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Occupation number formalism for arbitrary N_c baryons

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Abstract

A general method is presented for computing matrix elements of quark operators on baryonic states with low strangeness and arbitrary number of colors N_c. These results are useful in applications of the large N_c expansion to baryons and exotics. As an application we compute the matrix elements of strangeness changing operators contributing to kaon couplings to ground state baryons and pentaquarks, in broken $SU(3)$.

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1. Introduction

In the large N_c limit, with N_c the number of colors, a new symmetry emerges in the baryon sector of QCD. This is the contracted symmetry $SU(2F)_c$, with F the number of light quark flavors [1,2]. This symmetry can be used to organize the $1/N_c$ expansion of any quantity as an operator expansion. The nonrelativistic quark model is a convenient bookkeeping tool for implementing this expansion [3,4]. The baryons are constructed by placing the N_c quarks into one-body states. For example, taking $F = 3$ a possible basis of one-body states consists of

$$|u_\uparrow\rangle, |u_\downarrow\rangle, |d_\uparrow\rangle, |d_\downarrow\rangle, |s_\uparrow\rangle, |s_\downarrow\rangle, \tag{1}$$

transforming in the fundamental representation of $SU(6)$. The operators representing physical quantities such as masses, axial currents, etc., can be constructed from quark operators annihilating the basis states in Eq. (1). The building blocks are bilinears of the form $q^\dagger \Lambda^A q$ with Λ^A the generators of $SU(2F)$

$$J^i = q^\dagger\left(\frac{\sigma^i}{2}\otimes 1\right)q,$$

$$T^a = q^\dagger\left(1\otimes\frac{\lambda^a}{2}\right)q,$$

$$G^{ia} = q^\dagger\left(\frac{\sigma^i}{2}\otimes\frac{\lambda^a}{2}\right)q. \tag{2}$$

One important technical problem in the implementation of this program is the computation of the matrix elements of these operators on quark model states. Various methods have been discussed in the literature for this purpose. The computation of these matrix elements for $F \geqslant 3$ *and* arbitrary N_c turns out to be rather involved. For $F = 2$ a method for computing with any N_c was discussed in Refs. [5–7]. On the other hand, keeping $N_c = 3$ (or other small values) other methods are available, such as the holomorphic representation of the harmonic oscillator discussed in [8]. The difficulty is connected with the necessity of manipulating complicated expressions involving $SU(3)$ Clebsch–Gordan (CG) coefficients. Although not insurmontable (applications of the large N_c expansion in $SU(3)$ have been presented in Ref. [9] and analytic expressions for some arbitrary N_c CG coefficients have been computed recently in Ref. [10]), extracting the form of the result for arbitrary N_c remains a challenge.

In practice, we are interested only in baryon states with at most a few strange or heavy quarks. Therefore it is natural to expect that the large N_c dependence comes from the large number of light quarks in the u, d sector. In this Letter we give the

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details required for constructing explicitly any such states and computing matrix elements between them. The advantage of our method is that only usual $SU(2)$ CG coefficients are ever required, and closed form expressions are found for all matrix elements for arbitrary N_c.

2. States

Consider the spin-flavor group $SU(6)$, which contains as a subgroup (see, e.g. Ref. [11])

$$SU(6)_{SF} \supset SU(4)_{SI} \otimes SU(2)_K \otimes U(1)_{n_s}. \tag{3}$$

The factors on the right-hand side correspond to spin–isospin $SU(4)$, the strange quarks' spin $SU(2)_K$, and the $U(1)$ associated with strangeness $-n_s$.

We would like to construct the decomposition of a baryon state containing N_c quarks and transforming in the completely symmetric representation $\mathcal{S}_{N_c}$ of spin-flavor $SU(6)$ into irreducible representations of the subgroup on the rhs of Eq. (3). This is given by

$$\mathcal{S}_{N_c} = (S_{N_c}, \mathbf{1}, 0) \oplus (S_{N_c-1}, \mathbf{2}, 1) \oplus (S_{N_c-2}, \mathbf{3}, 2) \oplus \cdots. \tag{4}$$

We denoted the representations of $SU(2)_K$ by their multiplicity $2K+1$, with K the spin of the strange quarks. The terms written have $K = |n_s|/2$, which corresponds to the maximally possible value of the strange quark spin. This is required by Fermi statistics as applied to the system of the strange quarks, assuming that they are all in a completely symmetric orbital wave function. This is satisfied by ground state baryons, but not by orbitally excited states. We will comment on this case below.

The wave function of a hadron containing N_c quarks, of which n_s are strange quarks, factors according to Eq. (4) into a product of wave functions for its components. The nonstrange system has a symmetric spin-flavor wave function. For describing its state, it is convenient to use a Fock state formalism, familiar from the theory of many-body systems [12].

Since the orbital wave functions of all quarks are the same, we can label the state of a system of N_c identical quarks by giving the occupation numbers of the one-body spin-flavor states in Eq. (1). We introduce the "6n-symbol" defined as

$$\{n_1, n_2, n_3, n_4, n_5, n_6\} = \sqrt{\frac{n_1!n_2!n_3!n_4!n_5!n_6!}{N!}}\left(u_\uparrow^{n_1} u_\downarrow^{n_2} d_\uparrow^{n_3} d_\downarrow^{n_4} s_\uparrow^{n_5} s_\downarrow^{n_6} + \text{perms}\right), \tag{5}$$

with $N = \sum_{i=1}^{6} n_i$. These states are normalized as

$$\left\langle \{n'_1, n'_2, n'_3, n'_4, n'_5, n'_6\} \middle| \{n_1, n_2, n_3, n_4, n_5, n_6\} \right\rangle = \delta_{n_1 n'_1}\delta_{n_2 n'_2}\delta_{n_3 n'_3}\delta_{n_4 n'_4}\delta_{n_5 n'_5}\delta_{n_6 n'_6}. \tag{6}$$

Nonstrange hadrons have $n_5 = n_6 = 0$ and can be described by "4n-symbols" $\{n_1, n_2, n_3, n_4\}$. For simplicity we will use this notation when appropriate.

The nonstrange states have spin and isospin satisfying $I = J$. Their spin-flavor symmetric wave functions can be given in closed form as [6,7]

$$|I I_3 J_3; N_{ud}\rangle = \sum_i \left(\begin{matrix} \frac{N_u}{2} & \frac{N_d}{2} \\ i & J_3 - i \end{matrix} \middle| \begin{matrix} I \\ J_3 \end{matrix}\right) \times \left\{\frac{N_u}{2} + i, \frac{N_u}{2} - i, \frac{N_d}{2} + J_3 - i, \frac{N_d}{2} - J_3 + i\right\}, \tag{7}$$

where $N_{u,d}$ are the number of up and down quarks, respectively

$$N_u = \frac{N_{ud}}{2} + I_3, \qquad N_d = \frac{N_{ud}}{2} - I_3, \tag{8}$$

with $N_{ud} = N_c - n_s$. A few representative nonstrange $J_3 = +\frac{1}{2}$ states are

$$p_\uparrow = \sqrt{\frac{2}{3}}\{2,0,0,1\} - \frac{1}{\sqrt{3}}\{1,1,1,0\}, \tag{9}$$

$$\Delta_\uparrow^{++} = \{2,1,0,0\}. \tag{10}$$

Strange quarks are also straightforwardly added

$$\Sigma_\uparrow^{+} = \sqrt{\frac{2}{3}}\{2,0,0,0\}s_\downarrow - \frac{1}{\sqrt{3}}\{1,1,0,0\}s_\uparrow, \tag{11}$$

$$\Lambda_\uparrow^{0} = \left(\frac{1}{\sqrt{2}}\{1,0,0,1\} - \frac{1}{\sqrt{2}}\{0,1,1,0\}\right)s_\uparrow. \tag{12}$$

We could have equally well written these states in terms of the "6n-symbol" introduced above, but we gave them here in a form which is not symmetrized under the exchange of the s quark with the u, d quarks. While the two choices give the same results for one light s quark, the expressions Eqs. (11), (12) are appropriate in broken $SU(3)$ and for hadrons containing one heavy quark, with the replacement $s \to Q$.

Exotic states containing both quarks and antiquarks can also be constructed. We consider here only positive parity pentaquark-type states, containing $N_c + 1$ quarks and one antiquark. We take the $N_c + 1$ quarks to contain only u, d quarks in a spin-flavor symmetric state, as in Ref. [13], while the antiquark can be a strange or heavy quark. Representative states with $I = 0, 1$ can be chosen as

$$\Theta_\uparrow^{+} = \left(\frac{1}{\sqrt{3}}\{2,0,0,2\} - \frac{1}{\sqrt{3}}\{1,1,1,1\} + \frac{1}{\sqrt{3}}\{0,2,2,0\}\right)\bar{s}_\uparrow, \tag{13}$$

$$\Theta_{1\uparrow}^{++}(I_3 = +1) = \left(\frac{1}{\sqrt{2}}\{3,0,0,1\} - \frac{1}{\sqrt{6}}\{2,1,1,0\}\right)\bar{s}_\downarrow - \left(\frac{1}{\sqrt{6}}\{2,1,0,1\} - \frac{1}{\sqrt{6}}\{1,2,1,0\}\right)\bar{s}_\uparrow. \tag{14}$$

Next we consider the action of quark operators on these states. Any such operator can be constructed from one-body annihilation q_i and creation $q_i^\dagger$ operators, where $i = 1$–6 denotes one of the basis states in Eq. (1). Their action is given explicitly as

$$q_i\{\ldots, n_i, \ldots\} = \sqrt{n_i}\{\ldots, n_i - 1, \ldots\}, \tag{15}$$

$$q_i^\dagger\{\ldots, n_i, \ldots\} = \sqrt{n_i + 1}\{\ldots, n_i + 1, \ldots\} \tag{16}$$

with all occupation numbers $n_{j\neq i}$ unchanged. They satisfy the usual commutation relations for bosonic operators $[q_i, q_j^\dagger] = \delta_{ij}$.

As an example of their application, consider the isospin lowering and raising operators. Their action can be obtained by first writing them in terms of quark operators

$$I_- = d_\uparrow^\dagger u_\uparrow + d_\downarrow^\dagger u_\downarrow, \qquad I_+ = u_\uparrow^\dagger d_\uparrow + u_\downarrow^\dagger d_\downarrow \tag{17}$$

followed by the application of the rules Eq. (15). One finds

$$\begin{aligned} I_-\{n_1,n_2,n_3,n_4\} &= \sqrt{n_1(n_3+1)}\{n_1-1,n_2,n_3+1,n_4\} \\ &\quad + \sqrt{n_2(n_4+1)}\{n_1,n_2-1,n_3,n_4+1\}, \end{aligned} \tag{18}$$

$$\begin{aligned} I_+\{n_1,n_2,n_3,n_4\} &= \sqrt{n_3(n_1+1)}\{n_1+1,n_2,n_3-1,n_4\} \\ &\quad + \sqrt{n_4(n_2+1)}\{n_1,n_2+1,n_3,n_4-1\}. \end{aligned} \tag{19}$$

These relations are useful for obtaining the entire isospin multiplets from the representative states listed above.

3. Matrix elements

In broken $SU(3)$, the $SU(6)$ generators in Eq. (2) can be decomposed into generators of the subgroup Eq. (3) plus operators mediating transitions between sectors of different n_s. They can be chosen as

$$\begin{aligned} &J^i, \qquad I^a = T^a, \qquad G^{ia} = G^{ia} \quad (i, a = 1, \dots, 3), \\ &\tilde{t}^\alpha = q^{\dagger\alpha} s, \qquad t_\alpha = s^\dagger q_\alpha \qquad (\alpha = \pm 1/2), \\ &\tilde{Y}^{i\alpha} = q^{\dagger\alpha}\frac{\sigma^i}{2}s, \qquad Y_\alpha^i = s^\dagger \frac{\sigma^i}{2} q_\alpha, \\ &J_s^i = s^\dagger \frac{\sigma^i}{2} s, \qquad N_s = s^\dagger s, \end{aligned} \tag{20}$$

where $q^{\dagger\alpha} = (u^\dagger, d^\dagger)^\alpha$ and $q_\alpha = (u, d)_\alpha$. The adjoint of a spherical tensor operator $O^{j,m}$ is defined in terms of its components as $O^{\dagger j,m} \equiv (O^{j,-m})^\dagger(-1)^{j-m} = (O_m^j)^\dagger$, where indices are raised and lowered by contracting with the metric tensor [14]

$$O_m^j = \sum_{m'} \begin{pmatrix} j \\ m\ m' \end{pmatrix} O^{jm'}, \qquad O^{jm} = \sum_{m'} \begin{pmatrix} j \\ m'\ m \end{pmatrix} O_{m'}^j \tag{21}$$

defined as

$$\begin{pmatrix} j \\ m\ m' \end{pmatrix} = (-1)^{j-m}\delta_{m,-m'}. \tag{22}$$

The most general n-body operator can be constructed from the building blocks shown in Eq. (20).

The matrix elements of the strangeness conserving operators I^a, G^{ia}, N_s, J_s^i can be obtained using well-known $SU(2)$ methods [5,6]. The matrix element of G^{ia} on states with $N_{ud} = N_c - n_s$ up and down quarks are given by

$$\begin{aligned} &\langle I'I_3'J_3'|G^{ia}|II_3J_3\rangle \\ &\quad = \frac{1}{2I'+1} X^{(N_{ud})}(I', I) \left(\begin{array}{cc|c} I & 1 & I' \\ I_3 & a & I_3' \end{array}\right) \left(\begin{array}{cc|c} J & 1 & J' \\ J_3 & i & J_3' \end{array}\right) \end{aligned} \tag{23}$$

Table 1

Reduced matrix elements $\tilde{Y}$ and $\tilde{t}$ for $(sq^{N_c-1}) \to (q^{N_c})$ transitions

Transition	$(I'J', IJ)$	$\tilde{Y}(I'J'K', IJK)$	$\tilde{t}(I'K', IJK)$
$\Lambda \to N\bar{K}$	$(\frac{1}{2}\frac{1}{2}, 0\frac{1}{2})$	$\frac{\sqrt{3}}{2}\sqrt{N_c+3}$	$\frac{1}{2}\sqrt{N_c+3}$
$\Sigma \to N\bar{K}$	$(\frac{1}{2}\frac{1}{2}, 1\frac{1}{2})$	$-\frac{1}{2}\sqrt{N_c-1}$	$\frac{\sqrt{3}}{2}\sqrt{N_c-1}$
$\to \Delta\bar{K}$	$(\frac{3}{2}\frac{3}{2}, 1\frac{1}{2})$	$-\frac{1}{\sqrt{2}}\sqrt{N_c+5}$	–
$\Sigma^* \to N\bar{K}$	$(\frac{1}{2}\frac{1}{2}, 1\frac{3}{2})$	$\sqrt{2}\sqrt{N_c-1}$	–
$\to \Delta\bar{K}$	$(\frac{3}{2}\frac{3}{2}, 1\frac{3}{2})$	$\frac{1}{2}\sqrt{\frac{5}{2}}\sqrt{N_c+5}$	$\frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{N_c+5}$

with $X^{(N_{ud})}(I', I) = X^{(N_{ud})}(I, I')$, given explicitly as

$$\begin{aligned} X^{(N_{ud})}(I', I) &= \frac{1}{4}\sqrt{(2I'+1)(2I+1)} \\ &\quad \times \sqrt{(N_{ud}+2)^2 - (I'-I)^2(I'+I+1)^2}. \end{aligned} \tag{24}$$

For the matrix element of G^{ia} on general states $|JIn_s\rangle$ containing N_{ud} u, d quarks and n_s strange quarks we obtain

$$\begin{aligned} &\langle I'I_3', J'J_3'; n_s|G^{ia}|II_3, JJ_3; n_s\rangle \\ &\quad = \left(\begin{array}{cc|c} I & 1 & I' \\ I_3 & a & I_3' \end{array}\right) \left(\begin{array}{cc|c} J & 1 & J' \\ J_3 & i & J_3' \end{array}\right) X(I'J', IJ; K), \end{aligned} \tag{25}$$

with

$$\begin{aligned} &X(I'J', IJ; K) \\ &\quad = \sqrt{\frac{2J+1}{2I'+1}} X^{(N_{ud})}(I', I)(-)^{J+K+I'+1} \begin{Bmatrix} 1 & I & I' \\ K & J' & J \end{Bmatrix}. \end{aligned} \tag{26}$$

This matrix element has a $1/N_c$ expansion of the form

$$\begin{aligned} &X(I'J', IJ; K) \\ &\quad = N_c X_0(I'J', IJ; K) + X_1(I'J', IJ; K) + \cdots \end{aligned} \tag{27}$$

with the first two terms $X_{0,1}$ in agreement with the model-independent prediction following from the contracted symmetry [2].

In the following we compute also the matrix elements of the two strangeness lowering operators t^α, $Y^{i\alpha}$ (with raised indices), and the two strangeness raising operators $\tilde{t}^\alpha$, $\tilde{Y}^{i\alpha}$.

We define the reduced matrix elements as

$$\begin{aligned} &\langle I'I_3', J'J_3'; n_s-1|\tilde{Y}^{i\alpha}|II_3, JJ_3; n_s\rangle \\ &\quad = \left(\begin{array}{cc|c} I & \frac{1}{2} & I' \\ I_3 & \alpha & I_3' \end{array}\right) \left(\begin{array}{cc|c} J & 1 & J' \\ J_3 & i & J_3' \end{array}\right) \tilde{Y}(I'J'K', IJK) \end{aligned} \tag{28}$$

and similarly for $Y^{i\alpha}$, in terms of $Y(I'J'K', IJK)$. The reduced matrix elements of t^α and $\tilde{t}^\alpha$ are defined as

$$\begin{aligned} &\langle I'I_3', J'J_3'; n_s-1|\tilde{t}^\alpha|II_3, JJ_3; n_s\rangle \\ &\quad = \delta_{JJ'}\delta_{J_3J_3'} \left(\begin{array}{cc|c} I & \frac{1}{2} & I' \\ I_3 & \alpha & I_3' \end{array}\right) \tilde{t}(I'K', IJK) \end{aligned} \tag{29}$$

and similarly for $t(I'K', IJK)$.

The action of the strangeness changing operators on quark states can be obtained straightforwardly using the rules Eq. (15).

Table 2
Reduced matrix elements $\tilde{Y}$ and $\tilde{t}$ for $(ssq^{N_c-2}) \to (sq^{N_c-1})$ transitions

Transition	$(I'J', IJ)$	$\tilde{Y}(I'J'K', IJK)$	$\tilde{t}(I'K', IJK)$
$\Xi \to \Sigma\bar{K}$	$(1\frac{1}{2}, \frac{1}{2}\frac{1}{2})$	$\frac{5}{3\sqrt{2}}\sqrt{N_c+3}$	$\frac{1}{\sqrt{6}}\sqrt{N_c+3}$
$\to \Sigma^*\bar{K}$	$(1\frac{3}{2}, \frac{1}{2}\frac{1}{2})$	$-\frac{\sqrt{2}}{3}\sqrt{N_c+3}$	–
$\to \Lambda\bar{K}$	$(0\frac{1}{2}, \frac{1}{2}\frac{1}{2})$	$\frac{1}{\sqrt{2}}\sqrt{N_c-1}$	$\sqrt{\frac{3}{2}}\sqrt{N_c-1}$
$\Xi^* \to \Sigma\bar{K}$	$(1\frac{1}{2}, \frac{1}{2}\frac{3}{2})$	$-\frac{2}{3}\sqrt{N_c+3}$	–
$\to \Sigma^*\bar{K}$	$(1\frac{3}{2}, \frac{1}{2}\frac{3}{2})$	$\frac{\sqrt{10}}{3}\sqrt{N_c+3}$	$\sqrt{\frac{2}{3}}\sqrt{N_c+3}$
$\to \Lambda\bar{K}$	$(0\frac{1}{2}, \frac{1}{2}\frac{3}{2})$	$2\sqrt{N_c-1}$	–

Table 3
The reduced matrix elements Y and t for $\Theta \to NK, \Delta K$ transitions

Transition	$(I'J', IJ)$	$Y(I'J'K', IJK)$	$t(I'K', IJK)$
$\Theta_0(\frac{1}{2}) \to NK$	$(\frac{1}{2}\frac{1}{2}, 0\frac{1}{2})$	$-\frac{\sqrt{3}}{2}\sqrt{N_c+1}$	$\frac{1}{2}\sqrt{N_c+1}$
$\Theta_1(\frac{1}{2}) \to NK$	$(\frac{1}{2}\frac{1}{2}, 1\frac{1}{2})$	$\frac{1}{2}\sqrt{N_c+5}$	$\frac{\sqrt{3}}{2}\sqrt{N_c+5}$
$\to \Delta K$	$(\frac{3}{2}\frac{3}{2}, 1\frac{1}{2})$	$\frac{1}{\sqrt{2}}\sqrt{N_c-1}$	–
$\Theta_1(\frac{3}{2}) \to NK$	$(\frac{1}{2}\frac{1}{2}, 1\frac{3}{2})$	$-\sqrt{2}\sqrt{N_c+5}$	–
$\to \Delta K$	$(\frac{3}{2}\frac{3}{2}, 1\frac{3}{2})$	$-\frac{1}{2}\sqrt{\frac{5}{2}}\sqrt{N_c-1}$	$\frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{N_c-1}$
$\Theta_2(\frac{3}{2}) \to \Delta K$	$(\frac{3}{2}\frac{3}{2}, 2\frac{3}{2})$	$\frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{N_c+7}$	$\frac{1}{2}\sqrt{\frac{5}{2}}\sqrt{N_c+7}$
$\Theta_2(\frac{5}{2}) \to \Delta K$	$(\frac{3}{2}\frac{3}{2}, 2\frac{5}{2})$	$-\sqrt{\frac{3}{2}}\sqrt{N_c+7}$	–

Typical relations are

$$\begin{aligned}&(u^\dagger\sigma^3 s)\{n_1,n_2,n_3,n_4\}s_\downarrow = -\sqrt{n_2+1}\{n_1,n_2+1,n_3,n_4\},\\&(u^\dagger s)\{n_1,n_2,n_3,n_4\}s_\uparrow = \sqrt{n_1+1}\{n_1+1,n_2,n_3,n_4\}.\end{aligned}\tag{30}$$

Repeated application of the s creation operators leads to states containing multiple strange quarks. Typical matrix elements that appear, for example, in the computation of $\Xi \to \Sigma$ matrix elements are (in the notation of Eq. (7) for the nonstrange states)

$$\begin{aligned}&\langle 111; N_c-1|u_\uparrow^\dagger|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}; N_c-2\rangle\\&\quad=\sum_{i=-\frac{N_c-5}{4}}^{\frac{N_c-1}{4}}\left(\begin{array}{cc|c}\frac{N_c+1}{4} & \frac{N_c-3}{4} & 1\\ i+\frac{1}{2} & \frac{1}{2}-i & 1\end{array}\right)\left(\begin{array}{cc|c}\frac{N_c-1}{4} & \frac{N_c-3}{4} & \frac{1}{2}\\ i & \frac{1}{2}-i & \frac{1}{2}\end{array}\right)\\&\qquad\times\sqrt{\frac{N_c+3}{4}+i}=\sqrt{\frac{N_c}{3}+1}.\end{aligned}\tag{31}$$

We show in Tables 1, 2 the results for the strangeness raising transitions $\tilde{Y}(I'J'K', IJK)$ and $\tilde{t}(I'K', IJK)$ for all ground state baryons.

In Table 3 we give also the results for the kaon decays of pentaquarks with symmetric spin-flavor wave function. The computation of matrix elements of operators containing antiquarks requires some care. The antiquark spin doublet $\bar{s}$ has components $\bar{s}^\beta = (-\bar{s}_\downarrow, \bar{s}_\uparrow)^\beta$. This can be used to express the operators Y_α^i, t_α in terms of quark and antiquark one-body operators as

$$\begin{aligned}Y_\alpha^3 &= s^\dagger\frac{\sigma^3}{2}q_\alpha - \bar{s}\frac{\sigma^3}{2}q_\alpha\\&=\frac{1}{2}\big(s_\uparrow^\dagger q_{\alpha\uparrow} - s_\downarrow^\dagger q_{\alpha\downarrow} + \bar{s}_\downarrow q_{\alpha\uparrow} + \bar{s}_\uparrow q_{\alpha\downarrow}\big),\end{aligned}\tag{32}$$

$$\begin{aligned}t_\alpha &= s^\dagger q_\alpha - \bar{s}q_\alpha\\&= s_\uparrow^\dagger q_{\alpha\uparrow} + s_\downarrow^\dagger q_{\alpha\downarrow} + \bar{s}_\downarrow q_{\alpha\uparrow} - \bar{s}_\uparrow q_{\alpha\downarrow}.\end{aligned}\tag{33}$$

For example, the action of a typical strangeness changing operator on pentaquark states containing a $\bar{s}$ quark is given by

$$\bar{s}\sigma^3 u\{n_1,n_2,n_3,n_4\}\bar{s}_\uparrow = -\sqrt{n_2}\{n_1,n_2-1,n_3,n_4\}.\tag{34}$$

The results for the reduced matrix elements Y, $\tilde{Y}$ and t, $\tilde{t}$ take a simpler form in the large N_c limit, and can be given in analytical form. Expanding these operators as

$$Y = \sqrt{N_c}\left(Y_0 + \frac{1}{N_c}Y_1 + \cdots\right)\tag{35}$$

and similarly for t and the strangeness raising operators, the reduced matrix elements at leading order are given by (with $[I] \equiv 2I+1$)

$$\begin{aligned}\tilde{Y}_0(I'J'K', IJK) &= -Y_0(I'J'K', IJK)\\&= c(K,K')\sqrt{[I][J]}\begin{Bmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ I & J & K\\ I' & J' & K'\end{Bmatrix}\end{aligned}\tag{36}$$

and

$$\begin{aligned}\tilde{t}_0(I'K', IJK) &= t_0(I'K', IJK)\\&= d(K,K')(-)^{J+I+K}\sqrt{[I]}\begin{Bmatrix}K & K' & \frac{1}{2}\\ I' & I & J\end{Bmatrix}.\end{aligned}\tag{37}$$

The result Eq. (36) was found in Ref. [2] using the method of the induced representations for the contracted symmetry; the result Eq. (37) is new. From Tables 1, 2 and 3 we find $c(1,\frac{1}{2}) = 3\sqrt{2}$, $c(\frac{1}{2},0) = \sqrt{6}$ and $d(1,\frac{1}{2}) = \sqrt{3}$, $d(\frac{1}{2},0) = 1$.

4. Discussion and extensions

The applications discussed so far were limited to symmetric spin-flavor states. The methods of this Letter can be extended also to mixed-symmetric states, which are relevant for the application of the $1/N_c$ expansion to orbitally excited baryons in the $\mathbf{70}^-$ [9,15,16], and to negative parity exotic states [17,18].

The corresponding decomposition in broken $SU(3)$ of a state transforming in the mixed-symmetric $\mathcal{MS}_{N_c}$ representation of $SU(6)$ is more complicated than that for the symmetric state Eq. (4). Keeping only terms with $n_s = 0, 1$, this has the form

$$\begin{aligned}\mathcal{MS}_{N_c} &= (MS_{N_c}, \mathbf{1}, 0)\\&\quad\oplus(MS_{N_c-1}, \mathbf{2}, 1)\oplus(S_{N_c-1}, \mathbf{2}, 1)\oplus\cdots,\end{aligned}\tag{38}$$

where the two terms with one strange quark have the wave function of the nonstrange system in a MS and S representation of $SU(4)$, respectively. The states in $(MS_{N_c-1}, \mathbf{2}, 1)$ and $(S_{N_c-1}, \mathbf{2}, 1)$ that fall into $\mathcal{MS}_{N_c}$ can be constructed by requiring that they are eigenstates of the quadratic $SU(6)$ Casimir $C_2 = \sum_A \Lambda^A\Lambda^A$ with the eigenvalue of the $\mathcal{MS}_{N_c}$ representation.

The MS states of $N_c - 1$ nonstrange quarks can be represented in the usual way [7,9,15] as tensor products of a spin-flavor symmetric "core" of $N_c - 2$ quarks with one "excited" quark. This can be accommodated in our formalism in a similar way as we treated the strange quark. New annihilation and creation operators acting on this "excited" one-body state have to be introduced, from which transition operators can be constructed in the usual way.

In conclusion, we presented in this Letter a new general method which simplifies computations of matrix elements for ordinary and exotic baryon states, containing both light and strange or heavy quarks, for arbitrary number of colors N_c. The main advantage of the method is that only $SU(2)$ Clebsch–Gordan coefficients need to be used at any stage, and closed form results can be obtained for all matrix elements.

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