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An inequality for eigenvalues of quasilinear problems with monotonic weights

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ABSTRACT

In this work we extend an inequality of Nehari to the eigenvalues of weighted quasilinear problems involving the *p*-Laplacian when the weight is a monotonic function. We apply it to different eigenvalue problems.

1. Introduction

In this work we give a lower bound for the first eigenvalue of the following quasilinear problem involving the *p*-Laplacian,

$$\begin{aligned} &-(|u'|^{p-2}u')' = \lambda g(x)|u|^{p-2}u \quad x \in (a,b) \\ &u(a) = 0 \\ &u(b) = 0. \end{aligned} \tag{1.1}$$

Here, $1 < p < \infty$, λ is a real parameter, and $g \in L^1([a,b])$ is a monotonic function.

Let λ_g and u_g be the first eigenvalue and the corresponding eigenfunction. Our main result is the following:

Theorem 1.1. *Let $g \in L^1([a,b])$ be a non-negative monotonic function, and let λ_g be the first eigenvalue of problem (1.1). Then,*

$$\frac{\pi_p}{2} \le \lambda_g^{1/p} \int_a^b g^{1/p}(x)\mathrm{d}x. \tag{1.2}$$

Here, π_p is defined in terms of the first positive zero of the generalized sine function $\sin_p$, see [1] or [2] for details.

Inequality (1.2) was obtained first in the linear case $p = 2$ by Nehari in [3]. His proof follows by using an integral equation and Green functions. Here we give a different proof, based in the variational characterization of the first eigenvalue of problem (1.1). Let us recall from [4] this variational characterization,

$$\lambda_g^{-1} = \max_{\{u \in W_0^{1,p} : \|u'\|_p = 1\}} \int_a^b g(x)u^p(x)\mathrm{d}x.$$

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As a direct consequence, we have

$$\lambda_g^{-1} = \int_a^b g(x)u_g^p(x)\mathrm{d}x \ge \int_a^b g(x)u_r^p(x)\mathrm{d}x \tag{1.3}$$

for any normalized eigenfunction corresponding to a different weight r, a fact that we will use several times in Section 2 in the proof of Theorem 1.1.

Since the

$$\left(\int_a^b g^{1/p}(x)\mathrm{d}x\right)^p \le (b-a)^{p-1}\int_a^b g(x)\mathrm{d}x,$$

inequality (1.2) is related to the Lyapunov inequality, which was extended to nonlinear problems in the last years, with applications to inequalities, oscillation theory, and eigenvalue estimates (see [5–9] and the references therein). For problem (1.1), the Lyapunov inequality gives

$$\frac{2^p n^p}{(b-a)^{p-1}\int_a^b g(x)\mathrm{d}x} \le \lambda_n, \tag{1.4}$$

and we show in Section 3 some examples where inequality (1.2) improves the eigenvalue bounds obtained by using a Lyapunov inequality.

Then, we apply it to a singular problem in $(0,\infty)$ studied by Kusano and Naito [2]:

$$-(|u'|^{p-2}u')' = \lambda g(x)|u|^{p-2}u, \quad x \in (0,\infty), \tag{1.5}$$

with the boundary conditions

$$u(0) = 0, \qquad \lim_{x\to\infty}\frac{u(x)}{\sqrt{x}} = 0, \tag{1.6}$$

where $2 \le p < \infty$ and the weight g is a positive continuous function satisfying

$$(H1) \quad \int_0^\infty g(x)\mathrm{d}x < \infty, \qquad \lim_{t\to\infty} t^{p-1}\int_t^\infty g(x)\mathrm{d}x = 0.$$

This problem goes back to Einar Hille in the linear case [10], and there are several works devoted to its variants. He obtained the asymptotic behavior of the eigenvalues for $p = 2$ and decreasing weights g with $g^{1/2} \in L^1(0,\infty)$, and a different derivation was given in [11]. Remarkably, in both proofs the monotonicity of g plays a key role. We give here a lower bound of the eigenvalues using inequality (1.2).

We close the paper with some comments about future work and related problems.

2. The proof of Theorem 1.1

Proof of Theorem 1.1. We divide the proof in several steps.

Step 1: Let us show first that given $r_j \to g$ in $L^1([a,b])$, we have as $j \to \infty$,

(i) $\lambda_{r_j} \to \lambda_g$,

(ii) $\displaystyle\int_a^b r_j^{1/p}(x)\mathrm{d}x \to \int_a^b g^{1/p}(x)\mathrm{d}x.$

Let us prove both claims.

(i) We have

$$\lambda_g^{-1} = \int_a^b g(x)u_g^p(x)\mathrm{d}x = \int_a^b r_j(x)u_g^p(x)\mathrm{d}x + \int_a^b (g(x)-r_j(x))u_g^p(x)\mathrm{d}x.$$

Now, the first integral is bounded above by $\lambda_{r_j}^{-1}$, due to inequality (1.3).

Since the eigenfunctions are normalized such that $\|u_g'\|_p = \|u_{r_j}'\|_p = 1$, from Poincare's inequality and the embedding $W_0^{1,p} \hookrightarrow L^\infty([a,b])$, we get that these eigenfunctions are uniformly bounded in L^∞ by a certain positive constant C.

Then, the second integral can be bounded as

$$\int_a^b |g(x)-r_j(x)|\|u_g\|_\infty^p\mathrm{d}x \le C\int_a^b |g(x)-r_j(x)|\mathrm{d}x = O(\|g-r_j\|_1).$$

Interchanging the roles of g and r_j, we obtain

$$\left|\lambda_{r_j}^{-1} - \lambda_g^{-1}\right| = O(\|g - r_j\|_1),$$

and (i) is proved, since both are bounded away from zero.
(ii) From Minkowski's and Holder inequalities we get:

$$\int_a^b |r_j^{1/p}(x) - g^{1/p}(x)|\mathrm{d}x \le \int_a^b |r_j(x) - g(x)|^{1/p}\mathrm{d}x \le (b-a)^{1/p'}\|r_j - g\|_1^{1/p},$$

and the convergence is proved.

Step 1 is finished.

Step 2: Let us now show that given any non-negative simple function r, there exists a non-negative simple function s with at most one discontinuity, such that

$$\int_a^b r^{1/p}(x)\mathrm{d}x = \int_a^b s^{1/p}(x)\mathrm{d}x, \quad \text{and} \quad \lambda_s \le \lambda_r.$$

Let us take r such that $\int_a^b r^{1/p}(x)\mathrm{d}x = 1$ (the general case follows by scaling). We write $r^{1/p} = \sum_{i=1}^n c_i\sigma_i$, where each σ_i is a simple function with at most one discontinuity, $\int_a^b \sigma_i(x)\mathrm{d}x = 1$, and $\sum_i^n c_i = 1$.

We have

$$\lambda_r^{-1} = \int_a^b r(x)u_r^p(x)\mathrm{d}x = \int_a^b \left(\sum_{i=1}^n c_i\sigma_i(x)\right)^p u_r^p(x)\mathrm{d}x.$$

By using Jensen's inequality,

$$\int_a^b \left(\sum_{i=1}^n c_i\sigma_i(x)\right)^p u_r^p(x)\mathrm{d}x \le \int_a^b \sum_{i=1}^n c_i\sigma_i^p(x)u_r^p(x)\mathrm{d}x,$$

we obtain

$$\lambda_r^{-1} \le \sum_{i=1}^n c_i \int_a^b \sigma_i^p(x)u_r^p(x)\mathrm{d}x \le \sum_{i=1}^n c_i\lambda_{\sigma_i}^{-1} \le \max_{1\le i\le n}\{\lambda_{\sigma_i}^{-1}\} = \lambda_\sigma^{-1},$$

where σ is the one which gives the maximum eigenvalue. So, for $s = \sigma^p$ we get

$$\lambda_s^{1/p}\int_a^b s^{1/p}(x) \le \lambda_r^{1/p}\int_a^b r^{1/p}(x)\mathrm{d}x,$$

and Step 2 is finished.

Step 3: Let us bound the first eigenvalue λ_s when s is given by

$$s(t) = \begin{cases} \alpha^p & \text{if } t \in [a, t_1] \\ \beta^p & \text{if } t \in (t_1, b], \end{cases}$$

with α, β non-negatives, not both equal to zero.

Multiplying by u_s and integrating by parts we get

$$\int_a^b u_s'^p(x)\mathrm{d}x - \lambda_s\alpha^p\int_a^{t_1} u_s^p(x)\mathrm{d}x - \lambda_s\beta^p\int_{t_1}^b u_s^p(x)\mathrm{d}x = 0,$$

and we can write it as

$$\left[\int_a^{t_1} u_s'^p(x)\mathrm{d}x - \lambda_s\alpha^p\int_a^{t_1} u_s^p(x)\mathrm{d}x\right] + \left[\int_{t_1}^b u_s'^p(x)\mathrm{d}x - \lambda_s\beta^p\int_{t_1}^b u_s^p(x)\mathrm{d}x\right] = 0.$$

Now, one of those terms must be non-positive. Suppose that

$$\int_a^{t_1} u_s'^p(x)\mathrm{d}x - \lambda_s\alpha^p\int_a^{t_1} u_s^p(x)\mathrm{d}x \le 0,$$

the other case is similar. Then,

$$\lambda_s^{-1}\alpha^{-p} \le \frac{\int_a^{t_1} u_s^p(x)\mathrm{d}x}{\int_a^{t_1} u_s'^p(x)\mathrm{d}x}.$$

Since u_s belongs to the Sobolev space

$$W = \{u \in W^{1,p}([a, t_1]) : u(a) = 0\},$$

it is admissible in the variational characterization of the first eigenvalue of the following mixed problem

$$\begin{aligned} &-(|u'|^{p-2}u')' = \lambda g(x)|u|^{p-2}u \\ &u(a) = 0 \\ &u'(b) = 0, \end{aligned}$$

and we obtain

$$\lambda_s^{-1}\alpha^{-p} \le \max_{u\in W} \frac{\int_a^{t_1} u^p(x)\mathrm{d}x}{\int_a^{t_1} u'^p(x)\mathrm{d}x} = \frac{2^p(t_1 - a)^p}{\pi_p^p}.$$

Finally,

$$\lambda_s^{1/p} \int_a^b s^{1/p}(x)\mathrm{d}x \ge \lambda_s^{1/p} \int_a^{t_1} \alpha \mathrm{d}x > \frac{\pi_p}{2}.$$

Step 4: From Steps 1 and 2, and the density of simple functions in L^1, given any $\varepsilon > 0$ arbitrary small, and a non-negative monotonic function $g \in L^1([a, b])$, there exists a non-negative simple function s with a single discontinuity such that

$$\lambda_s^{1/p} \int_a^b s^{1/p}(x)\mathrm{d}x \le \lambda_g^{1/p} \int_a^b g^{1/p}(x)\mathrm{d}x + \varepsilon,$$

and the bound of Step 3 finishes the proof. □

3. Eigenvalue problems

3.1. A comparison with Lyapunov inequality

Let us consider problem (1.1) with $a = 0$ and $b = 1$.

When $g(x) = \mathrm{e}^{\alpha x}$, $\alpha > 0$, from the Lyapunov inequality (1.4) and inequality (1.2) we get, respectively,

$$\lambda_1 \ge \frac{2^p\alpha}{(\mathrm{e}^\alpha - 1)}, \qquad \lambda_1 \ge \left(\frac{\pi_p\alpha}{2p(\mathrm{e}^{\alpha/p} - 1)}\right)^p,$$

and the inequality (1.2) is better for α large.

When $g(x) = x^\alpha$, $\alpha > -1$, we get

$$\lambda_1 \ge 2^p(\alpha + 1), \qquad \lambda_1 \ge \left(\frac{\pi_p(\alpha + p)}{2p}\right)^p,$$

and we observe that the growth of the second bound is superlinear in α. Inequality (1.2) is better for α large, and the Lyapunov inequality is better for α close to zero. Moreover, for α close to -1, we have:

- If $p \le 2$, inequality (1.2) is better for
$$\alpha + 1 \le (p - 1)^{p+1} \left(\frac{\pi}{4}\right)^p.$$
- If $p \ge 2$, inequality (1.2) is better for
$$\alpha + 1 \le \frac{(p - 1)^{p+1}}{p^{2p}} \left(\frac{\pi}{2}\right)^p.$$

Both can be proved in the same way, by using that

$$\pi_p = 2(p - 1)^{1/p} \frac{\pi/p}{\sin(\pi/p)},$$

(see [1]), and

$$\frac{(\alpha + p)^p\pi_p^p}{2^p p^p} = \frac{(\alpha + p)^p(p - 1)\pi^p}{p^{2p} \sin^p(\pi/p)} \ge \frac{(p - 1)^{p+1}\pi^p}{p^{2p}}.$$

3.2. A singular problem

It was proved in [2] there exists that a sequence $\{\lambda_n\}_{n\geq 1}$ of eigenvalues of problem (1.5), with the boundary conditions (1.6), and g a continuous and positive function satisfying $(H1)$. The eigenfunction u_n corresponding to λ_n has exactly n zeros $0 = t_1 < t_2 < \cdots < t_n$.

Remark 3.1. We cannot apply the Lyapunov inequality to this problem, since the location of the zeros $\{t_j\}_j$ is unknown.

We also need to assume that $g^{1/p} \in L^1(0, \infty)$, and we have the following result:

Theorem 3.1. *Let $\{\lambda_n\}_n$ be the sequence of eigenvalues of problem (1.5)–(1.6), with $g^{1/p} \in L^1(0, \infty)$ satisfying $(H1)$. Then,*

$$\frac{\pi_p(n-1)}{2\int_0^\infty g^{1/p}(x)\mathrm{d}x} \leq \lambda_n^{1/p}.$$

Proof. Although this eigenvalue problem has no variational form, for any λ_n and the eigenfunction u_n, we can use the inequality (1.2) in the following way: given two consecutive zeros t_{j-1}, t_j, the restriction of u_n to $[t_{j-1}, t_j]$ is the first eigenfunction of the problem

$$-(|u'|^{p-2}u')' = \mu g(x)|u|^{p-2}u,$$
$$u(t_{j-1}) = 0 = u(t_j),$$

and $\mu_1 = \lambda_n$, since u_n is a solution of one sign.

Now, for $2 \leq j \leq n$ we have

$$\frac{\pi_p}{2} \leq \lambda_n^{1/p} \int_{t_{j-1}}^{t_j} g^{1/p}(x)\mathrm{d}x,$$

which gives

$$\frac{\pi_p(n-1)}{2} \leq \sum_{j=2}^{n} \lambda_n^{1/p} \int_{t_{j-1}}^{t_j} g^{1/p}(x)\mathrm{d}x < \lambda_n^{1/p} \int_0^\infty g^{1/p}(x)\mathrm{d}x,$$

and the Theorem is proved. □

4. Final remarks

Remark 4.1. The asymptotic behavior of the eigenvalues of problem (1.5)–(1.6) can be obtained now as in [11]. The case $g^{1/p} \notin L^1(0, \infty)$ is more subtle, although it is possible to obtain some lower bounds.

Remark 4.2. It would be interesting to extend the inequality (1.2) to more general weights. It is not clear that the convergence of eigenvalues in Step 1 can be obtained by using only that $r_j^{1/p} \to g^{1/p}$ in $L^1(a, b)$. The case $g(x) = x^{-1}$ is specially important due to the Kolodner result about rotating strings.

Remark 4.3. For unbounded intervals, the Lyapunov inequality needs some information on the location of zeros, since the separation between two of them appears in the lower bound. Here, Inequality (1.2) seems to be useful to derive a priori bounds on the location of zeros. Also, it can be reformulated to derive non-oscillation conditions or disconjugacy criteria. We have not explored this line of research.

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