J. Symbolic Computation (2002) **33**, 587–608
doi:10.1006/jsco.2002.0524
Available online at http://www.idealibrary.com on IDEAL®

Hybrid Sparse Resultant Matrices for Bivariate Polynomials

CARLOS D'ANDREA[†§] AND IOANNIS Z. EMIRIS[‡¶]

[†] *Departamento de Matemática, FCEyN, UBA (1428) Buenos Aires, Argentina*
[‡] *INRIA, B.P. 93, Sophia-Antipolis 06902, France*

We study systems of three bivariate polynomials whose Newton polygons are scaled copies of a single polygon. Our main contribution is to construct square resultant matrices, which are submatrices of those introduced by Cattani *et al.* (1998), and whose determinants are nontrivial multiples of the sparse (or toric) resultant. The matrix is hybrid in that it contains a submatrix of Sylvester type and an additional row expressing the toric Jacobian. If we restrict our attention to matrices of (almost) Sylvester-type and systems as specified above, then the algorithm yields the smallest possible matrix in general. This is achieved by strongly exploiting the combinatorics of sparse elimination, namely by a new piecewise-linear lifting. The major motivation comes from systems encountered in geometric modeling. Our preliminary Maple implementation, applied to certain examples, illustrates our construction and compares it with alternative matrices.

1. Introduction

Resultants (*or eliminants*) may be expressed by matrices whose determinants are nontrivial multiples of them. They eliminate several variables simultaneously and reduce system solving over the complex numbers to univariate polynomial factorization or an eigenproblem. There are two main classes of matrices, generalizing respectively Sylvester's and Bézout's construction. There are also hybrid matrices, such as Dixon's, which contain submatrices of each of the above types.

Motivated by problems encountered in geometric modeling and computer-aided design, we concentrate on systems of three bivariate polynomials whose Newton polygons are scaled copies of a single polygon; specific examples are worked out in Section 7. We propose a new piecewise-linear lifting, which yields a mixed subdivision with the desired properties, thus producing compact matrices of hybrid type.

Resultant matrices represent linear transformations in monomial bases; by convention, the rows shall correspond to the monomial basis of the domain. Our matrices contain all but one row of Sylvester-type, with one row containing the system's toric Jacobian. Moreover, they are square and their determinant is, generically, a nontrivial multiple of the sparse resultant. The resultant can then be obtained as the gcd of at most three such determinants, whereas the matrix from Cattani *et al.* (1998) was rectangular and all of its maximal minors had to be considered. The matrix dimension equals the number of

[§]E-mail: `cdandrea@dm.uba.ar`
[¶]E-mail: `Ioannis.Emiris@inria.fr`

0747–7171/02/050587 + 22 $35.00/0

integer points in the strict interior of the Newton polygons' Minkowski sum, which is smaller than the dimension of the matrix from Canny and Emiris (2000). It should be possible to derive a Macaulay-type formula for the resultant based on D'Andrea (2002).

The next section reviews related work. Section 3 discusses toric Jacobians and generalizes the corresponding approach of Cattani *et al.* (1998). Section 4 introduces our combinatorial geometric tools and relates them to the matrix construction. The lifting algorithm is stated in Section 5, along with some basic properties, whereas the following section derives the properties of the induced subdivision. Section 7 refers to experimental results obtained with our publicly available draft Maple implementation. Further applications to systems with scaled Newton polygons should emphasize the merits of our approach.

A preliminary version of this paper has appeared in D'Andrea and Emiris (2001).

2. Related Work

An introductory treatment of resultant theory can be found in Cox *et al.* (1998).

Let us start with univariate $f_0 := a_0+a_1x+\cdots+a_nx^n$, and $f_1 := b_0+b_1x+\cdots+b_mx^m$. If $a_n \neq 0 \neq b_m$, their resultant is a polynomial in the a_i, b_j that vanishes iff there is a common root of f_0 and f_1. Sylvester's matrix is of dimension $n+m$; one way to get a more compact formula is to use the affine Jacobian $\begin{vmatrix} nf_0 & mf_1 \\ \frac{\partial f_0}{\partial x} & \frac{\partial f_1}{\partial x} \end{vmatrix} = j_0 + j_1x + \cdots + j_{m+n-2}x^{m+n-2}$. Then cf. Jouanolou (1997) and Gelfand *et al.* (1994),

$$mn\,\mathrm{Res}(f_0,f_1) = \det\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots \\ & & \ddots & & & & \\ b_0 & b_1 & b_2 & \cdots & \cdots & b_m & \cdots \\ & & & & \ddots & & \\ j_0 & j_1 & j_2 & & \cdots & & j_{m+n-2} \end{pmatrix}. \qquad (1)$$

The matrix is of dimension $m+n-1$. Its first $m-1$ rows contain coefficients of f_0 and the following $n-1$ rows those of f_1.

Let us now consider three dense polynomials of total degrees d_0, d_1 and d_2, in x_1, x_2. The resultant $\mathrm{Res}_{d_0,d_1,d_2}(f_0,f_1,f_2)$ is again a polynomial in the coefficients, which vanishes whenever the system $f_0 = f_1 = f_2 = 0$ has a common root. Cayley gave an ingenious algorithm for computing $\mathrm{Res}_{d_0,d_1,d_2}(f_0,f_1,f_2)$ as a quotient of two determinants. This technique was generalized in Gelfand *et al.* (1994), using determinants of complexes. Macaulay presented a different way for computing the resultant as a quotient of two determinants, generalizing Sylvester's formula, where the denominator is a minor of the numerator. It was extended to the sparse case in D'Andrea (2002). In Jouanolou (1997), Jouanolou shows how to generalize formula (1) and get a Macaulay-style formula where the numerator is the determinant of a matrix having in one of its rows the coefficients of the *affine Jacobian*

$$\det\begin{pmatrix} d_0f_0 & d_1f_1 & d_2f_2 \\ \frac{\partial f_0}{\partial x_1} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_0}{\partial x_2} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

In order to generalize the previous situation, let $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2)$, with $\mathcal{A}_i \subset \mathbb{Z}^2$ being finite sets. Consider three generic Laurent polynomials (i.e. with integer exponents) in

$x = (x_1, x_2)$:

$$f_i(x) = \sum_{a \in \mathcal{A}_i} c_{ia}\, x^a,\ c_{ia} \neq 0,\ i = 0, 1, 2. \tag{2}$$

Let Q_i be the *Newton polygon* of f_i,, i.e. the convex hull of the exponent vectors in the *support* $\mathcal{A}_i$. Bernstein's (or BKK) theorem (Gelfand *et al.*, 1994; Cox *et al.*, 1998) bounds the number of isolated roots in $(\mathbb{C}^*)^2$ of f_i and f_j, by the *mixed volume* of the Newton polygons $\mathrm{MV}(Q_i, Q_j)$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Mixed volume generalizes Bézout's classical bound. This paper focuses on polynomials with all Q_i being scaled copies of one polygon P, assumed of positive Euclidean area. In addition, suppose the union of the supports spans the affine lattice $\mathbb{Z}^2$; this assumption can be eventually removed as in Sturmfels (1994) or shall be irrelevant if we apply the algorithm of Canny and Pedersen (1993).

The *sparse resultant* $\mathrm{Res}_{\mathcal{A}}(f_0, f_1, f_2)$ is an irreducible element of the factorial ring $\mathbb{Z}[c]$, where $c := (c_{ia})_{i,a}$, homogeneous in the coefficients of each f_i, with total degree equal to $\mathrm{MV}(Q_0, Q_1)+\mathrm{MV}(Q_0, Q_2)+\mathrm{MV}(Q_1, Q_2)$. It vanishes after a specialization of the coefficients if and only if the specialized system (2) has a solution in a compactification of the torus $(\mathbb{C}^*)^2$ specified by Q_i. This is the associated *toric variety* (Gelfand *et al.*, 1994; Cox *et al.*, 1998). Let us consider coefficients in a field whose algebraic closure is $\mathbb{C}$, though this field could be arbitrary. When the affine lattice generated by the union of the polygons has dimension ≤ 1, we define the sparse resultant to be identically 1.

Our goal is to present an algorithm for computing a square matrix having its determinant equal to a nonzero multiple of the sparse resultant, thus generalizing Jouanolou's method. Our matrix construction is based on the mixed-subdivision algorithm of Canny and Emiris (2000): they perturb the Minkowski sum $Q = Q_0 + Q_1 + Q_2$ by a small generic vector δ and associate to each integer point in $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^2$ a monomial multiple of some f_i. Here we use piecewise-linear lifting functions; nonlinear liftings were first used in Sturmfels (1994). The greedy variant of Canny and Pedersen (1993) leads usually to smaller matrices, without any *a priori* knowledge of the dimension, however; our implementation shall use this technique. Our matrix is indexed by the integer points in the strict *interior* of Q, which will be denoted by $\mathcal{F}$. This will typically give a significantly smaller matrix than (Canny and Emiris, 2000) (cf. Section 7 for examples) since the latter avoids only the boundary points outside $Q + \delta$.

Systems with all Q_i equal to scaled copies of a single polygon are considered in Cattani *et al.* (1998). The matrices defined are of Sylvester-type, except for one row which corresponds to a toric Jacobian (or a kind of Bezoutian), which is defined below. If $Q_{-i} := \sum_{j \neq i} Q_j, i = 0, 1, 2$ and $\mathcal{F}_i := int(Q_{-i}) \cap \mathbb{Z}^2$, then their matrix has dimensions $|\mathcal{F}|$ and $1+\sum_i |\mathcal{F}_i|$, where $int(\cdot)$ denotes the interior with respect to the Euclidean topology; the large dimension of the matrix can be considerably larger than $|\mathcal{F}|$. The sparse resultant is the gcd of all maximal minors. We construct *square* matrices of the same type to express the sparse resultant; they are maximal submatrices of the matrices of Cattani *et al.* (1998), i.e. with dimension equal to $|\mathcal{F}|$. The determinant has the same degree in the coefficients of f_0, as the resultant. Replacing index 0 by any other index $i = 1, 2$, gives an immediate way of obtaining $\mathrm{Res}_{\mathcal{A}}$ as the gcd of at most three determinants.

In fact, the obtained matrix has minimum dimension among those with all, except one, rows of Sylvester type. The reason is that $\mathcal{F}$ is in bijective correspondence with a graded piece of the homogeneous ring of the associated toric variety (Cattani *et al.*, 1998, Proposition 1.1). Modulo the ideal generated by f_0, f_1 and f_2, this graded piece has codimension 1, and the toric Jacobian is generically a nonzero element in this quotient.

Recall also that, in the dense case, $\mathcal{F}$ is in correspondence with the monomials of degree up to $\sum_i \deg f_i - 3$, which are all required in order to define a Sylvester-type matrix.

A larger class of matrices for the homogeneous case were studied in Jouanolou (1997), where the polynomials filling in the special rows were coefficients of the so-called "Morley form". These formulas have been generalized to the bivariate sparse case in Khetan (2002), where determinantal formulas are presented when all supports are equal.

Special interest has been shown for bivariate systems since they are crucial in applications such as computer-aided design and geometric modeling (Manocha, 1992; Zhang and Goldman, 2000). It is known that for bi-homogeneous systems, optimal Sylvester-type matrices are available. Zhang and Goldman (2000) constructs optimal Sylvester-type formulae for bihomogeneous systems with identical Newton polygons, equal to a rectangle from which smaller rectangles have been removed at the corners. For a class of bihomogeneous polynomials arising in implicitization, there are optimal hybrid matrices constructed in Aries and Senoussi (2001). Alternative resultant-based methods that use Bézout-type matrices are beyond the scope of this paper.

3. Toric Jacobians and Resultants

We review here, for three bivariate polynomials, some results which appeared in Cattani *et al.* (1998) in a more general setting; we propose more direct constructions. Recall that $\mathcal{F} := int(Q) \cap \mathbb{Z}^2$.

Proposition 3.1. (Cattani *et al.*, 1998, Proposition 1.2) *Consider the system $f = (f_0, f_1, f_2)$ with identical supports $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2$, of cardinality $|\mathcal{A}_0|$. Its toric Jacobian has support in $\mathcal{F}$ and equals*

$$J(f) := \det \begin{pmatrix} f_0 & f_1 & f_2 \\ x_1 \frac{\partial f_0}{\partial x_1} & x_1 \frac{\partial f_1}{\partial x_1} & x_1 \frac{\partial f_2}{\partial x_1} \\ x_2 \frac{\partial f_0}{\partial x_2} & x_2 \frac{\partial f_1}{\partial x_2} & x_2 \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

When $Q_i = k_i P$, for $k_i \in \mathbb{Q}$, the toric Jacobian is constructed in Cattani *et al.* (1998) via a P-homogenization. We present instead a direct way which reduces, in the bivariate case, to the following: Let p be any vertex of P, and e one of its incident edges. For $i = 0, 1, 2$, $\mathcal{A}_i^2 := \{k_i p\}$, $\mathcal{A}_i^1 := (k_i\, e \setminus k_i p) \cap \mathbb{Z}^2$ and let $f_i^j := \sum_{a \in \mathcal{A}_i^j} c_{ia} x^a$. Let σ be the dual cone generated by the inward normal vectors of the edges that contain vertex p. Proceeding as in Cox *et al.* (1998, chapter 7), the f_i may be transformed into "homogeneous" polynomials F_i, written (not in a unique way) as $F_i = A_{0i} \prod_{j>2} X_j + \sum_{j=1}^2 A_{ij} X_j$. We suppose that the variables X_1 and X_2 correspond to the edges whose intersection is p.

Definition 3.2. (Cattani *et al.*, 1997) $\Delta_\sigma := \det(A_{ij})_{0 \le i,j \le 2}$.

Proposition 3.3. *When each $Q_i = k_i P$ for polygon P, $k_i \in \mathbb{Q}$, the polynomial*

$$G(f) := \det \begin{pmatrix} f_0 & f_1 & f_2 \\ f_0^1 & f_1^1 & f_2^1 \\ f_0^2 & f_1^2 & f_2^2 \end{pmatrix}$$

has support in $\mathcal{F}$ and corresponds to a restriction in the torus of Δ_σ.

Proof. It is clear that $G(f)$ has its support contained in $(Q_0+Q_1+Q_2)\cap\mathbb{Z}^2$. We have to prove that the support is disjoint from the border of the Minkowski sum. Now,

$$f_i^* := f_i - \sum_{j=1}^{2} f_i^j, \qquad i=0,1,2 \Rightarrow G(f) = \det\begin{pmatrix} f_0^* & f_1^* & f_2^* \\ f_0^1 & f_1^1 & f_2^1 \\ f_0^2 & f_1^2 & f_2^2 \end{pmatrix}.$$

For any $\nu \in \mathbb{R}^n \setminus \{0\}$, consider the ν-border of Q. If we reach the ν-border in a sum $p_0 + p_1 + p_2,\ p_i \in k_i P$, then every p_i belongs to the ν-border of $k_i P$. But it is straightforward to check that $\exists j \in \{0,1,2\}$: $\mathcal{A}_i^j$ is disjoint from the ν-border of $k_i P$.

For the second part of the proposition, recall that toric homogenizations are 1–1 correspondences between affine monomials $x_1^a x_2^b$ and homogeneous monomials X^α. Recall that, for $j = 1,\ldots,s$, if X^α appearing in the expansion of F_i is a multiple of X_j, then the support of $x_1^a x_2^b$ is not on the edge $k_i\, e_j$. If e_1, e_2 are the edges incident to p, then it is easy to see that $f_i = f_i^* + f_i^1 + f_i^2$. Moreover, the support of f_i^* is disjoint from $k_i e_1$ and may be identified with A_{i1}; the support of f_i^1 is disjoint from $k_i e_2$ and may play the role of A_{i2}; finally, $f_i^2 = c_{ip} x^{k_i p}$, its support is disjoint from every edge $k_i e_j$, $j > 2$, and shall be A_{i3}. □

Δ_σ is equal to the toric Jacobian, within a constant, modulo the homogeneous ideal of f by Cattani *et al.* (1997, 1998). Hence, $G(f)$ can play the role that the toric Jacobian holds in Cattani *et al.* (1998).

Example 3.4. Let us consider the following system:

$$\begin{aligned} f_0 &= c_{00} + c_{01}x_1 + c_{02}x_2 + c_{03}x_1x_2 \\ f_1 &= c_{10} + c_{11}x_1 + c_{12}x_2 + c_{13}x_1x_2 \\ f_2 &= c_{20} + c_{21}x_1 + c_{22}x_2 + c_{23}x_1x_2. \end{aligned}$$

Here, $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2$ and $k_0 = k_1 = k_2 = 1$. Let $p = (0,0)$, $e = (p,(1,0))$, then

$$A_i^1 = (1,0)\ A_i^2 = (0,0), \qquad i = 0,1,2 \Rightarrow G(f) = \det\begin{pmatrix} f_0 & f_1 & f_2 \\ c_{01}x_1 & c_{11}x_1 & c_{21}x_1 \\ c_{00} & c_{10} & c_{20} \end{pmatrix}$$

which has a smaller support than $J(f)$.

Recall that $\mathcal{F}_i := int(Q_{-i}) \cap \mathbb{Z}^2$, let $S_{\mathcal{F}_i}$ be the $\mathbb{Z}[c]$—free module generated by the monomials $\{x^a\ : a \in \mathcal{F}_i\}$ and S_{Q_i} the free module generated by $\{x^a : a \in int(Q_i) \cap \mathbb{Z}^2\}$. Consider the following (Koszul) complex of modules:

$$\begin{aligned} &0 \to S_{Q_0} \oplus S_{Q_1} \oplus S_{Q_2} \xrightarrow{\psi} S_{\mathcal{F}_0} \oplus S_{\mathcal{F}_1} \oplus S_{\mathcal{F}_2} \oplus \mathbb{Z}[c] \xrightarrow{\phi} S_{\mathcal{F}} \to 0\ : \\ &\psi : (p_0,p_1,p_2) \to (p_1\, f_2 + p_2\, f_1, p_0\, f_2 - p_2\, f_0, -p_0\, f_1 - p_1\, f_0, 0), \\ &\qquad \phi : (g_0,g_1,g_2,\lambda) \to (g_0\, f_0 + g_1\, f_1 + g_2\, f_2 + \lambda\, J(f)). \end{aligned} \tag{3}$$

Proposition 3.5. (Cattani *et al.*, 1998, Proposition 2.1 & Theorem 2.2) *The complex* (3) *is generically exact, and after a specialization of the coefficients in* $\mathbb{C}$ *it will be exact iff* $\mathrm{Res}_{\mathcal{A}}(f_0,f_1,f_2) \neq 0$. *As a consequence,* ϕ *is generically surjective and every* *maximal minor of the matrix representing ϕ in the monomial bases is a multiple of the sparse resultant. Moreover,* $\mathrm{Res}_{\mathcal{A}}(f_0, f_1, f_2)$ *is the* gcd *of these maximal minors.*

Our goal will be to provide a square (maximal) submatrix of the matrix of ϕ in the monomial bases with nonzero determinant without computing the whole complex. This yields a nontrivial multiple of the resultant as well as an expression of the latter as the gcd of at most three determinants. In the following cases, we have an optimal formula.

Proposition 3.6. (Cattani *et al.*, 1998, Corollary 2.4) *Suppose P has no interior points and one of the following conditions holds: (i) All Q_i are identical, or (ii) $2\,P$ has no interior integer points and the sum of the scaling factors equals 4, or (iii) P is a triangle with no interior points, and no scaling factor exceeds 2. Then the computed matrix is square and its determinant equals* $\mathrm{Res}_{\mathcal{A}}$.

4. Resultant Matrix Construction

Sparse resultant matrices are computed by combinatorial geometric methods, the central construction being a coherent polyhedral subdivision defined by lifted Newton polygons (Sturmfels, 1994; Cox *et al.*, 1998; Canny and Emiris, 2000). The lifting functions are specified in the next section. Here, we describe the standard matrix construction assuming the liftings are given and, more precisely, we know the lifting values of the polygons' vertices.

Let $\omega = (\omega_0, \omega_1, \omega_2)$ be the lifting functions with $\omega_i \in \mathbb{Q}^{m_i}$, where m_i stands for the number of vertices W_i defining Q_i. Any point $a \in W_i$ is lifted to $a_{\omega_i} = (a, \omega_i(a)) \in \mathbb{Q}^3$, simply denoted by a_ω. Consider the *lifted polygons* $Q_{i,\omega}$, the convex hull of a_{ω_i}, $a \in W_i$, and their Minkowski sum

$$Q_\omega := Q_{0,\omega} + Q_{1,\omega} + Q_{2,\omega} \subset \mathbb{R}^3.$$

Taking the *lower* envelope $L(Q_\omega)$ of Q_ω, we get a coherent polyhedral (or mixed) *subdivision* Δ_ω of the Minkowski sum $Q := Q_0 + Q_1 + Q_2$, by means of the natural projection $\mathbb{R}^3 \to \mathbb{R}^2$ of Q_ω: suppose that ω is sufficiently generic (Sturmfels, 1994; Canny and Emiris, 2000), then we have a *tight* mixed decomposition, i.e. each facet in the lower envelope is of the form

$$F_\omega = F_{0,\omega} + F_{1,\omega} + F_{2,\omega} : \dim(F_{0,\omega}) + \dim(F_{1,\omega}) + \dim(F_{2,\omega}) = 2, \tag{4}$$

where $F_{i,\omega}$ is a face of $Q_{i,\omega}$. The projection of $F_{i,\omega}$ to Q_i defines a face or subset of a face in Q_i, with vertices in W_i. If we only consider this (sub)face, the lifting to $F_{i,\omega}$ is linear; cf. the next section.

Dimension considerations and the genericity of our lifting imply that at least one summand is zero-dimensional. Furthermore, sum (4), specialized at any point in $p_\omega \in F_\omega$, minimizes the aggregate lifting of any sum of three points in $Q_{i,\omega}$ equal to p_ω; so it is sometimes called an optimal sum. The cell F (the projection of F_ω) is said to be *mixed of type i* or *i-mixed* if $\dim F_i = 0,\ \dim F_j = 1,\ j \neq i$. Let Δ_ω be the subdivision of Q induced by the lower hull of the lifted Minkowski sum.

Proceeding as in Canny and Emiris (2000), we must choose a small vector $\delta \in \mathbb{Q}^2$ and consider $\mathbb{Z}^2 \cap (\delta + Q)$. We can define the *preimage* of every point in the latter set to be the point on the same vertical and the lower hull of the Minkowski sum of the lifted Newton polygons, after shifting them by δ. Vector δ should be small enough so that every perturbed integer point in Q lies in one of the maximal cells cobounding this point. Assume δ is generic enough not to be parallel to any edge of any lifted $Q_{i,\omega}$; for a random vector, (Canny and Emiris, 2000) bounds the error probability. Then,

$$\mathcal{F} = \mathbb{Z}^2 \cap (\delta + Q) \cap (-\delta + Q).$$

Definition 4.1. The *row content* $\mathrm{RC}(F)$ for any cell of Δ_ω will be a pair (i, a) which satisfies: if $F_\omega = F_{0,\omega} + F_{1,\omega} + F_{2,\omega}$ is the unique facet on the lower envelope of Q_ω projecting to F expressed by this optimal sum, then i is the *largest* index s.t. $\dim(F_i) = 0$, $F_i = \{a\}$. We also define the row content of all points $p \in \mathcal{F} : p \in F + \delta$ to be $\mathrm{RC}(p) = \mathrm{RC}(F)$.

Example 4.2. Consider the family of Example 3.4 and the following lifting, defined over the ordered sequence of vertices:

$$\begin{aligned}
&\omega_0 : ((0,0),(0,1),(1,0),(1,1)) \to (-M,1,1,M), \qquad \text{for } M \gg 1\\
&\omega_1 : ((0,0),(0,1),(1,0),(1,1)) \to (0,2,2,1),\\
&\omega_2 : ((0,0),(0,1),(1,0),(1,1)) \to (0,2,2,1).
\end{aligned}$$

Let T_1 be the triangle which is the convex hull of the points $\{(0,0),(1,0),(1,1)\}$. Two of the cells in the coherent mixed decomposition associated with the lifting are $F = \{(0,0)\} + T_1 + \{(0,0)\}$, with $\mathrm{RC}(F) = (2,(0,0))$, and $F' = T_1 + \{(1,1)\} + \{(1,1)\}$, with $\mathrm{RC}(F') = (2,(1,1))$.

Matrix $\mathbb{M}$ is of size $|\mathcal{F}| \times |\mathcal{F}|$, with rows and columns indexed by the points $p \in \mathcal{F}$ as follows. There will be a distinguished point $\mathbf{p}$, which will be defined below.

- If $p \in \mathcal{F} \setminus \{\mathbf{p}\}$ then, for every $p' \in \mathcal{F}$, the entry of $\mathbb{M}$ indexed by (p, p') will be the coefficient of $x^{p'}$ in the expansion of $x^{p-a} f_i(x)$. Here, $\mathrm{RC}(p) = (i, a)$.
- The entry indexed by $(\mathbf{p}, p')$ will be the coefficient of $x^{p'}$ of $G(f)$.

5. Lifting Algorithm

The originality of our approach consists in using piecewise-linear liftings. In other words, each Newton polygon will be subdivided to one or more cells that are lifted linearly. In the rest of the paper, we sometimes revert to the case of identical Newton polygons for simplicity. All arguments readily generalize to scaled copies of P by the fact that $\omega_i(k_i p) = k_i \omega_i(p)$ for any point $p \in Q_i = k_i P$.

Algorithm 5.1. (Step 1)

- *Let (a_r, a_{r+1}) be any edge of P. Here, $a_r < a_{r+1}$ in the clockwise sense.*
- *Let $\delta' \in \mathbb{Q}^2$ be a vector perpendicular to this edge and pointing to the exterior of P.*
- *If δ' is not parallel to any edge of the lifted Q_i's, then, vector $\delta \in \mathbb{Q}^2$ has the same slope as δ'; otherwise, the two slopes differ sufficiently little so that $\langle p, \delta \rangle < \langle p', \delta \rangle \Leftarrow \langle p, \delta' \rangle < \langle p', \delta' \rangle$, for any vertices p, p' of P. Finally, δ is scaled so that it becomes sufficiently small so that every perturbed point in Q lies in one of the maximal cobounding cells.*
- *Let b be any vertex of P minimizing the inner product with δ.*

Observe that $b+\delta$ does not necessarily lie inside P. Now, let us offer some intuition. First, we wish that Q_0 be raised more steeply than the other polygons in such a way that it always contributes b to optimal sums; a technique also used in D'Andrea (2002) and Emiris (1996). Q_1, Q_2 will be lifted in piecewise-linear fashion, so that they are broken into linearly lifted cells. The cells' normals can be linearly ordered when projected onto $\mathbb{Q}^2$.

The edges of P which are extremal in the δ direction will be called δ-*boundary* edges and will be labeled clockwise by $e_1, \ldots, e_g$. We denote by a_i the vertices of P which are extremal in the δ direction, i.e. belong to the δ-boundary, for $i = 0, \ldots, g$. Edge e_i has endpoints a_{i-1}, a_i. If P is not a triangle, a typical cell in Q_1 or Q_2 forms a quadrilateral with vertices b, a_{i-1}, a_i, a_{i+1}. We shall denote the latter by C_i. It is clear that a_r, as defined in Step 1, lies on the δ-boundary of P. We shall be writing sums with their summands ordered so as to indicate the polygon in which they belong. Let b_0 (resp. b_g) denote any vertex of P which may exist on the clockwise (resp. counter-clockwise) chain between b and a_0 (resp. a_g).

LEMMA 5.2. *Vertices on the δ-boundary of $Q_1 + Q_2$ (resp. Q) are always of the form $2a_i$(resp. $3a_i$), for some vertex a_i. Any other point on the δ-boundaries of the form $a_i + a_j$ (resp. $a_l + a_i + a_j$) must satisfy $\{l, i, j\} = \{\rho, \rho + 1\}$, for some $\rho \in \mathbb{N}$.*

Algorithm 5.1 (Step 2)**.** *For all possible values of $i \geq 0$, set*

$$\omega_1(b) := \omega_2(b) := 0, \qquad \omega_1(a_{2i+1}) := \omega_2(a_{2i}) := 1, \qquad \omega_1(a_0) := \omega_\gamma(a_g) := 2,$$

where $\gamma = (g \bmod 2) + 1$. In case b coincides with a_0 or a_g, then it is lifted as b, and $\omega_2(a_1) := 2$ or $\omega_{\gamma'}(a_{g-1}) := 2$ respectively, where $\gamma' = (\gamma \bmod 2) + 1$. The δ-boundary vertices of Q_1, Q_2, which are not mentioned above, are lifted in such a way so that the cells where they belong admit a linear lifting.

Any vertices b_0 (resp. b_g) in Q_2 (resp. $Q_{\gamma'}$) form, together with b and a_0 (resp. a_g) a linearly lifted cell denoted by C_{-1} (resp. C_{g+1}). The liftings are sufficiently high, with the precise values being determined by the proof of Lemma 6.3*, namely inequality* (11)*.*

For an illustration of the lifting see Figure 1. If $b \neq a_0$, cell C_0 exists in Q_1 and includes only e_1 among the δ-boundary edges. If $b \neq a_g$, cell C_g exists and includes e_{g-1}, e_g or just e_g depending on whether g is even or odd. The coherent polyhedral subdivision induced over Q_1 (resp. Q_2) contains cells C_0 (if it exists), $C_2, \ldots$ (resp. C_1, $C_3 \ldots$). These subdivisions also contain C_{-1} or C_{g+1} when they exist. Notice that any b_0 (resp. b_g) vertices in Q_1 (resp. Q_γ) are lifted into C_0 (resp. C_g). Let the cell edges in Q_1, Q_2 containing vertex b be called b-edges. Some properties of these cells can now be established, to be used later.

LEMMA 5.3. *$\omega_1(a_{2i}) > 1$ and $\omega_2(a_{2i+1}) > 1$ for all possible values $i \geq 0$, unless $a_{2i} = b$ or $a_{2i+1} = b$. If $b + \delta \in P$, then there exist unique cells C_i, C_{i+1} in Q_1, Q_2 respectively, s.t. the slope of δ lies between the slopes of b-edges defining each cell and, for every other cell, the slopes of both of its b-edges shall be either larger or smaller than that of δ.*

PROOF. The first claim follows from the convexity and Step 2 of Algorithm 5.1: consider cell C_{2i} of Q_1; an analogous argument holds for the cells of Q_2. Let h be the point of

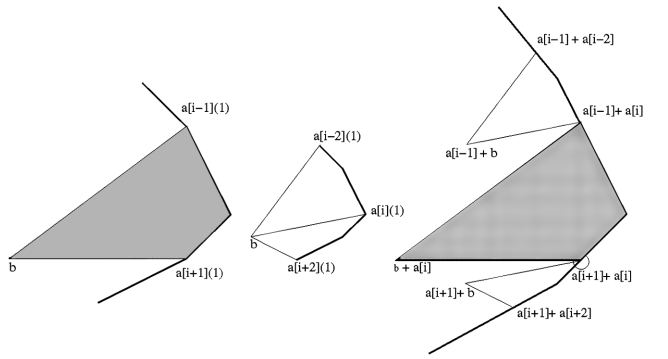

Figure 1. Q_1 is a scaled copy of Q_2. The vertex indices are shown inside square brackets and the liftings in parentheses, for i even. Shown are a typical cell of Q_1, including δ-boundary edges e_i, e_{i+1} for $i \geq 2$ and i even, and a cell of Q_2, including edges e_i, e_{i+1} for i odd. The unique point of the δ-boundary of Q_1+Q_2 perturbed outside all unmixed cells is circled, where δ lies in the direction of $(1, \epsilon)$, for $1 \gg \epsilon > 0$.

intersection of segments (b, a_{2i}) and (a_{2i-1}, a_{2i+1}). Then $\exists V \in \mathbb{Q}^2, c \in \mathbb{Q}$ s.t.

$$\begin{pmatrix} a_{2i-1} & 1 \\ a_{2i+1} & 1 \\ b & 0 \\ h & \omega_1(h) \\ a_{2i} & \omega_1(a_{2i}) \end{pmatrix} \begin{pmatrix} V \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \\ c \\ c \end{pmatrix}.$$

We can write $h = \lambda a_{2i-1} + (1-\lambda) a_{2i+1}$ for $0 < \lambda < 1$. Since $(h, \omega_1(h)) \in C_{2i,\omega}$, we have $\omega_1(h) = \lambda\omega_1(a_{2i-1}) + (1-\lambda)\omega_1(a_{2i+1}) = 1$. Now, $a_{2i} = \mu h + (1-\mu) b$ where $\mu > 1$ hence $\omega_1(a_{2i}) = \mu\omega_1(h) > 1$. The second claim follows by letting C_i, C_{i+1} be the cells containing $b + \delta$. □

Recall that the inward *normal cone* to a polygon vertex a is comprised of all inward vectors that minimize the inner product over the polygon vertices at a.

LEMMA 5.4. *There exist vectors $V^{(i)} \in \mathbb{Q}^2$, $i = 0, 1$ in the strict interior of the inward normal cone of a_{r+i}, s.t.*

$$\langle V^{(i)}, b\rangle > \langle V^{(i)}, p_i\rangle > \langle V^{(i)}, a_{r+i}\rangle, \tag{5}$$

for $i = 0$ (resp. $i = 1$) and any p_i in the clockwise (resp. counter-clockwise) vertex chain between b, a_{r+i}. Moreover, we may find these vectors satisfying the following additional inequalities:

$$\langle V^{(0)}, (b - a_{r+1})\rangle > 0, \qquad \langle V^{(1)}, (p - a_r)\rangle > 0, \qquad p \in W \setminus \{a_r, a_{r+1}\}. \tag{6}$$

PROOF. Let us refer to Step 1: $-\delta'$ is on the boundary of the inward cone of a_{r+i}, $i = 0, 1$. Let $V^{(0)}$ (resp. $V^{(1)}$) be obtained by increasing (resp. decreasing) the slope of $-\delta'$ sufficiently little. The bounds on these slope perturbations are obtained by considering each inequality of (5) in turn: on the one hand, $V^{(i)}$ must lie inside the normal cone of a_{r+i}. For this, consider vertices a_{r-1} (resp. a_{r+2}) for $V^{(0)}$ (resp. $V^{(1)}$); these are the first vertices where the second inequality might be violated if the slope of $V^{(i)}$ were not bounded. Note these vertices are adjacent to a_r (resp. a_{r+1}) but, despite notation, they are not necessarily on the δ-boundary. Once the $V^{(i)}$ lie in the interior of the normal cone, the second inequality of (5) follows.

Let us show that no new constraints are imposed by the first inequality of (5). It holds for $-\delta'$ in the place of $V^{(i)}$ and all p_i, except possibly one such vertex (a neighbor of b), where the inequality becomes an equality. If this p_i lies on the clockwise (resp. counter-clockwise) vertex chain, then inner product with $V^{(0)}$ (resp. $V^{(1)}$) satisfies the claim because the slope has increased (resp. decreased). This is true for any $V^{(i)}$ in the respective cone, so there are no extra constraints on the slope. For all p_i, with the possible exception of the neighbor of b discussed above, we have $\langle \delta', b\rangle > \langle \delta', p_i\rangle > 0$. To guarantee the claim, it suffices to require that the inner products be positive with $V^{(i)}$ instead of δ'. This is always true inside the normal cone.

Inequalities (6) yield additional constraints on $V^{(i)}$. The first inequality is satisfied for $-\delta'$, so it is possible to satisfy it with $V^{(0)}$. For the second, observe that $\langle \delta', a_{r+1} - a_r\rangle = 0$ so, by convexity, $\langle -\delta', p - a_r\rangle > 0$ for all vertices $p \notin \{a_r, a_{r+1}\}$. Therefore we can choose $V^{(1)}$ to satisfy (6). □

In short, the determining conditions for $V^{(i)}$ are that each belongs to the respective normal cone and that they satisfy inequalities (6).

Algorithm 5.1 (Step 3: Lifting Q_0)**.** *For Q_0, define lifting ω_0 s.t.*

$$\omega_0(b) = -M, \qquad \omega_0(a_r) = M, \qquad M \gg 1,$$

where M is a sufficiently large positive rational. In addition, ω_0 defines one or two linearly lifted cells. The cell including any vertices between b and a_r in the clockwise direction (resp. counter-clockwise) is specified by $\omega_0(b), \omega_0(a_r)$ and the condition that its inward normal is of the form $(V^{(i)}, v_3^{(i)})$ for $i = 0, 1$ respectively, where vector $V^{(i)} \in \mathbb{Q}^2$ is as in Lemma 5.4 *and*

$$v_3^{(i)} := \langle V^{(i)}, (b - a_r)\rangle / 2M \ \in \mathbb{Q}, \qquad i = 0, 1.$$

Deterministic lower bounds on M are obtained by the proofs of certain lemmas below. It is possible to bound the bit asymptotic complexity of the algorithm by a quasi-linear function in the largest support cardinality.

Proposition 5.5. *Step* 3 *of Algorithm* 5.1 *is self-coherent and defines at most two linearly lifted cells.*

Proof. When there are sufficiently few vertices to define exactly one cell in Q_0, the construction is trivially valid. Otherwise, there are cells C'_0, C'_1 and we wish to show that $(V, v_3)^i = (V^{(i)}, v_3^{(i)})$ minimizes inner product over Q_{0,ω_0} at C'_i for $i = 0, 1$. By Step 3 and Lemma 5.4, $v_3^{(i)} > 0$, $b_{\omega_0}, a_{r,\omega_0}$ lie on both lifted cells and for any vertex $p_i \in C'_i \setminus \{b, a_r\}$, $\omega_0(p_i) = M + 2M \frac{\langle a_r - p_i, V^{(i)}\rangle}{\langle b - a_r, V^{(i)}\rangle}$, $i = 0, 1$.

It suffices now to show that $\langle (V, v_3)^i, p_{j,\omega_0}\rangle > \langle (V, v_3)^i, a_{r,\omega_0}\rangle$, $\quad (i, j) \in \{(0, 1), (1, 0)\}$, which is equivalent to

$$\frac{\langle a_r - p_j, V^{(j)}\rangle}{\langle b - a_r, V^{(j)}\rangle} \langle b - a_r, V^{(i)}\rangle > \langle a_r - p_j, V^{(i)}\rangle, \qquad (i, j) \in \{(0, 1), (1, 0)\}.$$

Let us identify the origin with a_r w.l.o.g., then the above inequality becomes

$$\frac{\langle b, V^{(i)}\rangle}{\langle b, V^{(j)}\rangle} < \frac{\langle p_j, V^{(i)}\rangle}{\langle p_j, V^{(j)}\rangle}, \qquad (i,j) \in \{(0,1),(1,0)\}, \tag{7}$$

because $\langle b, V^{(i)}\rangle, \langle p_j, V^{(i)}\rangle > 0, \langle p_i, V^{(i)}\rangle > 0$ by Step 1 of the algorithm and Lemma 5.4, except that $\langle a_{r+1}, V^{(1)}\rangle < 0$. In the latter case, the inequality before (7) is clear because the left-hand side is negative and the right-hand side positive.

To prove (7), observe that all vectors lengths cancel out, so it is possible to write the right-hand side as follows, using $\angle(\cdot,\cdot)$ to represent the angle, in the counter-clockwise direction, between two vectors:

$$\frac{\cos(\angle(p_j, V^{(j)}) + \angle(V^{(j)}, V^{(i)}))}{\cos(\angle(p_j, V^{(j)}))} = \cos(\angle(V^{(j)}, V^{(i)})) - \sin(\angle(V^{(j)}, V^{(i)}))\tan(\angle(p_j, V^{(j)})).$$

In the same way we may write the left-hand side of the inequality as

$$\frac{\cos(\angle(b, V^{(j)}) + \angle(V^{(j)}, V^{(i)}))}{\cos(\angle(b, V^{(j)}))} = \cos(\angle(V^{(j)}, V^{(i)})) - \sin(\angle(V^{(j)}, V^{(i)}))\tan(\angle(b, V^{(j)})).$$

So, inequality (7) is equivalent to $\tan(\angle(p_j, V^{(j)})) < \tan(\angle(b, V^{(j)}))$, which follows straightforwardly just noting that $\tan x$ is strictly increasing in the interval $(-\pi/2, \pi/2)$ and the properties of $V^{(j)}$. □

LEMMA 5.6. *Lifting function* ω_0 *guarantees that* $\mathrm{RC}(p) = (0, p_0) \Rightarrow p_0 = b$ *for sufficiently large* M.

PROOF. The existence of M is proven in Emiris (1996) provided $\omega_0(b) < \omega_0(p), \forall p \in W \setminus \{b\}$. This is equivalent to $\langle b, V^{(i)}\rangle > \langle p_i, V^{(i)}\rangle$ for $p_i \in W \setminus \{b\}$ on the clockwise (resp. counter-clockwise) vertex chain from b to a_r for $i = 0$ (resp. $i = 1$). These hold by Lemma 5.4. □

6. Mixed Subdivision

This section establishes the properties of the combinatorial lifting, thus yielding the subdivision suitable to our ends. Intuitively, we prove that the subdivision of Q_1, Q_2 into cells allows us to avoid using any boundary point and, moreover, lets us assign the toric Jacobian to some special point $\mathbf{p}$. It is a novelty, and part of the paper's intricacy, that we classify all cells and determine their placement in the subdivision; a similar task was undertaken in D'Andrea (2002). This leads to our main result, namely a nontrivial sparse resultant matrix with the desired features.

DEFINITION 6.1. Let $V_{i,\omega} \in \mathbb{R}^3$ be the normalized inward normal vector corresponding to the lifted facet $C_{i,\omega}$ in $Q_{1,\omega}$ (resp. $Q_{2,\omega}$) if i is even (resp. odd). It is the unique inward normal vector with the third coordinate equal to 1.

Referring to Figure 1, let $V_{i,\omega} = (v_1, v_2, 1)$, then, for Q_1 and $\lambda := \omega_1(a_i) > 1$,

$$\begin{pmatrix} a_{i-1} - b & 1 \\ a_i - b & \lambda \\ a_{i+1} - b & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{8}$$

by translation of the reference frame, where the left-most entries in the matrix represent 1×2 vectors.

LEMMA 6.2. *Δ_ω contains unmixed cells, which are copies of each C_i for $i = 1, \dots, g-1$, against the δ-boundary of $b + Q_1 + Q_2$.*

PROOF. Suppose w.l.o.g. that i is an even integer between $1, \dots, g-1$ (the case for odd integer in this interval is similar). At the δ-boundary of Q_j, $\omega_j(a_{i+p}) \geq 1$ with equality iff p is odd (resp. even) and $j = 1$ (resp. $j = 2$) and $\omega_1(a_{i+1}) = \omega_2(a_i) = \omega_1(a_{i-1}) = 1$, $\omega_1(a_i) := \lambda$, $\omega_2(a_{i+1}), \omega_2(a_{i-1}) > 1$, by Lemma 5.3. As the points $a_{i-1} + a_i,\ a_i + a_i$ and $a_{i+1} + a_i$ belong to the δ-boundary of $Q_1 + Q_2$, they must be sum of two points of the δ-boundary of $Q_j,\ j = 1, 2$. On the lower hull $L(Q_{1,\omega} + Q_{2,\omega})$ the minimum height of each of the respective points is

$$\begin{aligned} \omega_1(a_{i-1}) + \omega_2(a_i) &= 1 + 1, \\ \omega_1(a_i) + \omega_2(a_i) &= \lambda + 1, \\ \omega_1(a_{i+1}) + \omega_2(a_i) &= 1 + 1. \end{aligned}$$

This implies that the points $(b + a_{i-1} + a_i, -M + 2)$, $(b + a_i + a_i, -M + 1 + \lambda)$ and $(b + a_{i+1} + a_i, -M + 2)$ belong to $L(Q_\omega)$, where the notation $(\cdot, \cdot)$ indicates a vector in $\mathbb{Q}^3$, with first component a vector in $\mathbb{Q}^2$. When multiplied by the vector $V_{i,\omega}$ of (8), the three points give the same value, namely the inner product $\langle (2b + a_i, -M + 1), V_{i,\omega} \rangle$. Hence these points lie on a flat with normal $V_{i,\omega}$ and their convex hull projects to $b + C_i + a_i$.

It must still be shown that this polygon is a facet on $L(Q_\omega)$, i.e. it minimizes inner product with $V_{i,\omega}$ over all vertices of $b_{\omega_0} + Q_{1,\omega} + Q_{2,\omega}$. Among the Q_1 vertices, this inner product is minimized at the vertices of C_i by definition, so it suffices to prove the claim $\langle (a_i, 1), V_{i,\omega} \rangle < \langle (a', \omega_2(a')), V_{i,\omega} \rangle \quad \forall a' \in W \setminus \{a_i\}$, where we have set $(b, 0) = (0, 0, 0)$, w.l.o.g., and W is the vertex set of P. Then (8) becomes

$$\langle a_{i-1}, v \rangle + 1 = \langle a_{i+1}, v \rangle + 1 = \langle a_i, v \rangle + \lambda = 0.$$

Therefore,

$$\begin{aligned} \langle a_{i-1}, v \rangle + \omega_2(a_{i-1}) &> 0, \\ \langle a_{i+1}, v \rangle + \omega_2(a_{i+1}) &> 0, \\ \langle (a_i, 1), V_{i,\omega} \rangle = \langle a_i, v \rangle + 1 &< 0. \end{aligned} \tag{9}$$

With $a' = b = (0, 0)$ or $a' \in \{a_{i-1}, a_{i+1}\}$ the claim is trivial by (9). Let us define open half-spaces $\mathcal{H}^+ := \{x \in \mathbb{Q}^3 : \langle x, V_{i,\omega} \rangle > 0\}$ and $\mathcal{H}^- := \{x \in \mathbb{Q}^3 : \langle x, V_{i,\omega} \rangle \leq 0\}$, which are disjoint and convex, and the fan $F := \{r(a_{i-1}, \omega_2(a_{i-1})) + s(a_{i+1}, \omega_2(a_{i+1})),\ r, s > 0\}$, which is completely contained in $\mathcal{H}^+$ by (9). Suppose $(a', \omega(a'))$ belongs to $\mathcal{H}^-$ for $a' \in W_2 \setminus \{b, a_i, a_{i-1}, a_{i+1}\}$. Since $(a_i, 1) \in \mathcal{H}^-$ by (9), then the segment S of $(a', \omega_2(a')), (a_i, 1)$ must be contained in $\mathcal{H}^-$. To get a contradiction one must check that $S \cap F \neq \emptyset$. To see this, consider that F divides $Q_{2,\omega}$ into two convex polytopes. The "lower" polytope contains $a_{i,\omega}$, whereas the "upper" one contains a' by the convexity of $Q_{2,\omega}$ (Figure 2). In fact, convexity implies a' lies between the hyper planes of $b_\omega, a_{i,\omega}, a_{i-1,\omega}$ (lifted cell C_{i-1}) and $b_\omega, a_{i,\omega}, a_{i+1,\omega}$ (lifted cell C_{i+1}), thus S intersects the triangle defined by $b_\omega, a_{i-1,\omega}, a_{i+1,\omega}$. □

LEMMA 6.3. *If cells C_i, for $i = 0, g$ exist, then Δ_ω contains unmixed cells, which are copies of them, against the δ-boundary of $b + Q_1 + Q_2$.*

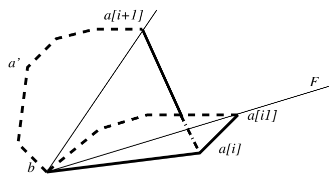

Figure 2. Last step in the proof of Lemma 6.2: bold (resp. dashed bold) are the lifted edges bounding C_i (resp. and the other edges) of Q_2, whereas the thin lines define fan F.

PROOF. We shall analyze $i = 0$ because $i = g$ can be treated similarly. Recall that in this situation, $\omega_1(a_0) = 2$. The proof is, for the most part, analogous to that of Lemma 6.2; here we present the additional part. By Definition 6.1, $V_{0,\omega} = (v, 1)$ with $v \in \mathbb{Q}^2$ is normal to $C_{0,\omega}$ hence, $\forall a' \in (W_1 \cap C_0) \setminus \{b, a_0, a_1\}$,

$$\begin{pmatrix} a_0 - b & \omega_1(a_0) \\ a_1 - b & 1 \\ a' - b & \omega_1(a') \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

It suffices to prove the claim

$$\langle (a_0, 1), V_{0,\omega} \rangle < \langle (a', \omega_2(a')), V_{0,\omega} \rangle, \qquad \forall a' \in W \setminus \{a_0\}.$$

W.l.o.g. we assume $b = 0$ and consider the possibility that a vertex $a' = b_0 \in W_2 \cap C_{-1}$ exists between b and a_0; b_0 is defined in Algorithm 5.1, Step 2. Using the fact that $b_0 \in W_1 \cap C_0$, one may compute $\omega_1(b_0)$ by (10). Then the claim reduces to

$$-\omega_1(a_0) + 1 < -\omega_1(b_0) + \omega_2(b_0) \Leftrightarrow \omega_2(b_0) > \omega_1(b_0) - \omega_1(a_0) + 1 = \omega_1(b_0) - 1. \tag{11}$$

This holds for $\omega_2(b_0)$ sufficiently large. An analogous bound is imposed on $\omega_{\gamma'}(b_g)$ for $\gamma' = ((g + 1) \bmod 2) + 1$. □

For ease of presentation we shall work with $\delta \in \mathbb{Q}^2_{>0}$ with coordinates $\delta_1 \gg \delta_2$ and shall let $\epsilon \in (0, 1)$ stand for its slope. The next step is to determine the point corresponding to the toric Jacobian.

LEMMA 6.4. *There exists a unique point* $p : \mathrm{RC}(p) = (0, b)$ *on the boundary of* $b + Q_1 + Q_2$ *iff* $b + \delta \in P$.

PROOF. If P contains $b + \delta$, then let the cells in Q_1, Q_2 containing $b + \delta$ be denoted by C_{ρ_1}, C_{ρ_2}, respectively, where $|\rho_1 - \rho_2| = 1$. All points in the δ-border of $b + Q_1 + Q_2$ are displaced inside a cell which is a copy of C_i for some i, unless they lie exactly in the intersection of two cells of this type, i.e. in $C_i \cap C_{i+1}$, $i \in \{0, \ldots, g - 1\}$. The latter are of the form $b + a_{i+1} + a_i$. We shall consider three different cases. Let us call $\sigma_{i-1}, \sigma_{i+1}$, the slopes of the b-edges of the cell C_i; σ_i, σ_{i+2} will be the slopes of C_{i+1}. Due to the special choice we have made on δ just before the statement of this lemma, we have that $\sigma_{i-1} > \sigma_i > \sigma_{i+1} > \sigma_{i+2}$.

- $\sigma_i > \epsilon > \sigma_{i+1}$: it is straightforward to check that the point is between cells C_{ρ_1} and C_{ρ_2}. This is the unique point claimed.

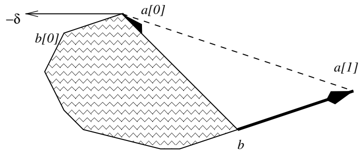

Figure 3. Cell C_0 of Q_1; the vertex chain to the left is where b_0 may belong. The shaded part is cell C_{-1} of Q_2. Notice that the vector $-\delta$ cannot be contained in C_0 since a_0 is, by hypothesis, the vertex closest to b which is extremal in the direction of δ.

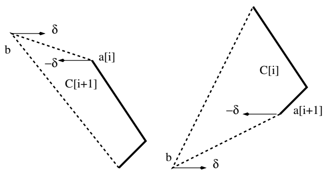

Figure 4. Cases of the proof of Lemma 6.4. Vertices a_i and cells C_i are denoted $a[i]$ and $C[i]$ respectively.

- $\epsilon > \sigma_i$: both slopes of C_{i+1} are smaller than ϵ, so $b + a_{i+1} + a_i - \delta \in b + a_{i+1} + C_{i+1}$ because $a_i - \delta \in C_{i+1}$; see the left part of Figure 4.
- $\sigma_{i+1} > \epsilon$: both slopes of C_i are strictly bigger than ϵ, then $b + a_{i+1} + a_i - \delta \in b + C_i + a_i$ because $a_{i+1} - \delta \in C_i$; see the right part of Figure 4.

This discussion also covers the points $a_0 + a_0, a_g + a_g$ which are first and last on the vertex chain corresponding to the δ-boundary. If points b_0 or b_g exist (defined in Algorithm 5.1, Step 2), then $b_0 + a_0$ and $b_g + a_g$ (or $a_g + b_g$) are boundary points of $b + Q_1 + Q_2$, closest but not on its δ-boundary.

If P does not contain $b + \delta$, then no cell in Q_1, Q_2 contains $b + \delta$ and all points on the δ-boundary of $b + Q_1 + Q_2$ are perturbed inside some cell. □

This point shall index the matrix row containing the toric Jacobian in the constructed matrix; its general definition follows. Recall that by Algorithm 5.1, Step 1, there are at least two points a_r, a_{r+1}, different than b, on the δ-boundary of P.

Definition 6.5. Let point $\mathbf{p} \in \mathcal{F}$ be on the δ-boundary of $b + Q_1 + Q_2$ such that: If P contains $b + \delta$, $\mathbf{p}$ is the unique point for which $\mathrm{RC}(\mathbf{p}) = (0, b)$. Otherwise, b is on the δ-boundary of P. If $b = a_0$ then $\mathbf{p} = b + a_2 + a_1$; otherwise $b = a_g$ and $\mathbf{p} = b + a_{g-\gamma} + a_{g-\gamma'}$ where $\gamma = (g \bmod 2) + 1$, $\gamma' = (\gamma \bmod 2) + 1$.

Example 6.6. Take polynomials $f_i = c_{i0} + c_{i1}x_1x_2 + c_{i2}x_1x_2^2,\ i = 0, 1, 2$ with $W = \{(0,0), (1,1), (1,2)\}$, $g = 2$. If $a_r = (1,2), a_{r+1} = (1,1)$, then $b = (0,0) = a_g$ and δ is horizontal. Moreover, $\gamma = 1, \gamma' = 2, \mathbf{p} = b + a_{g-\gamma} + a_{g-\gamma'}$. All twofold mixed volumes are equal to 1. Now $b + Q_1 + Q_2 \cap \mathcal{F}$ contains only point $a_r + a_{r+1} = (2,3)$. It lies on its δ-boundary, it is perturbed inside a cell and shall be identified with $\mathbf{p}$; see Figure 5. Furthermore, this is the only integer point in $\mathcal{F}$, hence the resultant matrix is 1×1 and yields the sparse resultant. It is easy to see that $G(f) = -x_1^2x_2^3 \det[c_{ij}]$, then

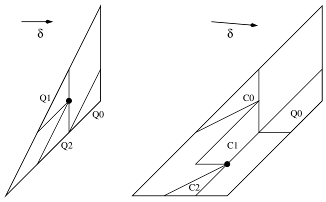

Figure 5. Example 6.6. Shown are certain cells of the subdivision for each system.

$M = [G(f)]$ and $\mathrm{Res}_{\mathcal{A}}(f_0, f_1, f_2) = \det M$. The second Figure in 5 concerns the system $f_i = c_{i0} + c_{i1}x + c_{i2}x^2y + c_{i3}x^2y^2$, $i = 0, 1, 2$, with $W = \{(0,0), (1,0), (2,1), (2,2)\}$, $g = 3$, $a_r = a_0 = (2,2), a_{r+1} = a_1 = (2,1)$, $b = (0,0) = a_g$ and $\mathbf{p} = b + a_{g-\gamma} + a_{g-\gamma'} = (3,1)$ where $\gamma = 2$, $\gamma' = 1$; $\delta = (1, -\epsilon)$.

For a well-defined matrix, we show that it suffices to consider all points in $\mathcal{F}$.

Lemma 6.7. *For any $p \in \mathcal{F}$, except for the point $\mathbf{p}$ of Definition* 6.5*, if* $\mathrm{RC}(p) = (i, a)$ *then $(p - a + Q_i) \cap \mathbb{Z}^2 \subset \mathcal{F}$, supposing p lies in cell $F_0 + F_1 + F_2 + \delta$,* $\dim(F_0) = 0$.

Proof. The first case is when p does not belong to a 0-mixed cell, i.e. $\exists i > 0 : \dim(F_i) = 0$. Due to ω_0, Lemma 5.6 implies $F_0 = b$, hence $p = b + p_i + p_j \in b + p_i + F_j + \delta$, $p_i \in W_i, p_j \in W_j$ $\dim(F_j) = 2$, $i \neq j$; here, exceptionally, the order of p_i, F_j is not important. So $\mathrm{RC}(p) = (i, p_i)$. We wish to arrive at a contradiction when assuming $\exists q = b + q_i + p_j \in p - p_i + Q_i : q \in \mathcal{E} \setminus \mathcal{F}$. This implies b, q_i, p_j lie on the δ-boundary of their respective polytope and also on the same edge, which is (b, a_1) or (b, a_{g-1}).

If $p_j = b$ then $p = b + p_i + b \in b + p_i + F_j + \delta \Rightarrow b - \delta \in F_j \subset Q_j$ which implies b does not maximize the inner product with $-\delta$, in contrast to the definition of b. Otherwise $p_j \in \{a_1, a_{g-1}\}$ so for F_j to be a cell that contains it, it must equal one of C_1, C_2 or C_{g-1}, C_g. For $p_i + F_j$ to give a cell of Δ_ω, we must have $F_j = C_t$ and $p_i = a_t$ for some $t \in \{1, 2, g-1, g\}$. There are four cases for p: $b + a_1 + a_1, b + a_{g-1} + a_{g-1} \notin \mathcal{F}$, and $b + a_2 + a_1, b + a_g + a_{g-1}$ which is precisely $\mathbf{p}$. All cases contradict the lemma's hypothesis.

The second case is that p is in a 0-mixed cell, so $\mathrm{RC}(p) = (0, b)$ and $p \in int(b + Q_1 + Q_2) = b + int(Q_1 + Q_2)$. Then, $p - b + Q_0 \subset b + int(Q_1 + Q_2) - b + Q_0$, which equals $Q_0 + int(Q_1 + Q_2) = int(Q_0 + Q_1 + Q_2)$. □

Lemma 6.8. *Δ_ω contains, against the δ-boundary of Q, one or two unmixed cells which are copies of the respective cell(s) of Q_0. Each cell contains exactly one of vertices $3a_r$, $3a_{r+1}$. The rest of the cells covering $Q \setminus (Q_1 + Q_2)$ are mixed of type* 1 *or* 2.

Proof. Consider the inward normals $V^{(i)}$ to the Q_0 cells; we shall show they are parallel to the inward normals of the corresponding cells of Δ_ω. By Proposition 5.5, the inner product is minimized over all vertices of Q_{0,ω_0} at a_{r+i,ω_0}, $i = 0, 1$. The same should hold at a_{r+i,ω_1} and a_{r+i,ω_2} over $Q_{1\omega_1}, Q_{2\omega_2}$ respectively. If W_j is the vertex set of Q_j, we wish to show that

$$\langle V^{(i)}, p_j \rangle + \omega_j(p_j) v_3^{(i)} > \langle V^{(i)}, a_{r+i} \rangle + \omega_j(a_{r+i}) v_3^{(i)},$$
$$\forall p_j \in W_j \setminus \{a_{r+i}\},\ i = 0, 1, \quad j = 1, 2.$$

The conclusion is immediate when $\omega_j(p_j) \geq \omega_j(a_{r+i})$ because $\langle V^{(i)}, p_j\rangle > \langle V^{(i)}, a_{r+i}\rangle$, since $V^{(i)}$ is in the inward cone of a_{r+i}, and $v_3^{(i)} > 0$ by the proof of Proposition 5.5. Otherwise, the conclusion is equivalent to

$$\frac{\langle V^{(i)}, p_j - a_{r+i}\rangle}{\omega_j(a_{r+i}) - \omega_j(p_j)} > v_3^{(i)} = \frac{\langle V^{(i)}, b - a_r\rangle}{2M} \Leftrightarrow M > (\omega_j(a_{r+i}) - \omega_j(p_j))\frac{\langle V^{(i)}, b - a_r\rangle}{2\langle V^{(i)}, p_j - a_{r+i}\rangle}. \tag{12}$$

The lower bound is positive: the first parenthesis is positive by hypothesis; the fraction denominator because $V^{(i)}$ is in the corresponding cone; and the numerator because $V^{(0)}$ is in the cone of a_r and $V^{(1)}$ satisfies (6). Hence, the lower bound is always satisfied for sufficiently large M.

The last statement holds straightforwardly by noting that mixed cells of types 1 and 2 are contained in $Q \setminus (Q_1 + Q_2)$, the area covered by mixed cells of type i is equal to $\mathrm{MV}(Q_j, Q_k)$, $j \neq i \neq k$, and we have that $vol(Q \setminus (Q_1 + Q_2))$ is equal to $\mathrm{MV}(Q_0, Q_1)$ $+\mathrm{MV}(Q_0, Q_2) + vol(Q_0)$. □

Lemma 6.9. *For any $p \in \mathcal{F}$, suppose p lies in cell $F_0 + F_1 + F_2 + \delta$, $\dim(F_0) > 0$: if $\mathrm{RC}(p) = (i, a)$, then $(p - a + Q_i) \cap \mathbb{Z}^2 \subset \mathcal{F}$.*

Proof. Point p is not that of Definition 6.5 because $\dim(F_0) > 0$. Being an integer point, p must belong to the cell $F_0 + F_1 + F_2$ (possibly to its δ-border) if δ is small enough. Then $F_1 + F_2$ lies on the boundary of $Q_1 + Q_2$, by the lifting.

Suppose w.l.o.g. that $RC(p) = (2, \tilde{a})$, so $p = p_0 + p_1 + \tilde{a}$, for $p_i \in Q_i$. If $(p - \tilde{a} + Q_2) \cap \mathbb{Z}^2$ is not contained in $\mathcal{F}$, it must intersect the δ-boundary of Q because it is certainly in $\mathcal{E}$. For a point h in the common intersection, $\exists p_2 \in Q_2 : h = p - \tilde{a} + p_2 = (p_0 + p_1 + \tilde{a}) - \tilde{a} + p_2 = p_0 + p_1 + p_2$. Then, each p_i, for $i = 0, 1, 2$, must belong to the segment (a_j, a_{j+1}) for some j (and hence F_0, F_1 intersect the δ-boundary of Q_0, Q_1 respectively). This is the key property of the p_i's, which shall lead to a contradiction.

If F_0 is a two-dimensional cell, say containing a_{r_1}, then by Lemma 6.8 and its proof, $F_1 = F_2 = a_i$, for $i \in \{r_1, r_2\}$. By the key property above, we would also have that $p_0 \in \{a_{i-1}, a_i, a_{i+1}\}$. Now p is a sum of three points on the δ-boundaries of the respective Q_i, hence $p \notin \mathcal{F}$, a contradiction. It remains to consider F_0 as an edge. F_0 and F_1 must be non-parallel edges, so that a (mixed) cell can be generated. If $F_1 = (a_j, a_{j+1})$ then $\tilde{a}$ is a vertex of F_1 and both $p_0, p_1 \in F_1$ which implies $p \notin \mathcal{F}$, a contradiction.

All remaining cases can be treated analogously to the case that F_1 is edge (a_{j-1}, a_j) and $p_1 = a_j$. If $\tilde{a}$ or p_0 is a_j, then we arrive at the same contradiction as before, so suppose $\tilde{a} = a_{j-1}$ and $p_0 = a_{j+1}$. If $F_0 = (b, a_r)$, Lemma 6.8 implies that F_1 is adjacent to F_0, hence p_0 must be their intersection a_r, which is in contrast to the assumption $p_0 = a_{j+1}$.

So F_0 lies entirely on the δ-boundary of Q_0. Letting p_0, p_1 sweep edges F_0, F_1 respectively, $h \in \mathcal{E} \setminus \mathcal{F}$ will sweep copies of F_0, F_1 on $\mathcal{E} \setminus \mathcal{F}$. These form cells, in the mixed subdivision, which are sums of vertices from Q_0, Q_1 and one edge of Q_2. Having two consecutive edges from Q_2 on the δ-boundary of Q implies they define a two-dimensional cell, which contradicts Lemma 6.8. □

All these lemmas prove the following:

Theorem 6.10. *For $p \in \mathcal{F}$, except the unique point of Definition 6.5, if $\mathrm{RC}(p) = (i, a)$ then $(p - a + Q_i) \cap \mathbb{Z}^2 \subset \mathcal{F}$.*

THEOREM 6.11. (MAIN) *Let* $\mathbb{M}$ *be the matrix constructed at the end of Section* 4 *with the lifting defined in Section* 5. *Then, we have that* $\det(\mathbb{M}) = \mathrm{Res}_{\mathcal{A}}(f_0, f_1, f_2)\, p_{\mathbb{M}}$*, where* $p_{\mathbb{M}}$ *does not depend of the coefficients of* f_0.

PROOF. Theorem 6.10 tells us that $\mathbb{M}$ is actually a minor of the matrix associated with the last morphism in the complex (3). The fact that $\det(\mathbb{M})$ is a multiple of the resultant follows from Cattani *et al.* (1998) and Proposition 3.3. The determinant has the same degree in the coefficients of f_0 as the resultant because the number of rows depending on the coefficients of f_0 is exactly $\mathrm{MV}(Q_1, Q_2)$, and all the rows depend linearly on those coefficients; (cf. Cox *et al.*, 1998). In order to see that $\det(\mathbb{M}) \neq 0$, we shall use the same convexity argument given in Canny and Emiris (2000) and Sturmfels (1994) with some care: we shall specialize some coefficients to zero and regard the structure of $G(f)$.

As the polynomial $G(f)$ is well defined modulo the homogeneous ideal, it is enough to show that the determinant of $\mathbb{M}$ is non-zero for a specific choice of it. In order to do this choice, we write $\mathbf{p} = b + a_i + a_{i+1}$. This can be done because of Lemma 6.4 and Definition 6.5. Taking as supports $\mathcal{A}_i^1 := e_{i+1} \setminus \{a_i\}$ and $\mathcal{A}_i^2 := \{a_i\}$, we compute $G(f)$. Let us specialize the polynomials as follows: $\tilde{f}_0 := c_{0b}x^b$, $\tilde{f}_1 := \sum_{a\in W} c_{1a}x^a$, $\tilde{f}_2 := \sum_{a\in W} c_{2a}x^a$. $G(\tilde{f}) = \pm c_{0b}(c_{1a_i}c_{2a_{i+1}} - c_{2a_i}c_{c_1 a_{i+1}})x^{\mathbf{p}}$. We shall denote by $\mathbb{M}'(\tilde{f})$ the matrix obtained by eliminating the rows and columns indexed by $\mathbf{p}$ in $\mathbb{M}(\tilde{f})$. By convexity, $\mathrm{init}_{-\omega}(\det(\mathbb{M}(\tilde{f})))$ will be equal to $\pm c_{0b}c_{1a_i}c_{2a_{i+1}}$ or $\pm c_{0b}c_{2a_i}c_{1a_{i+1}}$ times the product of the elements lying in the diagonal of $\mathbb{M}'(\tilde{f})$ which are always coefficients corresponding to the vertices of P if the polynomial to which they are associated belongs to $\{f_1, f_2\}$, otherwise it is c_{0b}. In all the cases, they are different from zero. Then, $init_{-\omega}(\det(\mathbb{M}(\tilde{f})))$ is nonzero. □

COROLLARY 6.12. *The extraneous factor* $p_{\mathbb{M}}$ *is a polynomial in* $\mathbb{Z}[c]$ *with content* 1.

PROOF. The proof of the previous theorem tells us that $init_{-\omega}(\det(\mathbb{M}(\tilde{f}))) = \pm c_{0b}^{\mathrm{MV}(Q_1,Q_2)} \prod_{i>0} c_{ia}^{\alpha_{ia}}$. On the other hand, $init_{-\omega}(\mathrm{Res}_{\mathcal{A}}(\tilde{f}_0, \tilde{f}_1), \tilde{f}_2) = \pm c_{0b}^{\mathrm{MV}(Q_1,Q_2)} \prod_{i>0} c_{ia}^{\alpha'_{ia}}$ with $0 \leq \alpha'_{ia} \leq \alpha_{ia}$ (Sturmfels, 1994). Now compute $init_{-\omega}(p_{\mathbb{M}}(\tilde{f}))$. □

7. Examples

Public MAPLE code, still under development, is available on the second author's web page. One heuristic of the code is to have the support of $G(f)$ in $b + e + Q_i$, where b is one of the endpoints of edge e; thus we are sure not to add any columns that would not have been included by the greedy matrix construction.

EXAMPLE 7.1. Let us compute the resultant of the family of Example 3.4. The Q_i are equal to the unitary square. Taking $\delta = (\epsilon, \epsilon - \epsilon_0)$, with $0 < \epsilon_0 \ll \epsilon$, $b = (0, 0)$, and using Algorithm 5.1, we get a partition of Q_1 in two triangles whose common side is the segment with vertices $(0, 0), (1, 1)$. Q_2 is not divided under the lifting. In this example, it does not matter which will be the lifting over Q_0 because there are no integer points outside $b + Q_1 + Q_2 + \delta$. We have that $\mathcal{F} = \{(1, 1),\ (1, 2),\ (2, 1),\ (2, 2)\}$. Expanding $G(f)$

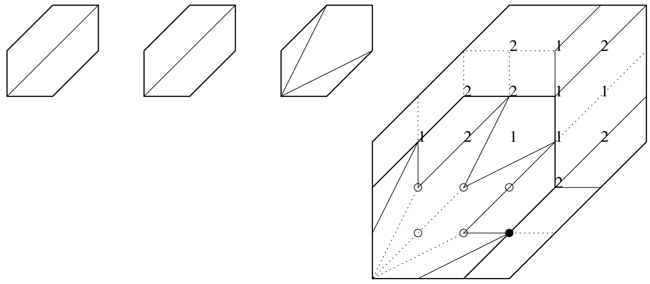

Figure 6. Example 7.2. Shown are the cells of the Q_i, certain cells of the subdivision, and the point $\mathbf{p}$. Circled points correspond to f_0 rows, whereas the rest of the points are marked according to the corresponding f_i, $i = 1, 2$.

as $\Delta_{(1,1)}x_1x_2 + \Delta_{(1,2)}x_1x_2^2 + \Delta_{(2,1)}x_1^2x_2 + \Delta_{(2,2)}x_1^2x_2^2$, we obtain the following matrix:

$$\mathbb{M} = \begin{pmatrix} c_{00} & c_{01} & c_{02} & c_{03} \\ \Delta_{(1,1)} & \Delta_{(1,2)} & \Delta_{(2,1)} & \Delta_{(2,2)} \\ c_{10} & c_{11} & c_{12} & c_{13} \\ c_{20} & c_{21} & c_{22} & c_{23} \end{pmatrix},$$

whose determinant equals $\pm\mathrm{Res}_{\mathcal{A}}(f_0, f_1, f_2)$. The following MAPLE session produces the above matrix, where the last two arguments of `hybridJac()` are δ and a_r:

```
read ‘bivar.mpl‘: read ‘hybridJac.mpl‘:
for i from 0 to 2 do f[i] := a[i] + x[1]*b[i] + x[2]*c[i]
  + x[1]*x[2]*d[i] od:
hybridJac([f[0],f[1],f[2]],[x[1],x[2]],vector([.2,.1]),vector([1,1]));
```

EXAMPLE 7.2. Consider the system where the unique Newton polygon, shown in Figure 6, is the convex hull of the points $\{(0,0), (0,1), (1,0), (1,2), (2,1), (2,2)\}$. $\mathcal{F}$ has 19 elements. Taking $a_r = (2,2), a_{r+1} = (2,1), \delta' = (1,0)$, we may take δ parallel to $\left(\frac{5}{6}, \frac{1}{3}\right)$. Then $b = (0,0)$ and ω_1, ω_2 give the following lifting values corresponding to the sequence of vertices above: $0, \frac{3}{2}, 1, 2, \frac{3}{2}, 1$ and $0, 1, 2, 1, 1, \frac{4}{3}$ where $\omega_2(0,1)$ can be any value larger than 1/2. The cells in the Q_i, point $\mathbf{p}$, and certain cells of the subdivision are shown in Figure 6. Our code yields a 16×16 matrix whose determinant equals $\mathrm{Res}_{\mathcal{A}}(f_0, f_1, f_2)$. Here is the MAPLE session, supposing the lifting is known and given as the last argument. The other arguments are, in order, the convex hulls and supports, the indices of the vertex and edge used to define the Jacobian, the scaling factors, and δ.

```
read ‘bivar.mpl‘: read ‘hybridJac.mpl‘
thesupp:=matrix([[0,0,1,1,2,2],[0,1,0,2,1,2]]): sort_vecs(0,thesupp):
THESUPPARR:=array(1..6,[%,%,%,%,%,%]):
bivar(eval(THESUPPARR),[[1,2],[1,2],[1,2]],[1,1,1],[5/600,1/300],
  [map(w->w*10^7,[-1,-100/101,1/99,1/101,100/99,1]),
  [0,3/2,1,2,3/2,1],[0,1,2,1,1,4/3]]):
```

Suppose that the last polygon is scaled by 2. Then the mixed volumes become 12, 12 and 6, so the total degree of the sparse resultant is 30. Our code constructs a 33×33 matrix, whereas the algorithm of Zhang and Goldman (2000) would give a matrix of dimension 84 and no guarantee that it is not singular; see also Table 1. The corresponding MAPLE session is the following, using the variables from the session above.

```
sort_vecs(0,thesupp):
THESUPPARR:=array(1..6,[%,%,scalarmul(sort_vecs(0,thesupp),2),
  %,%,scalarmul(sort_vecs(0,thesupp),2)]):
bivar(eval(THESUPPARR),[[1,2],[1,2],[1,2]],[1,1,2],[5/600,1/300],
  [map(w->w*10^7,[-1,-100/101,1/99,1/101,100/99,1]),
  [0,3/2,1,2,3/2,1],[0,2,4,2,2,8/3]]):
```

EXAMPLE 7.3. Galligo and Stillman arrive at the following system in order to study the self-intersections of a parametrized surface in $\mathbb{R}^3$:

$$f_0 = a_0 + a_1x + a_2y + a_3xy + a_4x^2y + a_5x^2y^2 + a_6x^3y + a_7x^3y^2,$$
$$f_1 = b_0 + b_1x + b_2y + b_3xy + b_4x^2y + b_5x^2y^2 + b_6x^3y,$$
$$f_2 = c_0 + c_1x + c_2y + c_3xy + c_4x^2y + c_5x^3y + c_6x^3y^2.$$

Here $Q_1, Q_2 \subset Q_0$. To apply our algorithm we take three polytopes equal to Q_0, and at the end specialize certain coefficients to zero. With $\delta = \left(\frac{1}{10}, \frac{1}{500}\right), b = (0,1), \omega_0(3,1) = \omega_0(2,2) = 100, \omega_0(1,0) = 0$ we obtain a 22×22 matrix. The matrix determinant has total degree 24, which equals the degree of the sparse resultant which is $8+8+8 = 24$, hence we obtain the exact resultant despite the fact that the input polygons were actually not identical. Furthermore, $|\mathcal{E}| = 35$ for some δ; the greedy version of Canny and Emiris (2000) may yield a matrix with dimension as small as 30.

EXAMPLE 7.4. Consider the system of Figure 7. Q_1, Q_2 are each subdivided into three linearly lifted cells. The vertex liftings which are not explicit can be deduced by the liftings of three points in the same cell. δ is a scaled-down multiple of vector $(50, 1)$ so that any point is perturbed to a maximal cobounding cell. This leaves the point $(6, 10)$, circled in the figure, corresponding to f_0 s.t. $(6, 10) - b + Q_0$ intersects the δ-boundary of Q; this precisely corresponds to the Jacobian. The supports of $G(f)$ and toric Jacobian $J(f)$ have cardinalities 5 and 30.

Table 1 starts with homogeneous systems of degree d. The method of Zhang and Goldman (2000) yields a matrix of dimension $6d^2 - 3s$, where s is the number of integer points in the unique rectangle that can be cut off, namely the rectangle with vertices (d, d) and $(\lfloor d/2 \rfloor + 1, \lceil d/2 \rceil)$. The next rows regard bihomogeneous systems of degrees $d_1, d_2 \geq 1$, and the Hirzebruch surface of Example 2.5 from Cattani *et al.* (1998). The row labeled (Zhang and Goldman, 2000) refers to the example of that paper, namely $f_0 = 2s + t, f_1 = st + st^2, f_2 = s^2t + 2t$; we study the equivalent system, in the toric context, with supports contained in $\{(0,0), (1,0), (0,1), (2,0)\}$. The next row regards surface implicitization from (Manocha, 1992, p. 64) with supports $\{(1,0), (2,0), (0,1)\}$, $\{(1,0), (0,1), (0,2)\}$, $\{(1,0), (0,1), (1,1)\}$, where the eliminated variables are the two parameters, after having set $u = 1$. We took the union of these supports as W, then specialized the missing coefficients to zero. The matrix from Zhang and Goldman (2000) has dimension $6 \cdot 2 \cdot 2 - 3(1+2) = 15$ because the degree in each variable is 2 and we can cut off two rectangles, with point cardinality 1 and 2 respectively. The last rows correspond to examples discussed in this paper.

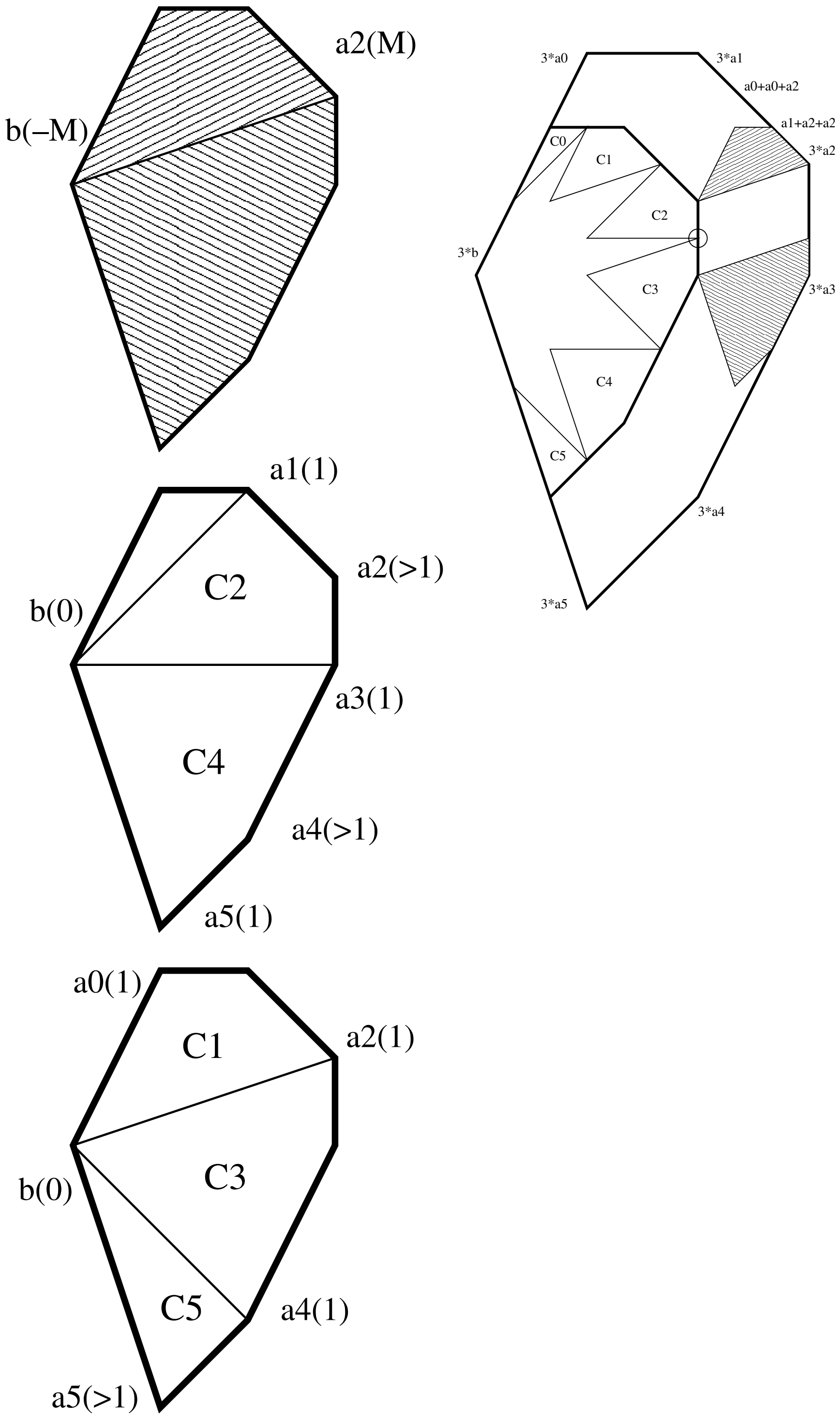

Figure 7. Example 7.4 with the lifting values shown in parentheses. Here, $r = 2$. To the right is Minkowski sum Q and certain of the Δ_ω cells. The two shaded cells are those from Q_0.

Table 1. Comparison of resultant matrices.

	deg$R$	Canny and Emiris (2000)	Cattani *et al.* (1998)	Zhang and Goldman (2000)	This paper
d-Homogeneous	$3d^2$	$\frac{9}{2}d^2 - \frac{3}{2}d$	$6d^2 - 9d + 4$	$6d^2 - 3\lceil \frac{d}{2} \rceil \lfloor \frac{d+2}{2} \rfloor$	$\frac{9}{2}d^2 - \frac{9}{2}d + 1$
(d_1, d_2)-Bihomog.	$6d_1d_2$	$9d_1d_2$	$[12d_1d_2$ $-6(d_1 + d_2) + 4]$	$6d_1d_2$	$[9d_1d_2$ $-3(d_1 + d_2) + 1]$
Cattani *et al.* (1998)	12	15	10	12	10
Zhang and Goldman (2000)	5	5	4	6	4
Manocha (1992)	7	12	7	15	7
Example 7.2	18	26	22	18	16
Example 7.2 scaled	30	46	47	84	33
Example 7.3	24	30	34	24	22
Example 7.4	57	92	97	63	64

The table columns give the total degree of the sparse resultant in the polynomial coefficients, an upper bound or the dimension of the matrix of Canny and Emiris (2000), the large dimension of the (rectangular) matrix from Cattani *et al.* (1998), namely $1 + \sum_i |\mathcal{F}_i|$, the dimension of the matrix from Zhang and Goldman (2000), and, finally, the dimension of the matrix built by our code or $|\mathcal{F}|$, which upper bounds this dimension. In some of these algorithms, it is necessary to fill the supports with zero coefficients in order to satisfy the hypotheses; in this case, there is no guarantee that the matrix determinant is nonzero. We plan further applications on scaled Newton polygons that should emphasize the merits of our construction.

An important parameter in comparisons is the degree of the matrix determinant. For all methods yielding square matrices, this equals the dimension of matrix $\mathbb{M}$. For our method, as well as Cattani *et al.* (1998), $\deg(\det \mathbb{M}) = 2 + \dim \mathbb{M}$. For homogeneous systems, Macaulay's matrix is obtained by Canny and Emiris (2000). For bihomogeneous systems, an optimal sparse resultant matrix exists.

Acknowledgements

We thank R. N. Goldman for specifying the corner-cutting method for d-homogeneous systems and the anonymous referees for their suggestions. The first author was partially supported by Action A00E02 of the ECOS-SeTCIP French-Argentina bilateral collaboration and UBACyT TX094 and ANPCyT 3-6568, Argentina. The second author was partially supported by Action A00E02 of the ECOS-SeTCIP French-Argentina bilateral collaboration and FET Open European Project IST-2001-35512 (GAIA-II).

References

Aries, F, Senoussi, R. (2001). An implicitization algorithm for rational surfaces with no base points. *J. Symb. Comput.*, **31**, 357–365.

Canny, J. F., Emiris, I. Z. (May 2000). A subdivision-based algorithm for the sparse resultant. *J. ACM 47*, **3**, 417–451.

Canny, J., Pedersen, P. (1993). An algorithm for the Newton resultant. In *Tech. Rep. 1394, Comp. Science Dept*, Cornell University.

Cattani, E., Cox, D., Dickenstein, A. (1997). Residues in toric varieties. *Composit. Math.*, **108**.

Cattani, E., Dickenstein, A., Sturmfels, B. (1998). Residues and resultants. *J. Math. Sci. Univ. Tokyo*, **5**, 119–148.

Cox, D., Little, J., O'Shea, D. (1998). *Using Algebraic Geometry,* volume 185 of *Graduate Texts in Mathematics*, New York, Springer.

D'Andrea, C. (2002). Macaulay-style formulas for the sparse resultant. *Trans. AMS*, to appear.
D'Andrea, C., Emiris, I. Z. (2001). Hybrid resultant matrices of bivariate polynomials. In *Proc. ACM Intern. Symp. on Symbolic and Algebraic Computation*, pp. 24–31. ACM Press.
Emiris, I. Z. (1996). On the complexity of sparse elimination. *J. Complexity*, **12**, 134–166.
Gelfand, I., Kapranov, M., Zelevinsky, A. (1994). *Discriminants and Resultants.* Boston, MA, Birkhäuser.
Jouanolou, J-P. (1997). Formes d'inertie et résultant: Un formulaire. *Adv. Math.*, **126**, 119–250.
Khetan, A. Determinantal formula for the Chow form of a toric surface. In *Proc. ACM Intern. Symp. on Symbolic and Algebraic Computation*, ACM Press, to appear.
Manocha, D. (May 1992). Algebraic and numeric techniques for modeling and robotics. Ph.D. Thesis, Comp. Science Div., Department of Electrical Engineering and Computer Science, University of California, Berkeley.
Sturmfels, B. (1994). On the Newton polytope of the resultant. *J. Algebr. Combin.*, **3**, 207–236.
Zhang, M., Goldman, R. (2000). Rectangular corner cutting and Sylvester A-resultants. In *Proc. ACM Intern. Symp. on Symbolic and Algebraic Computation*, pp. 301–308. ACM Press.

Received 12 November 2001
Accepted 30 March 2002