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On Nirenberg-type conditions for higher-order systems on time scales

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Abstract

We study the existence of periodic solutions for a nonlinear system of nth-order differential equations on time scales. Assuming a suitable Nirenberg-type condition, we prove the existence of at least one solution of the problem using Mawhin's coincidence degree.

Keywords: Time scales; Boundary value problems; Resonance; Nirenberg condition; Coincidence degree

1. Introduction

In this work, we investigate the existence of solutions $y : [0, \sigma^n(T)]_{\mathbb{T}} \to \mathbb{R}^N$ to the following nonlinear system of nth-order differential equations on time scales

$$y^{\Delta^n} = f(t, y, \dots, y^{\Delta^{n-1}}), \quad t \in [0, T]_{\mathbb{T}}; \tag{1}$$

under the time-scales periodic conditions:

$$y(0) = y(\sigma^n(T)), \qquad y^{\Delta}(0) = y^{\Delta}(\sigma^{n-1}(T)), \dots, y^{\Delta^{n-1}}(0) = y^{\Delta^{n-1}}(\sigma(T)). \tag{2}$$

We shall assume that the nonlinearity $f : [0, T]_{\mathbb{T}} \times \mathbb{R}^{n.N} \to \mathbb{R}^N$ is continuous and bounded with respect to y. However, in contrast with the systems of equations of pendulum type, in this work f will be typically a non-periodic function of y. More precisely, we shall study problem (1) under a generalization of the so-called Nirenberg condition for $n = 2$ which, in turn, generalizes the well-known Landesman–Lazer conditions for the case $N = 1$.

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There exists a vast literature on Landesman–Lazer-type conditions for resonant problems, starting with the pioneering work [1] for a (scalar) second-order elliptic differential equation under Dirichlet conditions (for a survey on conditions of this kind see e.g. [2]). In [3], Nirenberg extended the Landesman–Lazer conditions to a system of second-order elliptic equations. Nirenberg's result can be adapted for a system of periodic ODEs in the following way:

Theorem 1.1. *Let $p \in C([0, T], \mathbb{R}^N)$ and let $g : \mathbb{R}^N \to \mathbb{R}^N$ be continuous and bounded. Further, assume that the radial limits $g_v := \lim_{r\to+\infty} g(rv)$ exist uniformly with respect to $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. Then the problem*

$$y'' + g(y) = p(t)$$

has at least one T-periodic solution if the following conditions hold:

1. $g_v \neq \overline{p} := \frac{1}{T}\int_0^T p(t)\mathrm{d}t$ *for any* $v \in S^{N-1}$.
2. *The degree of the mapping $\theta : S^{N-1} \to S^{N-1}$ given by*

$$\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$$

is non-zero.

In this work, we generalize several aspects of this result. On the one hand, we do not deal with a system of classical ordinary differential equations but, more generally, with a system of dynamical equations on time scales. Let us recall that the concept of time scale, also known as measure chain, was introduced by Hilger in [4], with the aim of unifying continuous and discrete calculus. Thus, for a function $y : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T} \subset \mathbb{R}$ is an arbitrary closed set, a general derivative y^{Δ} is defined, in such a way that if $\mathbb{T} = \mathbb{R}$ (continuous case) then y^{Δ} is the usual derivative (i.e. $y^{\Delta} = y'$), and if $\mathbb{T} = \mathbb{Z}$, then the discrete derivative is retrieved, namely $y^{\Delta} = \Delta y$. For a detailed introduction to the theory of time scales see e.g. [5–7]. It is worth to remark that, although the field of boundary value problems for dynamic equations in time scales had a rapid growth in the last years, not much literature concerning *resonant* problems is known. A previous work dealing with this kind of situation on a time scale is [8], where a second-order multi-point boundary value problem is studied. We may also mention [9], where Landesman–Lazer conditions for a second-order periodic problem on time scales are obtained by variational methods.

On the other hand, our system consists of higher-order equations, for which some of the standard tools of the theory of second-order operators (e.g. maximum and comparison principles) are not applicable.

Finally, the nonlinearity f is more general, since it may also depend on the derivatives of y. In particular, even for the classical case $\mathbb{T} = \mathbb{R}$, this fact implies that the problem has non-variational structure, and motivates the use of topological methods instead: more precisely, we shall apply Mawhin's coincidence degree theory (see e.g. [10]). This powerful tool has been applied to many resonant boundary value problems. An application for a resonant problem on time scales is given in [8]; for periodic conditions, the continuation method has been firstly used in [11], and also in [12].

We shall assume that $f : [0, T]_{\mathbb{T}} \times \mathbb{R}^{n.N} \to \mathbb{R}^N$ is continuous and satisfies the linear growth condition

$$|f(t, y_0, \ldots, y_{n-1})| \le \varepsilon \sup_{1\le j\le n-1} |y_j| + M \tag{3}$$

for some ε to be specified, and some arbitrary constant M. In this situation, our condition concerning the existence of radial limits of the nonlinearity takes the following form:

Condition (F)

For each t the limit

$$\lim_{s\to+\infty} f(t, sv, y_1, \ldots, y_{n-1}) := f_v(t) \tag{4}$$

exists uniformly with respect to $v \in S^{N-1}$ and $|y_j| \le \frac{CM}{1-C\varepsilon}$ for $j = 1, \ldots, n-1$ where the constant $C > 0$ is defined in Lemma 2.2.

Thus, our main result reads:

Theorem 1.2. *Assume that condition* (F) *holds. Then the boundary value problem* (1) *and* (2) *admits at least one solution, provided that*

1. $\overline{f}_v := \frac{1}{\sigma(T)}\int_0^T f_v(t)\ \Delta t \neq 0$ *for any* $v \in S^{N-1}$.

2. *The degree of the mapping* $\theta : S^{N-1} \to S^{N-1}$ *given by*

$$\theta(v) = \frac{\overline{f}_v}{|\overline{f}_v|}$$

is non-zero.

For completeness, let us summarize the main aspects of coincidence degree theory. Let $\mathbb{V}$ and $\mathbb{W}$ be real normed spaces, $L : \mathrm{Dom}(L) \subset \mathbb{V} \to \mathbb{W}$ a linear Fredholm mapping of index 0, and $N : \mathbb{V} \to \mathbb{W}$ continuous. Moreover, set two continuous projectors $\pi_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}$ and $\pi_{\mathbb{W}} : \mathbb{W} \to \mathbb{W}$ such that $\mathrm{R}(\pi_{\mathbb{V}}) = \mathrm{Ker}(L)$ and $\mathrm{Ker}(\pi_{\mathbb{W}}) = \mathrm{R}(L)$, and an isomorphism $J : \mathrm{R}(\pi_{\mathbb{W}}) \to \mathrm{Ker}(L)$. It is readily seen that

$$L_{\pi_{\mathbb{V}}} := L|_{\mathrm{Dom}(L)\cap\mathrm{Ker}(\pi_{\mathbb{V}})} : \mathrm{Dom}(L) \cap \mathrm{Ker}(\pi_{\mathbb{V}}) \to \mathrm{R}(L)$$

is one-to-one; denote its inverse by $K_{\pi_{\mathbb{V}}}$. If Ω is a bounded open subset of $\mathbb{V}$, N is called L-compact on Ω if $\pi_{\mathbb{W}} N(\Omega)$ is bounded and $K_{\pi_{\mathbb{V}}}(I - \pi_{\mathbb{W}})N : \Omega \to \mathbb{V}$ is compact.

The following continuation theorem is due to Mawhin [10]:

Theorem 1.3. *Let L be a Fredholm mapping of index zero and N be L-compact on a bounded domain $\Omega \subset \mathbb{V}$. Suppose*
1. $Lx \neq \lambda Nx$ *for each* $\lambda \in (0, 1]$ *and each* $x \in \partial\Omega$.
2. $\pi_{\mathbb{W}} Nx \neq 0$ *for each* $x \in \mathrm{Ker}(L) \cap \partial\Omega$.
3. $d(J\pi_{\mathbb{W}} N, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, *where d denotes the Brouwer degree.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{Dom}(L) \cap \Omega$.

2. Proof of Theorem 1.2

Set $\mathbb{V} \subset C_{rd}([0, \sigma^n(T)]_{\mathbb{T}}, \mathbb{R}^N)$ given by

$$\mathbb{V} = \{y : \exists y^{\Delta^{n-j}} \in C_{rd}([0, \sigma^j(T)]_{\mathbb{T}}, \mathbb{R}^N) \text{ for } j = 1, \ldots, n-1, \text{ and } y \text{ satisfies (2)}\}$$

equipped with the norm

$$\|y\|_{\mathbb{V}} := \sup_{0\le j\le n-1} \|y^{\Delta^j}\|_{C_{rd}([0,\sigma^j(T)]_{\mathbb{T}},\mathbb{R}^N)}.$$

Moreover, let

$$D = \{y \in \mathbb{V} : \exists y^{\Delta^n} \in C_{rd}([0, T]_{\mathbb{T}}, \mathbb{R}^N)\},$$
$$\mathbb{W} = C_{rd}([0, T]_{\mathbb{T}}, \mathbb{R}^N),$$

and define the operators $L : D \to \mathbb{W}$, $N : \mathbb{V} \to \mathbb{W}$ given by

$$Ly = y^{\Delta^n}, \qquad Ny = f(\cdot, y, \ldots, y^{\Delta^{n-1}}).$$

A simple computation shows that $\mathrm{Ker}(L) = \mathbb{R}^N$, and

$$\mathrm{R}(L) = \left\{\varphi \in \mathbb{W} : \int_0^{\sigma(T)} \varphi(t)\Delta t = 0\right\}.$$

Thus, we may define the projectors

$$\pi_{\mathbb{V}}(y) = y(0), \qquad \pi_{\mathbb{W}}(\varphi) = \frac{1}{\sigma(T)} \int_0^{\sigma(T)} \varphi(t)\Delta t,$$

and consider $J : \mathrm{R}(\pi_{\mathbb{W}}) \to \mathrm{Ker}(L)$ as the identity of $\mathbb{R}^N$.

It is immediate to prove that N is continuous; furthermore, if $\varphi \in \mathrm{R}(L)$, then $K_{\pi_{\mathbb{V}}}(\varphi)$ is the unique solution $y \in D$ of the problem $y^{\Delta^n} = \varphi$ satisfying $y(0) = 0$.

Remark 2.1. The inverse operator $K_{\pi_{\mathbb{V}}}$ may be established in a more precise way. Indeed, if $y \in \mathbb{V}$ satisfies $y^{\Delta^n} = \varphi$, then

$$y^{\Delta^{n-1}}(t) = c_1 + \int_0^t \varphi(s)\Delta s := c_1 + I(\varphi)(t),$$

where the constant c_1 is uniquely determined by the boundary condition $y^{\Delta^{n-2}}(0) = y^{\Delta^{n-2}}(\sigma^2(T))$; namely

$$c_1 = -\frac{1}{\sigma^2(T)} \int_0^{\sigma^2(T)} I(\varphi)(s)\Delta s.$$

Inductively, it follows that

$$y(t) = P(t) + I^n(\varphi)(t),$$

where P is a generalized polynomial of order $n-1$ (i.e. an nth-order anti-derivative of 0), and the coefficients of P are uniquely determined by the successive integrals of φ.

The proof of the following lemma is immediate from the previous remark:

Lemma 2.2. *There exists a constant C such that*

$$\|K_{\pi_{\mathbb{V}}}(\varphi)\|_{C_{rd}^{n-1}} \leq C\|\varphi\|_{\mathbb{W}}$$

for any $\varphi \in \mathrm{R}(L)$.

If y belongs to a bounded set $\Omega \subset \mathbb{V}$, then $\varphi = (I - \pi_{\mathbb{W}})Ny$ is bounded, and from the Arzelá theorem and the previous lemma we deduce that $K_{\pi_{\mathbb{V}}}(I - \pi_{\mathbb{W}})N$ is compact. Thus, the L-compactness of N follows.

We claim that the solutions $y \in D$ of the equation $Ly = \lambda Ny$ with $0 < \lambda \leq 1$ are a priori bounded for the $\mathbb{V}$-norm. Indeed, otherwise there exists a sequence $\{y_k\} \subset D$ such that

$$y_k^{\Delta^n} = \lambda_k f(t, y_k, \ldots, y_k^{\Delta^{n-1}})$$

with $0 < \lambda_k \leq 1$ and $\|y_k\|_{\mathbb{V}} \to \infty$. Writing

$$y_k(t) = y_k(0) + K_{\pi_{\mathbb{V}}}(\lambda_k N y_k)$$

it follows that

$$\|y_k - y_k(0)\|_{C_{rd}^{n-1}} \leq C\|\lambda_k f(t, y_k, \ldots, y_k^{\Delta^{n-1}})\|_{\mathbb{W}} \leq C\varepsilon \sup_{1\leq j\leq n-1} \|y_k^{\Delta^j}\|_{C_{rd}} + CM.$$

Thus, if $\varepsilon < \min\{\frac{1}{C}, 1\}$, it follows that

$$\|y_k - y_k(0)\|_{C_{rd}^{n-1}} \leq \frac{CM}{1 - C\varepsilon}.$$

Then

$$|y_k^{\Delta^j}(t)| \leq \frac{CM}{1 - C\varepsilon} \quad \text{for } j = 1, \ldots n-1, 0 \leq t \leq \sigma^j(T),$$

and it follows that $|y_k(0)| \to \infty$. Taking a subsequence, we may assume that $\frac{y_k(0)}{|y_k(0)|} \to u$ for some $u \in S^{N-1}$, whence $z_k(t) := \frac{y_k(t)}{\|y_k\|_{\mathbb{V}}}$ also converges to u. Integrating the equation, we obtain that

$$0 = \int_0^{\sigma(T)} y^{\Delta^n}(t)\Delta t = \lambda_k \int_0^{\sigma(T)} f(t, y_k, \ldots, y_k^{\Delta^{n-1}})\Delta t.$$

Thus, writing $y_k = \|y_k\|_{\mathbb{V}}.z_k$, and using the dominated convergence theorem for the time scales integral (see [13]), we deduce from condition (F) that

$$\int_0^{\sigma(T)} f_u(t)\Delta t = 0,$$

a contradiction.

Thus, the first condition in Theorem 1.3 is fulfilled taking $\Omega = B_R(0)$ with R large enough.
Furthermore, if $y \in \mathrm{Ker}(L) \cap \partial\Omega$, then

$$\pi_{\mathbb{W}} N y = \frac{1}{\sigma(T)} \int_0^{\sigma(T)} f(t, y, 0, \ldots, 0) \Delta t,$$

and by the degree condition 2 it is easy to verify that the second and the third conditions of Theorem 1.3 are fulfilled.

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