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Non-Abelian Chern–Simons vortices

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Abstract

We consider the bosonic sector of an $\mathcal{N}=2$ supersymmetric Chern–Simons–Higgs theory in $2+1$ dimensions. The gauge group is $U(1)\times SU(N)$ and has N_f flavors of fundamental matter fields. The model supports non-Abelian (axially symmetric) vortices when $N_f \geqslant N$, which have internal (orientational) moduli. When $N_f > N$, the solutions acquire additional collective coordinates parameterizing their transverse size. We solve the BPS equations numerically and obtain local ($N_f = N$) and semi-local ($N_f > N$) string solutions.

1. Introduction

Chern–Simons (CS) theories are relevant in a quantum field theory context since they provide an alternative gauge-invariant procedure of mass generation [1]. Moreover, the high-temperature limit of quantum field theories in $d=4$ dimensions are effectively three dimensional and CS terms are precisely induced by fermions in $d=3$ dimensions through the parity anomaly [2]. CS actions play also a role in the analysis of interesting condensed matter phenomena [3–5], the computation of topological invariants of 3-manifolds [6] and they are connected with $d=2$ conformal field theories [7].

CS–Higgs models differ drastically from theories in which solely a Maxwell or Yang–Mills term governs the dynamics of the gauge fields. In particular, at the classical level, axially symmetric (vortex or string) solutions to the equations of motion necessarily carry electric charge [8–15] which, in the non-Abelian case, is quantized (for a complete review on CS theories and planar physics see [16]).

Bogomolny equations for the non-Abelian CS–Higgs system were first obtained in [15], where it was shown that, as in the Abelian case [12,13], a sixth-order potential has to be considered. Explicit vortex solutions were exhibited in [15], with the flux directed in the Cartan subalgebra of the non-Abelian group. It is the purpose of the present work to find genuine non-Abelian vortex configurations by proposing a more general ansatz as the one already considered for Yang–Mills-matter theories [17–19] with gauge group $U(1)\times SU(N)$ and $N_f \geqslant N$ flavors of fundamental matter multiplets. The resulting vortex configurations can be characterized by non-Abelian collective coordinates related to orientational degrees of freedom and, when $N_f > N$, to infinitesimal variations of the transverse size.

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2. Model and notation

We consider the truncated bosonic sector of the $\mathcal{N}=2$ SUSY $U(1)\times SU(N)$ Chern–Simons–Higgs action in $2+1$ dimensions (the complete SUSY Lagrangian can be found in [21])

$$\mathcal{S}=\int d^3x\left\{\frac{\kappa_1}{2}\epsilon^{\mu\nu\rho}F^0_{\mu\nu}A^0_\rho+\frac{\kappa_2}{2}\epsilon^{\mu\nu\rho}\left(F^I_{\mu\nu}A^I_\rho-\frac{1}{3}f^{IJK}A^I_\mu A^J_\nu A^K_\rho\right)+\left(D_\mu\phi^f\right)^\dagger\left(D^\mu\phi^f\right)-V\left[\phi,\phi^\dagger\right]\right\}, \tag{1}$$

where $\epsilon^{012}=1$, $g^{00}=1$, f^{IJK} are the structure constants of the non-Abelian group, and the quantization condition implies $\kappa_2=m/8\pi$. Our theory has a sixth-order potential which, forced by $\mathcal{N}=2$ supersymmetry, allows Bogomolny completion of the energy functional with all coupling constants at the Bogomolny point (see Ref. [22] and references therein for details on this point)

$$V\left[\phi,\phi^\dagger\right]=\frac{1}{16\kappa_1^2N^2}\phi_f^\dagger\phi^f\left(\phi_g^\dagger\phi^g-N\xi\right)^2+\frac{1}{4\kappa_2^2}\phi_f^\dagger\tau^I\tau^J\phi^f\left(\phi_g^\dagger\tau^I\phi^g\right)\left(\phi_h^\dagger\tau^J\phi^h\right)-\frac{1}{4\kappa_1\kappa_2N}\left(\phi_f^\dagger\tau^I\phi^f\right)^2\left(\phi_g^\dagger\phi^g-N\xi\right). \tag{2}$$

Here, $\mu,\nu,\rho=0,1,2$ are Lorentz indices, $I,J,K=1,\dots,N^2-1$ are the $SU(N)$ "color" group indices and τ_I are the anti-Hermitian generators of $SU(N)$. The complex scalar multiplets ϕ_i^f, besides the color index $i,j,k=1,\dots,N$, possess additional flavor index $f,g,h=1,\dots,N_f$ with $N_f\geqslant N$, thus can be written as $N\times N_f$ matrices. The covariant derivatives and field strengths are defined as

$$\begin{aligned}
&D_\mu\phi_i^f=\partial_\mu\phi_i^f+\left(A_\mu^{SU(N)}\right)_i^j\phi_j^f+\left(A_\mu^{U(1)}\right)_i^j\phi_j^f,\\
&A_\mu^{SU(N)}=A_\mu^I\tau_I,\qquad A_\mu^{U(1)}=A_\mu^0\tau_0,\\
&F^0_{\mu\nu}=\partial_{[\mu}A^0_{\nu]},\qquad F^I_{\mu\nu}=\partial_{[\mu}A^I_{\nu]}+f^{IJK}A^J_\mu A^K_\nu.
\end{aligned} \tag{3}$$

Up to gauge transformations, minima of the potential are given by

$$\textit{symmetric phase}:\qquad \phi^f=0, \tag{4}$$

$$\textit{asymmetric phase}:\qquad \phi^f\phi_f^\dagger=\xi\,\mathrm{diag}\{1,\dots,1\}. \tag{5}$$

In what follows we set, without loss of generality, $\xi=1$. The energy density is

$$\mathcal{H}=\left(D^0\phi^f\right)^\dagger\left(D^0\phi^f\right)+\left(D^i\phi^f\right)^\dagger\left(D^i\phi^f\right)+V\left[\phi,\phi^\dagger\right] \tag{6}$$

and the Euler–Lagrange equations of motion of the theory are

$$\kappa_1\epsilon_\mu^{\alpha\beta}F^0_{\alpha\beta}=J^0_\mu\equiv\phi_f^\dagger\tau^0D_\mu\phi^f-\left(D_\mu\phi^f\right)^\dagger\tau^0\phi^f,\qquad \kappa_2\epsilon_\mu^{\alpha\beta}F^I_{\alpha\beta}=J^I_\mu\equiv\phi_f^\dagger\tau^ID_\mu\phi^f-\left(D_\mu\phi^f\right)^\dagger\tau^I\phi^f, \tag{7}$$

$$D_\mu D^\mu\phi^f=\frac{\partial V}{\partial\phi_f^\dagger}. \tag{8}$$

Defining $D_\epsilon\equiv D_1+i\epsilon D_2$ with $\epsilon\equiv\pm$, and using Gauss' law, we can write the energy as a sum of squares

$$\begin{aligned}
H=\int d^2x\Bigg\{&\left[D_0\phi^f-i\epsilon\left(\frac{1}{4\kappa_1N}\left(\phi_g^\dagger\phi^g-N\right)\phi^f-\frac{1}{2\kappa_2}\phi_g^\dagger\tau^I\phi^g\tau^I\phi^f\right)\right]^\dagger\\
&\times\left[D_0\phi^f-i\epsilon\left(\frac{1}{4\kappa_1N}\left[\phi_g^\dagger\phi^g-N\right]\phi^f-\frac{1}{2\kappa_2}\phi_g^\dagger\tau^I\phi^g\tau^I\phi^f\right)\right]+\left(D_{-\epsilon}\phi^f\right)^\dagger\left(D_{-\epsilon}\phi^f\right)+\epsilon\sqrt{2N}F^0_{12}\Bigg\}
\end{aligned} \tag{9}$$

leading to the Bogomolny equations.

3. Non-Abelian local strings

In this section we investigate the so-called "local Z_N string-type solutions" as discussed for the Yang–Mills case in [17–19] (see also [20]).

We set $N_f=N$ so the matter fields can be arranged as a square matrix Φ. The Lagrangian (1) is then invariant under $SU(N)_{\mathrm{color}}\times SU(N)_{\mathrm{flavor}}$ rotations,

$$\Phi\to U\Phi V,\qquad A_\mu\to UA_\mu U^{-1}-(\partial_\mu U)U^{-1} \tag{10}$$

with $U\in U(N)_{\mathrm{local}}$ and $V\in SU(N)_{\mathrm{global}}$. We start from the trivial vacuum in the asymmetric phase

$$A_\mu^{\mathrm{vac}}=0,\qquad \Phi^{\mathrm{vac}}=\mathrm{diag}\{1,\dots,1\}. \tag{11}$$

After a $U(1) \times SU(N)$ gauge transformation we obtain a singular vortex configuration which, for the scalar field, takes the form

$$\Phi^{\mathrm{vac}} \to \Phi = \exp\bigl(\alpha\tau^0 + \beta\tau^{N^2-1}\bigr)\Phi^{\mathrm{vac}} = \mathrm{diag}\bigl\{1,\ldots,1,e^{-i\varphi n\epsilon}\bigr\} \tag{12}$$

with

$$\tau^0 = \frac{i}{\sqrt{2N}}\,\mathrm{diag}\{1,\ldots,1\}, \qquad \tau^{N^2-1} = \frac{i}{\sqrt{2N(N-1)}}\,\mathrm{diag}\{1,\ldots,1,1-N\},$$
$$\alpha = -\sqrt{2/N}\epsilon n\varphi, \qquad \beta = \sqrt{2(N-1)}\epsilon n\varphi. \tag{13}$$

Concerning the gauge fields we have

$$A_i^{\mathrm{vac}\,0} \to A_i^0 = -\sqrt{\frac{2}{N}}\varepsilon_{ij}\frac{x_j}{r^2}n\epsilon, \qquad A_i^{\mathrm{vac}\,N^2-1} \to A_i^{N^2-1} = \sqrt{\frac{2(N-1)}{N}}\varepsilon_{ij}\frac{x_j}{r^2}n\epsilon,$$
$$A_i^{\mathrm{vac}\,I} \to A_i^I = 0, \quad A_0^{\mathrm{vac}\,I} \to A_0^I = 0, \quad I = 1,2,\ldots,N^2-2. \tag{14}$$

In such configuration, the Z_N center of the gauge group $SU(N)$ has been combined with $U(1)$ elements to get a topologically stable string solution possessing both windings, in $SU(N)$ and in $U(1)$ (since $\pi_1(SU(N) \times U(1)/Z_N) \neq 0$, the topology is non-trivial). These kind of topological objects are also called "Z_N strings". In the present case the configurations (12), (14) represent a $(0,\ldots,0,n)$ singular string. A general $(n_1,\ldots,n_N)$ vortex configuration can be obtained following the same method. This suggests the following ansatz for the regular vortex configuration

$$\Phi = \mathrm{diag}\bigl\{\phi(r),\ldots,\phi(r),e^{-i\varphi n\epsilon}\phi_N(r)\bigr\}, \tag{15}$$

$$A_0^{N^2-1} = \sqrt{\frac{N-1}{2N}}f_0^{N^2-1}(r), \qquad A_i^{N^2-1} = \sqrt{\frac{2(N-1)}{N}}\varepsilon_{ij}\frac{x_j}{r^2}\bigl(n\epsilon + f^{N^2-1}(r)\bigr),$$
$$A_0^0 = \frac{1}{\sqrt{2N}}f_0(r), \qquad A_i^0 = -\sqrt{\frac{2}{N}}\varepsilon_{ij}\frac{x_j}{r^2}\bigl(n\epsilon + f(r)\bigr). \tag{16}$$

It should be noted that, contrary to the Yang–Mills–Higgs case [19], the A_0 fields are nontrivial. The boundary conditions for the fields are

$$\phi_N(0) = 0, \qquad f(0) = f^{N^2-1}(0) = -\epsilon\, n,$$
$$\phi(\infty) = \phi_N(\infty) = 1, \qquad f_0(\infty) = f_0^{N^2-1}(\infty) = f(\infty) = f^{N^2-1}(\infty) = 0, \tag{17}$$

where we required the solution to be singled-valued at the origin, and to be a pure gauge at infinity. With this ansatz both the flux and the energy of these solutions are quantized

$$\Phi \equiv \int d^2x\, F_{12}^0 = \frac{4\pi}{\sqrt{2N}}\epsilon n, \qquad E = 2\pi n. \tag{18}$$

Note that with our conventions n is always a positive integer, and ϵ determines whether the flux is positive ($\epsilon = +$), or negative ($\epsilon = -$).

Ansatz (15) corresponds, for the $n = 1$ case, to what is called an "elementary string". Composite strings can be constructed by introducing windings in several diagonal elements in the scalar field and can be seen as the superposition of elementary ones.

Substituting ansatz (15)–(16) into the Bogomolny and Gauss equations, we obtain the following system of non-linear first-order differential equations

$$r\partial_r\phi = -\frac{\epsilon}{N}\bigl(f - f^{N^2-1}\bigr)\phi, \qquad r\partial_r\phi_N = -\frac{\epsilon}{N}\bigl(f + (N-1)f^{N^2-1}\bigr)\phi_N, \tag{19}$$

$$\frac{1}{r}\partial_r f = -\frac{1}{4N\kappa_1}\bigl[f_0\bigl((N-1)\phi^2 + \phi_N^2\bigr) + f_0^{N^2-1}(N-1)\bigl(\phi^2 - \phi_N^2\bigr)\bigr], \tag{20}$$

$$\frac{1}{r}\partial_r f^{N^2-1} = \frac{1}{4N\kappa_2}\bigl[f_0\bigl(\phi^2 - \phi_N^2\bigr) + f_0^{N^2-1}\bigl(\phi^2 + (N-1)\phi_N^2\bigr)\bigr], \tag{21}$$

$$f_0 = \frac{\epsilon}{2\kappa_1}\bigl((N-1)\phi^2 + \phi_N^2 - N\bigr), \qquad f_0^{N^2-1} = \frac{\epsilon}{2\kappa_2}\bigl(\phi^2 - \phi_N^2\bigr). \tag{22}$$

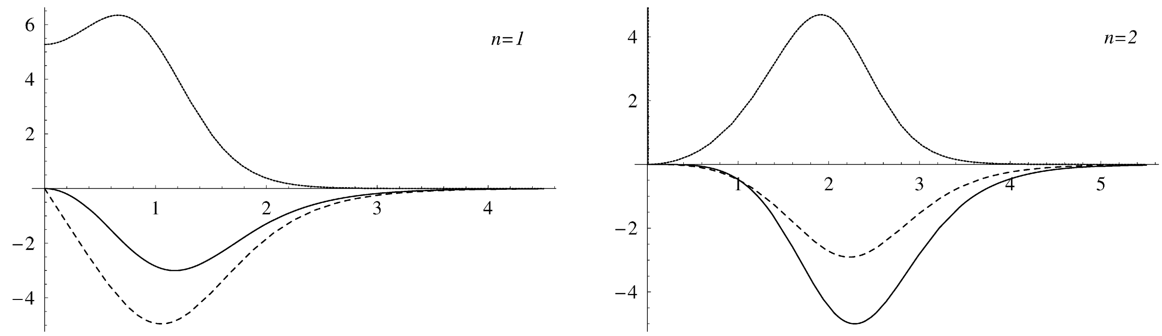

Fig. 1. Plot of the magnetic field B^0 (solid line), electric field E^0 (dashed line), and energy density $\mathcal{H}$ (dotted line) for local vortices with $\kappa_1/\kappa_2 = 1$. The distance between the energy density maximum and the origin increases with the winding number n.

Note that equations (22) can be used to eliminate f_0 and $f_0^{N^2-1}$ in (20) and (21). Eqs. (19)–(21) then correspond to the Bogomolny equations written in the standard form. In terms of the profile functions, the energy takes the form

$$E = 2\pi \int r\,dr \biggl\{ \frac{1}{8N^2\kappa_1^2}\bigl((N-1)\phi^2 + \phi_N^2 - N\bigr)^2\bigl((N-1)\phi^2 + \phi_N^2\bigr) + \frac{(N-1)}{8N^2\kappa_2^2}\bigl(\phi^2 - \phi_N^2\bigr)^2\bigl(\phi^2 + (N-1)\phi_N^2\bigr)$$
$$+ \frac{(N-1)}{4N^2\kappa_1\kappa_2}\bigl(\phi^2 - \phi_N^2\bigr)^2\bigl((N-1)\phi^2 + \phi_N^2 - N\bigr) + (N-1)(\partial_r\phi)^2 + (\partial_r\phi_N)^2$$
$$+ \frac{(N-1)}{r^2N^2}\phi^2\bigl(f - f^{N^2-1}\bigr)^2 + \frac{1}{N^2r^2}\phi_N^2\bigl(f + (N-1)f^{N^2-1}\bigr)^2 \biggr\}. \tag{23}$$

The magnetic field (which is a pseudoscalar in $2+1$ dimensions) and the electric field (with component only in the radial direction) take the form

$$B^0 \equiv F_{12}^0 = \sqrt{\frac{2}{N}}\frac{1}{r}\partial_r f, \qquad E^0 \equiv \sqrt{\bigl(F_{01}^0\bigr)^2 + \bigl(F_{02}^0\bigr)^2} = \frac{1}{2N}\partial_r f_0. \tag{24}$$

Let us study the solutions of Eqs. (19)–(22). We can distinguish two cases: first, when the $U(1)$ and $SU(N)$ coupling constants are equal and then the effective symmetry group is $U(N)$, and second, when the $U(1)$ and $SU(N)$ coupling constants are different.

3.1. $U(N)_{\mathrm{gauge}} \times SU(N)_{\mathrm{global}}$ solutions

If the $U(1)$ and $SU(N)$ coupling constants are equal $\kappa_1 = \kappa_2 \equiv \kappa$, the system of Eqs. (19)–(22) decouples into

$$r\partial_r\phi = -\epsilon g\phi, \qquad \frac{1}{r}\partial_r g = -\frac{\epsilon}{8\kappa^2}\phi^2\bigl(\phi^2 - 1\bigr), \tag{25}$$

$$r\partial_r\phi_N = -\epsilon g^{N^2-1}\phi_N, \qquad \frac{1}{r}\partial_r g^{N^2-1} = -\frac{\epsilon}{8\kappa^2}\phi_N^2\bigl(\phi_N^2 - 1\bigr), \tag{26}$$

where we have defined

$$g = \frac{1}{N}\bigl(f - f^{N^2-1}\bigr), \qquad g^{N^2-1} = \frac{1}{N}\bigl(f + (N-1)f^{N^2-1}\bigr). \tag{27}$$

In terms of the new functions, the boundary conditions are

$$g(0) = 0, \qquad g(\infty) = 0, \qquad g^{N^2-1}(0) = -\epsilon n, \qquad g^{N^2-1}(\infty) = 0, \tag{28}$$

$$\phi(0) = C, \qquad \phi(\infty) = 1, \qquad \phi_N(0) = 0, \qquad \phi_N(\infty) = 1. \tag{29}$$

Since the field ϕ has no winding, it does not necessarily vanish at the origin so C is an arbitrary parameter. Systems (25) and (26) are formally the same but the functions (ϕ, g) and (ϕ_N, g^{N^2-1}) obey different boundary conditions. Remarkably, both systems coincide with those arising in the Abelian case discussed in [14]. For the pair (ϕ, g), boundary conditions imply that

$$\phi \equiv 1, \qquad g \equiv 0. \tag{30}$$

The system (26) for the functions (ϕ_N, g^{N^2-1}) was solved numerically in [13]. We remark that these equations do not depend on N. At the origin, ϕ_N approaches to zero as r^n (this fact will be important for the semi-local vortex).

We show in Fig. 1 profiles of B^0, E^0 and the energy density $\mathcal{H}(r)$. As usually happens in CS theories, the sixth-order potential makes the maximum of the magnetic field to be away from the origin.

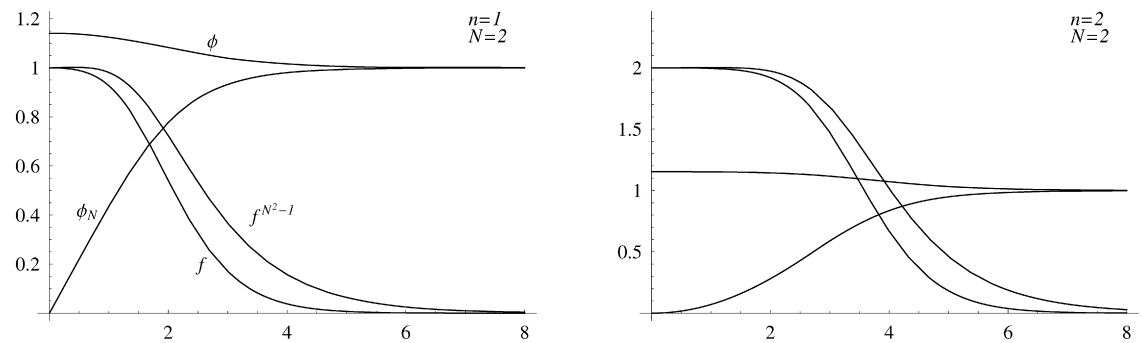

Fig. 2. Profile functions for local vortices, for negative magnetic flux and $\kappa_1/\kappa_2 = 1/2$.

3.2. $U(1)_{\text{gauge}} \times SU(N)_{\text{gauge}} \times SU(N)_{\text{global}}$ *solutions*

When $\kappa_1 \neq \kappa_2$, the complete set of Eqs. (19)–(22) has to be solved numerically. We used a relaxation method to find explicit numerical solutions. The ratio of coupling constants $k = \kappa_1/\kappa_2$ is in fact the only independent parameter of the theory, since κ_1 or κ_2 can be absorbed by a rescaling. In fact the energy can be expressed in terms of k only. We observe that as k departs from 1, f and f^{N^2-1} tend to separate from each other as well, forcing ϕ to be non-constant. As k goes from $k < 1$ to $k > 1$, the difference $f - f^{N^2-1}$ changes sign, forcing the derivative of ϕ to change sign. Qualitatively, the behavior resembles the $U(N)$ case (in which both coupling constant coincide). When varying the winding number n of the vortex, the profile functions change in a similar way as they do in the $U(N)$ case. For equal n and different N, solutions do not change considerably. We present in Fig. 2 some solutions for the case $k = 1/2$.

4. Non-Abelian semi-local strings

We have discussed non-Abelian vortex solutions in a $U(1)_{\text{gauge}} \times SU(N)_{\text{gauge}} \times SU(N)_{\text{flavor}}$ theory which are usually called *local* vortex solutions. In order to have *semi-local* vortex solutions, those for which $N_f > N$, one has to extend the matter content of the theory by adding N_e extra flavors so that $N_f = N + N_e$ [23,24]. For definiteness we take N_e equal to N (but a general case can be equally treated). Then, the symmetry of the model is $U(1)_{\text{gauge}} \times SU(N)_{\text{gauge}} \times SU(2N)_{\text{flavor}}$. We call $\mathcal{X}$ the extra matter fields.

In this case the trivial vacuum state is given by

$$\Phi^{\text{vac}} = \text{diag}\{1, \dots, 1\}, \qquad \mathcal{X}^{\text{vac}} = \text{diag}\{0, \dots, 0\}, \qquad A_\mu^{\text{vac}} = 0, \tag{31}$$

and it is invariant under the following transformation

$$\Phi^{\text{vac}} \to V^{-1}\Phi^{\text{vac}}V, \qquad \mathcal{X}^{\text{vac}} \to V^{-1}\mathcal{X}^{\text{vac}}\tilde{V} = 0, \qquad A_\mu^{\text{vac}} \to V^{-1}A_\mu^{\text{vac}}V = 0 \tag{32}$$

where we have chosen, as in the local case, a gauge element $U = V^{-1}$ with V an $N \times N$ flavor block and we have called $\tilde{V}$ the $N \times N$ flavor block acting on the extra $\mathcal{X}$ mater fields.

We follow now the same steps as in the local case: first we perform a rotation of the vacuum to find a singular vortex configuration and then propose an ansatz for a regular configuration which we write explicitly

$$\begin{aligned}
&\Phi = \text{diag}\big\{\phi(r), \dots, \phi(r), e^{-i\varphi n\epsilon}\phi_N(r)\big\}, \qquad \mathcal{X} = \text{diag}\big\{\chi(r), \dots, \chi(r), \chi_N(r)\big\},\\
&A_i = \frac{i}{N}\varepsilon_{ij}\frac{x_j}{r^2}\big(n\epsilon + f^{N^2-1}(r)\big)\,\text{diag}\{1, 1, \dots, 1-N\} - \frac{i}{N}\varepsilon_{ij}\frac{x_j}{r^2}\big(n\epsilon + f(r)\big)\,\text{Id},\\
&A_0 = \frac{i}{2N}f_0^{N^2-1}(r)\,\text{diag}\{1, 1, \dots, 1-N\} + \frac{i}{2N}f_0(r)\,\text{Id}
\end{aligned} \tag{33}$$

where Id is the $N \times N$ identity matrix. It should be stressed that more general ansatz could lead to solutions which exhaust the number N_ρ of collective coordinates related to the transverse moduli space [24]. Our ansatz corresponds to just two collective coordinates.

Substituting (33) into the Bogomolny equations, we arrive to a system of six differential equations for the fields ϕ, ϕ_N, χ, χ_N, f and f^{N^2-1}, and two constraints for the fields f_0 and $f_0^{N^2-1}$. The solutions to these equations will have the same energy and flux than the local ones (18), provided they satisfy the same boundary conditions (17). At infinity the solutions should reach the vacuum state, so we impose

$$\phi(\infty) = \phi_N(\infty) = 1, \qquad \chi(\infty) = \chi_N(\infty) = 0, \qquad f_0(\infty) = f_0^{N^2-1}(\infty) = f(\infty) = f^{N^2-1}(\infty) = 0. \tag{34}$$

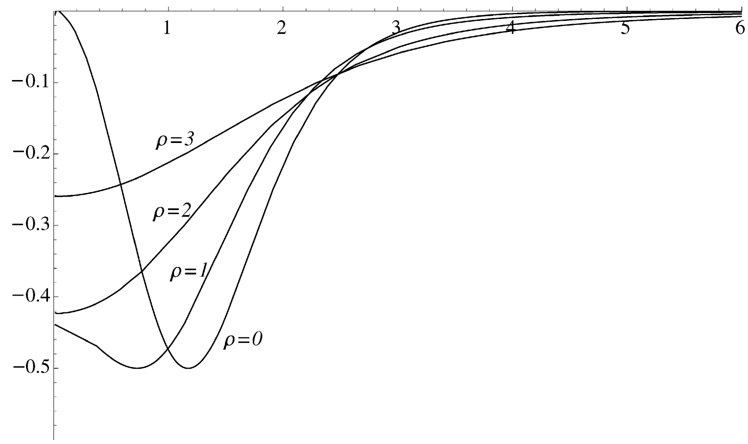

Fig. 3. The magnetic field for semi-local vortices with $\kappa_1 = \kappa_2$ for different ρ values.

Again we distinguish the case in which the $U(1)$ and $SU(N)$ gauge coupling constants are equal from the one in which they are different.

4.1. $U(N)_{\text{gauge}} \times SU(2N)_{\text{flavor}}$ semi-local solutions

As it happens for local vortex, when $\kappa_1 = \kappa_2 \equiv \kappa$ the equations of motion decouple into two sets of independent equations. One corresponds to a system with no winding

$$r\partial_r\phi = -\epsilon g\phi, \qquad r\partial_r\chi = -\epsilon g\chi, \qquad \frac{1}{r}\partial_r g = -\frac{\epsilon}{8\kappa^2}\left(\phi^2+\chi^2\right)\left(\phi^2+\chi^2-1\right), \tag{35}$$

and the other corresponds to a system with winding n,

$$r\partial_r\phi_N = -\epsilon g^{N^2-1}\phi_N, \qquad r\partial_r\chi_N = -\epsilon\left(g^{N^2-1}+\epsilon n\right)\chi_N, \qquad \frac{1}{r}\partial_r g^{N^2-1} = -\frac{\epsilon}{8\kappa^2}\left(\phi_N^2+\chi_N^2\right)\left(\phi_N^2+\chi_N^2-1\right) \tag{36}$$

(we have defined g and g^{N^2-1} as in (27)). As in the case of local strings, these equations coincide with those that arise in the Abelian case [25]. The system (35) admits a trivial solution

$$g = \chi = 0, \qquad \phi = 1, \tag{37}$$

while (36) can be solved numerically. Combining the equations for χ_N and ϕ_N, we get

$$\chi_N = \rho\frac{\phi_N}{r^n}, \tag{38}$$

where ρ is a (complex) integration constant. So, we finally have

$$r\partial_r\phi_N = -\epsilon g^{N^2-1}\phi_N, \qquad \frac{1}{r}\partial_r g^{N^2-1} = -\frac{\epsilon}{8\kappa^2}\left(\left(1+\frac{\rho^2}{r^{2n}}\right)\phi_N^2-1\right)\left(1+\frac{\rho^2}{r^{2n}}\right)\phi_N^2. \tag{39}$$

The same argument applied to χ and ϕ determines that $\chi = \alpha\phi$, but since $\phi(\infty) = 1$ and $\chi(\infty) = 0$, α must vanish, and so $\chi = 0$ everywhere. The solutions we obtain are then the most general ones. The energy is independent of the complex parameter ρ, which is then associated with the "size moduli". One has in general, n_i complex parameters ρ_i for arbitrary flavor $N_f > N$. For elementary strings one can see that $n_i = N_f - N$ (since we are considering $N_f = N + 1$, we have in this case just one complex parameter).

From Fig. 3 we see that the solutions spread when the parameter $|\rho|$ is incremented. As ρ increases, since the flux is conserved, the extremum of the magnetic field must approach to the origin to compensate the spread towards spatial infinity. A similar phenomenon occurs with the energy.

4.2. $U(1)_{\text{gauge}} \times SU(N)_{\text{gauge}} \times SU(N+1)_{\text{flavor}}$ semi-local solutions

We consider now the case in which $\kappa_1 \neq \kappa_2$. The same arguments as above lead to $\chi(r) = 0$, so that the flavor group is in fact reduced from $SU(2N)$ to $SU(N+1)$. Similarly, the relation (38) holds, and the differential equations do not decouple. The system is very analogous to the system (19)–(22) obtained for the local case, with the only difference that, in the equations for the f's, the field ϕ_N gets locally scaled

$$\phi_N^2 \to \phi_N^2\left(1+\frac{\rho^2}{r^{2n}}\right). \tag{40}$$

Then, semi-local solutions arising from these equations are similar to those shown in Fig. 2 for the local case, with the only difference that they are smoother since they decay as powers of r at spatial infinity.

In summary, our main task in this work was to solve the BPS equations for a non-Abelian Chern–Simons–Higgs theory. By proposing an axially symmetric ansatz we obtained non-Abelian vortex solutions and discussed their properties. A class of vortex solutions in non-Abelian CS theories were already known [9–11,15]. The solutions discussed here correspond to more fundamental vortices in the sense that they are genuinely non-Abelian while the former correspond to Z_N vortices with the gauge flux in the Cartan algebra of $SU(N)$).

The model discussed here is indeed related to the one analyzed by [17–19] except that in our case the dynamics of the gauge fields is governed by a CS action instead of a Yang–Mills one. This drastically changes the vortex properties, in particular forcing them to carry electric charge.

In the case of local vortices, our solutions generalize those discussed in [20] to the case in which the gauge group is $U(1) \times SU(N)$, with the Abelian and non-Abelian sectors having different gauge coupling constants. When both couplings are equal, the equations decouple into two sets of equations that coincide with those arising in the Abelian case. This is not the case when the couplings are different and the BPS equations do not decouple. Nevertheless we were able to construct explicit solutions and discuss their properties.

Further, we have also considered semi-local vortices, by allowing the flavor number N_f to be larger than the color number N_c. As already noted for the Yang–Mills–Higgs system, the main feature in this case is that the solutions develop an additional moduli ρ related to the vortex transverse size, thus modifying the asymptotic behavior of the fields, from exponential decay for the case of local vortices, to a decay as negative power of the radial coordinate for the semi-local ones. Interestingly enough, one can see from our explicit solutions how the size of the vortex grows with ρ.

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