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Neumann Casimir effect: A singular boundary-interaction approach

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ABSTRACT

Dirichlet boundary conditions on a surface can be imposed on a scalar field, by coupling it quadratically to a δ-like potential, the strength of which tends to infinity. Neumann conditions, on the other hand, require the introduction of an even more singular term, which renders the reflection and transmission coefficients ill-defined because of UV divergences. We present a possible procedure to tame those divergences, by introducing a minimum length scale, related to the nonzero 'width' of a *nonlocal* term. We then use this setup to reach (either exact or imperfect) Neumann conditions, by taking the appropriate limits. After defining meaningful reflection coefficients, we calculate the Casimir energies for flat parallel mirrors, presenting also the extension of the procedure to the case of arbitrary surfaces. Finally, we discuss briefly how to generalize the worldline approach to the nonlocal case, what is potentially useful in order to compute Casimir energies in theories containing nonlocal potentials; in particular, those which we use to reproduce Neumann boundary conditions.

1. Introduction

Material bodies can modify the vacuum structure of a quantum field theory, giving rise to interesting physical phenomena, like forces between neutral objects (Casimir effect) and changes in the decay rates of excited atoms [1]. In some cases, the effect of the bodies can be grossly described by assuming that the fields satisfy exact Dirichlet or Neumann boundary conditions on their surfaces. This idealization, as well as the perfect conductor approximation in QED, must of course be modified to cope with more realistic situations. Indeed, inside a real conductor the fields do not vanish. A more realistic description corresponds to a linear relation between the field (and its derivative) on one side of the conducting interface and the same objects on the other.

Boundary conditions are just an effective, approximate, macroscopic way of taking into account the effects of the interaction between the vacuum quantum fields and microscopic matter degrees of freedom inside the bodies. A more refined way of taking that interaction into account is to use linear response theory, whereby a generally nonlocal, quadratic effective action for the quantum field is obtained. The details about the microscopic interaction become then subsumed into the nonlocal kernels of this effective action term [2]. This procedure does not, in general, yield exact boundary conditions: on the one hand, the kernel is different from zero on a finite width region. On the other, in a realistic situation, it is a smooth bilocal function. Within the context of Casimir physics, there are several reasons to consider this and other kinds of 'imperfect' versions of the exact Dirichlet and Neumann boundary conditions. Firstly, phenomenology tells us that realistic models for the electromagnetic properties of neutral bodies can hardly be described by 'sharp' boundary conditions on the quantum fields. Secondly, the perfect conductor approximation presents difficulties, even from a purely theoretical standpoint: divergences of the vacuum energy density close to the surfaces [3] and, as a consequence, some ambiguities and even conceptual problems in the calculation of self-energies [4] and its concomitant gravitational effects [5]. Finally, there may be a practical reason to use a sharp but imperfect version of a boundary condition: Dirichlet Boundary Conditions (DBC) can be implemented numerically as interaction terms involving surface deltas, taking the strong coupling limit at the end. The analogue procedure is not known, however, for the case of Neumann Boundary Conditions (NBC). Note that an alternative approach to Neumann boundary conditions, in terms of a special kind of 'matching conditions' has been introduced in [6].

In this Letter we study the problem of imposing (NBC) on a massless real scalar field, by means of boundary interaction terms. There is, at first sight, a straightforward solution to this problem, namely, to adapt the approach followed for the Dirichlet case, to impose a different boundary condition. That would imply to use interaction terms proportional to the normal derivative of the δ-function. However, because these are highly singular objects, the situation becomes more subtle. Indeed, as we have shown in a previous paper [7], in order to compute the Casimir energy for potentials containing derivatives of the δ-function, it is necessary to

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introduce an ultraviolet regulator. We will show that the ultraviolet cutoff is needed to compute the transmission and reflection coefficients for a single mirror. We shall introduce that cutoff explicitly into the theory, by considering nonlocal interaction terms, showing also that the cutoff can be naturally interpreted as the inverse of the width of the mirror.

The structure of this Letter is as follows: In Section 2 we compute the reflection coefficients for theories described by local and nonlocal potentials, emphasizing the difficulties that the straightforward approach to the NBC has. We then calculate, in Section 3, the Casimir energy for flat mirrors using Lifshitz formula [8], both in the local and nonlocal cases. We finally present, in Section 4 the construction of nonlocal potentials for mirrors of arbitrary shape, and also a possible generalization of the worldline approach to cope with nonlocal potentials. This would allow one, in principle, to extend the worldline approach [9] to the calculation of Casimir energies in the Neumann case, by taking the appropriate limit.

2. Boundary conditions and interaction terms

Throughout this Letter, we shall consider a massless real scalar field $\varphi(x)$ in $d+1$ dimensions as the 'vacuum' (as opposed to matter) field. The aim of this section is to derive the boundary conditions for the vacuum field from the knowledge of the interaction term that accounts for its coupling to a mirror, by solving its equations of motion. In particular, we want to explore the nonlocal and NBC cases.

To that end, we first define the action S, which will be assumed to be of the form:

$$S(\varphi) = S_0(\varphi) + S_I(\varphi) \tag{1}$$

where the free action S_0 is given by

$$S_0(\varphi) = \frac{1}{2}\int d^{d+1}x\, \partial_\mu\varphi(x)\partial^\mu\varphi(x) \tag{2}$$

while the term that implements the interaction with the mirror has the structure

$$S_I(\varphi) = -\frac{1}{2}\int d^{d+1}x \int d^{d+1}x'\, \varphi(x)V(x,x')\varphi(x'), \tag{3}$$

with a real and symmetric kernel $V(x,x')$. The explicit form of the kernel depends on the details of the interaction between the quantum scalar field and the degrees of freedom in the mirror. The kernel $V(x,x')$ is assumed to be invariant under translations in time (x^0) and in the parallel ($x^i, i=1,\ldots,x^{d-1}$) spatial coordinates. Thus, denoting by $x_\parallel$ all the coordinates except x_d, we assume that:

$$V(x,x') = V(x_\parallel - x'_\parallel; x^d, x'^d). \tag{4}$$

Then the equations of motion adopt the nonlocal form:

$$\Box\varphi(x_\parallel, x^d) = -\int d^{d+1}x'\, V(x_\parallel - x'_\parallel; x^d, x'^d)\varphi(x'_\parallel, x'^d). \tag{5}$$

Then we take advantage of translation invariance in $x_\parallel$ to use the Fourier transformed version of (5):

$$(\partial_d^2 + k^2)\widetilde{\varphi}(k, x^d) = \int dx'^d\, \widetilde{V}(k; x^d, x'^d)\widetilde{\varphi}(k, x'^d), \tag{6}$$

where we introduced k^α, $\alpha = 0,1,\ldots,d-1$, and:

$$\varphi(x_\parallel, x^d) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x_\parallel}\widetilde{\varphi}(k, x^d),$$

$$V(x_\parallel - x'_\parallel, x^d) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x_\parallel - x'_\parallel)}\widetilde{V}(k; x^d, x'^d). \tag{7}$$

We are interested in solutions that look like plane waves far from the mirror, and we want to be able to extract from the solution the reflection and transmission coefficients. It is then natural to treat the system as a scattering problem, writing the solution by means of the corresponding Lippmann–Schwinger (L–S) equation:

$$\widetilde{\varphi}(k, x^d) = \widetilde{\varphi}^{(0)}(k, x^d) + \int dx'^d \times \int dx''^d\, \Delta(k; x^d, x'^d)\widetilde{V}(k; x'^d, x''^d)\widetilde{\varphi}(k, x''^d), \tag{8}$$

where $\widetilde{\varphi}^{(0)}$ is the (incident) free-particle wave, solution of

$$(\partial_d^2 + k^2)\widetilde{\varphi}^{(0)}(k, x^d) = 0 \tag{9}$$

and Δ is the retarded Green's function. We shall assume the free-particle solution to correspond to a wave incident from $x^d < 0$, namely: $\widetilde{\varphi}^{(0)}(k, x^d) = e^{ik^dx^d}$, where $k^d \equiv \sqrt{(k^0)^2 - \sum_{i=1}^{d-1}(k^i)^2} = k > 0$ and $k^0 > 0$, which are just the mass shell conditions.

On the other hand, the retarded Green's function satisfies:

$$(\partial_d^2 + k^2)\Delta(k; x^d - x'^d) = \delta(x^d - x'^d) \tag{10}$$

(with retarded boundary conditions) and may be written more explicitly as follows:

$$\Delta(k; x^d - x'^d) = \int \frac{d\nu}{2\pi} e^{i\nu(x^d - x'^d)} \frac{1}{-\nu^2 + (k^0 + i\eta)^2 - \sum_{i=1}^{d-1}(k^i)^2} = -\frac{i}{2k}e^{ik|x^d - x'^d|}. \tag{11}$$

Let us now solve the L–S equation in some particular cases.

2.1. Dirichlet-like boundary conditions

As a warming-up exercise, we consider a local kernel V_D which may be used to impose Dirichlet-like boundary conditions:

$$\widetilde{V}_D(k, x^d, x'^d) \equiv \mu_0(k)\delta(x^d)\delta(x'^d). \tag{12}$$

Inserting this into (8), we obtain:

$$\widetilde{\varphi}(k, x^d) = \widetilde{\varphi}^{(0)}(k, x^d) + \mu_0(k)\Delta(k; x^d, 0)\widetilde{\varphi}(k, 0), \tag{13}$$

whence we obtain, by evaluating the equation above at $x^d = 0$:

$$\widetilde{\varphi}(k, 0) = \frac{1}{1 + \frac{i\mu_0(k)}{2k}}, \tag{14}$$

and, finally:

$$\widetilde{\varphi}(k, x^d) = e^{ikx^d} - \frac{\frac{i\mu_0(k)}{2k}}{1 + \frac{i\mu_0(k)}{2k}} e^{ik|x^d|} \equiv \widetilde{\varphi}(k, x^d) = e^{ikx^d} + r(k)e^{ik|x^d|}. \tag{15}$$

When $\frac{|\mu_0(k)|}{2k} \to \infty$, we see that $\widetilde{\varphi}(k, x^d) = 0$ for $x^d > 0$, $r(k) = -1$, the wave is perfectly reflected and the field satisfies DBC at $x_d = 0$. An interesting particular case is $\mu_0(k) = \gamma k$, that produces a constant reflection coefficient and DBC in the limit $\gamma \to \infty$. This kind of potentials are generated by massless fermion fields confined to the mirror [10].

In the general case, the reflection (R) and transmission (T) coefficients are:

$$R = \left|r(k)\right|^2 = \frac{\frac{|\mu_0(k)|^2}{4(k^d)^2}}{1 + \frac{|\mu_0(k)|^2}{4k^2} - \frac{\mathrm{Im}(\mu_0)}{k}},$$

$$T = \left|1 + r(k)\right|^2 = \frac{1 - \frac{\mathrm{Im}(\mu_0)}{k}}{1 + \frac{|\mu_0(k)|^2}{4k^2} - \frac{\mathrm{Im}(\mu_0)}{k}}. \tag{16}$$

It is easy to check that $R + T = 1$.

2.2. Neumann-like boundary conditions

In this case, we consider a kernel:

$$\widetilde{V}_N\left(k, x^d, x'^d\right) \equiv \mu_2(k)\delta'\left(x^d\right)\delta'\left(x'^d\right). \tag{17}$$

Now we obtain the relation:

$$\widetilde{\varphi}\left(k, x^d\right) = \widetilde{\varphi}^{(0)}\left(k, x^d\right) + \mu_2(k)\left[\frac{\partial}{\partial x'^d}\Delta\left(k; x^d, x'^d\right)\right]_{x'^d=0}\widetilde{\varphi}'(k, 0), \tag{18}$$

which yields $\widetilde{\varphi}(k, 0) = \widetilde{\varphi}^{(0)}(k, 0)$ $(= 1)$, i.e., no effect on the value of the incident wave at the mirror. On the other hand, by taking a derivative with respect to x^d above, and setting $x^d = 0$:

$$\widetilde{\varphi}'(k, 0) = \frac{\widetilde{\varphi}'^{(0)}(k, 0)}{1 - \mu_2(k)D(k)}, \tag{19}$$

where:

$$D(k) \equiv \left[\frac{\partial^2}{\partial x^d \partial x'^d}\Delta\left(k; x^d, x'^d\right)\right]_{x_d=0, x'^d=0}. \tag{20}$$

This quantity is ill-defined. Indeed, we see that it is linearly divergent in the UV (large momenta in the x^d direction). Introducing a momentum cutoff Λ, we see that its regularized version, $D_{\mathrm{reg}}(k, \Lambda)$, behaves as follows:

$$D_{\mathrm{reg}}(k, \Lambda) \sim -\frac{\Lambda}{\pi} - i\frac{k}{2}. \tag{21}$$

There is a very clear physical meaning in this cutoff: indeed, as we shall show in the next subsection, it may be interpreted as due to a finite width ϵ for the mirror. In particular, it may be implemented by using a kernel similar to the one in this subsection, but with the derivatives of the deltas replaced by one of its approximants.

Keeping the cutoff $\frac{\Lambda}{\pi} \equiv \epsilon^{-1}$ finite, we find a relation involving the derivatives of the free and exact field configurations:

$$\widetilde{\varphi}'(k, 0) = \frac{\widetilde{\varphi}'^{(0)}(k, 0)}{1 + \mu_2(k)(\epsilon^{-1} + i\frac{k}{2})}. \tag{22}$$

It is possible (and convenient) in this context to hide the cutoff, by relating it to a quantity with a more direct physical meaning, playing the role of a renormalization condition. For example, introducing the ratio:

$$\alpha \equiv \left[\frac{\widetilde{\varphi}'(k, 0)}{\widetilde{\varphi}'^{(0)}(k, 0)}\right]_{k\to 0}, \tag{23}$$

we may write ϵ in terms of α and $\mu_2(0)$: $\epsilon^{-1} = \mu_2^{-1}(0)(\alpha^{-1} - 1)$.

Then we may write the general solution for the field in terms of the function μ_2 and the constant α:

$$\widetilde{\varphi}\left(k, x^d\right) = e^{ikx^d} - r(k)\,\mathrm{sign}\left(x^d\right)e^{ik|x^d|}, \tag{24}$$

where:

$$r(k) \equiv \frac{\frac{ik\mu_2(k)}{2}}{1 + \frac{\mu_2(k)}{\mu_2(0)}(\alpha^{-1} - 1) + \frac{ik\mu_2(k)}{2}} = \frac{\frac{ik\mu_2(k)}{2}}{1 + \mu_2(k)(\epsilon^{-1} + i\frac{k}{2})}. \tag{25}$$

It is clear that NBC emerge if $r(k) \to 1$, and this is the case for an infinitesimal $\mu_2(k) = -\epsilon$. Indeed, writing $\mu_2^{-1} = -\epsilon^{-1} + \Gamma^{-1}$ we have

$$r(k) = \frac{1}{1 - \frac{2i}{\Gamma k}}. \tag{26}$$

Note that, in the limit $\Gamma k \gg 1$, this corresponds to a "soft" NBC with $\widetilde{\varphi}'(k, 0) \sim \Gamma^{-1}$.

2.3. Nonlocal kernel

We consider here a case which includes the mirror's size into the game, albeit not in the most general form. It does allow one, however, to reach both the Dirichlet and Neumann cases as particular limits. Besides, it automatically introduces a regularization (related to a finite width) for the Neumann case.

This example corresponds simply to using the kernel:

$$\widetilde{V}_\epsilon\left(k; x^d, x'^d\right) = \sigma(k)g_\epsilon\left(x^d\right)g_\epsilon\left(x'^d\right) \tag{27}$$

where $g_\epsilon(x^d)$ is a function localized on a region of size ϵ. We shall assume, for the sake of concreteness, its support to be the interval $[-\epsilon/2, +\epsilon/2]$. It may even depend on k: everything we shall do in what follows would remain valid had one included such a dependence. The same holds true for an eventual dependence of σ on ϵ. As already stressed, one can think of the nonlocal kernel as coming from the integration of microscopic degrees of freedom living on the mirror and interacting with the quantum field φ. Although from this point of view the form of the kernel given in (27) may be nonrealistic, it will be sufficient in order to show the regularizing effect of a nonlocality in the normal direction.

We shall not use specific forms for the function $g_\epsilon(x^d)$ yet. However, one may think of size-ϵ approximants of the δ function or of its derivative, although the results we shall obtain will not depend on those assumptions.

The application of (8) to this case yields:

$$\widetilde{\varphi}\left(k, x^d\right) = \widetilde{\varphi}^{(0)}\left(k, x^d\right) + \sigma(k)\left[\int dx'^d\, \Delta\left(k; x^d, x'^d\right)g_\epsilon\left(x'^d\right)\right]\widetilde{\varphi}_g(k), \tag{28}$$

where $\widetilde{\varphi}_g(k) \equiv \int dx^d\, g_\epsilon(x^d)\widetilde{\varphi}(k, x^d)$.

Multiplying both members of (28) by $g_\epsilon(x^d)$ and integrating over x^d, we find the relation:

$$\widetilde{\varphi}_g(k) = \frac{\widetilde{\varphi}_g^{(0)}(k)}{1 - \sigma(k)\Delta_g(k, \epsilon)} \tag{29}$$

where:

$$\Delta_g(k, \epsilon) \equiv \int dx^d \int dx'^d\, g_\epsilon\left(x^d\right)\Delta\left(k; x^d, x'^d\right)g_\epsilon\left(x'^d\right). \tag{30}$$

Note that this object is finite for approximants that are square-integrable, although the limit when $\epsilon \to 0$ may be singular. For example, to reproduce the Neumann case, one may consider the function:

$$g_\epsilon\left(x^d\right) = -\frac{4}{\epsilon^2}\theta\left(\frac{\epsilon}{2} - \left|x^d\right|\right)\mathrm{sign}\left(x^d\right), \tag{31}$$

which yields a finite result for Δ_g whenever $\epsilon \neq 0$:

$$\Delta_g(k,\epsilon) = \frac{32}{k^2\epsilon^4}\left[\frac{\epsilon}{2} + \frac{1}{2ik}\left(e^{ik\epsilon}-1\right) - \frac{2}{ik}\left(e^{ik\frac{\epsilon}{2}}-1\right)\right]. \tag{32}$$

To proceed, we insert (29) into (28), obtaining:

$$\widetilde{\varphi}\left(k,x^d\right) = \widetilde{\varphi}^{(0)}\left(k,x^d\right) + \frac{\sigma(k)\widetilde{\varphi}_g^{(0)}(k)}{1-\sigma(k)\Delta_g(k,\epsilon)} \times \frac{1}{2ik}\int dx'^d\, e^{ik|x^d-x'^d|} g_\epsilon\left(x'^d\right). \tag{33}$$

Then we evaluate the equation above for two different situations, both corresponding to points outside of the mirror: either $x^d > \frac{\epsilon}{2}$, or $x^d < -\frac{\epsilon}{2}$, what yields:

$$\widetilde{\varphi}_>\left(k,x^d\right) = e^{ikx^d}\left[1 + \frac{\frac{\sigma(k)}{2ik}\widetilde{\varphi}_g^{(0)}(k)\widetilde{\varphi}_g^{*(0)}(k)}{1-\sigma(k)\Delta_g(k,\epsilon)}\right] \equiv t(k)e^{ikx^d},$$

$$\widetilde{\varphi}_<\left(k,x^d\right) = e^{ikx^d} + e^{-ikx^d}\frac{\frac{\sigma(k)}{2ik}\widetilde{\varphi}_g^{(0)}(k)\widetilde{\varphi}_g^{(0)}(k)}{1-\sigma(k)\Delta_g(k,\epsilon)} \equiv e^{ikx^d} + r(k)e^{-ikx^d}, \tag{34}$$

respectively. Note that $\widetilde{\varphi}_g^{(0)}(k) = \int dx^d\, e^{ikx^d} g_\epsilon(x^d)$, may be thought of as the Fourier transform of g_ϵ. Using the notation: $\int dx^d\, e^{-ikx^d} g_\epsilon(x^d) \equiv \widetilde{g}_\epsilon(k)$ to make that property explicit we write the T ant R coefficients as follows:

$$T(k,\epsilon) = \left|t(k)\right|^2 = \left|1 + \frac{\frac{\sigma(k)}{2ik}|\widetilde{g}_\epsilon(k)|^2}{1-\sigma(k)\Delta_g(k,\epsilon)}\right|^2,$$

$$R(k,\epsilon) = \left|r(k)\right|^2 = \left|\frac{\frac{\sigma(k)}{2ik}[\widetilde{g}_\epsilon^*(k)]^2}{1-\sigma(k)\Delta_g(k,\epsilon)}\right|^2. \tag{35}$$

After some calculations it is possible to show that $R + T = 1$. At this point it is worth to note that, generally speaking, nonlocality can induce violations to this relation, since the interaction with microscopic degrees of freedom in the mirror may affect unitarity. However, this is not the case for nonlocal kernels of the form (27).

For the Neumann-like case, we may consider the $\epsilon \to 0$ limit with the g_ϵ introduced in (31). This yields for Δ_g the asymptotic behaviour:

$$\Delta_g(k,\epsilon) \sim -\frac{4}{3}\epsilon^{-1} - i\frac{k}{2}, \quad \epsilon \sim 0, \tag{36}$$

as it may be seen from (32).

Note that, for the same approximant, we find:

$$\widetilde{g}_\epsilon(k) = -\frac{16}{ik\epsilon^2}\sin^2\left(\frac{k\epsilon}{4}\right). \tag{37}$$

To reach the exact NBC from the nonlocal case, one may use an appropriate dependence of σ, namely, choosing a $\sigma(k,\epsilon)$ such that condition is approached in the limit when $\epsilon \to 0$. To that end, we take the derivative of (33) at the origin, and see that, when $\epsilon \sim 0$:

$$\widetilde{\varphi}'(k,0) \simeq ik\left[1 + \frac{k}{2i}\frac{\sigma(k,\epsilon)}{1-\sigma(k,\epsilon)\Delta_g(k,\epsilon)}\right] \tag{38}$$

thus, using a σ such that:

$$\frac{\sigma(k,\epsilon)}{1-\sigma(k,\epsilon)\Delta_g(k,\epsilon)} = -\frac{2i}{k}, \tag{39}$$

yields NBC, i.e. $r(k) = 1$ and $t(k) = 0$. It is worth noting that, as in the local case, the NBC are obtained for an infinitesimal value of σ, that is $\sigma \simeq -3\epsilon/4$.

Finally, we stress that the Dirichlet-like case is obtained by considering g_ϵ to tend to a δ function. This implies $\widetilde{g}_\epsilon(k) \to 1$, and $\Delta_g(k\epsilon) \to \frac{1}{2ik}$, what (setting $\sigma \to \mu_0$) reproduces the Dirichlet-like case result.

3. Casimir energy

We will now compute the Casimir energy for the configuration of two identical mirrors centered at $x_d = 0$ and $x_d = a$ in $3+1$ dimensions. In principle, the Casimir energy can be obtained from the nonlocal effective action using path integral techniques [7,11]. However, it is simpler to use Lifshitz formula [8]. Indeed, we just need the analytic continuation to the imaginary frequency axis of the reflection coefficients already computed in the previous section. The reflection coefficients $r(k)$ depend on the wave vector $k^\alpha = (k^0, k^1, k^2)$. Denoting by $\bar{r}$ the analytic continuation $\bar{r} = r(k_0 = i\xi, k^1, k^2)$, according to Lifshitz formula, the Casimir energy then reads

$$E(a) = \frac{1}{2\pi}\int_0^\infty d\xi \int \frac{d^2k}{(2\pi)^2}\log\left(1-\bar{r}^2 e^{-2\kappa a}\right), \tag{40}$$

where $\kappa = \sqrt{k_1^2 + k_2^2 + \xi^2}$.

It is easy to see that, due to the symmetries we are assuming, $\bar{r}$ depends only on κ. Therefore, using spherical coordinates in momentum space Lifshitz formula can be written as

$$E(a) = \frac{1}{4\pi^2}\int_0^\infty d\kappa\, \kappa^2 \log\left(1-\bar{r}^2 e^{-2\kappa a}\right), \tag{41}$$

which is well defined as long as $\bar{r}^2 < 1$.

Let us now discuss the main properties of the reflection coefficients, for the local and nonlocal interaction terms previously considered. For the Dirichlet-like boundary conditions considered in Section 2.1, the analytic continuation of the reflection coefficient, which can be read from (15), is given by:

$$\bar{r}^2 = \left(\frac{\mu_0(\kappa)}{2\kappa + \mu_0(\kappa)}\right)^2. \tag{42}$$

Note that $\bar{r}$ is well defined and smaller than 1 for $\mu_0(\kappa) > 0$. Besides, it does reproduce the Dirichlet Casimir energy in the limit $\mu_0(\kappa)/\kappa \to \infty$. Inserting Eq. (42) into Eq. (41) we reproduce the usual result for δ-potentials [12].

For the local Neumann-like case described in Section 2.2, the reflection coefficient is given in (25), and its analytic continuation reads

$$\bar{r}^2 = \left[\frac{\mu_2(\kappa)\kappa}{2 + \mu_2(\kappa)(2\epsilon^{-1} + \kappa)}\right]^2, \tag{43}$$

and it is well defined except when $-\epsilon < \mu_2(\kappa) < 0$. The Neumann Casimir energy can be reproduced when $\mu_2 \to -\epsilon$ from below.

For the Neumann-like nonlocal boundary term considered in Section 2.3, Eqs. (34) and (37) allow we to derive

$$\bar{r}^2 = \left[\frac{128\bar{\sigma}}{\epsilon^4\kappa^3(1-\bar{\sigma}\bar{\Delta}_g)}\sinh^4\left(\frac{\kappa\epsilon}{4}\right)\right]^2, \tag{44}$$

where $\bar{\sigma}$ and $\bar{\Delta}_g$ denote the analytic continuations of σ and Δ_g respectively. In order to reproduce Neumann boundary conditions, $\bar{\sigma}$ must be a rather involved function of κ and ϵ. However, when computing the Casimir energy for mirrors separated by a distance $a \gg \epsilon$, we expect the relevant values of κ to satisfy $\kappa\epsilon \ll 1$. In this limit, that function simplifies to $\sigma = -\frac{3}{4}\epsilon$. If we assume that

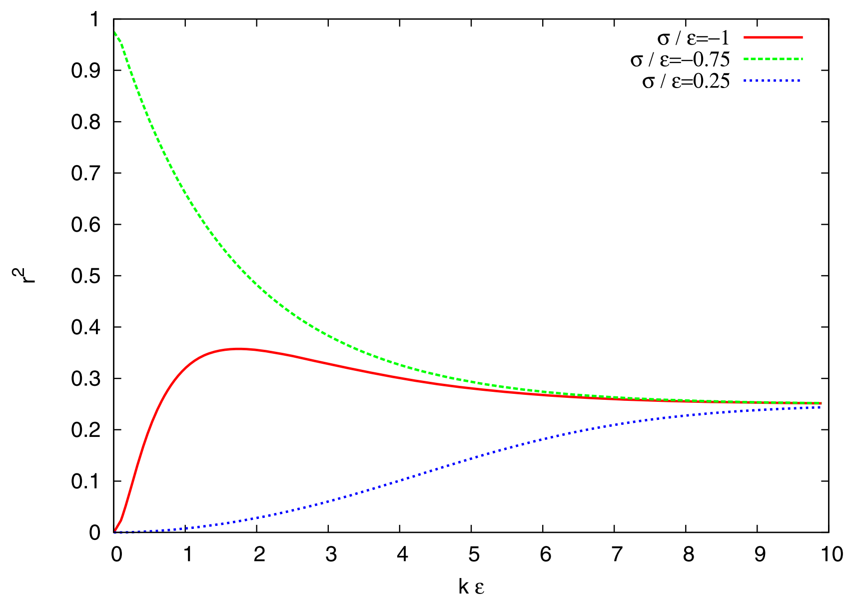

Fig. 1. The reflection coefficient given in (44) as a function of $x = \kappa\epsilon$, for different values of σ/ϵ. In the particular case $\sigma/\epsilon = -0.75$, the reflection coefficient tends to 1, reproducing NBC for $x \ll 1$.

σ is a constant, the Casimir energy for this reflection coefficient is well defined as long as $\sigma < -\frac{3}{4}\epsilon$ or $\sigma > 0$. These properties are illustrated in Fig. 1.

4. Generalization to arbitrary surfaces and worldline approach

In the previous sections, we have introduced nonlocal interaction terms, which may be regarded as 'regularized' versions of the terms one should introduce to impose Dirichlet or Neumann boundary conditions. Indeed, the singular terms involving Dirac's δ function or its derivatives disappear, at the price of introducing a finite length scale.

That step is not necessary for the Dirichlet case (unless one wanted to describe a medium with an intrinsic nonlocality). On the other hand, because of its highly singular nature, a *local* Neumann term should be approached as a special limit of a nonlocal term. Indeed, one should do so in order to tame the infinities that otherwise would pop up at a very early stage, namely, when calculating the reflection and transmission coefficients, *before* dealing with the Casimir energy.

The explicit construction of the nonlocal terms has, however, only been carried out for the simplest possible case, regarding the mirror's geometry. Indeed, we have considered terms which, in the local limit, would correspond to a plane mirror at $x_d = 0$.

Let us now extend the construction to more general surfaces. We have in mind a situation where one really wants to deal with a Neumann-like condition on a local (zero-width) surface and, in order to do that, one temporarily introduces a small nonlocality along the normal direction to the surface, on a length scale $\sim \epsilon$. That nonlocality is introduced to regularize the problem, and it should disappears in the end when one takes the 'zero width Neumann limit' $\epsilon \to 0$ (and the corresponding limit for the coupling constant). Because of this reason, we shall make use of some simplifications that stem from the fact that, even though it is different from zero, ϵ is very small in comparison with the other characteristic length scales in the system.

The only assumption about the surface will be that it is piecewise regular. Besides, it is sufficient to consider just one surface. Indeed, for more than one surface, one just have to add more terms to the action: one for every disconnected piece. Also, for the sake of simplicity, we shall first assume that the nonlocality only exists for the normal direction to the surface. Finally, we shall restrict our study to the case of surfaces in three-dimensional space.

Let us begin by writing the well-known *local* interaction term, corresponding to Dirichlet-like boundary conditions, for a zero-width static surface Σ, and generalize it to the nonlocal case afterwards. This surface will be assumed to be given in parametric form:

$$\Sigma : (\sigma_1, \sigma_2) \to \mathbf{Y}(\sigma), \quad \mathbf{Y}(\sigma) \in \mathbb{R}^{(3)}. \tag{45}$$

Then, the local Dirichlet-like interaction term $S_\Sigma(\sigma)$, is:

$$S_\Sigma(\varphi) = \frac{\mu_0}{2} \int dx_0\, d\sigma_1\, d\sigma_2 \sqrt{G(\sigma)} \big(\varphi\big[x_0, \mathbf{Y}(\sigma)\big]\big)^2 \tag{46}$$

where $G(\sigma) \equiv \det[G_{ab}(\sigma)]$, with $G_{ab}(\sigma) \equiv \mathbf{T}_a(\sigma) \cdot \mathbf{T}_b(\sigma)$, $a, b = 1, 2$, and $\mathbf{T}_a(\sigma) \equiv \partial_a \mathbf{Y}(\sigma)$.

We then introduce a nonlocality in this term, proceeding as follows: we first construct the finite volume region, Σ_ϵ, that results from 'dragging' Σ along the normal direction. The volume Σ_ϵ is spanned by introduced a new parameter, η (to be denoted also by σ_3), in such a way that:

$$\Sigma_\epsilon : (\sigma_1, \sigma_2, \eta) \to \mathbf{X}(\sigma, \eta), \tag{47}$$

such that $\mathbf{X}(\sigma, 0) = \mathbf{Y}(\sigma)$, $\forall \sigma$. η will have infinitesimal values around zero and we want it to introduce departures in the normal direction only. Then, to first-order in η, a parametrization for Σ_ϵ can be explicitly written:

$$\begin{aligned} \mathbf{X}(\sigma, \eta) &= \mathbf{X}(\sigma, 0) + \big[\partial_\eta \mathbf{X}(\sigma, \eta)\big]_{\eta=0} \eta + \mathcal{O}\big(\eta^2\big) \\ &= \mathbf{Y}(\sigma) + \widehat{\mathbf{N}}(\sigma)\eta + \mathcal{O}\big(\eta^2\big), \end{aligned} \tag{48}$$

where we introduced the unit normal vector field:

$$\widehat{\mathbf{N}}(\sigma) = \frac{\mathbf{T}_1(\sigma) \times \mathbf{T}_2(\sigma)}{|\mathbf{T}_1(\sigma) \times \mathbf{T}_2(\sigma)|}, \tag{49}$$

at each point of the surface (which is assumed to be regular).

In these coordinates, for small η, and using the index 3 for η, the metric tensor $G_{ij}(\sigma, \eta)$ $(i, j = 1, 2, 3)$ on Σ_ϵ becomes: $G_{ab}(\sigma, \eta) = G_{ab}(\sigma)$, $G_{aj}(\sigma, \eta) = G_{ja}(\sigma, \eta) = 0$, and $G_{33} = 1$.

With these conventions, we may write the local term above as follows:

$$S_\Sigma(\varphi) = \frac{\mu_0}{2}\int dx_0\, d\sigma_1\, d\sigma_2\, d\eta\, \sqrt{G(\sigma)}\delta(\eta)\big(\varphi\big[x_0, \mathbf{X}(\sigma,\eta)\big]\big)^2 = \frac{\mu_0}{2}\int dx_0\, d\sigma_1\, d\sigma_2\, \sqrt{G(\sigma)}\big(\varphi\big[x_0, \mathbf{Y}(\sigma)\big]\big)^2, \tag{50}$$

as it should be.

The nonlocal term (either Dirichlet- or Neumann-like) is then constructed in a quite straightforward way:

$$S_{\Sigma_\epsilon}(\varphi) = \frac{\lambda}{2}\int dx_0\, d\sigma_1\, d\sigma_2\, \sqrt{G(\sigma)}\, d\eta\, d\eta'\, g_\epsilon(\eta) g_\epsilon(\eta') \times \varphi\big[x_0, \mathbf{X}(\sigma,\eta)\big]\varphi\big[x_0, \mathbf{X}(\sigma,\eta')\big], \tag{51}$$

where g_ϵ has the form of an approximant of the δ in the Dirichlet case, and of its derivative in the Neumann case. Note that, with our conventions, η has the dimensions of a length. Thus we may effectively assume that because of the function g_ϵ, the relevant range of η is $\sim[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$.

As a concrete example, we may write S_{Σ_ϵ} for a sphere of radius R:

$$S_{\Sigma_\epsilon}(\varphi) = \frac{\lambda}{2}R^2\int dx_0\, d\theta\, d\varphi\, \sin^2\theta\, dr\, dr'\, g_\epsilon(r) g_\epsilon(r') \times \varphi\big[x_0, \mathbf{X}(\theta,\varphi,r)\big]\varphi\big[x_0, \mathbf{X}(\theta,\varphi,r')\big], \tag{52}$$

where we used spherical coordinates.

We conclude by presenting the implementation of this type of term within the worldline approach to Casimir effect [9], a very useful tool for the calculation of Casimir energies. The usual worldline applies, by construction, to Dirichlet-like boundary conditions, which emerge as the result of the introduction of a *local* potential term. On the other hand, as we have shown, Neumann conditions require the consideration of nonlocal terms, even when one is interested in imposing Neumann boundary conditions on a zero-width surface.

Let us consider here the changes one has to introduce in the worldline approach, to be able to deal with nonlocal potentials (we have in mind just the kind of nonlocality considered in the previous sections). For the Casimir energy of a (massive, for the sake of completeness) field in the worldline approach, the starting point is formally the same as in the local case; indeed, one first defines an effective action $\Gamma[V_\epsilon]$:

$$\Gamma[V_\epsilon] = \frac{1}{2}\operatorname{Tr}\ln\left[\frac{-\partial^2 + m^2 + V_\epsilon}{-\partial^2 + m^2}\right] \tag{53}$$

where now V_ϵ is an operator whose matrix elements, in the coordinate representation, are:

$$\mathcal{V}_\epsilon(x, x') = \langle x|V_\epsilon|x'\rangle = \langle x_0, \mathbf{x}|V_\epsilon|x'_0, \mathbf{x}'\rangle = \mu_0\delta(x_0 - x'_0)\int d^2\sigma\sqrt{G(\sigma)}\int d\eta\, d\eta'\, g_\epsilon(\eta)\delta^{(3)} \times \big(\mathbf{x} - \mathbf{X}(\sigma,\eta)\big)g_\epsilon(\eta')\delta^{(3)}\big(\mathbf{x}' - \mathbf{X}(\sigma,\eta')\big) \tag{54}$$

(we follow here the usual convention whereby Dirac's notation is used for the Hilbert space of functions of x).

Then, using Frullani's representation [13] for the logarithm of a ratio, one has:

$$\Gamma[V_\epsilon] = -\frac{1}{2}\int_{0+}^{\infty}\frac{dT}{T}\big[K(T) - K_0(T)\big], \tag{55}$$

where:

$$K(T) = \int d^4x\, K(x, T; x, 0), \tag{56}$$

$$K(x'', T; x', 0) \equiv \langle x''|e^{-TH}|x'\rangle, \tag{57}$$

with $H = p^2 + m^2 + V_\epsilon \equiv H_0 + V_\epsilon$, and

$$K_0(x'', T; x', 0) \equiv \langle x''|e^{-TH_0}|x'\rangle. \tag{58}$$

The only place where a departure with respect to the local case appears is in the path integral representation for $K(T)$. Partitioning the T interval into many equal steps and evaluating the transition amplitude for an infinitesimal evolution in each step, one realizes that, in the limit when the number of steps tends to infinity, the following path integral representation emerges:

$$K(T) = \mathcal{N}e^{-m^2T}\int_{x(T)=x(0)}\mathcal{D}x e^{-\mathcal{S}[x]} \tag{59}$$

where:

$$\mathcal{S}[x] \equiv \mathcal{S}_0[x] + \mathcal{S}_I[x] \tag{60}$$

with

$$\mathcal{S}_0[x] = \int_0^T d\tau\frac{1}{4}\dot{x}^2(\tau), \tag{61}$$

$$\mathcal{S}_I[x] = \int_0^T d\tau\int_0^T d\tau'\, \mathcal{V}_\epsilon\big[x(\tau), x(\tau')\big], \tag{62}$$

and $\mathcal{N}$ is a factor that comes from the path integration over momenta, and is independent of the potential.

Finally, coming back to the effective action, and using the known result for the free transition amplitude, one may write:

$$\Gamma[V_\epsilon] = -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d+1}{2}}}\int_{0+}^{\infty}\frac{dT}{T^{1+\frac{d+1}{2}}}e^{-m^2T}\big[\langle e^{-\mathcal{S}_I[x]}\rangle - 1\big], \tag{63}$$

where

$$\langle e^{-\mathcal{S}_I[x]}\rangle = \frac{\int_{x(0)=x(T)}\mathcal{D}x e^{-\mathcal{S}_I[x]}e^{-\mathcal{S}_0[x]}}{\int_{x(0)=x(T)}\mathcal{D}x e^{-\mathcal{S}_0[x]}}. \tag{64}$$

We do not dwell with the numerical evaluation of this kind of path integral, which may certainly be more difficult than its local counterpart, since the interaction term is not simply the integral of a local function of time. However, since the scale of nonlocality is assumed to be small, we expect the properties of the integral not to be dramatically different to the standard ones. Finally, a scaling to unit proper time may be of course implemented in a similar way to the local case, as well as the extraction of a 'center of mass' for the closed paths involved in the integral over paths.

5. Conclusions

We have shown that NBC can be obtained by coupling the field to an interaction term which, if one wanted to have well-defined transmission and reflection properties, has to include an intrinsic regularization. We have also provided an explicit mechanism to introduce that regularization in the calculation.

That regularization may be naturally thought of as due to two sources: a finite width, and a finite nonlocal coupling. Neumann conditions on a zero width surface emerge when one takes a special limit, whereby the coupling constant is related to the width.

One may then conclude that, to reach NBC, one needs essentially just one constant to control the UV behaviour, rather than two independent scales. Of course, if one wanted to deal with a phenomenological model where the nonlocal term is derived, one may obtain more that one scale, and the resulting boundary conditions may exhibit a richer structure.

We have presented a possible way to implement the kind of path integral that one would require if one wanted to use a worldline approach to nonlocal terms. We believe such an approach might shed light on the properties of the Casimir energy with NBC for arbitrary surfaces.

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