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Generalized polar varieties: geometry and algorithms☆

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Dedicated to Arnold Schönhage

Abstract

Let V be a closed algebraic subvariety of the n-dimensional projective space over the complex or real numbers and suppose that V is non-empty and equidimensional. The classic notion of a polar variety of V associated with a given linear subvariety of the ambient space of V was generalized and motivated in Bank et al. (Kybernetika 40 (2004), to appear). As particular instances of this notion of a generalized polar variety one reobtains the classic one and an alternative type of a polar variety, called *dual*. As main result of the present paper we show that for a generic choice of their parameters the generalized polar varieties of V are empty or equidimensional and smooth in any regular point of V. In the case that the variety V is affine and smooth and has a complete intersection ideal of definition, we are able, for a generic parameter choice, to describe locally the generalized polar varieties of V by explicit equations. Finally, we indicate how this description may be used in order to design in the context of algorithmic elimination theory a highly efficient, probabilistic elimination procedure for the following task: In case, that the variety V is $\mathbb{Q}$-definable and affine, having a complete intersection

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ideal of definition, and that the real trace of V is non-empty and smooth, find for each connected component of the real trace of V an algebraic sample point.
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1. Introduction

The notion of *generalized polar varieties* was introduced and motivated in [4]. The present paper is devoted to the study of their smoothness which entails important algorithmic consequences. It turns out that the classic polar varieties are special instances of the generalized ones. Classic polar varieties were used in [2,3] for the design of a highly efficient elimination procedure which, in the case of an affine, smooth and *compact* real hypersurface or, more generally, a complete intersection variety, produces an algebraic sample point for each connected component of the given real variety. The aim to generalize this algorithmic result to the *non-compact* case motivates the introduction of the concept of the generalized polar varieties of a given algebraic manifold associated with suitably generic linear subvarieties and a non-degenerate hyperquadric of the projective ambient space. In [4] it was shown that these generalized polar varieties become Cohen–Macaulay. This rather *algebraic* than geometric result suffices to solve the algorithmic task which motivates the consideration of generalized polar varieties. In this paper we go a step further: we show a *geometric* result saying that under the same genericity condition the generalized polar varieties of a given algebraic manifold are smooth, and we derive handy local equations for them.

Let $\mathbb{P}^n$ denote the n-dimensional projective space over the field of complex numbers $\mathbb{C}$ and let, for $0 \leqslant p \leqslant n$, V be a pure p-codimensional closed algebraic subvariety of $\mathbb{P}^n$.

Now we are going to outline the basic properties of the notion of a generalized polar variety of V associated with a given linear subspace K, a given non-degenerate hyperquadric Q and a given hyperplane H of the ambient space $\mathbb{P}^n$, subject to the condition that $Q \cap H$ is a non-degenerate hyperquadric of H. We denote this generalized polar variety by $\widehat{W}_K(V)$. It turns out that $\widehat{W}_K(V)$ is empty or a smooth subvariety of V having pure codimension i in V, if V is smooth and K is a “sufficiently generic”, $(n-p-i)$-dimensional, linear subspace of $\mathbb{P}^n$, for $0 \leqslant i \leqslant n-p$ (see Corollary 11 and the following comments).

In this paper we consider mainly the case that H is the hyperplane at infinity of $\mathbb{P}^n$, fixing in this manner an embedding of the complex n-dimensional affine space $\mathbb{A}^n$ into the projective space $\mathbb{P}^n$. Let $S := V \cap \mathbb{A}^n$ be the affine trace of V and suppose S is non-empty. Then S is a pure p-codimensional closed subvariety of the affine space $\mathbb{A}^n$.

The affine trace $\widehat{W}_K(S) := \widehat{W}_K(V) \cap \mathbb{A}^n$ is called the *affine* generalized polar variety of S associated with the linear subvariety K and the hyperquadric Q of $\mathbb{P}^n$. The affine generalized polar varieties of S give rise to classic and dual affine polar varieties.

Let us denote the field of real numbers by $\mathbb{R}$ and the real n-dimensional projective and affine spaces by $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{A}^n_{\mathbb{R}}$, respectively. Assume that V is $\mathbb{R}$-definable and let $V_{\mathbb{R}} :=$

$V \cap \mathbb{P}^n_{\mathbb{R}}$ and $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}} = V \cap \mathbb{A}^n_{\mathbb{R}}$ be the real traces of the complex algebraic varieties V and S. Similarly, let $H_{\mathbb{R}} := H \cap \mathbb{P}^n_{\mathbb{R}}$ be the hyperplane at infinity of the real projective space $\mathbb{P}^n_{\mathbb{R}}$. Suppose that the real codimension of $V_{\mathbb{R}}$ and $S_{\mathbb{R}}$ at any point is p, that $S_{\mathbb{R}}$ (and hence $V_{\mathbb{R}}$) is non-empty and that K and Q are $\mathbb{R}$-definable. Then the generalized *real* polar varieties $\widehat{W}_K(V_{\mathbb{R}}) := \widehat{W}_K(V) \cap \mathbb{P}^n_{\mathbb{R}}$ and $\widehat{W}_K(S_{\mathbb{R}}) := \widehat{W}_K(S) \cap \mathbb{A}^n_{\mathbb{R}} = \widehat{W}_K(V) \cap \mathbb{A}^n_{\mathbb{R}}$ are well defined and lead to the notions of a classic and a dual polar variety of $V_{\mathbb{R}}$ and $S_{\mathbb{R}}$. Suppose that $S_{\mathbb{R}}$ is smooth. Then "sufficiently generic" real dual polar varieties of $S_{\mathbb{R}}$ contain for each connected component of $S_{\mathbb{R}}$ at least one algebraic sample point. The same is true for the real classic polar varieties if additionally the ideal of definition of S is a complete intersection ideal and if $S_{\mathbb{R}}$ is compact (see Propositions 1 and 2).

Let $\mathbb{Q}$ be the field of rational numbers, let $X_1, \ldots, X_n$ be indeterminates over $\mathbb{R}$ and let a regular sequence $F_1, \ldots, F_p$ in $\mathbb{Q}[X_1, \ldots, X_n]$ be given such that $(F_1, \ldots, F_p)$ is the ideal of definition of the affine variety S. Then, in particular, S is a $\mathbb{Q}$-definable, complete intersection variety. Suppose that the hyperquadric Q is given by a non-degenerate quadratic form over $\mathbb{Q}$ and, in particular, that $Q \cap H_{\mathbb{R}}$ can be described by the standard, n-variate positive definite quadratic form (inducing on $\mathbb{A}^n_{\mathbb{R}}$ the usual euclidean distance). Assume that the projective linear variety K is spanned by $n-p-i+1$ rational points $(a_{1,0} : \cdots : a_{1,n}), \ldots, (a_{n-p-i+1,0} : \cdots : a_{n-p-i+1,n})$ of $\mathbb{P}^n$ with $a_{j,1}, \ldots, a_{j,n}$ generic for $1 \leqslant j \leqslant n-p-i+1$. Thus K has dimension $n-p-i$. Then, if S is smooth, the generalized affine polar variety $\widehat{W}_K(S)$ is empty or of pure codimension i in S. Moreover, $\widehat{W}_K(S)$ is smooth and its ideal of definition in $\mathbb{Q}[X_1, \ldots, X_n]$ is generated by $F_1, \ldots, F_p$ and by all $(n-i+1)$-minors of the polynomial $((n-i+1) \times n)$matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} - a_{1,0}X_1 & \cdots & a_{1,n} - a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0}X_n \end{bmatrix}$$

(see Theorem 10).

In [2,3], classic polar varieties were used in order to design a new generation of efficient algorithms for finding at least one algebraic sample point for each connected component of a given smooth, compact hypersurface or complete intersection subvariety of $\mathbb{A}^n_{\mathbb{R}}$.

Let us illustrate this comment in the case of the two-dimensional unit sphere S, which is a smooth, compact hypersurface of $\mathbb{A}^3_{\mathbb{R}}$ given by the polynomial $F := X_1^2 + X_2^2 + X_3^2 - 1$. Let Q be the hyperquadric of $\mathbb{P}^3_{\mathbb{C}}$ defined by the quadratic form $X_0^2 + X_1^2 + X_2^2 + X_3^2$ and let K^1 and K^0 be the linear subvarieties of $\mathbb{P}^3_{\mathbb{C}}$ defined by the linear equation systems $X_0 = 0, X_3 = 0$ and $X_0 = 0, X_3 = 0, X_2 = 0$, respectively. Observe that K^1 is spanned by the rational points $(0:1:0:0)$ and $(0:0:1:0)$, and that K^0 consists of the point $(0:0:0:1)$. One verifies easily that the real polar variety $\widehat{W}_{K^1}(S)$ is described by the vanishing of the polynomials F and $\frac{\partial F}{\partial X_3}$, whereas the polar variety $\widehat{W}_{K^0}(S)$ is described by the vanishing of F, $\frac{\partial F}{\partial X_3}$ and $\frac{\partial F}{\partial X_2}$. Therefore, $\widehat{W}_{K^1}(S)$ and $\widehat{W}_{K^0}(S)$ are the classic real polar varieties obtained by cutting the two-dimensional unit sphere S with the linear subspaces of

$\mathbb{A}^3_{\mathbb{R}}$ given by the equation systems $X_3 = 0$ and $X_3 = X_2 = 0$, respectively. In other words, $\widehat{W}_{K^0}(S)$ is the zero-dimensional algebraic variety $\{(1, 0, 0),\ (-1, 0, 0)\}$, and $\widehat{W}_{K^1}(S)$ is the unit circle of $\mathbb{A}^2_{\mathbb{R}}$. The elimination algorithm of [2] or [3] applied to the input equation $F = X_1^2 + X_2^2 + X_3^2 - 1 = 0$ consists in solving the zero-dimensional equation system $X_1^2 + X_2^2 + X_3^2 - 1 = 0$, $X_2 = 0$, $X_3 = 0$, which describes the polar variety $\widehat{W}_{K^0}(S)$.

In this paper we will use *dual* polar varieties for the same algorithmic task in the *non-compact* (but still smooth) case. This leads to a complexity result that represents the basic motivation of this paper: If the real variety $S_{\mathbb{R}}$ is non-empty and smooth and if S is given as before by a regular sequence $F_1, \ldots, F_p$ in $\mathbb{Q}[X_1, \ldots, X_n]$ such that, for any $1 \leqslant h \leqslant p$, the ideal generated by $F_1, \ldots, F_h$ is radical, then there exists a uniform, *probabilistic* algorithm which finds a (real algebraic) sample point for each connected component of $S_{\mathbb{R}}$ in expected sequential time $\tilde{O}(\binom{n}{p} L n^4 p^2 d^2 \delta^2)$ (counting arithmetic operations in $\mathbb{Q}$ at unit costs). Here, the $\tilde{O}$-notation (introduced in [45]) indicates that we neglect polylogarithmic factors in the complexity estimate, d is an upper bound for the degrees of the polynomials $F_1, \ldots, F_p$, L denotes the (sequential time) *arithmetic circuit complexity* of them and $\delta \leqslant d^n\, p^{n-p}$ is the (suitably defined) *degree of the real interpretation* of the polynomial equation system $F_1, \ldots, F_p$ (see Theorems 13 and 14 and for the pattern of elimination algorithm described here compare [21] and [22]). Although this complexity bound is *polynomial* in δ, it may become exponential with respect to the number of variables n, at least in the worst case. This exponential worst case complexity becomes unavoidable since $S_{\mathbb{R}}$ may contain exponentially many connected components. On the other hand, the elimination problem under consideration is intrinsically of non-polynomial character with respect to the syntactic input length for any reasonable continuous data structure (cf. [11,20]).

In view of [12] we may conclude that no numerical procedure (based on the bit representation of integers) is able to solve this algorithmic task more efficiently than our symbolic-seminumeric procedure.

2. Intrinsic aspects of polar varieties

For two given linear subvarieties A and B of the complex n-dimensional projective space $\mathbb{P}^n$ we denote by $\langle A, B \rangle$ the linear subvariety of $\mathbb{P}^n$ spanned by A and B. We say that A and B intersect transversally (in symbols: $A \pitchfork B$) if $\langle A, B \rangle = \mathbb{P}^n$ holds. In case that A and B do not intersect transversally, we shall write $A \not\pitchfork B$, (observe that $\dim A + \dim B < n$ implies $A \not\pitchfork B$). Let V be a projective algebraic subvariety of $\mathbb{P}^n$ and suppose that V is of pure codimension p for some $0 \leqslant p \leqslant n$ (this means that all irreducible components of V have the same codimension p). We denote by V_{reg} the set of all regular (smooth) points of V. Observe that V_{reg} is a complex submanifold of $\mathbb{P}^n$ of codimension p and that V_{reg} is Zariski-dense in V. We call $V_{\mathrm{sing}} := V \setminus V_{\mathrm{reg}}$ the singular locus of the projective variety V. Let V and W be two given pure codimensional projective subvarieties of $\mathbb{P}^n$ and let M be a given point of $\mathbb{P}^n$ belonging to the intersection of V_{reg} and W_{reg}. We say that V and W intersect transversally at the point M if the Zariski tangent spaces $T_M V$ and $T_M W$ of the algebraic varieties V and W at the point M intersect transversally (here we interprete $T_M V$ and $T_M W$ as linear subvarieties of the ambient space $\mathbb{P}^n$ that contain the point M).

We assume for the moment that the variety V is the projective closure of a given closed algebraic subvariety S of the affine space $\mathbb{A}^n$ and that S has pure codimension p. We call $S_{\mathrm{reg}} := V_{\mathrm{reg}} \cap \mathbb{A}^n$ and $S_{\mathrm{sing}} := V_{\mathrm{sing}} \cap \mathbb{A}^n$ the set of smooth (regular) points and the singular locus of the affine variety S, respectively. For any smooth point M of the affine variety S we interprete, as usual, the tangent space $T_M\, S$ of S at M as a linear subspace of $\mathbb{A}^n$ passing through the origin. Thus, if we interprete M as a point of the projective variety V, the affine trace of the tangent space $T_M\, V$ of V at M, namely $T_M\, V \cap \mathbb{A}^n$, turns out to be the affine linear subspace of $\mathbb{A}^n$ that is parallel to $T_M\, S$ and passes through M, namely $M + T_M\, S$. In the same sense we write $M + A := \langle M, A\rangle \cap \mathbb{A}^n$ for any linear subvariety A of $\mathbb{P}^n$.

For the rest of this paper let us fix integers $n \geqslant 0,\quad 0 \leqslant p \leqslant n$ and a projective subvariety V of $\mathbb{P}^n$ having pure codimension p. Using the projective setting, we introduce in Section 2.1 the notion of a *generalized* polar variety of V associated with a given linear subspace K, a given non-degenerate hyperquadric Q and a given hyperplane H of the ambient space $\mathbb{P}^n$, subject to the condition that $Q \cap H$ is a non-degenerate hyperquadric of H.

Restricting again our attention to the case that H is the hyperplane at infinity of $\mathbb{P}^n$, we may consider the complex n-dimensional affine space $\mathbb{A}^n$ as embedded in $\mathbb{P}^n$. In this context we may define the *affine* generalized polar varieties of the affine variety $S := V \cap \mathbb{A}^n$, which we suppose to be non-empty. It turns out that the *classic* polar varieties of S are special instances of the affine generalized ones. We finish Section 2.1 with the description of another type of special instance of affine generalized varieties, namely the *dual* polar varieties introduced in [4]. Finally, in Section 2.2 we introduce and discuss the *real* (generalized, affine, classic, dual) *polar varieties* of the real varieties $V_{\mathbb{R}} := V \cap \mathbb{P}^n_{\mathbb{R}}$ and $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ (supposing that $V_{\mathbb{R}}$ and $S_{\mathbb{R}}$ are non-empty). We will formulate two sufficient conditions for the non-emptyness of such real polar varieties.

2.1. *Generalized polar varieties*

Let Q be a non-degenerate hyperquadric defined in the projective space $\mathbb{P}^n$. For a linear subvariety $A \subset \mathbb{P}^n$ of dimension a, we denote by $A^\vee$ the dual of A with respect to Q. The dimension of $A^\vee$ is $n - a - 1$.

Further, let H be a hyperplane such that the intersection $Q \cap H$ is a non-degenerate hyperquadric of H (this means that at any point M of $Q \cap H$ the hyperplane H is not contained in the tangentspace $T_M Q$). If A is a linear subvariety of $\mathbb{P}^n$ contained in H, we denote by A^* its dual with respect to $Q \cap H$. The dimension of A^* is $n - a - 2$. Observe that the linear varieties A^* and $A^\vee \cap H$ coincide.

Now we are going to introduce the notion of a generalized polar variety contained in the projective space $\mathbb{P}^n$. Such polar varieties will be associated with a given flag of linear subvarieties, a non-degenerate hyperquadric and a hyperplane of $\mathbb{P}^n$, which is supposed not to be tangent to the hyperquadric. We consider this situation to be represented by a point of a suitable parameter space given as a Zariski open subset of the product of the corresponding flag variety, the space of hyperquadrics and the dual space of $\mathbb{P}^n$. We will denote a current point of this parameter space by $P = (\mathcal{K}, Q, H)$.

In view of the intended algorithmic applications to real polynomial equation solving, the principal aim of this paper is the proof of suitable smoothness results for generic polar varieties associated with the given projective variety V. For this purpose we will work locally (in the Zariski sense) in the variety V. This allows us to restrict our attention to locally closed conditions in the parameter space (instead of the more general constructible ones).

For a given point $P = (\mathcal{K}, Q, H)$ we define, for any member K of the flag $\mathcal{K}$, the *generalized polar variety* $\widehat{W}_K(V)$ associated with K as the Zariski-closure of the constructible set

$$\left\{M \in V_{\mathrm{reg}} \setminus (K \cup H) \mid T_M V \not\pitchfork \langle M, (\langle M, K\rangle \cap H)^*\rangle \text{ at } M\right\}. \tag{1}$$

Note that $\widehat{W}_K(V)$ is contained in V. Let us denote the given flag by

$$\mathcal{K}: \qquad \mathbb{P}^n \supset K^{n-1} \supset K^{n-2} \supset \cdots \supset K^{n-p-1} \supset \cdots \supset K^1 \supset K^0.$$

Then the generalized polar varieties associated with $\mathcal{K}$ are organized as a decreasing sequence as follows:

$$V = \widehat{W}_{K^{n-1}} = \cdots = \widehat{W}_{K^{n-p}} \supset \widehat{W}_{K^{n-p-1}} \supset \cdots \supset \widehat{W}_{K^1} \supset \widehat{W}_{K^0}.$$

In order to simplify notations, we write

$$\widehat{V}_i := \widehat{W}_{K^{n-p-i}}, \quad 1 \leqslant i \leqslant n-p.$$

We call $\widehat{V}_i$ the *ith generalized polar variety* of V associated with the parameter point P. The subscript i reflects the expected codimension of $\widehat{V}_i$ in V. Note that the *relevant part* of the flag $\mathcal{K}$ leading to non-trivial polar varieties ranges from K^{n-p-1} to K^0.

Let K be any member of the flag $\mathcal{K}$ and assume that H is the hyperplane at infinity of $\mathbb{P}^n$ (fixing an embedding of the n-dimensional affine space $\mathbb{A}^n$ into $\mathbb{P}^n$) and that V is the projective closure of a given pure p-dimensional closed subvariety S of the affine space $\mathbb{A}^n$. We denote by $\widehat{W}_K(S) := \widehat{W}_K(V) \cap \mathbb{A}^n$ the *affine generalized polar variety associated to K*.

Two particular choices of the parameter point $P = (\mathcal{K}, H, Q)$ are noteworth. Let us fix a non-degenerate hyperquadric Q and a hyperplane H not tangent to Q. Furthermore, let be given a flag

$$\mathcal{L}: \qquad L^0 \subset L^1 \cdots \subset L^{p-1} \subset \cdots \subset L^{n-2} \subset L^{n-1} \subset \mathbb{P}^n$$

organized as an *increasing* sequence of linear subvarieties of the n-dimensional projective space and suppose that $L^{n-1} = H$ holds.

We associate two new flags of linear subspaces of $\mathbb{P}^n$ with the flag $\mathcal{L}$, both organized as *decreasing* sequences. We call these two flags the internal and the external flag of $\mathcal{L}$ and denote them by $\underline{\mathcal{K}}$ and $\overline{\mathcal{K}}$, respectively.

We write the *internal flag* $\underline{\mathcal{K}}$ as

$$\underline{\mathcal{K}}: \qquad \mathbb{P}^n \supset \underline{K}^{n-1} \supset \underline{K}^{n-2} \supset \cdots \supset \underline{K}^{n-p-1} \supset \cdots \supset \underline{K}^1 \supset \underline{K}^0.$$

For i ranging from 1 to $n-p$, we define the relevant part of $\underline{\mathcal{K}}$ by $\underline{K}^{n-p-i} := (L^{p+i-2})^*$ (observe that the linear variety L^{p+i-2} is contained in the hyperplane H). The irrelevant part $\underline{K}^{n-1} \supset \underline{K}^{n-2} \supset \cdots \supset \underline{K}^{n-p}$ of $\underline{\mathcal{K}}$ may be chosen arbitrarily.

Consider now an arbitrary member $\underline{K}$ of the relevant part of the internal flag $\underline{\mathcal{K}}$. Furthermore, let L be the member of the flag $\mathcal{L}$ determined by the condition $\underline{K} = L^*$, and let M

be a point belonging to $V_{\mathrm{reg}} \setminus H$. Taking into account that $\underline{K}$ is contained in H, whereas M does not belong to H , we conclude that

$$\langle M, \underline{K}\rangle \cap H = \underline{K}$$

holds. This implies

$$\langle M, (\langle M, \underline{K}\rangle \cap H)^*\rangle = \langle M, \underline{K}^*\rangle = \langle M, L\rangle.$$

Provided that H does not contain any irreducible component of V, we infer from (1) that $\widehat{W}_{\underline{K}}(V)$ coincides with the Zariski-closure of the constructible set

$$\left\{M \in V_{\mathrm{reg}} \setminus L \quad | \quad T_M V \not\pitchfork \langle M, L\rangle \text{ at } M\right\}. \tag{2}$$

We denote this Zariski-closure by $W_L(V)$.

As before let H be the hyperplane at infinity of $\mathbb{P}^n$ and let V be the projective closure of a given pure p-codimensional closed subvariety S of the affine space $\mathbb{A}^n$. Then H does not contain any irreducible component of V and from (2) we deduce that the affine generalized polar variety $\widehat{W}_{\underline{K}}(S) = \widehat{W}_{\underline{K}}(V) \cap \mathbb{A}^n$ is the affine trace $W_L(S) := W_L(V) \cap \mathbb{A}^n$ of the polar variety $W_L(V)$. Since the linear subvariety L is contained in the hyperplane at infinity H, we may interprete L as a linear subspace I of the affine space $\mathbb{A}^n$. From (2) one infers easily that the affine polar variety $W_L(S)$ is the Zariski-closure of the constructible set

$$\left\{M \in S_{\mathrm{reg}} \setminus (L \cap \mathbb{A}^n) \quad | \quad M + T_M S \not\pitchfork M + L \text{ at } M\right\},$$

or, alternatively, of

$$\left\{M \in S_{\mathrm{reg}} \setminus (L \cap \mathbb{A}^n) \quad | \quad T_M S \not\pitchfork I \text{ at } M\right\} \tag{3}$$

(here we use freely the usual notion and notation of non-transversality of linear and affine linear subspaces of $\mathbb{A}^n$). This implies that $W_L(S)$ is exactly the classic polar variety of S associated with the linear space I. Finally, let us remark that any classic polar variety of S is obtained by a suitable choice of the flag $\mathcal{L}$ with $L^{n-1} = H$. For the definition and basic properties of classic polar varieties we refer to [40] and the references cited therein. More details on the relations between classic and generalized polar varieties can be found in [4].

We write the *external flag* $\overline{\mathcal{K}}$ as

$$\overline{\mathcal{K}}: \qquad \mathbb{P}^n \supset \overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p-1} \supset \cdots \supset \overline{K}^1 \supset \overline{K}^0.$$

For i ranging from 1 to $n-p$, we define the relevant part of $\overline{\mathcal{K}}$ by $\overline{K}^{n-p-i} := (L^{p+i-1})^\vee$. The irrelevant part $\overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p}$ of $\overline{\mathcal{K}}$ may be chosen arbitrarily.

Consider now an arbitrary member $\overline{K}$ of the relevant part of the external flag $\overline{\mathcal{K}}$. Further, let L be the member of the flag $\mathcal{L}$ determined by the condition $\overline{K} = L^\vee$, and let M be a point belonging to $V_{\mathrm{reg}} \setminus (\overline{K} \cup H)$. From $\overline{K}^0 \subset \overline{K}$ we deduce that $\overline{K}^0$ is contained in $\langle M, \overline{K}\rangle$. Taking into account that $\overline{K}^{0\vee} = L^{n-1} = H$ holds, we conclude that any element of $\langle M, \overline{K}\rangle^\vee$ belongs to the hyperplane H. Thus $\langle M, \overline{K}\rangle^\vee$ is contained in $(\langle M, \overline{K}\rangle \cap H)^\vee \cap H$. A straightforward dimension argument implies now

$$\langle M, \overline{K}\rangle^\vee = (\langle M, \overline{K}\rangle \cap H)^\vee \cap H = (\langle M, \overline{K}\rangle \cap H)^*.$$

Hence, from (1) we conclude that the generalized polar variety $\widehat{W}_{\overline{K}}(V)$ coincides with the Zariski-closure of the constructible set

$$\left\{M \in V_{\mathrm{reg}} \setminus (\overline{K} \cup H) \mid T_M V \not\pitchfork \langle M, \langle M, \overline{K}\rangle^{\vee}\rangle \text{ at } M\right\}. \tag{4}$$

Again, let us assume that the variety V is the projective closure of a given closed subvariety S of the affine space $\mathbb{A}^n$, that S has pure codimension p and that H is the hyperplane at infinity of $\mathbb{P}^n$. We call the affine polar variety $\widehat{W}_{\overline{K}}(S) := \widehat{W}_{\overline{K}}(V) \cap \mathbb{A}^n$ the *dual* polar variety of S associated with $\overline{K}$.

From (4) one easily deduces that the affine dual polar variety $\widehat{W}_{\overline{K}}(S)$ is nothing else but the Zariski-closure (in $\mathbb{A}^n$) of the constructible set

$$\left\{M \in S_{\mathrm{reg}} \setminus (\overline{K} \cap \mathbb{A}^n) \mid M + T_M S \not\pitchfork M + \langle M, \overline{K}\rangle^{\vee} \text{ at } M\right\}. \tag{5}$$

Let M be a regular point of S that does not belong to $\overline{K} \cap \mathbb{A}^n$. Since the linear subvariety $\langle M, \overline{K}\rangle^{\vee}$ is contained in the hyperplane at infinity of $\mathbb{P}^n$, we may interprete the affine cone of $\langle M, \overline{K}\rangle^{\vee}$ as a linear subspace $I_{M,\overline{K}}$ of $\mathbb{A}^n$. In the same way we may interprete the affine cone of the linear variety L as a linear subspace I of $\mathbb{A}^n$. Then the linear space $I_{M,\overline{K}}$ consists exactly of those elements of I that are orthogonal to the point M with respect to the bilinear form induced by $Q \cap H$. From (5) one easily deduces that the affine dual polar variety $\widehat{W}_{\overline{K}}(S)$ is the Zariski-closure of the constructible set

$$\left\{M \in S_{\mathrm{reg}} \setminus (\overline{K} \cap \mathbb{A}^n) \mid T_M S \not\pitchfork I_{M,\overline{K}}\right\}. \tag{6}$$

In conclusion: Internal flags lead to the classic polar varieties and external flags lead to a new type of polar varieties, namely the dual ones.

Classic and dual polar varieties play a fundamental role in the context of semialgebraic geometry, the main subject of this paper. In the next subsection we shall discuss *real* polar varieties.

2.2. Real polar varieties

Recall the following notation: $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{A}^n_{\mathbb{R}}$ for the real n-dimensional projective and affine spaces. Sometimes, we will also write $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{A}^n := \mathbb{A}^n_{\mathbb{C}}$ for n-dimensional complex projective and affine spaces.

Let a flag of real linear subvarieties of the projective space $\mathbb{P}^n_{\mathbb{R}}$ be given, namely

$$\mathcal{L}: \qquad L^0 \subset L^1 \subset \cdots \subset L^{n-1} \subset \mathbb{P}^n_{\mathbb{R}}.$$

Let H be the hyperplane at infinity of $\mathbb{P}^n_{\mathbb{C}}$, and let $H_{\mathbb{R}} := H \cap \mathbb{A}^n_{\mathbb{R}}$ be its real trace. Thus $H_{\mathbb{R}}$ fixes an embedding of the real affine space $\mathbb{A}^n_{\mathbb{R}}$ into $\mathbb{P}^n_{\mathbb{R}}$. Furthermore, let an $\mathbb{R}$-definable, non-degenerate hyperquadric Q of $\mathbb{P}^n_{\mathbb{C}}$ be given and suppose that $Q \cap H$ is also non-degenerate, and that $Q \cap H_{\mathbb{R}}$ can be described by means of a positive definite bilinear form. Observe that $Q \cap H_{\mathbb{R}}$ induces a Riemannian structure on the affine space $\mathbb{A}^n_{\mathbb{R}}$ and that $\mathcal{L}$ induces a flag of $\mathbb{R}$-definable linear subvarieties of the complex projective space $\mathbb{P}^n_{\mathbb{C}}$. We call this flag the complexification of $\mathcal{L}$. Suppose that we are given a purely

p-codimensional, $\mathbb{R}$-definable closed subvariety S of $\mathbb{A}^n_{\mathbb{C}}$ whose projective closure in $\mathbb{P}^n_{\mathbb{C}}$ is V. We denote by $V_{\mathbb{R}} := V \cap \mathbb{P}^n_{\mathbb{R}}$ and $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ the real traces of V and S (see [7]).

Suppose for the rest of this subsection that $L^{n-1} = H_{\mathbb{R}}$ holds, that p is the real codimension of $S_{\mathbb{R}}$ at any point and that $S_{\mathbb{R}}$ is non-empty (this implies that $S_{\mathbb{R}}$ contains at least one smooth point). For the given flag $\mathcal{L}$ of linear subvarieties of $\mathbb{P}^n_{\mathbb{R}}$ we may now define the notion of an internal and an external flag and the notion of a *real* generalized, affine, classic and dual polar variety of $V_{\mathbb{R}}$ and of $S_{\mathbb{R}}$ in the same way as in the Section 2.1. It turns out that these polar varieties are the real traces of their complex counterparts given by V, S and the complexification of $\mathcal{L}$ and its internal and external flag. All our comments on classic and dual polar varieties made in the Section 2.1 are valid mutatis mutandis in the real case. Again we denote the (real) internal and external flag associated with $\mathcal{L}$ by $\underline{\mathcal{K}}$ and $\overline{\mathcal{K}}$, respectively. For any member L of the flag $\mathcal{L}$, $\underline{K}$ of the flag $\underline{\mathcal{K}}$ and $\overline{K}$ of the flag $\overline{\mathcal{K}}$, we denote the corresponding real polar variety by

$$W_L(V_{\mathbb{R}}), \quad W_L(S_{\mathbb{R}}), \quad \widehat{W}_{\underline{K}}(V_{\mathbb{R}}), \quad \widehat{W}_{\underline{K}}(S_{\mathbb{R}}), \quad \widehat{W}_{\overline{K}}(V_{\mathbb{R}}) \text{ and } \widehat{W}_{\overline{K}}(S_{\mathbb{R}}).$$

We are now going to discuss the real affine polar varieties associated with the internal and external flags $\underline{\mathcal{K}}$ and $\overline{\mathcal{K}}$ of $\mathcal{L}$.

Let us first consider the case of the internal flag $\underline{\mathcal{K}}$. Let L be any member of the relevant part of the given flag $\mathcal{L}$ and let $\underline{K}$ be the member of the internal flag $\underline{\mathcal{K}}$ defined by $\underline{K} := L^*$. Observe that L and $\underline{K}$ are contained in the hyperplane at infinity $H_{\mathbb{R}}$. Hence, we may interprete L as a $\mathbb{R}$-linear subspace I of the real affine space $\mathbb{A}^n_{\mathbb{R}}$. From our considerations in the Section 2.1 we deduce that

$$\widehat{W}_{\underline{K}}(S_{\mathbb{R}}) = \widehat{W}_{\underline{K}}(V_{\mathbb{R}}) \cap \mathbb{A}^n_{\mathbb{R}} = W_L(V_{\mathbb{R}}) \cap \mathbb{A}^n_{\mathbb{R}} = W_L(S_{\mathbb{R}})$$

holds and from (3) we infer that the classic real polar variety $W_L(S_{\mathbb{R}})$ is the Zariski-closure of the semialgebraic set

$$\left\{ M \in (S_{\mathbb{R}})_{\text{reg}} \mid T_M\, S_{\mathbb{R}} \not\pitchfork \; I \right\}$$

in $\mathbb{A}^n_{\mathbb{R}}$.

In principle, the classic real polar variety $\widehat{W}_L(S_{\mathbb{R}})$ may be empty, even in case that S contains real smooth points. However, under certain circumstances, we may conclude that $\widehat{W}_L(S_{\mathbb{R}})$ is non-empty. This is the content of the following statement:

Proposition 1. *Suppose that S is a pure p-codimensional complete intersection variety given as the set of common zeros of p polynomials $F_1, \ldots, F_p \in \mathbb{R}[X_1, \ldots, X_n]$, where $X_1, \ldots, X_n$ are indeterminates over the reals. Suppose that the ideal generated by $F_1, \ldots, F_p$ is radical and that $S_{\mathbb{R}}$ is a pure p-codimensional, non-empty, smooth and compact real variety. Then $\widehat{W}_{\underline{K}}(S_{\mathbb{R}}) = W_L(S_{\mathbb{R}})$ contains at least one point of each connected component of $S_{\mathbb{R}}$.*

Proposition 1 is an easy consequence of the arguments used in [3, Section 2.4], which will not be repeated here.

Let us now consider the external flag $\overline{\mathcal{K}}$. Observe that $\overline{K}^0$ is a zero-dimensional linear subvariety of $\mathbb{P}^n_{\mathbb{R}}$, namely the origin of $\mathbb{A}^n$. Therefore any member of the external flag $\overline{\mathcal{K}}$ has a non-empty intersection with $\mathbb{A}^n_{\mathbb{R}}$. Assume now that the Riemannian metric of $\mathbb{A}^n$ induced by the hyperquadric Q is the ordinary euclidean distance. Under these assumptions we shall show the following result:

Proposition 2. *Suppose that $S_{\mathbb{R}}$ is a pure p-codimensional, non-empty, smooth real variety. Let $\overline{K}$ be any member of the external flag $\overline{\mathcal{K}}$ and suppose that $\overline{K} \cap \mathbb{A}^n_{\mathbb{R}}$ is not contained in $S_{\mathbb{R}}$. Then, the real affine dual polar variety $\widehat{W}_{\overline{K}}(S_{\mathbb{R}})$ is nonempty and contains at least one point of each connected component of $S_{\mathbb{R}}$.*

Observe that the statement of Proposition 2 becomes trivial for $\overline{K}$ belonging to the irrelevant part of $\overline{\mathcal{K}}$, since in this case $\widehat{W}_{\overline{K}}(S_{\mathbb{R}}) = S_{\mathbb{R}}$ holds.

Proof. Since $\overline{K} \cap \mathbb{A}^n_{\mathbb{R}}$ is not contained in $S_{\mathbb{R}}$, there exists a point P of $\overline{K} \cap \mathbb{A}^n$ that does not belong to $S_{\mathbb{R}}$. Consider now an arbitrary connected component C of $S_{\mathbb{R}}$. Then C is a smooth, closed subvariety of $\mathbb{A}^n_{\mathbb{R}}$ whose distance to the point P is realized by a point M of C. Since P does not belong to $S_{\mathbb{R}}$, one has $M - P \neq 0$.

The square of the euclidean distance of any point X of $\mathbb{A}^n_{\mathbb{R}}$ to the point P is a real valued polynomial function defined on $\mathbb{A}^n_{\mathbb{R}}$ whose gradient in X is $2(X - P)$. Applying now the Lagrangian Multiplier Theorem (see e.g. [46]) to this function and the polynomial equations defining $S_{\mathbb{R}}$ we deduce that $M - P$ belongs to the orthogonal complement of the real tangent space $T_M(S_{\mathbb{R}})$ (observe that M is a smooth point of $S_{\mathbb{R}}$). The real trace $I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}}$ of the linear space $I_{M,\overline{K}}$ introduced in Section 2.1 consists of all elements of the orthogonal complement of $\overline{K} \cap \mathbb{A}^n_{\mathbb{R}}$ that are also orthogonal to M. Observe now that the linear space $T_M(S_{\mathbb{R}}) + (I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}})$ is strictly contained in $\mathbb{A}^n_{\mathbb{R}}$, since otherwise any point of $\mathbb{A}^n_{\mathbb{R}}$ would be orthogonal to $M - P$. On the other hand, $T_M(S_{\mathbb{R}}) + (I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}}) \neq \mathbb{A}^n_{\mathbb{R}}$ implies that $T_M(S_{\mathbb{R}}) \not\pitchfork (I_{M,\overline{K}} \cap \mathbb{A}^n_{\mathbb{R}})$ holds. From (6) we finally deduce that the point M belongs to the real affine dual polar variety $\widehat{W}_{\overline{K}}(S_{\mathbb{R}}) = \widehat{W}_{\overline{K}}(S) \cap \mathbb{A}^n_{\mathbb{R}}$ and therefore we have $C \cap \widehat{W}_{\overline{K}}(S_{\mathbb{R}}) \neq \emptyset$. $\square$

3. Extrinsic aspects of polar varieties

In this section we will describe in more detail the generalized polar varieties of a closed, p-codimensional subvariety S of $\mathbb{A}^n$, which is given by a system of polynomial equations. We suppose that these polynomial equations form a regular sequence and generate the (radical) ideal of definition of S. Let K be a "sufficiently generic" linear subvariety of $\mathbb{P}^n$ of dimension at most $n - p$. We will show that the polar variety $\widehat{W}_K(S)$ of S is either empty or equidimensional of expected codimension in S. We will describe $\widehat{W}_K(S)$ locally by transversal intersections of explicitly given hypersurfaces of $\mathbb{A}^n$ and, in case that S is smooth, globally by explicit polynomial equations, which generate the ideal of definition of $\widehat{W}_K(S)$.

3.1. Explicit description of affine polar varieties

Let $\mathbb{P}^n$ and $\mathbb{A}^n$ be the n-dimensional projective or affine space over $\mathbb{C}$ or $\mathbb{R}$, according to the context. As above, we consider $\mathbb{A}^n$ to be embedded in $\mathbb{P}^n$ in the usual way. For given complex or real numbers $x_0, \ldots, x_n$ that are not all zero, $x := (x_0 : x_1 : \ldots : x_n)$ denotes the corresponding point of the projective space $\mathbb{P}^n$. Moreover, for $x_0 = 1$ we denote the corresponding point of the affine space $\mathbb{A}^n$ by $(x_1, \ldots, x_n) := (1 : x_1 : \ldots : x_n)$. Let $X_0, \ldots, X_n$ be indeterminates over $\mathbb{C}$ (or $\mathbb{R}$).

As of now we suppose that the given projective, purely p-codimensional variety V is defined by p non-zero forms $f_1, \ldots, f_p$ over $\mathbb{C}$ (or $\mathbb{R}$) in the variables $X_0, \ldots, X_n$. In other words, we suppose

$$V := V(f_1, \ldots, f_p),$$

where $V(f_1, \ldots, f_p)$ denotes the set of common zeros of $f_1, \ldots, f_p$ in $\mathbb{P}^n$. Therefore, the homogeneous polynomials $f_1, \ldots, f_p$ form a regular sequence in the polynomial ring $\mathbb{C}[X_0, , \ldots, X_n]$ (or $\mathbb{R}[X_0, , \ldots, X_n]$). Let $S := V \cap \mathbb{A}^n$ and assume that S is non-empty. The dehomogenizations of $f_1, \ldots, f_p$ are denoted by

$$F_1 := f_1(1, X_1, \ldots, X_n), \ldots, F_p := f_p(1, X_1, \ldots, X_n).$$

Observe that $F_1, \ldots, F_p$ are non-zero polynomials in the variables $X_1, \ldots, X_n$ over $\mathbb{C}$ (or $\mathbb{R}$). Thus we have

$$S = V \cap \mathbb{A}^n = V(F_1, \ldots, F_p),$$

where $V(F_1, \ldots, F_p)$ denotes the set of common zeros of $F_1, \ldots, F_p$ in $\mathbb{A}^n$. Note that the polynomials $F_1, \ldots, F_p$ form a regular sequence in $\mathbb{C}[X_1, , \ldots, X_n]$ (or in $\mathbb{R}[X_1, , \ldots, X_n]$).

The projective Jacobian (matrix) of $f_1, \ldots, f_p$ is denoted by

$$J(f_1, \ldots, f_p) := \left[\frac{\partial f_j}{\partial X_k}\right]_{\substack{1 \leqslant j \leqslant p \\ 0 \leqslant k \leqslant n}}.$$

For any point x of $\mathbb{P}^n$ we write

$$J(f_1, \ldots, f_p)(x) := \left[\frac{\partial f_j}{\partial X_k}(x)\right]_{\substack{1 \leqslant j \leqslant p \\ 0 \leqslant k \leqslant n}}$$

for the projective Jacobian of the polynomials $f_1, \ldots, f_p$ at the point x. Similarly we denote the affine Jacobian of the polynomials $F_1, \ldots, F_p$ by

$$J(F_1, \ldots, F_p) := \left[\frac{\partial F_j}{\partial X_k}\right]_{\substack{1 \leqslant j \leqslant p \\ 1 \leqslant k \leqslant n}},$$

and we write for any point x of $\mathbb{A}^n$:

$$J(F_1, \ldots, F_p)(x) := \left[\frac{\partial F_j}{\partial X_k}(x)\right]_{\substack{1 \leqslant j \leqslant p \\ 1 \leqslant k \leqslant n}}.$$

A point x of V (or of $V \cap \mathbb{A}^n$) is called $(f_1, \ldots, f_p)$-regular (or $(F_1, \ldots, F_p)$-regular) if the Jacobian $J(f_1, \ldots, f_p)(x)$ (or $J(F_1, \ldots, F_p)(x)$) has maximal rank p. Note that the $(f_1, \ldots, f_p)$-regular points of V are always smooth points of V, but not vice versa. For the sake of simplicity, we shall therefore suppose from now on that all smooth points of V are $(f_1, \ldots, f_p)$-regular. In other words, we suppose that $f_1, \ldots, f_p$ (and hence $F_1, \ldots, F_p$) generate a radical ideal of its ambient polynomial ring. Any smooth point of S is therefore $(F_1, \ldots, F_p)$-regular. On the other hand, by assumption, the polynomials $F_1, \ldots, F_p$ form a regular sequence in $\mathbb{C}[X_1, \ldots, X_n]$.

Suppose for the rest of this section that our ground field is $\mathbb{C}$. Next, we will generate local equations for the generalized polar varieties of the affine complete intersection variety S. To this end (and having in mind the algorithmic applications of our geometric considerations to real affine polar varieties described in Section 4) we may restrict our attention to the case where H is the hyperplane at infinity of $\mathbb{P}^n$ (defined by the equation $X_0 = 0$) and where the given non-degenerate hyperquadric Q is defined by a quadratic form R, which can be represented as follows:

$$R(X_0, \ldots, X_n) := {X_0}^2 + \sum_{k=1}^{n} 2c_k X_0 X_k + \sum_{k=1}^{n} {X_k}^2$$

with $c_1, \ldots, c_n$ belonging to $\mathbb{C}$ or $\mathbb{R}$, according to the context. Observe that this representation of R implies the hyperquadrics Q and $Q \cap H$ to be non-degenerate in $\mathbb{P}^n$ and H, respectively. Further, observe that $Q \cap H$ is defined by the quadratic form $R_0(X_1, \ldots, X_n) := \sum_{k=1}^{n} X_k^2 \in \mathbb{R}[X_1, \ldots, X_n]$. Therefore, the real variety $Q \cap H_{\mathbb{R}}$ is represented by a positive definite quadratic form that induces the usual euclidean distance on $\mathbb{A}^n_{\mathbb{R}}$. Let us note that the special shape of R (and, in particular, the positive definiteness of the quadratic form R_0 representing $Q \cap H_{\mathbb{R}}$) does not restrict the generality of the arguments which will follow. These may be applied mutatis mutandis to any non-degenerate hyperquadric whose intersection with the hyperplane at infinity H is still non-degenerate.

Fix now $1 \leqslant i \leqslant n-p$ and choose for each $1 \leqslant j \leqslant n-p-i+1$ a point $A_j = (a_{j,0} : \ldots : a_{j,n})$ of $\mathbb{P}^n$ with $a_{j,0} = 0$ or $a_{j,0} = 1$ and $a_{j,1}, \ldots, a_{j,n}$ generic (our genericity conditions will become evident in the sequel). By this choice, we may assume that the points $A_1, \ldots, A_{n-p-i+1}$ span an $(n-p-i)$-dimensional linear subvariety $K := K^{n-p-i}$ of the projective space $\mathbb{P}^n$.

Let us consider an $(f_1, \ldots, f_p)$-regular point $M = (x_0 : \ldots : x_n)$ of V with $x_0 \neq 0$ and $M \notin K$. Then one easily sees that the $(n-p-i)$-dimensional linear subvariety $\langle M, K\rangle \cap H$ is spanned by the $n-p-i+1$ linearly independent points

$$x_0\, A_1 - a_{1,0}\, M\,, \ldots,\; x_0\, A_{n-p-i+1} - a_{n-p-i+1,0}\, M.$$

Let $Y_1, \ldots, Y_n$ be new indeterminates and let $\Theta := \sum_{k=1}^{n} X_k Y_k$, $\Theta \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, denote the (polarized) bilinear form associated with the hyperquadric $Q \cap H$. For $1 \leqslant j \leqslant n-p-i+1$, let $\ell_j \in \mathbb{C}[X_1, \ldots, X_n]$ be defined by

$$\ell_j := \ell_j^{(x_0, \ldots, x_n)} := \Theta(x_0\, a_{j,1} - a_{j,0}\, x_1, \ldots, x_0\, a_{j,n} - a_{j,0}\, x_n,\; X_1, \ldots, X_n)$$

and $G_j \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ by

$$G_j := G_j^{(x_0,\ldots,x_n)} := x_0\, \ell_j^{(x_0,\ldots,x_n)}(X_1,\ldots,X_n) - X_0\, \ell_j^{(x_0,\ldots,x_n)}(x_1,\ldots,x_n).$$

Then the linear forms $\ell_1, \ldots, \ell_{n-p-i+1}$ define the $(p+i-2)$-dimensional linear variety $(\langle M, K\rangle \cap H)^*$ in H and are therefore linearly independent. Moreover, the linear forms $G_1, \ldots, G_{n-p-i+1}$ vanish at M and at any point of $(\langle M, K\rangle \cap H)^*$. Hence, they vanish at any point of the $(p+i-1)$-dimensional linear variety $\langle M, (\langle M, K\rangle \cap H)^*\rangle$. From the linear independence of $\ell_1, \ldots, \ell_{n-p-i+1}$ one easily deduces the linear independence of the linear forms $G_1, \ldots, G_{n-p-i+1}$. Therefore $G_1, \ldots, G_{n-p-i+1}$ describe the linear variety $\langle M, (\langle M, K\rangle \cap H)^*\rangle$ used in (1) to define the generalized polar variety $\widehat{W}_K(V)$ (see Section 2.1).

Observe now that for any $1 \leqslant j \leqslant n-p-i+1$ the linear form $G_j^{(x_0,\ldots,x_n)}$ can be written as

$$\begin{aligned} G_j^{(x_0,\ldots,x_n)} &= -(X_0 - x_0)\, \ell_j^{(x_0,\ldots,x_n)}(x_1,\ldots,x_n) \\ &\quad + x_0\, \ell_j^{(x_0,\ldots,x_n)}(X_1 - x_1, \ldots, X_n - x_n) \\ &= -(X_0 - x_0)\, \ell_j^{(x_0,\ldots,x_n)}(x_1,\ldots,x_n) \\ &\quad + x_0 \sum_{k=1}^{n} (x_0\, a_{j,k} - a_{j,0}\, x_k)(X_k - x_k). \end{aligned}$$

Without loss of generality suppose that $x_0 = 1$ holds. Then $x := (x_1, \ldots, x_n)$ is an $(F_1, \ldots, F_p)$-regular point of $S = V \cap \mathbb{A}^n$ and the polynomial $G_j^{(1,x_1,\ldots,x_n)}$ depends only on the variables $X_1, \ldots, X_n$. Therefore, it makes sense to consider the Jacobian

$$T^{(i)} := T^{(i)}(X_1,\ldots,X_n) := J(F_1,\ldots,F_p, G_1^{(1,x_1,\ldots,x_n)}, \ldots, G_{n-p-i+1}^{(1,x_1,\ldots,x_n)}),$$

whose entries belong to the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$. Observe that the polynomial matrix $T^{(i)}$ is of the following explicit form, namely

$$T^{(i)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} - a_{1,0} X_1 & \cdots & a_{1,n} - a_{1,0} X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - a_{n-p-i+1,0} X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0} X_n \end{bmatrix}$$

with $a_{1,0}, \ldots, a_{n-p-i+1,0}$ being elements of the set $\{1, 0\}$.

Moreover, observe that the condition

$$T_M(V) \not\pitchfork \langle M, (\langle M, K\rangle \cap H)^*\rangle$$

from (1) is equivalent to the vanishing of all $(n-i+1)$-minors of the $((n-i+1) \times n)$-matrix $T^{(i)}$ at the point x. Therefore the polynomials $F_1, \ldots, F_p$ and the $(n-i+1)$-minors of $T^{(i)}$ define the generalized affine polar variety $\widehat{W}_K(S)$ outside of the locus S_{sing} (recall that by assumption all smooth points of S are $(F_1, \ldots, F_p)$-regular). Let W be the closed subvariety of $\mathbb{A}^n$ defined by these equations. Then any irreducible component of $\widehat{W}_K(S)$ is an irreducible component of W. In particular, we have $\widehat{W}_K(S) \cap S_{\text{reg}} = W \cap S_{\text{reg}}$, and $\widehat{W}_K(S) = W$ if the affine variety S is smooth. Note, that i is the expected codimension of $\widehat{W}_K(S) = \widehat{W}_{K^{n-p-i}}(S)$ in S. These considerations lead to the following conclusion:

Lemma 3. *Any irreducible component of $\widehat{W}_K(S) = \widehat{W}_{K^{n-p-i}}(S)$ has codimension at most i in S.*

Proof. Let us denote by $\mathfrak{a}$ the ideal of the coordinate ring $\mathbb{C}[S]$ of the affine variety S, generated by all $(n-i+1)$-minors of the $((n-i+1)\times n)$-matrix induced by $T^{(i)}$ in $\mathbb{C}[S]$. Let C be a given irreducible component of the affine polar variety $\widehat{W}_K(S)$ and let $\mathfrak{p}$ be the ideal of definition of C in $\mathbb{C}[S]$. Then $\mathfrak{p}$ is an isolated prime component of the determinantal ideal $\mathfrak{a}$. From [17], Theorem 3 (see also [38, Theorem 13.10]) we deduce that the height of the prime ideal $\mathfrak{p}$ is bounded by i. This means that the codimension of C in S is at most i. $\square$

In the further analysis of the generalized affine polar variety $\widehat{W}_K(S)$ we shall distinguish from time to time two cases, namely the case that the linear projective variety $K = K^{n-p-i}$, spanned by the given points $A_1, \ldots, A_{n-p-i+1}$ of $\mathbb{P}^n$, is contained in the hyperplane at infinity H of $\mathbb{P}^n$, and the case that K is not contained in H. If K is contained in H, we have $a_{1,0} = \cdots = a_{n-p-i+1,0} = 0$ and if K is not contained in H, we may suppose without loss of generality that $a_{n-p-i+1,0} = 1$ holds.

Let us now discuss the particular case that $K = K^{n-p-i}$ is contained in the hyperplane at infinity H of $\mathbb{P}^n$. Let $\overline{S}$ be the Zariski-closure of the affine variety S in the projective space $\mathbb{P}^n$ and let $L := K^*$. Thus L is a $(p+i-2)$-dimensional linear projective subvariety of H, the projective variety $\overline{S}$ is of pure codimension p in $\mathbb{P}^n$ and none of the irreducible components of $\overline{S}$ is contained in H. Furthermore, we have $K = L^*$ and $S = \overline{S} \cap \mathbb{A}^n$. From (2) we deduce now that $\widehat{W}_K(\overline{S}) = W_L(\overline{S})$ holds. This implies

$$\widehat{W}_K(S) = \widehat{W}_K(\overline{S}) \cap \mathbb{A}^n = W_L(\overline{S}) \cap \mathbb{A}^n = W_L(S). \tag{7}$$

Therefore $\widehat{W}_K(S)$ is the *classic* polar variety associated with the $(p+i-2)$-dimensional linear subvariety L of the hyperplane at infinity H of $\mathbb{P}^n$.

We now return to the analysis of the general situation. Let be given a complex $((n-p-i+1)\times(n+1))$-matrix

$$b := \begin{bmatrix} b_{1,0} & \cdots & b_{1,n} \\ \vdots & \vdots & \vdots \\ b_{n-p-i,0} & \cdots & b_{n-p-i,n} \\ b_{n-p-i+1,0} & \cdots & b_{n-p-i+1,n} \end{bmatrix}$$

with $b_{n-p-i+1,0} = a_{n-p-i+1,0}, \ldots, b_{n-p-i+1,n} = a_{n-p-i+1,n}$ and with $b_{1,0}, \ldots, b_{n-p-i,0}$ being elements of the set $\{1, 0\}$ and suppose that b has maximal rank $n-p-i+1$ and that the entries $a_{n-p-i+1,n-i+1}, \ldots, a_{n-p-i+1,n}$ are generic with respect to the other entries of b (e.g., $a := (a_{j,k})_{\substack{1\leqslant j\leqslant n-p-i+1\\ 0\leqslant k\leqslant n}}$ is such a $((n-p-i+1)\times(n+1))$-matrix).

Let $K(b)$ be the linear subvariety of $\mathbb{P}^n$ spanned by the $n-p-i+1$ projective points

$$(b_{1,0}:\cdots:b_{1,n}),\ldots,(b_{n-p-i+1,0}:\cdots:b_{n-p-i+1,n}).$$

Observe that $K(b)$ is $(n-p-i)$-dimensional and that $K(a)=K^{n-p-i}$ holds. For the sake of notational succinctness let us use, for $1\leqslant j\leqslant n-p-i+1$ and $1\leqslant k\leqslant n$, the abbreviation

$$r_{j,k}^{(b)}(X_k):=b_{j,k}-b_{j,0}\,X_k.$$

We consider now the polynomial $((n-i+1)\times n)$-matrix

$$T_b^{(i)}=\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ r_{1,1}^{(b)}(X_1) & \cdots & r_{1,n}^{(b)}(X_n) \\ \vdots & \vdots & \vdots \\ r_{n-p-i+1,1}^{(b)}(X_1) & \cdots & r_{n-p-i+1,n}^{(b)}(X_n) \end{bmatrix}.$$

Observe that $T_a^{(i)}=T^{(i)}$ holds.

Let $s\in\{n-i,n-i+1\}$. For any ordered sequence $(k_1,\ldots,k_s)$ of different elements of the set $\{1,\ldots,n\}$ we denote by $M^{(b)}(\{k_1,\ldots,k_s\}):=M^{(b)}(k_1,\ldots,k_s)$ the minor that corresponds to the first s rows and to the columns $k_1,\ldots,k_s$ of the matrix $T_b^{(i)}$.

Let us fix an ordered sequence I of $n-i$ different elements of the set $\{1,\ldots,n\}$, say $I:=(1,\ldots,n-i)$, and let us consider the upper $(n-i)$-minor

$$m^{(b)}:=M^{(b)}(I):=\det\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} \\ r_{1,1}^{(b)}(X_1) & \cdots & r_{1,n}^{(b)}(X_{n-i}) \\ \vdots & \vdots & \vdots \\ r_{n-p-i,1}^{(b)}(X_1) & \cdots & r_{n-p-i,n}^{(b)}(X_{n-i}) \end{bmatrix}$$

of the matrix $T^{(i)}$.

Note that $m^{(b)}$ depends only on the entries $b_{j,k},\ 1\leqslant j\leqslant n-p-i,\ 0\leqslant k\leqslant n-i$, of the matrix b. In what follows, we will assume that b satisfies the additional condition $m^{(b)}\neq 0$. Let us assume, without loss of generality, that the polynomial $(p\times p)$-matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_p} \end{bmatrix}$$

is non-singular. Then, in particular, the genericity of the entries $a_{j,k}$ of the $((n-p-i+1)\times(n+1))$-matrix a implies that $m^{(a)}$ is a nonzero element of the polynomial ring $\mathbb{C}[X_1,\ldots,X_n]$. Therefore the matrix a satisfies this condition.

The Exchange Lemma of [3] implies that, for any ordered sequence $(k_1, \ldots, k_{n-i+1})$ of different elements of the set $\{1, \ldots, n\}$, the identity

$$\begin{aligned} m^{(b)}\ M^{(b)}(k_1, \ldots, k_{n-i+1}) \\ = \sum_{l \in \{k_1, \ldots, k_{n-i+1}\} \setminus \{1, \ldots, n-i\}} \mu_l\ \ M^{(b)}(\{k_1, \ldots, k_{n-i+1}\} \setminus \{l\}) \\ \times M^{(b)}(1, \ldots, n-i, l) \end{aligned} \tag{8}$$

holds with $\mu_l \in \{-1, 0, 1\}$, for any index $l \in \{k_1, \ldots, k_{n-i+1}\} \setminus \{1, \ldots, n-i\}$.

Let us abbreviate $M^{(b)}_{n-i+1} := M^{(b)}(1, \ldots, n-i+1)$, $M^{(b)}_{n-i+2} := M^{(b)}(1, \ldots, n-i, n-i+2), \ldots, M^{(b)}_n := M^{(b)}(1, \ldots, n-i, n)$. Assume now that there is given a point x of S satisfying the conditions $m^{(b)}(x) \neq 0$ and

$$M^{(b)}_{n-i+1}(x) = \cdots = M^{(b)}_n(x) = 0. \tag{9}$$

Then we infer from (8) that $M^{(b)}(k_1, \ldots, k_{n-i+1})(x) = 0$ holds for any ordered sequence $(k_1, \ldots, k_{n-i+1})$ of different elements of the set $\{1, \ldots, n\}$. This means that all $(n-i+1)$-minors of the matrix $T_b^{(i)}$ vanish at the point x. Since $m^{(b)}(x) \neq 0$ implies $x \in S_{\mathrm{reg}}$, we conclude that x belongs to the polar variety $\widehat{W}_{K(b)}(S)$. On the other hand, any point x of $\widehat{W}_{K(b)}(S)$ satisfies condition (9). Therefore, the polar variety $\widehat{W}_{K(b)}(S)$ is defined by the equations $F_1, \ldots, F_p, M^{(b)}_{n-i+1}, \ldots, M^{(b)}_n$ outside of the locus $V(m^{(b)})$.

Let $Z_{n-i+1}, \ldots, Z_n$ be new indeterminates and consider the $((n-i+1) \times n)$-matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} & \frac{\partial F_1}{\partial X_{n-i+1}} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} & \frac{\partial F_p}{\partial X_{n-i+1}} & \cdots & \frac{\partial F_p}{\partial X_n} \\ r^{(b)}_{1,1}(X_1) & \cdots & r^{(b)}_{1,n-i}(X_{n-i}) & r^{(b)}_{1,n-i+1}(X_{n-i+1}) & \cdots & r^{(b)}_{1,n}(X_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r^{(b)}_{n-p-i,1}(X_1) & \cdots & r^{(b)}_{n-p-i,n-i}(X_{n-i}) & r^{(b)}_{n-p-i,n-i+1}(X_{n-i+1}) & \cdots & r^{(b)}_{n-p-i,n}(X_n) \\ r^{(b)}_{n-p-i+1,1}(X_1) & \cdots & r^{(b)}_{n-p-i+1,n-i}(X_{n-i}) & Z_{n-i+1} - b_{n-p-i+1,0}X_{n-i+1} & \cdots & Z_n - b_{n-p-i+1,0}X_n \end{bmatrix}.$$

Let $\widetilde{M}^{(b)}_{n-i+1}, \widetilde{M}^{(b)}_{n-i+2}, \ldots, \widetilde{M}^{(b)}_n$ denote the $(n-i+1)$-minors of this matrix obtained by successively selecting the columns $1, \ldots, n-i,\ n-i+1$, then $1, \ldots, n-i, n-i+2$, up to, finally, the columns $1, \ldots, n-i,\ n$. Let $U_b := \mathbb{A}^n \setminus V(m^{(b)})$ and observe that U_b is non-empty since $m^{(b)}$, by assumption, is a non-zero polynomial.

Now we consider the following morphism of smooth, affine varieties

$$\Phi^{(b)}_i :\ U_b \times \mathbb{A}^i \to \mathbb{A}^p \times \mathbb{A}^i,$$

defined by

$$\Phi^{(b)}_i(x, z) := \Big(\ F_1(x), \ldots, F_p(x), \widetilde{M}^{(b)}_{n-i+1}(x, z), \ldots, \widetilde{M}^{(b)}_n(x, z)\ \Big)$$

for any pair of points $x \in U_b$, $z \in \mathbb{A}^i$.

Lemma 4. *The origin* $(0,\ldots,0)$ *of the affine space* $\mathbb{A}^p \times \mathbb{A}^i$ *is a regular value of the morphism* $\Phi_i^{(b)}$.

Proof. Without loss of generality, we may assume that the fibre $(\Phi_i^{(b)})^{-1}(0,\ldots,0)$ is non-empty. Consider an arbitrary point (x,z) of $(\Phi_i^{(b)})^{-1}(0,\ldots,0)$ with $x \in U_b$ and $z \in \mathbb{A}^i$ and observe that the Jacobian $J(\Phi_i^{(b)})(x,z)$ of $\Phi_i^{(b)}$ at the point (x,z) has the form

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1}(x) & \cdots & \frac{\partial F_1}{\partial X_n}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1}(x) & \cdots & \frac{\partial F_p}{\partial X_n}(x) & 0 & \cdots & 0 \\ * & \cdots & * & m^{(b)}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 0 & \cdots & m^{(b)}(x) \end{bmatrix}.$$

Since x belongs to U_b, we conclude that the first p rows of $J(\Phi_i^{(b)})(x,z)$ are $\mathbb{C}$-linearly independent and that $m^{(b)}(x) \neq 0$ holds. Therefore, $J(\Phi_i^{(b)})(x,z)$ has maximal rank $p+i$. Thus (x,z) is a regular point of $\Phi_i^{(b)}$. Since (x,z) is an arbitrary element of the fibre $(\Phi_i^{(b)})^{-1}(0,\ldots,0)$, we conclude finally that $(0,\ldots,0)$ is a regular value of $\Phi_i^{(b)}$. $\square$

Applying now the Weak–Transversality Theorem of Thom–Sard (see e.g. [15]) to $\Phi_i^{(b)}$, we deduce from Lemma 4 that there exists a residual dense set Ω of $\mathbb{A}^i$ such that, for any point $z \in \Omega$, the polynomials

$$F_1,\ldots,F_p,\ \widetilde{M}_{n-i+1}^{(b)}(X_1,\ldots,X_n,z),\ldots,\widetilde{M}_n^{(b)}(X_1,\ldots,X_n,z)$$

of $\mathbb{C}[X_1,\ldots,X_n]$ intersect transversally in any of their common zeros outside of the positive codimensional, Zariski closed locus $\mathbb{A}^n \setminus U_b$. From the genericity of the entries $b_{n-p-i+1,n-i+1} = a_{n-p-i+1,n-i+1},\ldots,b_{n-p-i+1,n} = a_{n-p-i+1,n}$ of the matrix $T_b^{(i)}$ we deduce that we may assume, without loss of generality, that the point $\alpha := (a_{n-p-i+1,n-i+1},\ldots,a_{n-p-i+1,n})$ belongs to the set Ω. Observing now that $M_{n-i+1}^{(b)} = \widetilde{M}_{n-i+1}^{(b)}(X_1,\ldots,X_n,\alpha),\ldots,M_n^{(b)} = \widetilde{M}_n^{(b)}(X_1,\ldots,X_n,\alpha)$ holds, we conclude that the equations $F_1,\ldots,F_p, M_{n-i+1}^{(b)},\ldots,M_n^{(b)}$ intersect transversally at any point of $\widehat{W}_{K(b)}(S)$ not belonging to the locus $V(m^{(b)})$ and that such points exist.

We have therefore shown the following statement:

Lemma 5. *Let the notations and assumptions be as before. Then the polynomial* $((n-i+1)\times n)$*-matrix* $T_b^{(i)}$ *satisfies the following condition*:

The equations $F_1,\ldots,F_p,\ M_{n-i+1}^{(b)},\ldots,M_n^{(b)}$ *define the generalized polar variety* $\widehat{W}_{K(b)}(S)$ *outside of the locus* $V(m^{(b)})$ *and intersect transversally in any point of the affine variety* $\widehat{W}_{K(b)}(S) \setminus V(m^{(b)})$. *In particular,* $\widehat{W}_{K(b)}(S) \setminus V(m^{(b)})$ *is empty or a smooth, complete intersection variety of dimension* $n-p-i$.

Observe that all upper $(n-i)$-minors of $T^{(i)}$ vanish at a given $(F_1,\ldots,F_p)$-regular point x of S if and only if x belongs to the polar variety $\widehat{W}_{K^{n-p-i-1}}(S)$ which is contained in $\widehat{W}_K(S)=\widehat{W}_{K^{n-p-i}}(S)$. Applying now Lemma 5 to any upper $(n-i)$-minor of the matrix $T^{(i)}=T_a^{(i)}$ we conclude:

Proposition 6. *For any $(F_1,\ldots,F_p)$-regular point x of $\widehat{W}_{K^{n-p-i}}(S)\setminus\widehat{W}_{K^{n-p-i-1}}(S)$ there exist indices $1\leqslant k_1<\cdots<k_{n-i}\leqslant n$ with the following property*:

Let $m:=M(\{k_1,\ldots,k_{n-i}\})$ be the upper $(n-i)$-minor of the polynomial $((n-i+1)\times n)$-matrix $T^{(i)}$ determined by the columns $(k_1,\ldots,k_{n-i})$, let $\{1,\ldots,n\}\setminus\{k_1,\ldots,k_{n-i}\}=\{k_{n-i+1},\ldots,k_n\}$ and let $M_{n-i+1}:=M(\{k_1,\ldots,k_{n-i},k_{n-i+1}\})$, $M_{n-i+1}:=M(\{k_1,\ldots,k_{n-i},k_{n-i+2}\}),\ldots,M_n:=M(\{k_1,\ldots,k_{n-i},k_n\})$. Then the minor m does not vanish at the point x and the equations $F_1,\ldots,F_p,M_{n-i+1},\ldots,M_n$ intersect transversally at x. Moreover, the polynomials $F_1,\ldots,F_p,M_{n-i+1},\ldots,M_n$ define the polar variety $\widehat{W}_{K^{n-p-i}}(S)$ outside of the locus $V(m)$.

Fix for the moment $1\leqslant j\leqslant n-p-i+1$ and let E_j be the $(n-p-i-1)$-dimensional linear projective subvariety of $\mathbb{P}^n$ spanned by the points $A_1,\ldots,A_{j-1},A_{j+1},\ldots,A_{n-p-i+1}$. In particular, we have $E_{n-p-i+1}=K^{n-p-i-1}$.

From the generic choice of the complex numbers $a_{j,k},\ 1\leqslant j\leqslant n-p-i+1,\ 1\leqslant k\leqslant n$ we infer that Proposition 6 remains still valid if we replace in its statement the upper $(n-i)$-minor $m=M(k_1,\ldots,k_{n-i})$ by the $(n-i)$-minor of $T^{(i)}$ given by the rows $1,\ldots,p+j-1,\ p+j+1,\ldots,n-p-i+1$ and the columns $k_1,\ldots,k_{n-i}$ and the polar variety $\widehat{W}_{K^{n-p-i-1}}(S)$ by $\widehat{W}_{E_j}(S)$.

Let $\Delta_i:=\bigcap_{1\leqslant j\leqslant n-p-i+1}\widehat{W}_{E_j}(S)$. Then Δ_i is contained in $\widehat{W}_{K^{n-p-i+1}}(S)$ and Proposition 6 implies that, outside of the locus Δ_i, the polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is smooth and of pure codimension i in S. It is not too difficult to deduce from Proposition 6 that the codimension of Δ_i in S is at least $2i+1$. Hence, for $\frac{n-p-1}{2}<i\leqslant n-p$, the algebraic variety Δ_i is empty and therefore, the polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is smooth in any of its $(F_1,\ldots,F_p)$-regular points. In the next subsection we will show this property of $\widehat{W}_{K^{n-p-i}}(S)$ for any $0\leqslant i\leqslant n-p$ (see Theorem 10 below).

Finally, let us consider the case $i:=n-p$. Observe that $T^{(n-p)}$ is a $((p+1)\times n)$-matrix which contains the Jacobian $J(F_1,\ldots,F_p)$ as its first p rows. Thus, for any $(F_1,\ldots,F_p)$-regular point x of $\widehat{W}_{K^0}(S)$, there exists an upper p-minor m of $T^{(n-p)}$ with $m(x)\neq 0$. Therefore, we define $\widehat{W}_{K^{-1}}$ as the empty set. Thus, in particular, Δ_{n-p} is empty and this implies that $\widehat{W}_{K^0}(S)$ is smooth and of pure codimension $(n-p)$ outside of the locus S_{sing} (cf. Lemma 7 below).

3.2. Geometric conclusions

The geometric main outcome of this section is Theorem 10 below, which is a basic result for generalized affine polar varieties in the reduced complete intersection case. The proof of this result requires three fundamental technical statements, namely Lemmas 7, 9 and Proposition 8 below. For the rest of this section let the assumptions and notations be as before.

Lemma 7. *The generalized polar variety* $\widehat{W}_{K^0}(S)$ *is empty or of* (*expected*) *codimension* $n-p$ *in* S (*i.e.,* $\widehat{W}_{K^0}(S)$ *contains at most finitely many points*). *Moreover,* $\widehat{W}_{K^0}(S)$ *is contained in* S_{reg}.

Proof. Without loss of generality, we may assume that $\widehat{W}_{K^0}(S)$ is non-empty. Consider an irreducible component C of $\widehat{W}_{K^0}(S)$. Since the $(F_1, \ldots, F_p)$-regular points of $\widehat{W}_{K^0}(S)$ are Zariski dense in $\widehat{W}_{K^0}(S)$, we conclude that $C \cap S_{\text{reg}}$ is non-empty.

Observe that $\widehat{W}_{K^{-1}}(S)$ is empty. From Proposition 6 we conclude now that any point of $\widehat{W}_{K^0}(S) \cap S_{\text{reg}}$ is isolated. Therefore C consists of a single point that belongs to S_{reg}. This implies that the algebraic variety $\widehat{W}_{K^0}(S)$ is contained in S_{reg} and of pure codimension $n-p$ in S. $\square$

Proposition 6 and Lemma 7 imply our next result.

Proposition 8. *Suppose that the generalized affine polar variety* $\widehat{W}_{K^{n-p-i}}(S)$ *is non-empty. Then* $\widehat{W}_{K^{n-p-i}}(S)$ *is of pure codimension* i *in* S (*and therefore, the codimension of* $\widehat{W}_{K^{n-p-i}}(S)$ *in* S *coincides with the expected one*). *Moreover, for each irreducible component* C *of* $\widehat{W}_{K^{n-p-i}}(S)$ *there exists an upper* $(n-i)$*-minor* m *of* $T^{(i)}$ *such that* m *does not vanish identically on* C. *In particular, no irreducible component of* $\widehat{W}_{K^{n-p-i}}(S)$ *is contained in* $\widehat{W}_{K^{n-p-i-1}}(S)$.

Proof. Let C be an irreducible component of $\widehat{W}_{K^{n-p-i}}(S)$. Suppose for the moment that all upper $(n-i)$-minors of $T^{(i)}$ vanish identically on C. Then Proposition 6 implies that C is contained in $\widehat{W}_{K^{n-p-i-1}}(S)$. From Lemma 7 we deduce that there exists an index $0 \leqslant j < n-p-i$ such that C is contained in $\widehat{W}_{K^j}(S)$, but not in $\widehat{W}_{K^{j-1}}(S)$. Since $C \cap S_{\text{reg}}$ is Zariski dense in C, there exists a point $x \in C \cap S_{\text{reg}} \setminus \widehat{W}_{K^{j-1}}(S)$. From Proposition 6 we infer that there is a single irreducible component C' of $\widehat{W}_{K^j}(S)$ that contains the point x. Moreover, this irreducible component has codimension $n-p-j > i$ in S and contains C. Thus the codimension of C in S is strictly larger than i. On the other hand, Lemma 3 implies that C has codimension at most i in S. From this contradiction we deduce that there exists an upper $(n-i)$-minor m of $T^{(i)}$ that does not vanish identically on C.

Therefore, $C \setminus V(m)$ is non-empty. From Proposition 6 we deduce now that the codimension of C in S is exactly i. Hence, the generalized affine polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is of pure codimension i in S.

Observe that the same arguments are valid for the polar variety $\widehat{W}_{K^{n-p-i-1}}(S)$. Thus $\widehat{W}_{K^{n-p-i-1}}(S)$ is of pure codimension $(i+1)$ in S. Consequently, the irreducible component C of $\widehat{W}_{K^{n-p-i}}$ cannot be contained in $\widehat{W}_{K^{n-p-i-1}}(S)$. $\square$

Let us remark that, for a generic choice of the parameters $a_{j,k}$, $1 \leqslant j \leqslant n-p$, $1 \leqslant k \leqslant n$, Propositions 6 and 8 yield a local description of the generalized polar varieties of a given complete intersection variety by polynomial equations.

Moreover, it is not too difficult to conclude from Propositions 6 and 8 that in the case, where S and $\widehat{W}_{K^{n-p-i}}(S)$ are non-empty and smooth, the ideal generated in $\mathbb{C}[X_1, \ldots, X_n]$ by $F_1, \ldots, F_p$ and all $(n-i+1)$-minors of $T^{(i)}$ is Cohen–Macaulay and radical. This is a consequence of the main result of [16]. On the other hand, this ideal theoretic conclusion of Propositions 6 and 8 ensures already the correctness of the main algorithmic results described in Section 4, namely Theorems 13 and 14 below (see [4] for details).

In what follows, we shall use the notations that we are going to introduce now.

Let $A_1^* := (a_{1,0}^* : \cdots : a_{1,n}^*), \ldots, A_{n-p-i}^* := (a_{n-p-i,0}^* : \cdots : a_{n-p-i,n}^*)$, $A_{n-p-i+1}^* := (a_{n-p-i+1,0}^* : \cdots : a_{n-p-i+1,n}^*)$ be the $n-p-i+1$ points of $\mathbb{P}^n$ whose coordinates are defined by $a_{1,0}^* := 0, \ldots, a_{n-p-i,0}^* := 0$, $a_{n-p-i+1,0}^* = a_{n-p-i+1,0}$ and, for $1 \leqslant j \leqslant n-p-i+1$ and $1 \leqslant k \leqslant n$, by

$$a_{j,k}^* := \begin{cases} a_{j,k} - a_{n-p-i+1,k} & \text{if } 1 \leqslant j \leqslant n-p-i \text{ and } a_{j,0} = 1, \\ a_{j,k} & \text{if } 1 \leqslant j \leqslant n-p-i \text{ and } a_{j,0} = 0, \\ a_{n-p-i+1,k} & \text{if } j = n-p-i+1. \end{cases}$$

Note that we have always $A_{n-p-i+1}^* = A_{n-p-i+1}$ and $A_1^* = A_1, \ldots, A_{n-p}^* = A_{n-p}$ in the case that $a_{1,0} = \cdots a_{n-p-i+1,0} = 0$ holds. Let us recall from Section 3.1 that the condition $a_{1,0} = \cdots a_{n-p-i+1,0} = 0$ is equivalent to the condition $K^{n-p-i} \subset H$ and that we have by assumption $a_{n-p-i+1,0} = 1$ in the case $K^{n-p-i} \not\subset H$. Observe that the $n-p-i+1$ points $A_1^*, \ldots, A_{n-p-i+1}^*$ span the linear projective subvariety K^{n-p-i} of $\mathbb{P}^n$ and that their coordinates $a_{j,k}^*$, $1 \leqslant j \leqslant n-p-i+1$, $1 \leqslant k \leqslant n$, are generic. Moreover, $\widehat{W}_{K^{n-p-i}}(S)$ is defined, outside of the singular locus of *S*, by the vanishing of the polynomials $F_1, \ldots, F_p$ and of all $(n-i+1)$-minors of the $((n-i+1) \times n)$-matrix

$$\Gamma^{(i)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1}^* & \cdots & a_{1,n}^* \\ \vdots & \vdots & \vdots \\ a_{n-p-i,1}^* & \cdots & a_{n-p-i,n}^* \\ a_{n-p-i+1,1}^* - a_{n-p-i+1,0}^* X_1 & \cdots & a_{n-p-i+1,n}^* - a_{n-p-i+1,0}^* X_n \end{bmatrix}.$$

We denote the polynomial $((n-i) \times n)$-matrix of the first $n-i$ rows of $\Gamma^{(i)}$ by $\check{\Gamma}^{(i)}$. From Proposition 8 we deduce that the Zariski closure of all $(F_1, \ldots, F_p)$-regular points of *S* at which all $(n-i)$-minors of $\check{\Gamma}^{(i)}$ vanish constitutes a (classic) polar variety $\check{W}_{i+1}$ which is contained in $\widehat{W}_{K^{n-p-i}}(S)$ and is empty or has pure codimension $i+1$ in *S*.

Let $(k_1, \ldots, k_p)$ be an arbitrary ordered sequence of p different elements of the set $\{1, \ldots, n\}$. We denote by $J^{(k_1,\ldots,k_p)}$ the p-minor of the Jacobian $J(F_1, \ldots, F_p)$ determined by the columns $k_1, \ldots, k_n$.

Lemma 9. *Suppose that the variables $X_1, \ldots, X_n$ are in general position with respect to the affine variety S and that the generalized polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is non-empty. For $n-i+1 \leqslant k \leqslant n$, we denote by N_k the $(p+1)$-minor determined by the columns $1, \ldots, p, k$ of the polynomial $((p+1) \times n)$-matrix*

$$\begin{bmatrix} & J(F_1(Z), \ldots, F_p(Z)) & \\ a_{n-p-i+1,1} - a_{n-p-i+1,0} X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0} X_n \end{bmatrix}.$$

Then there exists a point x of the affine variety S satisfying the conditions

$$J^{(1,\dots,p)}(x) \neq 0,\ N_{n-i+1}(x) = \cdots = N_n(x) = 0.$$

Proof. For $1 \leqslant j \leqslant n-p-i$ and $1 \leqslant k \leqslant n$, consider the generic linear form $\Lambda_j^* := \sum_{1\leqslant l\leqslant n} a_{j,l}^* X_l$ and let $\zeta_k = (\zeta_{k,1},\dots,\zeta_{k,n})$ be a point of $\mathbb{A}^n$ such that $\zeta_1,\dots,\zeta_{p+1}$ and $\zeta_1,\dots,\zeta_n$ constitute a $\mathbb{C}$-vector space basis of the $(p+1)$-dimensional subspace of $\mathbb{A}^n$ defined by the generic linear forms $\Lambda_1^*,\dots,\Lambda_{n-p-i}^*$ and of the affine space $\mathbb{A}^n$, respectively. We denote the transposed matrix of $(\zeta_{j,k})_{1\leqslant j,k\leqslant n}$ by B. From the genericity of the linear forms $\Lambda_1^*,\dots,\Lambda_{n-p-i+1}^*$ we deduce that $\zeta_1,\dots,\zeta_{p+i}$ form a generic set of points of $\mathbb{A}^n$. Without loss of generality, we may assume that the same is true for the points $\zeta_1,\dots,\zeta_n$. For $1\leqslant k\leqslant n$, let $Z_k := \sum_{1\leqslant l\leqslant n} \tilde{\zeta}_{k,l} X_l$, where $(\tilde{\zeta}_{k,1},\dots,\tilde{\zeta}_{k,n})$ is the kth row of the inverse of the matrix B.

Let $Z := (Z_1,\dots,Z_n)$ and observe that Z represents a generic coordinate transformation of the affine space $\mathbb{A}^n$. Following the context, we will consider $Z_1,\dots,Z_n$ to be new variables or linear forms in $X_1,\dots,X_n$. By $F_1(Z),\dots,F_p(Z)$ we denote the polynomials $F_1,\dots,F_p$ rewritten in the new variables $Z_1,\dots,Z_n$ and by

$$J(F_1(Z),\dots,F_p(Z)) := \left(\frac{\partial F_h(Z)}{\partial Z_k}\right)_{\substack{1\leqslant h\leqslant p\\ 1\leqslant k\leqslant n}}$$

the Jacobian of $F_1,\dots,F_p$ with respect to the variables $Z_1,\dots,Z_n$. For $1\leqslant h\leqslant p$ and $1\leqslant k\leqslant n$ we have then

$$\frac{\partial F_h}{\partial Z_k}(Z) \;=\; \sum_{l=1}^{n} \zeta_{k,l}\, \frac{\partial F_h}{\partial X_l}. \tag{10}$$

In that what follows, we shall consider the entries $Z_1,\dots,Z_n$ of Z to be linear forms in the variables $X_1,\dots,X_n$. Consequently, the entries of the Jacobian $J(F_1(Z),\dots,F_p(Z))$ will be considered elements of the polynomial ring $\mathbb{C}[X_1,\dots,X_n]$. Taking into account that $\zeta_1,\dots,\zeta_{p+i-1}$ form a basis of the $\mathbb{C}$-vector space defined by the linear forms

$$\Lambda_1^* = \sum_{l=1}^{n} a_{1,l}^* X_l\ ,\dots,\ \Lambda_{n-p-i}^* = \sum_{l=1}^{n} a_{n-p-i,l}^* X_l,$$

we deduce from identities (10) that the matrices $\check{\Gamma}^{(i)}B$ and $\Gamma^{(i)}B$ are of the form

$$\check{\Gamma}^{(i)}B = \begin{bmatrix} J(F_1(Z),\dots,F_p(Z)) \\ O_{n-p-i,\,p+i}\ \ (*)_{\,n-p-i,\,n-p-i} \end{bmatrix}$$

and

$$\Gamma^{(i)}B = \begin{bmatrix} & J(F_1(Z),\dots,F_p(Z)) & \\ O_{n-p-i,\,p+i} & & (*)_{\,n-p-i,\,n-p-i} \\ \beta_1-\gamma_1 Z_1 & \cdots & \beta_n-\gamma_n Z_n \end{bmatrix},$$

where $O_{n-p-i,\,p+i}$ denotes the $((n-p-i)\times(p+i))$-zero matrix, $(*)_{\,n-p-i,\,n-p-i}$ indicates a suitable $((n-p-i)\times(n-p-i))$-matrix with generic complex entries, $\beta_1,\dots,\beta_n$

are generic complex numbers and $\gamma_1, \ldots, \gamma_n$ are complex numbers satisfying the condition $\gamma_1 = \cdots = \gamma_n = 0$ in the case $a_{n-p-i+1} = 0$ and $\gamma_1 \neq 0, \ldots, \gamma_n \neq 0$ in the case $a_{n-p-i+1,0} = 1$. Without loss of generality we may assume that $\gamma_1 = \cdots \gamma_n = a_{n-p-i+1,0}$ holds. Let us therefore abbreviate $\gamma := \gamma_1$.

The genericity of the matrix $(*)_{n-p-i,\, n-p-i}$ implies that the $(n-i)$-minors of the matrix $\check{\Gamma}^{(i)} B$, that are not identically zero, are scalar multiples of the p-minors selected between the columns $1, \ldots, p+i$ of the Jacobian $J(F_1(Z), \ldots, F_p(Z))$, and vice versa. Hence, the ideal generated by these p-minors in $\mathbb{C}[X_1, \ldots, X_n]$ coincides with the ideal generated by all $(n-i)$-minors of $\check{\Gamma}^{(i)} B$, and therefore also with the ideal generated by all $(n-i)$-minors of the matrix $\check{\Gamma}^{(i)}$ (observe that the matrix B is generic and consequently regular).

Therefore all p-minors selected between the columns $1, \ldots, p+i$ of the Jacobian $J(F_1(Z), \ldots, F_p(Z))$ vanish at a $(F_1, \ldots, F_p)$-regular point $\check{x}$ of S if and only if all $(n-i)$-minors of $\check{\Gamma}^{(i)}$ do so, i.e., if and only if $\check{x}$ belongs to the classic polar variety $\check{W}_{i+1}$. Since $\check{W}_{i+1}$ is an affine subvariety of $\widehat{W}_{K^{n-p-i}}(S)$ which is either empty or of pure codimension $i+1$ in S and since by Proposition 8 the generalized polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is of pure codimension i in S, we conclude that there exists a point x of $\widehat{W}_{K^{n-p-i}}(S)$ which does not belong to $\check{W}_{i+1}$.

Since the point x does not belong to the polar variety $\check{W}_{i+1}$, we may assume without loss of generality that the p-minor $J_Z^{(1,\ldots,p)}$ of the matrix $J(F_1(Z), \ldots, F_p(Z))$ determined by the columns $1, \ldots, p$ does not vanish at x. Consider now the $((p+1) \times n)$-matrix

$$T^{(Z)} = \begin{bmatrix} & J(F_1(Z), \ldots, F_p(Z)) & \\ \beta_1 - \gamma Z_1 & \cdots & \beta_n - \gamma Z_n \end{bmatrix}$$

and denote by $N_{n-i+1}^{(Z)}, \ldots, N_n^{(Z)}$ the $(p+1)$-minors of $T^{(Z)}$ obtained by successively selecting the columns $1, \ldots, p, p+1$, then $1, \ldots, p, p+2$, up to, finally, the columns $1, \ldots, p, p+i$.

The same argument as above implies that the $(n-i+1)$-minors of the matrix $\Gamma^{(i)} B$ that are not identically zero, are scalar multiples of the $(p+1)$-minors selected between the columns $1, \ldots, p+i$ of the polynomial $((p+1) \times n)$-matrix $T^{(Z)}$. Therefore these $(p+1)$-minors generate the ideal of all $(n-i+1)$-minors of the polynomial $((n-i+1) \times n)$-matrix $\Gamma^{(i)}$ which on their turn define the generalized polar variety $\widehat{W}_{K^{n-p-i}}(S)$ outside of the singular locus of S. This implies that the polynomials $F_1(Z), \ldots, F_p(Z)$ and $N_{n-i+1}^{(Z)}, \ldots, N_n^{(Z)}$ vanish at the point $z := Z(x)$, whereas $J_Z^{(1,\ldots,p)}$ does not.

We may easily rearrange the coordinate transformation $Z = (Z_1, \ldots, Z_n)$ in such a way that the polynomials $N_{n-i+1}^{(Z)}, \ldots, N_n^{(Z)}$ become the $(p+1)$-minors of the matrix $T^{(Z)}$ obtained by successively selecting the columns $1, \ldots, p,\ n-i+1$, then $1, \ldots, p,\ n-i+2$ up to, finally, the columns $1, \ldots, p,\ n$. Since the entries $\beta_1, \ldots, \beta_n$ of the matrix $T^{(Z)}$ and the coefficients of the coordinate transformation $Z = (Z_1, \ldots, Z_n)$ are generic, the conclusion of the lemma follows now easily replacing the variables $Z_1, \ldots, Z_n$ by $X_1, \ldots, X_n$, the parameter γ by $a_{n-p-i+1,0}$, the parameters $\beta_1, \ldots, \beta_n$ by $a_{n-p-i+1,1}, \ldots, a_{n-p-i+1,n}$ and observing that $J_X^{(1,\ldots,p)} = J^{(1,\ldots,p)}$ and that $N_{n-i+1}^{(X)} = N_{n-i+1}, \ldots, N_n^{(X)} = N_n$ holds. $\square$

Theorem 10. *Let the assumptions and notations be as at the beginning of Section* 3. *Then the following assertions are true*:

(i) *The affine polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is smooth in any of its $(F_1, \ldots, F_p)$-regular points. Suppose that the variables $X_1, \ldots, X_n$ are in general position with respect to S. Then for any p-minor J of the Jacobian $J(F_1, \ldots, F_p)$ the ideal of definition of the affine variety $\widehat{W}_{K^{n-p-i}}(S) \setminus V(J)$ in $\mathbb{C}[X_1, \ldots, X_n]_J$, the localization of $\mathbb{C}[X_1, \ldots, X_n]$ by the polynomial J, is generated by $F_1, \ldots, F_p$ and all $(n-i+1)$-minors of the polynomial matrix $T^{(i)}$.*

(ii) *Suppose that the polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is non-empty. If any point of S is $(F_1, \ldots, F_p)$-regular, then $\widehat{W}_{K^{n-p-i}}(S)$ is smooth and its (radical) ideal of definition in $\mathbb{C}[X_1, \ldots, X_n]$ is generated by $F_1, \ldots, F_p$ and by all $(n-i+1)$-minors of the polynomial matrix $T^{(i)}$. In particular, this ideal is unmixed and regular.*

Proof. Assertion (ii) is an immediate consequence of assertion (i). Therefore we will show only assertion (i). Without loss of generality we may assume that $\widehat{W}_{k^{n-p-i}}(S)$ is non-empty and that the variables $X_1, \ldots, X_n$ are in general position with respect to S. For $1 \leqslant j \leqslant n-p-i$ and $1 \leqslant k \leqslant n$, let $Z_{j,k}$ be new indeterminates and let us consider the polynomial $((n-p-i+1) \times n)$-matrix

$$\widetilde{T}^{(i)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ Z_{1,1} & \cdots & Z_{1,n} \\ \vdots & \vdots & \vdots \\ Z_{n-i+1,1} & \cdots & Z_{n-i+1,n} \\ a_{n-p-i+1,1} - a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0}X_n \end{bmatrix}$$

whose entries belong to the polynomial $\mathbb{C}$-algebra

$$\mathcal{R}_i := \mathbb{C}[Z_{j,k},\ X_k \mid 1 \leqslant j \leqslant n-p-i,\ 1 \leqslant k \leqslant n]$$

We denote by $\mathfrak{a}_i$ the ideal generated by $F_1, \ldots, F_p$ and all $(n-p-i+1)$-minors of $\widetilde{T}^{(i)}$ in $\mathcal{R}_i$ and by rad $\mathfrak{a}_i$ the radical ideal of $\mathfrak{a}_i$. Let $\Sigma_i := V(\mathfrak{a}_i)$ be the closed subvariety of the affine space $\mathbb{A}^{(n-p-i)\times n} \times \mathbb{A}^n$ defined by the vanishing of $F_1, \ldots, F_p$ and of all $(n-p-i+1)$-minors of $\widetilde{T}^{(i)}$ and let $\pi_i : \Sigma_i \to \mathbb{A}^{(n-p-i)\times n}$ be the morphism of affine varieties induced by the canonical projection of $\mathbb{A}^{(n-p-i)\times n} \times \mathbb{A}^n$ onto $\mathbb{A}^{(n-p-i)\times n}$. Then the $\mathbb{C}$-algebra $\mathcal{R}_i/\text{rad}\ \mathfrak{a}_i$ is isomorphic to the coordinate ring $\mathbb{C}[\Sigma_i]$ of Σ_i.

Let us consider an arbitrary point $z = (z_{j,k})_{\substack{1 \leqslant j \leqslant n-p-i \\ 1 \leqslant k \leqslant n}}$ of $\mathbb{A}^{(n-p-i)\times n}$ and, in analogy with the notation of Lemma 5, let us write $K(z)$ for be the linear projective variety spanned by the $n-p-i+1$ points $(0 : z_{1,1} : \cdots : z_{1,n}), \ldots, (0 : z_{n-p-i,1} : \cdots : z_{n-p-i,n}), (a_{n-p-i+1,0} : a_{n-p-i+1,1} : \cdots : a_{n-p-i+1,n})$ in $\mathbb{P}^n$. Then the fibre $\pi_i^{-1}(z)$ is canonically isomorphic to the generalized polar variety $\widehat{W}_{K(z)}(S)$. In particular, $\widehat{W}_{K(z)}(S)$ is non-empty if and only if z belongs to the image of π_i.

We consider now the point $a^* = (a^*_{j,k})_{\substack{1 \leqslant j \leqslant n-p-i \\ 1 \leqslant k \leqslant n}}$ of $\mathbb{A}^{(n-p-i)\times n}$ whose coordinates are generic. Observe that $K(a^*) = K^{n-p-i}$ holds. Since $\widehat{W}_{K^{n-p-i}}(S)$ is by assumption non-empty, we conclude that the point a^* belongs to the image of π_i. From the genericity of the coordinates of the point a^* we deduce now that the morphism of affine varieties $\pi_i : \Sigma_i \to \mathbb{A}^{(n-p-i)\times n}$ is dominating and from Proposition 8 we infer that the dominating irreducible components of Σ_i are of pure codimension $p+i$ in $\mathbb{A}^{(n-p-i)\times n}$.

Let $\mathcal{A}_i := \mathbb{C}[Z_{j,k} \mid 1 \leqslant j \leqslant n-p-i,\ 1 \leqslant k \leqslant n]$ and $\mathcal{B}_i := \mathcal{R}_i/\mathfrak{a}_i$. Since $\pi_i : \Sigma_i \to \mathbb{A}^{(n-p-i)\times n}$ is dominating and $\mathcal{R}_i/\mathrm{rad}\,\mathfrak{a}_i$ is isomorphic to $\mathbb{C}[\Sigma_i]$, we may consider $\mathcal{A}_i$ as a $\mathbb{C}$-subalgebra of $\mathcal{B}_i$.

Let $(k_1, \ldots, k_p)$ be an arbitrary ordered sequence of p different elements of the set $\{1, \ldots, n\}$. Then

$$\Sigma_i^{(k_1,\ldots,k_p)} := \{(z, x) \in \Sigma_i \mid z \in \mathbb{A}^{(n-p-i)\times n},\ x \in \mathbb{A}^n,\ J^{(k_1,\ldots,k_p)}(x) \neq 0\}$$

is an open, affine subvariety of Σ_i whose coordinate ring is isomorphic to $(\mathcal{R}_i/\mathrm{rad}\,\mathfrak{a}_i)_{J(k_1,\ldots,k_p)}$. We denote by $\pi_i^{(k_1,\ldots,k_p)} : \Sigma_i^{(k_1,\ldots,k_p)} \to \mathbb{A}^{(n-p-i)\times n}$ the morphism of affine varieties induced by π_i and we write $\mathcal{B}_i^{(k_1,\ldots,k_p)}$ for the localization of the $\mathcal{R}_i$-algebra $\mathcal{B}_i$ by the polynomial $J^{(k_1,\ldots,k_p)}$ of $\mathcal{R}_i$.

For any point z in $\mathbb{A}^{(n-p-i)\times n}$ we denote by $\mathfrak{n}_z$ the ideal of definition of z in $\mathcal{A}_i$ and by $(\mathcal{A}_i)_{\mathfrak{n}_z}$, $(\mathcal{B}_i)_{\mathfrak{n}_z}$ and $(\mathcal{B}_i^{(k_1,\ldots,k_p)})_{\mathfrak{n}_z}$ the localizations of $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{B}_i^{(k_1,\ldots,k_p)}$ at the maximal ideal $\mathfrak{n}_z$, respectively (here we consider $\mathcal{B}_i$ and $\mathcal{B}_i^{(k_1,\ldots,k_p)}$ as $\mathcal{A}_i$-algebras). Let $\mathcal{U}_i^{(k_1,\ldots,k_p)}$ be the set of all points z of $\mathbb{A}^{(n-p-i)\times n}$ such that $(\mathcal{B}_i)_{\mathfrak{n}_z}$ is a smooth $(\mathcal{A}_i)_{\mathfrak{n}_z}$-algebra. From [33, Corollary 8.2], we infer that $\mathcal{U}_i^{(k_1,\ldots,k_p)}$ is a Zariski-open subset of $\mathbb{A}^{(n-p-i)\times n}$.

In order to simplify notations we suppose without loss of generality that $k_1 = 1, \ldots, k_p = p$ holds. Since, by assumption, the variables $X_1, \ldots, X_n$ are in general position with respect to S we have $J^{(1,\ldots,p)} \neq 0$.

We are going to show that $\mathcal{U}_i^{(1,\ldots,p)}$ is non-empty. To this end let us consider the complex $((n-p-i)\times n)$-matrix $b = (b_{j,k})_{\substack{1 \leqslant j \leqslant n-p-i \\ 1 \leqslant k \leqslant n}}$ defined for $1 \leqslant j \leqslant n-p-i$ and $1 \leqslant k \leqslant n$ by

$$b_{j,k} := \begin{cases} 0 & \text{if } k \neq p+j\,, \\ 1 & \text{if } k = p+j\,. \end{cases}$$

Let $\tilde{b} = (\tilde{b}_{j,k})_{\substack{1 \leqslant j \leqslant n-p-i+1 \\ 0 \leqslant k \leqslant n}}$ be the complex $((n-p-i+1)\times n)$-matrix defined for $1 \leqslant j \leqslant n-p-i$ and $1 \leqslant k \leqslant n$ by

$$\tilde{b}_{j,k} := \begin{cases} b_{j,k} & \text{if } 1 \leqslant j \leqslant n-p-i \text{ and } 1 \leqslant k \leqslant n\,, \\ 0 & \text{if } 1 \leqslant j \leqslant n-p-i \text{ and } k = 0\,, \\ a_{n-p-i+1,k} & \text{if } j = n-p-i+1 \text{ and } 0 \leqslant k \leqslant n\,. \end{cases}$$

One verifies easily, with the notations of Section 3.1, the identity $m^{(\tilde{b})} = J^{(1,\ldots,p)}$, that the complex $((n-p-i+1)\times n)$-matrix $\tilde{b}$ satisfies the assumptions of Lemma 5 and that $m^{(\tilde{b})}(x) = J^{(1,\ldots,p)}(x) \neq 0$ holds, for any point x of $\mathbb{A}^n$ with $(b, x) \in \Sigma_i^{(k_1,\ldots,k_p)}$. Observe now that the $\pi_i^{(1,\ldots,p)}$-fibre of the point b is canonically isomorphic to the affine variety

$\widehat{W}_{K(b)}(S) \setminus V(J(1,\ldots,p))$. Therefore we have

$$\widehat{W}_{K(b)}(S) \setminus V(m^{(b)}) \simeq (\pi_i^{(1,\ldots,p)})^{(-1)}(b).$$

From Lemma 5 we deduce now that $\mathfrak{n}_b$ and $\mathfrak{a}_i$ generate in $(\mathcal{R}_i)_{J(1,\ldots,p)}$ the trivial or a complete intersection ideal $\mathfrak{a}_i^{(b)}$ that defines the $\pi_i^{(1,\ldots,p)}$-fibre of the point $b \in \mathbb{A}^{(n-p-i)\times n}$ and that this fibre is empty or a smooth complete intersection variety of dimension $n-p-i$. Observe now that $(\pi_i^{(1,\ldots,p)})^{(-1)}(b)$ is defined by the vanishing of $F_1,\ldots,F_p$ and of all $(n-i+1)$-minors of the polynomial $((n-i+1)\times n)$-matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_p} & \frac{\partial F_1}{\partial X_{p+1}} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} & \frac{\partial F_1}{\partial X_{n-i+1}} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_p} & \frac{\partial F_p}{\partial X_{p+1}} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} & \frac{\partial F_p}{\partial X_{n-i+1}} & \cdots & \frac{\partial F_p}{\partial X_n} \\ 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \rho_1(X_1) & \cdots & \rho_p(X_p) & \rho_{p+1}(X_{p+1}) & \cdots & \rho_{n-i}(X_{n-i}) & \rho_{n-i+1}(X_{n-i+1}) & \cdots & \rho_n(X_n) \end{bmatrix},$$

where $\rho_k(X_k)$ denotes the linear form $a_{n-p-i+1,k} - a_{n-p-i+1,0}X_k$, for $1 \leqslant k \leqslant n$. From Lemmas 5 and 9 we deduce now easily that $(\pi_i^{(1,\ldots,p)})^{(-1)}(b)$ is non-empty. Therefore, the ideal $\mathfrak{a}_i^{(b)}$ is generated by a regular sequence which consists of the canonical generators of $\mathfrak{n}_b$ and suitable elements of $\mathfrak{a}_i$. Consider now an arbitrary point $y = (b, x)$ of $\Sigma_i^{(k_1,\ldots,k_p)}$ with x in $\mathbb{A}^n$. Let $\mathfrak{n}_y$ be the ideal of definition of y in $\mathcal{R}_i$ and observe that $\mathfrak{n}_b = \mathcal{A}_i \cap \mathfrak{n}_y$ and $\mathfrak{a}_i \subset \mathfrak{n}_y$ holds and that $J(1,\ldots,p)$ is not contained in $\mathfrak{n}_y$. Thus $(\mathcal{R}_i)_{J(1,\ldots,p)}$ is a subring of $(\mathcal{R}_i)_{\mathfrak{n}_y}$ and $(\mathfrak{a}_i^{(b)})_{\mathfrak{n}_y}$ is a complete intersection ideal of the local ring $(\mathcal{R}_i)_{\mathfrak{n}_y}$. From [38, 6.16, Corollary] and the Exchange Lemma of [3] we deduce now that $\mathfrak{a}_i$ generates in $(\mathcal{R}_i)_{\mathfrak{n}_y}$ a complete intersection ideal and that $(\mathcal{R}_i/\mathfrak{a}_i)_{\mathfrak{n}_y} = (\mathcal{B}_i)_{\mathfrak{n}_y}$ has Krull dimension $(n+1)(n-p-i)$. In particular, $(\mathcal{B}_i)_{\mathfrak{n}_y}$ is Cohen–Macaulay. Since y is an arbitrary element of the $\pi_i^{(1,\ldots,p)}$-fibre of the point b of $\mathbb{A}^{(n-p-i)\times n}$, we deduce that $(\mathcal{B}_i^{(1,\ldots,p)})_{\mathfrak{n}_b}$ is an equidimensional Cohen–Macaulay ring of Krull dimension $(n+1)(n-p-i)$. Moreover, $\mathcal{B}_i^{(1,\ldots,p)}/\mathcal{B}_i^{(1,\ldots,p)}\cdot\mathfrak{n}_b$ is an equidimensional regular $\mathbb{C}$-algebra of Krull dimension $n-p-i$, isomorphic to the coordinate ring of the $\pi_i^{(1,\ldots,p)}$-fibre of the point b of $\mathbb{A}^{(n-p-i)\times n}$. Observing that $(\mathcal{A}_i)_{\mathfrak{n}_b}$ is a regular local $\mathbb{C}$-algebra of Krull dimension $n(n-p-i)$ contained in $(\mathcal{B}_i)_{\mathfrak{n}_b}$, we deduce from [38, 8.23, Theorem 31.1] that $(\mathcal{B}_i^{(1,\ldots,p)})_{\mathfrak{n}_b}$ is a flat $(\mathcal{A}_i)_{\mathfrak{n}_b}$-algebra. Taking into account that $\mathcal{B}_i^{(1,\ldots,p)}/\mathcal{B}_i^{(1,\ldots,p)}\cdot\mathfrak{n}_y$ is an equidimensional regular $\mathbb{C}$-algebra we infer now from [33, Theorem 8.1] and Nakayama's Lemma (or from [38, 8.23, Theorem 37.7]) that $(\mathcal{B}_i)_{\mathfrak{n}_b}$ is a smooth $(\mathcal{A}_i)_{\mathfrak{n}_b}$-algebra. This implies that the point $b \in \mathbb{A}^{(n-p-i)\times n}$ belongs to the set $\mathcal{U}_i^{(1,\ldots,p)}$. Therefore, $\mathcal{U}_i^{(1,\ldots,p)}$ is a non-empty Zariski open subset of the affine space $\mathbb{A}^{(n-p-i)\times n}$.

Let $\mathcal{U}_i$ be the intersection of all (non-empty, Zariski open) sets $\mathcal{U}_i^{(k_1,\ldots,k_p)}$, where $1 \leqslant k_1 < \cdots < k_p \leqslant n$. Then $\mathcal{U}_i$ is a non-empty Zariski open subset of $\mathbb{A}^{(n-p-i)\times n}$. Since the coordinates of the point $a^* = (a^*_{j,k})$ of $\mathbb{A}^{(n-p-i)\times n}$ are generic, we may assume without loss of generality that a^* belongs to $\mathcal{U}_i$. Therefore, for any ordered sequence $(k_1, \ldots, k_p)$ of p different elements of the set $\{1, \ldots, n\}$, the ring $(\mathcal{B}_i^{(k_1,\ldots,k_p)})_{\mathfrak{n}_{a^*}}$ is a smooth $(\mathcal{A}_i)_{\mathfrak{n}_{a^*}}$-algebra. This implies that $\mathcal{B}_i^{(k_1,\ldots,k_p)}/\mathcal{B}_i^{(k_1,\ldots,k_p)} \cdot \mathfrak{n}_{a^*}$ is a regular $\mathbb{C}$-algebra and therefore reduced. This $\mathbb{C}$-algebra is the coordinate ring of the $\pi_i^{(k_1,\ldots,k_p)}$-fibre of the point a^* of $\mathbb{A}^{(n-p-i)\times n}$ which is canonically isomorphic to $\widehat{W}_{K(a^*)}(S) \setminus V(J(k_1, \ldots, k_p)) = \widehat{W}_{K^{n-p-i}}(S) \setminus V(J(k_1, \ldots, k_p))$.

Since $\mathcal{B}_i^{(k_1,\ldots,k_p)}$ is the localization of the $\mathcal{R}_i$-algebra $\mathcal{B}_i$ by the polynomial $J(k_1, \ldots, k_p)$ of $\mathcal{R}_i$, we infer now from the identity $\mathcal{B}_i = \mathcal{R}_i/\mathfrak{a}_i$ that the ideal of definition of the affine variety $\widehat{W}_{K^{n-p-i}}(S) \setminus V(J(k_1, \ldots, k_p))$ in $\mathbb{C}[X_1, \ldots, X_n]_{J(k_1,\ldots,k_p)}$ is generated by $F_1, \ldots, F_p$ and all $(n-i+1)$-minors of the polynomial matrix $T^{(i)}$ (observe that the ideal generated by the $(n-i+1)$-minors of $T^{(i)}$ equals the ideal generated by the $(n-i+1)$-minors of $\Gamma^{(i)}$ and that $\Gamma^{(i)}$ is obtained from $\widetilde{T}^{(i)}$ by specializing for each pair of indices $1 \leqslant j \leqslant n-p-i$ and $1 \leqslant k \leqslant n$ the variable $Z_{j,k}$ to the complex number $a^*_{j,k}$).

Moreover, the coordinate ring of the variety $\widehat{W}_{K^{n-p-i}}(S) \setminus V(J(k_1, \ldots, k_p))$ is isomorphic to the regular $\mathbb{C}$-algebra $\mathcal{B}_i^{(k_1,\ldots,k_p)}/\mathcal{B}_i^{(k_1,\ldots,k_p)} \cdot \mathfrak{n}_a$. Therefore, $\widehat{W}_{K^{n-p-i}}(S) \setminus V(J(k_1, \ldots, k_p))$ is a smooth, locally closed, affine subvariety of $\mathbb{A}^n$. This implies that the generalized polar variety $\widehat{W}_{K^{n-p-i}}(S)$ is smooth in any of its $(F_1, \ldots, F_p)$-regular points. □

In the case of affine, classic polar varieties (i.e., in the case $K^{n-p-i} \subset H$) Propositions 6 and 8 and Theorem 10 are nothing but a careful reformulation of [3, Theorem 1].

In terms of standard algebraic geometry, Theorem 10 implies the following result:

Corollary 11. *Let S be a smooth, pure p-dimensional closed subvariety of $\mathbb{A}^n$. Let K be a linear, projective subvariety of $\mathbb{P}^n$ of dimension $(n-p-i)$ with $1 \leqslant i \leqslant n-p$. Suppose that K is generated by $n-p-i+1$ many points $A_1 = (a_{1,0} : \cdots : a_{1,n}), \ldots, A_j = (a_{j,0} : \cdots : a_{j,n}), \ldots, A_{n-p-i+1} = (a_{n-p-i+1,0} : \cdots : a_{n-p-i+1,n})$ of $\mathbb{P}^n$ with $a_{j,0} = 0$ or $a_{j,0} = 1$ and $a_{j,1}, \ldots, a_{j,n}$ generic for any $1 \leqslant j \leqslant n-p-i+1$. Then $\widehat{W}_K(S)$ is either empty or a smooth variety of pure codimension i in S.*

Proof. By assumption, S is smooth and of pure codimension p in $\mathbb{A}^n$. Therefore, the algebraic variety S is locally definable by radical complete intersection ideals of height p in $\mathbb{C}[X_1, \ldots, X_n]$. Therefore we may apply a suitably adapted local version of Propositions 6 and 8 and Theorem 10 to the variety S. Thus $\widehat{W}_K(S)$ is locally empty or a smooth variety of pure codimension i in S. This implies the corresponding global properties of $\widehat{W}_K(S)$. □

Observe that Corollary 11 remains mutatis mutandis true if we replace in its formulation the affine variety S by the projective variety V and if V is smooth.

Remark 12. One obtains a slightly more elementary but less transparent proof of Theorem 10 (and hence of Corollary 11) by a suitable refinement of Lemma 9.

For an alternative proof of Corollary 11 for classic polar varieties we may reason as follows: Suppose for the moment that the linear projective variety K^{n-p-i} is contained in the hyperplane at infinity H of $\mathbb{P}^n$ and let $L^{p+i-2} := (K^{n-p-i})^*$. Then (7) implies the identity $\widehat{W}_{K^{n-p-i}}(S) = W_{L^{p+i-2}}(S)$. In other words, $\widehat{W}_{K^{n-p-i}}(S)$ is a classic polar variety.

In the Grassmannian of $(n-p)$-dimensional subspaces of $\mathbb{A}^n$ we consider now the Schubert variety σ_i associated with the multiindex $(1, \ldots, 1) \in \mathbb{Z}^{i-1}$ and with the linear projective variety L^{p+i-2}. Then it is not too difficult to see that the classic polar variety $\widehat{W}_{K^{n-p-i}}(S) = W_{L^{p+i-2}}(S)$ is obtained as the image under the Nash modification of the fibre product of the Gauss map of S_{reg} and the canonical embedding of σ_i into the corresponding Grassmannian. Applying Kleiman's Transversality Lemma [31] to this situation, one infers now easily Corollary 11 (cf. [47, Corollaire 1.3.2 and Définition 1.4] or [39] for this kind of reasoning).

In the context of the present paper, this elegant geometric argument has two disadvantages:

- We lose control over the equations which define the polar variety $\widehat{W}_{K^{n-p-i}}(S)$.
- The general case of polar varieties requires a different use of Schubert varieties.

4. Real polynomial equation solving

The geometric and algebraic results of Sections 2 and 3 allow us to enlarge the range of applications of the new generation of elimination procedures for real algebraic varieties introduced in [2,3].

Let S be a pure p-dimensional and $\mathbb{Q}$-definable, closed algebraic subvariety of the n-dimensional, complex, affine space $\mathbb{A}^n_{\mathbb{C}}$ and suppose that S is given by p polynomial equations $F_1, \ldots, F_p$ of degree at most d, forming a regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]$. Assume that, for any $1 \leqslant k \leqslant p$, $F_1, \ldots, F_k$ generate a radical ideal. Moreover, suppose that the real algebraic variety $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ is non-empty and smooth.

In this section we are going to explain how the geometric and algebraic results of Sections 2 and 3 together with [30], Theorem 4.4, lead to a *probabilistic* elimination procedure that finds an algebraic sample point for each connected component of $S_{\mathbb{R}}$. The complexity of this algorithm will be of *intrinsic* type, depending on the maximal geometric degree of the dual polar varieties of S that are associated with the external flag of a generic, $\mathbb{Q}$-definable flag contained in the hyperplane at infinity H of the n-dimensional, projective space $\mathbb{P}^n_{\mathbb{C}}$.

In order to describe this algorithm, let us first discuss these polar varieties and then the data structure and the algorithmic model we are going to use.

Let us choose a rational point $u = (u_1, \ldots, u_n)$ of $\mathbb{A}^n \setminus S_{\mathbb{R}}$ with generic coordinates $u_1, \ldots, u_n$ and, generically in the hyperplane at infinity H, a flag $\mathcal{L}$ of $\mathbb{Q}$-definable, linear subvarieties of $\mathbb{P}^n_{\mathbb{C}}$, namely

$$\mathcal{L}: \qquad L^0 \subset L^1 \subset \cdots \subset L^{p-1} \subset \cdots \subset L^{n-2} \subset L^{n-1} \subset \mathbb{P}^n_{\mathbb{C}}$$

with $L^{n-1} = H$. Let Q_u be the hyperquadric of $\mathbb{P}^n_{\mathbb{C}}$ defined by the quadratic form

$$R_u(X_0, X_1, \ldots, X_n) := X_0^2 - 2 \sum_{1 \leqslant k \leqslant n} u_k X_0 X_k + \sum_{1 \leqslant k \leqslant n} X_k^2.$$

Observe that the hyperquadrics Q_u and $Q_u \cap H$ are non-degenerate in $\mathbb{P}^n_{\mathbb{C}}$ and H, respectively, and that $Q_u \cap H_{\mathbb{R}}$ is represented by the positive definite quadratic form R_0

$(X_1, \ldots, X_n) = \sum_{1 \leqslant k \leqslant n} X_k^2$ that introduces the usual euclidean distance on $\mathbb{A}^n_{\mathbb{R}}$. One verifies immediately that the point $(1 : u_1 : \cdots : u_n) \in \mathbb{P}^n$ spans, with respect to the hyperquadric Q_u, the dual space of $L^{n-1} = H$.

Let us consider the external flag $\overline{\mathcal{K}}$ associated with $\mathcal{L}$, namely

$$\overline{\mathcal{K}}: \qquad \mathbb{P}^n_{\mathbb{C}} \supset \overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p-1} \supset \cdots \supset \overline{K}^1 \supset \overline{K}^0,$$

with $\overline{K}^{n-p-i} := (L^{p+i-1})^{\vee}$, for $1 \leqslant i \leqslant n-p$, and with an arbitrarily chosen irrelevant part

$$\overline{K}^{n-1} \supset \overline{K}^{n-2} \supset \cdots \supset \overline{K}^{n-p}.$$

Observe that $\overline{K}^0$ consists of the rational point $(1 : u_1 : \cdots : u_n) \in \mathbb{P}^n$.

Let $1 \leqslant i \leqslant n-p$ and recall that the $(p+i-1)$-dimensional, $\mathbb{Q}$-definable, linear subvariety L^{p+i-1} was chosen generically in the hyperplane at infinity H of $\mathbb{P}^n_{\mathbb{C}}$. Therefore, $\overline{K}^{n-p-i}$ is an $(n-p-i)$-dimensional, $\mathbb{Q}$-definable, linear subvariety of $\mathbb{P}^n_{\mathbb{C}}$, which we may imagine to be spanned by $n-p-i+1$ rational points

$$A_1 = (a_{1,0} : \cdots : a_{1,n}), \ldots, A_{n-p-i+1} = (a_{n-p-i+1,0} : \cdots : a_{n-p-i+1,n})$$

of $\mathbb{P}^n_{\mathbb{C}}$ with $a_{1,1} = u_1, \ldots, a_{1,n} = u_n$ and $a_{j,1}, \ldots, a_{j,n}$ generic, for $2 \leqslant j \leqslant n-p-i+1$, and $a_{1,0} = 1$, $a_{2,0} = \cdots = a_{n-p-i,0} = 0$. Observe that the point u belongs to $\overline{K}^{n-p-i} \cap \mathbb{A}^n$ and is not contained in $S_{\mathbb{R}}$. Thus Proposition 2 implies that the real affine dual polar variety $\widehat{W}_{\overline{K}^{n-p-i}}(S_{\mathbb{R}})$ contains at least one algebraic sample point of each connected component of $S_{\mathbb{R}}$.

In particular, the complex affine dual polar variety $\widehat{W}_{\overline{K}^{n-p-i}}(S)$ is not empty. From the generic choice of the point u and of the flag $\mathcal{L}$ we deduce now that Proposition 6, Lemma 7, Proposition 8 and Theorem 10 are applicable to the dual polar variety $\widehat{S}_i := \widehat{W}_{\overline{K}^{n-p-i}}(S)$. Observe that $\widehat{S}_i$ is $\mathbb{Q}$-definable and of pure codimension i in S. According to the terminology introduced in Section 2, we call $\widehat{S}_i$ the ith dual polar variety of S associated with the flag $\overline{\mathcal{K}}$. Observe that $\widehat{S}_i$ is non-empty and intersects each connected component of the real variety $S_{\mathbb{R}}$.

Thus, in particular, $\widehat{S}_{n-p}$ is a $\mathbb{Q}$-definable, zero-dimensional, algebraic variety that contains an algebraic sample point for any connected component of $S_{\mathbb{R}}$.

We will now analyse the dual polar variety $\widehat{S}_i$ more closely. For $2 \leqslant j \leqslant n-p$ let $\Lambda_j := \sum_{1 \leqslant l \leqslant n} a_{j,l} X_l$ and, for $1 \leqslant k \leqslant n$, let $\zeta_k = (\zeta_{k,1}, \ldots, \zeta_{k,n}) \in \mathbb{Q}^n$ such that ζ_1 is a zero of Λ_2, ζ_1 and ζ_2 are zeros of Λ_2 and Λ_3 , $\ldots$, $\zeta_1, \ldots, \zeta_{p+1}$ are zeros of $\Lambda_2, \ldots, \Lambda_{n-p}$ and such that $\zeta_1, \ldots, \zeta_n$ form a $\mathbb{Q}$-vector space basis of $\mathbb{Q}^n$ (recall that the coefficients of the forms $\Lambda_2, \ldots, \Lambda_{n-p}$ are generic). Let B be the transposed matrix of $(\zeta_{j,k})_{1 \leqslant j,\, k \leqslant n}$. For $1 \leqslant k \leqslant n$, let $Z_k = \sum_{1 \leqslant j \leqslant n} \tilde{\zeta}_{k,j} X_j$, where $(\tilde{\zeta}_{k,1}, \ldots, \tilde{\zeta}_{k,n})$ is the kth row of the inverse of the transposed matrix of B. Let $Z := (Z_1, \ldots, Z_n)$. As in Section 3, consider now the polynomial $((n-i+1)\times n)$-matrix

$$T^{(i)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} - a_{1,0}X_1 & \cdots & a_{1,n} - a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0}X_n \end{bmatrix}.$$

Observe that $T^{(i)}B$ is of the following form:

$$T^{(i)}B = \begin{bmatrix} & J(F_1(Z), \ldots, F_p(Z)) & \\ b_1 - c_1X_1 \cdots b_{p+i} - c_{p+i}X_{p+i} & & b_{p+i+1} - c_{p+i+1}X_{p+i+1} \cdots b_n - c_nX_n \\ O_{n-p-i,\ p+i} & & (*)_{n-p-i,\ n-p-i} \end{bmatrix},$$

where $b_1, \ldots, b_n$ and the entries of $(*)_{n-p-i,\ n-p-i}$ are all generic rational numbers and where $c_1, \ldots, c_n$ belong to $\mathbb{Q} \setminus \{0\}$. For the sake of simplicity we shall suppose that $c_1 = \cdots = c_n = 1$ (this assumption does not change the following argumentation substantially).

Thus the $(n-i+1)$-minors of the matrix $T^{(i)}B$, which are not identically zero, are scalar multiples of the $(p+1)$-minors selected among the columns $1, \ldots, p+i$ of the $((p+1)\times n)$-matrix

$$\theta := \begin{bmatrix} J(F_1(Z), \ldots, F_p(Z)) \\ b_1 - X_1 \ \cdots \ b_n - X_n \end{bmatrix}$$

and vice versa.

Consider now an arbitrary p-minor m of the Jacobian $J(F_1(Z), \ldots, F_p(Z))$. For the sake of definiteness let us suppose that m is given by the columns $1, \ldots, p$. For $p+1 \leqslant j \leqslant p+i$, let M_j be the $(p+1)$-minor of the matrix θ given by the columns $1, \ldots, p, j$.

Then we deduce from the Exchange Lemma of [3] that, for any point x of S with $m(x) \neq 0$, the condition $M_{p+1}(x) = \cdots = M_{p+i}(x) = 0$ is satisfied if and only if all $(p+1)$-minors of θ vanish at x.

Taking into account that $m(x) \neq 0$ implies the $(F_1, \ldots, F_p)$-regularity of the point $x \in S$, we conclude that the equations $F_1, \ldots, F_p, M_{p+1}, \ldots, M_{p+i}$ define the dual polar variety $\widehat{S_i}$ outside of the locus $V(m)$. Moreover, from Theorem 10 (i) we deduce that the (radical) ideal of definition of the affine variety $\widehat{S_i} \setminus V(m)$ in $\mathbb{Q}[X_1, \ldots, X_n]_m$ is generated by the polynomials $F_1, \ldots, F_p, M_{p+1}, \ldots, M_{p+i}$.

For $1 \leqslant h \leqslant p$, let S_h be the affine variety defined by the equations $F_1, \ldots, F_h$. Denote by $\deg S_h$ the geometric *degree* of S_h in the set-theoretic sense introduced in [25] (see also [18,48]). Thus, in particular, we do not take into account multiplicities and components at infinity for our notion of geometric degree. We call

$$\delta := \max\{\max\{\deg S_h \mid 1 \leqslant h \leqslant p\}, \max\{\deg \widehat{S_i} \mid 1 \leqslant i \leqslant n-p\}\}$$

the *degree of the real interpretation of the polynomial equation system* $F_1, \ldots, F_p$.

Taking into account the arguments used in the proof of Theorem 10 and the genericity of $\mathcal{L}$ in H we deduce from Propositions 6 and 8 that δ does not depend on the choice of the particular flag $\mathcal{L}$.

Since, by assumption, the degrees of the polynomials $F_1, \ldots, F_p$ are bounded by d, we infer from the Bézout-Inequality of [25] the degree estimates $\deg S \leqslant d^p$ and $\deg S_h \leqslant d^h \leqslant d^p$, for any $1 \leqslant h \leqslant p$.

Let $1 \leqslant i \leqslant n-p$ and recall from the beginning of Section 3.1 that each irreducible component of the polar variety $\widehat{S}_i = W_{\overline{K}^{n-p-i}}(S)$ is a $(n-p-i)$-dimensional irreducible component of the closed subvariety of $\mathbb{A}^n$ defined by the vanishing of $F_1, \ldots, F_p$ and of all $(n-i+1)$-minors of the polynomial $((n-i+1) \times n)$-matrix $T^{(i)}$. Taking generic linear combinations of these minors, one deduces easily from the Bézout inequality that $\deg \widehat{S}_i$ is bounded by

$$(\deg S) \cdot (p(d-1)+1)^i \leqslant d^{p+i} \ p^i \leqslant d^n \ p^{n-p}.$$

This implies the extrinsic estimate $\delta \leqslant d^n \ p^{n-p}$.

We are now going to introduce a data structure for the representation of polynomials of $\mathbb{Q}[X_1, \ldots, X_n]$ and describe our algorithmic model and complexity measures. Our elimination procedure will be formulated in the algorithmic model of (division-free) arithmetic circuits and networks (arithmetic-boolean circuits) over the rational numbers $\mathbb{Q}$.

Roughly speaking, a division-free arithmetic circuit β over $\mathbb{Q}$ is an algorithmic device that supports a step by step evaluation of certain (output) polynomials belonging to $\mathbb{Q}[X_1, \ldots, X_n]$, say $F_1, \ldots, F_p$. Each step of β corresponds either to an input from $X_1, \ldots, X_n$, either to a constant (circuit parameter) from $\mathbb{Q}$ or to an arithmetic operation (addition/subtraction or multiplication). We represent the circuit β by a labelled *directed acyclic graph (dag)*. The size of this dag measures the sequential time requirements of the evaluation of the output polynomials $F_1, \ldots, F_p$ performed by the circuit β.

A (division-free) arithmetic network over $\mathbb{Q}$ is nothing else but an arithmetic circuit that additionally contains decision gates comparing rational values or checking their equality, and selector gates depending on these decision gates.

Arithmetic circuits and networks represent non-uniform algorithms, and the complexity of executing a single arithmetic operation is always counted at unit cost. Nevertheless, by means of well known standard procedures our algorithm will be transposable to the uniform *random* bit model and will be implementable in practice as well. All this can be done in the spirit of the general asymptotic complexity bounds stated in Theorems 13 and 14 below.

Let us also remark that the depth of an arithmetic circuit (or network) measures the *parallel* time of its evaluation, whereas its size allows an alternative interpretation as “number of processors". In this context we would like to emphasize the particular importance of counting only *nonscalar* arithmetic operations (i.e., only essential multiplications), taking $\mathbb{Q}$-linear operations (in particular, additions/subtractions) for cost-free. This leads to the notion of nonscalar size and depth of a given arithmetic circuit or network β. It can be easily seen that the nonscalar size determines essentially the total size of β (which takes into account all operations) and that the nonscalar depth dominates the logarithms of degree and height of the intermediate results of β.

For more details on our complexity model and its use in the elimination theory we refer to [8,19,26,32,37] and, in particular, to [23,35] (where also the implementation aspect is treated).

Now we are ready to formulate the algorithmic main result of this paper.

Theorem 13. *Let n, p, d, δ, L and ℓ be natural numbers with $d \geqslant 2$ and $p \leqslant n$. Let $X_1, \ldots, X_n, Y$ be indeterminates over $\mathbb{Q}$. There exists an arithmetic network $\mathcal{N}$ over $\mathbb{Q}$, depending on certain random parameters, with size $\tilde{O}(\binom{n}{p} L n^4 p^2 d^2 \delta^2)$ and nonscalar depth $O(n(\ell + \log nd) \log \delta)$, such that $\mathcal{N}$ for suitable specializations of the random parameters has the following properties*:

Let $F_1, \ldots, F_p$ be a family of polynomials in the variables $X_1, \ldots, X_n$ of a degree at most d and assume that $F_1, \ldots, F_p$ are given by a division-free arithmetic circuit β in $\mathbb{Q}[X_1, \ldots, X_n]$ of size L and nonscalar depth ℓ. Suppose that the polynomials $F_1, \ldots, F_p$ form a regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]$ and that $F_1, \ldots, F_h$ generate a radical ideal for any $1 \leqslant h \leqslant p$. Moreover, suppose that the polynomials $F_1, \ldots, F_p$ define a closed, affine subvariety S of $\mathbb{A}^n_{\mathbb{C}}$ such that $S_{\mathbb{R}}$ is a pure p-codimensional, non-empty and smooth real variety. Assume that the degree of the real interpretation of the polynomial equation system is bounded by δ. Then the algorithm represented by the arithmetic network $\mathcal{N}$ starts from the circuit β as input and computes the coefficients of $n+1$ polynomials $P, P_1, \ldots, P_n$ in $\mathbb{Q}[Y]$ satisfying the following conditions:

- *P is monic and separable,*
- $1 \leqslant \deg P \leqslant \delta$,
- $\max\{\deg P_k \mid 1 \leqslant k \leqslant n\} < \deg P$,
- *the cardinality $\#\widehat{S}$ of the (non-empty) affine variety*

$$\widehat{S} := \{(P_1(y), \ldots, P_n(y)) \mid y \in \mathbb{C},\ P(y) = 0\}$$

  is at most $\deg P$, the affine variety $\widehat{S}$ is contained in S and at least one point of each connected component of $S_{\mathbb{R}}$ belongs to $\widehat{S}$.

Moreover, using sign gates the network $\mathcal{N}$ produces at most $\#\widehat{S}$ sign sequences of elements $\{-1, 0, 1\}$ such that these sign conditions encode the real zeros of the polynomial P "à la Thom" ([13]).

In this way, namely by means of the Thom encoding of the real zeros of P and by means of the polynomials $P_1, \ldots, P_n$, the arithmetic network $\mathcal{N}$ describes the finite, non-empty set

$$\widehat{S} \cap \mathbb{R}^n = \{(P_1(y), \ldots, P_n(y)) \mid y \in \mathbb{R},\ P(y) = 0\},$$

which contains at least one algebraic sample point for each connected component of the real variety $S_{\mathbb{R}}$. For a given specialization of the random parameters, the probability of failure of $\mathcal{N}$ is smaller than $\frac{1}{2}$ and tends rapidly to zero with increasing n. The parameters n, p, d, δ, L and ℓ represent an instance of a uniform procedure which produces the network $\mathcal{N}$ in time $\tilde{O}(\binom{n}{p} L n^4 p^2 d^2 \delta^2)$.

Taking into account the extrinsic estimate $\delta \leqslant d^n\, p^{n-d}$ of the beginning of this section and the straightforward estimates $L \leqslant d^{n+1}$ and $\ell \leqslant \log d$, we obtain the worst case bounds $\binom{n}{p}(np^{n-p}d^n)^{O(1)}$ and $O((n \log nd)^2)$ for the size and non-scalar depth of the network $\mathcal{N}$ of Theorem 13. Thus, our worst case sequential time complexity bound meets the standards of todays most efficient $d^{O(n)}$-time procedures for the problem under consideration (compare [5,6] and also [9,10,14,24,27–29,41,42]).

Proof. Since we are going to describe a probabilistic procedure, we may assume without loss of generality that we have already chosen a rational point u of $\mathbb{A}^n \setminus S_{\mathbb{R}}$ and a flag $\mathcal{L}$ of $\mathbb{Q}$-definable linear subvarieties $\mathbb{P}^n_{\mathbb{C}}$, satisfying the genericity conditions considered at the beginning of this section, and that the variables $X_1, \ldots, X_n$ are in general position with respect to S. Therefore, we may use freely the previous notations.

For any choice of p columns $1 \leqslant i_1 < \cdots < i_p \leqslant n$ and any index $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$ we denote by $m^{(i_1,\ldots,i_p)}$ the p-minor of $J(F_1(Z), \ldots, F_p(Z))$ given by the columns $i_1, \ldots, i_p$ and by $M^{(i_1,\ldots,i_p,j)}$ the $(p+1)$-minor of θ given by the columns $i_1, \ldots, i_p, j$.

Let us recall that the dual variety $\widehat{S}_{n-p}$ is $\mathbb{Q}$-definable and zero-dimensional. Moreover, $\widehat{S}_{n-p}$ is of degree (i.e., cardinality) δ and contains an algebraic sample point for each connected component of $S_{\mathbb{R}}$.

Let us consider an arbitrary point x of $\widehat{S}_{n-p}$. Since x is $(F_1, \ldots, F_p)$-regular, there exist indices $1 \leqslant i_1 < \cdots < i_p \leqslant n$ such that $m^{(i_1,\ldots,i_p)}(x) \neq 0$ holds. Let $i_{p+1}, \ldots, i_n$ be an enumeration of the set $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$. From Lemma 5 we deduce that the n equations $F_1, \ldots, F_p, M^{(i_1,\ldots,i_p,i_{p+1})}, \ldots, M^{(i_1,\ldots,i_p,i_n)}$ define the dual variety $\widehat{S}_{n-p}$ outside of the locus $V(m^{(i_1,\ldots,i_p)})$. Moreover, they intersect transversally in any point of $\widehat{S}_{n-p} \setminus V(m^{(i_1,\ldots,i_p)})$ and hence, also in the point x.

Let $1 \leqslant j \leqslant n-p$. Observe that our argumentation at the beginning of this section implies that the equations $F_1, \ldots, F_p, M^{(i_1,\ldots,i_p,i_{p+1})}, \ldots, M^{(i_1,\ldots,i_p,i_j)}$ define the dual polar variety $\widehat{S}_j$ outside of the locus $V(m^{(i_1,\ldots,i_p)})$. Moreover, the polynomials $F_1, \ldots, F_p$, $M^{(i_1,\ldots,i_p,i_{p+1})}, \ldots, M^{(i_1,\ldots,i_p,i_j)}$ generate a radical ideal in $\mathbb{Q}[X_1, \ldots, X_n]_{m^{(i_1,\ldots,i_p)}}$.

We are now in conditions to apply the main algorithm of [23] or [26] to the system

$$F_1 = 0, \ldots, F_p = 0,\ M^{(i_1,\ldots,i_p,i_{p+1})} = 0, \ldots, M^{(i_1,\ldots,i_p,i_n)} = 0,\ m^{(i_1,\ldots,i_p)} \neq 0,$$

and in fact, we are using a combination of both. This algorithm may be realized by an arithmetic network $\mathcal{N}_{(i_1,\ldots,i_p)}$ of size $\widetilde{O}(L\, n^4 p^2 d^2 \delta^2)$ and non-scalar depth $O(n\,(\ell + \log nd) \log \delta)$ which produces as output $(n+1)$ polynomials $P^{(i_1,\ldots,i_p)}$, $P_1^{(i_1,\ldots,i_p)}, \ldots,$ $P_n^{(i_1,\ldots,i_p)} \in \mathbb{Q}[Y]$ satisfying the following conditions:

- $P^{(i_1,\ldots,i_p)}$ is monic and separable,
- $1 \leqslant \deg P^{(i_1,\ldots,i_p)} \leqslant \delta$,
- $\max\{\deg P_k^{(i_1,\ldots,i_p)} \mid 1 \leqslant k \leqslant n\} < \deg P^{(i_1,\ldots,i_p)}$,
- $\widehat{S}_{n-p} \setminus V(m^{(i_1,\ldots,i_p)}) =$
  $= \{(P_1^{(i_1,\ldots,i_p)}(y), \ldots, P_n^{(i_1,\ldots,i_p)}(y)) \mid y \in \mathbb{C} : P^{(i_1,\ldots,i_p)}(y) = 0\}$

(see [23, Theorem 1]). Now we repeat this procedure for each index set $\{i_1, \ldots, i_n\}$ with $1 \leqslant i_1 < \cdots < i_p \leqslant n$ and simplify the outputs by iterated greatest common divisor computations in the polynomial $\mathbb{Q}$-algebra $\mathbb{Q}[Y]$.

The final outcome is an arithmetic network $\mathcal{N}_1$ of size $\widetilde{O}(\binom{n}{p} L\, n^4 p^2 d^2 \delta^2)$ and non-scalar depth $O(n\,(\ell + \log nd) \log \delta)$ which produces as output $(n+1)$ polynomials $P, P_1, \ldots, P \in$

$\mathbb{Q}[Y]$ satisfying the following conditions:

- P is monic and separable,
- $\deg P = \#\widehat{S}_{n-p} \leqslant \delta$,
- $\max\{\deg P_k \mid 1 \leqslant k \leqslant n\} < \deg P$,
- $\widehat{S} := \widehat{S}_{n-p} = \{(P_1(y), \ldots, P_n(y)) \mid y \in \mathbb{C} : \ P(y) = 0\}$.

We apply now to the polynomial $P \in \mathbb{Q}[Y]$ any of the known, well parallelizable Computer Algebra algorithms for the determination of all real roots of a given univariate polynomial, where these roots are thought to be encoded "à la Thom" (see e.g. [13]). This subroutine may be realized by an arithmetic network $\mathcal{N}$ which uses sign gates and extends the network $\mathcal{N}_1$. The size and the non-scalar depth of $\mathcal{N}$ are asymptotically the same as those of $\mathcal{N}_1$.

The estimation of the error probability of the probabilistic algorithm just described is cumbersome and contains no substantial new ideas. It is not difficult to derive such an estimation from the proof of [4, Theorem 11]. We omit these details here. $\square$

Some of the ideas contained in the proof of Theorem 13 are implicitly used in [1,43] for the purpose to find for any connected component of $S_{\mathbb{R}}$ an algebraic sample point. However, the algorithm developed in loc.cit. is rewriting based, and, although a rigorous complexity analysis is missing, its minor efficiency can easily be verified.

In the particular case that the real variety $S_{\mathbb{R}}$ is compact, our method produces the following alternative complexity result:

Theorem 14. *Let the notations and assumptions be as in Theorem* 13. *Suppose that the real variety $S_{\mathbb{R}}$ is not only of pure codimension p, non-empty and smooth, but also compact. For $1 \leqslant h \leqslant p$, let S_h be the closed subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by the equations $F_1, \ldots, F_h$. Let $\mathcal{L}$ be the generic flag of $\mathbb{Q}$-definable, linear subvarieties of $\mathbb{P}^n_{\mathbb{C}}$ introduced at the beginning of this section, namely*

$$\mathcal{L}: \qquad L^0 \subset L^1 \subset \cdots \subset L^{p-1} \subset \cdots \subset L^{n-2} \subset L^{n-1} \subset \mathbb{P}^n_{\mathbb{C}}$$

with $L^{n-1} = H$. Let $\underline{\mathcal{K}}$ be the internal flag associated with $\mathcal{L}$, namely

$$\underline{\mathcal{K}}: \qquad \mathbb{P}^n \supset \underline{K}^{n-1} \supset \underline{K}^{n-2} \supset \cdots \supset \underline{K}^{n-p-1} \supset \cdots \supset \underline{K}^1 \supset \underline{K}^0.$$

For $1 \leqslant i \leqslant n-p$, let $\widetilde{S}_i := \widehat{W}_{\underline{K}^{n-p-i}}(S) = W_{L^{p+i-2}}(S)$ be the ith classic polar variety of S associated with the internal flag $\underline{\mathcal{K}}$.

Finally, suppose that δ is an upper bound for the geometric degrees of the affine varieties $S_1, \ldots, S_p$ and $\widetilde{S}_1, \ldots, \widetilde{S}_{n-p}$. Under these assumptions the same conclusions as in Theorem 13 hold true.

Theorem 14 is the algorithmic main result of [3], where its statement is slightly different.

Motivated by the outcome of Theorems 14 and 13 above, the following geometric result was shown in [44]. This result is interesting on its own because of its mathematical and algorithmic consequences.

Let the notations and assumptions be as in Theorem 14, however, dropping the requirement that $S_{\mathbb{R}}$ is compact. Suppose that the variables $X_1, \ldots, X_n$ are in generic position with respect to the algebraic variety S. For each $1 \leqslant i < n-p$, let $\pi_i : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^{n-p-i}_{\mathbb{C}}$ be the *projection given by the variables* $X_1, \ldots, X_{n-p-i}$. *Furthermore, let* $\lambda = (\lambda_1, \ldots, \lambda_{n-p})$ *be a randomly chosen point of* $\mathbb{Z}^{n-p}$ *and let* $\lambda^{(i)} := (\lambda_1, \ldots, \lambda_{n-p-i})$. *Then, for each* $1 \leqslant i < n-p$, *the algebraic variety* $\pi_i^{-1}(\lambda^{(i)}) \cap \widetilde{S}_i$ *is zero-dimensional or empty and the finite set* $\widetilde{S}_{n-p} \cup \bigcup_{1 \leqslant i < n-p} (\pi_i^{-1}(\lambda^{(i)}) \cap \widetilde{S}_i)$ *intersects any connected component of* $S_{\mathbb{R}}$.

This geometric result allows to extend the validity of Theorem 14 to the non-compact case, however, its proof is somewhat different, because it requires the more general elimination procedure of [36] for polynomial equation and inequation systems defining (locally closed) algebraic subvarieties of $\mathbb{A}_{\mathbb{C}}^n$. From the point of practical computations it seems difficult to compare the algorithm of Theorem 13 with the elimination algorithm described in [44]. On the one hand, one may expect that the degree associated with the real interpretation of a polynomial equation system is typically smaller if this notion of degree is based on the concept of classic polar varieties as in Theorem 14. On the other hand, the use of a more general and intricate elimination algorithm may diminish this complexity gain.

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