Comment on “Two-dimensional potential theory for translating and rotating solids” [Phys. Fluids15, 3576 (2003)]

Hernán D. Reisin

Citation: Physics of Fluids (1994-present) **20**, 069101 (2008); doi: 10.1063/1.2930674
View online: http://dx.doi.org/10.1063/1.2930674
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/20/6?ver=pdfcov
Published by the AIP Publishing

PHYSICS OF FLUIDS **20**, 069101 (2008)

Comment on "Two-dimensional potential theory for translating and rotating solids" [Phys. Fluids 15, 3576 (2003)]

Hernán D. Reisin[a)]
Depto. de Física Juan José Giambiagi, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina

(Received 31 March 2008; accepted 19 April 2008; published online 13 June 2008)

The two-dimensional potential theory for translating and rotating solids developed in Minotti [Phys. Fluids **15**, 3576 (2003)] is briefly revised and formulas of the force and moment of the force are given in their correct form. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2930674]

Minotti[1] (M03) showed a clever formalism to deal with inviscid, incompressible fluid flows when a rigid body immersed in it performs arbitrary two-dimensional rotations with angular velocity $\vec{\boldsymbol{\Omega}}(t)=\Omega(t)\hat{\boldsymbol{e}}_z$ about a point which itself is moving with velocity $\vec{\boldsymbol{U}}(t)$, in the xy plane. In many aspects, that theory improved Milne-Thomson's treatment of the Blasius theorem,[2] allowing the existence of point singularities in the flow and facilitating the calculation of the hydrodynamic force. However, the accompanying formulas provided for the force [Eqs. (27) and (29) of M03] and for the moment of the force [Eqs. (31) and (32) of M03] must be modified. Since the hydrodynamic force is real, the pressure (p) can be calculated from the Euler equation in the noninertial frame of reference attached to the moving solid,[3]

$$\frac{D\vec{\boldsymbol{u}}}{Dt} = -\frac{\boldsymbol{\nabla} p}{\rho} + \vec{\boldsymbol{f}}_r - \frac{d\vec{\boldsymbol{U}}}{dt} - 2\vec{\boldsymbol{\Omega}}\times\vec{\boldsymbol{u}} - \vec{\boldsymbol{\Omega}}\times\vec{\boldsymbol{\Omega}}\times\vec{\boldsymbol{x}} - \frac{d\vec{\boldsymbol{\Omega}}}{dt}\times\vec{\boldsymbol{x}}, \tag{1}$$

where the velocity $\vec{\boldsymbol{U}}$ refers to the velocity of the moving frame relative to the Newtonian frame of reference and not relative to the rotating frame, as in Eq. (18) of M03 (see also p. 245 of Ref. 2). This is a more natural and practical choice, although is rather irrelevant for further calculations. The superposition principle applied to the velocity field $\vec{\boldsymbol{u}}$ (relative to the moving frame) allows us to separate the rigid rotation of the fluid from the remaining potential flow as (in complex notation) $u_x - iu_y = dW/dZ + i\Omega(t)Z^*$ (where the asterisk denotes complex conjugation). Following the formalism of M03,[1] the potential W is split into two contributions,

$$u_x - iu_y = \frac{dw}{dZ} + \frac{d\Lambda}{dZ} + i\Omega(t)Z^*, \tag{2}$$

where the potential $w=\tilde{\varphi}+i\tilde{\psi}$ defines a *fictitious* velocity field $\tilde{u}_x - i\tilde{u}_y \doteq dw/dZ$ that includes all possible singularities of the real flow, and $\Lambda = F + i\left[G - \frac{1}{2}\Omega(t)(x^2+y^2)\right]$. Using these definitions, the integration of the Euler equation (1) leads to the Bernoulli equation,

$$-\frac{p}{\rho} = \frac{\partial}{\partial t}(\tilde{\varphi}+F) + \frac{1}{2}|\boldsymbol{\nabla}\tilde{\varphi}|^2 + \frac{1}{2}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right] + \Omega\frac{\partial F}{\partial x}y - \Omega\frac{\partial F}{\partial y}x + \tilde{u}_x\frac{\partial G}{\partial y} - \tilde{u}_y\frac{\partial G}{\partial x} + \phi_r + \frac{d\vec{\boldsymbol{U}}}{dt}\cdot\vec{\boldsymbol{x}}, \tag{3}$$

with which the force (*per* unit length) is obtained by integration along the boundary of the solid body ($\mathcal{C}$),

$$F_x - iF_y = i\rho\frac{\partial}{\partial t}\left[\oint_{\mathcal{C}}(w+\boldsymbol{\Lambda})dZ\right]^* + \frac{i\rho}{2}\oint_{\mathcal{C}}\left[\frac{d}{dZ}(w+\Lambda)\right]^2 dZ + (F^{\mathrm{Arc}})^* - \rho\Omega\left[\oint_{\mathcal{C}} Z\frac{d}{dZ}(w+\boldsymbol{\Lambda})dZ\right]^*. \tag{4}$$

The Archimedes force $(F^{\mathrm{Arc}})^*$ gathers all the w-, F-, and G-independent terms,

$$(F^{\mathrm{Arc}})^* = \rho\int\int_A(\boldsymbol{\nabla}\phi_r)^* dS + \rho A\left(\frac{dU^*}{dt} - Z_c^*\Omega^2 - iZ_c^*\frac{d\Omega}{dt}\right), \tag{5}$$

where A is the area enclosed by curve $\mathcal{C}$, and Z_c is the centroid of the body: $AZ_c = \int\int_A Z dS$. Terms of Eq. (4) emphasized in bold indicate those considered in M03 as vanishing contributions, thus missing in the expression of the force [Eq. (27)] of M03. Yet, function Λ is holomorphic outside the solid boundary by hypothesis, which implies that it must have poles inside curve $\mathcal{C}$ that may lead to nonvanishing residues for $\Lambda(Z)$ and $Zd\Lambda/dZ$. An immediate interpretation of the force formula is attainable by simple inspection of the velocity [Eq. (2)]; loosely speaking, the first three terms in Eq. (4) resemble the usual forces for non-rotating bodies [$\Omega(t)=0$] replacing $w\to(w+\Lambda)$, while the last term emerges from the product of the irrotational $(d/dZ)(w+\Lambda)$ times the rotational $i\Omega Z^*$ part of the velocity. Nonetheless, it must be remarked that there are no purely translational nor purely rotational terms in the force; indeed, Λ is function of $\Omega(t)$ and, therefore, all terms in Eq. (4) are Ω-dependent. A com-

a)Electronic mail: hreisin@gmail.com.

plex form can also be provided for the moment of the force,

$$M = -\rho \,\mathrm{Re}\left[\frac{\partial}{\partial t}\oint_{\mathcal{C}} Z^{*}(w + \boldsymbol{\Lambda})dZ\right] + M^{\mathrm{Arc}} - \rho \,\mathrm{Re}\left[\frac{1}{2}\oint_{\mathcal{C}} Z\left(\frac{d}{dZ}(w+\Lambda)\right)^{2} dZ\right], \tag{6}$$

where the w- and Λ-independent term is

$$M^{\mathrm{Arc}} = 2\frac{d\Omega}{dt}I_{zz} + \rho \int\int_{A}\left(x\frac{\partial \phi_r}{\partial y} - y\frac{\partial \phi_r}{\partial x}\right)dS - \rho A \,\mathrm{Re}\left[i\frac{dU}{dt}Z_c^{*}\right], \tag{7}$$

where I_{zz} is the moment of inertia of the body along the z axis. The term in bold in Eq. (6) is not included as a part of M^{Arc} at variance to the original article[1] because the integral of $Z^{*}\Lambda(Z)$ along $\mathcal{C}$ does not necessarily tend toward zero when $\mathcal{C}$ contracts to a zero area curve, as should for an Archimedes' contribution.

It is worth noting that either the force [Eq. (4)] or the moment of the force [Eq. (6)] is a formula invariant under the interchange of w and Λ, a property lacking from Eqs. (27) and (31) of M03. Beyond any aesthetic matter, this property has two important consequences. First, it indicates that the split, $W \rightarrow w+\Lambda$, has no net incidence in the expression of dynamical magnitudes, as expected since this separation is mathematically convenient to include singularities in the flow and make easiest the calculation of both $w(Z)$ [see paragraph below Eq. (17) of M03] and Λ [Eqs. (15) and (16) of M03] but is rather arbitrary from an exclusively physical standpoint of view, as suggested by the author [see paragraph below Eq. (12) in Ref. 1]. Second, it assures the regularity of the force for solids with sharp points at their boundaries like cuspid edges. Considering that there exists an analytical complex function $Z=f(\zeta)$ that maps the circle $|\zeta|=a$ in the ζ plane to the boundary of the solid body, the velocity [Eq. (2)] is written as

$$u_x - iu_y = \left(\frac{d\omega}{d\zeta} + \frac{d\Pi}{d\zeta}\right)\left(\frac{dZ}{d\zeta}\right)^{-1} + i\Omega f^{*}(\zeta), \tag{8}$$

where the transformed functions are defined by $\omega(\zeta) = w[f(\zeta)]$ and $\Pi(\zeta)=\Lambda[f(\zeta)]$. If the function $f(\zeta)$ is not invertible over a set of isolated points $\{\zeta_i\}$ over the boundary $\mathcal{C}$, then $df/d\zeta$ will be zero at those points. Yet, since the fluid flow is regular everywhere over the solid by hypothesis (point singularities lie outside the body), and $i\Omega f^{*}(\zeta)$ is always finite, the velocity [Eq. (8)] can only be regular if the Kutta condition is satisfied at the zeros of $df/d\zeta$,

$$\left.\frac{d}{d\zeta}(\omega + \Pi)\right|_{\{\zeta_i\}} = 0. \tag{9}$$

Therefore, from Eq. (4), the Blasius force in the ζ plane,

$$F_x^B - iF_y^B = \frac{i\rho}{2}\oint_{|\zeta|=a}\left(\frac{d\omega}{d\zeta} + \frac{d\Pi}{d\zeta}\right)^{2}\left(\frac{df}{d\zeta}\right)^{-1} d\zeta, \tag{10}$$

which may be divergent because of the factor $(df/d\zeta)^{-1}$, is compensated by $(d\omega/d\zeta + d\Pi/d\zeta)^{2}$ that is zero because of the Kutta condition imposed to the velocity field. As a final remark, it must be noted that the Kutta condition [Eq. (9)] is ascribed to the sum, $\omega+\Pi$, and not to any of these functions alone, in accordance with the notion that the split $W \rightarrow w + \Lambda$ is not of physical origin.

The author thanks kind suggestions made by Dr. I. L. Reisin and M. S. I. C. Carretero.

[1] F. O. Minotti, "Two dimensional potential theory for translating and rotating solids," Phys. Fluids **15**, 3576 (2003).
[2] L. M. Milne-Thomson, *Theoretical Hydrodynamics*, 4th ed. (Macmillan, London, 1962).
[3] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1998).