



Exact meron black holes in four dimensional $SU(2)$ Einstein–Yang–Mills theory

Fabrizio Canfora^{a,b}, Francisco Correa^a, Alex Giacomini^{c,*}, Julio Oliva^{c,d}

^a Centro de Estudios Científicos (CECs), Valdivia, Chile

^b Universidad Andrés Bello, Av. República 440, Santiago, Chile

^c Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile

^d Universidad de Buenos Aires FCEN-UBA, Ciudad Universitaria, Pabellón 1, 1428, Buenos Aires, Argentina

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ABSTRACT

In this Letter an intrinsically non-Abelian black hole solution for the $SU(2)$ Einstein–Yang–Mills theory in four dimensions is constructed. The gauge field of this solution has the form of a meron whereas the metric is the one of a Reissner–Nordström black hole in which, however, the coefficient of the $1/r^2$ term is not an integration constant. Even if the stress-energy tensor of the Yang–Mills field is spherically symmetric, the field strength of the Yang–Mills field itself is not. A remarkable consequence of this fact, which allows to distinguish the present solution from essentially Abelian configurations, is the Jackiw–Rebbi–Hasenfratz–t Hooft mechanism according to which excitations of Bosonic fields moving in the background of a gauge field with this characteristic behave as Fermionic degrees of freedom.

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1. Introduction

The Yang–Mills (YM) action is one of the main ingredients of the standard model which up to now has been phenomenologically extremely successful. According to General Relativity, a Yang–Mills field contributes as any other field to the curvature of the space–time. There are several physically relevant situations (for instance, close to a neutron star or a black hole, in the early cosmology) where the gravitational fields are extremely strong and the effects of curvature on the propagation of matter fields, as well as the back-reaction of the Yang–Mills fields cannot be neglected.

The self-gravitating Yang–Mills field is of great theoretical interest in black hole physics. Indeed, non-Abelian gauge fields are known to violate the celebrated “no hair conjecture”. This means that there exist black hole configurations with a non-Abelian gauge field which however does not contribute to the conserved charges [1]. At least in four dimensional Einstein–YM theory such hairy black holes have been found only numerically. It has been shown that there exist also $SU(2)$ Reissner–Nordström-like black holes [2].

Up to now in the Einstein–YM system, in four dimensions, in spite of the great effort in constructing numerical solutions [1] and in making rigorous proofs of existence of genuine non-Abelian solutions [2] very few exact, intrinsically non-Abelian solutions are

known (for two detailed reviews, see Refs. [3] and [4]). In particular, within the family considered in [1] and [2] only one exact solution of a Reissner–Nordström black hole can be constructed in which the corresponding Yang–Mills field is gauge equivalent to a potential with only one of its three $SU(2)$ generators switched on. This means that the mentioned solution actually belongs to an Abelian sector of the theory.

On the other hand, it would be of great importance to have an exact Yang–Mills black hole solution which is genuinely non-Abelian and which therefore captures the most relevant characteristic features of Yang–Mills theory, since many of the available results are numerical [1,5–8].

A good strategy to construct non-Abelian black holes is to consider an ansatz for the Yang–Mills field which is both intrinsically non-Abelian and as simple as possible: in this Letter we will consider the meron ansatz. A meron is a field configuration which has the form $\mathbf{A} = \lambda \tilde{\mathbf{A}}$ where $\tilde{\mathbf{A}}$ is pure gauge field. In an Abelian theory a multiple of a pure gauge field is, of course, a pure gauge field as well. In a non-Abelian theory however, the field strength has also the commutator term. Hence, when $\lambda \neq 0, 1$, the meron configuration has a non-zero field strength: $\mathbf{F} = \lambda(\lambda - 1)[\tilde{\mathbf{A}}, \tilde{\mathbf{A}}]$. Therefore, the existence of merons is a genuine non-Abelian feature.

Merons, firstly introduced in [9], are configurations of the YM theory which in flat space–time attracted a lot of attention. They interpolate between different topological sectors and, in particular, it can be shown that instantons can be thought of as composed by a pair of a meron and an anti-meron [10–13]. Furthermore, at least on flat spaces, merons are quite relevant configurations as far as confinement is concerned [10,11] (for a recent discussion

* Corresponding author.

E-mail addresses: canfora@cecs.cl (F. Canfora), correa@cecs.cl (F. Correa), alexgiacomini@uach.cl (A. Giacomini), julio.oliva@docentes.uach.cl (J. Oliva).

see also [12]). It is also worth noting that the existence of merons is closely related to the presence of Gribov copies discovered in the seminal paper [14], in fact, it has been shown that one can interpret a meron as a tunneling between a two Gribov vacua.¹ Due to the fact that the pattern of appearance of Gribov copies on curved space–time may be quite different from the flat case as it has been shown in [16–18], it is natural to analyze how the curvature of space–time affects the presence of merons.

Merons are also important for another reason. A very deep feature of non-Abelian gauge theories first noticed in [19,20] is that, unlike what happens in Abelian theories, the non-Abelian field strength does not uniquely determine the non-Abelian gauge potential modulo gauge transformations and it is possible to construct many examples of *gauge potentials which are not gauge equivalent but have the same field strength* (of course, they are distinguished by higher order invariants). It has been shown in [20] that, using merons, it is easy to construct examples of non-equivalent gauge potential with the same curvature.

In this Letter we will construct an analytic black hole corresponding to the energy–momentum tensor of a meron in the case where the constant λ takes the value $1/2$, which turns out to be reminiscent of the original paper of de Alfaro, Fubini and Furlan [9]. The metric of the solution will be the one of a magnetically charged Reissner–Nordström black hole. Nevertheless, we will show that it is impossible to transform our meron black hole into the known analytic solution in [1] and [2] (which belongs to an Abelian sector) by any globally defined $SU(2)$ gauge transformation.

Furthermore, it is possible to disclose the genuine non-Abelian nature of the present black hole solution with a non-trivial physical effect. As it will be shown in the next sections, the Yang–Mills stress tensor is spherically symmetric but the field strength itself is not (unless one compensates a spatial rotation with an internal $SU(2)$ rotation). This fact is the physical origin of the *Jackiw–Rebbi–Hasenfratz–t Hooft effect* [21,22] according to which excitations of Bosonic fields charged under $SU(2)$ around the meron black hole solution are Fermionic despite to the fact that all the fundamental fields involved in the model are Bosonic. This phenomenon is not restricted to Yang–Mills theory: the earliest and most famous example is probably the (Bosonic) Skyrme field [23] (for a detailed review see [24]). Indeed, the excitations around the Skyrme soliton behaves as Fermions.

This gives the possibility of physically distinguishing our solutions from the analytic solution of the Abelian sector.

The structure of the Letter will be the following. In Section 2, a short review of the merons will be presented. Section 3 is devoted to the discussion of the hedgehog ansatz. In Section 4 the meron black hole will be analyzed. In Section 5 it will be shown that the present meron black hole is not continuously connected to any Abelian sector, the non-Abelian charges will be analyzed as well. In Section 6, we will show that the *Jackiw–Rebbi–Hasenfratz–t Hooft* mechanism is a nice observable effect able to distinguish the present black hole solution from an Abelian solution. In the final section some conclusions will be drawn.

2. A short review on merons

One of the most important features of Yang–Mills theory is the presence of topologically non-trivial configurations such as instantons, merons, monopoles and so on (see, for instance, [25]). In the present section we will focus on the computations of the energy–momentum tensor of the merons as well as of the corresponding

Yang–Mills equations. Let us consider the following action S_{YM} for the Yang–Mills system for the gauge group $SU(2)$,

$$S_{YM} = \frac{1}{2e^2} \int \sqrt{-g} d^4x \text{Tr}(F_{\mu\nu} F^{\mu\nu}), \quad (2.1)$$

where g is the determinant of the metric tensor and

$$\begin{aligned} A_\mu &= iA_\mu^i \sigma_i, \\ \sigma_i \sigma_j &= \delta_{ij} \mathbf{1} + i\epsilon_{ijk} \sigma_k, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \end{aligned}$$

Here σ_i are the Pauli matrices that we have to choose as the Hermitian generators of $su(2)$ ($[\sigma_i, \sigma_j] = 2i\epsilon_{ijk} \sigma_k$) and $\mathbf{1}$ is the 2×2 identity matrix. e is the coupling constant and the Latin letters (i, j, k) correspond to the gauge group indices. ϵ_{ijk} is the Levi-Civita symbol that fulfills the identity $\epsilon_{ijk} \epsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$.

As mentioned in the introduction, a meron is a configuration of the following form,

$$\begin{aligned} A_\mu &= \lambda U^{-1} \partial_\mu U, \quad \lambda \neq 0, 1, \\ U &= U(x^\mu) \in SU(2). \end{aligned} \quad (2.2)$$

Thus, a meron is proportional to a pure gauge term without being, of course, a pure gauge configuration. It is worth emphasizing that the existence of merons is an *intrinsically non-Abelian feature* since, obviously, in an Abelian gauge theory a gauge field which is proportional to a pure gauge is itself a pure gauge. Thus, merons only exist in non-Abelian sectors of gauge theories.²

The most famous meron configuration on flat space–time have been constructed by de Alfaro, Fubini and Furlan [9] and it has $\lambda = 1/2$. In principle λ could take any value different from zero and one. However, using a purely topological argument, we will show why $\lambda = 1/2$ is indeed a special value, even in curved space–time. Soon afterwards its discovery, it was recognized that merons are very important to explain, at least at a qualitative level, confinement [10,11]. The close relations between merons and confinement has been recently confirmed in [12] (for a review of the original arguments see [13]).

The field strength $F_{\mu\nu}$ of the meron in Eq. (2.2) is proportional to the commutator,

$$F_{\mu\nu} = \lambda(\lambda - 1) [U^{-1} \partial_\mu U, U^{-1} \partial_\nu U]. \quad (2.3)$$

In the following we will use the following standard parametrization of the $SU(2)$ -valued functions $U(x^\mu)$:

$$U(x^\mu) = Y^0 \mathbf{1} + iY^i \sigma_i, \quad U^{-1}(x^\mu) = Y^0 \mathbf{1} - iY^i \sigma_i, \quad (2.4)$$

$$Y^0 = Y^0(x^\mu), \quad Y^i = Y^i(x^\mu), \quad (2.5)$$

$$(Y^0)^2 + Y^i Y_i = 1, \quad (2.6)$$

where the sum over repeated indices is understood also in the case of the group indices (in which case the indices are raised and lowered with the flat metric δ_{ij}). Therefore, the meron gauge field in Eq. (2.2) can be written as follows,

$$A_\mu = i\lambda P_\mu^k \sigma_k, \quad (2.7)$$

$$P_\mu^k = \epsilon_{ijk} Y_i \partial_\mu Y_j + Y^0 \partial_\mu Y^k - Y^k \partial_\mu Y^0. \quad (2.8)$$

² This fact has the following practical advantage: when one searches for exact solutions of the Einstein–Yang–Mills system, there is always the risk that, by simplifying too much the gauge potential, at the end one reduces A_μ to an Abelian gauge field (namely, a configuration in which the commutator in the field strength vanishes). This point will be analyzed in more details in the next sections.

¹ See the reviews [13] and [15].

In order to determine the energy–momentum tensor of the meron field it is useful to compute the following quadratic combination,

$$\delta_{mn} P_\mu^m P_\alpha^n = G_{ij} \partial_\mu Y^i \partial_\alpha Y^j, \quad (2.9)$$

$$G_{ij} = G_{ij}(\vec{Y}) = \left(\delta_{ij} + \frac{Y_i Y_j}{1 - Y^k Y_k} \right), \quad (2.10)$$

where G_{ij} is the metric corresponding to the group manifold which, in the present case is S^3 . It is worth to note here that if one considers a configuration in which Y^0 vanishes, then the internal metric $G_{ij}(\vec{Y})$ reduces to the δ_{ij} ,

$$Y^0 = 0 \Rightarrow Y^k Y_k = 1 \Rightarrow \quad (2.11)$$

$$G_{ij}(\vec{Y}) = \delta_{ij}. \quad (2.12)$$

The energy–momentum tensor for the Yang–Mills field reads

$$T_{\mu\nu} = \frac{1}{e^2} \text{Tr} \left(-F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} + \frac{g^{\mu\nu}}{4} F_{\lambda\sigma} F^{\lambda\sigma} \right), \quad (2.13)$$

and using Eqs. (2.9) and (2.10) the energy–momentum tensor for the merons reduces to

$$T_{\mu\nu} = \xi \left[(g^{\alpha\beta} G_{ij} \partial_\alpha Y^i \partial_\beta Y^j) G_{mn} \partial_\mu Y^m \partial_\nu Y^n - g^{\alpha\beta} (G_{ij} G_{mn} \partial_\mu Y^i \partial_\beta Y^j \partial_\nu Y^m \partial_\alpha Y^n) - \frac{g^{\mu\nu}}{4} ((g^{\alpha\beta} G_{ij} \partial_\alpha Y^i \partial_\beta Y^j)^2 - g^{\alpha\beta} g^{\lambda\sigma} (G_{ij} G_{mn} \partial_\lambda Y^i \partial_\beta Y^j \partial_\sigma Y^m \partial_\alpha Y^n)) \right], \quad (2.14)$$

where

$$\xi = \frac{8(\lambda(\lambda - 1))^2}{e^2}.$$

Finally, the Yang–Mills equations for the meron field read

$$\varepsilon_{lmn} \nabla^\nu (P_\mu^m P_\nu^n) - 2\lambda (\varepsilon_{kjm} \varepsilon_{lik}) P^{iv} P_\mu^j P_\nu^m = 0. \quad (2.15)$$

3. The hedgehog ansatz

In the following we will consider the spherically symmetric hedgehog ansatz for the meron field in terms of a group valued function U . The notion of spherical symmetry in which one gets *spherical symmetry only up to an internal $SU(2)$ rotation* is the one introduced³ in [26] (see also [27]) and, as it will be explained in Section 6, it is responsible for the appearance of the Jackiw–Rebbi–Hasenfratz–’t Hooft effect [21,22]. In terms of the group element U it reads

$$\begin{aligned} U &= \mathbf{1} \cos f(r) + i \hat{\chi}^i \sigma_i \sin f(r), \\ U^{-1} &= \mathbf{1} \cos f(r) - i \hat{\chi}^i \sigma_i \sin f(r), \\ \delta_{ij} \hat{\chi}^i \hat{\chi}^j &= 1, \end{aligned} \quad (3.1)$$

where $\hat{\chi}^j$ is the unit radial vector (normalized with respect to the internal metric δ_{ij}). The hedgehog ansatz corresponds to the following choice,

$$\begin{aligned} Y^0 &= \cos f(r), & Y^i &= \hat{\chi}^i \sin f(r), \\ \hat{\chi}^1 &= \sin \theta \cos \phi, & \hat{\chi}^2 &= \sin \theta \sin \phi, & \hat{\chi}^3 &= \cos \theta. \end{aligned} \quad (3.2)$$

³ This definition is locally but not globally equivalent to the one which is commonly adopted in the analysis of colored black holes.

Thus, the meron gauge field in Eq. (2.7) in this case reads as follows,

$$A_\mu = i\lambda P_\mu^k \sigma_k, \quad (3.3)$$

$$P_\mu^k = \sin^2 f \varepsilon_{ijk} \hat{\chi}^i \partial_\mu \hat{\chi}^j + \hat{\chi}^k \partial_\mu f + \frac{\sin(2f)}{2} \partial_\mu \hat{\chi}^k. \quad (3.4)$$

As it will be discussed in the next section, one can obtain an exact solution of the Einstein–Yang–Mills system in the case in which

$$f(r) = \frac{\pi}{2} \Rightarrow P_\mu^k = \varepsilon_{ijk} \hat{\chi}^i \partial_\mu \hat{\chi}^j. \quad (3.5)$$

In this case the field strength $F_{\mu\nu} = iF_{\mu\nu}^k \sigma_k$ of the non-Abelian field is purely magnetic and reads

$$F_{\mu\nu}^i = Y^i \Pi_{\mu\nu}, \quad (3.6)$$

$$\Pi_{\mu\nu} = 2\lambda(\lambda - 1) (\varepsilon_{mnq} Y^m \partial_{[\mu} Y^q \partial_{\nu]} Y^n), \quad (3.7)$$

so we define the two form Π ,

$$\Pi := \frac{1}{2} \Pi_{\mu\nu} dx^\mu \wedge dx^\nu = \lambda(\lambda - 1) \varepsilon_{mnq} Y^m dY^q \wedge dY^n,$$

with $\delta_{ij} Y^i Y^j = 1$. The functions Y^i are define since Eq. (3.5) which implies

$$\begin{aligned} Y^1 &= \hat{\chi}^1 = \sin \theta \cos \phi, \\ Y^2 &= \hat{\chi}^2 = \sin \theta \sin \phi, \\ Y^3 &= \hat{\chi}^3 = \cos \theta, \end{aligned} \quad (3.8)$$

where θ and ϕ are the coordinates on the two sphere corresponding to the metric in Eq. (4.3).

3.1. The geometrical meaning of $\Pi_{\mu\nu}$

It is worth emphasizing that $\Pi_{\mu\nu}$ has the same form as an effective Abelian magnetic field strength. It is easy to see that the energy–momentum tensor corresponding to the non-Abelian field strength in Eq. (3.6) coincides with twice the (Maxwell) energy–momentum tensor of $\Pi_{\mu\nu}$. This is due to the fact that the trace over the group indices in the energy–momentum tensor eliminates the explicit factor Y^i which multiplies $\Pi_{\mu\nu}$ in Eq. (3.6) thanks to $Y_i Y_i = 1$. Thus,

$$T_{\mu\nu} = \frac{1}{e^2} \text{Tr} \left(-g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} + \frac{g^{\mu\nu}}{4} F_{\alpha\beta} F^{\alpha\beta} \right)$$

reduces to

$$T_{\mu\nu} = \frac{2}{e^2} \left(\Pi_{\mu\alpha} \Pi_{\nu\beta} - \frac{g^{\mu\nu}}{4} \Pi_{\alpha\beta} \Pi^{\alpha\beta} \right).$$

For any triple of functions Y^i satisfying the relation in $Y_i Y_i = 1$ the expression in Eq. (3.7) represents the pull-back of the area form on S^2 and its integral represents the $\pi_3(S^2)$. This implies that the two-form Π is closed,

$$d\Pi = \lambda(\lambda - 1) d(\varepsilon_{mnq} Y^m dY^q \wedge dY^n) = 0 \Rightarrow$$

$$\Pi = dA \quad \text{locally,}$$

and, therefore, $\Pi_{\mu\nu}$ satisfies the first set of Maxwell equations. Furthermore, as it is well known, the field strength F_D of the Dirac monopole reads

$$F_D = \frac{g}{4\pi} \sin \theta d\theta \wedge d\phi, \quad (3.9)$$

where g is the magnetic charge and the field strength F_D is proportional to the volume form of S^2 . Thus, it turns out that the effective Abelian field defined in Eq. (3.7) is proportional to F_D ,

$$\frac{\Pi}{2\lambda(\lambda - 1)} = -\frac{4\pi}{g}F_D = -\sin\theta d\theta \wedge d\phi, \quad (3.10)$$

and so the effective Abelian field strength $\Pi_{\mu\nu}$ defined in Eq. (3.7) automatically satisfies also the second set of Maxwell equations with a δ -like source.

4. The black hole solution

The Einstein equations derived from the action,

$$S[g_{\mu\nu}, A_\mu] = \int d^4x \sqrt{-g} \left(\frac{R - 2\Lambda}{\kappa} + \frac{1}{2e^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \right), \quad (4.1)$$

read

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\kappa}{e^2} T_{\mu\nu} \quad (4.2)$$

(κ and e being the Newton and Yang–Mills coupling constants respectively). The energy–momentum tensor $T_{\mu\nu}$ is given in Eq. (2.13). Let us consider a four dimensional metric of the form

$$ds^2 = -\exp(2a(r)) dt^2 + \exp(2b(r)) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad 0 \leq r < \infty, \quad 0 \leq t < \infty. \quad (4.3)$$

Since, as it has been already explained, the energy–momentum tensor corresponding to the above meron field strength in Eqs. (3.6), (3.7) coincides with the energy–momentum tensor of a Dirac monopole, the coupled Einstein–Yang–Mills system of equations (both with and without cosmological constant) is solved, for any value of the meron parameter λ , by the magnetic Reissner–Nordström black hole metric

$$ds^2 = -\left(1 - \frac{\kappa M}{8\pi r} + \frac{4\kappa\lambda^2(\lambda - 1)^2}{e^2 r^2} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{1 - \frac{\kappa M}{8\pi r} + \frac{4\kappa\lambda^2(\lambda - 1)^2}{e^2 r^2} - \frac{\Lambda r^2}{3}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.4)$$

where M is the ADM mass. However, the Yang–Mills equations have not been solved yet: it will be now shown that the Yang–Mills equations corresponding to the present meron ansatz fix the value of λ as in the original de Alfaro–Fubini–Furlan paper, namely $\lambda = 1/2$. Hence, unlike the Abelian case, λ is not an integration constant but is fixed to be $1/2$.

The Yang–Mills equations read:

$$YM_\mu = \nabla^\nu F_{\mu\nu} + [A^\nu, F_{\mu\nu}] = 0,$$

$$[A^\nu, F_{\mu\nu}]^i = -2i\lambda \Pi_{\mu\nu} \nabla^\nu Y^i$$

(∇^ν being the Levi-Civita covariant derivative corresponding to the metric in Eq. (4.3)) so that they reduce to

$$\Pi_{\mu\nu} \nabla^\nu Y^i - 2\lambda \nabla^\nu (\Pi_{\mu\nu} Y^i) = 0. \quad (4.5)$$

Furthermore, $\Pi_{\mu\nu}$ is proportional to the field strength of the Dirac monopole and so it satisfies the Maxwell equations (outside the δ -source)

$$\nabla^\nu \Pi_{\mu\nu} = 0,$$

therefore Eq. (4.5) can be written as

$$(1 - 2\lambda)\Pi_{\mu\nu} \nabla^\nu Y^i = 0, \quad (4.6)$$

whose non-trivial components are

$$YM_\theta^1 = r^{-2} \sin\phi \left(\lambda - \frac{1}{2}\right),$$

$$YM_\phi^1 = r^{-2} \cos\phi \sin\theta \cos\theta \left(\lambda - \frac{1}{2}\right), \quad (4.7)$$

$$YM_\theta^2 = r^{-2} \cos\phi \left(\lambda - \frac{1}{2}\right),$$

$$YM_\phi^2 = r^{-2} \cos\phi \sin\theta \cos\theta \left(\lambda - \frac{1}{2}\right), \quad (4.8)$$

$$YM_\phi^3 = r^{-2} \sin^2\theta \left(\lambda - \frac{1}{2}\right). \quad (4.9)$$

Therefore, all the above equations are simultaneously satisfied if and only if

$$\lambda = \frac{1}{2}. \quad (4.10)$$

To the best of authors knowledge, the above argument provides an additional explanation of why the value $\lambda = 1/2$ is special in the present Lorentzian meron which is reminiscent of the more well-known Euclidean ones.

It is worth emphasizing that, even on flat spaces, merons present singularities (see, for instance, [10,11,13]) and so they play an important although indirect role as building blocks of the instantons but they cannot be observed directly due to their singularities (which, in the present case, are manifest in Eq. (5.5)). However, one of the most interesting results of the present analysis is that the meron singularity is hidden behind the black hole horizon and, consequently, in a gravitational context, merons could be observed directly in principle.

The causal structure of the black hole solution corresponds to the one of the Reissner–Nordström–(A)dS space–time. However, an essential difference between the present and the Abelian Reissner–Nordström solutions is that in this case the coefficient of the $1/r^2$ term in the lapse function is not an integration constant. Its value is fixed to $1/4$ which is the square of the non-Abelian magnetic charge of the configuration.

For $\Lambda = -\frac{3}{l^2} < 0$, there is an event horizon provided $M \geq M_c$, where M_c is a critical mass. M_c is related with the minimum radius of the horizon r_c as follows,

$$r_c^2 = \frac{l^2}{6} \left[\sqrt{1 + \frac{3\kappa}{e^2 l^2}} - 1 \right], \quad M_c = \frac{16\pi r_c}{k} \left[1 + \frac{2r_c^2}{l^2} \right].$$

When the lower bound of the mass is achieved, the event and the Cauchy horizons coincide and the black hole is extremal. In the asymptotically flat case ($\Lambda = 0$) the causal structure is similar to the asymptotically AdS case with

$$r_c^{\text{flat}} = \frac{\sqrt{\kappa}}{2e}, \quad M_c^{\text{flat}} = \frac{4\pi}{e^2 r_c^{\text{flat}}}.$$

Finally, in the asymptotically de Sitter case (with $\Lambda = \frac{3}{l^2} > 0$), if we define r_{\min} and r_{\max} respectively by

$$r_{\min}^2 = \frac{l^2}{6} \left[1 - \sqrt{1 - \frac{3\kappa}{e^2 l^2}} \right], \quad r_{\max}^2 = \frac{l^2}{6} \left[1 + \sqrt{1 - \frac{3\kappa}{e^2 l^2}} \right],$$

the minimum and maximum masses read

$$M_{\min} = \frac{16\pi r_{\min}}{k} \left[1 - \frac{2r_{\min}^2}{l^2} \right],$$

$$M_{\max} = \frac{16\pi r_{\max}}{k} \left[1 - \frac{2r_{\max}^2}{l^2} \right].$$

When $M < M_{\min}$ the space-time represents a naked singularity. For $M = M_{\min}$ the space-time represents an extremal black hole surrounded by a cosmological horizon, giving rise to what is known as a “lukewarm” black hole. When $M_{\min} < M < M_{\max}$ there is a Cauchy and an event horizon, both surrounded by the cosmological horizon. Finally for $M = M_{\max}$ the event and the cosmological horizons coincide. For masses above this value, the space-time represents again a naked singularity.

5. The present meron-black hole as a genuine non-Abelian configuration

We have seen that both the metric of the meron black hole and the corresponding energy-momentum tensor look like a magnetically charged Reissner-Nordström black hole. This fact, at a first glance, may give the impression that this solution is gauge equivalent to an Abelian solution.

In particular, within the family of configurations analyzed in [2] to prove non-trivial existence theorems (as well as in [1] to construct numerically hairy colored black hole), there is a configuration which allows to construct an analytic black hole solution but it belongs to an Abelian sector of Yang-Mills theory. The corresponding magnetic non-Abelian field strength reads

$$\mathbf{F}_{\mu\nu} = i\Pi_{\mu\nu}\sigma_3 \tag{5.1}$$

where $\Pi_{\mu\nu}$ is the one defined in Eq. (3.9) and σ_3 is a fixed generator of the algebra of $SU(2)$. Thus, to obtain such a field strength it is enough to consider a gauge potential with only one generator (namely, σ_3) turned on in such a way that the commutators both in the Yang-Mills and in the Einstein equations vanish and the solution reduces globally to an Abelian black hole.

On the other hand, as far as the solution constructed in the present Letter is concerned, the gauge field is a meron: an intrinsically non-Abelian object. This observation by itself strongly suggests that present solution cannot be gauge transformed to an Abelian sector. We will now present two rigorous arguments which proves that there is no continuous gauge transformation connecting our solution with an Abelian sector.

The first argument is the following: let us compare the field strength in Eq. (5.1) with the field strength of the present solution (see Eqs. (3.6) and (3.7)) which reads

$$\mathbf{F}_{\mu\nu} = iF_{\mu\nu}^i\sigma_i = i\Pi_{\mu\nu}Y^i\sigma_i \equiv i\Pi_{\mu\nu}\sigma_r, \tag{5.2}$$

$$i\Pi_{\mu\nu}Y^i\sigma_i \equiv i\Pi_{\mu\nu}\sigma_r, \tag{5.3}$$

where we have introduced the radial Pauli matrix $\sigma_r \equiv Y^i\sigma_i$ and the Y^i are defined in Eq. (3.8). At a first glance, the two field strengths in Eqs. (5.1) and (5.2) look similar since they are both proportional to $\Pi_{\mu\nu}$ (which is the field strength of a Dirac monopole). However, in the first case in Eq. (5.1) $\Pi_{\mu\nu}$ multiplies a constant generator of the algebra of $SU(2)$ while in the meron case $\Pi_{\mu\nu}$ multiplies the radial Pauli matrix σ_r which is a non-trivial and non-constant combination of the generators of $SU(2)$. Therefore, if the solution constructed in the present Letter would be equivalent to an Abelian configuration then one should be able to find a smooth gauge transformation $U(x) \in SU(2)$ such that

$$U^{-1}(\Pi_{\mu\nu}\sigma_r)U = \Pi_{\mu\nu}\sigma_3 \Leftrightarrow U^{-1}\sigma_r U = \sigma_3, \tag{5.4}$$

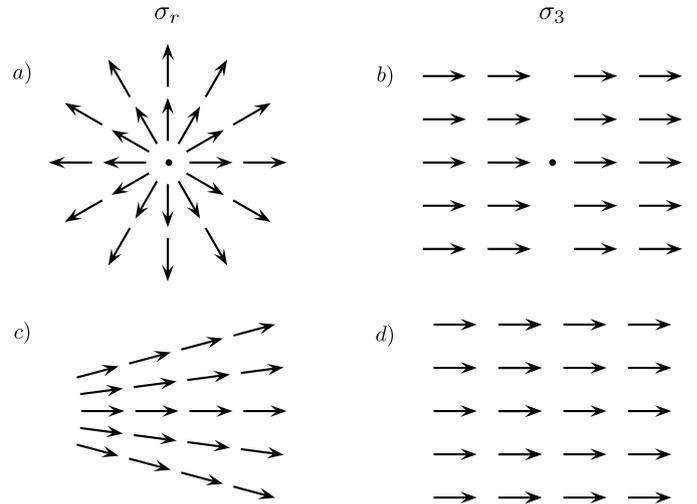


Fig. 1. Orientation of the field strength in the isospin space. (a) and (b) represent the neighborhoods of the origin of the present meron solution and an effective Abelian solution, respectively. It is not possible to transform (a) into (b) by a local continuous rotation. On the other hand, in a neighborhood which does not include the origin ((c) and (d)) it is possible to transform one field strength into the other by a local rotation.

where we have used the fact that in the expression of $\Pi_{\mu\nu}$ in Eq. (3.7) all the internal indices are contracted so that the gauge transformation only acts on σ_r . We will now show that no such U can exist.

The radial Pauli matrix in Eq. (5.3) points outwards in the radial direction of the inner space at every point of the physical space. The σ_3 generator (corresponding to the field strength in Eq. (5.1)) at every point of the physical space points in the same direction of the inner space (see Fig. 1). Hence, if one considers a small enough neighborhood of the origin in the physical space, one can see that any gauge transformation $U(x, y, z)$ satisfying Eq. (5.4) is necessarily discontinuous. Indeed (introducing a Cartesian coordinates system around the origin) one can see that, for instance, the radial Pauli matrix in Eq. (5.3) behaves as follows,

$$\forall \varepsilon > 0: \sigma_r(0, 0, \varepsilon) = \sigma_3, \quad \sigma_r(0, 0, -\varepsilon) = -\sigma_3, \tag{5.5}$$

where $(0, 0, \varepsilon)$ and $(0, 0, -\varepsilon)$ are two points (symmetrically placed with respect to the origin) along the z -axis. Therefore, due to Eq. (5.4), one should require the following condition on $U(x, y, z)$ (see Fig. 1):

$$\forall \varepsilon > 0: (U(0, 0, \varepsilon))^{-1}\sigma_3(U(0, 0, \varepsilon)) = \sigma_3, \tag{5.6}$$

$$(U(0, 0, -\varepsilon))^{-1}\sigma_3(U(0, 0, -\varepsilon)) = -\sigma_3, \tag{5.7}$$

so that $U(x, y, z)$ cannot be continuous since the quantity $(U(0, 0, \varepsilon))^{-1}\sigma_3(U(0, 0, \varepsilon))$ is not continuous with respect to ε around $\varepsilon = 0$. This obviously implies that the field strengths in Eqs. (5.1) and (5.2) are not continuously connected.

Since in the present case the origin of the coordinates is a singularity, it is natural to wonder whether if, when removing a small spherical region around the singularity, the gauge transformation leading from one configuration to the other remains singular. Indeed, for topological reasons (see for instance [26]) the transformation must be singular at least on one point of each sphere of fixed radius r . A more direct way to see this is the following. Any gauge transformation which transforms the field strength in Eq. (5.2) into the one in Eq. (5.1) and at the same time the meron gauge potential in Eqs. (3.3), (3.4) and (3.5) into an Abelian gauge

potential with only one generator turned is actually singular along the whole z -axis and not only at the origin. The reason is that the meron gauge potential is only singular at $r = 0$ while the gauge potential of the Dirac monopole corresponding to the field strength in Eq. (5.1) is singular along the whole z -axis. Therefore, any gauge transformation of the above type must be singular along the z -axis as well. A very famous example in General Relativity of two configurations which are connected by a gauge transformation which is not globally well defined are the $2+1$ dimensional AdS space-time and the BTZ black hole [28]. This example clearly shows that improper gauge transformations have in general non-trivial physical effects since $2+1$ AdS is the vacuum of the theory while the BTZ black hole has temperature and entropy and therefore they cannot be gauge equivalent.

A second easier way to prove that the present Einstein–Yang–Mills configuration is genuinely non-Abelian is the following. Once one fixes $f = \pi/2$ in Eqs. (3.3) and (3.4), the field strength of the meron configuration has the form in Eqs. (3.6) and (5.2) for any value of λ . However, if it would exist a regular gauge transformation transforming the present meron gauge potential in Eqs. (3.3), (3.4) and (3.5) into an Abelian gauge potential with only one generator turned on, then one could solve trivially the Yang–Mills equations for any value of λ (since they would reduce to the Maxwell equations) and then one could go back to the meron form in Eqs. (3.3), (3.4) and (3.5). However, in the previous section we showed that the Yang–Mills equations are satisfied if and only if $\lambda = 1/2$.

5.1. The non-Abelian charges

The fact that the meron field and the Abelian configuration produce the same stress tensor and therefore the same metric would lead to think that physics is not able to distinguish between the Abelian and meron black hole configuration. However, globally the Abelian and the non-Abelian black hole configurations are different as it is apparent in the computations of non-local quantities like Wilson loops (in particular, the radial Pauli matrix at a point does not commute in general with the radial Pauli matrix at another point). In the next section we will discuss a more direct physical effect which is able to reveal the non-Abelian nature of the present solution distinguishing it from an Abelian black hole configuration. Here we will discuss the non-Abelian charges of the configuration.

It is worth emphasizing here that the non-Abelian charges are gauge invariant only under *proper gauge transformations* (namely, everywhere smooth gauge transformations which approach the center of the gauge group at spatial infinity). Indeed, in the non-Abelian case, if the gauge transformation does not approach the center of the gauge group at infinity, the charges defined as surface integrals of the non-Abelian fluxes at infinity may change (for a detailed review on the concept of charges in Yang–Mills theory see [29]). Therefore, in order for the concept of non-Abelian hair to be well defined, the only allowed gauge transformations have to be proper gauge transformations. Of course, any transformation which maps the field strength (3.6) into an Abelian one is improper for two reasons: it is singular at the origin and it cannot approach the center of the gauge group at infinity.

The classic definition of non-Abelian charge is in [30] (see also [31,32]). The first step is to find an $SU(2)$ -valued covariantly constant scalar ξ^i ,

$$D_\mu \xi^i = 0,$$

where D_μ is the $SU(2)$ covariant derivative. Then, with this covariantly constant scalar one can construct fluxes which are conserved in the ordinary sense by contracting the field strength (or its dual)

with ξ^i . In the present case, it is easy to see that the Y^i in Eq. (3.8) are covariantly constant with respect to the gauge field in Eqs. (3.3) and (3.5). Thus, following [30] and [32], the charge $Q = Q(Y^i)$ is the integral over the 2-sphere at infinity of the non-Abelian magnetic field B_μ^i contracted with Y^i ,

$$Q = \frac{1}{4\pi} \int_{S_\infty^2} B_\mu^i Y_i n^\mu = -\frac{1}{8\pi} \int_{S_\infty^2} \Pi_{\theta\phi} d\theta d\phi = -\frac{1}{2},$$

where n^μ is the unit normal to S_∞^2 and we used Eqs. (3.9) and (3.10). On the other hand, the electric charges vanish identically.

It is interesting to note that if one would compute the non-Abelian magnetic charges Q_M^i as surface integrals at infinity without projecting the magnetic field along the covariantly constant scalar Y_i as

$$Q_M = Q_M^i \sigma_i = \frac{1}{4\pi} \int_{S_\infty^2} F, \tag{5.8}$$

one would get a different result. In the case of the field strength in Eqs. (3.6), (3.7) and (3.8), the above expression reduces to

$$Q_M^i \sigma_i = \frac{1}{2\pi} \sigma_i \int_{S_\infty^2} Y^i(\theta, \phi) \Pi_{\theta\phi} d\theta d\phi, \tag{5.9}$$

and due to the presence of the functions $Y^i(\theta, \phi)$ (whose expressions are in Eq. (3.8)) the Q_M^i would vanish for all the components of the internal $su(2)$ index i .

However, here it is more appropriate the first approach. From the physical point of view, the idea to project the magnetic field along the Y_i corresponds to measure the charge with respect to the radial Pauli matrix defined in Eq. (5.3).

6. Jackiw–Rebbi–Hasenfratz–’t Hooft mechanism

We have already discussed that the physical origin behind the genuine non-Abelian nature of the present solution is the non-trivial realization of spherical symmetry. Namely, even if the energy–momentum tensor is spherically symmetric, the field strength in Eq. (3.6) is not spherically symmetric since a spatial rotation does not change $\Pi_{\mu\nu}$ (which is the pull-back of the volume form of the two-sphere) but it *does change* the unit radial vector \vec{Y} in Eq. (3.8). Unlike what happens in Abelian sectors in which the field strength is directly spherically symmetric (see Eq. (5.1)), the present meron field strength is spherically symmetric only up to an internal $SU(2)$ rotation which compensates for the spatial rotation in order to keep Eq. (3.6) invariant.

To see this one can look at the field strength of the meron field in Eq. (3.6): such curvature is composed by two factors. The first factor is the field strength of the Dirac monopole which is invariant under spatial rotations. The second factor however is the combination $\sigma_r = Y^i \sigma_i$ which is not invariant under spatial rotations since the Y^i transform as a vector

$$Y^i \rightarrow R_j^i Y^j, \tag{6.1}$$

where R_j^i is the element of $SO(3)$ corresponding to the spatial rotation. Consequently, σ_r transforms as follows under a spatial rotation,

$$\sigma_r \rightarrow R_j^i Y^j \sigma_i \neq \sigma_r.$$

Hence, in any equation (such as the $SU(2)$ covariant Klein–Gordon and Dirac equations) in which the field strength or the corresponding gauge potential appear explicitly as background fields, the orbital angular momentum \vec{l} will not be a symmetry operator. On the

other hand, it is possible to compensate the rotation in Eq. (6.1) with a corresponding rotation (generated by R^{-1}) of the $SU(2)$ generators in such a way to keep σ_r invariant. This means that the symmetry operator in any equation such as the $SU(2)$ covariant Klein–Gordon equation will be the total angular momentum $\vec{J} = \vec{L} + \vec{\sigma}$.

It is precisely the *spherical symmetric up to an internal $SU(2)$ rotation* which gives rise to the *Jackiw–Rebbi–Hasenfratz–t Hooft* mechanism [21,22] according to which the excitations of a Bosonic field charged under $SU(2)$ around a background gauge field with the above characteristics behave as Fermions.⁴ Indeed, following exactly the same arguments in [21,22], one can analyze the Klein–Gordon equation for a scalar field Φ (which will be assumed to belong to the fundamental representation) charged under $SU(2)$,

$$g^{\mu\nu}(\nabla_\mu - A_\mu)(\nabla_\nu - A_\nu)\Phi = 0 \quad (6.2)$$

where ∇_μ are the Levi-Civita corresponding to the metric in Eq. (4.4) and A_μ is the meron gauge potential in Eq. (3.3). Due to the fact that the metric is static, one can Fourier-expand the scalar field Φ with respect to the time

$$\Phi = \exp(iEt)\psi(r, \theta, \phi). \quad (6.3)$$

The clearest way to disclose the above mechanism is to change coordinates from the Schwarzschild-like in Eq. (4.4) to Cartesian-like coordinates as follows:

$$\delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2 = r^2 d\Omega^2 + dr^2,$$

$$r^2 d\Omega^2 = \delta_{ij} dx^i dx^j - dr^2, \quad r = \sqrt{\delta_{ij} x^i x^j},$$

$$dr = \frac{1}{r}(\delta_{ij} x^i dx^j), \quad dr^2 = \frac{1}{r^2}(x^i x^j dx^i dx^j).$$

In this coordinate system, the metric in Eq. (4.4) reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\exp(2a) dt^2 + dx^i dx^j \left[\delta_{ij} + (\exp(-2a) - 1) \frac{x^i x^j}{r^2} \right], \quad (6.4)$$

$$g^{tt} = -\exp(-2a), \quad g^{ij} = \left[\delta_{ij} + h_2(r) \frac{x^i x^j}{r^2} \right],$$

$$g^{0j} = 0,$$

$$\exp(2a) = \left(1 - \frac{\kappa M}{8\pi r} + \frac{4\kappa\lambda^2(\lambda - 1)^2}{e^2 r^2} - \frac{\Lambda r^2}{3} \right),$$

$$h_2 = -\frac{r(\exp(-2a) - 1)}{1 + r^2(\exp(-2a) - 1)}, \quad (6.5)$$

while the meron gauge potential reads (see Eq. (2.7))

$$Y^0 = 0, \quad Y^i = \hat{x}^i, \quad (6.6)$$

$$\hat{x}^1 = \frac{x}{r}, \quad \hat{x}^2 = \frac{y}{r}, \quad \hat{x}^3 = \frac{z}{r}, \quad (6.7)$$

$$A_\mu = i\lambda P_\mu^k \sigma_k, \quad (6.8)$$

$$P_\mu^k = \varepsilon_{ijk} \hat{x}^i \partial_\mu \hat{x}^j = \frac{1}{r^2} \varepsilon_{ijk} x^i \delta^{j\mu}. \quad (6.9)$$

If one replaces the ansatz in Eq. (6.3) into Eq. (6.2) one gets an effective system of coupled effective Schrodinger equations for the

components of ψ . Explicitly, using the above Cartesian-like coordinates system, it reads:

$$0 = (\nabla^i \nabla_i) \psi - \exp(-2a) E^2 \psi - 2(i\lambda) \frac{\sigma^k}{r^2} \varepsilon_{ijk} x^i g^{jl} \partial_l \psi + \mathbf{V} \psi, \quad (6.10)$$

$$\mathbf{V} = (i\lambda)^2 [\delta_{ij} g^{\mu\nu} (\partial_\mu \hat{x}^i) (\partial_\nu \hat{x}^j)] \mathbf{1},$$

where $\mathbf{1}$ is the identity of the gauge group in the fundamental representation. Using the explicit expression of the inverse metric in Eq. (6.5), Eq. (6.2) reads

$$0 = (\nabla^i \nabla_i) \psi - \exp(-2a) E^2 \psi - 2(i\lambda) \frac{1}{r^2} \sigma^k \varepsilon_{ijk} x^i \partial_j \psi + \mathbf{V} \psi. \quad (6.11)$$

Indeed, the above equation has exactly the form required for the realization of the *Jackiw–Rebbi–Hasenfratz–t Hooft* mechanism since the operator

$$\sigma^k \varepsilon_{ijk} x^i \partial_j$$

appearing in Eq. (6.11) can be written as

$$\sigma^k \varepsilon_{ijk} x^i \partial_j = \vec{\sigma} \cdot \vec{L}, \quad \vec{L} = \vec{r} \times \vec{\nabla}$$

($\vec{\nabla}$ being the flat gradient) which is nothing but the “isospin–orbit” term able to shift the eigenvalue of the angular momentum from integer to half-integer. Hence, the final form of Eq. (6.11) is

$$0 = (\nabla^i \nabla_i) \psi - \exp(-2a) E^2 \psi - 2(i\lambda) \frac{1}{r^2} \vec{\sigma} \cdot \vec{L} \psi + \mathbf{V} \psi. \quad (6.12)$$

In particular, in the asymptotically flat case and in the approximation in which r is very large Eq. (6.11) reduces to

$$(\nabla^i \nabla_i - a(r)(\vec{\sigma} \cdot \vec{L}) - b(r) + E^2) \psi = 0,$$

where the explicit forms of $a(r)$ and $b(r)$ are not important as far as the present discussion is concerned. Hence, the Schrodinger equations are only invariant under a spatial rotation plus an internal $SU(2)$ rotation generated by the total angular momentum operator \vec{J} . Thus, the eigenvalues of the total angular momentum operator \vec{J} are good quantum numbers. The key observation in [21, 22] is that the eigenvalues of \vec{J} can be both integers and half-integers so that the excitations of Φ can behave as Fermions and should be quantized accordingly. This phenomenon is very interesting in the asymptotic region when r is very large. In this case the background metric is approximately flat or (A)dS so that the fields can be quantized using the standard techniques. In particular, in the asymptotically AdS case which is relevant in the AdS/CFT correspondence, one can have Fermionic excitations charged under the gauge group on the boundary without having any Fermionic field in the bulk.

7. Conclusions and perspectives

In the present Letter we have constructed a non-Abelian black hole configuration for the $SU(2)$ Einstein Yang–Mills theory. Even if the metric coincides with the magnetic Reissner–Nordström black hole in which, however, the coefficient of the $1/r^2$ term is not an integration constant, the solution is intrinsically non-Abelian since it cannot be transformed to an Abelian sector by any globally defined gauge transformation. The gauge field of the solution has the form of a meron.

An important feature of the present black hole solution when compared with solutions of Abelian sectors is that the present black hole–meron configuration is spherically symmetric *only up*

⁴ An effect which is very similar to the *Jackiw–Rebbi–Hasenfratz–t Hooft* mechanism occurs for Skyrmions [23] (for a detailed review see [24]). Indeed, the excitations around the Skyrme soliton with winding number equal to one can behave as Fermions.

to an internal $SU(2)$ rotation. A consequence of this is the realization of the *Jackiw–Rebbi–Hasenfratz–’t Hooft* mechanism according to which excitations of Bosonic fields charged under the gauge group can behave as Fermionic excitations.

The present results can be quite relevant in the context of the AdS/CFT correspondence since one could have Fermionic excitations at the boundary without having any Fermionic field in the bulk. It would be interesting to further explore the consequences of the *Jackiw–Rebbi–Hasenfratz–’t Hooft* effect within the context of the AdS/CFT correspondence. For instance, a nice issue to analyze is how to distinguish, by just looking at the boundary theory, a Fermionic excitation coming from a Fermionic field in the bulk from a Fermionic excitation generated with the *Jackiw–Rebbi–Hasenfratz–’t Hooft* effect as it has been proposed here.

It is well known that on flat spaces merons play an important but indirect role in providing a qualitative explanation of confinement as building blocks of the instantons. However, they cannot be observed directly due to their singularities. Indeed, one of the most interesting results of the present analysis is that the meron singularity is hidden behind the black hole horizon and, consequently, in a gravitational context, merons could be observed directly in principle.

The Yang–Mills equations in the present black hole solution force the proportionality factor of the meron to be $\lambda = 1/2$. In the Euclidean solutions of de Alfaro, Fubini and Furlan [9], the value $\lambda = 1/2$ is related to the fact that merons behave as half-instantons. It would be interesting to find an analogous interpretation in the Lorentzian case.

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