



## On the $b$ -coloring of $P_4$ -tidy graphs

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### ABSTRACT

A  $b$ -coloring of a graph is a coloring such that every color class admits a vertex adjacent to at least one vertex receiving each of the colors not assigned to it. The  $b$ -chromatic number of a graph  $G$ , denoted by  $\chi_b(G)$ , is the maximum number  $t$  such that  $G$  admits a  $b$ -coloring with  $t$  colors. A graph  $G$  is  $b$ -continuous if it admits a  $b$ -coloring with  $t$  colors, for every  $t = \chi(G), \dots, \chi_b(G)$ , and it is  $b$ -monotonic if  $\chi_b(H_1) \geq \chi_b(H_2)$  for every induced subgraph  $H_1$  of  $G$ , and every induced subgraph  $H_2$  of  $H_1$ . In this work, we prove that  $P_4$ -tidy graphs (a generalization of many classes of graphs with few induced  $P_4$ s) are  $b$ -continuous and  $b$ -monotonic. Furthermore, we describe a polynomial time algorithm to compute the  $b$ -chromatic number for this class of graphs.

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### 1. Introduction

In this paper we deal with finite undirected graphs, without loops or multiple edges. A *coloring* of a graph  $G$  is an assignment of colors (represented by natural numbers) to the vertices of  $G$  such that no two adjacent vertices are assigned the same color. The minimum number  $k$  such that there exists a coloring of  $G$  with  $k$  colors is the *chromatic number* of  $G$ , denoted by  $\chi(G)$ .

When we try to color the vertices of a graph using the minimum number of colors, we can start from a given coloring and try to decrease the number of colors by eliminating color classes. One possible such procedure consists in trying to reduce the number of colors by taking a color class such that we can recolor every vertex from that class with a different color that is not used by any of its neighbors, if any such class exists. A vertex  $v$  of a colored graph  $G$  is *dominant* if it has at least one neighbor of every color, except the one assigned to  $v$ . A dominant vertex cannot be recolored with this procedure. A  $b$ -coloring of a graph is a coloring with dominant vertices in each color class, i.e., a coloring where we cannot apply the strategy above to decrease the number of colors. The  $b$ -chromatic number of a graph  $G$ , denoted by  $\chi_b(G)$ , is the maximum number  $t$  such that  $G$  admits a  $b$ -coloring with  $t$  colors [15]. Thus,  $\chi_b(G) \geq \chi(G)$ , and every coloring with  $\chi(G)$  colors is a  $b$ -coloring. A graph  $G$  is  $b$ -perfect if  $\chi_b(H) = \chi(H)$  for every induced subgraph  $H$  of  $G$  [12].  $b$ -perfect graphs were recently characterized by a finite list of forbidden induced subgraphs [14]. Note that  $b$ -perfect graphs can be colored with a minimum number of colors in polynomial time, by simply applying the decreasing algorithm, starting from an arbitrary coloring.

The behavior of the  $b$ -chromatic number can be surprising. For example, the values of  $k$  for which a graph admits a  $b$ -coloring with  $k$  colors do not necessarily form an interval of the set of integers; in fact any finite subset of  $\mathbb{N}_{\geq 2}$  can be the set of these values for some graph [9]. A graph  $G$  is  $b$ -continuous if it admits a  $b$ -coloring with  $t$  colors, for every  $t = \chi(G), \dots, \chi_b(G)$ . In [21] (see also [9]) it is proved that chordal graphs and some planar graphs are  $b$ -continuous.

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Another atypical property is that the  $b$ -chromatic number can increase when taking induced subgraphs. A graph  $G$  is defined to be  $b$ -monotonic if  $\chi_b(H_1) \geq \chi_b(H_2)$  for every induced subgraph  $H_1$  of  $G$ , and every induced subgraph  $H_2$  of  $H_1$  [3].

Irving and Manlove [15] proved that determining  $\chi_b(G)$  is NP-hard for general graphs, but polynomial-time solvable for trees. In [24], Kratochvíl et al. show that determining  $\chi_b(G)$  is NP-hard even if  $G$  is a connected bipartite graph. More results on algorithmic aspects and bounds for some graph classes can be found in [3,7,8,16,23].

An induced path on  $k$  vertices shall be denoted by  $P_k$ . Vertices of degree one (resp. two) in  $P_k$  will be called *endpoints* (resp. *midpoints*). An induced subgraph of  $G$  isomorphic to  $P_k$  is simply said to be a  $P_k$  in  $G$ . A chordless cycle on  $k$  vertices is denoted by  $C_k$ .

A *cograph* is a graph that does not contain  $P_4$  as an induced subgraph [5]. Several generalizations of cographs have been defined in the literature, such as  $P_4$ -sparse [11],  $P_4$ -lite [17],  $P_4$ -extendible [19] and  $P_4$ -reducible graphs [18]. A graph class generalizing all of them is the class of  $P_4$ -tidy graphs [10]. Let  $G$  be a graph and  $A$  a  $P_4$  in  $G$ . A *partner* of  $A$  is a vertex  $v$  in  $G - A$  such that  $A \cup \{v\}$  induces at least two  $P_4$ s in  $G$ . A graph  $G$  is  $P_4$ -sparse if no induced  $P_4$  has a partner and  $P_4$ -tidy if every induced  $P_4$  has at most one partner. Another generalization of  $P_4$ -sparse graphs are  $(q, q4)$ -graphs. A graph is a  $(q, q4)$ -graph if no set of at most  $q$  vertices induces more than  $q - 4$  distinct  $P_4$ 's [1]. There is no containment relationship between the classes  $P_4$ -tidy and  $(q, q4)$ -graphs.

In [3], it was proved that  $P_4$ -sparse graphs are  $b$ -continuous and  $b$ -monotonic and a dynamic programming algorithm to compute their  $b$ -chromatic number was presented. Recently, some of the results on  $P_4$ -sparse graphs were also extended for the class of  $(q, q4)$ -graphs, with fixed  $q$  [4]. In this paper, we extend these results to the class of  $P_4$ -tidy graphs.

### 1.1. Definitions and preliminary results

Let  $G = (V, E)$  be a graph. We will denote by  $V(G)$  the vertex set  $V$ , by  $E(G)$  the edge set  $E$ , and by  $\bar{G}$  the complement graph of  $G$ . Given a subset of vertices  $X \subset V$ , we will denote by  $G[X]$  the subgraph of  $G$  induced by  $X$ . The complete graph on  $n$  vertices will be denoted by  $K_n$  and the stable set of  $n$  vertices by  $S_n$ . Two vertices will be said to be *true twins* if they are adjacent and have the same neighborhood, and *false twins* if they are non-adjacent but have the same neighbors. A vertex is *simplicial* if its neighbors induce a complete subgraph. A vertex  $v$  *controls* a vertex  $w$  if  $v$  and  $w$  are non-adjacent and all the neighbors of  $w$  are neighbors of  $v$ .

**Lemma 1** ([13]). *Let  $G$  be a graph and  $\varphi$  a coloring of  $G$ . If  $v$  and  $w$  are false twins in  $G$ , then either none of them is dominant, or  $\varphi(v) = \varphi(w)$ .*

This can be extended straightforward to the following one.

**Lemma 2.** *Let  $G$  be a graph and  $\varphi$  a coloring of  $G$ . If  $v$  controls  $w$ , then if  $w$  is dominant, so is  $v$  and  $\varphi(v) = \varphi(w)$ .*

**Lemma 3** ([13]). *Let  $G$  be a graph and  $\varphi$  a coloring of  $G$  with more than  $\chi(G)$  colors. Then no simplicial vertex of  $G$  is dominant.*

Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ . The *union* of  $G_1$  and  $G_2$  is the graph  $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ . The union is clearly an associative operation and, for each nonnegative integer  $t$ , we will denote by  $tG$  the union of  $t$  disjoint copies of  $G$ . The *join* of  $G_1$  and  $G_2$  is the graph  $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$ . That is, the vertex set of  $G_1 \vee G_2$  is  $V_1 \cup V_2$  and its edge set is  $E_1 \cup E_2$  plus all the possible edges with an endpoint in  $V_1$  and the other one in  $V_2$ .

Cographs can be built from isolated vertices by using these two operations.

**Theorem 1** ([5]). *Every non-trivial cograph is either a union or join of two smaller cographs.*

Thus, the chromatic number of a cograph can be recursively calculated due to the following result.

**Theorem 2** ([6]). *If  $G$  is the trivial graph, then  $\chi(G) = 1$ . Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ . Then,*

- i.  $\chi(G_1 \cup G_2) = \max\{\chi(G_1), \chi(G_2)\}$ .
- ii.  $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$ .

A similar result holds for the  $b$ -chromatic number, but the relation between the  $b$ -chromatic number of two graphs and the  $b$ -chromatic number of their union is weaker.

**Theorem 3** ([22]). *If  $G$  is the trivial graph, then  $\chi_b(G) = 1$ . Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ . Then,*

- i.  $\chi_b(G_1 \cup G_2) \geq \max\{\chi_b(G_1), \chi_b(G_2)\}$ .
- ii.  $\chi_b(G_1 \vee G_2) = \chi_b(G_1) + \chi_b(G_2)$ .

$P_4$ -tidy graphs have also a useful decomposition theorem. We will use it extensively in this work to inductively prove our results. A brief description of the theorem follows.

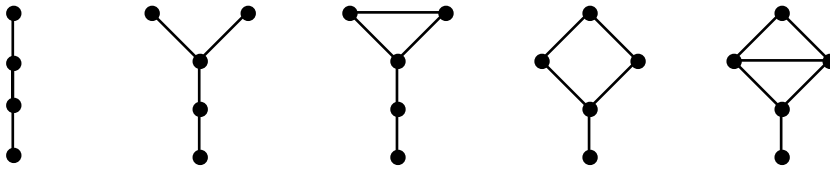


Fig. 1. Possible quasi-starfishes of size two. From left to right:  $P_4$ , fork,  $\bar{P}$ ,  $P$  and kite.

Let  $G = (V, E)$  be a graph. Let  $F = \{e \in E \mid e \text{ belongs to an induced } P_4 \text{ of } G\}$ . Let  $G_p = (V, F)$ . A connected component of  $G_p$  having exactly one vertex is called a *weak vertex*. Any connected component of  $G_p$  distinct from a weak vertex is called a *p-component* of  $G$ . A graph  $G$  is *p-connected* if it has only one *p-component* and no weak vertices [2].

A *p-connected* graph  $G = (V, E)$  is *p-separable* if  $V$  can be partitioned into two sets  $(C, S)$  such that each  $P_4$  that contains vertices from  $C$  and from  $S$  has its midpoints in  $C$  and its endpoints in  $S$ . We will call it a *p-partition*. If such a partition exists, then it is unique [20].

An *urchin* (resp. *starfish*) of size  $k$ ,  $k \geq 2$ , is a *p-separable* graph with *p-partition*  $(C, S)$ , where  $C = \{c_1, \dots, c_k\}$  is a clique;  $S = \{s_1, \dots, s_k\}$  is a stable set;  $s_i$  is adjacent to  $c_i$  if and only if  $i = j$  (resp.  $i \neq j$ ).

A *quasi-urchin* (resp. *quasi-starfish*) of size  $k$  is a graph obtained from an urchin (resp. starfish) of size  $k$  by replacing at most one vertex by  $K_2$  or  $S_2$ . Note that the new vertices result in true or false twins, respectively, and they are in the same set of the new *p-partition*  $(C^*, S^*)$ . The elements of  $S^*$  are called the *legs* and  $C^*$  is called the *body* of the quasi-starfish or quasi-urchin.

Note that there are five possible quasi-starfishes of size two, and they are also the five possible quasi-urchins of size two:  $P_4$ ,  $P$ ,  $\bar{P}$ , fork and kite (see Fig. 1). To avoid ambiguity, we will consider these five graphs as quasi-starfishes, while quasi-urchins will be always of size at least three.

When considering quasi-urchins and quasi-starfishes, we have ten kinds of them. We will call *type 1* (resp. *type 2*) the urchins (resp. starfishes); *type 3* (resp. *type 4*) the urchins (resp. starfishes), where a vertex in the body was replaced by  $K_2$ ; *type 5* (resp. *type 6*) the urchins (resp. starfishes), where a vertex in the body was replaced by  $S_2$ ; *type 7* (resp. *type 8*) the urchins (resp. starfishes), where a leg was replaced by  $K_2$ ; and *type 9* (resp. *type 10*) the urchins (resp. starfishes), where a leg was replaced by  $S_2$ . Recall that graphs of odd type have always size at least three and, with this condition, the ten types form a partition over the family of quasi-urchins and quasi-starfishes.

Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs with  $V_1 \cap V_2 = \emptyset$ , such that  $G_1$  is *p-separable* with partition  $(V_1^1, V_1^2)$ . Consider the graph with vertex set  $V_1 \cup V_2$  and edge set  $E_1 \cup E_2 \cup \{xy \mid x \in V_1^1, y \in V_2\}$ . We shall denote this graph by  $G_1 \vee G_2$ .

**Theorem 4** ([20]). *Every graph  $G$  either is  $p$ -connected or can be obtained uniquely from its  $p$ -components and weak vertices by a finite sequence of  $\cup$ ,  $\vee$  and  $\vee$  operations.*

**Proposition 1** ([10]). *A graph  $G$  is  $P_4$ -tidy if and only if every  $p$ -component is isomorphic to either  $P_5$  or  $\bar{P}_5$  or  $C_5$  or a quasi-starfish or a quasi-urchin. Quasi-starfishes and quasi-urchins are the  $p$ -separable  $p$ -components of  $G$ .*

**Remark 1.** Let  $G_1$  be a quasi-urchin or a quasi-starfish, and  $G_2$  be a graph. If  $G_1$  is type 7 or 8, all the legs are simplicial vertices both in  $G_1$  and in  $G_1 \vee G_2$ . Otherwise, both in  $G_1$  and in  $G_1 \vee G_2$ , each leg of  $G_1$  is controlled by a vertex in the body of  $G_1$ . Then, by Lemmas 2 and 3, for every coloring of  $G_1$  (resp.  $G_1 \vee G_2$ ) with more than  $\chi(G_1)$  (resp.  $\chi(G_1 \vee G_2)$ ) colors, if there is a dominant vertex of color  $c$  in  $V(G_1)$ , then there is a dominant vertex of color  $c$  in the body of  $G_1$ .

**Lemma 4.** *Let  $G$  be a quasi-starfish or quasi-urchin of size  $k$ . Then,*

- i. *If  $G$  is type 1, 2, 5, 6, 7, 9 or 10, then  $\chi(G) = k$ .*
- ii. *If  $G$  is type 3, 4 or 8, then  $\chi(G) = k + 1$ .*
- iii.  $\chi_b(G) = \chi(G)$ .

**Proof.** Items i. and ii. are easy to prove, since a coloring of the maximum clique of  $G$  can be extended to the whole graph without increasing the number of colors. Let  $(C^*, S^*)$  be the *p-partition* of  $G$ . To prove item iii., suppose on the contrary that we have a  $b$ -coloring  $\varphi$  of  $G$  with more than  $\chi(G)$  colors. By Remark 1, if there is a dominant vertex of color  $c$  in  $G$ , then there is a dominant vertex of color  $c$  in  $C^*$ . If  $G$  is neither type 5 nor type 6, then  $|C^*| \leq \chi(G)$ , a contradiction. If  $G$  is type 5 or 6, then there is a pair of false twins in  $C^*$ , so by Lemma 1, at most  $|C^*| - 1$  different colors can have dominant vertices and  $|C^*| - 1 \leq \chi(G)$ , a contradiction.  $\square$

**Lemma 5.** *Let  $G_1 = (V_1, E_1)$  be a  $p$ -separable  $P_4$ -tidy graph, and  $G_2 = (V_2, E_2)$  a graph such that  $V_1 \cap V_2 = \emptyset$ . Then,*

- i. *If  $G_1$  is not type 8, then  $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$ ; if  $G_1$  is type 8, then  $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2) - 1$ .*
- ii. *If  $G_1$  is not type 8, then  $\chi_b(G_1 \vee G_2) = \chi_b(G_1) + \chi_b(G_2)$ ; if  $G_1$  is type 8, then  $\chi_b(G_1 \vee G_2) = \chi_b(G_1) + \chi_b(G_2) - 1$ .*

**Proof.** Let  $G = G_1 \vee G_2$ . By Proposition 1,  $G_1$  is a quasi-urchin or a quasi-starfish. Let  $(C^*, S^*)$  be its *p-partition*. Then  $G$  contains  $G_1[C^*] \vee G_2$  as an induced subgraph, thus  $\chi(G) \geq \chi(G_1[C^*] \vee G_2)$ . On the other hand, every coloring of  $G_1[C^*] \vee G_2$

can be extended to  $G$  without adding new colors, by giving to each vertex in  $S^*$  either a color used by a non-neighbor of it in  $C^*$  or a color used in  $G_2$ . Hence,  $\chi(G) = \chi(G_1[C^*] \vee G_2)$ . By [Theorem 2](#),  $\chi(G_1[C^*] \vee G_2) = \chi(G_1[C^*]) + \chi(G_2)$ . By [Lemma 4](#), if  $G_1$  is type 8 then  $\chi(G_1[C^*]) = \chi(G_1) - 1$ , otherwise  $\chi(G_1[C^*]) = \chi(G_1)$ . This concludes the proof of item i.

In order to prove item ii., we will show that  $\chi_b(G) = \chi_b(G_2) + \chi(G_1[C^*])$ . Any  $b$ -coloring of  $G_2$  can be extended to a  $b$ -coloring of  $G$  by assigning  $\chi(G_1[C^*])$  new colors to  $C^*$  and giving to each vertex in  $S^*$  either a color used by a non-neighbor of it in  $C^*$  or a color used in  $G_2$ . So,  $\chi_b(G) \geq \chi_b(G_2) + \chi(G_1[C^*])$ .

If  $\chi_b(G) = \chi(G)$ , by item i.,  $\chi_b(G) = \chi(G_2) + \chi(G_1[C^*]) \leq \chi_b(G_2) + \chi(G_1[C^*])$ . So, we may suppose  $\chi_b(G) > \chi(G)$ . Now let  $\varphi$  be a  $b$ -coloring of  $G$  with more than  $\chi(G)$  colors. By [Remark 1](#), if there is a dominant vertex of color  $c$  in  $G$ , then there is a dominant vertex of color  $c$  in  $C^* \cup V(G_2)$ . Notice that the set of colors used by vertices in  $G_2$  and the set of colors used in  $C^*$  are disjoint, so  $C^*$  should contain dominant vertices for all the colors used in  $V(C^*)$ . In particular, if  $G_1$  is type 5 or 6, by [Lemma 1](#), it follows that the two non-adjacent vertices in  $C^*$  receive the same color, thus  $C^*$  is colored with  $\chi(G_1[C^*])$  colors. On the other hand, it is easy to see that  $\varphi$  restricted to  $V(G_2)$  is a  $b$ -coloring of  $G_2$ . So  $\chi_b(G) \leq \chi_b(G_2) + \chi(G_1[C^*])$ .

We have proved that  $\chi_b(G) = \chi_b(G_2) + \chi(G_1[C^*])$ . By [Lemma 4](#), if  $G$  is type 8 then  $\chi(G_1[C^*]) = \chi(G_1) - 1 = \chi_b(G_1) - 1$ , otherwise  $\chi(G_1[C^*]) = \chi(G_1) = \chi_b(G_1)$ . This concludes the proof of item ii.  $\square$

## 2. $b$ -continuity in $P_4$ -tidy graphs

In [3], a family of cographs with arbitrarily large difference between their  $b$ -chromatic number and their chromatic number was shown. Therefore, it makes sense to analyze  $b$ -continuity in  $P_4$ -tidy graphs. In this section we prove that  $P_4$ -tidy graphs are  $b$ -continuous, by using the decomposition theorem for this class of graphs.

**Lemma 6.** *If  $G$  is  $P_5, \overline{P_5}, C_5$ , a quasi-urchin or a quasi-starfish, then  $G$  is  $b$ -continuous.*

**Proof.** If  $G = P_5$ , then  $\chi(G) = 2$  and  $\chi_b(G) = 3$  and, for the remaining cases, by [Lemma 4](#),  $\chi_b(G) = \chi(G)$ . So, they are trivially  $b$ -continuous.  $\square$

**Lemma 7** ([3]). *Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ . If  $G_1$  and  $G_2$  are  $b$ -continuous and  $G = G_1 \cup G_2$ , then  $G$  is  $b$ -continuous.*

**Lemma 8** ([3]). *Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ . If  $G_1$  and  $G_2$  are  $b$ -continuous and  $G = G_1 \vee G_2$ , then  $G$  is  $b$ -continuous.*

**Lemma 9.** *Let  $G_1 = (V_1, E_1)$  be a  $p$ -separable  $P_4$ -tidy graph and  $G_2 = (V_2, E_2)$  be a graph such that  $V_1 \cap V_2 = \emptyset$ . If  $G_2$  is  $b$ -continuous and  $G = G_1 \vee G_2$ , then  $G$  is  $b$ -continuous.*

**Proof.** By [Proposition 1](#),  $G_1$  is a quasi-starfish or a quasi-urchin. Let  $(C^*, S^*)$  be the  $p$ -partition of  $G_1$ . Suppose first that  $G_1$  is not type 8. Any  $b$ -coloring of  $G_2$  with  $t$  colors  $\{1, \dots, t\}$  can be extended to a  $b$ -coloring of  $G$  with  $t + \chi(G_1)$  colors, in the following way. If we color  $G_1$  using colors  $\{t + 1, \dots, t + \chi(G_1)\}$ , then every dominant vertex in  $G_2$  will have now also neighbors with colors  $t + 1, \dots, t + \chi(G_1)$ , and every dominant vertex in  $C^*$  will now also have neighbors with colors  $1, \dots, t$ . Since  $C^*$  contains dominant vertices of all colors in  $\{t + 1, \dots, t + \chi(G_1)\}$ , the resulting coloring is a  $b$ -coloring of  $G$  with  $t + \chi(G_1)$  colors.

Suppose now that  $G_1$  is type 8. Any  $b$ -coloring of  $G_2$  with  $t$  colors  $\{1, \dots, t\}$  can be extended to a  $b$ -coloring of  $G$  with  $t + \chi(G_1) - 1$  colors, in the following way. If we color  $G_1$  using colors  $\{t, \dots, t + \chi(G_1) - 1\}$  in such a way that  $C^*$  uses colors from  $t + 1$  to  $t + \chi(G_1) - 1$ , then every dominant vertex in  $G_2$  will now also have neighbors with colors  $t + 1, \dots, t + \chi(G_1) - 1$ , and every dominant vertex in  $C^*$  will now also have neighbors with colors  $1, \dots, t$ . Since  $C^*$  contains dominant vertices of all colors in  $\{t + 1, \dots, t + \chi(G_1) - 1\}$ , this results in a  $b$ -coloring of  $G$  with  $t + \chi(G_1) - 1$  colors.

Since  $G_2$  is  $b$ -continuous, we can obtain  $b$ -colorings for  $G$  with each color  $t'$ , where  $\chi(G_2) + \chi(G_1) \leq t' \leq \chi_b(G_2) + \chi_b(G_1)$  in the first case, and  $\chi(G_2) + \chi(G_1) - 1 \leq t' \leq \chi_b(G_2) + \chi_b(G_1) - 1$  in the second case. By [Lemma 5](#),  $\chi(G) = \chi(G_2) + \chi(G_1)$  and  $\chi_b(G) = \chi_b(G_2) + \chi_b(G_1)$  in the first case, while  $\chi(G) = \chi(G_2) + \chi(G_1) - 1$  and  $\chi_b(G) = \chi_b(G_2) + \chi_b(G_1) - 1$  in the second case, so  $G$  is  $b$ -continuous.  $\square$

**Theorem 5.**  *$P_4$ -tidy graphs are  $b$ -continuous.*

**Proof.** Immediate by an inductive argument using the decomposition [Theorem 4](#), [Proposition 1](#), [Lemmas 7–9](#) and [6](#) for the base case of the induction.  $\square$

## 3. Computation of the $b$ -chromatic number in $P_4$ -tidy graphs

The inequality in part i. of [Theorem 3](#) can be strict, and this fact prevents us from using this result for directly computing the  $b$ -chromatic number of  $P_4$ -tidy graphs by using the decomposition [Theorem 4](#). In fact, it is not difficult to build examples showing that the  $b$ -chromatic number of the graph  $G_1 \cup G_2$  does not depend only on the  $b$ -chromatic numbers of  $G_1$  and  $G_2$ . To overcome this problem, we follow the approach in [3] in the definition of the *dominance sequence*  $\text{dom}_G \in \mathbb{Z}^{\mathbb{N} \geq \chi(G)}$  of a graph  $G$ , where  $\text{dom}[t]$  is the maximum number of distinct color classes that admit dominant vertices in any coloring of  $G$  with  $t$  colors, for  $\chi(G) \leq t \leq |V(G)|$ . We will compute this sequence recursively on  $P_4$ -tidy graphs by using the decomposition theorem. Then we will obtain the  $b$ -chromatic number of  $G$  as the maximum  $t$  such that  $\text{dom}_G[t] = t$ .

**Lemma 10.** Let  $G$  be  $P_5$ ,  $\overline{P_5}$ ,  $C_5$ , a quasi-urchin or a quasi-starfish. The dominance sequence for  $G$  can be obtained in linear time.

**Proof.** It is easy to see that  $\text{dom}_{P_5}[2] = 2$ ,  $\text{dom}_{P_5}[3] = 3$ , and  $\text{dom}_{P_5}[t] = 0$  for  $t \geq 4$ ;  $\text{dom}_{\overline{P_5}}[3] = 3$ ,  $\text{dom}_{\overline{P_5}}[4] = 1$ , and  $\text{dom}_{\overline{P_5}}[5] = 0$ ;  $\text{dom}_{C_5}[3] = 3$ , and  $\text{dom}_{C_5}[t] = 0$  for  $t \geq 4$ . Now, let  $G = (C^*, S^*)$  be a quasi-urchin or quasi-starfish of size  $k$ . Let  $(\overline{C}, \overline{S})$  be the  $p$ -partition of the urchin or starfish,  $S = \{s_1, \dots, s_k\}$ ,  $C = \{c_1, \dots, c_k\}$ . If a vertex in  $S$  (resp.  $C$ ) was replaced by two vertices, we will assume that the vertex was  $s_1$  (resp.  $c_1$ ) and that it was replaced by vertices  $s'_1, s''_1$  (resp.  $c'_1, c''_1$ ). Recall that, for every graph  $G$ ,  $\text{dom}_G[\chi(G)] = \chi(G)$ . Consider now colorings of  $G$  with more than  $\chi(G)$  colors. By Remark 1, if there is a dominant vertex of color  $c$  in  $G$ , then there is a dominant vertex of color  $c$  in  $C^*$ . So, for  $t > \chi(G)$ ,  $\text{dom}_G[t] \leq |C^*|$ .

If  $G$  is type 1, then  $\text{dom}_G[k] = \text{dom}_G[k+1] = k$  and  $\text{dom}_G[t] = 0$  for  $t \geq k+2$ ; if  $G$  is type 2, then  $\text{dom}_G[k+s] = \min\{k, 2k-2s\}$  for  $0 \leq s \leq k$ , and  $\text{dom}_G[t] = 0$  for  $t > 2k$  [3].

We start by analyzing the different kinds of quasi-urchins.

**Claim 1.** If  $G$  is type 3, then  $\text{dom}_G[k+1] = \text{dom}_G[k+2] = k+1$ ,  $\text{dom}_G[t] = 0$  for  $t \geq k+3$ .

In  $G$  there are  $k+1$  vertices of degree  $k+1$  and no vertex of degree at least  $k+2$ , so the upper bounds for each value of  $\text{dom}_G$  are clear (a dominant vertex in a coloring with  $t$  colors must have degree at least  $t-1$ ). A coloring with  $k+2$  colors and  $k+1$  dominant vertices of different colors can be obtained by coloring all the vertices in  $S^*$  with the same color, different from the colors used in  $C^*$ .  $\square$

**Claim 2.** If  $G$  is type 5, then  $\text{dom}_G[k] = \text{dom}_G[k+1] = k$ ,  $\text{dom}_G[k+2] = k-1$ ,  $\text{dom}_G[t] = 0$  for  $t \geq k+3$ .

Since  $k \geq 3$ , in  $G$  there are  $k-1$  vertices of degree  $k+1$ , 2 vertices of degree  $k$ , and no vertex of degree at least  $k+2$ . So, the upper bounds on  $\text{dom}_G[t]$  for  $t \geq k+2$  are clear. The upper bound for  $\text{dom}_G[k+1]$  holds by Lemma 1. Two colorings attaining the upper bounds for  $\text{dom}_G[k+1]$  and  $\text{dom}_G[k+2]$  are defined as follows. Vertices  $c_2, \dots, c_k$  receive colors  $1, \dots, k-1$ ; vertices  $s_1, \dots, s_k$  receive color  $k+1$ ; vertices  $c'_1, c''_1$  receive both color  $k$  or colors  $k$  and  $k+2$ , respectively.  $\square$

**Claim 3.** If  $G$  is type 7 or type 9, then  $\text{dom}_G[k] = \text{dom}_G[k+1] = k$ ,  $\text{dom}_G[k+2] = 1$ ,  $\text{dom}_G[t] = 0$  for  $t \geq k+3$ .

Since  $k \geq 3$ , in  $G$  there are  $k-1$  vertices of degree  $k$ , one vertex of degree  $k+1$ , and no vertex of degree at least  $k+2$ . So, the upper bounds on  $\text{dom}_G[t]$  are clear. Two colorings attaining the upper bounds for  $\text{dom}_G[k+1]$  and  $\text{dom}_G[k+2]$  are defined as follows. Vertices  $c_1, \dots, c_k$  receive colors  $1, \dots, k$ ; vertices  $s'_1, s_2, \dots, s_k$  receive color  $k+1$ ; vertex  $s''_1$  receives color 2 or  $k+2$ , respectively.  $\square$

We will now analyze the different kinds of quasi-starfishes.

**Claim 4.** If  $G$  is type 4, then  $\text{dom}_G[k+1+s] = \min\{k, 2k-2s\} + 1$  for  $0 \leq s < k$ , and  $\text{dom}_G[t] = 0$  for  $t > 2k$ .

Since  $\chi(G) = k+1$ , then  $\text{dom}_G[k+1] = k+1$ . Since the maximum degree of  $G$  is  $2k-1$ , it is clear that  $\text{dom}_G[t] = 0$  for  $t > 2k$ . Let  $t = k+1+s$  such that  $1 \leq s \leq k-1$  and let  $\varphi$  be a coloring of  $G$  with  $t$  colors and maximum number of colors with dominant vertices. At least one of  $c'_1, c''_1$  has a color different from  $\varphi(s_1)$ . Suppose without loss of generality that  $\varphi(c''_1) \neq \varphi(s_1)$ , then  $\varphi(c'_1) \neq \varphi(v)$  for every  $v \in V(G)$ . Let  $G' = G - \{c''_1\}$ . Thus the restriction of  $\varphi$  to  $G'$  is a coloring with  $t-1$  colors, and dominant vertices of  $G$  are still dominant in  $G'$ , therefore  $\text{dom}_G[t] \leq \text{dom}_{G'}[t-1] + 1$ . Conversely, let  $\psi$  be a coloring of  $G'$  with  $t-1$  colors (namely, colors  $1, \dots, t-1$ ) and maximum number of colors with dominant vertices. We can extend  $\psi$  to a  $t$ -coloring of  $G$  by defining  $\psi(c''_1) = t$ . Since  $t-1 \geq k+1$ , no vertex in  $S^*$  was dominant in  $G'$ , so every dominant vertex of  $G'$  is still dominant in  $G$ . Besides,  $c''_1$  is now dominant in  $G$  if and only if  $\psi(s_1) = \psi(v)$  for some vertex  $v$  of  $G'$ , different from  $s_1$ , and this happens if and only if  $c'_1$  was dominant in  $G'$ . By symmetry of  $G'$ , we may assume that if  $\text{dom}_{G'}[t-1] > 0$  then  $c'_1$  was dominant in  $G'$ . So, if  $\text{dom}_{G'}[t-1] > 0$ , we have that  $\text{dom}_G[t] = \text{dom}_{G'}[t-1] + 1$ . Since  $G'$  is type 2, we already know that  $\text{dom}_{G'}[k+s] = \min\{k, 2k-2s\}$ . Since  $s \leq k-1$ ,  $\text{dom}_{G'}[t-1] > 0$ , and  $\text{dom}_G[k+1+s] = \min\{k, 2k-2s\} + 1$ .  $\square$

**Claim 5.** If  $G$  is type 6, then  $\text{dom}_G[k+s] = k$  for  $0 \leq s \leq \lfloor \frac{k}{2} \rfloor$ ,  $\text{dom}_G[k+s] = \min\{k-1, 2k-2s+2\}$  for  $\lfloor \frac{k}{2} \rfloor \leq s \leq k$ , and  $\text{dom}_G[k+s] = 0$  for  $s > k$ .

Since  $\chi(G) = k$ , then  $\text{dom}_G[k] = k$ . Since the maximum degree of  $G$  is  $2k-1$ , it is clear that  $\text{dom}_G[t] = 0$  for  $t > 2k$ . Let  $t = k+s$  with  $1 \leq s \leq k$  and let  $\varphi$  be a coloring of  $G$  with  $t$  colors and maximum number of colors with dominant vertices.

Suppose first that  $\varphi(c'_1) = \varphi(c''_1)$ . Then the number of colors with dominant vertices in  $G$  is the same as the number of colors with dominant vertices when restricting  $\varphi$  to  $G' = G - \{c'_1\}$ . Conversely, any coloring of  $G'$  can be extended to a coloring of  $G$  by giving to  $c''_1$  the color used by  $c'_1$ , thus preserving the dominant vertices. Then, if  $\varphi(c'_1) = \varphi(c''_1)$ , it follows that  $\text{dom}_G[k+s] = \text{dom}_{G'}[k+s]$  and, since  $G'$  is type 2,  $\text{dom}_G[k+s] = \min\{k, 2k-2s\}$ .

Suppose now that  $\varphi(c'_1) \neq \varphi(c''_1)$ . By Lemma 1, none of  $c'_1, c''_1$  is dominant. So, in this case, the number of colors with dominant vertices is at most  $k-1$ . We may assume  $2s > k$ , otherwise, by the arguments above, we can find a coloring  $\varphi'$  of  $G$  with  $\varphi'(c'_1) = \varphi'(c''_1)$  and such that there are  $k$  colors with dominant vertices. Since  $k \geq 2$ , this implies  $s > 1$ , hence  $t > k+1$ . Since  $\varphi(c'_1) \neq \varphi(c''_1)$ , at least one of them has a color different from  $\varphi(s_1)$ . Suppose without loss of generality that  $\varphi(c'_1) \neq \varphi(s_1)$ , then  $\varphi(c'_1) \neq \varphi(v)$  for every  $v \in V(G)$ . Let  $G' = G - \{c'_1\}$ . Thus the restriction of  $\varphi$  to  $G'$  is a coloring with  $t-1$  colors, and dominant vertices of  $G$  are still dominant in  $G'$ . Since  $c'_1$  was not dominant in  $G$ , the number of colors with dominant vertices does not decrease. Conversely, let  $\psi$  be a coloring of  $G'$  with  $t-1$  colors (namely, colors  $1, \dots, t-1$ ) and maximum number of colors with dominant vertices. By Lemma 3, all the dominant vertices are in  $C^*$ . We can extend  $\psi$  to a  $t$ -coloring of  $G$  by defining  $\psi(c'_1) = t$ . All dominant vertices in  $\{c_2, \dots, c_k\}$  are still dominant. If there were less than  $k$  dominant vertices, we may assume

by symmetry of  $G'$  that they were in  $\{c_2, \dots, c_k\}$ . If there were  $k$  dominant vertices in  $G'$ , vertex  $c_1$  is no longer dominant, still  $c_2, \dots, c_k$  are dominant, and we know that, if  $\varphi(c'_1) \neq \varphi(c''_1)$ , then in  $G$  there cannot be more than  $k - 1$  colors with dominant vertices. So, in that case,  $\text{dom}_G[k + s] = \min\{k - 1, \text{dom}_{G'}[k + s - 1]\}$ . Since  $G'$  is type 2,  $\text{dom}_G[k + s] = \min\{k - 1, 2k - 2s + 2\}$ .

So, if  $2s \leq k$ , then  $\text{dom}_G[k + s] = k$  and the optimum is attained by a coloring where  $c'_1$  and  $c''_1$  receive the same color. If  $2s > k$ , then  $\text{dom}_G[k + s] = \min\{k - 1, 2k - 2s + 2\}$  and the optimum is attained by a coloring where  $c'_1$  and  $c''_1$  receive different colors.  $\square$

**Claim 6.** If  $G$  is type 8, then  $\text{dom}_G[k + 1] = k + 1$ ; then  $\text{dom}_G[k + s] = k$  for  $2 \leq s \leq \lfloor \frac{k+1}{2} \rfloor$ ;  $\text{dom}_G[k + s] = k - 1$  for  $s = \frac{k+2}{2}$  (when  $k$  is even);  $\text{dom}_G[k + s] = 2k - 2s + 2$  for  $\lfloor \frac{k+3}{2} \rfloor \leq s \leq k$ ; and  $\text{dom}_G[t] = 0$  for  $t > 2k$ .

Since  $\chi(G) = k + 1$ , then  $\text{dom}_G[k + 1] = k + 1$ . Since the maximum degree of  $G$  is  $2k - 1$ , it is clear that  $\text{dom}_G[t] = 0$  for  $t > 2k$ . Let  $t = k + s$  with  $2 \leq s \leq k$  and let  $\varphi$  be a  $t$ -coloring of  $G$  with maximum number of colors with dominant vertices. For  $i \geq 2$ , vertex  $c_i$  will be dominant if and only if color  $\varphi(s_i)$  is used by some other vertex in  $G$ , and vertex  $c_1$  will be dominant if and only if colors  $\varphi(s'_1)$  and  $\varphi(s''_1)$  are used by some other vertices in  $(c_1, s_2, \dots, s_k)$ . We may assume without loss of generality that  $\varphi(c_i) = i$ , for  $i = 1, \dots, k$ , and that vertices in  $S^*$  use colors  $k + 1, \dots, k + s$ . If some vertex  $s_i$  uses a color at most  $k$ , we can always recolor it with a color from  $k + 1, \dots, k + s$  that is already used in  $S^*$ . Since  $s \geq 2$ , we can do it also for  $s'_1$  and  $s''_1$ . If  $2s \leq k + 1$ , we can assign colors  $k + 1, \dots, k + s$  to vertices in  $S^*$ , repeating each of them at least once, in such a way that all the vertices in  $C^*$  are dominant. If  $2s > k + 1$ , this is not possible. Since  $\varphi(s'_1) \neq \varphi(s''_1)$  and all the colors  $k + 1, \dots, k + s$  are used in  $S^*$ , we may assume without loss of generality that  $\varphi(s'_1) = k + 1$ ,  $\varphi(s''_1) = k + 2$ , and  $\varphi(s_i) = k + 1 + i$  for  $i = 2, \dots, s - 1$  (when  $s \geq 3$ ). To each of the  $k + 1 - s$  remaining vertices we can assign different colors from  $k + 1, \dots, k + s$ . If we assign color  $k + 1 + i$  to vertex  $s_j$ , with  $s \leq j \leq k$  and  $2 \leq i \leq s - 1$ , both  $c_i$  and  $c_j$  become dominant. If we assign color  $k + 1$  (resp.  $k + 2$ ) to some vertex  $s_j$  with  $s \leq j \leq k$ , then  $c_j$  will be dominant but  $c_1$  will be dominant only if some other vertex  $s_{j'}$ ,  $s \leq j' \leq k$ , receives  $k + 2$  (resp.  $k + 1$ ). So, as we have less than  $s$  remaining vertices, the optimum  $2(k + 1 - s)$  is attained by assigning to  $s_s, \dots, s_k$  different colors from  $k + 3$  to  $k + s$  when  $k + 1 - s \leq s - 2$ . The last case is when  $k + 1 - s = s - 1$ , that is,  $k$  is even and  $2s = k + 2$ . In this case we can assign to  $s_s, \dots, s_{k-1}$  different colors from  $k + 3$  to  $k + s$  and to vertex  $s_k$  color  $k + 1$ . In this case, all the vertices of  $C^*$  but  $c_1$  are dominant, and this is optimal.  $\square$

**Claim 7.** If  $G$  is type 10, then  $\text{dom}_G[k + s] = k$  for  $0 \leq s \leq \lfloor \frac{k+1}{2} \rfloor$ ;  $\text{dom}_G[k + s] = k - 1$  for  $s = \frac{k+2}{2}$  (when  $k$  is even);  $\text{dom}_G[k + s] = 2k - 2s + 2$  for  $\lfloor \frac{k+3}{2} \rfloor \leq s \leq k$ ; and  $\text{dom}_G[t] = 0$  for  $t > 2k$ .

Since  $\chi(G) = k$ , then  $\text{dom}_G[k] = k$ . A coloring with  $k + 1$  colors and  $k$  dominant vertices is obtained by giving colors  $1, \dots, k$  to vertices in  $C^*$  and color  $k + 1$  to each vertex in  $S^*$ . Since the maximum degree of  $G$  is  $2k - 1$ , it is clear that  $\text{dom}_G[t] = 0$  for  $t > 2k$ . The arguments for  $k + 2 \leq s \leq 2k$  are very similar to those in the proof of Claim 6, and are omitted.  $\square$

In all the cases, given the type of the graph, the dominance sequence can be computed in linear time. The type of the graph can be also determined in linear time [10].  $\square$

**Theorem 6** ([3]). Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ . Let  $G = G_1 \cup G_2$  and  $t \geq \chi(G)$ . Then

$$\text{dom}_G[t] = \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}.$$

**Theorem 7** ([3]). Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ . Let  $G = G_1 \vee G_2$  and  $\chi(G) \leq t \leq |V(G)|$ . Let  $a = \max\{\chi(G_1), t - |V(G_2)|\}$  and  $b = \min\{|V(G_1)|, t - \chi(G_2)\}$ . Then  $a \leq b$  and

$$\text{dom}_G[t] = \max_{a \leq j \leq b} \{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t - j]\}.$$

**Theorem 8.** Let  $G_1 = (V_1, E_1)$  be a quasi-urchin or a quasi-starfish of size  $k$  and  $G_2 = (V_2, E_2)$  be a graph such that  $V_1 \cap V_2 = \emptyset$ ,  $V_2 \neq \emptyset$ . Let  $G = G_1 \vee G_2$ . Then, the following statements hold.

- i. If  $G_1$  is type 1, 2, 7, 9 or 10, then
  - a.  $\text{dom}_G[k + r] = k + \text{dom}_{G_2}[r]$ , for  $\chi(G_2) \leq r \leq |V_2|$ ;
  - b.  $\text{dom}_G[k + |V_2| + s] = \text{dom}_{G_1}[k + s]$ , for  $1 \leq s \leq |V_1| - k$ .
- ii. If  $G_1$  is type 3 or 4, then
  - a.  $\text{dom}_G[k + 1 + r] = k + 1 + \text{dom}_{G_2}[r]$ , for  $\chi(G_2) \leq r \leq |V_2|$ ;
  - b.  $\text{dom}_G[k + 1 + |V_2| + s] = \text{dom}_{G_1}[k + 1 + s]$ , for  $1 \leq s \leq |V_1| - k - 1$ .
- iii. If  $G_1$  is type 5 or 6, then
  - a.  $\text{dom}_G[k + \chi(G_2)] = k + \chi(G_2)$ ;
  - b.  $\text{dom}_G[k + r] = \max\{k + \text{dom}_{G_2}[r], k - 1 + \text{dom}_{G_2}[r - 1]\}$ , for  $\chi(G_2) < r \leq |V_2|$ ;
  - c.  $\text{dom}_G[k + 1 + |V_2|] = \max\{k, k - 1 + \text{dom}_{G_2}[|V_2|]\}$ ;
  - d.  $\text{dom}_G[k + |V_2| + s] = \text{dom}_{G_1}[k + s]$ , for  $2 \leq s \leq |V_1| - k$ .
- iv. If  $G_1$  is type 8, then
  - a.  $\text{dom}_G[k + r] = k + \text{dom}_{G_2}[r]$ , for  $\chi(G_2) \leq r \leq |V_2|$ ;
  - b.  $\text{dom}_G[k + 1 + |V_2|] = k$ ;
  - c.  $\text{dom}_G[k + |V_2| + s] = \text{dom}_{G_1}[k + s]$ , for  $2 \leq s \leq |V_1| - k$ .

**Proof.** Recall that  $\text{dom}[\chi(G)] = \chi(G)$ , and that the chromatic number of each type of quasi-starfish or quasi-urchin is described in Lemma 4. Let  $(C^*, S^*)$  be the  $p$ -partition of  $G_1$ . Notice first that in any coloring of  $G$ , the set of colors used by  $V_2$  and  $C^*$  are disjoint. Let  $\varphi$  be a coloring of  $G$  with  $t$  colors,  $t > \chi(G)$ . Vertices in  $S^*$  are either simplicial or have degree at most  $\chi(G) - 1$  (recall that  $V_2 \neq \emptyset$ ). So no vertex in  $S^*$  can be dominant. If some vertex of  $S^*$  has a color that is used neither in  $V_2$  nor in  $C^*$ , then no vertex in  $V_2$  is dominant. We start the case analysis. If  $G_1$  is type 1, 2, 7, 9 or 10, then  $C^*$  is a clique of size  $k$ . Every vertex in  $C^*$  is dominant when the colors used by  $S^*$  are used also in  $C^* \cup V_2$ , and they are still dominant if we consider  $\varphi$  restricted to  $G[V_2 \cup C^*]$ . By Theorem 7,  $\text{dom}_G[k+r] = k + \text{dom}_{G_2}[r]$ , for  $\chi(G_2) \leq r \leq |V_2|$ . If  $t > k + |V_2|$ , at least some color must be used only in  $S^*$ . So the only candidates to be dominant vertices are vertices in  $C^*$ . Since they are adjacent to all the vertices in  $V_2$ , we may assume that no vertex in  $S^*$  uses a color used in  $V_2$ , and each vertex of  $C^*$  is dominant if and only if it is dominant in  $G[V_1]$ , so  $\text{dom}_G[k+|V_2|+s] = \text{dom}_{G_1}[k+s]$ , for  $1 \leq s \leq |V_1| - k$ . If  $G_1$  is type 3 or 4, the analysis is the same but taking into account that  $C^*$  is a clique of size  $k+1$ . If  $G_1$  is type 5 or 6, then  $C^*$  is not a clique. We may assume that the original set was  $C = \{c_1, \dots, c_k\}$  and vertex  $c_1$  was replaced by two false twins  $c'_1, c''_1$ . Item iii.a holds because  $\chi(G) = k + \chi(G_2)$ . Most of the observations for the previous cases still hold. So, when  $\chi(G_2) < t - k \leq |V_2| + 1$ , we have two possibilities to color  $C^*$ : we can either use  $k$  colors, being  $\varphi(c'_1) = \varphi(c''_1)$ , and in that case  $k$  vertices of different colors will be dominant in  $C^*$ , or use  $k+1$  colors and, by Lemma 1, only  $k-1$  vertices in  $C^*$  will be dominant. This leads to the expressions iii.b and iii.c. Finally, when  $t > k + |V_2| + 1$ , at least some color must be used only in  $S^*$ . The analysis in (\*) leads to the expression iii.d. Finally, if  $G_1$  is type 8, then  $C^*$  is a clique of size  $k$  but  $\chi(G_1) = k+1$ . In this case, if  $\chi(G_2) \leq r \leq |V_2|$ , necessarily one color in  $V_2$  will be used also in  $S^*$ , but the analysis is the same as in case i.a. Also the case iv.c is similar to i.b. The only difference is when  $t = k + 1 + |V_2|$ . We cannot say that  $\text{dom}_G[k + 1 + |V_2|] = \text{dom}_{G_1}[k + 1] = k + 1$ , because we know that we have dominant vertices only in  $C^*$ , so  $\text{dom}_G[k + 1 + |V_2|] \leq k$ . A coloring with  $k$  dominant vertices in  $C^*$  is attainable by giving colors  $1, \dots, k$  to vertices in  $C^*$ , color  $k+1$  to vertices in  $S^* \setminus \{s'_1\}$ , color  $k+2$  to  $s'_1$ , and colors  $k+2, \dots, k+1+|V_2|$  to vertices in  $V_2$ .  $\square$

**Theorem 9.** *The dominance vector and the  $b$ -chromatic number of a  $P_4$ -tidy graph  $G$  can be computed in  $O(n^3)$  time.*

**Proof.** The previous results give a dynamic programming algorithm to compute the dominance sequence of a  $P_4$ -tidy graph from its decomposition tree, that can be computed in linear time [10]. By Theorems 6–8 and 4, Proposition 1 and the fact that  $P_4$ -tidy graphs are hereditary, we can compute recursively the dominance vector and consequently the  $b$ -chromatic number of  $G$  in  $O(n^3)$  time. Indeed, if  $G = G_1 \cup G_2$ , by Theorem 6, the value for  $\text{dom}_G[t]$  is obtained from  $\text{dom}_{G_1}[t]$  and  $\text{dom}_{G_2}[t]$  directly. By Theorem 8, the same case holds for  $G = G_1 \vee G_2$ . If  $G = G_1 \vee G_2$ , we must examine at most  $n$  values of  $j$  for each value of  $t$ , by Theorem 7. We have at most  $n$  of these reduction steps, because in each case we must compute two disjoint subgraphs. The base case, computing the dominance sequence of the trivial graph and the five elementary subgraphs in the decomposition, can be done in  $O(1)$  by Lemma 10. So the total computation time is  $O(n^3)$ . Once we have computed the dominance sequence of  $G$ , we obtain the  $b$ -chromatic number as the maximum value  $t$  such that  $\text{dom}_G[t] = t$ .  $\square$

#### 4. $b$ -monotonicity in $P_4$ -tidy graphs

In this section, we will show that  $P_4$ -tidy graphs are  $b$ -monotonic. To this end, we will prove the following property.

**Theorem 10.** *For every  $P_4$ -tidy graph  $G$ , every induced subgraph  $H$  of  $G$  and every  $t \geq \chi(G)$ ,  $\text{dom}_H[t] \leq \text{dom}_G[t]$  holds.*

We first state some necessary results.

**Lemma 11.** *Let  $G$  be a  $P_5$ , a  $\overline{P_5}$ , a  $C_5$ , a quasi-urchin or a quasi-starfish. Then, for every  $t \geq \chi(G)$  and every vertex  $v$  of  $G$ ,  $\text{dom}_{G-\{v\}}[t] \leq \text{dom}_G[t]$  holds.*

**Proof.** The cases  $P_5$ ,  $\overline{P_5}$  and  $C_5$  are easy to verify. Let  $G = (C^*, S^*)$  be a quasi-urchin or quasi-starfish of size  $k$ . Let  $(C, S)$  be the  $p$ -partition of the urchin or starfish,  $S = \{s_1, \dots, s_k\}$ ,  $C = \{c_1, \dots, c_k\}$ . If a vertex in  $S$  (resp.  $C$ ) was replaced by two vertices, we will assume that the vertex was  $s_1$  (resp.  $c_1$ ) and that it was replaced by vertices  $s'_1, s''_1$  (resp.  $c'_1, c''_1$ ). Let  $t \geq \chi(G)$ , and let  $v$  be a vertex of  $G$ . Let  $\varphi$  be a  $t$ -coloring of  $G - \{v\}$  that maximizes the number of color classes with dominant vertices. Suppose first that  $v$  is a leg of  $G$  and either  $G$  is not type 8 or  $v$  is different from  $s'_1, s''_1$ . Then  $\varphi$  can be extended to a  $t$ -coloring of  $G$  with the same number of dominant vertices by giving to  $v$  the color of some vertex in the body non-adjacent to it. If  $G$  is type 8 and  $v = s'_1$ , since  $t \geq \chi(G) = k + 1$ , we can give to  $s'_1$  either a color that is not used in the body of  $G$  or the color  $\varphi(c_1)$  (depending on whether  $\varphi(s''_1) = \varphi(c_1)$  or not). Now, suppose that  $v$  is a vertex in the body of  $G$ . If  $v$  has a false twin, we can color  $v$  with the color used by its false twin. Otherwise, since  $t \geq \chi(G)$ , there is some color  $c$  that is not used in the body of  $G$ . We will extend  $\varphi$  to a  $t$ -coloring of  $G$  with at least the same number of dominant vertices by setting  $\varphi(v) = c$ . If some leg  $w$  of  $G$  adjacent to  $v$  was colored  $c$ , then all its neighbors are also neighbors of  $v$ , so we can recolor  $w$  with the color of some vertex in the body non-adjacent to it, and all dominant vertices will still be dominant. The only case in which we cannot do this is when  $G$  is type 8,  $v$  is not  $c_1$ , one of  $s'_1, s''_1$  uses color  $c$  and the other one uses color  $\varphi(c_1)$ . But, in that case, since  $t \geq \chi(G) = k + 1$ , there are in fact at least two colors  $c, c'$  not used in the body of  $G$ . So we can give color  $c'$  to  $v$ , and recolor as mentioned above all the legs adjacent to it (note that neither  $s'_1$  nor  $s''_1$  uses  $c'$  in the case we are dealing with). Hence,  $\text{dom}_{G-\{v\}}[t] \leq \text{dom}_G[t]$ .  $\square$

**Lemma 12.** Let  $G_1 = (V_1, E_1)$  be a quasi-starfish or a quasi-urchin and  $G_2 = (V_2, E_2)$  be a  $b$ -continuous graph such that  $V_1 \cap V_2 = \emptyset$  and, for every  $t \geq \chi(G_2)$  and every induced subgraph  $H$  of  $G_2$ ,  $\text{dom}_H[t] \leq \text{dom}_{G_2}[t]$ . Let  $G = G_1 \vee G_2$ . Then, for every  $t \geq \chi(G)$  and every vertex  $v$  of  $G$ ,  $\text{dom}_{G-\{v\}}[t] \leq \text{dom}_G[t]$  holds.

**Proof.** If  $t = \chi(G)$  the statement is clearly true. Let  $t > \chi(G)$ , and let  $v$  be a vertex of  $G$ . Let  $\varphi$  be a  $t$ -coloring of  $G - \{v\}$  that maximizes the number of color classes with dominant vertices. We will extend  $\varphi$  to a  $t$ -coloring of  $G$  with the same number of color classes with dominant vertices. Let  $(C^*, S^*)$  be the  $p$ -partition of  $G_1$ . Notice that, since  $t > \chi(G) \geq \chi(G - \{v\})$ , no vertex in  $S^*$  is dominant.

Suppose first that  $v$  is a vertex of  $S^*$ . We can extend  $\varphi$  by giving to  $v$  a color not used by any of its neighbors (it is always possible because  $t > \chi(G)$ ).

Suppose now that  $v$  is a vertex of  $V_2$ . If  $|V_2| = 1$  then the lemma holds by Theorem 8 and the claims in the proof of Lemma 10. If  $|V_2| > 1$ , let  $r$  be the number of colors used by  $V_2 - \{v\}$  in  $\varphi$ . If  $r \geq \chi(G_2)$ , since  $\text{dom}_{G_2}[r] \geq \text{dom}_{G_2-\{v\}}[r]$ , we can replace  $\varphi$  restricted to  $V_2 - \{v\}$  by an  $r$ -coloring of  $G_2$  with  $\text{dom}_{G_2}[r]$  color classes with dominant vertices, thus obtaining a  $t$ -coloring of  $G$  with at least the same dominant color classes as before. Otherwise,  $r = \chi(G_2 - \{v\}) = \chi(G_2) - 1$ . Since  $t > \chi(G)$ , it follows that  $t - r \geq \chi(G_1)$ . Notice that the equality can hold only if  $G_1$  is type 8. Then we can replace  $\varphi$  restricted to  $V_2 - \{v\}$  by an  $(r + 1)$ -coloring of  $G_2$  and, by Lemma 10,  $\varphi$  restricted to  $V_1$  by a coloring of  $G_1$  with at most  $t - r - 1$  new colors and at least the same dominant color classes as before (if  $G_1$  is type 8 and  $t - r = \chi(G_1)$ , we can assign to one of the true twin vertices in  $S^*$  a color used in  $V_2$ ).

Finally, suppose that  $v$  is a vertex in  $C^*$ . If  $v$  has a false twin  $v'$  in  $C^* - \{v\}$ , we are done by setting  $\varphi(v) = \varphi(v')$ . If there are two false twins  $w, w'$  in  $C^* - \{v\}$  using different colors, we can assign to  $v$  color  $\varphi(w')$  and to  $w'$  color  $\varphi(w)$  (possibly recoloring in a suitable way vertices in  $S^*$ ), obtaining a  $t$ -coloring of  $G$  with at least the same dominant color classes as before. Otherwise,  $v$  is adjacent to all vertices in  $C^* - \{v\}$  and they are colored with  $\chi(G[C^* - \{v\}])$  colors. Let  $r$  be the number of colors used by  $V_2$  in  $\varphi$ . If  $\varphi$  restricted to  $V_2$  is not a  $b$ -coloring, we can eliminate one color class from  $V_2$  without decreasing the number of color classes with dominant vertices, and give that color to  $v$ , that will be adjacent to all the vertices that were dominant, thus obtaining the desired  $t$ -coloring for  $G$ . If  $\varphi$  restricted to  $V_2$  is a  $b$ -coloring and  $r > \chi(G_2)$ , since  $G_2$  is  $b$ -continuous, we can replace  $\varphi$  restricted to  $V_2$  by a  $b$ -coloring of  $G_2$  with  $r - 1$  colors, thus giving the remaining color to  $v$  as before. Finally, by Lemma 5 and being  $t > \chi(G)$ , if  $r = \chi(G_2)$  then  $t - r \geq \chi(G_1)$ . In that case, it is easy to see that we can replace  $\varphi$  restricted to  $V_1$  by a  $t - r$ -coloring of  $G_1$ , maintaining or increasing the number of color classes with dominant vertices, thus obtaining the desired  $t$ -coloring of  $G$ .  $\square$

**Lemma 13** ([3]). Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs such that  $V_1 \cap V_2 = \emptyset$ , and let  $G = G_1 \cup G_2$ . Assume that for every  $t \geq \chi(G_i)$  and every induced subgraph  $H$  of  $G_i$  we have  $\text{dom}_H[t] \leq \text{dom}_{G_i}[t]$ , for  $i = 1, 2$ . Then, for every  $t \geq \chi(G)$  and every induced subgraph  $H$  of  $G$ ,  $\text{dom}_H[t] \leq \text{dom}_G[t]$  holds.

**Lemma 14** ([3]). Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two  $b$ -continuous graphs such that  $V_1 \cap V_2 = \emptyset$ , and let  $G = G_1 \vee G_2$ . Assume that for every  $t \geq \chi(G_i)$  and every induced subgraph  $H$  of  $G_i$  we have  $\text{dom}_H[t] \leq \text{dom}_{G_i}[t]$ , for  $i = 1, 2$ . Then, for every  $t \geq \chi(G)$  and every induced subgraph  $H$  of  $G$ ,  $\text{dom}_H[t] \leq \text{dom}_G[t]$  holds.

**Lemma 15** ([3]). Let  $G$  be a graph. The maximum value of  $\text{dom}_G[t]$  is attained in  $t = \chi_b(G)$ .

**Proof of Theorem 10.** Let us consider a minimal counterexample for the theorem, that is, a  $P_4$ -tidy graph  $G$  and an induced subgraph  $H$  of  $G$  such that  $\text{dom}_H[t] > \text{dom}_G[t]$  for some  $t \geq \chi(G)$ , but such that  $\text{dom}_{H_2}[t] \leq \text{dom}_{H_1}[t]$  for every induced subgraph  $H_1$  of  $H$ , every induced subgraph  $H_2$  of  $H_1$ , and every  $t \geq \chi(H_1)$ . By Lemmas 13 and 14,  $G$  is neither the union nor the join of two smaller graphs. Let  $W = V(G) \setminus V(H)$ , namely,  $W = \{w_1, \dots, w_s\}$ . If we consider the sequence of graphs defined by  $G_0 = G$ ,  $G_i = G_{i-1} - \{w_i\}$  for  $1 \leq i \leq s$ , it turns out that  $G_s = H$ . Since  $\text{dom}_H[t] > \text{dom}_G[t]$ , for some  $i \geq 1$ , it holds  $\text{dom}_{G_i}[t] > \text{dom}_{G_{i-1}}[t]$ . Since the counterexample was minimal, it should be  $i = 1$ , thus  $H = G - \{w\}$  for some vertex  $w$  of  $G$ . By Lemma 11, Theorem 5 and Lemma 12, Theorem 4 and Proposition 1, such a counterexample does not exist.  $\square$

**Corollary 1.**  $P_4$ -tidy graphs are  $b$ -monotonic.

**Proof.** Since  $P_4$ -tidy graphs are hereditary, it suffices to show that given a  $P_4$ -tidy graph  $G$ ,  $\chi_b(G) \geq \chi_b(H)$  for every induced subgraph  $H$  of  $G$ . Let  $G$  be a  $P_4$ -tidy graph, and let  $H$  be an induced subgraph of  $G$ . If  $\chi_b(H) < \chi(G)$ , then  $\chi_b(H) < \chi_b(G)$ . Otherwise, by Theorem 10,  $\chi_b(H) = \text{dom}_H[\chi_b(H)] \leq \text{dom}_G[\chi_b(H)]$  and, by Lemma 15,  $\text{dom}_G[\chi_b(H)] \leq \text{dom}_G[\chi_b(G)] = \chi_b(G)$  implying that  $\chi_b(G) \geq \chi_b(H)$ .  $\square$

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