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On a variational principle for Beltrami flows

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In a previous paper [R. González, L. G. Sarasua, and A. Costa, "Kelvin waves with helical Beltrami flow structure," Phys. Fluids **20**, 024106 (2008)] we analyzed the formation of Kelvin waves with a Beltrami flow structure in an ideal fluid. Here, taking into account the results of this paper, the topological analogy between the role of the magnetic field in Woltjer's theorem [L. Woltjer, "A theorem on force-free magnetic fields," Proc. Natl. Acad. Sci. U.S.A. **44**, 489 (1958)] and the role of the vorticity in the equivalent theorem is revisited. Via this analogy we identify the force-free equilibrium of the magnetohydrodynamics with the Beltrami flow equilibrium of the hydrodynamic. The stability of the last one is studied applying Arnold's theorem. We analyze the role of the enstrophy in the determination of the equilibrium and its stability. We show examples where the Beltrami flow equilibrium is stable under perturbations of the Beltrami flow type with the same eigenvalue as the basic flow one. The enstrophy variation results invariant with respect to a uniform rotating and translational frame and the stability is conserved when the flow experiences a transition from a Beltrami axisymmetric flow to a helical one of the same eigenvalue. These results are discussed in comparison with that given by Moffatt in 1986 [H. K. Moffatt, "Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2. Stability considerations," J. Fluid Mech. **166**, 359 (1986)]. © *2010 American Institute of Physics*. [doi:10.1063/1.3460297]

I. INTRODUCTION

In a previous paper,[1] hereafter Paper I, we analyzed the dynamics of an inviscid axisymmetric Rankine flow which experiences a soft expansion from an initial radius b_0 to a larger one. For given values of the Rossby number $\vartheta = U_0/\Omega b_0$ (Ω and U_0, the Rankine rotating and translational speeds, respectively), we showed that the downstream resulting flow can be expressed as the addition of a Rankine vortex and a Beltrami flow, i.e., $\boldsymbol{\omega}_B = \nabla \times \mathbf{v}_B = \alpha \mathbf{v}_B$, $\alpha = 2\Omega/U_0$.

Woltjer[2] has formulated a variational principle for force-free magnetic fields in ideal magnetohydrodynamics (MHD). The force-free magnetic equilibrium equation is $\nabla \times \mathbf{B} = \kappa \mathbf{B}$, being $\mathbf{B}$ the magnetic field and κ the eigenvalue. In a closed system of volume D the principle states that the force-free field with constant eigenvalue is the state of minimum magnetic energy. He demonstrated that in D, the helicity is conserved,

$$\int_D \mathbf{A} \cdot \mathbf{B} dV = \text{const}, \tag{1}$$

being $\mathbf{A}$ the potential vector and $\nabla \times \mathbf{A} = \mathbf{B}$. When the magnetic energy is minimized under constraint (1), the magnetic field of the resulting equilibrium state is inhibited to produce movement, which is equivalent to the cancellation of the Lorentz force, i.e., a force-free equilibrium.

It was shown in literature[3] that if similar boundary conditions are satisfied, i.e., if for a field $\mathbf{C}$ (where $\mathbf{C}$ replaces the magnetic field or the velocity field) that accomplishes $\nabla \cdot \mathbf{C} = 0$ in the closed domain D, we have $\hat{n} \cdot \mathbf{C} = 0$ on the contour ∂D, then an identification between the ideal hydrodynamic (HD) and the ideal MHD equilibria is possible, in the sense explained in Sec. II.

The identification of the ideal HD Beltrami flow equilibrium with the MHD force-free one is a particular case of the general one. This suggests that a variational principle, analogous to Woljter's principle, could be obtained for the Beltrami flow. In order to determine to what extent the analogy proposed is valid, we study the similarities and differences between the properties that lead to the equilibria and we compare the stabilities of these particular HD and MHD equilibria. To this end we specially focus in the common topological properties of the HD and MHD.

The concept of helicity in fluids, defined as $\int \mathbf{v} . \boldsymbol{\omega} dV$, which expresses topological properties of the flow, was first introduced by Moffatt.[4,5] He demonstrated the conservation of the ideal fluid helicity in two cases: when the vorticity vanishes at the boundaries or when the vorticity is perpendicular to the surface of the fluid.

On the other hand, a direct consequence of Kelvin's circulation conservation law is that the vorticity lines are "fro-

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zen" to the fluid. Due to the induction equation, this is also a property of the ideal MHD magnetic field lines.

The conservation of the helicity (in the HD and MHD cases) and the frozen character of the magnetic ($\mathbf{B}$) and the vorticity ($\boldsymbol{\omega}$) fields are both topological and dynamical properties. Woljter used these properties, which are satisfied in the ideal case (see Paper I), to demonstrate his theorem. In addition, we identify the enstrophy, defined by $\Phi \equiv \frac{1}{2}\int \omega^2 dV$, as the topological analog of the magnetic energy $M=\frac{1}{2}\int B^2 dV$, through the association $\mathbf{B} \leftrightarrow \boldsymbol{\omega}$. Extremizing the enstrophy, subject to the conservation of the helicity constraint, we obtain a Beltrami flow (see the Appendix of Paper I),

$$\delta\left\{\frac{1}{2}\int \boldsymbol{\omega}^2 - \alpha\int \mathbf{v}\,.\,\boldsymbol{\omega}\right\}dV = 0 \Rightarrow \boldsymbol{\omega} = \alpha\mathbf{v}. \tag{2}$$

However, this identification cannot be extended to the stability of the equilibrium. Contrary to the force-free stability case, which is determined by the minimum character of the magnetic energy, stability is not derived from Eq. (2). Thus, the physical meaning and the role of the enstrophy in the equilibrium state determination remain unclear.

The starting point of this work is a result of Paper I. We showed that an axisymmetric Beltrami flow, in a uniform rotating and translating frame, has a marginal stability under helical Beltrami flow perturbations with the same eigenvalue as the basic flow. In fact, a conjecture of Paper I is that this equal eigenvalue condition is a sufficient condition for the Beltrami flow stability.

The paper is organized as follows. In Sec. II, we trace the HD-MHD analogies that we are considering. In Sec. III, we use Arnold's variational stability principle to study the stability of the Beltrami flow and we show its application to the transition from an axisymmetric flow to a helical one. In Sec. IV, we discuss the physical meaning and the role of the enstrophy in both the equilibrium and stability determination. Section V is devoted to the conclusions. Detailed calculations are relegated to Appendices A and B, in order not to deviate the text from the principal line of reasoning.

II. ANALOGY BETWEEN THE IDEAL MHD AND HD: STATIC AND DYNAMIC CASES

Assuming the following identifications: (1) the magnetic field and the speed of the fluid $\mathbf{B} \leftrightarrow \mathbf{v}$, (2) the current and the vorticity vector $\mathbf{j} \leftrightarrow \boldsymbol{\omega}$, and (3) minus the pressure with the total head $-p \leftrightarrow h$, the following MHD and HD equilibrium equations are equivalent provided similar boundary conditions are satisfied,

$$\begin{gathered}\mathbf{j}\times\mathbf{B} = \nabla p, \quad \mathbf{j} = \nabla\times\mathbf{B}, \quad \nabla\cdot\mathbf{B} = 0 \quad \text{in } D,\\ \hat{n}\cdot\mathbf{B} = 0 \quad \text{on } \partial D,\end{gathered} \tag{3}$$

and

$$\begin{gathered}\boldsymbol{\omega}\times\mathbf{v} = -\nabla h, \quad \boldsymbol{\omega} = \nabla\times\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0 \quad \text{in } D,\\ \hat{n}\cdot\mathbf{v} = 0 \quad \text{on } \partial D.\end{gathered} \tag{4}$$

We note that a force-free equilibrium with $\mathbf{j}=\kappa\mathbf{B}$ corresponds to a Beltrami flow type equilibrium, i.e., $\boldsymbol{\omega}=\alpha\mathbf{v}$. However, the stability is not covered by the analogy because—via this identification—there is no correspondence between the dynamic equations,

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) \tag{5}$$

and

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{v}\times\boldsymbol{\omega}). \tag{6}$$

We now consider the analogy that results from the identification $\mathbf{B}\leftrightarrow\boldsymbol{\omega}$ based on dynamical *topological properties*. First, we consider the analogy associated with Eqs. (5) and (6) that express the frozen in condition to the ideal fluid of the magnetic field and of the vorticity, respectively. Second, we take into account the invariance of the dynamic helicities, defined as

$$H_{\mathbf{B}} = \int \mathbf{A}\,.\,\mathbf{B}dV \quad \text{in } D \tag{7}$$

for the MHD case, and as

$$H_{\boldsymbol{\omega}} = \int \mathbf{v}\,.\,\boldsymbol{\omega}dV \quad \text{in } D \tag{8}$$

for the HD case. These invariances are also manifestations of topological properties, such as the linkage of the magnetic field lines and the vorticity lines.[4]

Woltjer[2] showed that a minimum of the magnetic energy, $M=\frac{1}{2}\int B^2 dV$, subject to the conservation of $H_{\mathbf{B}}$, is obtained when the equilibrium field is of the force-free type. That is,

$$\nabla\times\mathbf{B} = \kappa\mathbf{B}, \quad \kappa = \text{const}. \tag{9}$$

Moffatt[4] has demonstrated that $H_{\boldsymbol{\omega}}$ is a constant for ideal fluids when $\hat{n}\,.\,\boldsymbol{\omega}=0$ on ∂D.

Based on common topological properties, this identification implies that the magnetic energy corresponds to the enstrophy,

$$\Phi = \frac{1}{2}\int \omega^2 dV. \tag{10}$$

In Paper I, we showed that the enstrophy is an extreme—with the condition that the HD helicity is constant—when the hydrodynamic equilibrium is a Beltrami flow,

$$\nabla\times\mathbf{v} = \alpha\mathbf{v}, \quad \alpha = \text{const}, \tag{11}$$

equivalent to (see Paper I),

$$\nabla\times\boldsymbol{\omega} = \alpha\boldsymbol{\omega}, \quad \alpha = \text{const}, \tag{12}$$

with the boundary conditions considered.

As was already mentioned, this result does not assure stability due to the fact that canceling $\boldsymbol{\omega}\times\mathbf{v}$ does not have the same physical meaning as the cancellation of the Lorentz force in the force-free case.

In comparison with the magnetic energy role in Woljter's theorem, one of our aims is to determine the role of the enstrophy, both in the equilibrium and the stability determination. Thus, we establish our comparison defining a *topological analogy* which identifies (i) the dynamical properties: the frozen character [Eqs. (5) and (6)], the topological properties of the helicities [Eqs. (7) and (8)], and (ii) the force-free and Beltrami flow equilibria [Eqs. (9) and (12)] which, subject to the conservation of the helicities, are extremes of the magnetic energy and of the enstrophy, respectively.

III. THE STABILITY OF THE BELTRAMI FLOW

To study the stability of the Beltrami equilibrium flow we apply Arnold's variational principle for steady inviscid circulation-preserving flows.[6–8] Following the usual procedure we perturb the equilibrium with a virtual incompressible displacement $\boldsymbol{\eta}$ that follows the dynamics [Eq. (6)], that is,

$$\delta\boldsymbol{\omega} = \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\omega}^E), \quad \delta\mathbf{v} = (\boldsymbol{\eta} \times \boldsymbol{\omega}^E), \quad \boldsymbol{\omega}^E = \alpha \mathbf{v}^E, \tag{13}$$
$$\nabla \cdot \boldsymbol{\eta} = 0, \quad \text{in } D, \quad \hat{n} \cdot \boldsymbol{\eta} = 0, \quad \text{in } \partial D,$$

where $\boldsymbol{\omega}^E$ represents the Beltrami flow equilibrium. The kinetic energy $K = \frac{1}{2}\int v^2 dV$ is stationary at the equilibrium, i.e., $\delta K = \int \mathbf{v}^E \cdot \delta\mathbf{v} dV = 0$. The second variation of the kinetic energy is[9]

$$\delta^2 K = \frac{1}{2}\int [(\delta\mathbf{v})^2 + (\mathbf{v}^E \times \boldsymbol{\eta}) \cdot \delta\boldsymbol{\omega}] dV, \tag{14}$$

thus, according to Arnold's variational principle, the equilibrium is stable if $\delta^2 K$ has a definite sign.

If we replace $\delta\mathbf{v}$ and $\delta\boldsymbol{\omega}$ in Eq. (14) by the expressions given in Eq. (13), the second variation of the kinetic energy is

$$\delta^2 K = \frac{1}{2}\int [(\boldsymbol{\eta} \times \boldsymbol{\omega}^E)^2 + (\mathbf{v}^E \times \boldsymbol{\eta}) \cdot \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\omega}^E)] dV, \tag{15}$$

using $\boldsymbol{\omega}^E = \alpha \mathbf{v}^E$ and changing the order of the factors in the cross product $\mathbf{v}^E \times \boldsymbol{\eta}$, we obtain

$$\delta^2 K = \frac{1}{2}\int [(\boldsymbol{\eta} \times \alpha\mathbf{v}^E)^2 - (\boldsymbol{\eta} \times \mathbf{v}^E) \cdot \nabla \times (\boldsymbol{\eta} \times \alpha\mathbf{v}^E)] dV, \tag{16}$$

which can be written in a compact form as

$$\delta^2 K = \frac{1}{2}\int \left[(\boldsymbol{\Lambda})^2 - \frac{1}{\alpha}\boldsymbol{\Lambda} \cdot (\nabla \times \boldsymbol{\Lambda}) \right] dV, \quad \text{with } \boldsymbol{\Lambda} \equiv \alpha\boldsymbol{\eta} \times \mathbf{v}^E. \tag{17}$$

A particular virtual disturbance which satisfies the Beltrami flow condition, with eigenvalue γ=const,[10] is

$$\nabla \times \boldsymbol{\eta} = \gamma\boldsymbol{\eta}. \tag{18}$$

Following this procedure, Moffatt[3] showed that if $\gamma \neq \alpha$, being α the eigenvalue of the Beltrami flow equilibrium, defined as in Eq. (13), the perturbation is generally unstable. Moreover, our conjecture states that if $\gamma = \alpha$ the stability is guaranteed.

In order to determine up to what extent our conjecture is valid, we now apply Eq. (17) to the important flow transition where a cylindrical axisymmetric Beltrami flow results in a Beltrami flow with helical symmetry,[1,11–13] i.e., the transition of the flow from the $m=0$ mode to the $m=1$ mode. The Beltrami flow equilibrium chosen is

$$v_r^E = 0, \quad v_\theta^E = \alpha A J_1(\alpha r), \quad v_z^E = \alpha A J_0(\alpha r), \tag{19}$$

where A=const and, as a perturbation with helical Beltrami flow structure with eigenvalue $\gamma=\alpha$, we take[1,11–13]

$$\boldsymbol{\eta} = \eta_0 \{f(r), g(r), h(r)\}\{\cos\phi, \sin\phi\}, \tag{20}$$

where η_0=const and $\phi = \theta - kz$ is the helical coordinate with k=const. The helix pass is defined as $\lambda = 2\pi/k$, i.e., the change of z when the angle ϕ varies 2π. Also, $f(r), g(r), h(r)$ are given by

$$f(r) = \left[\frac{\mu}{k} J_1'(\mu r) - \frac{\alpha}{k^2 r} J_1(\mu r) \right], \tag{21}$$

$$g(r) = \left[\frac{1}{kr} J_1(\mu r) - \frac{\mu\alpha}{k^2} J_1'(\mu r) \right], \tag{22}$$

$$h(r) = \frac{\mu^2}{k^2} J_1(\mu r), \tag{23}$$

where

$$\mu^2 = \alpha^2 - k^2. \tag{24}$$

These flows are present in tubes[11] of radius b_0, considering frames with rotational speed Ω and translational speed U_0. They also appear as Kelvin waves in tubes with a soft expansion,[1] e.g., from b_0 to another radius b. As Beltrami flows are characteristic of Kelvin waves, it is convenient to obtain dimensionless variables using the magnitudes Ω, U_0, and b_0. Thus, the flow is characterized by the Rossby number: $\vartheta = U_0/\Omega b_0$. On the other hand, as we are leading with a Beltrami eigenvalue of a Beltrami flow, generated by the expansion of a Rankine vortex[1,14] and given by $\alpha = 2\Omega/U_0$, this expression can be written as $\tilde{\alpha} = 2/\vartheta$ (where $\tilde{\alpha} \equiv \alpha b_0$), i.e., a dimensionless quantity.

Note that, if we define $\tilde{\mu} \equiv \mu b_0$ and $\tilde{k} \equiv k b_0$, Eq. (24) can be rewritten as $\tilde{\mu}^2 = (2/\vartheta)^2 - \tilde{k}^2$. So, if the flows represented by Eqs. (19)–(24) are replaced as $\alpha \to 2/\vartheta$, $\mu^2 \to (2/\vartheta)^2 - \tilde{k}^2$, the variable r and the amplitudes A, η_0 are replaced by the dimensionless quantities $\tilde{r} \equiv r/b_0$, $\tilde{A} \equiv A/U_0 b_0$, and $\widetilde{\eta_0} \equiv \eta_0/b_0$, respectively. From now onward all the relevant quantities of the analysis, i.e., kinetic energy and its variations, are dimensionless.

Taking into account that a characteristic longitude of the Beltrami flow variation is $\alpha^{-1} = \vartheta/2$, the integral of Eq. (17) can be performed varying ϕ between 0 and 2π and averaging radially over α^{-1}. From Eq. (17) we obtain $\delta^2 K$ as a function with dependence on the variables ϑ and k, related

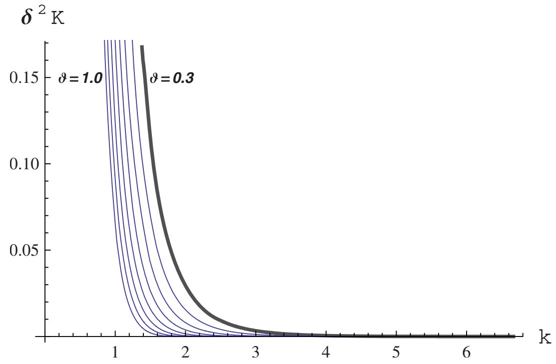

FIG. 1. (Color online) Second variation of the kinetic energy divided by the square product of the amplitudes $(A\eta_0)^2$ using quantities defined in transition equations (19)–(24), as a function of k for the Rossby numbers $\vartheta=0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0$.

with the helix pass. For a fixed value of ϑ, the k values vary in the range $k\leq 2/\vartheta$ where the functions, Eqs. (21)–(23), are defined.

In Paper I we were able to show that the transition defined by Eqs. (19)–(24) is associated with the marginal stability of the Beltrami axisymmetric flow against Beltrami flow type perturbations with equal eigenvalues: $\gamma=\alpha$. In fact, Eqs. (21)–(23) represent the most general form of the Beltrami flow type perturbations allowed with this symmetry (i.e., the helical symmetry with $m=1$). We then apply Eq. (17) to transition equations (19)–(24) and we obtain Fig. 1. The figure shows $\delta^2 K$ as a function of k using quantities defined in Eqs. (19)–(24) for different values of the Rossby number. As we can see $\delta^2 K$ is positive, in agreement with the stability found in Paper I.

As noted previously Moffatt[3] has also restricted the application of Arnold's theorem to an ABC Beltrami flow with eigenvalue α and a Beltrami flow perturbations of the type Eq. (18), with eigenvalue γ. In general, $\gamma\neq\alpha$ and in this case $\delta^2 K$ is not sign definite. However, if in Moffatt's work we restrict the general case to the class of disturbances with $\gamma=\alpha$ the sign of $\delta^2 K$ is definite, in agreement with our conjecture.

IV. THE ENSTROPHY ROLE

We now turn to the role of the enstrophy in the stability and equilibrium determination. We showed that Eq. (2) leads to a Beltrami flow type equilibrium. Following an inviscid circulation-preserving flow dynamics $(\delta\boldsymbol{\omega}=\nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega}^E))$, the enstrophy first order variation for the equilibrium is

$$\begin{aligned}\delta\Phi &= \int \boldsymbol{\omega}^E\cdot\delta\boldsymbol{\omega}dV\\ &= \int \boldsymbol{\omega}^E\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega}^E)dV\\ &= \int [(\boldsymbol{\eta}\times\boldsymbol{\omega}^E)\cdot\nabla\times\boldsymbol{\omega}^E-\nabla\cdot(\boldsymbol{\omega}^E\times(\boldsymbol{\eta}\times\boldsymbol{\omega}^E))]dV,\end{aligned} \tag{25}$$

where the last term can be written as

$$\begin{aligned}&\int[\nabla\cdot(\boldsymbol{\omega}^E\times(\boldsymbol{\eta}\times\boldsymbol{\omega}^E))]dV\\ &\quad=\int[(\hat{n}\cdot\boldsymbol{\eta})(\omega^E)^2-(\hat{n}\cdot\omega^E)(\boldsymbol{\eta}\cdot\boldsymbol{\omega}^E)]dS=0.\end{aligned} \tag{26}$$

For a Beltrami flow type equilibrium with $\nabla\times\boldsymbol{\omega}^E=\alpha\boldsymbol{\omega}^E$,

$$\begin{aligned}\delta\Phi &= \int[(\boldsymbol{\eta}\times\boldsymbol{\omega}^E)\cdot\nabla\times\boldsymbol{\omega}^E]dV\\ &= \int[(\boldsymbol{\eta}\times\boldsymbol{\omega}^E)\cdot\alpha\boldsymbol{\omega}^E]dV=0,\end{aligned} \tag{27}$$

so the enstrophy is stationary for this type of equilibrium.

We are now interested in the second order variation of the enstrophy and its relation with stability. Consider the evolution equation of the enstrophy,[15]

$$\frac{d\Phi}{dt}=-\nu\int[(\nabla\times\boldsymbol{\omega})^2]dV+\int[\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}]dV. \tag{28}$$

For the inviscid case the enstrophy is conserved if the second term of Eq. (28) vanishes. This is true for the two dimensional cases but is not assured for the three dimensional ones.[15] However, note that in the Beltrami flow case ($\boldsymbol{\omega}=\iota\mathbf{v}$, ι=const), the term vanishes in accordance with the boundary conditions considered here. In fact,

$$\begin{aligned}\int[\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}]dV &= \int\left[\frac{1}{\iota}\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{\omega}\right]dV\\ &= \int\left[\frac{1}{2\iota}\nabla\cdot(\boldsymbol{\omega}\omega^2)\right]dV\\ &= \int\left[\frac{1}{2\iota}((\boldsymbol{\omega}\cdot\hat{n})\omega^2)\right]dS=0\end{aligned} \tag{29}$$

because

$$\begin{aligned}\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{\omega} &= \boldsymbol{\omega}\cdot\left(\tfrac{1}{2}\nabla\omega^2-\boldsymbol{\omega}\times\nabla\times\boldsymbol{\omega}\right)\\ &= \boldsymbol{\omega}\cdot\left(\tfrac{1}{2}\nabla\omega^2-\boldsymbol{\omega}\times\alpha\boldsymbol{\omega}\right)\\ &= \nabla\cdot(\boldsymbol{\omega}\omega^2)-\omega^2\nabla\cdot\boldsymbol{\omega}=\nabla\cdot(\boldsymbol{\omega}\omega^2).\end{aligned}$$

Hence, Φ is an invariant proportional to the kinetic energy,[16] and their behavior is the same, i.e., they reach their extremes simultaneously; the second order variations have the same sign so the stability of the equilibrium is established by this second order quantity. Note that the second order variation of the enstrophy for the case studied in Sec. III, with the equilibrium and the virtual displacement defined in Eq. (13), is

$$\begin{aligned}\delta^2\Phi &= \int\left[(\nabla\times\boldsymbol{\Lambda})^2-\frac{1}{\alpha}\boldsymbol{\Lambda}\cdot(\nabla\times\boldsymbol{\Lambda})\right]dV,\\ &\text{with }\ \boldsymbol{\Lambda}=\alpha\boldsymbol{\eta}\times\mathbf{v}^E.\end{aligned} \tag{30}$$

Figure 2 shows the result of applying last formula, for $\delta^2\Phi$, to the transition defined by Eqs. (19) and (20) of Sec. III. Within scalar factors, note that the figure is the same as Fig. 1 showing the symmetric role played by the kinetic energy and the enstrophy in this transition.

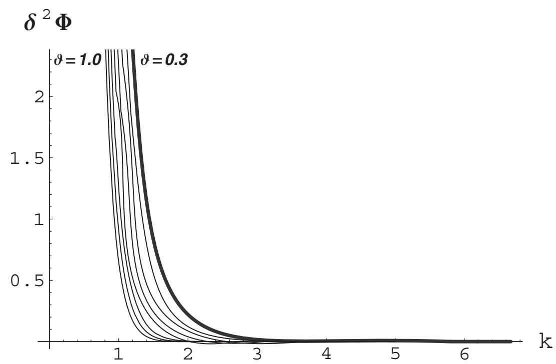

FIG. 2. Second variation of the enstrophy divided by the square product amplitudes $(A\eta_0)^2$, applied to transition equations (19)–(24), as a function of k for the Rossby numbers $\vartheta=0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$.

V. DISCUSSION AND CONCLUSIONS

We can now summarize our results.

(1) Considering the topological analogy between the HD and the MHD, we showed that the enstrophy plays the same role as in the magnetic energy in Woljter's theorem, in the sense that the Beltrami flow equilibrium with constant eigenvalue is obtained when the enstrophy is extremized with the constraint that the helicity is conserved.

(2) For the Beltrami flow equilibrium, the enstrophy is stationary.

(3) The stability of the Beltrami flow equilibrium is not a direct consequence of the process given in Sec. II. This is a difference with Woljter's theorem, where a minimum of the magnetic energy is obtained for a force-free equilibrium with constant α.

(4) The variation of Φ is defined irrespective of a uniform rotation and uniform translation that can be thought of as a surface integral, which can be eliminated through boundary conditions (Appendix A). Then, in Eq. (2) the enstrophy makes these frames to appear as the natural ones, in the sense that some variational principles hold in these systems.[17]

(5) Using Arnold's theorem stability principle, we showed (Sec. III) that a Beltrami axisymmetric flow is stable under helical perturbations with $m=1$ and with the same eigenvalue that the basic flow. In agreement with Paper I results, where we showed its marginal stability, Moffatt has applied the same procedure to study the stability of a Beltrami flow of the ABC type. A point in accordance with our conjecture is that in Moffatt's work $\delta^2 K$ has a definite sign if the perturbations are Beltrami flow perturbations with the same eigenvalue that the basic flow.

(6) In Paper I [Eq. (10)], we showed that the addition of a Rankine and a Beltrami axisymmetric flow, similar to that described in Sec. III, is marginally stable under Beltrami helical perturbations with the same eigenvalue. On the other hand, in Appendix B, we showed that a Rankine flow plus a Beltrami flow satisfies the Euler equation on a system that rotates uniformly. Thus, the Beltrami flow can be considered as a $m=0$ perturbation not restricted to small amplitudes, i.e., it can be considered as a finite amplitude perturbation of the Rankine. Batchelor[14] and Chandrasekhar[18] have shown that finite amplitude plane or axisymmetric waves, propagating in a rotating fluid, are possible. It can be verified that in both cases the flow is of the Rankine flow plus Beltrami flow type.

(7) We have verified that an analog to Fig. 1 is obtained applying Arnold's theorem to a basic flow subject to a Beltrami flow $m=1$ helical perturbation as the one given in Eq. (10) of Paper I. Then, this basic flow is *formally stable*. The formal stability such as that considered in Arnold's theorem is a stronger condition than the linear stability one, but it is weaker than the nonlinear stability[8] condition.

(8) In the special case of a Beltrami flow with the boundary conditions here considered, the enstrophy is a dynamic invariant. Moreover, the second order variation of the enstrophy, $\delta^2\Phi$, coincides with the second order variation of the kinetic energy, $\delta^2 K$, and thus they have a symmetric role in the determination of the stability of the Beltrami flow equilibrium.

Finally, how can we then interpret the perturbation process of a Beltrami basic flow by another Beltrami flow, of equal eigenvalue? Supposing a Beltrami axisymmetric flow of eigenvalue α plus a Rankine one, the theorem presented in Appendix B indicates that a finite amplitude is allowed if we consider the axisymmetric Beltrami flow as the perturbation of the Rankine. So, as suggested by Batchelor,[14] finite amplitude rotating axisymmetric waves are possible. Consider the axisymmetric Beltrami flow in a rotating frame as the basic flow and an infinitesimal helical perturbation of the Beltrami flow type with the same α eigenvalue. We know that stability is marginal (Paper I). If $m=1$, when the Rossby ϑ reaches the ϑ_{cc} (Ref. 19) value, the basic flow subject to finite perturbations of this type is unstable. A Hopf bifurcation occurs leading to a final rotating Beltrami flow wave state of helical symmetry and equal α eigenvalue. A symmetry breaking transition has occurred. In Sec. III we see that the basic flow is formally stable upon perturbations of the same type.

In Paper I [Eq. (28)] we saw that the Beltrami flow can be expanded in a Chandrasekhar–Kendall[20] basis, each term being of a Beltrami flow type of the same eigenvalue α. Appendix B theorem shows that the expansion coefficients can be of finite amplitude. Thus, we can take the addition of the $m=0$ and $m=1$ modes as the basic flow and $m=2$ mode as the Beltrami flow type perturbation of the same α eigenvalue. Hence, we can consider that we have a process of successive bifurcations with increasing angular speed or decreasing Rossby number ϑ. The formal stability of the flow upon perturbations could be an indication of a new equilibrium state reached due to the finite amplitude of the perturbation.

An important issue is to understand the way transitions between Beltrami flows of the same eigenvalue α occur. This will be a subject of further research work.

APPENDIX A: ENSTROPHY VARIATION WITH RESPECT TO A UNIFORM ROTATION AND TRANSLATION

In a frame with rotational velocity $\mathbf{\Omega}$ and translational velocity $\mathbf{U}_0$, the variation of the enstrophy is given by

$$\delta\int (\nabla\times(\mathbf{v}+\mathbf{U}_0)+\mathbf{\Omega})^2 dV$$
$$= \int 2(\boldsymbol{\omega}+\mathbf{\Omega})\cdot\delta\boldsymbol{\omega} dV$$
$$= \delta\Phi + \mathbf{\Omega}\cdot\int[\boldsymbol{\eta}(\mathbf{n}\cdot\boldsymbol{\omega})-\boldsymbol{\omega}(\mathbf{n}\cdot\mathbf{v})]dS, \qquad \text{(A1)}$$

where

$$\delta\boldsymbol{\omega} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega}), \qquad \text{(A2)}$$

was used. The surface term in Eq. (A1) vanishes due to the boundary conditions.

APPENDIX B: ADDITION OF A RANKINE FLOW AND A BELTRAMI FLOW

If $\mathbf{v}_R$ is the relative velocity field of a Rankine vortex (the rotating frame turning with $\tilde{\Omega}=\tilde{\Omega}\mathbf{e}_z$) and $\mathbf{v}_B$ is a Beltrami flow defined as $\boldsymbol{\omega}_B=\nabla\times\mathbf{v}_B=\alpha\mathbf{v}_B$, then the composed flow $\mathbf{v}=\mathbf{v}_B+\mathbf{v}_R$ satisfies the Euler equation,

$$\boldsymbol{\omega}\times(\mathbf{v}+2\tilde{\Omega}\mathbf{e}_z) = -\nabla H, \qquad \text{(B1)}$$

with

$$H = \frac{P}{\rho}+\frac{(\mathbf{v}^2-r^2\tilde{\Omega}^2)}{2}. \qquad \text{(B2)}$$

The Beltrami flow pressure P_B satisfies

$$\nabla P_B = -\rho(\mathbf{v}_B\,.\,\nabla)\mathbf{v}_B. \qquad \text{(B3)}$$

Choosing H as

$$H = H_B + H_R + \frac{(P-P_R-P_B)}{\rho}+\mathbf{v}_R\,.\,\mathbf{v}_B \qquad \text{(B4)}$$

and

$$H_R = \frac{P_R}{\rho}+\frac{({\mathbf{v}_R}^2-r^2\tilde{\Omega}^2)}{2}, \quad H_B = \frac{P_B}{\rho}+\frac{({\mathbf{v}_B}^2)}{2}, \qquad \text{(B5)}$$

and

$$\nabla P_R = -\rho(\mathbf{v}_R\,.\,\nabla)\mathbf{v}_R - \tilde{\mathbf{\Omega}}\times(\tilde{\mathbf{\Omega}}\times\mathbf{r}) - 2\tilde{\mathbf{\Omega}}\times\mathbf{v}_R, \qquad \text{(B6)}$$

where H_R and H_B are the Bernoulli functions for $\mathbf{v}_R$ and $\mathbf{v}_B$, respectively, being P_R the Rankine pressure.

Since the Rankine vortex obeys the Euler equation, Eq. (B1), and the Beltrami flow satisfies the same equation with $\tilde{\Omega}=0$, being

$$\boldsymbol{\omega}_B\times\mathbf{v}_B = 0 \qquad \text{(B7)}$$

(so the relation $\nabla H_B=0$ holds), we obtain

$$\nabla H = \mathbf{v}_R\times\boldsymbol{\omega}_R + \nabla\left[\frac{(P-P_R-P_B)}{\rho}+\mathbf{v}_R\,.\,\mathbf{v}_B\right], \qquad \text{(B8)}$$

where $\boldsymbol{\omega}_R=2\Omega_R\mathbf{e}_z$ is the Rankine vorticity measured from the laboratory frame. Following the procedure to obtain Eq. (B6) for the Rankine flow, using the Euler equation, it results

$$(\mathbf{v}\,.\,\nabla)\mathbf{v} = -\frac{\nabla P}{\rho} - \tilde{\mathbf{\Omega}}\times(\tilde{\mathbf{\Omega}}\times\mathbf{r}) - 2\tilde{\mathbf{\Omega}}\times\mathbf{v}. \qquad \text{(B9)}$$

From this equation, Eq. (B3), and the vectorial identity

$$\nabla(\mathbf{v}_R\,.\,\mathbf{v}_B) = \mathbf{v}_R\times(\nabla\times\mathbf{v}_B)+\mathbf{v}_B\times(\nabla\times\mathbf{v}_R) + (\mathbf{v}_R\,.\,\nabla)\mathbf{v}_B + (\mathbf{v}_B\,.\,\nabla)\mathbf{v}_R, \qquad \text{(B10)}$$

we obtain

$$-\boldsymbol{\omega}_B\times\mathbf{v}_R - \boldsymbol{\omega}_R\times\mathbf{v}_B = \nabla\left[\frac{(P-P_R-P_B)}{\rho}+\mathbf{v}_R\,.\,\mathbf{v}_B\right]. \qquad \text{(B11)}$$

Therefore, using Eqs. (B11), (B7), and (B8), we prove that $\mathbf{v}$ satisfies the Euler equation, Eq. (B1).

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[9] Strictly, the procedure requires the addition of the gradient of a potential function: $\delta\mathbf{v}=(\eta\times\boldsymbol{\omega}^E)+\nabla\varphi$. As the disturbance proposed is conditioned by the potential, we can choose φ to be a disturbance of the Beltrami flow type given in Eq. (18). Another option is to chose φ satisfying $\varphi=0$ in ∂D adding the term $\int(\nabla\varphi)^2 dV$. To guarantee stability the total sum must be of a definite sign. In our case, the remaining terms of $\delta^2 K$ will be positive definite for the transition represented in Eqs. (19)–(24) of a Beltrami axisymmetric flow perturbed by a helical Beltrami flow in a cylindrical geometry that satisfies $\gamma=\alpha$.

[10] To denote the eigenvalues of the perturbation and equilibrium Beltrami flows, we use γ and α, respectively.

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[16] For a Beltrami flow $\boldsymbol{\omega}=\iota\mathbf{v}$, then $\Phi=\frac{1}{2}\int(\boldsymbol{\omega})^2 dV=\frac{1}{2}\int(\iota\mathbf{v})^2 dV=\iota^2 K$. So, Φ is proportional to K.

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[19] ϑ_{cc} introduced in Paper I (p. 5) is the value of the Rossby number for which the basic flow is marginally stable, i.e., for this transition case, $m=0$ to $m=1$, if $\vartheta < \vartheta_{cc}$ the flow is unstable and if $\vartheta > \vartheta_{cc}$ it is stable.

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