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Nonlinear Differential Equations and Applications NoDEA

Periodic motions in forced problems of Kepler type

Pablo Amster, Julián Haddad, Rafael Ortega,
and Antonio J. Ureña

Abstract. A Newtonian equation in the plane is considered. There is a central force (attractive or repulsive) and an external force $\lambda h(t)$, periodic in time. The periodic second primitive of $h(t)$ defines a planar curve and the number of periodic solutions of the differential equation is linked to the number of loops of this curve, at least when the parameter λ is large.

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1. Introduction and main results

Consider the second order equation in the plane

$$\ddot{z} \pm \frac{z}{|z|^{q+1}} = \lambda h(t), \quad z \in \mathbb{C}\backslash\{0\} \tag{1}$$

where $q \geq 2$, $\lambda \geq 0$ is a parameter and $h : \mathbb{R} \to \mathbb{C}$ is a continuous and 2π-periodic function satisfying

$$\int_0^{2\pi} h(t)dt = 0.$$

This equation models the motion of a particle under the action of a central force $F(z) = \mp\frac{z}{|z|^{q+1}}$ and an external force $\lambda h(t)$. The force F can be attractive or repulsive depending on the sign $+$ or $-$ in the Eq. (1). For $q = 2$ the vector field F becomes the classical gravitational or Coulomb force. For general information on this type of problems we refer to [1].

For the repulsive case it is known that (1) has no 2π-periodic solutions when λ is small enough (see [8] and [2]). In this paper we will discuss the existence of 2π-periodic solutions when λ is large. Before stating the main result we recall the notion of index as it is usually employed in Complex Analysis (see [5]). Given a continuous and 2π-periodic function $\gamma : \mathbb{R} \to \mathbb{C}$ and a point z lying in $\mathbb{C}\backslash\gamma(\mathbb{R})$, the index of z with respect to the circuit γ is an integer denoted by $j(z, \gamma)$. When γ is smooth, say C^1, this index can be expressed as an integral,

$$j(z,\gamma) = \frac{1}{2\pi\imath}\int_\gamma \frac{d\xi}{z-\xi} = \frac{1}{2\pi\imath}\int_0^{2\pi} \frac{\dot\gamma(t)}{z-\gamma(t)}dt.$$

It is well known that $z \mapsto j(z, \gamma)$ is constant on each connected component Ω of $\mathbb{C}\backslash\gamma(\mathbb{R})$. From now on we write $j(\Omega, \gamma)$ for this index. Let $\phi(t)$ be a 2π-periodic solution of (1), the index $j(0, \phi)$ is well defined and can be interpreted as the winding number of the solution ϕ around the singularity $z = 0$.

Theorem 1.1. *Let $H(t)$ be a 2π-periodic solution of*

$$\ddot H(t) = -h(t)$$

and let $\Omega_1, \ldots, \Omega_r$ be bounded components of $\mathbb{C}\backslash H(\mathbb{R})$. Then there exists $\lambda_* > 0$ such that the Eq. (1) has at least r different solutions $\phi_1(t), \ldots, \phi_r(t)$ of period 2π if $\lambda \geq \lambda_*$. Moreover,

$$j(0,\phi_k) = j(\Omega_k, H), \quad k = 1,\ldots,r.$$

Next we discuss the applicability of the theorem in three simple cases.

Example 1.2. *$h(t) \equiv 0$. We also have $H(t) \equiv 0$ and so $\mathbb{C}\backslash H(\mathbb{R}) = \mathbb{C}\backslash\{0\}$. This set has no bounded components and so the theorem is not applicable. This is reasonable since the equation $\ddot z - \frac{z}{|z|^{q+1}} = 0$ has no periodic solutions. This is easily checked since all solutions satisfy*

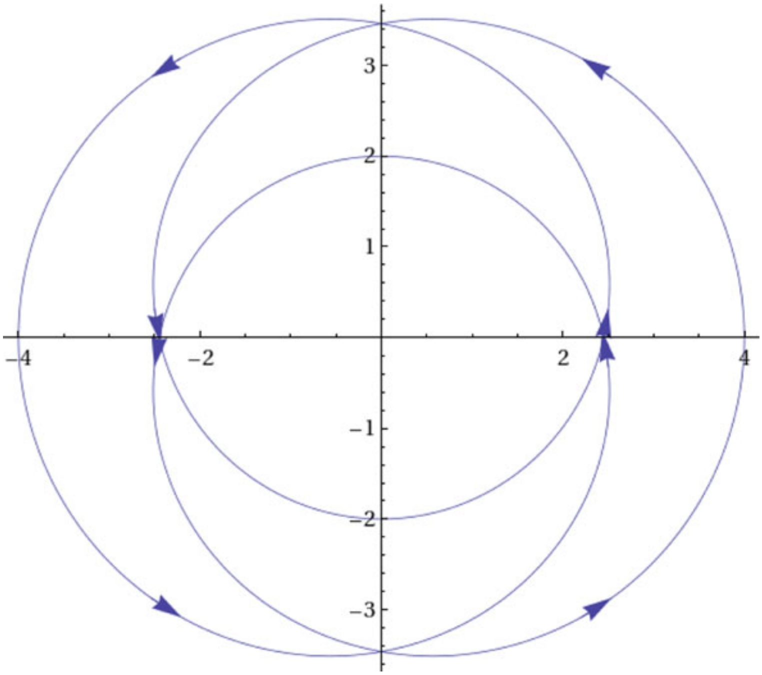
$$\frac{1}{2}\frac{d^2}{dt^2}(|z|^2) = |\dot z|^2 + \frac{1}{|z|^{q-1}} > 0.$$

On the contrary, in the attractive case the Eq. (1) has many periodic solutions for $h \equiv 0$. Notice that $\phi(t) = e^{\imath(t+c)}$ is a 2π periodic solution for any $c \in \mathbb{R}$. In particular this shows that the number of bounded components r is just a lower bound of the number of periodic solutions.

Example 1.3. *$h(t) = e^{\imath t}$. The second primitive of $-h$ is $H(t) = e^{\imath t}$ and $\mathbb{C}\backslash H(\mathbb{R})$ has one bounded component, the open disk $\{|z| < 1\}$. The theorem asserts the existence of a 2π-periodic solution $\phi_1(t)$ with $j(0, \phi_1) = 1$ for λ large enough. Indeed this result can be obtained using very elementary techniques. The change of variables $z = e^{\imath t}w$ transforms (1) into*

$$\ddot w + 2\imath\dot w - w \pm \frac{w}{|w|^{q+1}} = \lambda.$$

This equation has, for large λ, two equilibria w_+ and w_- with $|w_+| \to \infty$ and $|w_-| \to 0$ as $\lambda \to \infty$. These equilibria become 2π-periodic solutions with index one in the z-plane. Our method of proof can be seen as a continuation from infinity and this explains why we cannot detect the small solution. After lengthy computations it is possible to find the spectrum of the linearization of



the w equation around the equilibria. This allows to apply Lyapunov center theorem in some cases to deduce the existence of sub-harmonic and quasi-periodic solutions in the z-plane (*see* [7] *for more details on this technique*).

Example 1.4. $h(t) = e^{\imath t} + 27e^{3\imath t}$.

The function $H(t) = e^{\imath t} + 3e^{3\imath t}$ is a parametrization of an epicycloid.

We observe that $\mathbb{C}\backslash H(\mathbb{R})$ has five bounded connected components with corresponding indices $3, 2, 2, 1, 1$. Hence we obtain five 2π-periodic solutions.

For some forcings $h(t)$ the set $\mathbb{C}\backslash H(\mathbb{R})$ has infinitely many bounded components. In such a case the previous result implies that the number of 2π-periodic solutions grows arbitrarily as $\lambda \to \infty$.

2. Brouwer degree and weakly nonlinear systems

This section is devoted to describe a well known result on the existence of periodic solutions of the system

$$\dot{x} = \varepsilon g(t, x; \varepsilon), \quad x \in U \subseteq \mathbb{R}^d \tag{2}$$

where U is an open and connected subset of $\mathbb{R}^d$, $\varepsilon \in [0, \varepsilon_*]$ is a small parameter and $g : \mathbb{R} \times U \times [0, \varepsilon_*] \to \mathbb{R}^d$ is continuous and 2π-periodic with respect to t. Later it will be shown that our original system (1) can be transformed into a system of the type (2). Following the ideas of the averaging method, we define the function

$$G(c) = \frac{1}{2\pi}\int_0^{2\pi} g(t, c; 0)dt, \quad c \in U.$$

Next we assume that G does not vanish on the boundary of a certain open set W, whose closure $\overline{W}$ is compact and contained in U. In such a case the degree of G on W is well defined.

Proposition 2.1. *In the above conditions assume that*

$$deg(G, W, 0) \neq 0.$$

Then the system (2) *has at least one* 2π*-periodic solution* $x_\varepsilon(t)$ *lying in* W *for* $\varepsilon > 0$ *sufficiently small.*

This result is essentially contained in Cronin's book [4]. We also refer to the more recent paper by Mawhin [6] containing more general results and some history.

Before applying this Proposition to (1) it will be convenient to have some information on the behaviour of $x_\varepsilon(t)$ as $\varepsilon \searrow 0$. The function g is bounded on the compact set $[0, 2\pi] \times \overline{W} \times [0, \varepsilon_*]$ and so

$$\|\dot{x_\varepsilon}\|_\infty = O(\varepsilon) \ as \ \varepsilon \searrow 0.$$

Let $\varepsilon_n \searrow 0$ be a sequence such that $x_{\varepsilon_n}(0)$ converges to some point c in $\overline{W}$. Then $x_{\varepsilon_n}(t)$ converges uniformly to the constant c in $[0, 2\pi]$. Integrating the Eq. (2) over a period we obtain

$$\int_0^{2\pi} g(t, x_{\varepsilon_n}(t); \varepsilon_n)dt = 0$$

and letting $n \to \infty$ we deduce that $G(c) = 0$. In other words, as $\varepsilon \searrow 0$ the solutions $x_\varepsilon(t)$ given by the previous Proposition must accumulate on $G^{-1}(0)$, the set of zeros of G.

3. Reduction to a problem with small parameters

Let us start with the original Eq. (1) and consider the change of variables

$$z = \lambda(w - H(t))$$

where $w = w(t)$ is the new unknown. Then (1) is transformed into

$$\ddot{w} = \mp\varepsilon^2 \frac{w - H(t)}{|w - H(t)|^{q+1}} \tag{3}$$

with $\varepsilon^2 = \frac{1}{\lambda^{q+1}}$.

In principle this equation can have solutions passing through $H(\mathbb{R})$ but we will look for solutions lying in one of the components Ω_k of $\mathbb{C}\backslash H(\mathbb{R})$. On this domain the Eq. (3) is equivalent to a first order system of the type (2) with $x = (w, \xi) \in \mathbb{C}^2$, $U = \Omega_k \times \mathbb{C}$ and

$$\dot{w} = \varepsilon\xi, \quad \dot{\xi} = \mp\varepsilon \frac{w - H(t)}{|w - H(t)|^{q+1}}.$$

The averaging function is

$$G(c_1, c_2) = (c_2, \Phi(c_1)), \ \ c_1 \in \Omega_k, \ c_2 \in \mathbb{C}$$

and

$$\Phi(c_1) = \mp\frac{1}{2\pi}\int_0^{2\pi} \frac{c_1 - H(t)}{|c_1 - H(t)|^{q+1}} dt.$$

In the next section we will prove the following

Claim 3.1. *For each $k = 1, \ldots, r$ there exists an open and bounded set Ω_k^*, whose closure is contained in Ω_k, and such that*

$$\Phi(c_1) \neq 0 \quad if\ c_1 \in \partial\Omega_k^*, \quad deg(\Phi, \Omega_k^*, 0) = 1.$$

Assuming for the moment that this claim holds, we notice that G does not vanish on the boundary of $W = \Omega_k^* \times B$ where B is the unit disk $|c_2| < 1$. Moreover G can be expressed as

$$G = L \circ (\Phi \times id)$$

where $L : \mathbb{C}^2 \to \mathbb{C}^2$ is the linear map $(c_1, c_2) \mapsto (c_2, c_1)$ and id is the identity in $\mathbb{C}$. If we interpret L as an endomorphism of $\mathbb{R}^4$ then it can be represented by the 4×4 matrix $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$. Hence, if L is understood as a $\mathbb{R}$-linear map, the value of the determinant is one. The general properties of degree imply that

$$\begin{aligned} deg(G, W, (0, 0)) &= sign(detL) \cdot deg(\Phi \times id, \Omega_k^* \times B, (0, 0)) \\ &= deg(\Phi, \Omega_k^*, 0) = 1. \end{aligned}$$

In consequence Proposition 2.1 is applicable and we have proved the first part of the theorem 1.1. Namely, the existence of 2π-periodic solutions $\phi_1(t), \ldots, \phi_r(t)$ for large λ (or small ε).

Notice that $\phi_k(t) = \lambda(\psi_k(t) - H(t))$, where ψ_k is a 2π-periodic solution of (3) lying in Ω_k^*. For convenience we make explicit the dependence of ϕ_k with respect to ε and write $\phi_k(t) = \phi_k(t, \varepsilon)$.

To prove the identity

$$j(0, \phi_k(., \varepsilon)) = j(\Omega_k, H)$$

when ε is small enough, we proceed by contradiction. Let us assume that for some sequence $\varepsilon_n \searrow 0$, $j(0, \phi_k(., \varepsilon_n)) \neq j(\Omega_k, H)$. After extracting a subsequence of ε_n we can assume that $\psi_k(t, \varepsilon_n) \to z$, $\dot{\psi}_k(t, \varepsilon_n) \to 0$, uniformly in t, where z is some point in $\Omega_k^* \subset \Omega_k$ with $\Phi(z) = 0$. This is a consequence of the discussion after Proposition 2.1. Computing indexes via integrals and passing to the limit

$$\begin{aligned} j(0, \phi_k(\cdot, \varepsilon_n)) &= -\frac{1}{2\pi\imath}\int_0^{2\pi} \frac{\dot{\psi}(t, \varepsilon_n) - \dot{H}(t)}{\psi(t, \varepsilon_n) - H(t)} dt \\ &\to \frac{1}{2\pi\imath}\int_0^{2\pi} \frac{\dot{H}(t)}{z - H(t)} dt = j(z, H) = j(\Omega_k, H). \end{aligned}$$

Since we are dealing with integer numbers, $j(0, \phi_k(., \varepsilon_n))$ and $j(\Omega_k, H)$ must coincide for large n. This is a contradiction with the definition of ε_n. By now the proof of the main theorem is complete excepting for the above claim.

4. Degree of gradient vector fields

The purpose of this section is to prove the claim concerning the function Φ. To do this we first prove a result valid for general gradient maps in the plane.

Proposition 4.1. *Let Ω be a bounded, open and simply connected subset of $\mathbb{R}^2$ and let $V : \Omega \to \mathbb{R}$ be a continuously differentiable function. In addition assume that*

$$V(z) \to +\infty \quad \textit{as } z \to \partial\Omega. \tag{4}$$

Then there exists an open set Ω^*, whose closure is contained in Ω, such that

1. $\nabla V(z) \neq 0$ *for each* $z \in \partial\Omega^*$
2. $deg(\nabla V, \Omega^*, 0) = 1.$

Remark. *The condition (4) says that V blows up in the boundary of Ω. More precisely, given $r > 0$ there exist $\delta > 0$ such that if $z \in \Omega$ with $dist(z, \partial\Omega) < \delta$ then $V(z) > r$.*

Notice also that, by the properties of degree in two dimensions,

$$deg(\nabla V, \Omega^*, 0) = deg(-\nabla V, \Omega^*, 0).$$

Proof. By Sard lemma we know that V has many regular values in the interval $]\min_\Omega V, +\infty[$. Let us pick one of these values, say α. Then the set $M = V^{-1}(\alpha)$ is a one-dimensional manifold of class C^1. Since V blows up at the boundary, M is compact and so it has to be composed by a finite number of disjoint Jordan curves. Let γ be one of these Jordan curves and let us define Ω^* as the bounded component of $\mathbb{R}^2 \backslash \gamma$. Notice that the closure of Ω^* is contained in Ω because Ω is simply connected.

We know that

$$V(z) = \alpha \quad and \quad \nabla V(z) \neq 0 \quad if\ z \in \gamma$$

and so $\nabla V(z)$ must be colinear to $n(z)$, the outward unitary normal vector to the curve γ. This implies that $\langle \nabla V(z), n(z) \rangle$ does not vanish on the curve γ. Assume for instance that

$$\langle \nabla V(z), n(z) \rangle > 0 \quad if\ z \in \gamma,$$

the other case being similar. Then it is easy to prove that $\nabla V(z)$ is linearly homotopic to any continuous vector field which is tangent to γ on every point of this curve. The proof is complete because it is well known that these tangent vector fields have degree one. See for instance Th. 4.3 (Ch. 15) of [3]. □

We are ready to prove the claim concerning the function

$$\Phi : \Omega_k \to \mathbb{C}, \ \Phi(z) = \pm \frac{1}{2\pi} \int_0^{2\pi} \frac{z - H(t)}{|z - H(t)|^{q+1}} dt$$

where Ω_k is a bounded component of $\mathbb{C} \backslash H(\mathbb{R})$.

To do this we will apply Proposition 4.1 and the crucial observation is that Φ is a gradient vector field. Namely

$$\Phi = \mp\nabla V \quad on\ \Omega_k$$

where V is the real analytic function on Ω_k,

$$V(z) = \frac{1}{2\pi(q-1)}\int_0^{2\pi}\frac{dt}{|z-H(t)|^{q-1}}.$$

To check the assumptions of Proposition 4.1 we must prove that Ω_k is simply connected. This is done using very standard arguments of planar topology.

Lemma 4.1. *Let Γ be a closed and connected subset of $\mathbb{R}^2$ and let Ω be a bounded, connected component of $\mathbb{R}^2\backslash\Gamma$. Then Ω is simply connected.*

Proof. Given a Jordan curve γ in the plane, the bounded and unbounded components of $\mathbb{R}^2\backslash\gamma$ are denoted by $R_i(\gamma)$ and $R_e(\gamma)$ respectively. The set Ω is open and connected and it is sufficient to prove that, for any Jordan curve γ contained in Ω, the bounded component $R_i(\gamma)$ is also contained in Ω. Since $\gamma\subset\Omega\subset\mathbb{R}^2\backslash\Gamma$, we deduce that either $\Gamma\subset R_i(\gamma)$ or $\Gamma\subset R_e(\gamma)$. Here we are using that Γ is connected. Assume first that $\Gamma\subset R_i(\gamma)$. Then $\gamma\cup R_e(\gamma)$ is a connected subset of $\mathbb{R}^2\backslash\Gamma$ and so it must be contained in one of the components. Since γ is contained in Ω we deduce that that also $R_e(\gamma)$ is contained in this component. This is impossible because Ω is bounded. We conclude that the second alternative must hold. Once we know that $\Gamma\subset R_e(\gamma)$ we repeat the previous reasoning, after changing the roles of $R_i(\gamma)$ and $R_e(\gamma)$, to conclude that $\gamma\cup R_i(\gamma)$ is inside Ω. □

It remains to check that (4) holds. We finish this paper with a proof of this fact.

Lemma 4.2. *In the above setting,*

$$V(z)\to+\infty \quad as\ z\to\partial\Omega_k.$$

Proof. By a contradiction argument assume the existence of a sequence $\{z_n\}$ in Ω_k with $dist(z_n,\partial\Omega_k)\to 0$ and such that $V(z_n)$ remains bounded. Since Ω_k is bounded it is possible to extract a subsequence (again z_n) converging to some point $p\in\partial\Omega_k$. Let us define the set $A=\{t\in[0,2\pi]: H(t)=p\}$ and the function

$$\mu(t)=\begin{cases}\frac{1}{|H(t)-p|^{q-1}}, & t\in[0,2\pi]\backslash A\\ +\infty, & t\in A.\end{cases} \tag{5}$$

Then the sequence of functions $\frac{1}{|H(t)-z_n|^{q-1}}$ converges to μ pointwise. By Fatou's Lemma

$$\int_0^{2\pi}\mu(t)dt \le \liminf_{n\to\infty}\int_0^{2\pi}\frac{dt}{|H(t)-z_n|^{q-1}} = 2\pi(q-1)\liminf_{n\to\infty}V(z_n)<\infty.$$

Hence $\mu(t)$ is integrable in the Lebesgue sense. In particular the set A has measure zero. Since the boundary of Ω_k is contained in $H(\mathbb{R})$, the set A is non-empty and we can fix $\tau \in [0, 2\pi]$ with $H(\tau) = p$. The previous discussion shows that

$$\mu(t) = \frac{1}{|H(t) - H(\tau)|^{q-1}}, \quad a.e.\ t \in [0, 2\pi].$$

Let $L > 0$ be a Lipschitz constant for H, then

$$\mu(t) \geq \frac{1}{L^{q-1}|t - \tau|^{q-1}} \quad a.e.\ t \in [0, 2\pi].$$

At this point the condition $q \geq 2$ plays a role,

$$\int_0^{2\pi} \mu(t)dt \geq \frac{1}{L^{q-1}} \int_0^{2\pi} \frac{dt}{|t - \tau|^{q-1}} = +\infty$$

and this is a contradiction with the integrability of μ. □

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Pablo Amster, Julián Haddad
Departamento de Matemática,
Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires. Ciudad Universitaria,
Pabellón I, 1428 Buenos Aires, Argentina
e-mail: pamster@dm.uba.ar

J. Haddad
e-mail: jhaddad@dm.uba.ar

Pablo Amster, Julián Haddad
Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET),
Buenos Aires,
Argentina.

Rafael Ortega, Antonio J. Ureña
Departamento de Matemática Aplicada, Facultad de Ciencias,
Universidad de Granada, 18071 Granada, Spain.
e-mail: rortega@ugr.es

Antonio J. Ureña
e-mail: ajurena@ugr.es

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