Physics Letters B 692 (2010) 206–211

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Born–Infeld determinantal gravity and the taming of the conical singularity in 3-dimensional spacetime

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ARTICLE INFO

Article history:
Received 5 May 2010
Received in revised form 2 July 2010
Accepted 9 July 2010
Available online 23 July 2010
Editor: M. Cvetič

Keywords:
Born–Infeld
Singularities
3-dimensional gravity

ABSTRACT

In the context of Born–Infeld *determinantal* gravity formulated in an n-dimensional spacetime with absolute parallelism, we found an exact 3-dimensional *vacuum* circular symmetric solution without cosmological constant consisting in a rotating spacetime with non-singular behavior. The space behaves at infinity as the conical geometry typical of 3-dimensional General Relativity without cosmological constant. However, the solution has no conical singularity because the space ends at a minimal circle that no freely falling particle can ever reach in a finite proper time. The space is curved, but no divergences happen since the curvature invariants vanish at both asymptotic limits. Remarkably, this very mechanism also forbids the existence of closed timelike curves in such a spacetime.

1. Introduction

Nowadays it is widely accepted by high energy physicists that Einstein's theory must represent a low energy limit of a more fundamental (quantum) theory of gravity. This suggests that the transition between both regimes must be ruled by an ultraviolet deformation of GR which, presumably, could solve many of the puzzles present in Einstein's theory. In this direction, special interest has been put on 3-dimensional gravity as an attempt to understand many of the conceptual and technical problems associated with the quantization of spacetime in the realistic 4-dimensional scenario [1]. In this process, it was suddenly realized that three-dimensional Einstein gravity has a number of peculiar facts; it contains no propagating degrees of freedom, and does not reduce to 2-dimensional Newtonian gravity in the weak-field limit. Moreover, the spacetime is flat outside matter and hence there exists no static interaction between sources [2].

Not long after the first investigations in 3-dimensional General Relativity have appeared [3], several generalizations were proposed in order to make 3-dimensional dynamics more alike the realistic $(3+1)$-gravity. Among the plethora of theories that are not constrained to exist only in 3 dimensions we can mention $(2+1)$-dilaton gravity [4–6], conformal gravity [7,8] and the newcomer New Massive Gravity (NMG) [9,10]. On the other hand, some constructions that are unique to $2+1$ dimensions have been also considered. One that has attracted much attention in the last years is the so-called Topological Massive Gravity (TMG), which adds to Einstein action a Chern–Simons term free of torsion [11,12] (see [13] for a comprehensive review of solutions).

The singularities inherent to Einstein theory had been matter of research since the early days of General Relativity (GR). Thought the concept of singularity encounter its *raison d'être* in the geodesic incompleteness [14], it historically came into light associated with the divergences of physical quantities. Regarding this matter, most of the major achievements in the subject have arisen from examination of two fundamental issues: the question of the origin of the Universe and the final state occurring in the gravitational collapse of massive stars. In the former issue (leaving aside ontological discussions about the origin of time), physical quantities such as the energy density and pressure of matter fields, become infinite in the Big Bang. In the latter, the unfortunate destiny of the infalling observer who goes beyond the Schwarzschild radius, is to experiment infinite tidal forces as he/she approaches $r=0$, due to the very infiniteness of the Riemann curvature tensor at that point.

As is well known, vacuum solutions for 3-dimensional GR are free of curvature singularities, because the Einstein tensor is just the double dual of the curvature and so, essentially, it is proportional to the stress–energy tensor. However, due to non-trivial topological properties, the massive circular symmetric solutions of vacuum Einstein equations in 3 spacetime dimensions displays a conical singularity at the origin. In the case without cosmological constant, which might be considered the three-dimensional analogue of the exterior Kerr metric, the solution exhibits closed time-

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0370-2693
doi:10.1016/j.physletb.2010.07.040

like curves (CTC). More realistic four-dimensional cosmic string solutions inherit all these properties [15,16].

In papers [17,18] we have introduced the so-called Born–Infeld (BI) gravity with the aim of smoothing the curvature singularities characterizing the cosmological (Friedmann–Robertson–Walker) solutions of $n = 3 + 1$ GR. In the present work, in turn, we are pursuing a different task by asking whether it is possible to remove the singularities of topological nature existent in vacuum $2 + 1$ Einstein Gravity. It is worth of mention that none of the above referred approaches to gravity in 3 dimensions have supplied a non-singular behavior in its circular symmetric vacuum solutions. For this purpose we extend the construction presented in the articles [17,18] by working with a determinantal form of the action. This new approach to the subject has the benefit of being in more close correspondence with the original BI construction. For this new scheme we have found a circular symmetric vacuum solution in three-dimensional spacetime without cosmological constant. We have obtained that the angular momentum J not only controls the global properties of the spacetime, but it has an impact on the local physics through the curvature of the manifold. Remarkably, the curvature invariants are bounded functions of the radial coordinate. When the BI parameter λ tends to infinity, the conical geometry characterizing the elementary solution of Einstein's theory in $n = 3$ is restored. Particularly interesting is the fact that the theory provides a minimum attainable circle whose circumference is $\pi J/M$, where M is a constant related with the mass of the spinning source. As a consequence, the spacetime structure becomes geodesically complete because no free falling particles can ever reach this minimum circle in a finite proper time. Another feature of this natural cutoff on the radial coordinate is that, unlike its low energy (i.e. GR) version, there are not closed timelike curves in this geometry.

2. Born–Infeld gravity in Weitzenböck spacetime

In order to motivate the construction we will work out, let us briefly examine Born–Infeld electrodynamics. As is well known, this non-linear theory for the electromagnetic field was able to tame the infinite self energy of the point-like charged particle. In its first version [19,20], BI theory deformed the Maxwell Lagrangian $L_M \propto (\mathbf{E}^2 - \mathbf{B}^2)$ according to the rule

$$\mathcal{I}_M \to \mathcal{I}_{\mathrm{BI0}} = \lambda \int d^4x \left[\sqrt{1 + \lambda^{-1} L_M} - 1\right]. \tag{1}$$

The scheme (1) is not as unnatural as it seems at first glance; the same technique can be used for going from the classical free particle action to the relativistic one; in such case, the scale is $\lambda = -mc^2$, which smooths the particle velocity by preventing its unlimited growing. In the regime where $L = m\mathbf{V}^2 \ll \lambda$ the relativistic physics restore its low energy (Newtonian) realm.

Soon after its advent [21–23], Born and Infeld generalized their construction by considering the generally covariant determinantal action

$$\mathcal{I}_{\mathrm{BI}} = \lambda \int d^4x \left[\sqrt{|g_{\mu\nu} + \lambda^{-1} F_{\mu\nu}|} - \sqrt{|g_{\mu\nu}|}\right], \tag{2}$$

which implicitly includes also the pseudo-invariant $\mathbf{E} \cdot \mathbf{B}$ (| | stands for the absolute value of the determinant). Expressions (1) and (2) are coincident only in pure electrostatic or magnetostatic situations, or in electrodynamical phenomena concerning plane waves (where the two field invariants are null). In this last case, the scale λ plays not role at all, hence the field configurations are exactly the same than those of Maxwell's theory. BI electrodynamics reduces to Maxwell's theory for small amplitudes, both of them having causal propagation and absence of birefringence. Remarkably, after a long exile, BI action came back again to the stage in the context of more modern developments; the quartic terms implicit in (2) reproduce the effective action of one-loop supersymmetric QED [24], and the structure (2) emerge naturally in the low energy limit of string theory as the action governing the electromagnetic field of D-branes [25].

The above mentioned remarkable features of the BI program, together with its well-known curative properties concerning singularities, invites to search for gravitational analogues with the structure (2). This matter has attracted some attention in the past [26–32], where several deformations à la Born–Infeld combining higher order invariants related to the curvature in a Riemannian context were tried. More recently, a thorough analysis of cosmological models by means of dynamical systems techniques was performed in [33]. All these constructions, however, lead to troublesome four order field equations for the metric. Actually, within these frameworks, exact solutions were never found. In spite of this, the importance of BI-like actions for the gravitational field was revisited very recently in connection with the problem of quantum gravity [34,35]. In a different direction, BI-like actions were explored also in Refs. [36–38] using the Palatini formalism, where metric and connection are taken as independent entities. In this article, we shall follow a different path by considering a BI deformation in Weitzenböck spacetime.

General Relativity can be formulated in a spacetime possessing absolute parallelism. This approach is usually known as teleparallel equivalent of General Relativity TEGR [39,40], and relies on the existence of a set $\{e^a(x)\}$ of n one-forms that turn out to be autoparallel for the Weitzenböck connection $\Gamma^{\lambda}_{\mu\nu} = e^{\lambda}_a \partial_\nu e^a_\mu$ (e^{λ}_a makes up the inverse matrix of e^a_μ). This connection is compatible with the metric $\mathrm{g}(x) = \eta_{ab} e^a(x) \otimes e^b(x)$ and curvature free: Weitzenböck spacetime is flat though it possesses torsion $T^a = de^a$, which is the agent where the gravitational degrees of freedom are encoded. The structure of the torsion tensor resembles the one of the electromagnetic field tensor $F = dA$ and, like Maxwell's, teleparallel Lagrangian density is quadratic in this tensor. In fact, TEGR action with cosmological constant Λ is [41]

$$\mathcal{I}_{\mathrm{GR}\parallel} = \frac{1}{16\pi G} \int d^n x \sqrt{|g_{\mu\nu}|} (\mathbb{S} \cdot \mathbb{T} - 2\Lambda), \tag{3}$$

where $\mathbb{S} \cdot \mathbb{T} \doteq S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}$, $T^\rho{}_{\mu\nu} = e^\rho_a (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)$ and $S_\rho{}^{\mu\nu}$ is defined as

$$S_\rho{}^{\mu\nu} = -\frac{1}{4}\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right) + \frac{1}{2}\left(\delta^\mu_\rho T^{\theta\nu}{}_\theta - \delta^\nu_\rho T^{\theta\mu}{}_\theta\right).$$

The equivalence between GR and the theory (3) comes from the fact that the GR Lagrangian – i.e. the curvature scalar R of the Levi-Civita connection – is $R = \mathbb{S} \cdot \mathbb{T} + \textit{Surface Terms}$. In this expression, the surface terms encompass all the second derivatives entering the scalar curvature R. In fact, Weitzenböck torsion T contains just first derivatives of the fields $e^a(x)$. This distinctive feature makes Weitzenböck torsion a privileged geometric structure to formulate modified theories of gravitation, since it guarantees that any modified Lagrangian in this language will assure second order field equations.

In Ref. [17] we followed the spirit of Eq. (1) by studying the *deformed* action

$$\mathcal{I}_{\mathrm{BI0}} = \lambda \int d^n x \sqrt{|g_{\mu\nu}|} \left[\sqrt{1 + 2\lambda^{-1} \mathbb{S} \cdot \mathbb{T}} - \alpha\right], \tag{4}$$

which proved to be capable of smoothing the GR cosmological singularity, providing a natural inflationary stage (without the mediation of an inflaton) and bounding the dynamics of the Hubble parameter.[2] Apart from this cosmological success, the scheme (4) was unable to deform the 3-dimensional circular symmetric solutions, in particular the BTZ black hole [43]. This inability is a consequence of the fact that the scalar Lagrangian in (4) is constant on the BTZ solution: $\mathbb{S}\cdot\mathbb{T} = -2\Lambda$ [18].

Here we will follow the spirit of (2), so we shall propose the general n-dimensional BI action in Weitzenböck spacetime

$$\mathcal{I}_{\mathrm{BIG}} = \frac{\lambda/(A+B)}{16\pi G}\int d^n x\left[\sqrt{\left|g_{\mu\nu}+2\lambda^{-1}\mathcal{F}_{\mu\nu}\right|}-\alpha\sqrt{|g_{\mu\nu}|}\right], \quad (5)$$

where $\mathcal{F}_{\mu\nu}$ is quadratic in the Weitzenböck torsion, and reads $\mathcal{F}_{\mu\nu} = AS_{\mu\lambda\rho}T_{\nu}{}^{\lambda\rho} + BS_{\lambda\mu\rho}T^{\lambda}{}_{\nu}{}^{\rho}$, A and B being non-dimensional constants. Such a combination ensures the correct GR limit since both terms in $\mathcal{F}_{\mu\nu}$ have trace proportional to $\mathbb{S}\cdot\mathbb{T}$. In fact, we can factor out $\sqrt{|g_{\mu\nu}|}$ from expression (5) and use the expansion of the determinant,

$$\det(\mathbb{I}-\epsilon\mathbb{F}) = 1 + p_1\epsilon + p_2\epsilon^2 + \cdots + p_{n-1}\epsilon^{n-1} + p_n\epsilon^n,$$

where

$$p_1 = -s_1$$
$$p_2 = -\frac{1}{2}(s_2 + p_1 s_1)$$
$$\vdots$$
$$p_n = -\frac{1}{n}(s_n + p_1 s_{n-1} + \cdots + p_{n-1}s_1),$$

and $s_i = \mathrm{Tr}(\mathbb{F}^i)$. In our case it is $\epsilon = -2\lambda^{-1}$ and $\mathbb{F} \equiv \mathcal{F}_{\mu}{}^{\nu}$. Thus the Lagrangian density in $\mathcal{I}_{\mathrm{BIG}}$ is

$$\begin{aligned}\mathcal{L}_{\mathrm{BIG}} &= \frac{\lambda/(A+B)}{16\pi G}\sqrt{|g_{\mu\nu}|}\bigg[1+\lambda^{-1}\mathcal{F}_{\mu}{}^{\mu} \\ &\quad + \lambda^{-2}\left(\frac{1}{2}\left(\mathcal{F}_{\mu}{}^{\mu}\right)^2 - \mathcal{F}_{\mu}{}^{\nu}\mathcal{F}_{\nu}{}^{\mu}\right) - \alpha\bigg] + \mathcal{O}\left(\lambda^{-2}\right) \\ &= \frac{\sqrt{|g_{\mu\nu}|}}{16\pi G}\bigg[\mathbb{S}\cdot\mathbb{T} + \frac{A+B}{2\lambda}(\mathbb{S}\cdot\mathbb{T})^2 \\ &\quad - \frac{1}{\lambda(A+B)}\mathcal{F}_{\mu}{}^{\nu}\mathcal{F}_{\nu}{}^{\mu} - \frac{\lambda(\alpha-1)}{A+B}\bigg] + \mathcal{O}\left(\lambda^{-2}\right).\end{aligned}$$

At the lowest order we retrieve the low energy regime described by the Einstein theory (3) with cosmological constant $\Lambda = \lambda(\alpha-1)/[2(A+B)]$. The following term $\lambda^{-1}(\mathbb{S}\cdot\mathbb{T})^2$ is also present in the expansion of action (4). However we get now a new term $\mathcal{F}_{\mu}{}^{\nu}\mathcal{F}_{\nu}{}^{\mu}$ at the order λ^{-1}, so $\mathcal{I}_{\mathrm{BIG}}$ departs from $\mathcal{I}_{\mathrm{BI0}}$ even at the order λ^{-1}. Whether the action (5) can be regarded as an effective (low energy) action for gravity coming from a more fundamental quantum theory is unknown at present, perhaps because the very quantum theory of gravity is yet a tale to be unfolded. Nevertheless, the experience acquired with its electromagnetic analogue suggest that theory (5) would constitute a slope worth to be explored. Action (5) shows us that the framework (4) is, among the whole Born–Infeld catalogue, just the top of the iceberg. The use of a Lagrangian which is not a mere deformation of the one in action (3) opens the possibility of finding a high energy modification for the GR spherically symmetric solutions. In the next section we show that this is indeed the case.

[2] See Ref. [42] for a brief summary of these results in 4 dimensions

3. Taming the conical singularity and erasing CTC's

We will investigate the properties of action (5) in the more accessible environment of (2 + 1)-gravity. In particular, let us work under the assumption of spherically (circular) symmetric spacetimes, and propose the following dreibein written down in standard polar coordinates (t, r, θ)

$$\begin{aligned}e^0 &= N(r)\,dt, \\ e^1 &= \left(Y(r)/N(r)\right)dr, \\ e^2 &= r\left(N^{\theta}(r)\,dt + d\theta\right),\end{aligned} \quad (6)$$

which implies the metric tensor

$$ds^2 = N^2(r)\,dt^2 - \frac{Y^2(r)}{N^2(r)}dr^2 - r^2\left(N^{\theta}(r)\,dt + d\theta\right)^2.$$

As is known, the vacuum solution for the GR ($\lambda \to \infty$) limit is

$$N_o^{\theta}(r) = -\frac{J}{2r^2}, \qquad N_o^2(r) = -M - \Lambda r^2 + \frac{J^2}{4r^2}, \qquad Y = 1 \quad (7)$$

which becomes the rotating BTZ black hole when $\Lambda < 0$.

We will try the dreibein (6) in the dynamical equations coming from the action (5), for the particular case $B = 0$ (constant A will be absorbed in λ). In terms of the natural variables defined as

$$X = -\frac{(N^2)'}{\lambda r Y^2}, \qquad Z = \frac{r^2(N^{\theta\prime})^2}{2\lambda Y^2}, \quad (8)$$

the dynamical equations read

$$\frac{1 - X + Z/2}{\sqrt{\mathcal{U}(X,Z)}} = KY, \quad (9)$$

$$\frac{\sqrt{2\lambda Z}(1 - X/2)}{\sqrt{\mathcal{U}(X,Z)}} = \frac{J}{r^2}, \quad (10)$$

$$(1 + 2\Lambda/\lambda)\sqrt{\mathcal{U}(X,Z)} = 1 - X^2 + XZ, \quad (11)$$

with

$$\mathcal{U}(X,Z) = 1 - 2X + X^2 + 2Z - ZX, \quad (12)$$

K and J being two integration constants. Actually K can be absorbed in Y by redefining the variables N, N^{θ} and the coordinate t (without affecting X, Z); so, we will use $K = 1$. Eqs. (9)–(11) are three coupled algebraic equations. In spite of its apparent harmlessness, they are quite hard to solve in its full generality.

In the case $\Lambda = 0$ it is not difficult to find an exact solution for the system (9)–(11). Notice that the GR solution (7) satisfies the relation $X = Z$, which in turn leads to $\mathcal{U}(X = Z, Z) = 1$ (see Eq. (12)). If $\Lambda = 0$ then λ does not explicitly appear in Eq. (11). So the relation $X = Z$ is still suitable to solve Eq. (11). The remaining equations are cast in the form

$$1 - \frac{Z}{2} = Y, \quad (13)$$

$$ZY^2 = \frac{J^2}{2\lambda r^4} \doteq 2\Delta. \quad (14)$$

From Eq. (14), the definitions (8) for X, Z and the relation $X = Z$ one gets:

$$N^{\theta}(r) = -\frac{J}{2r^2}, \qquad N^2(r) = M^2 + \frac{J^2}{4r^2}, \quad (15)$$

where M^2 is an integration constant. Thus the interval takes the form

$$\begin{aligned} ds^2 &= \left(J^2/(4r^2)+M^2\right)dt^2-\left(\frac{Y(r)^2}{J^2/(4r^2)+M^2}\right)dr^2 \\ &\quad -r^2\left(-\frac{J}{2r^2}\,dt+d\theta\right)^2 \\ &= \left[d\left(Mt+J\theta/(2M)\right)\right]^2-\left(\frac{Y(r)^2}{J^2/(4r^2)+M^2}\right)dr^2 \\ &\quad -\frac{r^2}{M^2}\left(J^2/(4r^2)+M^2\right)d\theta^2. \end{aligned} \tag{16}$$

By performing the changes

$$r\to\rho=M^{-2}\left(J^2/4+M^2r^2\right)^{1/2}, \tag{17}$$

$$t\to T=Mt+J\theta/(2M), \tag{18}$$

the interval (16) is cast in the form

$$ds^2=dT^2-Y(\rho)^2\,d\rho^2-M^2\rho^2\,d\theta^2. \tag{19}$$

In the TEGR limit ($\lambda\to\infty$) it is $\Delta\to 0$; then $Z\to 0$ and $Y\to 1$ in Eqs. (13)–(14). Thus the flat spacetime is locally recovered (notice that the constant M could be absorbed by redefining θ). From a global viewpoint, Eq. (19) with $Y=1$ could be regarded as a conical structure: the slices $T=constant$, $0\leqslant\theta<2\pi$, $0\leqslant\rho<\infty$ are planes where a wedge was cut off and its opposite sides were identified. The deficit angle is $\beta=2\pi(1-M)$ (β is related with the mass m of a source at the origin: $m=\beta/(2\pi G)$ [44]). Actually, the coordinate ρ is not allowed to reach the value $\rho=0$ in Eq. (17). However, in TEGR this is not a real limitation of coordinate ρ but a consequence of the chosen dreibein. In fact ρ can be effectively extended up to $\rho=0$, as is apparent in Eq. (19) with $Y=1$.

As is well known, TEGR theory (3) is invariant under local Lorentz transformations of the vielbein; therefore the geometry (19) with $Y=1$ could be derived not only from the dreibein (6) but from the inertial dreibein $\{E^0=dT, E^1=d\rho=e^1, E^2=M\rho\,d\theta\}$. Both dreibeins are related by the Lorentz transformation

$$e^0=\frac{E^0-J/(2M^2\rho)E^2}{\sqrt{1-J^2/(4M^4\rho^2)}},\qquad e^2=\frac{E^2-J/(2M^2\rho)E^0}{\sqrt{1-J^2/(4M^4\rho^2)}},$$

which is a boost tangent to the circle $\rho=constant$ with velocity $V=J/(2M^2\rho)$. So J measures the rotation of the dreibein (6) with respect to the inertial frame. The boost velocity increases from infinity to reach the maximum value at $\rho=J/(2M^2)$, i.e. at $r=0$ (see Eq. (17)). However, as a consequence of the gauge freedom, the geometry (19) with $Y=1$ is not imprinted with the value of J. Thus one can fix the gauge by choosing $J=0$, which amounts to the choice of the inertial dreibein, so extending the range of ρ from infinity up to zero.

On the contrary, the modified teleparallel actions are invariant only under *global* Lorentz transformations of the vielbein [17], which pre-announce a different role of J in these theories and a geometrical meaning for the bound $\rho>J/(2M^2)$. In fact, whereas a local Lorentz transformation of the vielbein adds a divergence term to $\mathbb{S}\cdot\mathbb{T}$, which is not physically significant in action (3), instead such a divergence term does affect the modified actions (4) and (5). This loss of gauge freedom means that the modified teleparallel theories govern more dynamical variables that TEGR does. Thus, the parameters characterizing the lost gauge transformations become integration constants associated with the recovered degrees of freedom. Therefore, the family of metrics resulting from the solutions is enlarged. Because of this, J plays a very different role in modified teleparallelism; since dreibeins with different values of J are not related through global Lorentz transformations, then they represent genuine different solutions of the theory. J enters the metric (19) to make Y a function of the radial coordinate, so labeling different (curved) solutions. In fact, according to Eqs. (13)–(14), the function $Y(r)$ is obtained from the cubic equation

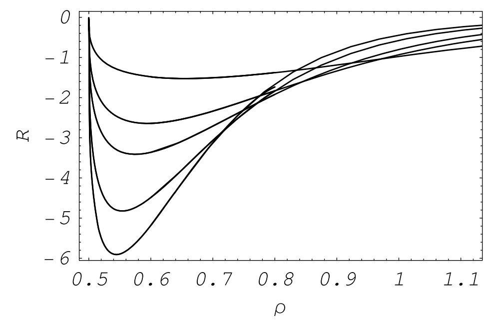

Fig. 1. Scalar curvature R as a function of the radial coordinate ρ, for $J/M=1$. Following the minimum of the curves from bottom to top, it is $-\lambda=15,10,5,3,1$.

$$Y^2-Y^3=\frac{J^2}{4\lambda r^4}=\Delta. \tag{20}$$

In the modified theory (i.e., for finite values of λ), the flat solution $Y=1$ can only be obtained when $J=0$, otherwise the space is curved. The integration constant J is the source of the curvature. According to Eq. (20), $J/\sqrt{|\lambda|}$ is the squared length scale for such deformation of flat spacetime. Alternatively, the spatial curvature could be regarded as a variable deficit angle (just perform the coordinate change $d\xi=Y(\rho)\,d\rho$ in (19)). Summarizing, the modified theory not only contains the GR solution but a family of curved spacetimes parametrized by the integration constant J. As we are going to show, the curvature of the solutions with $J\neq 0$ softens the conical singularity by replacing it with an unreachable minimal circle of radius $\rho_o\equiv J/(2M^2)$.

Among the three solutions of Eq. (20), we will keep the one going to 1 when $\Delta\to 0$, since it contains both the GR limit and the proper behavior at infinity. This solution is:

$$\begin{aligned} 3Y &= 1+\left(1-\frac{27}{2}\Delta-\frac{3}{2}\sqrt{3\Delta(27\Delta-4)}\right)^{-1/3} \\ &\quad +\left(1-\frac{27}{2}\Delta-\frac{3}{2}\sqrt{3\Delta(27\Delta-4)}\right)^{1/3}. \end{aligned} \tag{21}$$

If $\lambda<0$ then $\Delta<0$ and the function Y is defined for $0<r<\infty$ (i.e., $J/(2M^2)<\rho<\infty$); so hereafter we shall focus in the case with $\lambda<0$. We can characterize the geometry (19) by computing its curvature invariants:

$$\begin{aligned} R &= \frac{2Y(\rho)'}{\rho Y(\rho)^3}=\frac{2Y(r)'}{rY(r)^3},\qquad R^{\mu\nu}R_{\mu\nu}=\frac{1}{2}R^2, \\ K &\equiv R^{\alpha}_{\beta\gamma\delta}R_{\alpha}^{\beta\gamma\delta}=R^2. \end{aligned} \tag{22}$$

Of course, they go to zero for r (or ρ) going to infinity. In this case, due to the fact that $Y\to 1$ when $\rho\to\infty$, the metric (19) describes the conical (locally flat) GR spacetime.

The invariants (22) also go to zero for $r\to 0$ (or $\rho\to J/(2M^2)$). In fact, according to Eq. (20) Y behaves as $(-\Delta)^{1/3}$ for $\Delta\to\infty$, what implies

$$R\sim-\frac{16}{3}\left(\frac{\sqrt{2}|\lambda|r}{J^2}\right)^{2/3}, \tag{23}$$

when $r\to 0$. As was said before, the coordinate change $d\xi=Y(\rho)\,d\rho$ in the metric (19) allows to regard this curved geometry as a space of a variable deficit angle ranging from $2\pi(1-M)$

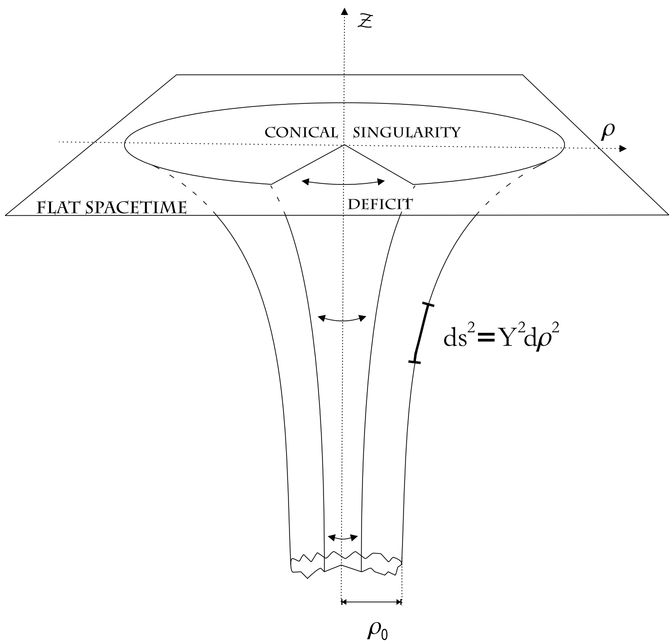

Fig. 2. Schematic representation of the spacetime (19) as embedded in three-dimensional Euclidean space.

at spatial infinity, to 2π at ρ_o ($r=0$). In this last limit, since the deficit cover the whole range of the angular variable, the metric describes a cylinder of radius ρ_o, which is obtained by identifying points in opposite sides of the *total* wedge. Thus, the geometry (19) is also asymptotically locally flat when $\rho \to \rho_o$. Fig. 1 depicts the scalar curvature $R(\rho)$, for $J/M = 1$ and several negative values of the Born–Infeld parameter λ. The minimum curvature is reached at a position $\rho_{\min}$ that depends only on the combination $\mathcal{J} = J/\sqrt{|\lambda|}$. In the highly deformed regime $\mathcal{J} \approx 1$ the curvature effects can be felt at positions very distant from the origin. Instead, as long as the low energy limit is restored ($\mathcal{J} \ll 1$), such effects are confined to small neighborhoods of $\rho_o = J/2M^2$ (i.e. $r = 0$). In the GR limit $\lambda \to \infty$ (equivalently, $Y \to 1$) there are no effect at all, because the manifold becomes flat.

The lower bound for the radial coordinate means that the space ends at a minimal circle of circumference $2\pi M\rho_o = \pi J/M$. However this boundary requires an infinite proper time to be reached, which implies that the conical singularity is smoothed. In fact the radial light rays satisfy $dT = Y\,d\rho$, so the coordinate time T diverges when a light ray approaches the minimal circle (because Y diverges). On the other hand, since the metric components in Eq. (19) do not depend on T, then $p_T = g_{TT}p^T = p^T \propto dT/d\tau$ is conserved on geodesics. This means that the proper time τ of a freely falling particle is proportional to the coordinate time T. Since timelike geodesics remain inside the light cones, then a particle needs an infinite proper time to reach the minimal circle. In Fig. 2 we have schematically depicted the spacetime (19) with $T = constant$ as embedded in three-dimensional Euclidean space with coordinates (ρ, θ, z). The *funnel*-like structure appearing in the figure comes from the function $z(\rho)$ which is

$$z = \int \sqrt{Y^2(\rho) - 1}\, d\rho, \tag{24}$$

so the Euclidean squared interval on that curve is given by $ds^2 = dz^2 + d\rho^2 = Y^2\,d\rho^2$. In the asymptotic region we have $Y \to 1$ and then, z becomes constant there (we set $z = 0$ in the figure).

The obtained geometry not only succeeds in smoothing the conical singularity of the GR ($Y = 1$) solution but avoids another unpleasant feature of Einstein theory in $n = 2+1$ that was posed in early works [44–47]: the existence of closed timelike curves (see also Ref. [48] where additional physical criteria was discussed in order to avoid CTC's). Such a undesirable property appears when coordinate t is considered continuous instead of T. This condition forces a jump $\Delta T = J\pi/M$ along the circle ($\Delta\theta = 2\pi$). While a jump of θ (deficit angle) is related with the mass m of the solution, a jump of T provides the solution with angular momentum. In fact, by replacing the solution (19) whit $Y = 1$ in $(2+1)$-Einstein equations it results that the energy–momentum tensor of the source is $T^{tt} \propto m\delta^2(\mathbf{r})$, $T^{ti} \propto J\varepsilon^{ij}\partial_j\delta^2(\mathbf{r})$; i.e., a spinning massive particle is at the origin $\rho = 0$ [44]. In this spacetime we can consider the closed curve with constant (t, ρ) in the interval (16) under the coordinate change given in (17). It then becomes

$$ds^2 = \left[\left(\frac{J}{2M^2}\right)^2 - \rho^2\right] M^2\, d\theta^2. \tag{25}$$

For $\rho < J/(2M^2)$ the closed curve in θ would be time-like. GR allows this possibility, since $Y = 1$ and no restrictions appears for the coordinate ρ. In the determinantal theory, instead, ρ is constrained to be greater than $J/(2M^2)$, so excluding CTC. The same mechanism responsible for the taming of the conical singularity at the origin seems to provide a natural chronological protection.

4. Concluding comments

Born–Infeld determinantal action (5) could be seen as a natural ultraviolet deformation of Einstein gravity which operates at scales of order $\ell \sim |\lambda|^{-1/2}$. For the theory (4), which could be considered the simplest structure among the BI program, it was shown in Ref. [18] that this scale plays an important role in n-dimensional cosmological scenarios, because it works as an effective initial vacuum energy driving the inflationary stage. Moreover, the invariants are bounded by the BI parameter λ, ruling in this way not only the behavior of the inflationary phase, but also establishing a maximum attainable spacetime curvature, with its subsequent singularity avoidance.

In the present context we witness a similar behavior; while action (4) was unable of deforming three-dimensional vacuum solutions, its extension (5) contains non-constant curvature states in empty space. The example considered here, the one given by metric (16), is particularly interesting because it represents a circular symmetric spacetime with bounded curvature invariants and $\alpha = 1$, i.e., without cosmological constant. The relevant parameter in the deformation is $\mathcal{J} = J/\sqrt{|\lambda|}$, so extremely high energy regimes leads to strongly rotating systems ($J^2 = \mathcal{O}(|\lambda|)$). The asymptotic spacetime is the conical geometry (19) with $Y = 1$ typical of three-dimensional GR solutions without cosmological constant. However, while the GR solution has $Y = 1\ \forall\rho$, the determinantal action leads to the behavior (21) for the function Y. In this way the singularity is removed and replaced with an unreachable asymptotic minimal circle. Both asymptotic regions are flat, but the space between them is curved. So, unlike GR, the angular momentum not only affects the global properties of the spacetime, but also has an effect on its curvature. Furthermore, its presence is crucial in order to erase the conical singularity at the origin, and to give rise an spacetime free of CTC.

It is worth mentioning that the results here obtained are clearly extensible to the four-dimensional cosmic string solution, whose metric reads

$$ds^2 = dT^2 - Y(\rho)^2\, d\rho^2 - M^2\rho^2\, d\theta^2 - dz^2, \tag{26}$$

where now the slices $T = constant$ are described in cylindrical coordinates (ρ, θ, z). As another remarkable physical consequence, BI gravity seems to forbid the possibility of packing energy in arbitrarily small regions. Differing from GR, any junction of the vacuum solution (26) with an inner solution has to be made at a radius bigger than $\rho_o = J/(2M^2)$.

Finally, we can mention that the increasing of the deficit angle (coming from the change $d\xi = Y(\rho)\, d\rho$ in (26)) as the string is closer, might have important observational implications on the lensing effect.

Additional solutions for a wider set of parameters (A, B), and the search for non-singular black hole fields coming from (5), will be matter of future works.

Acknowledgements

F.F. is indebted to Alina Fiorini for the encouragement afforded during the time of writing these lines. We would like to thank G. Giribet for his valuable comments. This research was supported by CONICET and Universidad de Buenos Aires.

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