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Casimir effect with dynamical matter on thin mirrors

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ABSTRACT

We calculate the Casimir energy for scalar and gauge fields in interaction with zero-width mirrors, including quantum effects due to the matter fields inside the mirrors. We consider models where those fields are either scalar or fermionic, obtaining general expressions for the energy as a function of the vacuum field 1PI function. We also study, within the frame of a concrete model, the role of the dissipation induced by those degrees of freedom, showing that, after integration of the matter fields, the effective theory for the electromagnetic field contains modes with complex energies. As for the case of Lifshitz formula, we show that the formal result obtained by neglecting dissipation coincides with the correct result that comes from the quantum fluctuations of both bulk and matter fields.

It has become increasingly clear that the use of sensible models for the description of the media that constitute the mirrors is an unavoidable step for the refinement of Casimir energy calculations [1]. In particular, the theory of quantum open systems is the natural approach to clarify the role of dissipation in Casimir physics. In this framework, dissipation and noise appear in the effective theory of the relevant degrees of freedom (the electromagnetic field) after integration of the matter degrees of freedom.

In this Letter we consider mirrors described by thin films, equipped with matter fields confined to them; these localized fields shall induce an effective action for the coupling of the vacuum fields to the mirrors. It will turn out that the resulting effective action falls into a class of model that we had studied in a previous work [2]. We then apply those results here in a quite straightforward way, obtaining the Casimir energy for different cases: regarding the vacuum fields, we shall consider the cases of a massless (real) scalar field and of an Abelian gauge field. As we shall see, under our general assumptions, the Casimir energy of the latter may be obtained as a derived result of the former, after performing some identifications.

The Casimir effect produced by thin plasma sheets has been previously considered by other authors (see [3,4] and references therein) imposing appropriate boundary conditions on the position of the sheets [3], considering a fluid of non-relativistic electrons confined to the surface [4], or describing the matter inside thin films using the particle in a box model [5]. We will extend these results to the case of relativistic bosonic or fermionic degrees of freedom confined to the mirrors. Besides, for a particular fermionic model, we will show that, in the effective theory for the electromagnetic field, the modes have complex energies, due to the possibility of transferring energy from the bulk to the thin surfaces. The Casimir energy is of course always real, and given by the sum of eigenfrequencies of the full system (bulk and matter fields).

Let us first consider the effective action, $S_{\rm eff}^{(I)}$, for a real scalar field φ, which results from its interaction with matter confined to two identical, flat, parallel and zero-width mirrors in d spatial dimensions, defined by the equations $x_d = 0$ and $x_d = a$. The total Euclidean action S for φ will then be of the form:

$$S(\varphi) = S_0(\varphi) + S_{\rm eff}^{(I)}(\varphi), \tag{1}$$

where $S_0 \equiv \frac{1}{2}\int d^{d+1}x\, \partial_\mu\varphi\partial_\mu\varphi$, defines the free theory. $S_{\rm eff}^{(I)}$ do, of course, depends on the nature of the fields confined to the mirrors and on the structure of their interaction with φ. The functional form of $S_{\rm eff}^{(I)}$ will be assumed to be quadratic in φ. This approximation is justified by the following reason: since the media are assumed to impose (approximate) Dirichlet boundary conditions, the relevant configurations correspond to *small* values for the fields on the mirrors (otherwise there would not be approximate boundary conditions there). It is then legitimate to assume that higher-order terms in those fields shall be strongly suppressed in comparison with the lowest, non-trivial one (quadratic), since they will have a smaller weight in the functional integral.

With this remark in mind, taking into account the fact that the matter fields are confined to the mirrors, and recalling that the interaction is assumed to be local, we have:

$$S_{\rm eff}^{(I)}(\varphi) = S_{\rm eff}^{(1)}(\varphi) + S_{\rm eff}^{(2)}(\varphi) \tag{2}$$

where:

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$$S_{\rm eff}^{(\alpha)}(\varphi) = \frac{1}{2}\int dx_0 \int dx'_0 \int d^{d-1}x_\parallel \int d^{d-1}x'_\parallel \int dx_d \int dx'_d \times \varphi(x_0, \mathbf{x}_\parallel, x_d)\lambda(x_0 - x'_0; \mathbf{x}_\parallel - \mathbf{x}'_\parallel)\delta(x_d - a_\alpha)\delta(x'_d - a_\alpha) \times \varphi(x'_0, \mathbf{x}'_\parallel, x_d), \tag{3}$$

with $a_1 \equiv 0$ and $a_2 \equiv a$. $\mathbf{x}_\parallel$ denotes the $d-1$ coordinates parallel to the mirror: $x_1, x_2, \ldots, x_{d-1}$.

The interaction term will be obtained as an effective action coming from the integration of the degrees of freedom confined to the walls that interact *locally* with the scalar field $\varphi(x)$ at $x_d = 0$ and $x_d = a$. To simplify the notation in the following calculations, we shall denote by y_a the spacetime coordinates for the matter fields, with $y_a \equiv x_a$, for $a = 0, 1, \ldots, d-1$.

Using a Fourier transformation with respect to the y_a coordinates,

$$S_{\rm eff}^{(\alpha)}(\varphi) = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\tilde{\varphi}^*(k, a_\alpha)\tilde{\lambda}(k)\tilde{\varphi}(k, a_\alpha) \tag{4}$$

where $k \equiv (k_a)_{a=0}^{d-1}$, are the Fourier space variables associated to the mirror's spacetime coordinates. Thus, the effect of the matter modes can, for this case, be encoded in a single function $\tilde{\lambda}(k)$. Moreover, depending on whether the dynamics of the matter fields is assumed to be relativistic or non-relativistic, it will depend on $\kappa \equiv \sqrt{k_a k_a}$, or on k_0 and $|\mathbf{k}_\parallel|$, respectively.

We see that the term (4) is the $\epsilon \to 0$ version of the models we have introduced in [2]. Thus, the result of the derivation presented there for the Casimir energy has direct applicability here. That is true, regardless of the microscopic model assumed for the matter field; however, before writing the expression for the energy, let us derive an expression for $\tilde{\lambda}$ from a concrete model. Note that, since the structure of $S_{\rm eff}^{(\alpha)}$ depends trivially on the value of a_α, we just need to consider the $\alpha = 1$ case, shifting the field afterwards.

A first model emerges from the assumption that $S_{\rm eff}^{(1)} \equiv S_{\rm eff}$ is due to the linear coupling of φ to a microscopic real scalar field $\xi(y)$, in d spacetime dimensions, equipped with an action $S_m(\xi)$:

$$e^{-S_{\rm eff}(\varphi)} = \frac{\int \mathcal{D}\xi\, e^{-S_m(\xi) + ig\int d^d y\, \xi(y)\varphi(y,0)}}{\int \mathcal{D}\xi\, e^{-S_m(\xi)}}, \tag{5}$$

where g is a coupling constant. The matter field ξ may have a self-interaction, controlled by an independent coupling constant, implicit in S_m.

To proceed, we denote by $W(J)$ the generating functional of connected correlation functions of ξ (in a d-dimensional spacetime), related to $\mathcal{Z}(J)$, the one for the full correlation functions:

$$\mathcal{Z}(J) = \int \mathcal{D}\xi\, e^{-S_m(\xi) + \int d^d y\, J(y)\xi(y)}, \tag{6}$$

by $W = \ln \mathcal{Z}$. We then have that $S_{\rm eff}(\varphi) = -W[ig\varphi(y,0)]$. On the other hand, since only the quadratic part will be retained,

$$S_{\rm eff}(\varphi) = -W\big[ig\varphi(y,0)\big] \simeq \frac{1}{2}g^2 \int d^d y \int d^d y'\, \varphi(y,0) W^{(2)}(y,y')\varphi(y',0), \tag{7}$$

where $W^{(2)}$ is the connected 2-point function. Finally, since we assume translation invariance along the mirrors' surface, we conclude that:

$$\tilde{\lambda}(\kappa) = g^2 \tilde{W}^{(2)}(\kappa). \tag{8}$$

Thus, for this first example, $\tilde{\lambda}$ may be read from the full propagator for the material field on the mirror. In the relativistic case, we know that:

$$\tilde{W}^{(2)}(\kappa) = \frac{z(\kappa)}{\kappa^2 + \Pi(\kappa)} \tag{9}$$

where z and Π denote the wave function renormalization function and the full self-energy, respectively. Of course, we assume that the 2-point function has been renormalized, so that there is no remaining ambiguity in either one of these functions, which depend only on the d-scalar κ. Since the full propagator may be related (via analytic continuation) to response functions, we see that the renormalization conditions (when needed) can be naturally identified with the *static* response functions for the field, due to the microscopic degrees of freedom on the mirrors.

Thus, unless quantum effects introduced qualitatively important changes in the form of the full propagator, the conclusion is that, in this example, one should expect to have a good description by using the free propagator, namely,

$$\tilde{\lambda}(\kappa) \sim \frac{g^2}{\kappa^2 + m^2}, \tag{10}$$

with the obvious changes if the matter field were described by a different dispersion relation.

The zero-point energy density $\mathcal{E}_0$ can be written as

$$\mathcal{E}_0 \equiv \lim_{T,L\to\infty} \frac{1}{L^{d-1}T}\ln\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\right), \tag{11}$$

where

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \frac{\int \mathcal{D}\varphi\, e^{-S(\varphi)}}{\int \mathcal{D}\varphi\, e^{-S_0(\varphi)}}. \tag{12}$$

This expression for the vacuum energy is of course equivalent to the sum over eigenfrequencies of the full quantum system. For the case of two mirrors, one is interested not in $\mathcal{E}_0$, but rather in the subtracted quantity, $\tilde{\mathcal{E}}_0$, defined as the difference

$$\tilde{\mathcal{E}}_0(a) \equiv \mathcal{E}_0 - \mathcal{E}_0(\infty) \tag{13}$$

where $\mathcal{E}_0(\infty)$ denotes the surface energy density when the mirrors are separated by an infinite distance. Using the results of Ref. [2], we can write the general expression for the Casimir energy:

$$\tilde{\mathcal{E}}_0(a) = \frac{1}{2^d \pi^{d/2}\Gamma(d/2)} \times \int_0^\infty d\kappa\, \kappa^{d-1} \ln\left\{1 - \left[1 + \frac{2\kappa}{\tilde{\lambda}(\kappa)}\right]^{-2} e^{-2\kappa a}\right\}. \tag{14}$$

The standard behaviour for $\tilde{\lambda}(\kappa)$ is $\sim \kappa^{-2}$ for large values of κ; this implies that the UV behaviour shall be softened with respect to the perfect mirror case, as expected. The IR region of the integrand is, on the other hand, strongly affected by the inclusion of the 2-point function; this is more evident for the case of a massless microscopic field.

Let us now consider the case of an Abelian gauge field A_μ in $d+1$ dimensions, whose free action is of the usual Maxwell type

$$S_0(A) = \int d^{d+1}x \frac{1}{4}F_{\mu\nu}F_{\mu\nu}. \tag{15}$$

The coupling to the charged fields on the mirror, whose (total) electric current is $J_a(y)$ will be assumed to be of the standard $J_a A_a$ type, where the current component J_d does not appear (we assume that no charge can escape from the mirror). Under these assumptions, the effective action can be written in the following fashion:

$$e^{-S_{\rm eff}(A)} = \frac{\int \mathcal{D}\mu\, e^{-S_m(\mu) + ie\int d^d y\, J_a(y)A_a(y,0)}}{\int \mathcal{D}\mu\, e^{-S_m(\mu)}}, \tag{16}$$

where we use "μ" to denote an unspecified field (or fields) which carry a conserved current, and e is the coupling constant. It is straightforward to verify that this defines a gauge-invariant functional of A_a: $S_{\rm eff}(A) = S_{\rm eff}(A + \partial\omega)$, for every smooth function ω, as a consequence of current conservation.

Then, in the quadratic approximation for $S_{\rm eff}(A)$,

$$S_{\rm eff}(A) \simeq \frac{1}{2}e^2 \int d^d y \int d^d y'\, A_a(y,0)\Pi_{ab}(y,y')A_b(y',0), \tag{17}$$

the kernel Π_{ab} is given by the current–current correlation function:

$$\Pi_{ab}(y,y') = \langle J_a(y,0)J_b(y',0)\rangle, \tag{18}$$

where the average symbols correspond to matter fields functional averaging,

$$\langle \ldots \rangle = \frac{\int \mathcal{D}\mu \cdots e^{-S_m(\mu)}}{\int \mathcal{D}\mu\, e^{-S_m(\mu)}}. \tag{19}$$

Gauge invariance implies that Π_{ab} is transverse; this is more easily formulated in Fourier space: $k_a\tilde{\Pi}_{ab}(k) = 0$.

When the matter fields action is relativistic, we may write $\tilde{\Pi}_{ab}$ in terms of just a scalar function of κ,

$$\tilde{\Pi}_{ab}(k) = \tilde{\Pi}(\kappa)\kappa^2\delta^{\perp}_{ab}(k), \tag{20}$$

where $\delta^{\perp}_{ab}(k)$ is the transverse tensor: $\delta^{\perp}_{ab}(k) = \delta_{ab} - k_a k_b/\kappa^2$. With this information, we can then produce more convenient expressions for $S_{\rm eff}(A)$,

$$\begin{aligned} S_{\rm eff}(A) &\simeq \frac{1}{2}e^2 \int \frac{d^d k}{(2\pi)^d}\tilde{A}^*_a(k,0)\tilde{\Pi}_{ab}(k)\tilde{A}_a(k,0) \\ &= \frac{1}{4}e^2 \int \frac{d^d k}{(2\pi)^d}\tilde{F}^*_{ab}(k,0)\tilde{\Pi}(\kappa)\tilde{F}_{ab}(k,0), \end{aligned} \tag{21}$$

which shows that $\kappa^2\tilde{\Pi}$ plays a similar role to $\tilde{\lambda}$ for the real scalar field.

When matter is *non-relativistic*, the corresponding effective action, $S^{nr}_{\rm eff}$, will have the same structure as in the first line of the previous equation,

$$S^{nr}_{\rm eff}(A) = \frac{1}{2}e^2 \int \frac{d^d k}{(2\pi)^d}\tilde{A}^*_a(k,0)\tilde{\Pi}^{nr}_{ab}(k)\tilde{A}_a(k,0) \tag{22}$$

with an, in principle, different vacuum polarization tensor $\tilde{\Pi}^{nr}_{ab}$. This tensor will still be transverse (current conservation) but it will fail to be just a scalar times the transverse δ. Lack of relativistic invariance means that its structure will be more complex, depending on *two* independent functions, which in turn will depend on $|k_0|$ and $|\mathbf{k}|$ separately (note that $|\mathbf{k}|$ is the norm of a $(d-1)$-dimensional vector).

Assuming rotation invariance on the $(d-1)$-dimensional spatial plane of each mirror, we can decompose $\tilde{\Pi}^{nr}_{ab}$, using now *two* independent tensors: $P^{\perp}_{ab}$ and $P^{\parallel}_{ab}$, that can be constructed having in mind that only the Galilean group of symmetries is relevant here. A convenient choice for those tensors is such that $P^{\perp}_{ab}$ is transverse in the $(d-1)$-dimensional sense, namely,

$$P^{\perp}_{00} = 0 = P^{\perp}_{0i} = P^{\perp}_{i0}, \qquad P^{\perp}_{ij} = \delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|}, \tag{23}$$

where i, j denote run over the values $1, \ldots, d-1$, and

$$P^{\parallel}_{ab} = \delta^{\perp}_{ab} - P^{\perp}_{ab}. \tag{24}$$

Note that, with the conventions above, they verify:

$$\left(P^{\perp}\right)^2 = P^{\perp}, \quad \left(P^{\parallel}\right)^2 = P^{\parallel}, \quad P^{\perp}P^{\parallel} = 0, \quad P^{\perp} + P^{\parallel} = \delta^{\perp}, \tag{25}$$

and, on the other hand, each one of them is transverse to k_a.

Then the general structure of $\tilde{\Pi}^{nr}_{ab}$ is:

$$\tilde{\Pi}^{nr}_{ab} = \tilde{\Pi}^{\perp}P^{\perp}_{ab} + \tilde{\Pi}^{\parallel}P^{\parallel}_{ab}, \tag{26}$$

where $\tilde{\Pi}^{\perp}$ and $\tilde{\Pi}^{\parallel}$ are functions of $|k_0|$ and $|\mathbf{k}|$. These two functions are related to different properties of the medium. In the $T > 0$ case, for example, the function $\tilde{\Pi}^{\parallel}$ is responsible for the Debye screening of static electric fields, with a screening length determined by $\tilde{\Pi}^{\parallel}(k_0 = 0, |\mathbf{k}| \to 0)$. Static magnetic field, on the other hand, are not screened, although there is non-vanishing screening when time-dependent magnetic fields are considered. They are also related to plasma oscillations, with plasma frequencies determined also by the coefficients above.

Finally, to find the Casimir energy corresponding to the gauge field case, we recall that that energy can be derived from the Euclidean functional:

$$\mathcal{Z} = \int [\mathcal{D}A]e^{-S(A)} \tag{27}$$

where $[\mathcal{D}A]$ denotes the integration measure including gauge fixing and, as in the scalar case, S is the sum of the free action plus the effective actions concentrated on the two mirrors:

$$\begin{aligned} S^{(I)}_{\rm eff} = \frac{1}{2}e^2 \int dx_d \int dx'_d \int \frac{d^d k}{(2\pi)^d}\tilde{A}^*_a(k,x_d)\tilde{\Pi}(\kappa)\kappa^2 \\ \times \delta^{\perp}_{ab}(k)\sum_{\alpha=1}^{2}\delta(x_d - a_\alpha)\delta(a_\alpha - x'_d)\tilde{A}_b(k,x'_d), \end{aligned} \tag{28}$$

where we assumed relativistic matter fields.

The path integral must be gauge-fixed; we adopt the ‘axial’ gauge $A_d \equiv 0$. This is a convenient choice, since it yields a quite simple form for the action:

$$S = \frac{1}{2}\int dx_d \int dx'_d \int \frac{d^d k}{(2\pi)^d}\tilde{A}^*_a(k,x_d)M_{ab}(k;x_d,x'_d)\tilde{A}_b(k,x'_d), \tag{29}$$

where

$$\begin{aligned} M_{ab}(k;x_d,x'_d) = -\partial_d^2\delta(x_d - x'_d)\delta^{\parallel}_{ab}(k) + \Big\{\left(-\partial_d^2 + \kappa^2\right)\delta(x_d - x'_d) \\ + e^2\kappa^2\tilde{\Pi}(\kappa)\left[\sum_{\alpha=1}^{2}\delta(x_d - a_\alpha)\delta(a_\alpha - x'_d)\right]\Big\}\delta^{\perp}_{ab}(k) \end{aligned} \tag{30}$$

and $\delta^{\parallel}_{ab} \equiv \delta^{\perp}_{ab} - \delta_{ab}$.

The formal result for $\mathcal{Z}$ is

$$\mathcal{Z} = \left(\det\left[M_{ab}(k;x_d,x'_d)\right]\right)^{-\frac{1}{2}}, \tag{31}$$

where the determinant is evaluated over all the (continuum and discrete) indices. Since each variable in the path integral may be uniquely decomposed into transverse and longitudinal components: $\tilde{A}_a = \tilde{A}^{\perp}_a + \tilde{A}^{\parallel}_a$, we have that:

$$\mathcal{D}\tilde{A} = \mathcal{D}\tilde{A}^{\perp}\mathcal{D}\tilde{A}^{\parallel}, \tag{32}$$

and, as a consequence,

$$\mathcal{Z} = \left(\det\left[M^{\perp}(k;x_d,x'_d)\right]\right)^{-\frac{d-1}{2}}\left(\det\left[M^{\parallel}(k;x_d,x'_d)\right]\right)^{-\frac{1}{2}}, \tag{33}$$

where

$$\begin{aligned} M^{\perp}(k;x_d,x'_d) &= \left(-\partial_d^2 + \kappa^2\right)\delta(x_d - x'_d) \\ &\quad + e^2\kappa^2\tilde{\Pi}(\kappa)\left[\sum_{\alpha=1}^{2}\delta(x_d - a_\alpha)\delta(a_\alpha - x'_d)\right], \\ M^{\parallel}(k;x_d,x'_d) &= \left(-\partial_d^2\right)\delta(x_d - x'_d). \end{aligned} \tag{34}$$

Note the power of $d-1$ in (33), which appears as a because $\delta^{\perp}_{ab}$ is the identity on the space of transverse fields, a $(d-1)$-dimensional space.

From the previous expressions, we see that $\mathcal{Z}$ is equivalent to the product of the vacuum functional for a massless free field which sees no mirrors, times $d-1$ identical factors corresponding to scalar fields which interact with the walls with a (common) coupling $\tilde{\lambda} = e^2\kappa^2\tilde{\Pi}(\kappa)$. The free scalar field factor is irrelevant to the Casimir energy, since we subtract the energy when the mirrors are infinitely distant, while the others yields $(d-1)$ times a scalar field like energy

$$\tilde{\mathcal{E}}_0(a) = \frac{d-1}{2^d\pi^{d/2}\Gamma(d/2)}\int_0^\infty d\kappa\,\kappa^{d-1}\ln\left\{1-\left[1+\frac{2}{e^2\kappa\Pi(\kappa)}\right]^{-2}e^{-2\kappa a}\right\}. \tag{35}$$

We specialize the previous study to an interesting concrete example: that of $d=3$ with matter described by a massless Dirac field in a reducible $4n$ component representation (which preserves parity), we have [6]:

$$\tilde{\Pi}(\kappa) = \frac{n}{8\kappa}, \tag{36}$$

so that, after introducing the dimensionless variable $x = \kappa a$,

$$\tilde{\mathcal{E}}_0(a) = \frac{1}{8\pi a^3}\int_0^\infty dx\,x^2\ln\left\{1-\left[1+\frac{16}{e^2 n}\right]^{-2}e^{-2x}\right\} = -\frac{C(e^2 n)}{a^3}. \tag{37}$$

Incidentally, this is one of the cases studied in [2]. This expression has the remarkable feature that the dependence of the Casimir energy with the distance is exactly equal to the one for a pair of perfect mirrors, albeit with a different global factor $C(e^2n)$ that interpolates between zero, for vanishing coupling constant $e^2 = 0$, and the result for perfect mirrors for $e^2 \to \infty$.

This is to be contrasted to the non-relativistic case which, using the same gauge fixing as before, yields a result composed of two contributions, corresponding to the modes that are *spatially* transverse or longitudinal:

$$\begin{aligned}\tilde{\mathcal{E}}_0(a) &= \frac{d-2}{2^d\pi^{d/2}\Gamma(d/2)}\int_0^\infty d\kappa\,\kappa^{d-1} \\ &\quad\times\ln\left\{1-\left[1+\frac{2}{e^2\kappa\tilde{\Pi}^{\perp}(\kappa)}\right]^{-2}e^{-2\kappa a}\right\} \\ &\quad+\frac{1}{2^d\pi^{d/2}\Gamma(d/2)}\int_0^\infty d\kappa\,\kappa^{d-1} \\ &\quad\times\ln\left\{1-\left[1+\frac{2}{e^2\kappa\tilde{\Pi}^{\parallel}(\kappa)}\right]^{-2}e^{-2\kappa a}\right\}.\end{aligned} \tag{38}$$

Finally, we attempt to reinterpret the previous results for the Casimir energy as emerging from the existence of a discrete set of modes, reflecting the existence of (imperfect) boundary conditions at $x_d = 0, a$. We shall focus, for the sake of simplicity, on the case of a massless scalar vacuum field in $d+1$ dimensions. After integrating out the matter degrees of freedom, one can evaluate the in–in, Schwinger–Keldysh, or closed time path effective action $S_{\rm eff}^{\rm CTP}$ [7]. On general grounds, this effective action has non-local kernels, denoting the existence of dissipative and noise effects. Taking the functional variation of the effective action, one can write a Langevin equation for the system field as

$$\begin{aligned}&\Box\varphi + g^2\sum_\alpha\int dt'\,d^{d-1}x'_\parallel\,G_{\rm ret}(t,\mathbf{x}_\parallel,t',\mathbf{x}'_\parallel)\delta(x_d-x_\alpha)\varphi(t',\mathbf{x}'_\parallel,x_\alpha) \\ &\quad= \sum_\alpha K(t,\mathbf{x}_\parallel)\delta(x_d-x_\alpha)\end{aligned} \tag{39}$$

where $G_{\rm ret}$ is the dissipation kernel, and K is a stochastic force whose correlation function is related to $G_{\rm ret}$ via the fluctuation–dissipation relation.

If we neglect the effects of the noise, the dissipative system will have complex eigenfrequencies. In order to compute them, we put the thin mirrors inside a larger box of length L, and impose periodic boundary conditions. Setting $K=0$ in Eq. (39), it is possible to read the discontinuity of the field φ around the defect at $x_d = 0$ and $x_d = a$ as

$$\begin{aligned}&\varphi'|_{x_\alpha^+} - \varphi'|_{x_\alpha^-} \\ &\quad= g^2\sum_\alpha\int dt'\,d^{d-1}x'_\parallel\,G_{\rm ret}(t,\mathbf{x}_\parallel,t',\mathbf{x}'_\parallel)\varphi(t',\mathbf{x}'_\parallel,x_\alpha),\end{aligned} \tag{40}$$

with $\alpha = 0, a$.

The condition that defines the eigenfrequencies is $f(\omega, a, L) = 0$, with

$$\begin{aligned}f(\omega,a,L) &= \frac{1}{4}\left(e^{4ik_dL}-e^{2ik_da}\right)\left(e^{-2ik_da}-1\right) - \frac{ik_d}{\lambda_{ret}}\left(e^{4ik_dL}-1\right) \\ &\quad+\left(\frac{k_d}{\lambda_{\rm ret}}\right)^2\left(e^{2ik_dL}-1\right)^2 = 0,\end{aligned} \tag{41}$$

where $k_d = \sqrt{\omega^2 - \mathbf{k}_\parallel^2}$ and $\lambda_{\rm ret} = \lambda_{\rm ret}(\omega, \mathbf{k}_\parallel)$ is the Fourier transform of the retarded Green function $G_{\rm ret}$. Note that $\lambda_{\rm ret}$ is the appropriate analytic continuation of the Euclidean function $\tilde{\lambda}(\kappa)$ to Minkowski space.

Let us first consider the case in which $\lambda_{\rm ret}$ is real. When $\lambda_{\rm ret}$ is constant, it corresponds to a non-dissipative situation, i.e. a free field with massive terms concentrated on the position of the mirrors [8]. On the other hand, when $\lambda_{\rm ret}$ depends on frequency, it is unrealistic to assume that it does not have an imaginary part, because of causality. In spite of this, one can compute the Casimir energy following a standard procedure. From Eq. (41), it is easy to prove that, in the limit $L \to \infty$, the admitted eigenfrequencies in the cavity are real. In this particular case the Casimir energy can be obtained as a sum-over-modes,

$$\mathcal{E} = \frac{1}{2}\sum_i(\omega_{0,i} - \omega_{\infty,i}), \tag{42}$$

which may be computed using the so-called generalized argument theorem

$$\mathcal{E} = \frac{1}{4\pi i}\int\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\oint dz\,z\frac{d}{dz}\ln\frac{f(z,a,L)}{f(z,L/2,L)}. \tag{43}$$

The integral in the complex plane reduces to an integral on the imaginary axis $z = iy$, and therefore, when $\lambda_{\rm ret}$ is an even function of ω, one gets

$$\begin{aligned}\mathcal{E}(a) &= \frac{1}{2\pi}\int\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \\ &\quad\times\int_0^\infty dy\ln\left\{1-e^{-2\sqrt{y^2+\mathbf{k}_\parallel^2}a}\left(1+\frac{2\sqrt{y^2+\mathbf{k}_\parallel^2}}{\lambda_{\rm ret}(iy,\mathbf{k}_\parallel)}\right)^{-2}\right\},\end{aligned} \tag{44}$$

which is consistent with the result obtained in Eq. (14) from the Euclidean effective action.

However, when the effective strength of the coupling between field and mirrors comes from virtual effects of a matter field, one expects to have a *complex* spectrum. For example, for the case of massless Dirac fields in $d=3$, the Euclidean function $\tilde{\lambda}$ is given by $\tilde{\lambda}(\kappa) = ne^2\kappa/8$ (see Eq. (36)). Therefore, the retarded analytic continuation to Minkowski spacetime is [9]

$$\lambda_{\rm ret}(\omega, \mathbf{k}_\parallel) = -\frac{ne^2}{8}\big(i\,\mathrm{sign}(\omega)\sqrt{\omega^2 - \mathbf{k}_\parallel^2}\,\theta\big(\omega^2 - \mathbf{k}_\parallel^2\big) - \sqrt{\mathbf{k}_\parallel^2 - \omega^2}\,\theta\big(\mathbf{k}_\parallel^2 - \omega^2\big)\big), \tag{45}$$

introducing a dissipative term into Eq. (39). It is possible to show that, in this particular case, Eq. (41) only admits propagating ($\omega^2 > \mathbf{k}_\parallel^2$) complex solutions. In the limit $L \to \infty$, they are given by

$$\omega_m^2 = \left(\frac{m\pi}{a} - \frac{i}{a}\ln\left|1 - \frac{16}{ne^2}\right|\right)^2 + \mathbf{k}_\parallel^2 \tag{46}$$

where m is a natural number.

As expected, in order to evaluate the Casimir interaction energy it is not correct to use the sum-over-modes formula (42), since the modes of the effective theory have complex frequencies [10]. The correct procedure is to consider the sum over the real eigenfrequencies of the full system, as implicitly done in the Euclidean approach. Alternatively, one should include the effects of noise in the effective theory, as done by Lifshitz in the derivation of the well-known formula for the Casimir interaction in the presence of real media [11]. However, it is remarkable that the correct result derived using the Euclidean approach (14) formally coincides with the one derived from the sum over eigenfrequencies assuming an unphysical $\lambda_{\rm ret}$ which is real and non-constant Eq. (44). A similar situation holds for Lifshitz formula, that has been derived using a sum over modes for unphysical permittivities with no imaginary parts,[1] and also for the Van der Walls interaction between atoms immersed in an absorptive medium [12].

We conclude with some comments about the previous models, the assumptions made to motivate them, and the results obtained therefrom. An important point has to do with the relationship between the symmetries of the model and the resulting (approximate) boundary conditions. We have seen that, assuming relativistic invariance and parity conservation on the mirrors' world volume, gauge invariance implies that the physics depends on only one function. This function, the scalar part of the vacuum polarization function, is only effective to impose a particular kind of boundary condition, which affects the components of the electric field that are parallel to the mirrors. Had one wanted to impose more general boundary conditions, one should have had to relax some assumptions regarding symmetries. This has been, indeed, explicitly shown for the case of a non-relativistic system, where there appear *two* different functions. This situation also arises, for example, when the mirrors have a finite width. Finally, parity symmetry could have been broken, for example, by considering mirrors which are permeated by a strong magnetic field pointing along the direction of the x_d coordinate. This set up would produce exotic boundary conditions, mixing the transverse electric field with the normal *magnetic* field.

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