Non-trivial frames for $f(T)$ theories of gravity and beyond

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ABSTRACT

Some conceptual issues concerning $f(T)$ theories – a family of modified gravity theories based on absolute parallelism – are analyzed. Due to the lack of local Lorentz invariance, the autoparallel frames satisfying the field equations are evasive to an *a priori* physical understanding. We exemplify this point by working out the vierbein (tetrad) fields for closed and open Friedmann–Robertson–Walker cosmologies.

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1. Introduction: Teleparallelism in Weitzenböck spacetime

There exists a general consensus that the description of the gravitational field provided by general relativity (GR) is doomed at scales of the order of the Planck length, where the spacetime structure itself must be represented in terms of a quantum regime. In the opposite extreme of the physical phenomena, GR also faces an intriguing dilemma in connection with the late cosmic speed up stage of the Universe. For these reasons, and for other purely conceptual ones, GR has been the object of many extensions that have tried to provide a more satisfactory description of the gravitational field in the above mentioned extreme regimes. One of the newest extended theories of gravity is the so-called $f(T)$ gravity, which is a theory formulated in a spacetime possessing absolute parallelism [1,2]. We started this idea in Refs. [3–5] by working out an ultraviolet deformation of Einstein gravity. There we proposed a Born–Infeld-like action with the aim of smoothing singularities, namely the initial singularity of Friedmann–Robertson–Walker (FRW) cosmological models. The proposal was successful in replacing the initial singularity with an inflationary stage, so providing a geometrical mechanism for the exponential increasing of the scale factor without resorting to an inflaton. After that, the attention was focused in low energy deformations of Einstein gravity, to tackle those aspects of the cosmological evolution connected with the late speed up stage of the Universe [6–19]. Quite more recently, some fundamental aspects of $f(T)$ theories, like the presence of

extra degrees of freedom and the lack of Lorentz invariance, had been addressed in Refs. [20] and [21].

In order to gain a deep insight into these and other features of the $f(T)$ approach to modified gravity, it is mandatory to enlarge the narrow number of physically relevant spacetimes hitherto considered. In Section 2 we explain the lack of invariance of $f(T)$ theories under local Lorentz transformation of the field of vierbeins in a cosmological context, and discuss about the meaning of this feature. In Section 3, we work out the proper vierbein for closed and open FRW cosmologies. The discussion here stressed, in spite of its conceptual character, leads to practical conclusions that will allow the comparison of the cosmological consequences coming from different ways of modified gravity. Finally, in Section 4 we display the conclusions.

The cornerstone of (four-dimensional) $f(T)$ theories is that gravity can be described by providing the spacetime with a torsion $T^a = de^a$, $a = 0, \ldots, 3$, where $\{e^a\}$ is a vierbein (a basis of the cotangent space) in a 4-dimensional manifold.[2] The vierbein $\{e^a\}$ is the coframe of an associated basis $\{e_a\}$ in the tangent space. If e^a_μ and e^μ_a are respectively the components of the 1-forms e^a and the vectors e_a in a given coordinate basis, then the relation between frame and coframe is expressed as

$$e^a_\mu \, e^\mu_b = \delta^a_b. \tag{1}$$

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[2] $f(T)$-like gravities are non-trivial for dimensions higher than two. The torsion tensor coming from an arbitrary *diad* $e^a(t, x)$ in $1 + 1$ dimensions has only two independent components, $T^t{}_{tx}$ and $T^x{}_{tx}$. However, the tensor $S_{\rho\mu\nu}$ that takes part in the teleparallel Lagrangian of Eq. (11) is identically null. This property, in the light of Eq. (13), is consistent with the very known fact that the Einstein–Hilbert action in two spacetime dimensions is just the Euler characteristic class of the manifold.

Contracting with e_a^ν one also gets

$$e_a^\nu e_\mu^a = \delta_\mu^\nu. \tag{2}$$

The components $T^\lambda{}_{\mu\nu}$ of the torsion tensor in the coordinate basis is related to the 2-forms T^a through the equation

$$T^\lambda{}_{\mu\nu} \equiv e_a^\lambda T^a{}_{\mu\nu} = e_a^\lambda(\partial_\nu e_\mu^a - \partial_\mu e_\nu^a). \tag{3}$$

This means that the spacetime is endowed with a connection

$$\Gamma^\lambda_{\mu\nu} = e_a^\lambda \partial_\nu e_\mu^a + \text{terms symmetric in } \mu\nu \tag{4}$$

(since $T^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}$). The first term in Eq. (4) is the Weitzenböck connection. The metric is introduced as a subsidiary field given by

$$g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x), \tag{5}$$

where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. Eq. (5) can be inverted with the help of Eq. (1) to obtain

$$\eta_{ab} = g_{\mu\nu}(x) e_a^\mu(x) e_b^\nu(x), \tag{6}$$

which means that the vierbein is orthonormal.

The relation $T^a = de^a$ displays a remarkable analogy with the electromagnetic field: the e^a's play the role of potentials and the T^a's are the fields. The torsion T^a is invariant under a gauge transformation $e^a \to e^a + d\chi^a$; the symmetric terms in the connection (4) are the imprint of such gauge transformation. Instead, the metric (5) does not enjoy such gauge invariant meaning since it is built from the "potentials".

Teleparallelism uses the Weitzenböck spacetime, where the connection is chosen as

$$\Gamma^\lambda_{\mu\nu} = e_a^\lambda \partial_\nu e_\mu^a. \tag{7}$$

Thus, the gauge invariance has been frozen. As a consequence of the choice of the Weitzenböck connection (7), the Riemann tensor is identically null. So the spacetime is flat: the gravitational degrees of freedom are completely encoded in the torsion $T^a = de^a$.

On other hand, the metric (5) does possess invariance under local Lorentz transformations: $e^a \to e^{a'} = \Lambda_b^{a'}(x) e^b$; however the torsion $T^a = de^a$ transforms as

$$T^a \to T^{a'} = \Lambda_b^{a'} T^b - e^b \wedge d\Lambda_b^{a'}, \tag{8}$$

which means that the exterior derivative of the vierbein is not covariant under Lorentz transformations of the vierbein, unless the Lorentz transformations be *global*. This feature could be healed by using a covariant exterior derivative to define T^a (i.e., by introducing a spin connection). This procedure would restore the local Lorentz freedom of the vierbein. In 4 dimensions, this local freedom would reduce the 16 components e_μ^a to only 10 physically relevant ones, but we should add the new degrees of freedom encoded in the spin connection. However, this strategy turned out to be inviable, see Ref. [21].

In terms of parallelism, the choice of the Weitzenböck connection has a simple meaning. In fact, the covariant derivative of a vector yields

$$\nabla_\nu V^\lambda = \partial_\nu V^\lambda + \Gamma^\lambda_{\mu\nu} V^\mu = e_a^\lambda \partial_\nu(e_\mu^a V^\mu) \equiv e_a^\lambda \partial_\nu V^a. \tag{9}$$

In particular, Eq. (6) implies that $\nabla_\nu e_b^\lambda = 0$; so, the Weitzenböck connection is metric compatible. In general, Eq. (9) means that a given vector is parallel transported along a curve if its projections on the coframe remain constant. So, the vierbein parallelizes the spacetime. Of course, this nice criterion of parallelism would be destroyed if local Lorentz transformations of the coframe were allowed in the theory.

Teleparallelism is a dynamical theory for the vierbein, which is built from the torsion $T^a = de^a$. According to Eq. (5), a set of dynamical equations for the vierbein also implies a dynamics for the metric. This dynamics coincides with Einstein's dynamics for the metric when the teleparallel Lagrangian density is chosen as [22,23]

$$\mathcal{L}_\mathbf{T}[e^a] = \frac{1}{16\pi G} e T, \tag{10}$$

where $e \equiv \det e_\mu^a = \sqrt{-\det(g_{\mu\nu})}$, and

$$T = S^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu}. \tag{11}$$

The tensor $S^\rho{}_{\mu\nu}$ appearing in the last equation is defined according to

$$S^\rho{}_{\mu\nu} = \frac{1}{4}(T^\rho{}_{\mu\nu} - T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho)$$
$$+ \frac{1}{2}\delta_\mu^\rho T_{\sigma\nu}{}^\sigma - \frac{1}{2}\delta_\nu^\rho T_{\sigma\mu}{}^\sigma. \tag{12}$$

In fact, the Lagrangian (10) just differs from the Einstein–Hilbert Lagrangian $\mathcal{L}_\mathbf{GR} = -(16\pi G)^{-1}\sqrt{-g}\,R$ in a divergence

$$-eR[e^a] = eT - 2\partial_\nu(eT_\sigma{}^{\sigma\nu}), \tag{13}$$

where R is the scalar curvature for the Levi-Civita connection. When GR dresses this costume, the gravitational degrees of freedom are gathered in the torsion instead of the Levi-Civita curvature. It is a very curious and fortunate fact that both pictures enable to construct a gravitational action with the same physical content. However, it is remarkable that the Lagrangian (10) involves just first derivatives of its dynamical field, the vierbein. In some sense, the teleparallel Lagrangian picks up the essential dynamical content of Einstein theory without the annoying second order derivatives appearing in the last term of Eq. (13). Such Lagrangian is a better starting point for considering modified gravity theories, since any deformation of its dependence on T will always lead to second order dynamical equations. On the contrary, the so-called $f(R)$ theories lead to fourth order equations.

The teleparallel Lagrangian (10) can be rephrased in geometrical language. Since $S^\rho{}_{\mu\nu}$ is antisymmetric in $\mu\nu$, then $e_\rho^a S^\rho{}_{\mu\nu}$ is a set of four 2-forms S^a. Noticing that

$$T_{\nu\mu}{}^\rho = g^{\rho\sigma} g_{\nu\lambda} e_b^\lambda T^b{}_{\mu\sigma} = \eta^{df} e_d^\rho e_f^\sigma \eta_{bc} e_\nu^b T^c{}_{\mu\sigma},$$

then we have

$$4S^a = T^a - \eta^{ac}\eta_{bd} e_c \rfloor T^b \wedge e^d + 2e_b \rfloor T^b \wedge e^a,$$

where $e_c \rfloor T^b$ stands for the 1-form whose components are $e_c^\sigma T^b{}_{\sigma\mu}$. Thus, the teleparallel Lagrangian density can be written as

$$\mathcal{L}_\mathbf{T}[e^a] = \frac{1}{16\pi G} \eta_{ab} S^a \wedge {}^*T^b, \tag{14}$$

where $*$ is the Hodge star operator.

2. The $f(T)'s$ uncovered

Analogously to the $f(R)$ scheme, a $f(T)$ theory replaces the Weitzenböck invariant T in Eq. (10) with a general function $f(T)$. So, the dynamics is described by the action

$$\mathcal{I} = \frac{1}{16\pi G} \int d^4x\, e\, f(T) + \int d^4x\, \mathcal{L}_\mathcal{M}, \tag{15}$$

where $\mathcal{L}_\mathcal{M}$ is the matter Lagrangian density. Undoubtedly, the whole family of actions gathered in (15) constitutes a vast territory worth to be explored, specially when one is aware that the dynamical equations arising by varying the action (15) with respect to the vierbein components $e_\mu^a(x)$ are of second order. This

distinctive feature makes Weitzenböck spacetime a privileged geometric structure to formulate modified theories of gravitation. In fact, the dynamical equations for the vierbein are

$$e^{-1}\partial_\mu(eS_a{}^{\mu\nu})f' + e_a{}^\lambda S_\rho{}^{\nu\mu}T^\rho{}_{\mu\lambda}f' +$$
$$+ S_a{}^{\mu\nu}\partial_\mu(T)f'' + \frac{1}{4}e_a{}^\nu f = 4\pi G\mathbb{T}_a{}^\nu, \qquad (16)$$

where $\mathbb{T}_a{}^\nu = e_a{}^\mu T_\mu^\nu$ refers to the matter energy–momentum tensor $T_{\mu\nu}$, and the primes denote differentiation respect to T. These equations tell how the matter distribution organizes the orientation of the vierbein e^a at each point, in such a way that the field lines of $e^a(x)$ realize the parallelization of the manifold. After this vierbein field is obtained, one uses the assumption of orthonormality to get the metric (5). Instead, GR is a theory for the metric; so it is invariant under local Lorentz transformations of the vierbein. The equivalence between Teleparallelism (10) and GR dynamics, expressed in Eq. (13), implies that T changes by a boundary term under local Lorentz transformations. Because of this reason, the teleparallel equivalent of GR does not provide the manifold with a parallelization but only with a metric. On the contrary, in a $f(T)$ theory the "boundary term" in T will remain encapsulated inside the function f. This means that a $f(T)$ theory is not invariant under local Lorentz transformations of the vierbein. So, a $f(T)$ theory will determinate the vierbein field almost completely (up to global Lorentz transformations). In other words, a $f(T)$ theory will describe more degrees of freedom than the teleparallel equivalent of GR. This is an important issue in the search for solutions to the $f(T)$ dynamical equations, since every two pair of vierbeins connected by a local Lorentz transformation (i.e. leading to the same metric tensor) are *inequivalent* from the point of view of the theory. We will address this topic in the next section by considering the vierbeins that are suitable for closed and open FRW universes.

So far, the totality of the works alluding to $f(T)$ theories in cosmological spacetimes deals with spatially flat FRW cosmologies. This is not only because this geometry seems to be the more appropriate for the description of the large scale structure of the Universe, but also for technical reasons. In fact, one could think that the starring field – the vierbein $\{e^a\}$, related to the metric g via $g = \eta_{ab}\, e^a \otimes e^b$ – is the naive square root of g. Thus, in a spatially flat FRW universe described by the line element

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \qquad (17)$$

one would replace in Eqs. (16) the diagonal vierbein

$$e^0 = dt, \qquad e^1 = a(t)\,dx,$$
$$e^2 = a(t)\,dy, \qquad e^3 = a(t)\,dz. \qquad (18)$$

This is really a good guess because Eqs. (16) become a set of consistent dynamical equations for the scale factor $a(t)$. Moreover, the Weitzenböck invariant for $T^a = de^a = \{0, \dot{a}\,dt \wedge dx, \dot{a}\,dt \wedge dy, \dot{a}\,dt \wedge dz\}$ is

$$T = -6H^2, \qquad (19)$$

where $H = \dot{a}/a$ is the Hubble parameter. Additionally, it is easy to check that the field equations (16) for the vierbein (18) can be also obtained from a (minisuperspace) reduced action constructed by replacing this specific form of the Weitzenböck invariant in the general action (15).

It is not difficult to trace back the geometrical meaning of the diagonal vierbein (18). Actually, the autoparallel curves of flat Euclidean space are given by straight lines, which can be generated by the coordinate basis ∂_i, whose dual co-basis is just dx_i. Then, modulo a time-dependent conformal factor, the frames describing the autoparallel lines are just dx_i, as Eq. (18) shows.

However, things are not so easy in the context of closed and open FRW universes, whose line element can be described in hyper-spherical coordinates as

$$ds^2 = dt^2 - k^2 a^2(t)\big[d(k\psi)^2 + \sin^2(k\psi)\big(d\theta^2 + \sin^2\theta\,d\phi^2\big)\big], \quad (20)$$

where $k = 1$ for the closed universe and $k = i$ for the open universe. Here, one is also tempted to think that the vierbein that solves the dynamical equations (16) could have the form

$$e^{0'} = dt,$$
$$e^{1'} = k\,a(t)\,d(k\psi),$$
$$e^{2'} = k\,a(t)\,\sin(k\psi)\,d\theta,$$
$$e^{3'} = k\,a(t)\,\sin(k\psi)\,\sin\theta\,d\phi. \qquad (21)$$

In the teleparallel equivalent of GR, any choice of the vierbein reproducing the metric is valid because of the local Lorentz symmetry. On the contrary, the lack of this local invariance, which is inherent to $f(T)$ theories, makes this naive choice to be incompatible with the dynamical equations (16). In other words, the vierbein (21) does not correctly parallelize the spacetime. The symptom that the choice (21) will not work is the form acquired by the Weitzenböck invariant in such case, which turn out to be

$$T = 2\big[(k\,a)^{-2}\cot^2(k\psi) - 3H^2\big]. \qquad (22)$$

This form of T would be unable of giving a proper reduced Lagrangian for the dynamics of the scale factor $a(t)$, as a consequence of its dependence on ψ. This ψ-dependent Weitzenböck invariant is not consistent with the isotropy and homogeneity of the FRW cosmological models.

3. Vierbeins for spatially curved FRW universes

3.1. Closed universes

Let us discuss in detail the closed universe ($k = 1$) with topology $R \times S^3$. In order to parallelize the S^3 sphere, let us consider S^3 as embedded in a 4-dimensional Euclidean space with Cartesian coordinates (X, Y, Z, W), so

$$X^2 + Y^2 + Z^2 + W^2 = 1. \qquad (23)$$

At each point of the sphere there exists a Cartesian (canonical) orthonormal coframe basis of the host Euclidean space, $\{dX^a\}$, where X^a stands for (X, Y, Z, W). We will rotate this coframe in such a way that one of the resulting covectors be normal to the S^3 sphere, being the other three covectors automatically tangent to S^3. This tangent vierbein will prove to be the proper spatial part of the vierbein for the closed FRW cosmology, in the sense that it will lead to consistent dynamical equations for the scale factor of the closed universe. So, let us introduce a smooth coframe field $\{\mathring{E}^a\}$ on the S^3 sphere by rotating the canonical frame $\{dX^b\}$, i.e.,

$$\mathring{E}^a = R^a{}_b\,dX^b. \qquad (24)$$

It is not difficult to verify that the matrix

$$R = \begin{pmatrix} Y & -X & -W & Z \\ W & -Z & Y & -X \\ -Z & -W & X & Y \\ X & Y & Z & W \end{pmatrix}, \qquad (25)$$

constitute a local rotation on the sphere (it fulfills $\det R = 1$, and $R^T = R^{-1}$ on the S^3 sphere (23)). Then, the rotated coframe (24) turns out to be

$$\mathring{E}^1 = Y\,dX - X\,dY - W\,dZ + Z\,dW$$

$$\mathring{E}^2 = W\,dX - Z\,dY + Y\,dZ - X\,dW$$

$$\mathring{E}^3 = -Z\,dX - W\,dY + X\,dZ + Y\,dW$$

$$\mathring{E}^4 = X\,dX + Y\,dY + Z\,dZ + W\,dW. \tag{26}$$

Clearly, the covector $\mathring{E}^4 = (1/2)d(X^2 + Y^2 + Z^2 + W^2)$ is normal to the S^3 sphere; from now on we shall focus on the tangent orthonormal coframe $\{\mathring{E}^1, \mathring{E}^2, \mathring{E}^3\}$. For convenience, we will parametrize the S^3 sphere by using hyper-spherical coordinates, which are related with the Cartesian coordinates of the host Euclidean space in the usual manner

$$X = \sin\psi \sin\theta \cos\phi$$

$$Y = \sin\psi \sin\theta \sin\phi$$

$$Z = \sin\psi \cos\theta$$

$$W = \cos\psi. \tag{27}$$

The angular coordinates range in the intervals $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \psi \leqslant \pi$. Thus, we can expand the dreibein $\{\mathring{E}^1, \mathring{E}^2, \mathring{E}^3\}$ in the coordinate basis $\{d\psi, d\theta, d\phi\}$ to obtain

$$\mathring{E}^1 = -\cos\theta\,d\psi + \sin\psi \sin\theta(\cos\psi\,d\theta - \sin\psi \sin\theta\,d\phi),$$

$$\mathring{E}^2 = \sin\theta \cos\phi\,d\psi - \sin\psi\big[(\sin\psi \sin\phi - \cos\psi \cos\theta \cos\phi)\,d\theta$$
$$+ (\cos\psi \sin\phi + \sin\psi \cos\theta \cos\phi)\sin\theta\,d\phi\big],$$

$$\mathring{E}^3 = -\sin\theta \sin\phi\,d\psi - \sin\psi\big[(\sin\psi \cos\phi + \cos\psi \cos\theta \sin\phi)\,d\theta$$
$$+ (\cos\psi \cos\phi - \sin\psi \cos\theta \sin\phi)\sin\theta\,d\phi\big]. \tag{28}$$

Finally, let us consider the vierbein

$$e^0 = dt, \qquad e^1 = a(t)\mathring{E}^1, \qquad e^2 = a(t)\mathring{E}^2, \qquad e^3 = a(t)\mathring{E}^3. \tag{29}$$

It is worth noticing that this vierbein can be directly obtained from the naive vierbein (21) by means of a local rotation whose Euler angles are ψ, θ, ϕ; in fact, both frames are related via the Euler matrix

$$e^a = \mathcal{R}^a_{a'}\,e^{a'}, \tag{30}$$

where

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi & \sin\psi \\ 0 & 0 & -\sin\psi & \cos\psi \end{pmatrix}.$$

The fact that the naive vierbein (21) and (29) are connected by a local rotation guarantees that the latter actually describes the closed FRW metric given in Eq. (20) for $k = 1$; i.e., it leads to the interval

$$ds^2 = dt^2 - a^2(t)\big[d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\big].$$

The applied procedure is successful because S^3, like all the 3-manifolds, is a parallelizable manifold, which means that it accepts a globally well defined set of three smooth (C^∞) orthonormal vector fields, that serve as a global basis of the tangent bundle TM.

The Weitzenböck invariant associated to the vierbein (29) is

$$T = 6\big(a^{-2} - H^2\big), \tag{31}$$

which foretell that the vierbein (29) will be adequate to solve the dynamical equations. In fact, by replacing the vierbein (29) in Eqs. (16) one obtain the modified version of Friedmann equation (for $a = 0 = \nu$):

$$12H^2 f'(T) + f(T) = 16\pi G\rho. \tag{32}$$

The equations for the spatial sector ($a, \nu = 1, 2, 3$), are equal to

$$4\big(a^{-2} + \dot{H}\big)\big(12H^2 f''(T) + f'(T)\big) - f(T)$$
$$- 4f'(T)\big(2\dot{H} + 3H^2\big) = 16\pi Gp. \tag{33}$$

Note that Eq. (32) is of first order in time derivatives of the scale factor, irrespective of the function f. Eqs. (32) and (33) are two differential equations for just one unknown function $a(t)$; so, they are not independent. The way to see that this is indeed the case, is to take the time derivative of (32) and combine it with the conservation equation,

$$\dot{\rho} = -3H(\rho + p), \tag{34}$$

to obtain Eq. (33). Conversely, if the system (32) and (33) is consistent, then the conservation of energy in the matter sector is given automatically and Eq. (34) holds.

3.2. Open universes

Eq. (30) also shows a way to find a proper vierbein for the open universe: take the same rotation starting from the naive vierbein (21) with $k = i$, but replace the Euler angle ψ with $i\psi$. The aspect of the so locally rotated frame is now

$$\check{E}^1 = \cos\theta\,d\psi + \sinh\psi \sin\theta(-\cosh\psi\,d\theta + i\sinh\psi \sin\theta\,d\phi),$$

$$\check{E}^2 = -\sin\theta \cos\phi\,d\psi$$
$$+ \sinh\psi\big[(i\sinh\psi \sin\phi - \cosh\psi \cos\theta \cos\phi)\,d\theta$$
$$+ (\cosh\psi \sin\phi + i\sinh\psi \cos\theta \cos\phi)\sin\theta\,d\phi\big],$$

$$\check{E}^3 = \sin\theta \sin\phi\,d\psi$$
$$+ \sinh\psi\big[(i\sinh\psi \cos\phi + \cosh\psi \cos\theta \sin\phi)\,d\theta$$
$$+ (\cosh\psi \cos\phi - i\sinh\psi \cos\theta \sin\phi)\sin\theta\,d\phi\big]. \tag{35}$$

Then the vierbein

$$e^0 = dt, \qquad e^1 = a(t)\check{E}^1, \qquad e^2 = a(t)\check{E}^2, \qquad e^3 = a(t)\check{E}^3, \tag{36}$$

with the choice (35), leads to the metric for the open FRW cosmology:

$$ds^2 = dt^2 - a^2(t)\big[d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\big]. \tag{37}$$

In this case the Weitzenböck invariant is given by the expression (compare with Eq. (31))

$$T = -6\big(a^{-2} + H^2\big). \tag{38}$$

The modified Friedmann equation arising from the vierbein (36) is again Eq. (32). Hence, the equations coming from the spatial sector are (33) but with the change $a^{-2} \to -a^{-2}$ in the first term of the expression.

4. Final remarks

In the context of $f(T)$ theories, the spacetime structure is materialized in the coframe field $\{e^a\}$ which defines an orthonormal basis in the cotangent space $T_p^* M$ of the manifold M at each point $p \in M$. When $f(T) = T$, i.e., when one consider general relativity in Weitzenböck spacetime, the basis $\{e^a\}$ at two different points of the manifold are completely uncorrelated, and it is not
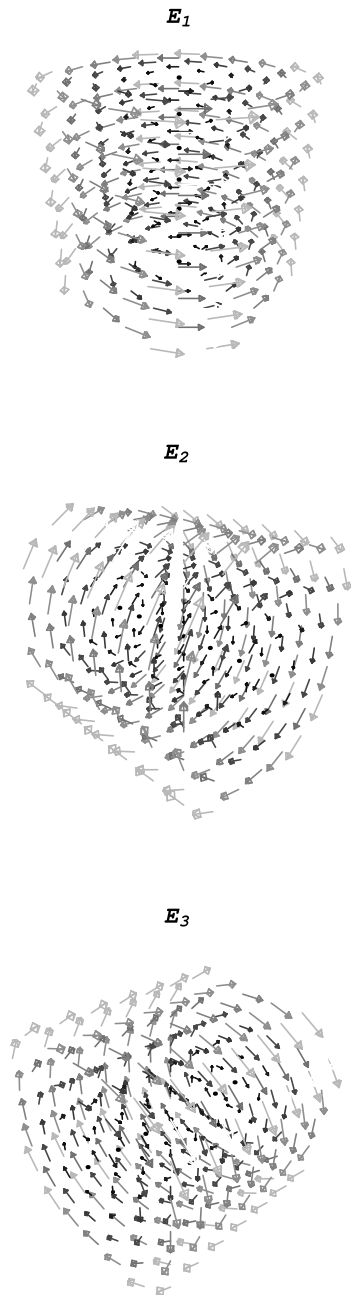
Fig. 1. The parallel vector fields dual to the forms (28), on the hyper-equator of the three sphere.

possible to define a global smooth field of basis unambiguously. This is so because the theory is invariant under the local Lorentz group acting on the coframes $\{e^a\}$. In turn, when $f(T) \neq T$, local Lorentz rotations and boosts are not symmetries of the theory anymore. Because of this lack of local Lorentz symmetry, the theory picks up a preferential global reference frame constituted by the coframe field $\{e^a\}$ that solves the field equations. In such case, the bases at two different points become strongly correlated in order to realize the parallelization of the manifold, as can be seen in the *fish shoal*-like pattern of Fig. 1, showing the vector fields E_1, E_2 and E_3, dual to the one-forms of Eq. (28). In this figure, the coordinate X_4 was set to zero, so the pictures represent the parallel vector fields of S^3 in the hyper-equator defined by

$\psi = \pi/2$, immersed in three-dimensional Euclidean space. The appearance of a preferred reference frame is a property coming from the symmetries of the spacetime, and it is not ruled by the specific form of the function $f(T)$. For instance, when one is dealing with FRW cosmological spacetimes, the field of frames that will lead to consistent field equations will be given by (18), (29) or (36) depending on whether the Universe is flat, closed or open respectively, whatever the function $f(T)$ is. Additionally, these fields are also valid in more general theories with absolute parallelism which are not related with the $f(T)$ schemes. See, for instance, Ref. [24].

Let us finish with some additional remarks. Let be $\{e^a(x)\}$ a vierbein field satisfying the vacuum Einstein equations, i.e., $\{e^a(x)\}$ is a solution of Eqs. (16) with $f(T) = T$ and $\mathbb{T}_\mu{}^\nu = 0$. Besides, suppose one can find a local Lorentz transformation $\bar{e}^a(x) = \Lambda^a_b(x)e^b(x)$ such that the Weitzenböck invariant $\bar{T}$ – which is invariant under diffeomorphisms, but not under local Lorentz transformations of the vierbein – becomes globally zero. Then $\{\bar{e}^a(x)\}$ is a solution not only for GR but for any ultraviolet deformations of GR, i.e., for any theory described by a function $f(T)$ verifying the condition

$$f(T) = T + \mathcal{O}(T^2), \quad \text{i.e., } f(0) = 0, \text{ and } f'(0) = 1. \tag{39}$$

In fact, $\{\bar{e}^a(x)\}$ fulfills Eqs. (16) for any function $f(T)$ satisfying the high energy conditions (39). This is because the prescription (39) make Eqs. (16) to be exactly the same that the GR ones whenever $T = 0$. In other words, this means that the original solution of the vacuum Einstein field equations remains as a solution of the deformed $f(T)$ theories described by the conditions (39). The so obtained non-trivial coframe $\{\bar{e}^a(x)\}$ could serve as the starting point to search for vacuum solutions of arbitrary infrared $f(T)$ deformations, like the ones considered in the literature in order to explain the late time cosmic speed up of the Universe [25]. This practice will enable to further reduce the set of physically relevant $f(T)$ models by using the well established post Newtonian constraints, and hopefully, will shed some light in the question of which are the extra degrees of freedom hidden behind $f(T)$ gravities. We shall have the opportunity to deal with these matters in a future work.

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