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Remarks on an optimization problem for the p-Laplacian

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ABSTRACT

In this note we give some remarks and improvements on our recent paper [5] about an optimization problem for the p-Laplace operator that were motivated by some discussion that we had with Prof. Cianchi.

1. Introduction

In this note, we want to give some remarks and improvements on our recent [1] about an optimization problem for the p-Laplace operator. These remarks were motivated by some discussion that we had with Prof. Cianchi and we are grateful to him.

Let us recall the problem analyzed in [1]. Given a domain $\Omega \subset \mathbb{R}^N$ (bounded, connected, with smooth boundary) and some class of admissible loads $\mathcal{A}$, in [1] we studied the following problem:

$$\mathcal{J}(f) := \int_{\partial\Omega} f(x) u_f \, dS \to \max \tag{1}$$

for $f \in \mathcal{A}$, where u is the (unique) solution to the nonlinear problem with load f

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Here $p \in (1, \infty)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian, $\frac{\partial}{\partial\nu}$ is the outer normal derivative and $f \in L^q(\partial\Omega)$ with $q > \frac{p'}{N'}$, where $r' = \frac{r}{r-1}$ for all $1 < r < \infty$.

In [1], we worked with three different classes of admissible functions $\mathcal{A}$

- The class of rearrangements of a given function f_0.
- The (unit) ball in $L^q(\partial\Omega)$.
- The class of characteristic functions of sets of given measure.

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For each of these classes, we proved existence of a maximizing load (in the respective class) and analyzed properties of these maximizer.

When we worked in the unit ball of L^q, we explicitly found the (unique) maximizer for $\mathcal{J}$, namely, the first eigenfunction of a Steklov-like nonlinear eigenvalue problem. Whereas when we worked with the class of characteristic functions of set of given boundary measure, besides to prove that there exists a maximizer function we could give a characterization of set where the maximizer function is supported. Moreover, in order to analyze properties of this maximizer, we computed the first variation with respect to perturbations on the set where the characteristic function was supported. See [1], Section 5.

The aim of this work is to generalize the results obtained for the class of characteristic functions of set of given boundary measure to the class of rearrangements of a given function f_0.

Recall that if f_0 is a characteristic function of a set of $\mathcal{H}^{N-1}$-measure α, then *every* characteristic function of a set of $\mathcal{H}^{N-1}$-measure α is a rearrangement of f_0.

We refer the reader to our work [1] for notation and more on problem (1).

2. Characterization of maximizer function

In this section we give characterization of a maximizer function relative to the class of rearrangements of a given function f_0.

We begin by observing that, for any given $f \in L^q(\partial\Omega)$, problem (2) has a unique weak solution u_f, for which the characterization holds

$$\int_{\partial\Omega} f u_f \, dS = \sup_{u\in W^{1,p}(\Omega)} \frac{1}{p-1}\left\{p\int_{\partial\Omega} f u \, dS - \int_{\Omega}\left(|\nabla u|^p + |u|^p\right) dx\right\}. \tag{3}$$

Let $f_0 \in L^q(\partial\Omega)$, with $q > \frac{p'}{N'}$ and let $\mathcal{R}_{f_0}$ be the class of rearrangements of f_0. We are interested in finding

$$\sup_{f\in\mathcal{R}_{f_0}} \mathcal{J}(f). \tag{4}$$

In [1], Theorem 3.1, we prove that there exists $\hat{f} \in \mathcal{R}_{f_0}$ such that

$$\mathcal{J}(\hat{f}) = \sup_{f\in\mathcal{R}_{f_0}} \mathcal{J}(f).$$

We begin by giving a characterization of this maximizer $\hat{f}$ in the spirit of [2].

Theorem 2.1. *$\hat{f}$ is the unique maximizer of linear functional $L(f) := \int_{\partial\Omega} f\hat{u} \, dS$, relative to $f \in \mathcal{R}_{f_0}$, where $\hat{u}$ is the solution to (2) with load $\hat{f}$. Therefore, there is an increasing function ϕ such that $\hat{f} = \phi \circ \hat{u}$, $\mathcal{H}^{N-1}$-a.e.*

Proof. By [3], Theorem once we show that $\hat{f}$ is the unique maximizer of L in the class $\mathcal{R}_{f_0}$, the existence of the function ϕ follows.

Now, we proceed in two steps.

Step 1. First we show that $\hat{f}$ is a maximizer of $L(f)$ relative to $f \in \mathcal{R}_{f_0}$.

In fact, let $h \in \mathcal{R}_{f_0}$, since $\int_{\partial\Omega} \hat{f}\hat{u} \, dS = \sup_{f\in\mathcal{R}_{f_0}} \int_{\partial\Omega} f u_f \, dS$, we have that

$$\begin{aligned}\int_{\partial\Omega} \hat{f}\hat{u} \, dS &\geq \int_{\partial\Omega} h u_h \, dS = \sup_{u\in W^{1,p}(\Omega)} \frac{1}{p-1}\left\{p\int_{\partial\Omega} h u \, dS - \int_{\Omega}\left(|\nabla u|^p + |u|^p\right) dx\right\} \\ &\geq \frac{1}{p-1}\left\{p\int_{\partial\Omega} h\hat{u} \, dS - \int_{\Omega}\left(|\nabla \hat{u}|^p + |\hat{u}|^p\right) dx\right\},\end{aligned}$$

and, since

$$\int_{\partial\Omega} \hat{f}\hat{u} \, dS = \frac{1}{p-1}\left\{p\int_{\partial\Omega} \hat{f}\hat{u} \, dS - \int_{\Omega}\left(|\nabla \hat{u}|^p + |\hat{u}|^p\right) dx\right\},$$

we have

$$\int_{\partial\Omega} \hat{f}\hat{u} \, dS \geq \int_{\partial\Omega} h\hat{u} \, dS.$$

Therefore,

$$\int_{\partial\Omega} \hat{f}\hat{u} \, dS = \sup_{f\in\mathcal{R}_{f_0}} L(f).$$

Step 2. Now, we show that $\hat{f}$ is the unique maximizer of $L(f)$ relative to $f \in \mathcal{R}_{f_0}$.

We suppose that g is another maximizer of $L(f)$ relative to $f \in \mathcal{R}_{f_0}$. Then

$$\int_{\partial\Omega} \hat{f}\hat{u}\mathrm{d}S = \int_{\partial\Omega} g\hat{u}\mathrm{d}S.$$

Thus

$$\begin{aligned}\int_{\partial\Omega} g\hat{u}\mathrm{d}S &= \int_{\partial\Omega} \hat{f}\hat{u}\mathrm{d}S \geq \int_{\partial\Omega} gu_g\mathrm{d}S \\ &= \sup_{u\in W^{1,p}(\Omega)} \frac{1}{p-1}\left\{p\int_{\partial\Omega} gu\mathrm{d}S - \int_{\Omega}\left(|\nabla u|^p + |u|^p\right)\mathrm{d}x\right\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\partial\Omega} g\hat{u}\mathrm{d}S &= \int_{\partial\Omega} \hat{f}\hat{u}\mathrm{d}S = \frac{1}{p-1}\left\{p\int_{\partial\Omega} \hat{f}\hat{u}\mathrm{d}S - \int_{\Omega}\left(|\nabla \hat{u}|^p + |\hat{u}|^p\right)\mathrm{d}x\right\} \\ &= \frac{1}{p-1}\left\{p\int_{\partial\Omega} g\hat{u}\mathrm{d}S - \int_{\Omega}\left(|\nabla \hat{u}|^p + |\hat{u}|^p\right)\mathrm{d}x\right\}.\end{aligned}$$

Then

$$\int_{\partial\Omega} g\hat{u}\mathrm{d}S = \sup_{u\in W^{1,p}(\Omega)} \frac{1}{p-1}\left\{p\int_{\partial\Omega} gu\mathrm{d}S - \int_{\Omega}\left(|\nabla u|^p + |u|^p\right)\mathrm{d}x\right\}.$$

Therefore $\hat{u} = u_g$ and as a consequence, $\hat{u}$ is the unique weak solution to (2) with load g. Since, moreover, $\hat{u}$ is the unique weak solution to (2) with load $\hat{f}$ it follows that $\hat{f} = g$, $\mathcal{H}^{N-1}$-a.e.

The proof is now complete. □

3. Derivative with respect to the load

Now we compute the derivative of the functional $\mathcal{J}(\hat{f})$ with respect to perturbations in $\hat{f}$. We will consider regular perturbations and assume that the function $\hat{f}$ has bounded variation in $\partial\Omega$.

We begin by describing the kind of variations that we are considering. Let V be a regular (smooth) vector field, globally Lipschitz, with support in a neighborhood of $\partial\Omega$ such that $\langle V, \nu\rangle = 0$ and let $\psi_t: \mathbb{R}^N \to \mathbb{R}^N$ be defined as the unique solution to

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = V(\psi_t(x)) & t > 0, \\ \psi_0(x) = x & x \in \mathbb{R}^N.\end{cases} \tag{5}$$

We have $\psi_t(x) = x + tV(x) + o(|t|)$ for all $x \in \mathbb{R}^N$.

Thus, if $f \in \mathcal{R}_{f_0}$, we define $f_t = f \circ \psi_t^{-1}$. Now, let

$$I(t) := \mathcal{J}(f_t) = \int_{\partial\Omega} u_t f_t \mathrm{d}\mathcal{H}^{N-1}$$

where $u_t \in W^{1,p}(\Omega)$ is the unique solution to (2) with load f_t.

Lemma 3.1. *Given $f \in L^q(\partial\Omega)$ then*

$$f_t = f \circ \psi_t^{-1} \to f \quad \text{in } L^q(\partial\Omega), \text{ as } t \to 0.$$

Proof. Let $\varepsilon > 0$, and let $g \in C_c^\infty(\partial\Omega)$ fixed such that $\|f - g\|_{L^q(\partial\Omega)} < \varepsilon$. By the usual change of variables formula, we have,

$$\|f_t - g_t\|_{L^q(\partial\Omega)}^q = \int_{\partial\Omega} |f-g|^q J_\tau\psi_t \mathrm{d}S,$$

where $g_t = g \circ \psi_t^{-1}$ and $J\psi_t$ is the tangential Jacobian of ψ_t. We also know that

$$J_\tau\psi_t := 1 + t\operatorname{div}_\tau V + o(|t|).$$

Here $\operatorname{div}_\tau V$ is the tangential divergence of V over $\partial\Omega$. Then

$$\|f_t - g_t\|_{L^q(\partial\Omega)}^q = \int_{\partial\Omega} |f-g|^q (1 + t\operatorname{div}_\tau V + o(|t|))\mathrm{d}S.$$

There exist $t_1 > 0$ such that if $0 < t < t_1$ then

$$\|f_t - g_t\|_{L^q(\partial\Omega)} \leq C\varepsilon$$

where C is a constant independent of t. Moreover, since $\psi_t^{-1} \to Id$ in the C^1 topology when $t \to 0$ then $g_t = g \circ \psi_t^{-1} \to g$ in the C^1 topology and therefore there exists $t_2 > 0$ such that if $0 < t < t_2$ then

$$\|g_t - g\|_{L^q(\partial\Omega)} < \varepsilon.$$

Finally, we have for all $0 < t < t_0 = \min\{t_1, t_2\}$ then

$$\|f_t - f\|_{L^q(\partial\Omega)} \leq \|f_t - g_t\|_{L^q(\partial\Omega)} + \|g_t - g\|_{L^q(\partial\Omega)} + \|g - f\|_{L^q(\partial\Omega)} \leq C\varepsilon$$

where C is a constant independent of t. □

Lemma 3.2. *Let u_0 and u_t be the solutions of (2) with loads f and f_t respectively. Then*

$$u_t \to u_0 \quad \text{in } W^{1,p}(\Omega), \text{ as } t \to 0^+.$$

Proof. The proof follows exactly as the one in Lemma 4.2 in [2]. The only difference being that we use the trace inequality instead of the Poincaré inequality. □

Remark 3.3. It is easy to see that, as $\psi_t \to Id$ in the C^1 topology, then from Lemma 3.2 it follows that $w_t := u_t \circ \psi_t \to u_0$ strongly in $W^{1,p}(\Omega)$.

With these preliminaries, the following theorem follows exactly as Theorem 5.5 of [1].

Theorem 3.4. *With the previous notation, we have that $I(t)$ is differentiable at $t = 0$ and*

$$\left.\frac{dI(t)}{dt}\right|_{t=0} = \frac{1}{p-1}\left\{p\int_{\partial\Omega} u_0 f \operatorname{div}_\tau V \, dS + p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0^T, V'\nabla u_0^T\rangle dx - \int_\Omega (|\nabla u_0|^p + |u_0|^p)\operatorname{div} V \, dx\right\}$$

where u_0 is the solution of (2) with load f.

Proof. For the details see the proof of Theorem 5.5 of [1]. □

Now we try to find a more explicit formula for $I'(0)$. For this, we consider $f \in L^q(\partial\Omega) \cap BV(\partial\Omega)$, where $BV(\partial\Omega)$ is the space of functions of bounded variation. For details and properties of BV functions we refer to the book [4].

Theorem 3.5. *If $f \in L^q(\partial\Omega) \cap BV(\partial\Omega)$, we have that*

$$\left.\frac{\partial I(t)}{\partial t}\right|_{t=0} = \frac{p}{p-1}\int_{\partial\Omega} u_0 V \, d[Df]$$

where u_0 is the solution of (2) with load f.

Proof. In the course of the computations, we require the solution u_0 to be C^2. However, this is not true. As it is well known (see, for instance, [5]), u_0 belongs to the class $C^{1,\delta}$ for some $0 < \delta < 1$.

In order to overcome this difficulty, we proceed as follows. We consider the regularized problems

$$\begin{cases} -\operatorname{div}\left((|\nabla u_0^\varepsilon|^2 + \varepsilon^2)^{(p-2)/2}\nabla u_0^\varepsilon\right) + |u_0^\varepsilon|^{p-2}u_0^\varepsilon = 0 & \text{in } \Omega, \\ (|\nabla u_0^\varepsilon|^2 + \varepsilon^2)^{(p-2)/2}\dfrac{\partial u_0^\varepsilon}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases} \tag{6}$$

It is well known that the solution u_0^ε to (6) is of class $C^{2,\rho}$ for some $0 < \rho < 1$ (see [6]).

Then, we can perform all of our computations with the functions u_0^ε and pass to the limit as $\varepsilon \to 0+$ at the end.

We have chosen to work formally with the function u_0 in order to make our arguments more transparent and leave the details to the reader. For a similar approach, see [7].

Now, by Theorem 3.4 and since

$$\operatorname{div}(|u_0|^p V) = p|u_0|^{p-2}u_0\langle \nabla u_0, V\rangle + |u_0|^p \operatorname{div} V,$$
$$\operatorname{div}(|\nabla u_0|^p V) = p|\nabla u_0|^{p-2}\langle \nabla u_0 D^2 u_0, V\rangle + |\nabla u_0|^p \operatorname{div} V,$$

we obtain

$$\begin{aligned} I'(0) &= \frac{1}{p-1}\left\{p\int_{\partial\Omega} u_0 f\ \mathrm{div}_\tau V\,\mathrm{d}S + p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0\,, ^{\mathrm{T}} V'\nabla u_0^{\mathrm{T}}\rangle \mathrm{d}x - \int_\Omega (|\nabla u_0|^p + |u_0|^p)\ \mathrm{div}\, V\,\mathrm{d}x\right\} \\ &= \frac{1}{p-1}\left\{p\int_{\partial\Omega} u_0 f\ \mathrm{div}_\tau V\,\mathrm{d}S + p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0\,, ^{\mathrm{T}} V'\nabla u_0^{\mathrm{T}}\rangle \mathrm{d}x - \int_\Omega \mathrm{div}\,((|\nabla u_0|^p + |u_0|^p)V)\mathrm{d}x \right. \\ &\quad \left. + p\int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0 D^2 u_0, V\rangle \mathrm{d}x + p\int_\Omega |u_0|^{p-2}u_0\langle \nabla u_0, V\rangle \mathrm{d}x\right\}. \end{aligned}$$

Hence, using that $\langle V, \nu\rangle = 0$ in the right hand side of the above equality we find

$$\begin{aligned} \frac{p-1}{p} I'(0) &= \int_{\partial\Omega} u_0 f\ \mathrm{div}_\tau V\,\mathrm{d}S + \int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0\,, ^{\mathrm{T}} V'\nabla u_0^{\mathrm{T}} + D^2 u_0 V^{\mathrm{T}}\rangle \mathrm{d}x + \int_\Omega |u_0|^{p-2}u_0\langle \nabla u_0, V\rangle \mathrm{d}x \\ &= \int_{\partial\Omega} u_0 f\ \mathrm{div}_\tau V\,\mathrm{d}S + \int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0, \nabla(\langle \nabla u_0, V\rangle)\rangle \mathrm{d}x + \int_\Omega |u_0|^{p-2}u_0\langle \nabla u_0, V\rangle \mathrm{d}x. \end{aligned}$$

Since u_0 is a week solution of (2) with load f we have

$$\begin{aligned} I'(0) &= \frac{p}{p-1}\left\{\int_{\partial\Omega} u_0 f\ \mathrm{div}_\tau V\,\mathrm{d}S + \int_{\partial\Omega}\langle \nabla u_0, V\rangle f\,\mathrm{d}S\right\} \\ &= \frac{p}{p-1}\int_{\partial\Omega} \mathrm{div}_\tau (u_0 V) f\,\mathrm{d}S. \end{aligned}$$

Finally, since $f \in BV(\partial\Omega)$ and $V \in C^1(\partial\Omega; \mathbb{R}^N)$,

$$I'(0) = \frac{p}{p-1}\int_{\partial\Omega} \mathrm{div}_\tau (u_0 V) f\,\mathrm{d}S = \frac{p}{p-1}\int_{\partial\Omega} u_0 V\,\mathrm{d}[Df].$$

The proof is now complete. □

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