Influence of the aspect ratio of a drop in the spreading process over a horizontal surface

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We study in this paper the influence of the aspect ratio of an axisymmetric drop on the spreading rate. For very small values of aspect ratio, the spreading rate is proportional to the cube of the aspect ratio as stated by Tanner's law. However, as the value of the aspect ratio increases, the proportionality constant shows a weak dependence on the aspect ratio, first decreasing and then increasing after reaching a minimum. Due to the fact that the aspect ratio of the drop decreases with time in the spreading drop, its influence decreases as time increases. [S1063-651X(98)09109-0]

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I. INTRODUCTION

In Ref. [1] we deduce a simple first-order differential equation for the spreading evolution of a drop over or under a horizontal surface. Both gravity as well the disjoining pressure effects were considered. We assumed a fluid with a positive spreading parameter $S=\sigma_{SG}-\sigma_{SL}-\sigma>0$, where σ_{SG}, σ_{SL}, and σ are the solid-gas, solid-liquid, and liquid-gas free energy per unit area, respectively. σ is also called the surface tension. We show how the spreading velocity depends on the ratio of the van der Waals influence length to the actual drop size. Due to the changing drop size with time, the spreading rate also changes with time, modifying the drop size evolution. We also considered very thin drops (very small aspect ratio of drops δ, $\delta=h/R$, where h is the thickness of the fluid drop and R is its radius), with a linearized form of the surface tension pressure gradients.

Using very simple arguments and assuming a small drop shape as a spherical cap, it is possible to derive a very simple theory for the spreading process [2]. The pressure gradient generated by surface tension must be balanced by the viscous force, i.e., $\mu v/h^2\sim\sigma h/R^3$, where μ is the viscosity of the fluid and v is the radial velocity of the fluid. It follows that $\mu v/\sigma=\mathrm{Ca}\sim\delta^3$. For $\delta\ll 1$, $\delta\sim\theta$, where θ is the angle of the surface profile and Ca is the usual capillary number. This is the so-called Tanner's law [3] written as $\theta=C_T\mathrm{Ca}^{1/3}$, where C_T is the Tanner's constant. Replacing v by dR/dt and h by V/C_VR^2 where V is the volume of the drop and C_V a constant of order unity, we obtain from Tanner's law a first-order ordinary differential equation for $R(t)$, which gives the well-known asymptotic behavior $R\sim t^{1/10}$ for $t\to\infty$. However, the constant C_T is not universal and depends globally on the specific problem [4–8]. In most of the analyses on the spreading process of small drops, the lubrication approximation has been employed, where the inertial terms are negligibly small compared with the surface tension and viscous terms. This is true if the Reynolds number associated with the spreading velocity and drop thickness is much lower than the inverse of the aspect ratio of the drop, $\mathrm{Re}=\rho vh/\mu\ll 1/\delta$ where ρ is the fluid density. From the order of magnitude written lines above, we know that the fluid velocity is $v\sim\sigma\delta^3/\mu$, therefore the associated Reynolds number will be $\mathrm{Re}\sim\delta^4/\mathrm{Oh}^2$, where Oh is the well-known Ohnesorge number defined by $\mathrm{Oh}=\mu/\sqrt{\rho R\sigma}$. Thus, the lubrication approximation is still valid for values of δ such as $\delta\ll\mathrm{Oh}^{2/5}$. Typical values for silicon oils are [9] $\rho\approx 841\ \mathrm{kg/m^3}$, $\mu\approx 0.0225\ \mathrm{kg/ms}$, and $\sigma\approx 0.035\ \mathrm{kg/s^2}$. With these values, the Ohnesorge number ranges from 0.415 for $R=10^{-4}$ m to 0.0415 for $R=10^{-2}$ m. Therefore, the lubrication approximation is valid for $\delta\ll 0.7$ for $R=10^{-4}$ m and $\delta\ll 0.3$ for $R=10^{-2}$ m.

The main purpose of this paper is to extend the analysis in [1], considering the full term arising from the Young-Laplace equation for the surface tension in order to evaluate the influence of a small but finite aspect ratio of the drop.

II. FORMULATION

Using the lubrication approximation, the nondimensional form of the equation for the evolution of a free surface of a fluid under gravity and capillary forces is given by [2]

$$\left(\frac{F}{G}\right)^4\left(\frac{\partial(\phi G)}{\partial\tau}-\frac{G}{F}\eta\frac{\partial\phi}{\partial\eta}\frac{dF}{d\tau}\right)=-\frac{1}{\eta}\frac{\partial}{\partial\eta}\left[\eta\phi^3\frac{\partial}{\partial\eta}\left(D\phi+\frac{\varepsilon^2}{\phi^3}-B\phi\right)\right],\tag{1}$$

where D is a differential operator given by

$$D\phi=\frac{\partial^2\phi/\partial\eta^2+(1/\eta)(\partial\phi/\partial\eta)[1+\delta^2(\partial\phi/\partial\eta)^2]}{[1+\delta^2(\partial\phi/\partial\eta)^2]^{3/2}},$$

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$\delta = \delta_0 G/F$ with $\delta_0 = H_0/R_0$ representing the aspect ratio of the drop at time $t=0$, ε is the ratio of the van der Waals length $a = \sqrt{|A|/6\pi\sigma}$ to the size of the drop, $\varepsilon = \varepsilon_0 F/G^2$, which is assumed to be very small compared with unity, with $\varepsilon_0 = aR_0/H_0^2$ being its value at time $t=0$. $B = \rho g R^2/\sigma$ corresponds to the Bond number, which relates the gravity to the surface tension forces. Here, $B = B_0 F^2$, with B_0 being the corresponding value at time $t=0$. A is the Hamaker constant, which is negative for a wetting liquid. Here we used the following nondimensional variables:

$$\phi = \frac{h}{H(\tau)}, \quad \eta = \frac{r}{R(\tau)}, \quad \tau = \frac{\sigma}{3\mu}\frac{H_0^3}{R_0^4}t, \tag{2}$$

where $H = H_0 G(\tau)$ and $R = R_0 F(\tau)$ are the thickness at the center of the drop and the macroscopic radius of the drop, respectively. H_0 and R_0 are the corresponding values at time $t=0$. In a nondimensional form, the radial averaged velocity is then

$$K_r = \frac{3\mu}{\sigma}\left(\frac{R_0 F}{H_0 G}\right)^3 v = \phi^2 \frac{\partial (D-B)\phi}{\partial \eta}. \tag{3}$$

At the macroscopic edge of the drop, the nondimensional radial velocity $K = K_f$ is therefore

$$K = \left(\frac{F}{G}\right)^3 \frac{dF}{d\tau} = \lim_{\eta \to 1}\left[\phi^2 \frac{\partial (D-B)\phi}{\partial \eta}\right], \tag{4}$$

where K is related to the usual capillary number by $K = 3\mathrm{Ca}/\delta_0^3$, with $\mathrm{Ca} = v_f \mu/\sigma$. The total volume of the drop that remains invariant during the spreading process is written by

$$V = 2\pi R_0^2 H_0 F^2 G I \quad \text{with} \quad I = \int_0^1 \phi \eta \, d\eta. \tag{5}$$

To solve Eq. (1) with the corresponding boundary and initial conditions, we divide the problem in a macroscopic (surface tension–viscous-gravity) region where $\phi/\varepsilon \gg 1$, and a thin region of order ε close to the edge of the drop, $(1-\eta) \sim \varepsilon$, where the effect of the van der Waals forces must be considered in the analysis. Due to the disparity in the two spatial scales, $\varepsilon \to 0$, the solution in both regions is to be obtained and properly matched.

A. Macroscopic region $[(1-\eta) \gg \varepsilon]$

Assuming a quasisteady self-similar solution to the macroscopic problem (where the effect of the van der Waals forces can be neglected), $\phi = \phi(\eta)$, from the overall volume conservation (5), it follows that $F^2 G = 1$ and $V = 2\pi R_0^2 H_0 I$. In this case, the macroscopic equation (3) reduces to

$$\phi^2 \frac{d(D-B)\phi}{d\eta} = K\eta, \tag{6}$$

where the averaged radial velocity related to its value at the edge drop is found to increase linearly with the radial coordinate, $K_r = K\eta$. The nondimensional boundary conditions needed to solve this equation are given by

$$\phi(1) \to 0, \quad \phi(0) - 1 = \left.\frac{d\phi}{d\eta}\right|_0 = 0, \tag{7}$$

together with the result from matching with the precursor region.

B. Inner or precursor region $[(1-\eta) \sim \varepsilon]$

At the edge of the drop, where ϕ is of order ε, there is a very thin region with $(1-\eta) \sim \varepsilon$, where the nonretarded van der Waals forces cannot be neglected. Introducing for this region the following inner variables of order unity,

$$y = \frac{K^{1/3}\phi}{3^{1/2}\varepsilon} \quad \text{and} \quad x = \frac{K^{2/3}(\eta - 1)}{3^{1/2}\varepsilon}, \tag{8}$$

the inner equation takes the nondimensional form

$$C_1 y^2 y''' - \frac{y'}{y^2} - 1 = C_1^2 C_2, \tag{9}$$

where

$$C_1 = \frac{1}{1+\alpha(y')^2}, \quad C_2 = 2\alpha y'(yy'')^2, \quad \alpha = (\delta K^{1/3})^2. \tag{10}$$

Here $y' = dy/dx$. In the precursor film, $y \to 0$ as $x \to \infty$. To the left, the boundary conditions are to be properly matched with the macroscopic region.

III. RESULTS

We transform Eq. (6) to a nonlinear equation given by

$$\phi^2 \frac{d}{d\zeta}\left(\frac{d^2\phi/d\zeta^2 + (1/\zeta)d\phi/d\zeta[1+\delta^{*2}(d\phi/d\zeta)^2]}{[1+\delta^{*2}(d\phi/d\zeta)^2]^{3/2}} - B^*\phi\right) = \zeta, \tag{11}$$

where $\zeta = K^{1/4}\eta$, δ^* is a reduced aspect ratio of the drop, $\delta^* = \delta K^{1/4}$, and B^* is a reduced Bond number given by $B^* = B/K^{1/2}$. The boundary conditions now take the form

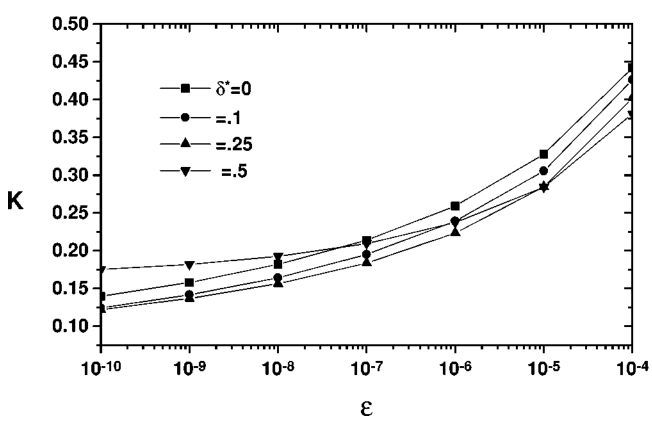

FIG. 1. Nondimensional spreading rate K as a function of ε, for different values of the reduced aspect ratio of the drop δ^*.

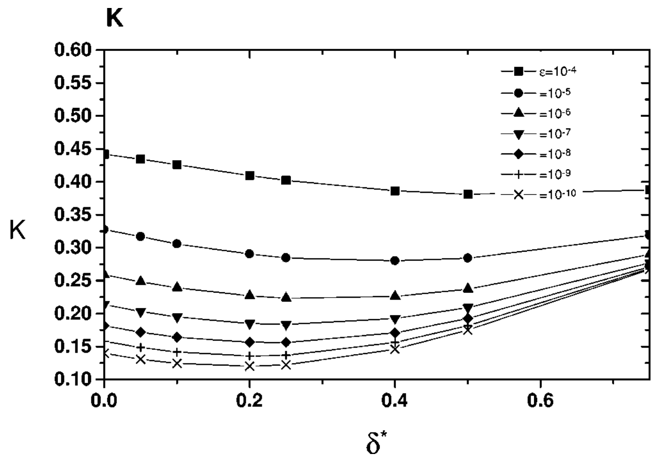

FIG. 2. Nondimensional spreading rate K as a function of the reduced aspect ratio of the drop δ^*, for different values of ε.

$$\phi(K^{1/4})\rightarrow 0, \quad \phi(0)-1=\left.\frac{d\phi}{d\zeta}\right|_0=0 \tag{12}$$

together with the matching condition to the precursor region. We integrate numerically Eqs. (11) and (12) using a fourth-order Runge-Kutta equation with an initial guess of $d^2\phi/d\zeta^2|_0$ until we reach the precursor film with $y\rightarrow 0$ as $x\rightarrow\infty$. The appropriate solution is obtained as we get the final condition in the precursor film, $y\rightarrow 0$ for $x\rightarrow\infty$. We used a step size of $\Delta\eta=10^{-10}$ for the macroscopic region and $\Delta x=10^{-4}$ for the inner region. For simplicity, we do not consider gravity effects here, included in Ref. [1].

Figure 1 shows the reduced capillary number K, as a function of ε, for different values of the reduced aspect ratio δ^*. For very small values of δ^*, the reduced spreading rate K is lower than that corresponding to $\delta^*=0$. However, there is a value of δ^* around 0.25, where the above tendency inverts, generating a minimum on K. For relatively large values of δ^*, the solution shows an asymptotic behavior for $\varepsilon\rightarrow 0$ with constant K, not depending on ε. The same results are presented in Fig. 2, but with K plotted as a function of δ^*, for different values of ε. This figure clearly shows the minimum of K and how the gap between the different values of ε is reduced as the value of δ^* increases. Therefore, it is shown that the spreading rate does not depend on ε, for large values of δ^*. In Fig. 3 similar results are plotted for the normalized spreading rate K/K_0 as a function of the aspect ratio of the drop δ, for the two limiting values of ε considered here. K_0 corresponds to the value obtained with $\delta=0$ [1],

$$K_0\simeq K^*[1+0.01991\ \ln(\varepsilon/\varepsilon^*)+0.008121\ \ln^2(\varepsilon/\varepsilon^*)]. \tag{13}$$

Here, $K^*(\varepsilon^*)$ is a reference value. In our case $K^*(10^{-10})=0.13963$. In this figure it is clear that the universal behavior represented by the similar minimum value of K/K_0, which suggests to introduce $\delta/\varepsilon^{0.063}$ instead of δ as shown in Fig. 4. A very good correlation for K is given by

$$K\simeq K_0(\varepsilon)\left[1-0.2167\frac{\delta}{\varepsilon^{0.063}}+0.0829\left(\frac{\delta}{\varepsilon^{0.063}}\right)^2\right]. \tag{14}$$

For large values of δ^*, the macroscopic region of the drop dictates the value of K without taking care of the precursor layer structure. To study this effect, we plotted in Fig. 5 the drop slope at the surface for different values of $d^2\phi/d\zeta^2|_0$, around the value that produces $d\phi/d\zeta=0$ with $\phi=0$ and $\zeta=\zeta_0$, neglecting the disjoining pressure effects. The drop slopes are plotted as a function of $(\zeta-\zeta_0)/\zeta_0$ for different values of δ^*. As the value of δ^* increases, $d\phi/d\zeta$

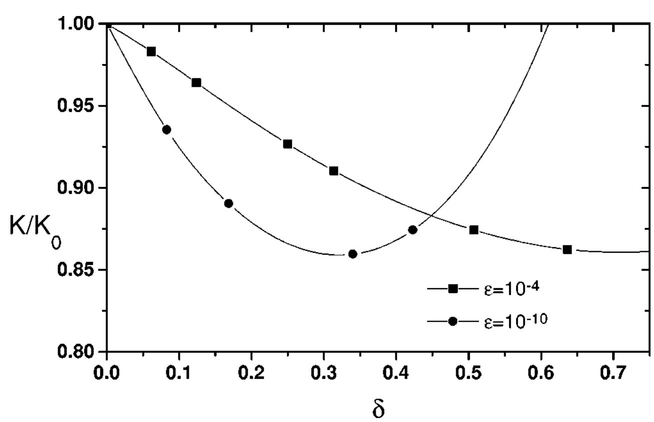

FIG. 3. Nondimensional spreading rate K as a function of the aspect ratio of the drop δ, for two different values of ε.

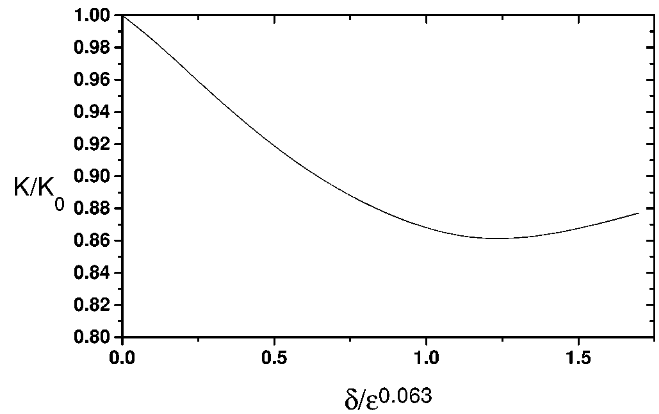

FIG. 4. Universal behavior of K/K_0 as a function of reduced aspect ratio of the drop $\delta/\varepsilon^{0.063}$.

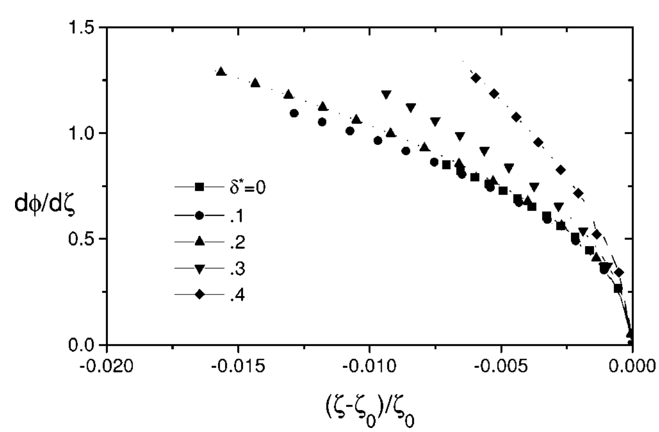

FIG. 5. Normalized macroscopic drop gradient $d\phi/d\zeta$ obtained at the wall as a function of the reduced coordinate $(\zeta-\zeta_0)/\zeta_0$, for different values of the reduced aspect ratio of the drop δ^*.

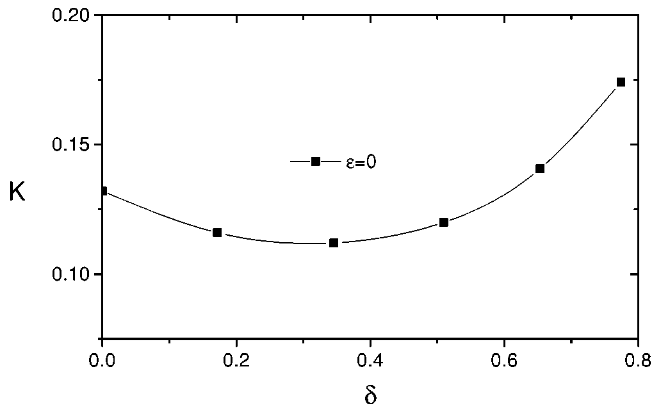

FIG. 6. Nondimensional spreading rate K as a function of the aspect ratio of the drop δ, for $\varepsilon=0$.

increases also very rapidly, thus indicating that the macroscopic shape dictates the form that can be managed in the precursor region to reach the appropriate condition $y\to 0$ as $x\to\infty$. Finally in Fig. 6 we show the asymptotic behavior of the nondimensional spreading rate as a function of δ, for $\varepsilon=0$, without considering the disjoining pressure effects. This in fact is very similar to that obtained with $\varepsilon=10^{-10}$ in Fig. 3.

In summary, for values of $\delta \ll 1$, the nondimensional spreading rate or capillary number can be well represented by

$$\mathrm{Ca}\simeq K_0(\varepsilon)\delta^3\left[1-0.2167\frac{\delta}{\varepsilon^{0.063}}+0.0829\left(\frac{\delta}{\varepsilon^{0.063}}\right)^2\right]. \tag{15}$$

Replacing Eqs. (13) and (14) into the definition of $K, K=F^9 dF/d\tau$, we obtain the evolution equation for the drop radius as

$$F^9\frac{dF}{d\tau}\simeq K_0(\varepsilon_0)\Omega(F)\Delta(F), \tag{16}$$

where

$$\Omega(F)=1+0.01991\ \ln(F)+0.008121\ \ln^2(F)$$

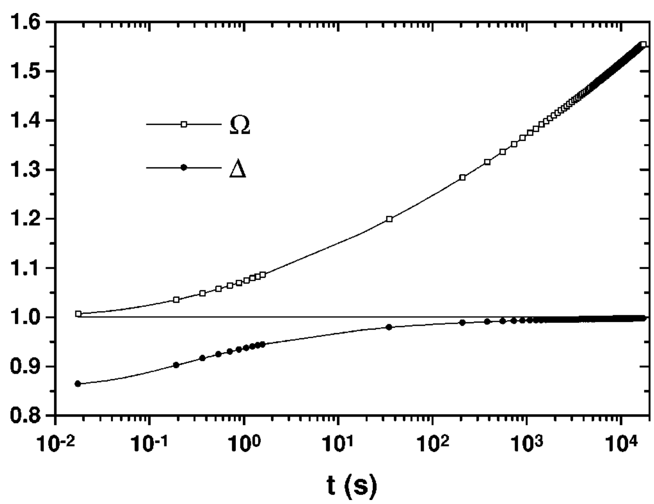

FIG. 7. Evolution of the nondimensional functions Ω and Δ as a function of time.

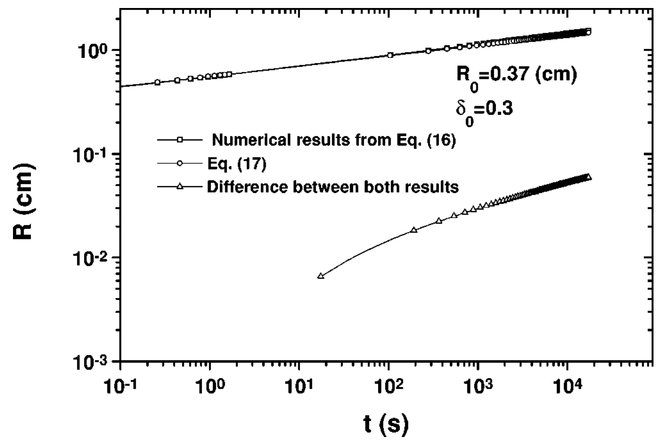

FIG. 8. Radius of a spreading drop of silicon oil as a function of time, for an initial volume of 0.024 cm^3 and initial aspect ratio of $\delta_0=0.3$.

measures the effect of changing ε in the spreading process and

$$\Delta(F)=1-\frac{0.2167\delta_0}{\varepsilon_0^{0.063}}\frac{1}{F^{3.315}}+\frac{0.0829\delta_0^2}{\varepsilon_0^{0.126}}\frac{1}{F^{6.630}}$$

measures the effect of the aspect ratio of the drop. Equation (16) must be integrated numerically. For silicon oils, using the data reported in [9] for a drop with an initial volume of 0.024 cm^3 and an initial aspect ratio of the drop, $\delta_0=0.3$, Eq. (16) is numerically integrated to give $R=R_0F$ as a function of time, $t=(3\mu R_0/\sigma\delta_0)\tau$. The results are plotted in Figs. 7 and 8. Figure 7 shows the values of Ω and Δ as a function of time. At the beginning, the actual aspect ratio of the drop is relatively large and the influence of Δ is strong compared with the influence of ε. As the drop radius increases the actual aspect ratio of the drop decreases, decreasing its influence in the spreading process. Therefore $\Delta\to 1$ for $\tau\to\infty$. The influence of the aspect ratio is negligible for times larger than 10^3 s. The contrary occurs with the influence of ε, which increases always as the time increases. The asymptotic behavior of the solution for $\tau\to\infty$ is therefore almost the same as that without considering the aspect ratio effects. The solution to Eq. (16) is shown in Fig. 8. This result is compared with the solution obtained by neglecting any contribution of the aspect ratio of the drop and changing values of ε, that is, with $\Omega=\Delta=1$, given by the classical form

$$R=R_0\left[\frac{10K_0\sigma\delta_0}{3\mu R_0}\right]^{1/10}. \tag{17}$$

Using the log-log plot in Fig. 8 it is difficult to show any big difference between both curves. However, when plotting the difference between them we see that Eq. (17) underestimates the solution for the drop radius on 4% at times around the hour. The influence of the initial aspect ratio is negligible small at large times.

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[1] C. Treviño, C. Ferro-Fontán, and F. Méndez, Physical Review E **58**, 4478 (1998).
[2] S. Middleman, *Modeling Axisymmetric Flows* (Academic, San Diego, 1995), p. 251.
[3] L. H. Tanner, J. Phys. D **12**, 1473 (1979).
[4] P. G. de Gennes, Rev. Mod. Phys. **57**, 827 (1985).
[5] G. Friz, Z. Angew. Phys. **19**, 374 (1965).
[6] J. D. Chen and N. Wada, J. Colloid Interface Sci. **148**, 207 (1988).
[7] J. A. Diez, R. Gratton, L. Thomas, and B. Marino, Phys. Fluids **6**, 24 (1994).
[8] S. Kalliadasis and H-C. Chang, Phys. Fluids **6**, 12 (1994).
[9] E. S. Mickaily, Ph.D. dissertation, University of California, San Diego (1992).