f(x,u_n)u_n - \int_{\partial\Omega} h(x,u_n)u_n.$$

We have that

$$\lim_{n\to\infty} p\int_\Omega \frac{F(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}} + p\int_{\partial\Omega} \frac{H(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}} - \int_\Omega f(x,u_n)v_n - \int_{\partial\Omega} h(x,u_n)v_n$$

$$= (p-1)\left(\int_{\{v>0\cap\Omega\}} f^+v + \int_{\{v>0\cap\partial\Omega\}} h^+v + \int_{\{v<0\cap\Omega\}} f^-v + \int_{\{v<0\cap\partial\Omega\}} h^-v\right) > 0.$$

Then,

$$\lim_{n\to\infty} \frac{pc_n}{\|u_n\|_{W^{1,p}(\Omega)}} > 0,$$

which contradicts the fact that $\{c_n\}$ is bounded from above.

Then we have that $\{u_n\}$ is bounded. Hence there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weak in $W^{1,p}(\Omega)$, using the compactness of B and C and the continuity of A^{-1} we have $u_n \to u$ strong in $W^{1,p}(\Omega)$.

Case $\lambda = \lambda_1$. Theorem 1.2.

Now we prove Theorem 1.2 which improves the conditions on f and h in the case where $\lambda = \lambda_1$. To prove this theorem we use ideas from [3]. First we need some estimates.

Lemma 2.7. *Let u be a positive solution of* (1.1)*. Then*

$$-\frac{\displaystyle\int_{\partial\Omega} h(x,u)\frac{\phi_1^p}{|u|^{p-2}u} + \int_\Omega f(x,u)\frac{\phi_1^p}{|u|^{p-2}u}}{\displaystyle\int_{\partial\Omega}\phi_1^p} \le \lambda_1 - \lambda \le -\frac{\displaystyle\int_{\partial\Omega} h(x,u)u + \int_\Omega f(x,u)u}{\displaystyle\int_{\partial\Omega}|u|^p}.$$

Proof. Taking in the weak form $v = u$ we have that

$$-\int_{\partial\Omega} g(x,u)u - \int_\Omega f(x,u)u = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p$$
$$-\lambda\int_{\partial\Omega}|u|^p \ge (\lambda_1-\lambda)\int_{\partial\Omega}|u|^p,$$

then we get the second inequality.

If we take $v = \dfrac{{\phi_1}^p}{|u|^{p-2}u}$ we have,

$$\begin{aligned}
&-\int_{\partial\Omega} h(x,u)\frac{{\phi_1}^p}{|u|^{p-2}u} - \int_\Omega f(x,u)\frac{{\phi_1}^p}{|u|^{p-2}u} - (\lambda_1-\lambda)\int_{\partial\Omega}{\phi_1}^p \\
= \;& \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\left(\frac{\phi_1^p}{|u|^{p-2}u}\right) \\
&+\int_\Omega |u|^{p-2}u\frac{\phi_1^p}{|u|^{p-2}u} - \int_\Omega |\nabla\phi_1|^p - \int_{\partial\Omega}|\phi_1|^p \\
= \;& \int_\Omega p|\nabla u|^{p-2}\frac{{\phi_1}^{p-1}}{|u|^{p-2}u}\nabla u\nabla\phi_1 - \int_\Omega (p-1)\frac{{\phi_1}^p}{|u|^p}|\nabla u|^p - \int_\Omega|\nabla\phi_1|^p \\
\le \;& \int_\Omega p\frac{{\phi_1}^{p-1}}{|u|^{p-1}}|\nabla u|^{p-1}|\nabla\phi_1| - \int_\Omega (p-1)\frac{{\phi_1}^p}{|u|^p}|\nabla u|^p - \int_\Omega|\nabla\phi_1|^p.
\end{aligned}$$

Using that

$$pt^{p-1}s - (p-1)t^p - s^p \le 0, \quad \forall t,s\ge 0$$

with $t = \dfrac{\phi_1}{|u|}|\nabla u|$ and $s = |\nabla\phi_1|$ we have that

$$-\int_{\partial\Omega} h(x,u)\frac{\phi_1}{|u|^{p-2}u} - \int_\Omega f(x,u)\frac{\phi_1}{|u|^{p-2}u} - (\lambda_1-\lambda)\int_{\partial\Omega}{\phi_1}^p \le 0,$$

the result follows. □

Now, let us proceed with the proof of the theorem.

Proof of Theorem 1.2. Let us suppose that f and h satisfy conditions (G_α^-) and (G_β^+). We will prove that there exists (λ_n, u_n) solutions of problem (1.1) with $\lambda_n \to \lambda_1$ such that $\|u_n\|_{W^{1,p}(\Omega)} \le K$.

Let $\lambda_n \searrow \lambda_1$, and u_n be the solutions of (1.1). Remark that Theorem 1.1 shows the existence of u_n for every λ_n close but not equal to λ_1 (as λ_1 is isolated the conditions on f and h of Theorem 1.1 are trivially verified for any λ_n close to λ_1).

Using the previous lemma it is not difficult to see that if

$$\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 < 0$$

or

$$\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 < 0$$

then any bifurcation from infinity must be subcritical hence $\{u_n\}$ is bounded (see [3] for the details).

We only have to consider the case where

$$\begin{aligned} &\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 = 0, \\ &\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 = 0, \end{aligned} \tag{2.7}$$

and

$$\int_{\partial\Omega} \overline{A}_\alpha \phi_1^{1-\alpha} + \int_\Omega A_\alpha \phi_1^{1-\alpha} < 0,$$

$$\int_{\partial\Omega} \overline{B}_\alpha \phi_1^{1-\alpha} + \int_\Omega B_\alpha \phi_1^{1-\alpha} > 0,$$

Let us assume by contradiction that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$. Then by Lemma 2.6 we have

$$\frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \to \phi_1 \quad \text{or} \quad -\phi_1.$$

The convergence is uniform by regularity results that show that $u_n \in C^\alpha$, see [15]. Using the previous lemma we get,

$$0 > (\lambda_1 - \lambda_n)\int_{\partial\Omega} \phi_1^p \geq -\int_{\partial\Omega} h(x,u_n)\frac{\phi_1^p}{|u_n|^{p-2}u_n} - \int_\Omega f(x,u_n)\frac{\phi_1^p}{|u_n|^{p-2}u_n},$$

using (2.7) we have

$$\begin{aligned} 0 < \quad & \int_{\partial\Omega} (h(x,u_n)\phi_1^{p-1}\frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n} - h^+(x))\phi_1 + \\ & \int_\Omega (f(x,u_n)\phi_1^{p-1}\frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n} - f^+(x))\phi_1 \\ = \quad & \int_{\partial\Omega} (h(x,u_n) - h^+(x))\phi_1^{p-1}\frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}\phi_1 \\ & + \int_{\partial\Omega} h^+(x)\phi_1(1 - \phi_1^{p-1}\frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}) \\ & + \int_\Omega (f(x,u_n) - f^+(x))\phi_1^{p-1}\frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}\phi_1 \\ & + \int_\Omega f^+(x)\phi_1(1 - \phi_1^{p-1}\frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}). \end{aligned}$$

If
$$\frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \to \phi_1$$
we have that for any n large enough, the sign of
$$\int_{\partial\Omega} (h(x,u_n) - h^+(x))\phi_1^p \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n} + \int_{\Omega} (f(x,u_n) - f^+(x))\phi_1^p \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}$$
is positive. Then
$$\|u_n\|_{W^{1,p}(\Omega)}^{-\alpha} \left(\int_{\partial\Omega} (h(x,u_n) - h^+(x)) u_n^\alpha \phi_1^p \frac{u_n^{-1-\alpha}}{\|u_n\|_{W^{1,p}(\Omega)}^{-1-\alpha}} \frac{|u_n|^{2-p}}{\|u_n\|_{W^{1,p}(\Omega)}^{2-p}} \right.$$
$$\left. + \int_{\Omega} (f(x,u_n) - f^+(x)) u_n^\alpha \phi_1^p \frac{u_n^{-1-\alpha}}{\|u_n\|_{W^{1,p}(\Omega)}^{-1-\alpha}} \frac{|u_n|^{2-p}}{\|u_n\|_{W^{1,p}(\Omega)}^{2-p}} \right) \geq 0.$$
Taking limit we have
$$\int_{\partial\Omega} \overline{A}_\alpha \phi_1^{1-\alpha} + \int_{\Omega} A_\alpha \phi_1^{1-\alpha} \geq 0,$$
which contradicts the hypothesis on f and h.

On the oder hand if $\dfrac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \to -\phi_1$ we get
$$\int_{\partial\Omega} \overline{B}_\beta \phi_1^{1-\beta} + \int_{\Omega} B_\beta \phi_1^{1-\beta} \leq 0,$$
which contradicts the hypothesis on f and h. Hence $\{u_n\}$ must be bounded. If f and h satisfy condition (G_α^+) and (G_β^-), using the other inequality we prove that if we take (λ_n, u_n) solutions of (1.1) with $\lambda_n \nearrow \lambda_1$ then $\{u_n\}$ must be bounded. Using the same argument as in the previous theorem we see that there exists $u \in W^{1,p}(\Omega)$ such that $u_n \to u$ and u is a solution for (1.1) with $\lambda = \lambda_1$. This ends the proof of the theorem. □

We can observe that in the proof of the previous theorem we prove that if f and h satisfy the condition (G_α^-) and (G_β^+) then any bifurcation from infinity must be subcritical, and in the second case any bifurcation from infinity must be supercritical.

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