Available online at www.sciencedirect.com
ScienceDirect
Journal of Complexity 22 (2006) 738–751
Journal of COMPLEXITY
www.elsevier.com/locate/jco

Randomness and universal machines

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Received 9 December 2005; accepted 22 May 2006
Available online 24 July 2006

Abstract

The present work investigates several questions from a recent survey of Miller and Nies related to Chaitin's Ω numbers and their dependence on the underlying universal machine. Furthermore, the notion $\Omega_U[X] = \sum_{p:U(p)\downarrow\in X} 2^{-|p|}$ is studied for various sets X and universal machines U. A universal machine U is constructed such that for all x, $\Omega_U[\{x\}] = 2^{1-H(x)}$. For such a universal machine there exists a co-r.e. set X such that $\Omega_U[X]$ is neither left-r.e. nor Martin-Löf random. Furthermore, one of the open problems of Miller and Nies is answered completely by showing that there is a sequence U_n of universal machines such that the truth-table degrees of the Ω_{U_n} form an antichain. Finally, it is shown that the members of hyperimmune-free Turing degree of a given Π^0_1-class are not low for Ω unless this class contains a recursive set.

Keywords: Recursion theory; Algorithmic randomness; Kolmogorov complexity; Halting probability; Ω-numbers; Universal machines; Truth-table degrees

1. Introduction

Chaitin [7,8] started to investigate the halting probability of prefix-free Turing machines M, that is, of machines which never halt on programs p, q where q is an extension of p viewed as a

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[1] S. Figueira is partially supported by a Grant of Fundación YPF. He also acknowledges CONICET.
[2] F. Stephan is supported in part by NUS Grant number R252–000–212–112.
[3] G. Wu is partially supported by the Start-Up Grant from Nanyang Technological University and the International Joint Project no. 60310213 of NSFC of China.

0885-064X/$ - see front matter
doi:10.1016/j.jco.2006.05.001

binary string. The halting probability Ω_M is then the probability that a randomly drawn infinite sequence extends a program p of M such that $M(p)$ halts. This is equivalent to the sum

$$\Omega_M = \sum_{p \in \mathrm{dom}(M)} 2^{-|p|},$$

where $|p|$ denotes the length of the binary string p. The value of Ω_M can recursively be approximated from below via a recursive increasing sequence of rational numbers. The word *left-r.e.* is used to denote this property. As a set R represents the number $\sum_{n \in R} 2^{-n-1}$, real numbers between 0 and 1 are identified with the sets representing them. Chaitin was mainly interested in the halting probability of universal machines which are defined as follows.

Definition 1. A prefix-free Turing machine $U : \{0,1\}^* \to \{0,1\}^*$ is *universal* if and only if

$$\forall \text{ prefix-free Turing machine } M\ \exists c\ \forall \sigma \in \mathrm{dom}(M)\ \exists \sigma'\ [U(\sigma') = M(\sigma)\ \wedge\ |\sigma'| \leqslant |\sigma| + c].$$

Furthermore, a prefix-free machine U is called *universal by adjunction* iff for every prefix-free machine V there is a fixed word p such that $U(pq) = V(q)$ for all $q \in \{0,1\}^*$ where pq is the concatenation of p and q and $U(pq) = V(q)$ means that either both sides are defined and equal or both are undefined.

The importance of universal Turing machines is that they can be used to define the Kolmogorov complexity in an optimal way; that is, the definitions based on two different machines differ at most by a constant. Given a prefix-free Turing machine M and an element x of its range, let

$$H_M(x) = \min\{|p| : M(p) = x\}$$

be the length of the shortest description of x with respect to M. If U is a universal Turing machine, then H_U is a total function and for every further machine M, there is a constant c such that for all x in the range of M, $H_U(x) \leqslant H_M(x) + c$. In this case, H_U is referred to as the *prefix-free Kolmogorov complexity based on U*. If there is no need to refer to the underlying universal machine U, one just writes H for H_U and Ω for Ω_U.

Martin-Löf [20] introduced a notion of randomness which became quite accepted in the field and is known as Martin-Löf randomness. Schnorr [27] found a characterization in terms of Kolmogorov complexity which is here used in place of the original definition:

$$A \text{ is Martin-Löf random } \Leftrightarrow \exists c\ \forall n\ [H(A(0)\ldots A(n)) \geqslant n - c].$$

Hence Martin-Löf random sets have highly incompressible prefixes.

Chaitin proved that the halting probability of a universal machine is Martin-Löf random. Further research [6,18] provided the following equivalence: a left-r.e. set is Martin-Löf random iff it is the halting probability of some universal machine.

The notion of Martin-Löf randomness can easily be relativized to oracles: A is Martin-Löf random relative to B iff there is a constant c such that for all n, $H^B(A(0)\ldots A(n)) \geqslant n - c$. Here H^B is defined as H, but based on an oracle machine which is universal for any oracle B among the prefix-free machines using the same oracle. Obviously, there is a constant c such that $\forall B\ \forall x\ [H^B(x) \leqslant H(x) + c]$, thus if A is Martin-Löf random relative to B then A is already Martin-Löf random. In case Ω is Martin-Löf random relative to B, B is called *low for Ω*. By a result of Kučera and Slaman [18], this definition does not depend on the choice of the universal machine.

Investigations on this topic continues and in a recent survey, Miller and Nies [21] listed a lot of interesting open questions related to the halting probability Ω. The present work addresses some of these questions from the eighth chapter of the survey of Miller and Nies, namely the following three questions:

Question 8.1. Given a nonrecursive A which is low for Ω, does A then have hyperimmune Turing degree?
Question 8.9. Are there universal machines U, V such that $\Omega_U \not\equiv_{tt} \Omega_V$?
Question 8.10. Given a machine U which is universal by adjunction, is there a co-r.e. set X such that $\sum_{p:U(p)\in X} 2^{-|p|}$ is not Martin-Löf random? Can such an X be taken to be many-one complete?

The present work answers Question 8.9 and obtains some results on the way to settle Questions 8.1 and 8.10. For Question 8.1, it is shown that for every Π^0_1 class without recursive members, every member which is low for Ω is also hyperimmune. For Question 8.10, it is shown that there is some universal machine U for which there is such an X, but this U is not universal by adjunction. Furthermore, it is open whether there is a Π^0_1-complete X with the same property.

For convenience, strings in $\{0,1\}^*$, are identified with natural numbers. More precisely, the string $b_0 b_1 \ldots b_{n-1}$ is identified with $2^n - 1 + \sum_{m<n} 2^m \cdot b_m$. For infinite sequences, $A \in \{0,1\}^\infty$ stands for both the set $\{n : A(n) = 1\}$ and the real number $\sum_{n=0,1,\ldots} A(n) \cdot 2^{-n-1}$. So the relation $A < B$ can be transferred from numbers to sets with the additional convention, that for the two representations of numbers of the form $n \cdot 2^{-m}$ the one ending with $011111\ldots$ is below the one ending with $100000\ldots$ so that $<$ becomes a linear ordering on sets. It is a convention to write strings of the same length in alphabetical order from the left to the right like $000, 001, 010, 011, \ldots, 111$. One can do the same with infinite strings being the characteristic functions of approximations to A. If $A_0, A_1, \ldots$ approximates A from below (viewed as reals), one can say that it also approximates A from the left (viewed as infinite strings of symbols). This explains the term "left-r.e." to denote such reals. Furthermore, A is called recursively enumerable or just r.e. iff it is recursively enumerable as a set. This definition is important since many authors call all left-r.e. reals just "r.e." which produces a conflict between the notations used for real numbers and the ones used for the sets representing them. Note that there are further synonyms for "left-r.e." like "nearly computable" and "left-computable". More information on recursion theory and algorithmic randomness can be found in the standard textbooks [9,10,19,25,28,30].

2. On the probability that U outputs an element of X

Becher and Grigorieff [3] proposed to study the probability to halt with a value in a given set X. This version of a halting probability is formally defined as follows:

Definition 2. For given X, let

$$\Omega_U[X] = \sum_{p:U(p)\downarrow\in X} 2^{-|p|}$$

denote the probability for U to halt and output an element of X.

Given an infinite r.e. set X and a universal machine U, the probability $\Omega_U[X]$ that U halts with the output being an element of X is a left-r.e. Martin-Löf random number. This can easily be seen as follows: there is a partial-recursive one–one function f from the domain X onto all natural numbers with a total and recursive inverse g. Now the machine V given as

$$V(p) = \begin{cases} f(U(p)) & \text{if } U(p) \downarrow\in X, \\ \uparrow & \text{otherwise} \end{cases}$$

is universal: there is a constant c such that $H_U(g(x)) \leqslant H_U(x) + c$ for all x. The shortest program p with $U(p) = g(x)$ then satisfies $V(p) = x$. Thus $H_V(x) = H_U(g(x)) \leqslant H_U(x) + c$ for all x. So V is a universal machine and $U(p) \downarrow\in X$ iff $V(p)\downarrow$. Therefore $\Omega_U[X] = \Omega_V$. The number $\Omega_U[X]$ is left-r.e. and Martin-Löf random since Ω_V is.

Miller and Nies [21] note that $\Omega_U[F]$ is Martin-Löf random for any finite set F whenever U is universal by adjunction. But the following example shows that the randomness of $\Omega_U[F]$ depends in general on the universal machine.

Proposition 3. *There is a universal Turing machine U such that $\Omega_U[\{x\}] = 2^{1-H(x)}$ for all $x \in \{0,1\}^*$ where H is the Kolmogorov complexity based on U.*

Proof. Let $\tilde{U}$ be a universal Turing machine and let $\tilde{H}$ be the Kolmogorov complexity for machine $\tilde{U}$.

The new universal machine U and the Kolmogorov complexity H based on U are constructed such that for any x, $\Omega_U[\{x\}] = 2^{1-H(x)}$ and $H(x) = \tilde{H}(x) + 1$. So the goal is to ensure that $\Omega_U[\{x\}] = 2^{-\tilde{H}(x)}$. This is done by constructing the U such that it maps for every length greater than $\tilde{H}(x)$ exactly one string to x.

For obtaining this goal, one chooses for every x and every $n > \tilde{H}(x)$ the first string $p \in \{0,1\}^n$ found such that there are q, m with $p = q0^m1$ and $\tilde{U}(q) = x$. U remains undefined on all those strings which were not chosen for any n, x in this way.

As $\{(x,n) : n > \tilde{H}(x)\}$ is recursively enumerable, U is partial recursive. Furthermore, for every $p \in \mathrm{dom}(U)$ there is a $q \in \mathrm{dom}(\tilde{U})$ with $U(p) = U(q)$. So, as $\tilde{U}$ is prefix-free, U cannot be defined in a contradictory way. As $p = q0^m1$ for some m, one can see that whenever there are several extensions of q in the domain of U then these extensions are incomparable as strings; thus U is prefix-free. Also, it is easy to see that there is exactly one $p \in \{0,1\}^n$ with $U(p) = x$ in the case that $\tilde{H}(x) < n$ and no such p in the case that $\tilde{H}(x) \geqslant n$. Thus, one has for every $x \in \{0,1\}^*$ and all sets X the following three equalities:

$$H(x) = \tilde{H}(x) + 1,$$

$$\Omega_U[\{x\}] = \sum_{p:U(p)\downarrow=x} 2^{-|p|} = \sum_{n>\tilde{H}(x)} 2^{-n} = 2^{-\tilde{H}(x)} = 2^{1-H(x)},$$

$$\Omega_U[X] = \sum_{x\in X} 2^{1-H(x)}.$$

The proof is completed by noting that the first of these three equalities guarantees that U is universal. □

Notice that by the observation of Miller and Nies [21], the constructed U cannot be universal by adjunction since $\Omega_U[\{x\}]$ is rational.

The following proposition shows the existence of an infinite co-r.e. set X such that for any two elements $x, y \in X$ with $x < y$, the Kolmogorov complexity of y and beyond is guaranteed to be much larger than the one of the strings x and the elements smaller than x. The basic idea of the construction is to check in every stage for every current elements x, y with $x < y$ whether the strings beyond y are much more complicated than x in the way specified below and to enumerate y into the complement of X whenever it turns out that this is not the case. Note that the construction of X does not make any requirements on U and works for every universal machine.

Proposition 4. *There is an infinite co-r.e. set X such that there are no x, y, v, w with $x, y \in X$, $x < y \leqslant w$, $H(w) \leqslant v$ and $H(v) \leqslant x$.*

Proof. One constructs the complement Y of X as follows. Let Y_s denote all the elements enumerated into Y before stage s, so $Y_0 = \emptyset$. Let H_s denote a recursive approximation of H from above.

> At stage s, a number $y \in \{0, 1, \dots, s\} - Y_s$ is enumerated into Y iff there are $x, v, w \leqslant s$ with $x < y$, $x \notin Y_s$, $y \leqslant w$, $H_s(w) \leqslant v$ and $H_s(v) \leqslant x$.

It is easy to see that the so constructed enumeration is recursive and thus X is a co-r.e. set. Furthermore, if $x, y \in X$ and $x < y$, there cannot be any v, w such that $y \leqslant w$, $H(w) \leqslant v$ and $H(v) \leqslant x$, since there is a stage s where $H_s(v) \leqslant x$ and $H_s(w) \leqslant v$ and then y would be enumerated into Y at that stage s at the latest.

It remains to show that X is infinite. So assume by way of contradiction that X is finite and let a_0 be an upper bound of all elements of X. Then the following maxima and minima exist:

$$a_1 = \max\{u : H(u) \leqslant a_0\},$$

$$a_2 = \max\{u : H(u) \leqslant a_1\},$$

$$s = \min\{t : \{0, 1, \dots, a_2\} \subseteq X \cup Y_t\},$$

$$y = \min\{z : z \notin X \cup Y_s\}.$$

By assumption y is enumerated into Y at some stage $t \geqslant s$ and so there are $x, v, w \leqslant t$ witnessing this fact in the sense that $x \notin Y_t$, $x < y$, $H_t(v) \leqslant x$, $H_t(w) \leqslant v$ and $y \leqslant w$. Since $H_t(v) \geqslant H(v)$ and $H_t(w) \geqslant H(w)$ one has $H(v) \leqslant a_0$, $v \leqslant a_1$, $H(w) \leqslant v \leqslant a_1$, $w \leqslant a_2$ and $y \leqslant a_2$ in contradiction to $\{0, 1, \dots, a_2\} \subseteq X \cup Y_s \subseteq X \cup Y_t$ and $y \notin X \cup Y_t$. From this contradiction one can conclude that X is infinite. □

Using the above propositions, one obtains a partial result for Question 8.10 of Miller and Nies [21]. But this is not an answer to this question since Question 8.10 considers only machines which are universal by adjunction. Such machines are more difficult to handle.

Theorem 5. *There is a universal machine U and a co-r.e. set Y such that $\Omega_U[Y]$ is neither left-r.e. nor Martin-Löf random.*

Proof. The theorem is proven by choosing U, H as in Proposition 3 and

$$Y = \{x \in X : \forall x' \leqslant x\ [H(x') < (x - 1)/2]\}$$

for the set X from Proposition 4. Recall that x', as a binary string, has length at most $\log x' + 1$ and so there is a constant c such that $H(x') \leqslant 2 \log x' + c$. Hence Y contains all sufficiently large elements of X and so Y is an infinite subset of X.

First, assume by way of contradiction that $\Omega_U[Y]$ is left-r.e. via an approximation $b_0, b_1, \ldots$ and consider any $x \in Y$. Let $a = \sum_{y \in Y \cap \{0,1,\ldots,x\}} 2^{1-H(y)}$. One can compute from (x, a) numbers s, v_x such that s is the first number with $b_s > a$ and v_x the least number with $b_s > a + 2^{2-v_x}$.

Let y_x be the next element of Y after x. Note that $\Omega_U[Y] < a + 2^{2-H(y_x)}$ and thus $2^{2-H(y_x)} > 2^{2-v_x}$. It follows that $H(y_x) < v_x$.

By the choice of Y and by $x \in Y$, one has $H(x') < x/2$ for all $x' \leqslant x$. Thus, one can compute a from a description of x and of the first $x/2$ bits of a's binary representation. So $H(v_x) < x/2 + H(x) < x$ for all sufficiently large x. Using $H(v_x) < x$ and taking $w_x = y_x$, the numbers v_x and w_x witness that y_x is eventually enumerated into the complement of X according to the definition of X in Proposition 4. Thus, $y_x \notin Y$ in contradiction to the choice of y_x. This gives that $\Omega_U[Y]$ cannot be left-r.e. in contradiction to the above assumption.

Second it is shown that $\Omega_U[Y]$ is not random. Note that if $x < y$ and $x, y \in Y$ then $x < H(H(y))$ and thus $H(x) < x < H(y)$; the first relation $H(x) < x$ holds since all x with $H(x) \geqslant (x-1)/2$ had been removed from Y. So $\Omega_U[Y] = \sum_{x \in Y} 2^{1-H(x)}$ has the properties that all ones in its binary representation correspond to some term $2^{1-H(x)}$ and that between two ones there is at least one zero, namely, the one corresponding to 2^{1-x}. Thus, one knows that after every one in the binary representation comes a 0 and so $\Omega_U[Y]$ is not Martin-Löf random. □

There might be an alternative approach to prove this result. If one succeeds to construct U, Y such that $\Omega_U[Y]$ is neither left-r.e. nor right-r.e., then $\Omega_U[Y]$ is not Martin-Löf random: as $\Omega_U[Y]$ is the difference of the two left-r.e. reals Ω_U and $\Omega_U[\overline{Y}]$, this follows from a result of Rettinger and Zheng [26].

Becher et al. [2] show that for every universal machine U and for each sufficiently small but positive recursive real number R there is a set X such that X is recursive relative to the halting problem K and $\Omega_U[X] = R$. If one can choose the universal machine freely then one can even get that the corresponding X is a co-r.e. set.

Recall that A is H-trivial iff there is a c such that $\forall n\ [H(A(0)\ldots A(n)) \leqslant H(n) + c]$. Hence, the prefix-free Kolmogorov complexity of an H-trivial real is as low as possible. Every H-trivial real is K-recursive and the class of H-trivial reals is closed under $\oplus$ and contains a nonrecursive r.e. set [11]. Furthermore, X is H-trivial iff the relativized Kolmogorov complexity H^X and H differ at most by a constant [23].

Proposition 6. *There is a universal machine U and an integer m such that for every H-trivial real R between* 0 *and* 2^{-m} *there is a co-r.e. set X with $R = \Omega_U[X]$.*

Proof. Given a universal machine $\tilde{U}$ which outputs within s steps only numbers smaller than 2^s, one can construct a new universal machine $\tilde{V}$ with the following property: if $\tilde{U}$ outputs on input p a number x after s steps then $\tilde{V}(p00) = x$ and $\tilde{V}(p1^k0) = 2^s \cdot 3^k$ for $k > 0$. Note that $\tilde{V}$ is also prefix-free. In particular, the complexity $\tilde{H}$ based on $\tilde{V}$ has an approximation $\tilde{H}_s$ such that for almost all n there is an x such that $\tilde{H}_x(x) = \tilde{H}(x) = n$. By the way, this x is the largest number with $\tilde{H}(x) = n$. If in Proposition 3 one constructs U from $\tilde{V}$ instead of constructing from $\tilde{U}$, this property is preserved to U and the complexity H based on U: there is an approximation H_s such that the numbers

$$x_n = \max\{z : z = 0 \vee H(z) \leqslant n\}$$

satisfy $H_{x_n}(x_n) = H(x_n)$ for all n. Furthermore, $H(x_n) = n$ for almost all n; one now chooses the constant m for the proposition such that $m \geqslant 2$ and $\forall n \geqslant m\ [H(x_n) = n]$.

Let R be an H-trivial real with $0 < R < 2^{-m}$. The aim is now to build a co-r.e. set X such that

$$R = \sum_{r \in R} 2^{-1-r} = \sum_{x \in X} 2^{1-H(x)} = \Omega_U[X],$$

where this goal is met by choosing $X \subseteq \{x_m, x_{m+1}, \ldots\}$ such that

$$x_n \in X \Leftrightarrow n - 2 \in R.$$

The further construction makes use of the fact that there is an r.e. H-trivial set $Q \geqslant_T R$ [23, Theorem 7.4]. This fact guarantees that R has a recursive approximation $R_0, R_1, \ldots$ such that the function c_R defined as

$$c_R(n) = \max\{s : s = 1 \vee \exists m < n\ [R_{s-1}(m) \neq R(m)]\}$$

can be computed relative to Q. For all n let

$$y_n = \max\{z \leqslant c_R(n) : z = 0 \vee H_z(z) = n\}.$$

Note that the sequence $y_0, y_1, \ldots$ can be computed relative to Q. Nies [23] showed that A is H-trivial if and only if there is a constant d such that $H(z) \leqslant H^A(z) + d$ for all z. Since Q is H-trivial and therefore H^Q differs from H only by a constant, one has that $H^Q(y_n)$ and $H(n)$ also differ at most by a constant. Thus, for almost all n, $H^Q(y_n) < H^Q(x_n)$ and $y_n < x_n$. For these n it holds that $R_{x_n}(n-2) = R(n-2)$. Without loss of generality one can assume this property for all $n \geqslant m$ since a finite modification of the approximation $R_0, R_1, \ldots$ would enforce it. After ensuring this property, one defines the co-r.e. set

$$X = \{x : H_x(x) \geqslant m \wedge R_x(H_x(x) - 2) = 1 \wedge \forall y > x\ \forall t\ [H_t(y) > H_x(x)]\}.$$

Now the connection between X and R is verified. On one hand, consider any $x \in X$ and let $n = H(x)$. Then the condition $\forall y > x\ \forall t\ [H_t(y) > H_x(x)]$ enforces that $H(y) > H(x)$ for all $y > x$ and thus $x = x_n$. Since $H_{x_n}(x_n) = H(x_n)$, one furthermore has that $n \geqslant m$. So $X \subseteq \{x_m, x_{m+1}, \ldots\}$. On the other hand, if $n \geqslant m$, then x_n satisfies $H_{x_n}(x_n) = H(x_n) = n$ and $R_{x_n}(H_{x_n}(x_n) - 2) = R(n-2)$. So one has for $n \geqslant m$ that $x_n \in X \Leftrightarrow n - 2 \in R$. Since by the choice of R no number below m is in R, the equivalence $x_n \in X \Leftrightarrow n - 2 \in R$ holds for all n and one has the following equalities:

$$\begin{aligned}\Omega_U[X] &= \sum_{x \in X} R_x(H_x(x) - 2) 2^{1-H(x)} \\ &= \sum_{x \in X} R(H(x) - 2) 2^{1-H(x)} \\ &= \sum_{n=m}^{\infty} R(n-2) 2^{1-n} = R.\end{aligned}$$

This completes the proof. □

3. Halting probability and truth-table reducibility

In Question 8.9, Miller and Nies [21] asked whether there are two different universal machines such that the corresponding Ω numbers are not tt-equivalent. One can show that the tt-degrees of the Ω numbers contain even an infinite antichain. Note that the resulting universal machines are universal by adjunction whenever the starting machine U is universal by adjunction; thus, Question 8.9 is answered completely by the theorem below.

Theorem 7. *Given a universal machine U, one can construct a whole sequence $U_1, U_2, \ldots$ of universal machines and an r.e. real X such that the Ω numbers $\Omega_{U_1}, \Omega_{U_2}, \ldots$ defined by $\Omega_{U_m} = 2^{-1} \cdot \Omega_U + 2^{-m} X$ form an antichain.*

Proof. Let X be a creative subset of the odd natural numbers. Recall that X is creative if it is r.e. and there is a recursive function f with $f(e) \notin X \cup W_e$ whenever W_e is disjoint from X. Now define U_m such that

$$U_m(ap) = \begin{cases} U(p) & \text{if } a = 0 \text{ and } U(p)\downarrow, \\ 0 & \text{if } ap = 1^{m+n-1}0 \text{ for an } n \in X, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then one easily sees that $\Omega_{U_m} = 2^{-1} \cdot \Omega_U + 2^{-m} X$ and that U_m is prefix-free. Furthermore, all programs of U are translated into U_m by placing a 0 in front, hence U_m is universal. Finally, it is easy to see that $\Omega_{U_i} - \Omega_{U_j} = (2^{-i} - 2^{-j})X$ for all i, j. Assume that $i \neq j$. Then $X = (\Omega_{U_i} - \Omega_{U_j})/(2^{-i} - 2^{-j})$ and X is truth-table reducible to Ω_{U_j} whenever $\Omega_{U_i} \leqslant_{tt} \Omega_{U_j}$ and to Ω_{U_i} whenever $\Omega_{U_j} \leqslant_{tt} \Omega_{U_i}$. But since X is creative and not truth-table reducible to a Martin-Löf random set [4,5], it cannot happen that Ω_{U_i} and Ω_{U_j} are tt-comparable and $\Omega_{U_1}, \Omega_{U_2}, \ldots$ is an infinite antichain for truth-table reducibility. □

A related question is whether Ω numbers of incomparable tt-degrees form a minimal pair. This is still unknown, but at least one knows that they do not have to.

Theorem 8. *There are Ω numbers which are incomparable but do not form a minimal pair with respect to truth-table reducibility.*

Proof. Recall that a set Z is identified with the real $\sum_{n \in Z} 2^{-1-n}$ and by the same way any real is identified with a set. One chooses universal machines U, V such that there is a recursive set Y and a creative set X satisfying

- $X \subseteq Y$;
- $\Omega_U + X = \Omega_V$;
- for all $x, y \in Y$ with $x < y$ there is a $z \notin \Omega_U$ with $x + 1 < z < y$.

Ω_U and Ω_V have different tt-degree since otherwise $X \leqslant_{tt} \Omega_U$ although no creative set is truth-table reducible to a random set. Furthermore, for every $x \in Y$, the digits $\Omega_U(x+1)$ and $\Omega_V(x+1)$ coincide as the nonelement z of Ω_U between $x+1$ and the next $y \in Y$ absorbs any eventual carry-bit in the addition $\Omega_U + X$. It follows that the set-theoretic intersection $\Omega_U \cap \{y + 1 : y \in Y\}$ is equal to $\Omega_V \cap \{y + 1 : y \in Y\}$ and therefore tt-reducible to these two sets. But this intersection is not recursive as Ω_U is random and $\{y + 1 : y \in Y\}$ an infinite recursive set. □

Call a tt-reduction M order-preserving iff $M(X) \leqslant M(Y)$ for all reals X, Y with $X \leqslant Y$. The next result shows that for order-preserving tt-reducibility, all Ω numbers are either equivalent or incomparable; thus together with the previous result one has that their degrees form an infinite antichain.

Proposition 9. *Let U, V be universal machines. If $\Omega_U \leqslant_{tt} \Omega_V$ via an order-preserving tt-reduction then $\Omega_U \equiv_{tt} \Omega_V$ and the reverse tt-reduction is also order-preserving.*

Proof. Let M denote the order-preserving tt-reduction from Ω_U to Ω_V and let f be its recursive use. For every length n there are with respect to the ordering $<$ on real numbers a least set $X_n \subseteq \{0, 1, \dots, f(2n)\}$ and a greatest set $Y_n \subseteq \{0, 1, \dots, f(2n)\}$ such that both $M(X_n)$ and $M(Y_n)$ coincide with Ω_U on the first $2n$ bits. If n is sufficiently large, then there is no finite set $F \subseteq \{0, 1, \dots, n\}$ such that $X_n < F \leqslant Y_n$ as real numbers since otherwise the first $2n$ bits of Ω_U could be computed from the $n + 1$ bits coding F and some code for n using $H(n)$ bits in contradiction to Ω_U being random. As a consequence, the first n bits of X_n and Y_n must be the same and both coincide with those of Ω_V. Since only the first $2n$ bits of Ω_U are relevant for these considerations, one can, for almost all n, compute the first n bits of Ω_V from the first $2n$ of Ω_U and patch the remaining finitely many cases from a table. So there is a tt-reduction from Ω_V to Ω_U. It is easy to see that this reverse tt-reduction is also order-preserving. □

4. On low for Ω sets

Recall that a set X is low for Ω iff the set Ω is Martin-Löf random relative to X. Furthermore, a set X has hyperimmune degree if there is a function $f \leqslant_T X$ not majorized by any recursive function and X has hyperimmune-free degree otherwise.

In the following let T be an infinite recursive tree, that is, let $T \subseteq \{0, 1\}^*$ be recursive and have the property that $\sigma \in T$ whenever $\sigma\tau \in T$ for $\sigma, \tau \in \{0, 1\}^*$. A set A is an infinite branch of T iff all nodes of the form $A(0)A(1)\dots A(n)$ are members of T. For recursive trees the effective analogue of König's Lemma fails and T may fail to have recursive infinite branches. But several results guarantee that some infinite branches of T are near to being recursive: Jockusch and Soare [17] showed that every infinite recursive tree has infinite branches of low degree and infinite branches of hyperimmune-free degree; Downey et al. [13] showed that every infinite recursive tree has an infinite branch which is low for Ω. If a tree has only recursive branches, then all of its branches are low for Ω. But one might ask under which conditions all infinite branches of an infinite recursive tree are low for Ω. The next result shows that this cannot happen if all infinite branches are nonrecursive; indeed in that case the infinite branches of hyperimmune-free degree are not low for Ω.

Theorem 10. *If T is an infinite recursive binary tree without infinite recursive branches then every A on T which is low for Ω also has hyperimmune Turing degree.*

Proof. Assuming that the theorem would be wrong, it is shown that then a set D would exist which is low for Ω, K-recursive and not H-trivial. This gives then a contradiction as by a result of Hirschfeldt et al. [15] such a set D does not exist.

So assume now by way of contradiction that T is an infinite recursive binary tree without infinite recursive branches and A is an infinite branch of T which is low for Ω and has hyperimmune-free Turing degree. Let U be a prefix-free universal oracle Turing machine, that is, for every oracle A and every prefix-free partial recursive V^A there is a constant c such that $H^A_U(x) \leqslant H^A_V(x) + c$

for all x in the range of V^A. From now on, let H^A denote H_U^A and let H_s^A be an A-recursive approximation from above to H^A where for the sth approximation the oracle A is queried only below s.

Since A is low for Ω, there is a constant c such that $H^A(\Omega(0)\dots\Omega(n))\geqslant n-c$ for all n. Let Ω_s be an approximation of Ω from the left. For every n there is an $s\geqslant n$ such that

$$\forall m\leqslant n\ [H_s^A(\Omega_s(0)\dots\Omega_s(m))\geqslant m-c]$$

and since A has hyperimmune-free Turing degree, there is a recursive function f such that this s is between n and $f(n)$ for all n. Now one can construct a new recursive binary tree $S\subseteq T$ such that

an infinite branch B of T is also an infinite branch of S iff
$\forall n\ \exists s\in\{n,n+1,\dots,f(n)\}\ \forall m\leqslant n\ [H_s^B(\Omega_s(0)\dots\Omega_s(m))\geqslant m-c]$.

The listed condition is a Π_1^0 condition since the first quantifier is unbounded and universal and all other quantifiers have recursively bounded range. Thus, there is such a recursive tree S. Furthermore, A is on this tree S and all infinite branches on the tree S are low for Ω.

Since $S\subseteq T$, S does not have any infinite recursive branch. It follows that no infinite branch of S is isolated, indeed through every node of S go either no or uncountably many infinite branches. Using the oracle K one can decide for any node which of these two cases applies. If a set B is not H-trivial then there is for every n a number u_n such that $H^B(u_n)+n<H(u_n)$. Since there are only countably many sets which are H-trivial, one can construct a sequence $\sigma_0,\sigma_1,\dots$ of nodes of S with the following properties:

- σ_n is above all nodes σ_m with $m<n$;
- there are infinitely many nodes in S above σ_n;
- there is a number u_n such that $H^B_{|\sigma_n|}(u_n)+n<H(u_n)$ for all oracles B extending σ_n.

Note that the construction does not terminate: given σ_n and assuming that there is one infinite branch of T through σ_n, one can conclude that there are uncountably many infinite branches through σ_n since T has no recursive infinite branches. As there are only countably many H-trivial sets, one infinite branch B of T through σ_n is not H-trivial. Thus, there is a number u_{n+1} with $H^B(u_{n+1})<H(u_{n+1})-n+1$ and σ_{n+1} is a prefix of B longer than σ_n such that the computation witnessing $H^B(u_{n+1})<H(u_{n+1})-n+1$ halts in less than $|\sigma_{n+1}|$ steps.

The so constructed sequence $\sigma_0,\sigma_1,\dots$ can be constructed using the halting problem and defines a K-recursive branch D of S. Furthermore, $H^D(u_n)+n<H(u_n)$ for each n since D and the B above both extend σ_n and the approximation $H^B_{|\sigma_n|}(u_n)$ evaluates B only at places which actually belong to the domain of the string σ_n. It follows that D is low for Ω, K-recursive and not H-trivial. As said before, such a set D does not exist. □

Nies et al. [24] showed that every set which is Martin-Löf random and low for Ω is already Martin-Löf random relative to K and thus does not have hyperimmune-free degree. This result can be generalized to diagonally nonrecursive degrees as follow. Recall that a set A has diagonally nonrecursive degree iff there is a total function $f\leqslant_T A$ such that $f(x)\neq\varphi_x(x)$ whenever $\varphi_x(x)$ is defined. In the case that A is in addition of hyperimmune-free degree, one can even find a function φ_e such that $f=\varphi_e^A$ and φ_e^B is total for all oracles B, that is, φ_e is a tt-reduction computing f relative to A. Now the tree $T=\{B:\forall x\in K\ [\varphi_e^B(x)\neq\varphi_x(x)]\}$ has only diagonally nonrecursive and thus no recursive infinite branch. Therefore, one obtains the following corollary.

Corollary 11. *If a set has diagonally nonrecursive degree and is low for Ω then it also has hyperimmune degree.*

Although Question 8.1 is not completely answered, one can use Theorem 10 to get a weaker but related result. Here, a Turing degree is hyperimmune relative to K if it can compute a function which is not dominated by any K-recursive function. Note that Example 13 and Theorem 14 show that Theorem 12 covers some but not all degrees in question.

Theorem 12. *Let A be a nonrecursive set such that the Turing degree of A is (unrelativized) hyperimmune-free and the Turing degree of $A \oplus K$ is hyperimmune-free relative to K. Then A is on a recursive binary tree without infinite recursive branches; in particular, A is not low for Ω.*

Proof. Given a nonrecursive set A such that the Turing degree of A is (unrelativized) hyperimmune-free and the Turing degree of $A \oplus K$ is hyperimmune-free relative to K, a recursive binary tree T which has A but no recursive set as an infinite branch is constructed. The construction uses the auxiliary functions F, G, P, Q considered in the next paragraphs. Since A is not recursive, the function

$$F^A(e) = \min\{x : A(x) \neq \varphi_e(x) \vee \varphi_e(x) \uparrow\}$$

is total. F can be computed relative to $A \oplus K$. There is a K-recursive function majorizing F which is recursively approximated by a function G_s. Let G be the maximum of the G_s so that

$$G(e) = \max\{G_s(e) : s = 0, 1, \ldots\} \geqslant F^A(e)$$

holds for all e. Now G also majorizes F. For every e, s, one can compute relative to A the value $P^A(e, s)$ which is defied as the least t such that the following formula $\Psi^A(e, s, t)$ is true:

$\Psi^A(e, s, t)$ is true iff one of the following three conditions holds:

- $\exists u \leqslant t\ [G_{s+u}(e) > \max\{G_0(e), G_1(e), \ldots, G_s(e)\}]$;
- $\exists r \leqslant G_s(e)\ [\varphi_{e,s+t}(r) \uparrow]$;
- $\exists r \leqslant G_s(e)\ [\varphi_{e,s+t}(r) \downarrow \neq A(r)]$.

Since A has hyperimmune-free Turing degree and the function P is A-recursive, there is a recursive function Q majorizing P. Now one can define an infinite recursive binary tree T such that B is an infinite branch of T iff

$$\forall e\ \forall s\ \exists t \leqslant Q(e, s)\ [\Psi^B(e, s, t)].$$

It is clear that A is on the tree T. If B is recursive then there is an index e such that φ_e is the characteristic function of B. Let s be so large such that $\varphi_{e,s}(x)$ is defined for every $x \leqslant G(e)$ and $G_u(e) = G(e)$ for some $u \in \{0, 1, \ldots, s\}$. Then there is no $t \leqslant Q(e, s)$ such that $\Psi^B(e, s, t)$ is defined and thus B is not an infinite branch of T.

So it follows that A is an infinite branch of a recursive binary tree without recursive infinite branches. Since A has hyperimmune-free Turing degree, it follows from Theorem 10 that A is not low for Ω. □

The following example shows that this result does not capture all hyperimmune-free Turing degrees. The basic idea of the construction is from Miller and Martin [22]. More precisely, one combines the just mentioned construction of Miller and Martin as presented by Odifreddi [25, Propositions V.5.5 and V.5.6] with the idea of forcing out of a given list of trees. The method of Miller and Martin was improved by Jockusch [16] who built a nonrecursive set of biimmune-free degree. Recall that a set A is biimmune if neither A nor $\overline{A}$ has an infinite recursive subset, that a biimmune-free degree is a Turing degree not containing any biimmune set and that every biimmune-free degree is also hyperimmune-free but not vice versa [16].

Example 13. Given a list $T_0, T_1, \ldots$ of trees without recursive infinite branches, there is a non-recursive set A of biimmune-free degree such that A is not on any of these trees.

Proof. A perfect recursive tree is a nonempty recursive tree $T \subseteq \{0, 1\}^*$ such that at least two infinite branches go through every node. Now one constructs a sequence of recursive perfect trees $P_0, P_1, \ldots$ such that P_0 is the full tree $\{0, 1\}^*$ and P_{e+1} is chosen from P_e as follows:

- choose a node $\sigma_e \in P_e$ such that $\sigma_e \notin T_e$ and the eth partial-recursive function does not compute an extension of σ_e;
- choose $P_{e+1} \subseteq P_e$ such that σ_e is below all branching nodes in P_{e+1} and either φ_e^A is partial or φ_e^A is not $\{0, 1\}$-valued or φ_e^A is not biimmune for all infinite branches of P_{e+1}.

Since every perfect recursive tree has infinite recursive branches, $P_e \not\subseteq T_e$ and so there are several incomparable nodes of P_e outside T_e. At least one of them is not extended by φ_e and so one can choose σ_e according to the given requirements.

For the second step, the reader is referred to Jockusch's construction [16] which is not reproduced here; it is an improved method of the corresponding one for hyperimmune-free degrees [25, Proposition V.5.5].

The set A is then the unique infinite branch which is on all the trees P_e. The construction gives that A has biimmune-free degree [16] and furthermore A is not on any tree T_e. □

So known methods easily give the existence of a set of hyperimmune-free degree whose jump has hyperimmune degree relative to K. Indeed the function $F \leqslant_T A \oplus K$ given by

$$F(e) = \min\{x : A(x) \neq \varphi_e(x) \vee \varphi_e(x) \uparrow\}$$

is not bounded by any total K-recursive function. Downey and Mileti [14] announced the following result which shows that Theorem 12 is not directly implied by a restriction on the jumps of the hyperimmune-free degrees.

Theorem 14 (*Downey and Milet [14]*). *There is a nonrecursive set of hyperimmune-free degree such that the degree of its jump is hyperimmune-free relative to K.*

Acknowledgments

The authors would like to thank Rod Downey, Carl Jockusch, Joseph Mileti, André Nies and Robert Rettinger for correspondence. A. Nies independently observed that every infinite recursive binary tree without recursive branches has an infinite K-recursive branch which is not H-trivial; this construction is implicitly used in Theorem 10.

The authors are grateful to the anonymous referees for many helpful comments. Furthermore, some of the anonymous referees pointed out that there is an alternative name for the notion of prefix-free Kolmogorov complexity which is traditionally called "*H*" in papers like "Program size complexity for possibly infinite computations" by Becher, Figueira, Nies and Picchi [1] and "Trivial Reals" by Downey, Hirschfeldt, Nies and Stephan [11]. Indeed, some of these authors recently prefer the name "*K*" as for example in the survey by Downey, Hirschfeldt, Nies and Terwijn [12], the list of open problems by Nies and Miller [21] and an upcoming book by Downey and Hirschfeldt [10]. Somehow, the authors of the present work preferred to keep their traditional notion "*H*" from [1,11] and other previous publications.

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