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Some Remarks about Compactly Supported Spline Wavelets

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In this paper we propose an extended family of almost orthogonal spline wavelets with compact support. These functions provide snug bases for $L^2(\mathscr{R})$, preserving semiorthogonal properties. As it is well known, orthogonality is a desirable quality while finite support has attractive features for numerical applications. This work represents an effort to combine these conditions in the spline case and to enhance previous results of Chui and Unser *et al.* We start by reviewing the concept of semiorthogonal wavelets and we discuss their performance. Next, we give a brief description of the general technique for computing compactly supported spline wavelets. Finally we expose these functions. We also develop several formulas in accord with our purposes. © 1996 Academic Press, Inc.

1. INTRODUCTION

Polynomial spline wavelets have played a significant role in the development of the Wavelet Transform Theory. They now provide powerful tools for many applications in Functional and Harmonic Analysis, Numerical Analysis, and several branches of applied sciences. An extensive literature about these topics can be found in the bibliography. We can mention the works of Battle [2], Lemarié [8], Meyer [12–14], Mallat [9, 10], Daubechies [5, 6], Chui [4], and Unser *et al.* [19–21]. For more about splines we refer to Schoenberg [18], Chui [3], Powell [17], Ahberg *et al.* [1] and Wahba [23].

Depending on the application or problem at hand, different choices for the basic spline wavelet are available. Orthonormal wavelets, proposed by Battle [2] and Lemarié in [8], provide a suitable representation and the direct estimation of the L^2 norm for any analyzed function. Also orthogonality is a desirable property in theoretical applications. However, although these wavelets have exponential decay, they are infinitely supported. This is an important problem in practice because the Battle–Lemarié filters cannot be implemented exactly. For this reason, we must approximate them in computational applications. The traditional approach is to simply truncate these wavelet filters. However, the filter truncation, such as the one used by Mallat [9], may present some problems because the underlying basis functions are not necessarily splines any more. Then, the convergence of the iterated filter band to a continuous function is no longer guaranteed.

On the other hand, nonorthogonal spline wavelets with minimal compact support have been recently proposed by Chui [4] and Unser *et al.* [19]. These elementary functions have attractive features for signal processing implementations. Their remarkable advantage consists in the use of finite discrete filters for computing the wavelet coefficients. For this purpose, the discrete convolutions with the signal data are computed with a generalized Mallat's scheme [9, 19]. However, nonorthogonality has its cost. Some distortion in the primal information appears, just as the image in an oblique mirror. Then, we need to use the dual wavelets, as the complementary mirror, to obtain the exact reconstruction or to evaluate the signal's energy. In consequence, the computational efforts increase. Moreover, the dual wavelets are not compactly supported and they might be badly localized in the time–frequency domain [13].

To overcome these problems, here we propose an extended family of almost orthogonal spline wavelets with compact support. These elementary functions allow us to combine, in a suitable way, both desirable properties. They are obtained by appropriate linear combinations of the compactly supported β-spline wavelets, and we approach orthogonality as the support increases. Moreover, since these wavelets are true splines, there are no smoothness or convergence problems.

The outline of this paper is as follows. In Section II we review the concept of semiorthogonal wavelets and give several considerations about their performance. We also propose there a formula for estimating the frame bounds, associated with semiorthogonal spline wavelet bases. In Section III, we summarize the general technique for com-

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puting compactly supported spline wavelets. Finally, in Section IV we propose a sequence of almost orthonormal wavelets with compact support converging on the orthonormal wavelets. Besides, we obtain a formula for computing the involved coefficients, and we expose for illustration purposess several cubic spline wavelets.

Nomenclature

$\mathscr{R}$	Set of real numbers.
$\mathscr{Z}$	Set of integer numbers.
$\mathscr{N}$	Set of positive integer numbers.
$L^2(\mathscr{R})$	Hilbert Space of square integrable functions.
V_{m0}	Hilbert Space of polynomial spline functions of order m and knots in $\mathscr{Z}$, belonging to $L^2(\mathscr{R})$.
$\mathscr{V}_m$	Multiresolution Analysis of $L^2(\mathscr{R})$. (generated by V_{m0})
$\langle\cdot,\cdot\rangle$	Inner product in $L^2(\mathscr{R})$.
$\|\cdot\|$	Norm in $L^2(\mathscr{R})$.
$\hat{f}$	Fourier Transform of f.
$\tilde{f}$	Dual function associated to f.
$\chi(x)$	Characteristic function of [0, 1).
$*$	Convolution operator.

2. SEMIORTHOGONAL SPLINE WAVELETS

We denote $\mathscr{V}_m = \{V_{mj}, j \in \mathscr{Z}\}$ the Multiresolution Analysis associated with the *polynomial spline wavelets* of order $m, m \in \mathscr{N}$ and by W_{mj} the orthogonal complement of V_{mj} in $V_{m(j+1)}$ [10, 14]. The fundamental subspace V_{m0} consists of all functions of $L^2(\mathscr{R})$ which are $m-1$ times continuously differentiable and equal to a polynomial of degree m on each interval $[k, k+1)$. As it is well known, there exists an unique function $\psi_m \in W_{m0}$, centered on $x_0 = 0.5$, with exponential decay and minimum dispersion [12], such that the family

$$\psi_{mjk}(x) = 2^{j/2}\psi_m(2^j x - k) \quad j,k \in \mathscr{Z} \tag{1}$$

constitutes an orthonormal basis for $L^2(\mathscr{R})$. These functions are called *orthonormal wavelets.*

Apart from orthonormal bases, we have other alternatives (see [4, 5, 7, 22]).

Here we consider only semiorthogonal bases. Let us recall that a function $\varphi_m \in W_{m0}$ is called *semiorthogonal wavelet* if the family $\{\varphi_m(x-k), k \in \mathscr{Z}\}$ constitutes an unconditional basis for W_{m0} [12]. From this definition it follows that

$$\langle\varphi_m(2^j x - k), \varphi_m(2^l x - n)\rangle = 0 \quad \text{if } j \neq l \tag{2}$$

and the complete family

$$\varphi_{mjk}(x) = 2^{j/2}\varphi_m(2^j x - k) \quad j,k \in \mathscr{Z} \tag{3}$$

is an unconditional basis for $L^2(\mathscr{R})$.

Associated with φ_m there exists a *dual function* $\tilde{\varphi}_m \in W_{m0}$ such that the family $\{\tilde{\varphi}_m(x-k), k \in \mathscr{Z}\}$ is an unconditional basis for this subspace and the following biorthonormal condition holds [4, 7]:

$$\langle\varphi_m(x-k), \tilde{\varphi}_m(x-n)\rangle = \delta_{kn}. \tag{4}$$

Note that $\tilde{\varphi}_m$ is also semiorthogonal. Given $f \in L^2(\mathscr{R})$ we have the well known formulas

$$Q_j f = \sum_k \langle f, \tilde{\varphi}_{mjk}\rangle\varphi_{mjk} = \sum_k \langle f, \varphi_{mjk}\rangle\tilde{\varphi}_{mjk} \tag{5}$$

$$\|Q_j f\|^2 = \sum_k \langle f, \varphi_{mjk}\rangle\langle f, \tilde{\varphi}_{mjk}\rangle, \tag{6}$$

where $Q_j f$ is the orthonormal projection of f onto W_{mj} and $\tilde{\varphi}_{mjk} = 2^{j/2}\tilde{\varphi}_m(2^j x - k)$.

As we have already mentioned, semiorthogonal spline wavelets with minimal compact support are proposed in the literature. At this point let us make some discussion about the performance of these bases. As the function φ_m generates by translations in $\mathscr{Z}$ and unconditional basis for W_{m0} there exist positive numbers $0 < A \leqslant B < \infty$, such that [4, 13 14]:

$$A\|h\|^2 \leqslant \sum_k |\langle h, \varphi_{m0k}\rangle|^2 \leqslant B\|h\|^2, \quad \text{for all } h \in W_{m0} \tag{7}$$

and, since the family $\{\varphi_m(x-k), k \in \mathscr{Z}\}$ is complete in W_{m0}, this condition is equivalent to:

$$A \leqslant \sum_k |\hat{\varphi}_m(\omega + 2k\pi)|^2 \leqslant B. \tag{8}$$

The *frame bounds* A and B give us information about the performance of the semiorthogonal wavelet basis. More precisely, we can consider the ratio $r = B/A - 1$ as a measure of nonorthogonality, or *slope* of the basis. Assuming that $\|\varphi_m\| = 1$ it is easy to show that

$$A \leqslant 1 \leqslant B \tag{9}$$

$$\|\varphi_m - \tilde{\varphi}_m\|^2 \leqslant r \tag{10}$$

$$\sum_{k\neq 0} |\langle\varphi_m(x), \varphi_m(x-k)\rangle|^2 \leqslant r, \tag{11}$$

and for any $h \in W_{m0}$ we have [6]

$$h(x) = \frac{2}{A+B}\sum_k \langle h(x), \varphi_m(x-k)\rangle\varphi_m(x) + \Delta h(x) \tag{12}$$

with $\|\Delta h\| \leqslant r/(r+2)\|h\|$.

Daubechies also shows in [6, Theorem 3.5.1] that the ratio $\sqrt{B/A}$ plays a significant role when one considers the wavelets φ_m as time–frequency location operators.

We conclude that if $r \ll 1$ the elementary functions φ_m can be considered as orthonormal wavelets for practical purposes. As we will see below, the semiorthogonal wavelets with minimal support do not verify this desirable property.

Next we will obtain an efficient tool in order to characterize any semiorthogonal spline wavelet and to estimate the associated frame bounds. For this purpose we consider the $\mathcal{B}$-*spline* of degree m defined as [18]:

$$\beta_m(x) = \underbrace{\chi *, \ldots, * \chi}_{m+1 \text{ times}}(x). \tag{13}$$

We recall some remarkable properties of these functions [4, 18]:

- the family $\{\beta_m(x-k) k \in \mathcal{Z}\}$ is an unconditional basis for V_{m0}.
- β_m is supported on $[0; m+1]$ and centered on $x_c = (m+1)/2$. Moreover it is a symmetric and positive function in this interval.
- the following *two scaling equation* holds

$$\beta_m(x/2) = \sum_k h_m(k)\beta_m(x-k), \tag{14}$$

where the sequence $h_m \in l^1(\mathcal{Z})$ verifies

$$h_m(k) \begin{cases} > 0 & \text{if } 0 \leqslant k \leqslant m+1 \\ = 0 & \text{otherwise} \end{cases} \tag{15}$$

$$\sum_k h_m(k) = 2 \tag{16}$$

$$\sum_k h_m(2k) = \sum_k h_m(2k+1). \tag{17}$$

- the trigonometric polynomials

$$P_{2m+1}(\omega) = b_m(0) + 2\sum_{n=1}^{m} b_m(n)\cos n\omega \tag{18}$$

where $b_m(n) = \langle \beta_m(x), \beta_m(x-n)\rangle$, are positive and 2π-periodic functions. Moreover, $b_m(n) = \beta_{2m+1}(n+m+1)$.

- if m is odd, the trigonometric polynomials:

$$H_m(\omega) = h_m\left(\frac{m+1}{2}\right) + 2\sum_{n=(m+3)/2}^{m+1} h_m(n)\cos n\omega \tag{19}$$

are 2π-periodic functions, they do not vanish for $-\pi < \omega < \pi$ and $H_m(\pi) = 0$.

Now consider any compactly supported function $p \in W_{m0}$. Since $W_{m0} \subset V_{m1}$ we can write

$$\rho(x) = \sum_n r(n)\beta(2x-n) \tag{20}$$

for a certain finite sequence r. Then the Fourier transform is given by

$$\hat{\rho}(\omega) = R(\omega)\left(\frac{\sin w/4}{\omega/4}\right)^{m+1} e^{-i(m+1)\omega/4} \tag{21}$$

where

$$R(\omega) = \frac{1}{2}\sum_n r(n)e^{-i\omega n/2}, \tag{22}$$

and we can derive the formula:

PROPOSITION 1.

$$\sum_k |\hat{\rho}(\omega+2k\pi)|^2 = |R(\omega)|^2 P_{2m+1}(\omega/2) + |R(\omega+2\pi)|^2 P_{2m+1}(\omega/2+\pi). \tag{23}$$

Proof. Since

$$|\hat{\rho}(\omega)|^2 = |R(\omega)|^2\left(\frac{\sin\omega/4}{\omega/4}\right)^{2m+2} \tag{24}$$

for all ω, and $R(\omega)$ is a 4π-periodic function, we can write:

$$\sum_k |\hat{\rho}(\omega+4k\pi)|^2 = |R(\omega)|^2\left(\frac{\sin\omega/4}{\omega/4}\right)^{2m+2} \times \sum_k\left(\frac{\omega}{\omega+4k\pi}\right)^{2m+2}. \tag{25}$$

On the other hand we have the remarkable identity

$$\left(\sum_k\left(\frac{\omega}{\omega+2k\pi}\right)^{2m+2}\right)^{-1} = \left(\frac{\sin\omega/2}{\omega/2}\right)^{2m+2} P_{2m+1}^{-1}(\omega), \tag{26}$$

taking these equalities in the sense of the limit when $\omega = 2n\pi$. Then we rewrite (25) as:

$$\sum_k |\hat{\rho}(\omega+4k\pi)|^2 = |R(\omega)|^2 P_{2m+1}(\omega/2). \tag{27}$$

Now, recalling that $P_{2m+1}(\omega)$ is a 2π-periodic function we complete the proof writing:

$$\sum_k |\hat{\rho}(\omega+2k\pi)|^2 = \sum_k |\hat{\rho}(\omega+4k\pi)|^2 + \sum_k |\hat{\rho}((\omega+2\pi)+4k\pi)|^2. \quad \blacksquare \tag{28}$$

Since P_{2m+1} is a positive function, we immediately deduce that condition (8) is equivalent to:

$$c_1 \leqslant |R(\omega)|^2 + |R(\omega+2\pi)|^2 \leqslant c_2 \tag{29}$$

for all $\omega \in [0, 2\pi]$ and for some positive numbers c_1, c_2.

Remark. Given a closed subspace $H \subset L^2(\mathcal{R})$ and $f \in H$, condition (8) by itself does not imply that the family $\{f(x-k), k \in \mathcal{Z}\}$ is a frame for H [14].

In our case, under the hypothesis $p \in W_{m0}$, (8) also implies that the family $\{\rho(x-k), k \in \mathcal{Z}\}$ is dense in W_{m0} [4, 7, 13].

Using these results, one can easily estimate the values A, B or decide if any function in W_{m0} is a semiorthogonal wavelet. The associated frame bounds A, B are given by the infimum and the supremum, respectively, in formula (23) for $\omega \in [0, 2\pi]$. We will employ this technique in the next sections.

3. COMPACTLY SUPPORTED SPLINE WAVELETS

We give here a brief description of the technique for computing polynomial spline wavelets with compact support. A rigorous treatment of this matter can be found in the literature above mentioned (see [4, 19]).

Given $m \in \mathcal{N}, m$ odd, we look for a compactly supported function $\varphi_m \in \mathcal{W}_{m0}$ such that the family of its translations in $\mathcal{Z}$ constitutes an unconditional basis for this subspace.

(a1) *Representation.*

$$\varphi_m(x) = \sum_n a_m(n)\beta_m(2x - m) \tag{30}$$

for a finite sequence a.

(a2) *Orthogonality.*

$$\langle \varphi_m(x), \beta_m(x - l)\rangle = 0 \quad \text{for all } l \in \mathcal{Z} \tag{31}$$

(a3) *Frame Bounds.* There are positive numbers A, B such that:

$$A \leqslant \sum_k |\hat{\varphi}_m(\omega + 2k\pi)|^2 \leqslant B. \tag{32}$$

Now denoting the discrete convolution $h_m * b_m$ as

$$q_m(n) = \sum_{k=0}^{m+1} h_m(k) b_m(n - k) \tag{33}$$

and using (30), we rewrite condition (31) as

$$\sum_n a_m(n) q_m(n - 2l) = 0 \quad \text{for all } l \in \mathcal{Z}. \tag{34}$$

A collection $\{a_m^{(i)}, i \in \mathcal{Z}\}$ of nontrivial solutions for this difference equation is given by:

$$a_m^{(i)}(n) = (-1)^n q_m(1 - n + 2i), \quad n \in \mathcal{Z}. \tag{35}$$

One can easily verify that these basic solutions $a_m^{(i)}$ satisfy:

- They are linearly independent sequences in $l^2(\mathcal{Z})$.
- They have zero mean, that is:

$$\sum_n a_m^{(i)}(n) = 0 \quad \text{for all } i. \tag{36}$$

- $a_m^{(i)}(n)$ is non-zero if and only if $-2m + i \leqslant n \leqslant m + 1 + 2i$. Moreover, there is not any nontrivial solution with shorter length.

We remark that the last property follows from the fact that the $\mathcal{B}$-spline functions have compact support (e.g., (13)–(15)) and consequently the sequences q_m are nonzero if and only if $-m \leqslant n \leqslant 2m + 1$. We also remark that any finite linear combination of the basic solutions $a_m^{(i)}$ is also a finite solution for Eq. (34).

Next, with the choice $i = 0$, we compute:

$$\varphi_m(x) = \sum_{n=-2m}^{m+1} a_m^{(0)}(n)\beta_m(2x - n). \tag{37}$$

Clearly the function φ_m satisfies conditions (a1) and (a2). It is supported on $[-m, m + 1]$, centered on $x = \frac{1}{2}$ and it has minimal support. In order to check condition (a3), we recall formula (29). Since:

$$\hat{\varphi}_m(\omega) = A_m(\omega) \cdot \hat{\beta}_m(\omega/2) \tag{38}$$

with

$$A_m(\omega) = \frac{1}{2} \sum_n a_m^{(0)} e^{-in\omega/2}, \tag{39}$$

then it is enough to observe that the Z-Transform of the sequence q_m

$$\hat{q}_m(z) = \hat{h}_m(z)\hat{b}_m(z) \tag{40}$$

verifies, following (18), (19), and (33)

$$\hat{q}_m(z^{1/2}) + \hat{q}_m(-z^{1/2}) > 0 \quad \text{for } |z| = 1, \tag{41}$$

and the last condition implies

$$c_1 \leqslant |A_m(\omega)|^2 + |A_m(\omega + 2\pi)|^2 \leqslant c_2 \quad \text{for all } \omega \in [0, 2\pi] \tag{42}$$

and for $0 < c_1 \leqslant c_2 < \infty$. Therefore, we conclude that the function φ_m, above defined, is a semiorthogonal wavelet with minimal compact support.

Let us give an example. For $m = 3$ we compute

$$\begin{aligned}
a(n) &= 0 & n &< -6\\
a(-6) &= a(4) & &= -1/40320\\
a(-5) &= a(3) & &= 124/40320\\
a(-4) &= a(2) & &= -1677/40320\\
a(-3) &= a(1) & &= 7904/40320\\
a(-2) &= a(0) & &= -18482/40320\\
a(-1) &= 24264/40320 & &\\
a(n) &= 0 & n &> 4
\end{aligned}$$

and the well-known semiorthogonal cubic spline wavelet given in [19] is obtained (Fig. 1a). Now, using formula (23) we obtain, after normalizing the wavelet:

$$\begin{aligned}
A &= 0.242\\
B &= 1.429\\
r &= 4.903.
\end{aligned}$$

TABLE 1
Almost Orthogonal Cubic Spline Wavelets

N	Support	A	B	r	Notes
0	[−3, 4]	0.242	1.429	4.903	Fig. 1
2	[−5, 6]	0.816	1.218	0.491	—
4	[−7, 8]	0.950	1.048	0.103	Fig. 2
6	[−9, 10]	0.978	1.022	0.044	—
8	[−11, 12]	0.982	1.015	0.034	Fig. 3
21	[−24, 25]	0.999^+	1.000^+	0.000^+	—

Clearly, the function φ_m cannot be considered an almost orthogonal wavelet. Moreover, note that its Fourier transform (Fig. 1b) hardly seems to be a band pass filter. Also, note that the wavelet φ_m resembles a modulated Gaussian function [19]. We refer to Daubechies [6] for useful comments in this point.

4. ALMOST ORTHOGONAL SPLINE WAVELETS

An extended family of semiorthogonal spline wavelets in W_{m0}, with compact support can be obtained by appropriate linear combinations of the basic sequences $a_m^{(i)}$ defined in (35). One can design the adequate analyzing wavelet according to the application or problem on hand or any desirable pattern.

Here we look for symmetric almost orthogonal functions. To determine the coefficients in (30) for these purposes seems a hard problem; therefore we will improve an alternative technique.

Assume that $m \in \mathcal{N}$ is an odd number. Since the orthonormal spline wavelet $\psi_m \in W_{0m}$ can be represented as

$$\psi_m(x) = \sum_k c_m(k)\varphi_m(x-k) \tag{43}$$

where φ_m is the wavelet defined in (37), $\tilde{\varphi}_m$ the associated dual function of $c_m(k) = \langle \psi_m(x), \tilde{\varphi}_m(x-k)\rangle$, we can define the sequence in W_{m0}

$$\psi_m^N(x) = \sum_{k=-N}^{N} c_m(k)\varphi_m(x-k). \tag{44}$$

Note that these functions have compact support and the orthonormal wavelet is approached as N increases. Let us formalize these properties:

PROPOSITION 2. *The functions defined by* (44) *verify:*

- *(a)* ψ_m^N *is supported on the interval* $[-(N+m), N+m+1]$, *is centered on* $x = \frac{1}{2}$ *and it is symmetric.*
- *(b) For* $N > m$, $\psi_m^N(x) = \psi_m(x)$ *for all* $x \in [m-N, N-m+1]$.
- *(c)* $\lim_{N\to\infty} \|\psi_m^N - \psi_m\| = 0$.
- *(d)* $\lim_{N\to\infty} \psi_m^N(x) = \psi_m(x)$ *pointwise.*

Proof. Recall that φ_m is supported on $[-m, m+1]$, centered on $x = \frac{1}{2}$ and symmetric. Moreover, using formula (5), with $f = \tilde{\varphi}_m$ and $j = 0$, we deduce that the dual function is also symmetric. Then the properties (a) can be easily checked.

Now, observe that the values of ψ_m in the interval $[m-N, N-m+1]$ are determined only for the functions $\varphi_m(x-k)$ for $-N \leqslant k \leqslant N$, hence it agrees with ψ_m^N in this interval. This proves (b).

The convergence (c), clearly, follows immediately from (43) and (44). Finally, the property (d) can be easily deduced from (b). ■

Remark. Since ψ_m^N has compact support and ψ_m has exponential decay, it follows that the L^2-error in (c), also has exponential decay.

Next we will obtain an explicit formula for computing the coefficients c_m. According to Chui [4], we have

$$\hat{\psi}_m(\omega) = \left(\frac{\sin \omega/4}{\omega/4}\right)^{m+1} S_m^{1/2}(\omega)e^{-i\omega/2} \tag{45}$$

where

$$S_m(\omega) = P_{2m+1}^{-1}(\omega/2) - P_{2m+1}^{-1}(\omega)\cos^{2m+2}(\omega/4) \tag{46}$$

with the trigonometric polynomial P_{2m+1} already defined in (18). On the other hand, denoting by

$$G_m(\omega) = \sum_{k=-2m-1}^{2m+1} \langle \varphi_m(x)\varphi_m(x-k)\rangle e^{-ik\omega} \tag{47}$$

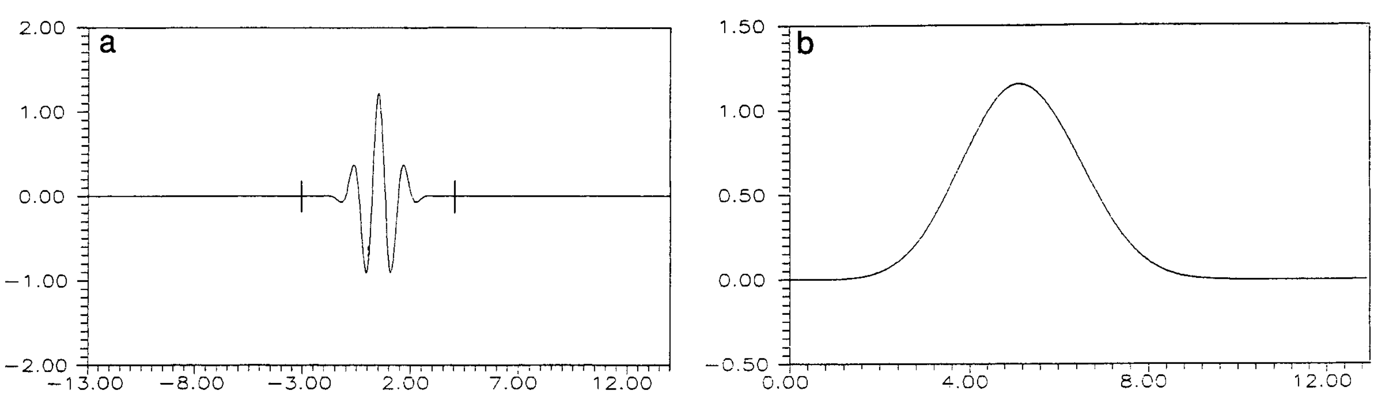

FIG. 1. Spline wavelets: (a) $\psi_3^0(x)$; (b) $|\hat{\psi}_3^0(\omega)|$.

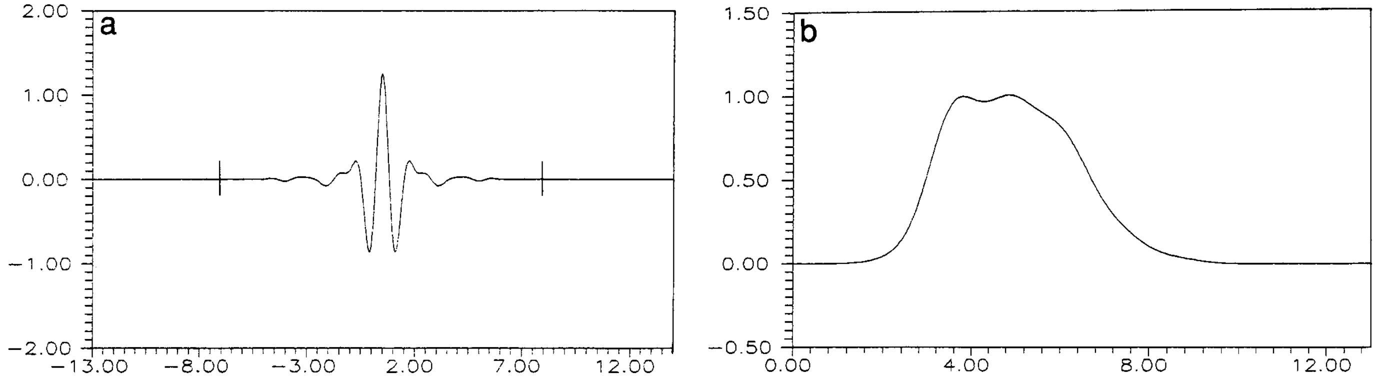

FIG. 2. Spline wavelets: (a) $\psi_3^4(x)$; (b) $|\hat{\psi}_3^4(\omega)|$.

and recalling the trigonometric polynomial A_m defined in (39), we can obtain the following expression for the dual wavelets $\tilde{\varphi}_m$

$$\hat{\tilde{\varphi}}_m(\omega) = \left(\frac{\sin \omega/4}{\omega/4}\right)^{m+1} \frac{A_m(\omega)}{G_m(\omega)} e^{-i(m+1)\omega/4}. \tag{48}$$

Note that the coefficients $a_m^{(0)}$ are centered on $(1-m)/2$ and that they are symmetric. Then we define

$$F_m(\omega) = A_m(\omega) e^{-i(m-1)\omega/4} \tag{49}$$

and we rewrite (48) as

$$\hat{\tilde{\varphi}}_m(\omega) = \left(\frac{\sin \omega/4}{\omega/4}\right)^{m+1} \frac{F_m(\omega)}{G_m(\omega)} e^{-i\omega/2}. \tag{50}$$

We remark that G_m is a trigonometric polynomial, symmetric and 2π-periodic. On the other hand, we have

$$F_m(\omega) = \frac{1}{2} a_m^{(0)}\left(\frac{1-m}{2}\right) + \sum_{(3-m)/2}^{m+1} a_m^{(0)}(n) \cos\left(\frac{m-1}{2} + n\right)\omega/2, \tag{51}$$

that is, a symmetric, 4π-periodic trigonometric polynomial.

Now we can formulate:

PROPOSITION 3.

$$c_m(k) = \frac{1}{\pi}\int_0^{2\pi} \frac{F_m(\omega)}{G_m(\omega)} S_m^{1/2}(\omega) P_{2m+1}(\omega/2) \cos k\omega \, d\omega. \tag{52}$$

Proof. Using (48), we write:

$$\begin{aligned} c_m(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\tilde{\varphi}}_m(\omega) \bar{\hat{\psi}}_m(\omega) e^{-ik\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{F_m(\omega)}{G_m(\omega)} \left(\frac{\sin \omega/4}{\omega/4}\right)^{2m+2} S_m^{1/2}(\omega) e^{-ik\omega} d\omega. \end{aligned} \tag{53}$$

Next we need again the relations:

$$\left(\frac{\sin(\omega + 4k\pi)/4}{(\omega + 4k\pi)/4}\right)^{2m+2} = \left(\frac{\sin \omega/4}{\omega/4}\right)^{2m+2} \left(\frac{\omega}{\omega + 4k\pi}\right)^{2m+2} \tag{54}$$

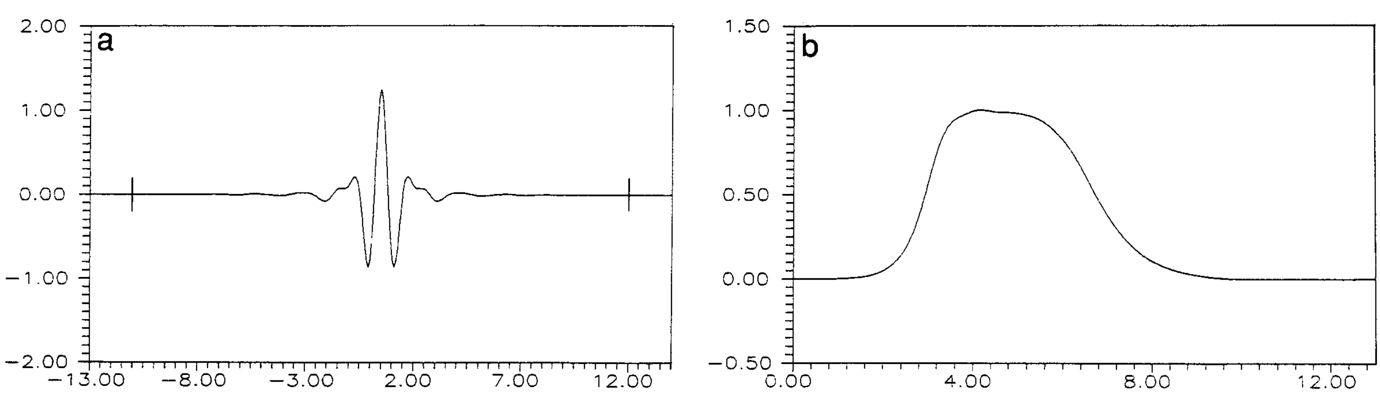

FIG. 3. Spline wavelets: (a) $\psi_3^8(x)$; (b) $|\hat{\psi}_3^8(\omega)|$.

and

$$\sum_k \left(\frac{\omega}{\omega+2\pi}\right)^{2m+2} = \left(\frac{\sin\omega/2}{\omega/2}\right)^{-(2m+2)} P_{2m+1}(\omega) \quad (55)$$

for all $\omega \in [-\pi, \pi]$.

Now, recalling that F_m is 4π-periodic and symmetric, and P_{2m+1}, G_m are 2π-periodic and symmetric functions, after some algebraic manipulations in (53) we obtain formula (52). ■

Remark. Note that, owing to the symmetric conditions of the involved functions, the kernel in the formula (52) is a function of cosines. The integral can be computed efficiently by an appropriate numerical method.

Analogous formulas for computing spline wavelets, dual functions or associated discrete filters, can be obtained by similar procedures.

Next we will exhibit some almost orthogonal wavelets. For $m = 3$, we compute cubic spline wavelets in $W_{3,0}$ as it is showed in Table 1. The associated frame bounds are estimated by using formula (23). Note that for relatively low values of N, a good performance in the ratio r is achieved, without increasing excessively the original minimal support. In the table, we assume $\psi_3^0 = \varphi_3/\|\varphi_3\|$. In Figs. 1–3 we show some of the computed wavelets and their Fourier transforms.

Remark. Using an analogous technique, one can compute almost orthogonal scaling functions ϕ_m^N, with compact support, such that the family of translations in $\mathscr{Z}$ constitutes an unconditional basis for V_{m0}. Then, Mallat's algorithm can be used with the pair of finite discrete filters [9]:

$$h_m^N(k) = \frac{1}{2}\langle \phi_m^N(x/2), \phi_m^N(x-k)\rangle \quad (56)$$

$$g_m^N(k) = \frac{1}{2}\langle \psi_m^N(x/2), \phi_m^N(x-k)\rangle. \quad (57)$$

Assume, as usual, that we start the analysis with a compactly supported signal $s \in V_{m0}$, given by a finite representation

$$s(x) = \sum_k s(k)\phi_m^N(x-k) \quad (58)$$

and we compute the recursive scheme for $j = -1, \ldots, j_{\min}$. Then we obtain almost orthogonal projections $P_j^N s$ and $Q_j^N s$ onto V_{mj} and W_{mj}, respectively. These are mutually orthogonal functions, with compact support, and they approach the exact projections. Note that some information is lost in each step

$$P_j^N s \oplus Q_j^N s = P_{j+1}^N s + \Delta_{j+1} s \quad (59)$$

but the global error can be neglected, owing to formula (12).

In sum, the filters h_m^N and g_m^N lead to a decomposition of the signal (58) in orthogonal and compactly supported projections, but this scheme only gives us almost perfect reconstruction. Of course, for higher precision one can implement in each step, a corrective method, just as Meyer [14] or Daubechies [6] has proposed. A deep discussion of these questions is beyond our present purposes.

At this point, we also remark that the dual functions $\tilde{\phi}_m^N$ and $\tilde{\psi}_m^N$ are not compactly supported.

5. CONCLUSION

As it is well known, there is not any regular, symmetric and compactly supported wavelet, such that it generates an orthonormal basis for $L^2(\mathscr{R})$. These desirable properties are opposite in some way [5–7]. Therefore, we must make some concessions in order to combine, in a suitable proportion, these conditions. The word *almost* is then required.

We have proposed an alternative solution for approximating orthogonal spline wavelets. The proposal method is to represent them in term of the compactly supported $\mathscr{B}$-spline wavelets and to truncate the corresponding sequence of coefficients. The advantage is that the so-constructed wavelets span the same subspace with frame bounds that can be easily determined. Moreover, since these wavelets are true splines functions, there are no smoothness or convergence problems.

Summing up, we have proposed an extended family of spline wavelets, covering the full range between the minimal supported and the orthonormal wavelets. In addition to the suitable properties of the spline functions, semiorthogonality, symmetry and compact support qualities are preserved, and almost orthogonality is given with a relative small interval. They lead to an orthogonal decomposition of any signal using an unique pair of finite discrete filters, in the Mallat's scheme, with almost perfect reconstruction.

The proposed family give us some reminiscences of the compactly support wavelets of Daubechies [5]. When more regularity and symmetry are required for these functions, they resemble splines.

We also design several formulas, according to our purposes. We hope that they may be useful for applications. We remark that other wavelets can be computed by adapting the formulas (43), (44) to any desirable pattern. In our opinion, this is an interesting perspective for future research.

Finally, we recall the opportune comment of Meyer in [16]:

> It seems impossible to decide what the best choice is. In my opinion, this decision should be left to the scientist who is using a wavelet analysis.

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