Low-frequency fluctuations in semiconductor lasers with optical feedback are induced with noise

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Semiconductor lasers with optical feedback present a regime in which irregular dropout events are observed (low-frequency fluctuations). This phenomenon has been interpreted in terms of a purely deterministic model. Recently, it was proposed that these lasers behave as an excitable medium and that the low frequency fluctuations are anticipated by noise. We study analytically and numerically the statistics of the firing processes predicted by this dynamical scenario and discuss how it compares with the results reported in the literature. [S1063-651X(98)02207-7]

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I. INTRODUCTION

The dynamics of semiconductor lasers with optical feedback has been extensively studied [1]. The dynamical behavior observed when these devices are studied can be roughly classified in the following way: (1) Time-independent intensity, (2) low frequency fluctuations (LFF), and (3) coherent collapse regime. The LFF regime is characterized by the existence of irregular pulses at large time intervals (large compared to the typical time scales of the system).

The analysis of the behavior of this system has been mostly carried out in terms of a single mode model that assumes low levels of feedback (the Lang and Kobayashi model). In this model, the dynamical origin of the low frequency fluctuations is of a purely deterministic nature [2,3]. Recently, a different dynamical scenario was proposed to account for the appearance of the LFF. According to this scenario, there is a range in parameter space in which the semiconductor laser with optical feedback behaves as an excitable system, and the noise plays a crucial role in anticipating the LFF [4].

Refuting one of the two interpretation paradigms on the onset of LFF is not trivial. The system presents noise (spontaneous emission) and there are different time scales in the problem, making some of the common strategies proposed from the field of nonlinear dynamics truly hard to implement [5]. On the other hand, there is a natural observable in the measurements: the time interval between dropout events. In the LFF regime, these time intervals are irregular, and therefore a statistical description should be performed. In fact, in a work by Sacher, Elsasser, and Gobel [6] a most interesting experimental result has been reported: the dependence of the average time between dropout events as a function of the parameters of the system. It is then only natural to ask whether the recently proposed dynamical scenario is compatible with the previously reported measurements. Are they compatible within a region of the parameter space? Can an eventual departure be used in order to refute either of the proposed scenarios?

The purpose of this paper is to answer those questions by studying the dynamics of a ''normal form'' of excitable dynamics in the presence of noise and studying the statistics of time intervals between dropout events.

The idea of computing escape rates from a deterministic model driven by noise was already implemented by Henry and Kazarinov, in the context of a particular model [7]. We will build a dynamical picture that is consistent with that idea and enlightens the dynamical nature of the reinjection within a recently proposed framework [4].

This paper is organized as follows. Section II describes the equations under study, and how they give rise to excitable behavior. Section III deals with the statistics of firing events when noise is applied to our dynamical model. Analytical and numerical results are obtained. In Sec. IV we compare our results with the experimental results reported in the literature. Section V contains a summary of our results and our conclusions.

II. EXCITABLE SYSTEMS

A system is characterized as excitable whenever its response to an excitation has the following feature: if the stimulus is larger than a certain threshold value, the response of the system is independent of the size of excitation. This somewhat vague definition allows us to recognize excitability in a wide variety of systems, most remarkably in biology [8].

It is a typical strategy in nonlinear dynamics to find paradigmatical equations, if possible the simplest ones, whose solutions present a desired feature. This allows one to make predictions about the behavior of a particular problem, regardless of its intrinsic details. In order to study excitability, it is usual to study the phase space organization of the flow of a dynamical system close in parameter space to a point in which an Andronov bifurcation takes place. This bifurcation is a saddle-node one in which the unstable manifolds of the saddle are stable manifolds of the node.

A simple model with the desired features is

$$x' = y, \tag{2.1}$$

$$y' = x - y - x^3 + xy + \epsilon_1 + \epsilon_2 x^2, \tag{2.2}$$

with $(x,y) \in \mathbb{R}^2$, and $\epsilon_1,\ \epsilon_2 \in \mathbb{R}^+$.

In Fig. 1 we display the qualitatively different behaviors of the flow in the different regions of the parameter space. Those regions are limited by curves in which either local or

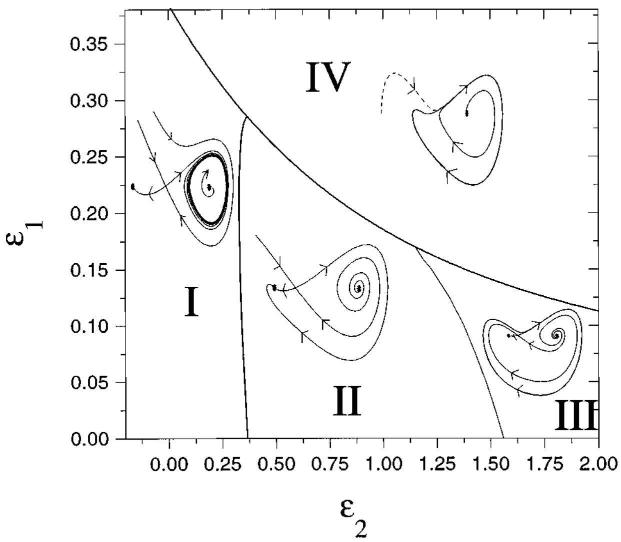

FIG. 1. Bifurcation diagram and phase portraits for the system described by Eqs. (2.1) and (2.2). In regions I, II, and III there are three fixed points: a node, a saddle, and a repulsor. Crossing the separatrix to region IV, the saddle and the attractor collapse. The three lower regions display qualitatively different behavior. In regions I and III the unstable manifold of the saddle approaches a limit cycle. In region II the unstable manifold of the saddle is the stable manifold of the node and the system behaves as an excitable one.

global bifurcations take place. Within region II, the unstable manifold of the saddle is the stable manifold of the node, and the saddle is connected to the repulsive fixed point. There is no limit cycle in this region. Notice that crossing the separatrix in parameter space from region II to region IV implies undergoing an Andronov bifurcation. As mentioned, the system described by Eqs. (2.1)–(2.2) with parameter values within what we have called region II behaves as an excitable one [9]. The response of the system to a perturbation of an initial state located at the node will be independent of the size of the perturbation, provided that it places the system beyond the stable manifold of the saddle. As the unstable manifold of the saddle is attractive, the system will evolve close to it no matter the size of the perturbation.

III. THE STATISTICS OF THE DROPOUT EVENTS

Let us follow the dynamics of an initial condition in the neighborhood of the attractor for parameter values within region II. Under the influence of noise, this state might be eventually taken beyond the stable manifold of the saddle. Associating the large excursion in the phase space along the unstable manifold of the saddle with a dropout event, we recover the behavior described in [4] for the onset of the LFF regime.

We derive an analytic expression for the rate of dropout events in our model, as a function of the parameters, following the seminal work by Kramers [10]. Although his theory was originally developed for the Brownian motion of a particle in a double potential well, we will be able to map our dynamical system to this problem. In Kramers' picture the particle is placed in a potential well and it may escape over the potential barrier due to the action of noise. Its averaged rate of escape is

$$\langle r\rangle = \frac{\sqrt{U''(x_{\min})|U''(x_{\max})|}}{2\pi\eta}\exp\left[-\frac{U(x_{\max})-U(x_{\min})}{D\eta}\right], \tag{3.3}$$

where $x_{\min}$ and $x_{\max}$ are the coordinates of the bottom of the well and the barrier for potential $U(x)$. The viscosity is denoted by η and D stands for the diffusive constant associated to the Brownian motion.

Let us consider the equations (2.1) and (2.2) in the spirit of Kramers' theory. We can associate the escape from the well over the potential barrier with the dropout event in our model. The node corresponds to the bottom of the potential well whereas the barrier is located at the saddle point. Hence, we may define a potential in a neighborhood of the node such as $U(x)=x^4/4-\epsilon_2 x^3/3-x^2/2-\epsilon_1 x$. Moreover, to take into account the nonlinear xy term and the linear damping $-y$ in this region, we introduce an effective viscosity $\eta_{\mathrm{eff}}=1-x_{\min}$. This choice is justified since we are taking an average of the x variable within the potential well. With the potential given above and adding a noise term, Eqs. (2.1) and (2.2) are completely analogous to the equations of motion for a particle undergoing a Brownian motion in a potential well. The noise term is taken with zero mean and variance $2D$.

Now we derive an analytic expression for the Kramers' rate formula near the separatrix in parameter space. We fix ϵ_2 and study the dependence of the mean rate on ϵ_1 and D. To this end we define: $f(x)=x^3-\epsilon_2 x^2-x$. From Eqs. (2.1) and (2.2), we obtain the coordinates of the potential well and the barrier as the two roots of $f(x)-\epsilon_1=0$ of smaller value, while the saddle node curve corresponds to the double root at $x=x_0$. As we are slightly below the saddle node separatrix we can approximate $f(x)\approx f(x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2$. This allows us to express the coordinates of the potential well and the barrier as a function of x_0, $f(x_0)$, and $f''(x_0)$ (note that these are explicit functions of ϵ_2 only).

Using that $\epsilon_1-f(x_0)\ll f''(x_0)$ near the separatrix and neglecting terms of second order, the form in the Kramers equation (3.3) becomes

$$\langle r\rangle(\epsilon_1)=\frac{\sqrt{2f''(x_0)[\epsilon_1-f(x_0)]}}{2\pi\eta_{\mathrm{eff}}(\epsilon_1)} \times\exp\left[-\frac{4[f(x_0)-\epsilon_1]^{3/2}}{3D\eta_{\mathrm{eff}}(\epsilon_1)|f''(x_0)/2|^{1/2}}\right], \tag{3.4}$$

$$\eta_{\mathrm{eff}}(\epsilon_1)=1-x_0+\sqrt{2[\epsilon_1-f(x_0)]/f''(x_0)}. \tag{3.5}$$

In order to obtain results for our excitable system, Eqs. (2.1) and (2.2) were integrated numerically. After each integration step, the x variable was perturbed by a noise term $\sqrt{2D\delta t}\phi$, with δt the time step and ϕ a Gaussian white noise with zero mean and variance one. For a given value of D and taking $\epsilon_2=0.7$, we simulated our system in the way described above for various values of ϵ_1. In Fig. 2 we compare, in a double logarithmic plot, the mean period of the dropout events $\langle T\rangle$ obtained numerically with the analytical result in Eq. (3.4)–(3.5), for different values of D. The mean period was obtained by averaging up to 10 000 dropout events. It is clear that the theoretically predicted mean period

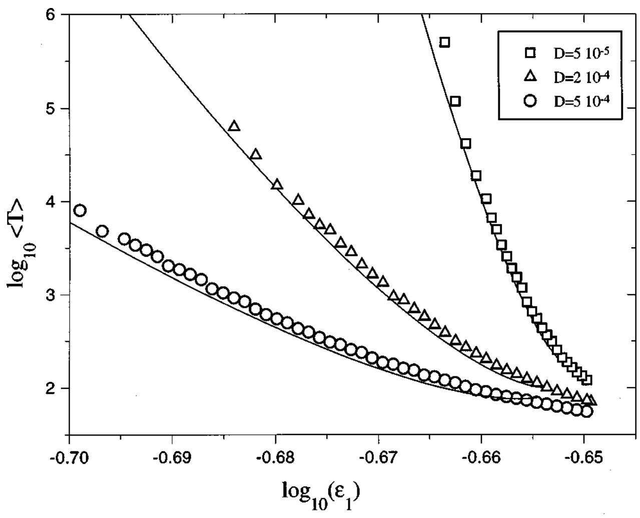

FIG. 2. Average time between dropout events $\langle T\rangle$ plotted against ϵ_1 in a log-log scale for three different noise levels and $\epsilon_2 = 0.7$. The symbols represent numerical data and the solid curves are the analytical results derived in Eqs. (3.4)–(3.5) from Kramers' rate theory. The values of D are chosen within the validity range of Kramers' formula (3.3).

fits the data extremely well. As expected, if we move towards the lower values of ϵ_1, the mean period rises faster as the level of noise decreases.

IV. COMPARISON WITH THE RESULTS REPORTED IN THE LITERATURE

The parameter dependence of the time intervals between dropout events in semiconductor lasers with optical feedback was reported for the first time in 1989 by Sacher, Elssaser, and Gobel. They fitted their results with the following expression: $\langle T\rangle \sim (I - I_{\mathrm{threshold}}/I_{\mathrm{threshold}})^{-\lambda}$, where $I_{\mathrm{threshold}}$ stands for the threshold of the solitary laser (measured) and λ is a fitted parameter (in their reported observations, $\lambda = -1.05$). In principle, it seems that there is a serious disagreement between the functional form of the $\langle T\rangle$ dependence with the parameters derived under the excitability scenario and the fitted function reported in [6]. Indeed, in the dynamical picture under study, it would not be appropriate to include a threshold value for the $I = \mathrm{const}$ regime, since noise can always trigger dropout events for a sufficiently long observation time. Yet, we will show that finite measurements give rise to compatible results.

We integrated our dynamical model in the presence of noise for different amounts of time. Once a total integration time was fixed, we searched for the lowest ϵ_1 value in which dropout events were found. Several numerical integrations were performed (for a fixed value of the total time of integration) in order to estimate this ''virtual threshold'' value. In other words, we are dealing with a dynamical scenario in which the threshold is not well defined, but we are trying to reproduce an experimental observation (the existence of a threshold value) as a consequence of a finite observation time.

Now we are able to try to reproduce the reported observations [6] in the framework of our dynamical scenario. For a finite integration time, the ''virtual threshold'' is a well

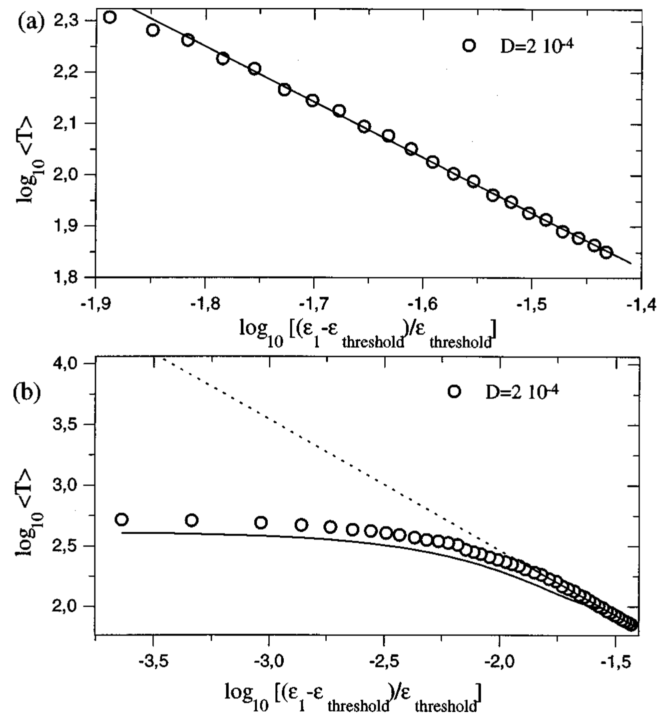

FIG. 3. Average time between dropout events $\langle T\rangle$ as a function of the normalized parameter $(\epsilon_1 - \epsilon_{\mathrm{threshold}}/\epsilon_{\mathrm{threshold}})$ in a double logarithmic plot. (a) Power-law scaling region with slope $\lambda = -1.079$. The arrow indicates the parameter value at which the Andronov bifurcation takes place. The left side corresponds to the region II in Fig. 2. (b) Departure from the power-law dependence at lower values of ϵ_1. The solid curve is the analytic result (3.4)–(3.5).

defined quantity, and we might want to display and fit the $\langle T\rangle$ dependence with the model parameters following the analytical expression derived in the previous section. In Fig. 3(a) we display the result for a specific integration time. We also found $\lambda = -1.079$, with $\chi^2 = 2\times 10^{-5}$. This dependence holds beyond the Andronov bifurcation (region IV in Fig. 1), which also corresponds to the LFF regime.

In Fig. 3(b) we display the $\langle T\rangle$ dependence with the parameters as predicted by our analytical results in the same scale in which, for finite integration times, we fitted a power law. Clearly, there is a departure between those curves. It is suggestive that a departure from the power law, qualitatively similar to the one found in our simulations and analytic calculations has been presented in the literature [11,12].

V. CONCLUSIONS

In this work, we studied a dynamical scenario recently proposed by Giudici [4] to account for the onset of LFF in semiconductor lasers with optical feedback. It has been claimed that these systems can behave as excitable media. Under that hypothesis, noise can eventually trigger large excursions in phase space. We derived an expression for the parameter dependence of the rate of excursion for a simple model of this dynamical scenario, and showed that these results are not incompatible with previously reported measurements. These pointed to the existence of a power law relating the average time interval between dropouts and a range of properly scaled parameters. The excitability scenario implies a dependence of the rate of events with the parameters that is given by the product of rational functions and exponentials: we showed that there is a range in parameter space close to the ones in which the Andronov bifurcation takes place in which the power law is a good approximation, but a departure when the events are rare (i.e., when the saddle and the node are well apart) is predicted. This departure has been measured and reported in [12], and therefore those measurements are consistent with the claims in [4] and the predictions in [7].

Finally, the functional form for the departure will depend on the details of the underlying model, but following Kramers it is clear that the excitability hypothesis explains in an elegant and simple way why the power law is bounded to be valid only within a region of the parameter space.

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